The braid group and the arc complex
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Contents

Introduction

We fix a point Q on the boundary ∂D of the closed unit disk D and a set S of n distinct points in the interior of D. An arc (based at Q) of $\Sigma = D \setminus S$ is a smooth injective path $\alpha\colon I \to D$ such that $\alpha(I) \cap \partial D = \{\alpha(0)\} = \{Q\}$ and $\alpha(I) \cap S = \{\alpha(1)\}$. We define the arc complex $\mathcal{A}$ as the simplicial complex whose q-simplices are $(q+1)$-tuples of homotopy classes of arcs of Σ intersecting only in Q.

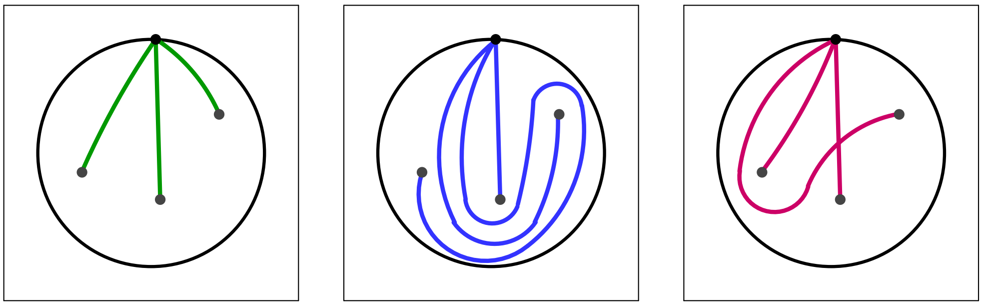


Figure 1: Three examples of triples of arcs in the case $n = 3$. The first two represent 2-simplices, while the third does not.

The present thesis has two principal results. The first is a combinatorial description of $\mathcal{A}$ in terms of the braid group B_n. The second can be resumed in the statement

Theorem (See 2.48)**.** *The geometric realization $|\mathcal{A}|$ of $\mathcal{A}$ is contractible.*

Hatcher and Wahl have shown [9, Proposition 7.2] that $|\mathcal{A}|$ is $(n-2)$-connected. This result is used by Ellenberg, Venkatesh and Westerland to prove instances of the Cohen-Lenstra conjecture over function fields [4]. In this thesis we analyse the topology of $|\mathcal{A}|$. In particular we present Hatcher and Wahl's proof that $\pi_i(|\mathcal{A}|) = 0$ for all $i \leq n-2$ providing more details. Moreover, we use the combinatorial description of $\mathcal{A}$ to strengthen that result and to show the above theorem.

The n-th braid group B_n is defined to be the fundamental group of the moduli space $\mathscr{C}$ parametrizing subsets of the open disk $\mathring{D}$ of cardinality n. Artin [2] has given explicit generators $\sigma_1, \ldots, \sigma_{n-1}$ of B_n and has shown that the group has a presentation with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all i and j with $|i - j| \geq 2$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for all $i \in \{1, \ldots, n-2\}$. We refer to Section 1.3 for more details.

Let $\mathscr{G}$ be the topological group of homeomorphisms of D to itself that fix the boundary ∂D point-wise. Let $\mathscr{H} \subset \mathscr{G}$ be the stabilizer of $S \subset D$. The mapping class group of Σ, denoted $\Gamma(\Sigma)$, is defined as $\pi_0(\mathscr{H})$.

In Chapter 1 we construct Artin's isomorphism between the braid group B_n and the mapping class group $\Gamma(\Sigma)$. The construction goes roughly as follows. First we show that the map

$$\mathscr{G} \to \mathscr{C},\, g \mapsto gS$$

is a fibration with fibre $\mathscr{H}$ (see Theorem 1.16). We then show that $\pi_i(\mathscr{G}) = 0$ for all i. The long exact sequence of homotopy groups then gives an isomorphism

$$B_n \cong \pi_1(\mathscr{C}) \to \pi_0(\mathscr{H}) \cong \Gamma(\Sigma).$$

The same result holds if we replace $\mathscr{G}$ and $\mathscr{H}$ by their subgroups $\mathscr{G}_d$ respectively $\mathscr{H}_d$ of *diffeomorphisms* of D.

Chapter 2 concerns the arc complex. If α is an arc of Σ and $h \in \mathscr{H}_d$, then the composition $h\alpha$ is also an arc of Σ. This induces a well defined action of B_n on $\mathcal{A}$. Studying this action we obtain the aforementioned combinatorial description of $\mathcal{A}$:

Theorem (See 2.20)**.** *Let H_q be the subgroup of B_n generated by $\{\sigma_{q+2}, \ldots \sigma_{n-1}\}$. The complex $\mathcal{A}$ is B_n-equivariantly isomorphic to the $(n-1)$-dimensional simplicial complex whose q-simplices are the left cosets of H_q in B_n and such that for every $b \in B_n$ the vertices of bH_q are*

$$bH_0,\, b\sigma_1^{-1}H_0,\, \ldots,\, b\sigma_q^{-1} \cdots \sigma_1^{-1}H_0.$$

In order to describe the homotopy type of $|\mathcal{A}|$ we give Hatcher and Wahl's proof that $\pi_j(|\mathcal{A}|) = 0$ for all $j \leq n-2$ which uses purely topological tools. Since the dimension of $\mathcal{A}$ is $n-1$, it follows from Hurewicz and Whitehead's theorems that in order to prove the contractibility of $|\mathcal{A}|$ it suffices to show $H_{n-1}(|\mathcal{A}|) = 0$. The combinatorial description of $\mathcal{A}$ allows us to give an explicit description of this homology group and with a direct computation we conclude that it is trivial.

1 | Braid groups and mapping class groups

The closed disk $D := \{z \in \mathbb{C} : |z| \leq 1\}$ is a compact subspace of $\mathbb{C}$ with the Euclidean topology. Its interior is denoted by $\overset{\circ}{D}$ and its boundary by ∂D. The symmetric group on n letters $\{1, \ldots, n\}$ is denoted by S_n. All the spaces of functions are endowed with the compact-open topology.

1.1 The configuration space and the braid group

Definition 1.1. Define the space $\mathscr{C}'$ to be

$$\mathscr{C}' = \mathscr{C}'_n = \{(P_1, \ldots, P_n) \mid P_i \in \overset{\circ}{D} \text{ for all } i \text{ and } P_i \neq P_j \text{ for all } i \neq j\}$$

with the topology induced by the product topology on D^n.

Definition 1.2. The *n-th configuration space* of D is given by the topological quotient space

$$\mathscr{C} = \mathscr{C}_n = \mathscr{C}'/S_n$$

where S_n acts on the right on $\mathscr{C}'$ permuting the n-points, i.e. for every $\sigma \in S_n$ the action is

$$\sigma(P_1, \ldots, P_n) = \left(P_{\sigma(1)} \ldots, P_{\sigma(n)}\right).$$

We identify the elements of $\mathscr{C}$ with subsets of $\overset{\circ}{D}$ of cardinality n.

Definition 1.3. Let X be a topological space on which a group G acts. G is said to *act freely and properly discontinuously* on X if given any point $x \in X$, there exists an open set U in X such that $x \in U$ and $g(U) \cap U = \emptyset$ for all $g \in G \setminus \{1\}$.

Proposition 1.4. *Let G be a group acting freely and properly discontinuously on a topological space X. Then the quotient map $q\colon X \to X/G$ is a covering map.*

Proof. We refer to [16, Proposition 4.20]. □

Corollary 1.5. *$\mathscr{C}'$ is a Galois covering of $\mathscr{C}$ with group S_n.*

Proof. The action of S_n on $\mathscr{C}'$ is free because if $\sigma(\mathcal{P}) = \tau(\mathcal{P})$ then necessarily $\sigma(i) = \tau(i)$ for all $i \in \{1, \dots, n\}$ and so $\sigma = \tau$. Moreover, since $\mathscr{C}'$ is a Hausdorff space and S_n is a finite group, the action is also properly discontinuous, hence Proposition 1.4 allows us to conclude. □

The following proposition describes the homotopy type of the spaces $\mathscr{C}'$ and $\mathscr{C}$.

Proposition 1.6. *Let $\mathcal{P} \in \mathscr{C}'$. Then we have $\pi_i(\mathscr{C}', \mathcal{P}) = 0$ and $\pi_i(\mathscr{C}, [\mathcal{P}]) = 0$ for every $i \neq 1$.*

Proof. Since $\mathscr{C}'$ is a covering of $\mathscr{C}$ we only need to check it for $\mathscr{C}'$. The proof is by induction on n, so we will stress the dependence on n in the notation using $\mathscr{C}'_n$ in place of $\mathscr{C}' \subset D^n$. The case $n = 1$ is clear since $\mathscr{C}'_1 = \overset{\circ}{D}$, so assume that $\pi_i(\mathscr{C}'_j, \mathcal{P}) = 0$ for every $j < n$ and $i \neq 1$. The map $\phi\colon \mathscr{C}'_n \to \mathscr{C}'_{n-1}$ defined by $(Q_1, \dots, Q_n) \to (Q_1, \dots, Q_{n-1})$ is a fibration whose fiber is homeomorphic to $\overset{\circ}{D} \setminus \{P_1, \dots, P_{n-1}\}$. For more details we refer to [5, Theorem 1.1]. Since $\overset{\circ}{D} \setminus \{P_1, \dots, P_{n-1}\}$ is homotopy equivalent to a bouquet of n circles its only non trivial homotopy group is the fundamental group. Thus the long exact sequence in homotopy groups implies that

$$\pi_i(\mathscr{C}'_n, \mathcal{P}) \to \pi_i(\mathscr{C}'_{n-1}, \phi(\mathcal{P}))$$

is an isomorphism for $i \neq 1, 2$ and

$$\pi_2(\mathscr{C}'_n, \mathcal{P}) \to \pi_2(\mathscr{C}'_{n-1}, \phi(\mathcal{P}))$$

is an injection. Since by induction hypothesis $\pi_i(\mathscr{C}'_{n-1}, \phi(\mathcal{P})) = 0$ for all $i \neq 1$ we can conclude that $\pi_i(\mathscr{C}'_n, \mathcal{P}) = 0$ for all $i \neq 1$. □

Remark 1.7. In particular Proposition 1.6 tells us that $\mathscr{C}'$ and $\mathscr{C}$ are path-connected, and from the fact that $\mathscr{C}' \to \mathscr{C}$ is a Galois covering we get the following exact sequence

$$0 \longrightarrow \pi_1(\mathscr{C}', \mathcal{P}) \longrightarrow \pi_1(\mathscr{C}, [\mathcal{P}]) \longrightarrow S_n \longrightarrow 0$$

Definition 1.8. The *n-th pure braid group* is $B'_n := \pi_1(\mathscr{C}', \mathcal{P})$, and the *n-th braid group* is $B_n := \pi_1(\mathscr{C}, [\mathcal{P}])$.

Notice that B_n and B'_n depend on the choice of the base point $\mathcal{P}$ and $[\mathcal{P}]$. However, since $\pi_0(\mathscr{C}) = 0 = \pi_0(\mathscr{C}')$ this dependence is only up to non canonical isomorphisms. With this terminology the short exact sequence above becomes

$$0 \longrightarrow B'_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 0$$

We give now a more geometrical interpretation of the braid group. From now on we consider fixed $\mathcal{P} = (P_1, \ldots, P_n) \in \mathscr{C}'$ and $S = \{P_1, \ldots, P_n\} \subset \overset{\circ}{D}$ corresponds to $[\mathcal{P}] \in \mathscr{C}$.

Definition 1.9. An *n-string* (based at $\mathcal{P}$) is a n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \colon I \to \overset{\circ}{D}$ such that

1. $\alpha_i(0) = P_i$;
2. $\alpha_i(1) = \sigma(P_i)$ for some $\sigma \in S_n$;
3. $\alpha_i(t) \neq \alpha_j(t)$ for all $i \neq j$ and $t \in I$.

Denote by $Str_{\mathcal{P}}$ the space of n-strings endowed with the compact-open topology.

The definition implies that the graphs of α_i and α_j seen as subsets of $D \times I$ are disjoint as long as $i \neq j$. We can then identify every n-string α with the union of the graphs of its components. In this way we can depict α as n disjoint paths from $\overset{\circ}{D} \times \{0\}$ to $\overset{\circ}{D} \times \{1\}$ as shown in Figure 1.1.

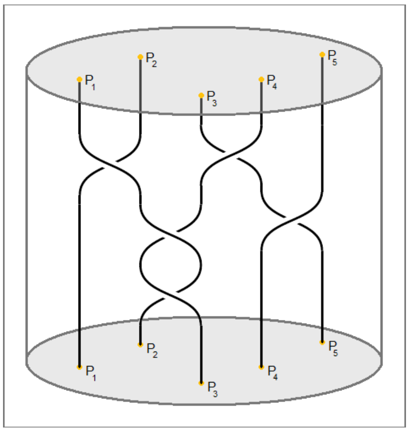


Figure 1.1: An example of 5-string.

Theorem 1.10. *The loop space $\Omega(\mathscr{C}, \mathcal{P})$ is homeomorphic to $Str_{[\mathcal{P}]}$.*

Proof. Notice that every path in $\mathscr{C}'$ is described as a n-tuple $(\alpha_i)_{i=1}^n$ with $\alpha_i \colon I \to \overset{\circ}{D}$ such that $\alpha_i(t) \neq \alpha_j(t)$ when $i \neq j$ and for all $t \in I$. Moreover every $\alpha' \in \Omega(\mathscr{C})$ is lifted to a unique $\alpha \colon I \to \mathscr{C}'$ such that $\alpha'(0) = \mathcal{P}$, and $\alpha'(1) = \sigma(\mathcal{P})$ for a necessarily unique $\sigma \in S_n$. Comparing this to the definition of $Str_{\mathcal{P}}$ we get the stated identification. □

In this way $Str_{\mathcal{P}}$ becomes an H-space where the composition of α and β can be described, as depicted in Figure 1.2, by putting the graphs of β_i under the graphs of α_i and then shrinking the height of the cylinder to the unitary interval I. In particular $\pi_0(Str_{\mathcal{P}})$ is a group.

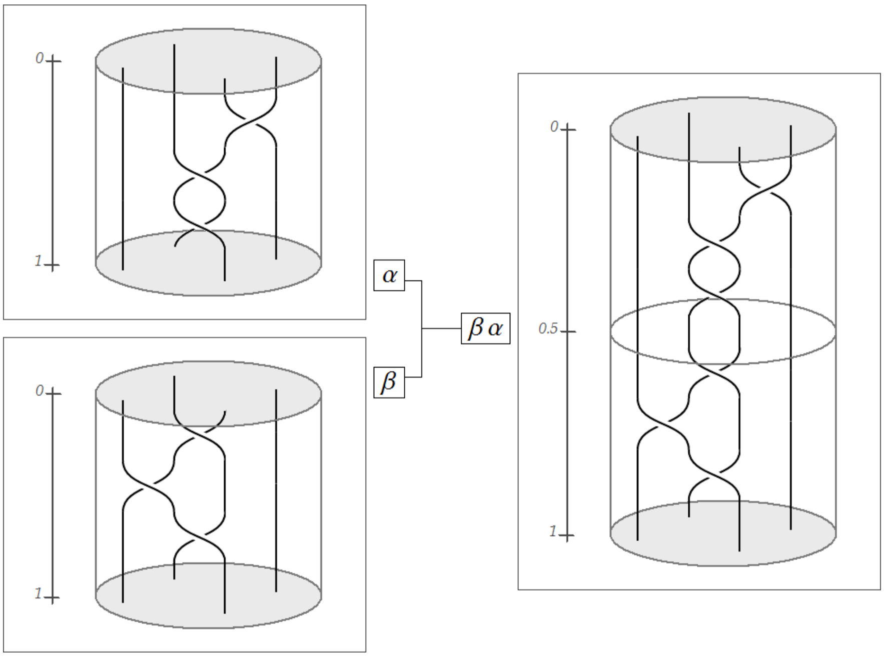


Figure 1.2: The graphical representation of the composition of two 4-strings.

Corollary 1.11. $B_n \cong \pi_0(Str_{\mathcal{P}})$. □

1.2 The fundamental isomorphism $B_n \cong \Gamma(\Sigma)$

Let $S = \{P_1, \dots, P_n\} \subseteq \mathring{D}$ correspond to $[\mathcal{P}] \in \mathscr{C}$. We define the space $\Sigma = D \setminus S$ with the topology induced by D.

Definition 1.12. The *mapping class group* of Σ is

$$\Gamma(\Sigma) := \pi_0(\text{Homeo}(\Sigma))$$

where $\text{Homeo}(\Sigma)$ is space of homeomorphisms $g \colon \Sigma \to \Sigma$ such that $g|_{\partial D} = \text{Id}_{\partial D}$.

The goal of this section is to construct an isomorphism between the braid group and the mapping class group.

Definition 1.13. Denote with $\mathscr{G}$ the topological space

$$\mathscr{G} = \{g \colon D \to D \mid g \text{ is a homeomorphism and } g|_{\partial D} = \text{Id}_{\partial D}\}$$

$\mathscr{G}$ has two important subspaces depending on the set S:

$$\mathscr{H}' := \{g \in \mathscr{G} \mid g(P) = P \text{ for all } P \in S\}$$

$$\mathscr{H} := \{g \in \mathscr{G} \mid g(S) = S\}$$

Remark 1.14. Notice that since D is metric and compact, the compact-open topology in $\mathscr{G}$ coincides with the one induced by the distance $d_{\mathscr{G}}$ defined, for every $f, g \in \mathscr{G}$ as:

$$d_{\mathscr{G}}(f,g) = \sup_{P \in D} |f(P) - g(P)| = \max_{P \in D} |f(P) - g(P)|$$

The space $\mathscr{G}$ has a natural structure of group given by the composition. Since D satisfies the properties of the proposition below we can conclude that it is a topological group.

Proposition 1.15. *Let X be a compact and Hausdorff topological space. Then the group G of homeomorphisms from X to X with the compact-open topology is a topological group.*

Proof. We refer to [1, Theorem 3]. □

The group $\mathscr{G}$ acts on $\mathscr{C}'$ in the obvious way

$$\mathscr{G} \times \mathscr{C}' \to \mathscr{C}',\ (g, (Q_1, \dots, Q_n)) \mapsto (g(Q_1), \dots, g(Q_n))$$

and the induced action on $\mathscr{C}$

$$\mathscr{G} \times \mathscr{C} \to \mathscr{C},\ (g, [(Q_1, \dots, Q_n)]) \mapsto [(g(Q_1), \dots, g(Q_n))]$$

is well defined.

The main results in order to exhibit an isomorphism between B_n and $\Gamma(\Sigma)$ are the following statements.

Theorem 1.16. *The map $\epsilon_{[\mathcal{P}]} \colon \mathscr{G} \to \mathscr{C}$ defined as $g \mapsto g([\mathcal{P}])$ is a fibration with fibre $\mathscr{H}$.*

Corollary 1.17. *The connecting map $\delta \colon \pi_1(\mathscr{C}) \to \pi_0(\mathscr{H})$ is an isomorphism.*

Remark 1.18. The isomorphism in Corollary 1.17 is a group isomorphism. Indeed even if usually π_0 is only a pointed set, in this case the group structure on $\mathscr{G}$ induces a group structure on $\pi_0(\mathscr{G}, Id)$, and the same holds for $\pi_0(\mathscr{H}, Id)$. The multiplication μ and the inverse ι are defined as

$$\mu \colon \pi_0(\mathscr{G}, Id) \times \pi_0(\mathscr{G}, Id) \to \pi_0(\mathscr{G}, Id),\ ([f],[g]) \mapsto [fg]$$

$$\iota \colon \pi_0(\mathscr{G}, Id) \to \pi_0(\mathscr{G}, Id),\ [f] \mapsto [f^{-1}]$$

Lemma 1.19. *Every homeomorphism $f \colon D \setminus \{0\} \to D \setminus \{0\}$ can be extended to a unique homeomorphism $f^{ext} \colon D \to D$.*

Proof. Since f^{ext} must agree with f on $D \setminus \{0\}$ we can only set $f^{ext}(0) = 0$. Let U be an open subset of D. If $U \subseteq D \setminus \{0\}$, then ${f^{ext}}^{-1}(U) = f^{-1}(U)$ which is an open. Otherwise $U = \{0\} \sqcup (D \setminus \{0\} \setminus K)$ where K is a compact of $D \setminus \{0\}$. Then

$${f^{ext}}^{-1}(\{0\} \sqcup (D \setminus \{0\} \setminus K)) = \{0\} \sqcup f^{-1}(D \setminus \{0\} \setminus K) = \{0\} \sqcup \left(D \setminus \{0\} \setminus f^{-1}(K)\right)$$

and since f is a homeomorphism we have that $f^{-1}(K)$ is a compact subset of $D \setminus \{0\}$, hence we proved the continuity. □

Proposition 1.20. *The map $res\colon \mathscr{H} \to Homeo(\Sigma)$ which restricts h to Σ is an isomorphism of groups. Moreover we have the isomorphism $\pi_0(res)\colon \pi_0(\mathscr{H}) \cong \pi_0(Homeo(\Sigma))$.*

Proof. Let $D_i := \{Q \in D \mid |Q - P_i| \leq \epsilon\}$ with $\epsilon > 0$ such that $D_i \subseteq \overset{\circ}{D}$ for all $i \in \{1, \ldots, n\}$ and $D_i \cap D_j = \emptyset$ for $i \neq j$. Use the notation $\Sigma_i := D_i \setminus \{P_i\}$. By definition each Σ_i is homeomorphic to a closed disk without the origin. Let $f \in \text{Homeo}(\Sigma)$ and since it is a homeomorphism $f(\Sigma_i)$ is homeomorphic to the closed disk without the origin. For every $i \in \{1, \ldots, n\}$ we can apply Lemma 1.19 to $f|_{\Sigma_i}$ and hence we find a unique extension of $f|_{\Sigma_i}$ to ${f|_{\Sigma_i}}^{ext}$. The maps glue to a homeomorphism $ext(f)\colon D \to D$ which extends f. It follows that $ext\colon \text{Homeo}(\Sigma) \to \mathscr{H}$ which map f to $ext(f)$ realizes the inverse of res. It is only a matter of computation to check that this bijection is a group isomorphism.

Since res is continuous the map $\pi_0(res)\colon \pi_0(\mathscr{H}) \to \Gamma(\Sigma)$ is well defined and a homomorphism.

Let $M\colon I \times \Sigma \to \Sigma$ be a continuous map such that for every $t \in I$ we have $M_t \in \text{Homeo}(\Sigma)$. It follows that M_t is extended to $ext(M_t)$. We define

$$ext(M)\colon I \times D \to D,\ (t, Q) \mapsto ext(M_t)(Q).$$

For every $t \in I$ the map $ext(M)(t, -)$ is continuous since it coincides with $ext(M_t)$. Let $P \in D$ be fixed and let $t \in I$ vary. When $P \in \Sigma$ the path $ext(M)(-, P)$ is continuous since it coincides with $M(-, P)$. If $P \in S$ we have that $ext(M)(-, P)$ is the constant path at $ext(M)(0, P)$, so it is still continuous. We can conclude that $\pi_0(res)$ has an inverse and hence it is a group isomorphism. □

We can then conclude that the following theorem holds.

Theorem 1.21. *There exists an isomorphism between B_n and $\Gamma(\Sigma)$ given by the composition of the isomorphisms δ and $\pi_0(res)$:*

$$B_n = \pi_1(\mathscr{C}) \overset{\delta}{\cong} \pi_0(\mathscr{H}, Id) \overset{\pi_0(res)}{\cong} \pi_0(Homeo(\Sigma), Id) = \Gamma(\Sigma)$$

The last part of this section contains the proofs of Theorem 1.16 and Corollary 1.17.

Lemma 1.22. *There exists a continuous map $h\colon \overset{\circ}{D} \to \mathscr{G}$, $Q \mapsto h_Q$ such that $h_Q(0) = Q$ for all $Q \in \overset{\circ}{D}$.*

Proof. For each $Q \in D$ define h_Q to be the map

$$h_Q\left(\alpha e^{i\theta}\right) = \alpha e^{i\theta} - (\alpha - 1)Q$$

which is a continuous bijection of D which fixes the boundary. The inverse can be computed explicitly in a similar way, interchanging the roles of O and Q, so h_Q belongs to $\mathscr{G}$. □

In the case $n = 1$, the theorem above implies that the action of $\mathscr{G}$ on $\mathscr{C}'$ is transitive. The following theorem gives a similar result for the general case.

Proposition 1.23. *For all $\mathcal{P} \in \mathscr{C}'$, there exists a neighbourhood U of $\mathcal{P}$ and a continuous map $F\colon U \to \mathscr{G}$ such that for all $\mathcal{Q} \in U$ we have $F(\mathcal{Q})(\mathcal{P}) = \mathcal{Q}$. Moreover the action of $\mathscr{G}$ on $\mathscr{C}'$ is transitive.*

Proof. (Sketch) We refer to the proof of [2, Theorem 6], for more details.

Use the Hausdorff property of D to find non intersecting disks $D_i \subseteq \overset{\circ}{D}$ such that $P_i \in D_i$ for each component P_i of $\mathcal{P}$. Define $U := \prod D_i$ and for all $\mathcal{Q} = (Q_1, \ldots, Q_n) \in U$, define the map $F(\mathcal{Q})$ to be the identity on $D \setminus \cup D_i$. For the other points the definition of $F(\mathcal{Q})$ reduces to the case $n = 1$ since the disks are disjoint and $\mathscr{C}'$ has the product topology. Lemma 1.22 allows us to conclude since the boundary of the disks is fixed and hence the definitions glue.

Let $\mathcal{P}_1$ and $\mathcal{P}_2 \in \mathscr{C}'$, we need to find a map $F \in \mathscr{G}$ such that $F(\mathcal{P}_1) = \mathcal{P}_2$. Since $\mathscr{C}'$ is path connected there exists a path $\alpha\colon I \to \mathscr{C}'$ such that $\alpha(0) = \mathcal{P}_1$ and $\alpha(1) = \mathcal{P}_2$. The compactness of $\alpha(I)$ allows us to find $r > 0$ such that $\overline{B_r(\alpha_i(t))} \cap \overline{B_r(\alpha_j(s))} = \emptyset$ for all $i \neq j$ and $s, t \in I$. Call $U_t = \prod_{i=1}^{n} B_r(\alpha_i(t))$ and $\{\widetilde{U}_{t_j} := \alpha(I) \cap U_t\}_{t \in I}$ is a covering of $\alpha(I)$. Using again the compactness of $\alpha(I)$ we can find a finite subset $J = \{t_0, \ldots, t_m\}$ of I such that $t_0 = 0$ and $\alpha(t_j) \in \widetilde{U}_{t_j} \cap \widetilde{U}_{t_{j-1}}$ for all $j \in \{1, \ldots, m\}$.

For all $j \in \{0, \ldots, m-1\}$ let $F_j \in \mathscr{G}$ be the map such that $F_j(\alpha(t_j)) = \alpha(t_{j+1})$ and $F_m(\alpha(t_m)) = \mathcal{P}_2$. Notice that the existence of such maps is guaranteed by the first part of the theorem. Define $F := F_m \cdots F_0$ and by construction it satisfies $F(\mathcal{P}_1) = \mathcal{P}_2$. □

Proposition 1.24. *$\mathscr{H}$ and $\mathscr{H}'$ are closed subgroups of $\mathscr{G}$. Moreover $\mathscr{H}'$ is normal in $\mathscr{H}$ with $\mathscr{H} / \mathscr{H}' \cong S_n$ as discrete topological groups. The canonical projection map $\rho\colon \mathscr{G} / \mathscr{H}' \to \mathscr{G} / \mathscr{H}$ is a Galois covering with group S_n.*

Proof. It is clear that $\mathscr{H}$ and $\mathscr{H}'$ are subgroups of $\mathscr{G}$. For the closedness consider the continuous maps

$$\epsilon_{\mathcal{P}} \colon \mathscr{G} \to \mathscr{C}', g \mapsto g(\mathcal{P}) \quad \text{and} \quad \epsilon_{[\mathcal{P}]} \colon \mathscr{G} \to \mathscr{C}, g \mapsto g[\mathcal{P}]$$

We deduce that $\mathscr{H}'$ and $\mathscr{H}$ are closed since they are the inverse images of the closed points $\mathcal{P}$ and $[\mathcal{P}]$.

To prove the normality notice that $\mathscr{H}$ acts continuously on S and $\mathscr{H}'$ is the kernel of the action.

In the quotient $\mathscr{H}/\mathscr{H}'$ two elements h and g are the same if and only if h and g act in the same way on the elements of S. The map ϕ which associates h to the unique $\sigma \in S_n$ such that $h(P_i) = P_{\sigma(i)}$ is a well defined group homomorphism which is injective by definition of $\mathscr{H}'$. For each $\sigma \in S_n$ Proposition 1.23 exhibits the map $F(\sigma(\mathcal{P}))$ as a preimage of such permutation, so ϕ turns out to be surjective. Endowing S_n with the discrete topology this open bijection becomes continuous since $\mathscr{H}'$ is closed in $\mathscr{G}$ and hence in $\mathscr{H}$. It follows that the spaces are homeomorphic.

The projection ρ corresponds to the quotient by $\mathscr{H}/\mathscr{H}' \cong S_n$. Since $\mathscr{G}$ is Hausdorff and S_n is finite and acts freely we conclude that it is a Galois covering. □

Proposition 1.25. *$\mathscr{G}/\mathscr{H}' \cong \mathscr{C}'$ and $\mathscr{G}/\mathscr{H} \cong \mathscr{C}$.*

Proof. Notice that the transitivity of the action of $\mathscr{G}$ on $\mathscr{C}'$ implies that $\mathscr{G}$ acts transitively also on $\mathscr{C}$. The orbit-stabilizer theorem gives then the continuous bijections:

$$\mathscr{C}' \cong \mathscr{G}/\mathrm{Stab}(\mathcal{P}) \quad \text{and} \quad \mathscr{C} \cong \mathscr{G}/\mathrm{Stab}([\mathcal{P}])$$

Proposition 1.23 guarantees that those bijections are homeomorphisms and since by definition $\mathrm{Stab}(\mathcal{P}) = \mathscr{H}'$ and $\mathrm{Stab}([\mathcal{P}]) \cong \mathscr{H}$ we are done. □

The homotopy type of $\mathscr{G}$ is completely determined by the following statement.

Proposition 1.26 (Alexander's trick)**.** *$\pi_k(\mathscr{G}, Id) = 0$ for all $k \geq 0$.*

Proof. The proof generalizes the one given in [6] from $k = 1$ to $k \geq 0$. We prove that any continuous map $\alpha \colon (I^k, \partial I^k) \to (\mathscr{G}, \mathrm{Id}_D)$ is homotopy equivalent to the map $k_{\mathrm{Id}} \colon I^k \to \mathscr{G}$ with constant value Id_D throughout maps sending the boundary of I^k to Id_D. Such a homotopy is given by a map $H \colon I \times I^k \times D \to D$ defined as

$$H(s, \underline{t})(P) = \begin{cases} (1-s)\alpha(\underline{t})\left(\dfrac{P}{1-s}\right) & \text{if } 0 \leq |P| < 1-s \\ P & \text{if } 1-s \leq |P| \leq 1 \\ P & \text{if } s = 1 \end{cases}$$

The map is continuous in each interval of definition. We need to check that the continuity in $s = 1$. When s tends to 1^- we see that $|P|$ tends to $s-1$. Write $P = |P|e^{i\pi\theta}$ for some θ. Hence we have that $(1-s)\alpha(t)\left(|P|e^{i\pi\theta}/(1-s)\right)$ tends to $(1-s)\alpha(t)\left(e^{i\pi\theta}\right)$. Since $e^{i\pi\theta} \in \partial D$ we have that $(1-s)\alpha(t)\left(e^{i\pi\theta}\right) = (1-s)e^{i\pi\theta} = P$. It follows that H is continuous everywhere. It is clear by definition that $H(0,\underline{t})(P) = \alpha(\underline{t})P$ and $H(1,t)(P) = P$. Moreover, fixing s and t, we can see that $H(s,\underline{t})$ is actually a homeomorphism because it is bijective and $I \times I^k \times D$ is compact and D Hausdorff. Moreover, when $\underline{t} \in \partial I^k$ we have that $\alpha(\underline{t}) = \mathrm{Id}$. This yields that $H|_{I\times\partial I^k\times D}$ is the identity, so H realizes the wanted homotopy. □

Proof of Theorem 1.16. Thanks to the identification $\mathscr{C} \cong \mathscr{G}/\mathscr{H}$ we need to prove that the projection map $\mathscr{G} \to \mathscr{G}/\mathscr{H}$ is a fibration. Since the map ρ is a covering with finite fibre, it is enough to show that the projection map $\mathscr{G} \to \mathscr{G}/\mathscr{H}'$ is a fibration with fiber $\mathscr{H}'$. We noticed earlier that $\mathscr{H}'$ is a closed subspace of $\mathscr{G}$ thus, according to [16, Theorem 4.13], it is sufficient to prove that the projection map $p\colon \mathscr{G} \to \mathscr{G}/\mathscr{H}'$ has enough local sections. This means that for every $g\mathscr{H}' \in \mathscr{G}/\mathscr{H}'$ there exists a neighbourhood U of $g\mathscr{H}'$and a map $s\colon U \to \mathscr{G}$ such that $ps\colon U \to \mathscr{G} \to \mathscr{G}/\mathscr{H}'$ is the identity on U. Thanks to Proposition 1.25 the projection corresponds to the map $\epsilon_{\mathcal{P}}\colon \mathscr{G} \to \mathscr{C}'$ which associates to g the element $g(\mathcal{P})$. Proposition 1.23 exhibits the existence of such sections, hence the theorem is proved. □

Proof or Corollary 1.17. Since $\epsilon_{[\mathcal{P}]}$ is a fibration we obtain the long exact sequence of homotopy groups

$$\cdots \to \pi_1(\mathscr{G},\mathrm{Id}) \to \pi_1(\mathscr{C},[\mathcal{P}]) \xrightarrow{\delta} \pi_0(\mathscr{H},\mathrm{Id}) \to \pi_0(\mathscr{G},\mathrm{Id}) \to \dots$$

hence we get

$$\cdots \to \pi_1(\mathscr{G},\mathrm{Id}) \to B_n \xrightarrow{\delta} \pi_0(\mathscr{H},\mathrm{Id}) \to \pi_0(\mathscr{G},\mathrm{Id}) \to \dots$$

Proposition 1.26 states that $\pi_0(\mathscr{G},\mathrm{Id}) = 0 = \pi_1(\mathscr{G},\mathrm{Id})$, so we can conclude that δ is an isomorphism. □

Remark 1.27. In particular Alexander's trick shows that $\mathscr{G}$ is path connected. Moreover for every $h \in \mathscr{H}$ there exists a continuous map $\alpha\colon I \times D \to D$ such that $\alpha(0,P) = P$ and $\alpha(1,P) = h(P)$. It follows that it is possible to give a graphical representation of h as the union of the graphs Γ_P of $\alpha(-,P)$ for all $P \in D$. Let Γ be an n-string corresponding to $\delta^{-1}h$ viewed as subspace of $D \times I$. The explicit construction of δ guarantees that α can be chosen such that $\Gamma = \bigcup_{P_i\in S}\Gamma_{P_i}$.

Remark 1.28. Let $\mathscr{H}_d$ be the subgroup of $\mathscr{H}$ of diffeomorphisms of D. As proved in [6, §2.1] the inclusion $\mathscr{H}_d \subset \mathscr{H}$ induces the group isomorphism $\pi_0(\mathscr{H},\mathrm{Id}) \cong$

$\pi_0(\mathscr{H}_d, \mathrm{Id})$. It follows that B_n is also isomorphic to $\pi_0(\mathscr{H}_d, \mathrm{Id})$. This smooth version will be used in the second chapter. Notice that Proposition 1.20 cannot be extended to the differential case because there are diffeomorphisms of Σ that cannot be extended to diffeomorphisms of D.

1.3 The action of B_n on the fundamental group of Σ

Let $\mathcal{P} = (P_1, \dots, P_n) \in \mathscr{C}'$, and let Q be a fixed point belonging to ∂D. Since Σ can be retracted to a bouquet of n circles $\pi_1(\Sigma, Q)$ is a free group on n generators. By definition $\mathscr{H}$ fixes S, thus the group acts on $\Sigma = D \setminus S$. Moreover this action induces an action of $\mathscr{H}$ to $\Omega(\Sigma, Q)$, the loop space of Σ with preferred point Q because the preferred point Q is fixed. The action

$$\mathscr{H} \times \Omega(\Sigma, Q) \to \Omega(\Sigma, Q),\ (h, \alpha) \mapsto h\alpha \colon t \to h(\alpha(t))$$

induces

$$\pi_0\left(\mathscr{H} \times \Omega(\Sigma, Q)\right) \to \pi_0\left(\Omega(\Sigma, Q)\right)$$

and since π_0 commutes with finite products this is

$$\pi_0(\mathscr{H}) \times \pi_0\left(\Omega(\Sigma, Q)\right) \to \pi_0\left(\Omega(\Sigma, Q)\right)$$

which defines an action of B_n on $\pi_1(\Sigma, Q)$.

The aim of this section is to describe in a combinatorial way this action. For this purpose we define generators of B_n and $\pi_1(\Sigma, Q)$ we can easily work with.

Recall that $D \subset \mathbb{C}$. Assume that $Q = 1$, and also

$$P_j = \frac{n+1-2j}{n+1} \cdot i \in \mathbb{C}$$

so that P_k, P_j and Q are not collinear when $k \neq j$.

We define the loops $\gamma_j \colon I \to \Sigma$ to be

$$\gamma_j(t) = \begin{cases} 3t(P_j + \delta) + (1-3t)Q & 0 \leq t \leq 1/3 \\ P_j + \delta e^{i2\pi(3t-1)} & 1/3 \leq t \leq 2/3 \\ (3t-2)Q + (3-3t)(P_j+\delta) & 2/3 \leq t \leq 1 \end{cases}$$

with $\delta > 0$ such that $\gamma_k(t) \neq \gamma_j(s)$ for all $t, s \in (0,1)$ and $k \neq j$.

Since each P_j is encircled by exactly one loop γ_j the n-tuple

$$\gamma := ([\gamma_1], \dots, [\gamma_n])$$

is a basis for $\pi_1(\Sigma, Q)$ which is called the *standard system of generators*.

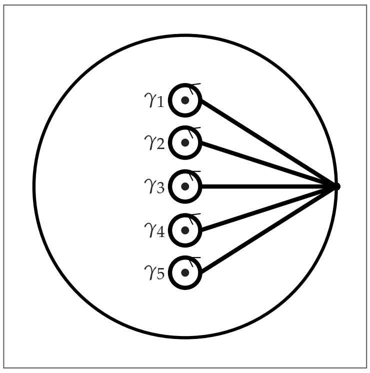


Figure 1.3: The standard system of generators in the case $n = 5$.

Definition 1.29. For all $k \in \{1, \ldots, n-1\}$ we define the element $\sigma_k \in B_n$ as the n-string whose j-th component is

$$(\sigma_k)_j(t) = \begin{cases} \dfrac{P_j + P_{j+1}}{2} + i\dfrac{1}{n+1}e^{i\pi t} & \text{if } j = k \\ \dfrac{P_j + P_{j+1}}{2} - i\dfrac{1}{n+1}e^{i\pi t} & \text{if } j = k+1 \\ P_j & \text{otherwise} \end{cases}$$

Theorem 1.30. *The group B_n has $\sigma_1, \ldots, \sigma_{n-1}$ as generators and has a presentation with relations*

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all i and j with $|i - j| \geq 2$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for all $i \in \{1, \ldots, n-2\}$.

Proof. Two proofs can be found in [3, Theorem 1.8] or [2, Theorem 16]. □

Example 1.31. The Figures 1.4, 1.5 and 1.6 show some generators and relations of B_n in terms of n-strings in the case $n = 5$. The composition is from the top to the bottom.

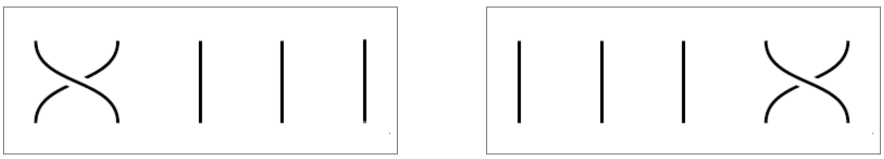


Figure 1.4: The generators σ_1 and σ_4.

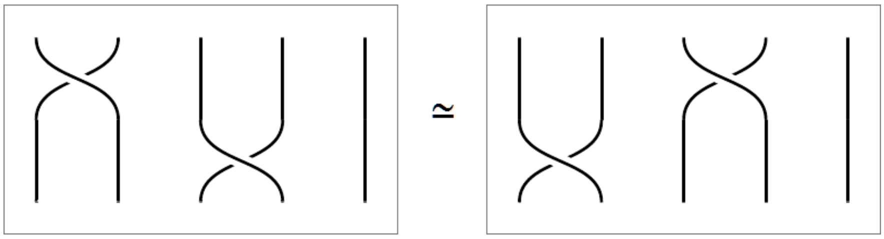


Figure 1.5: The relation $\sigma_3\sigma_1 = \sigma_1\sigma_3$.

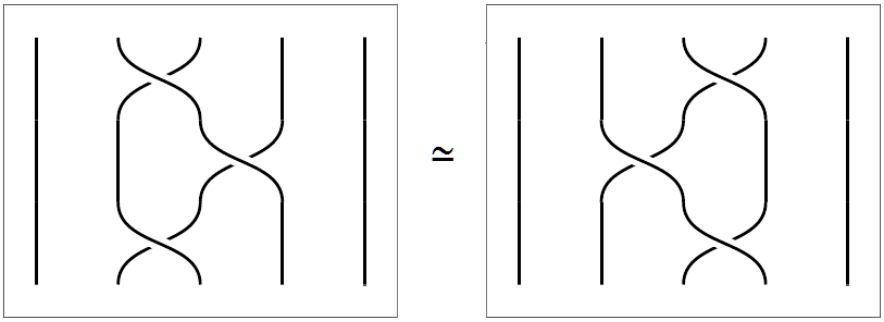


Figure 1.6: The relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$.

Theorem 1.32. *The action of $\pi_0(Homeo(\Sigma)) \simeq B_n$ on $\pi_1(\Sigma)$ satisfies:*

$$\sigma_j[\gamma_i] = \begin{cases} [\gamma_i] & \text{if } i \neq j, j+1 \\ [\gamma_j^{-1}\gamma_{j+1}\gamma_j] & \text{if } i = j \\ [\gamma_j] & \text{if } i = j+1 \end{cases}$$

Proof. By definition σ_j acts trivially on the P_i's for all $i \neq j, j+1$. So we can assume that the corresponding element in $\Gamma(\Sigma)$ acts in a non trivial way only on a connected neighbourhood of the loops γ_j and γ_{j+1} which is homeomorphic to the disk D.

In this way the proof is reduced to the case $n = 2$. The presentation of B_n shows that B_2 is the free group generated by σ_1, while the fundamental group of Σ is freely generated by γ_1 and γ_2. We are left to prove that

$$\sigma_1([\gamma_1]) = [\gamma_1^{-1}\gamma_2\gamma_1] \quad \text{and} \quad \sigma_1([\gamma_2]) = [\gamma_1].$$

Assuming $\delta < 1/4$ the action of σ_1 on the points P_i can be extended to the whole disk by the following homeomorphism:

$$s_1(\alpha e^{i\vartheta}) = \begin{cases} \alpha e^{i(\vartheta+\pi)} = -\alpha e^{i\vartheta} & 0 \leq \alpha \leq 3/4 \\ e^{i\vartheta} \cdot \left(\frac{1}{8} + \frac{7}{8} \cdot e^{4i\pi(1-\alpha)}\right) & 3/4 < \alpha \leq 1 \end{cases}$$

Using the explicit descriptions of the loops γ_i and of s_1 one can deduce the stated action.

Remark 1.27 showed how to give a graphical representation of the elements of $\mathscr{H}$. We can then use the following picture to convince ourselves of the truthfulness of the statement.

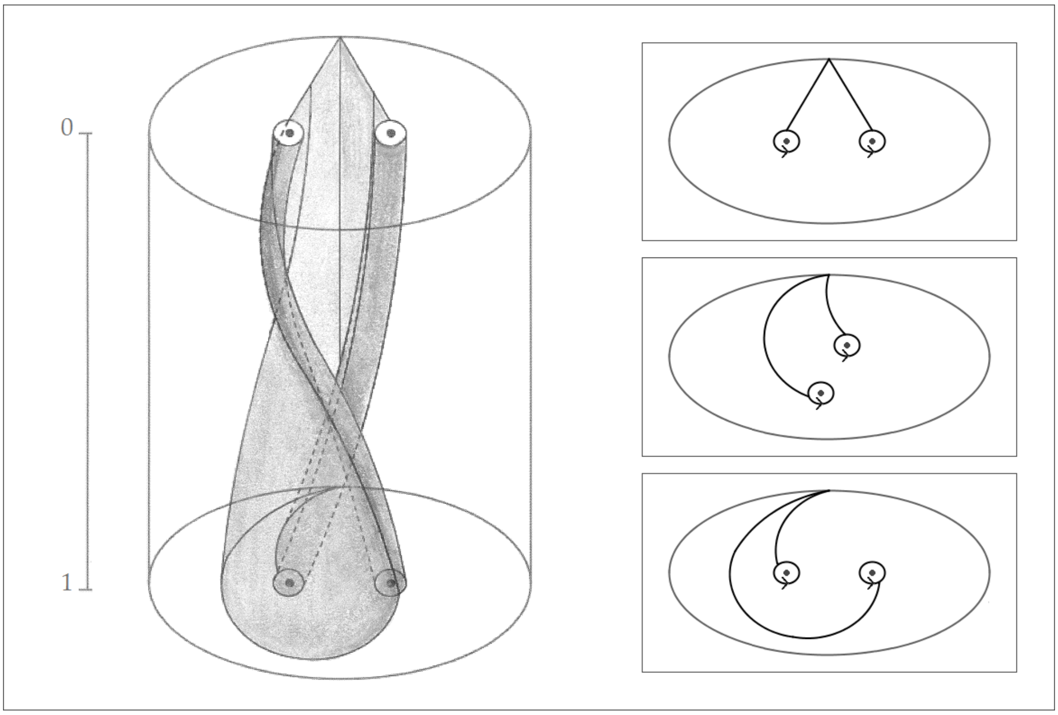


Figure 1.7: The graphical representation of s_1 restricted to the images of γ_1 and γ_2 together with the sections of the path connecting Id to s_1 at levels 0, 0.5 and 1.

□

2 | The arc complex

As in Section 1.3 the setting consists of a fixed point $Q \in \partial D$ and a set S of n distinct points $P_i \in \overset{\circ}{D}$ which defines the punctured disk $\Sigma = D \setminus S$.

2.1 The geometric definition

According to [15] we give the following definition of simplicial complex.

Definition 2.1. A *simplicial complex* is a collection $\mathcal{C}$ of finite non-empty sets, such that if A is an element of $\mathcal{C}$, so is every non-empty subset of A.

Definition 2.2. We say that $A \in \mathcal{C}$ is a *q-simplex* and has *dimension* q if it has $q+1$ elements. The set of all q-simplices is denoted by $\mathcal{C}_q$. The 0-simplices are also called vertices.

Definition 2.3. Let $n \in \mathbb{N}$. The simplicial complex $\mathcal{C}$ has dimension n if $\mathcal{C}_q = \emptyset$ for all $q > n$ and $\mathcal{C}_n \neq \emptyset$.

Definition 2.4. We say that $\mathcal{C}$ is spanned by $\mathcal{C}_0$ if A is an element of $\mathcal{C}$ for every non-empty $A \subseteq \mathcal{C}_0$.

Definition 2.5. An *arc* of Σ is a smooth and injective map $\alpha\colon I \to D$ such that

1. $\alpha(0) = Q$;
2. $\alpha(1) \in S$;
3. $\alpha(t) \in \overset{\circ}{\Sigma}$ for all $t \in (0,1)$.

Denote with *Arc* the topological space of all the arcs of Σ.

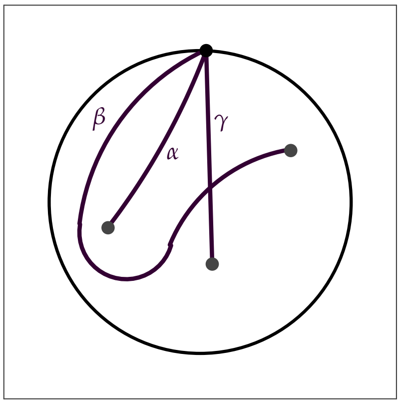
Definition 2.6. Two arcs α and β are *isotopic* if $[\alpha] = [\beta]$ as elements of $\pi_0(Arc)$.

Definition 2.7. Let $a_0, \dots, a_q$ be $q+1$ distinct elements of $\pi_0(Arc)$. We say that they are *non intersecting* if there exist representatives $\alpha_0, \dots, \alpha_q$ such that $\alpha_i(I) \cap \alpha_j(I) = Q$ for all i and j with $i \neq j$.

Definition 2.8. The *arc complex* $\mathcal{A}$ is the simplicial complex whose set of vertices is $\mathcal{A}_0 = \pi_0(Arc)$ and whose q-simplices are subsets $A \subseteq \mathcal{A}_0$ of $q+1$ non intersecting isotopy classes of arcs.

From the definition it follows that the set of vertices of $\mathcal{A}$ is $\mathcal{A}_0$, but $\mathcal{A}$ is not necessarily the complex spanned by $\mathcal{A}_0$ since in $\mathcal{A}_q$ we require non intersecting conditions. Since $\mathcal{A}_q = \emptyset$ for $q \geq n$ and $\mathcal{A}_{n-1} \neq \emptyset$ the dimension of $\mathcal{A}$ is equal to $n-1$.

Example 2.9. Let $n = 3$. The figure below represents three arcs of Σ.



The sets $\{[\alpha], [\beta]\}$ and $\{[\alpha], [\gamma]\}$ are elements of $\mathcal{A}_1$, while $\{[\beta], [\gamma]\}$ is not since every pair of representatives of $([\beta], [\gamma])$ intersect.

Remark 2.10. Suppose that the q-simplex A is represented by both the sets of arcs $\{\alpha_0, \dots, \alpha_q\}$ and $\{\beta_0, \dots, \beta_q\}$ with the property that $\alpha_i(I) \cap \alpha_j(I) = Q$ and $\beta_i(I) \cap \beta_j(I) = Q$ for all $i \neq j$. After reordering we can assume that for all $i \in \{0, \dots, q\}$ there exists a continuous map $H_i \colon I \to Arc$ such that $H_i(0) = \alpha_i$ and $H_i(1) = \beta_i$. Moreover we can assume that for all $t \in I$ the set of arcs $\{H_0(t), \dots, H_q(t)\}$ represents the q-simplex S.

Remark 2.11. Let $A \in \mathcal{A}_q$. We define an order relation on A in the following way. Let $a, b \in A$ and α and β be representatives of a respectively b with the property that $\alpha(I) \cap \beta(I) = Q$. We say that $a < b$ if and only if there exists an $\epsilon \in I$ such that for all $t \in (0, \epsilon)$ the set $\alpha([0, t])$ is before the set $\beta([0, t])$ according to the counter-clockwise order around Q. Thanks to Remark 2.10 the order does not depend on the choice of the representatives, so it is well defined. It follows that we can associate to every element $A \in \mathcal{A}_q$ an ordered $(q+1)$-tuple $(a_0, \dots, a_q)$ of elements of $\mathcal{A}_0$ such that $a_i < a_j$ if and only if $i < j$.

2.2 The action of B_n

We denote by $\mathscr{G}_d$ the group of diffeomorphisms of D which fix the boundary pointwise, and $\mathscr{H}_d$ its subgroup which stabilizes the set S. As stated in Remark 1.28 there is a canonical isomorphism $B_n \cong \pi_0(\mathscr{H}_d)$. The group $\mathscr{H}_d$ acts on the left on *Arc* by composition:

$$\mathscr{H}_d \times Arc \to Arc,\ (h, \alpha) \mapsto h\alpha$$

and by applying the functor π_0 this induces an action of $\pi_0(\mathscr{H}) \cong B_n$ on $\mathcal{A}_0$. The action on the vertices induces an action on the simplicial complex since the elements of $\mathscr{H}_d$ preserve the non-intersecting condition.

The action of B_n on $\mathcal{A}$ gives us a way to describe the arc complex in a combinatorial way.

Theorem 2.12. *B_n acts transitively on $\mathcal{A}_q$ for all q.*

Lemma 2.13. *The space of injective smooth paths $\alpha\colon I \to D$ such that $\alpha(0) = Q$ and $\alpha(t) \in \mathring{D}$ for all $t \neq 0$ is path connected.*

It may be intuitively clear that the lemma holds, but for completeness we give a proof.

Proof. Let α and β satisfy the hypothesis of the lemma. We need to find a homotopy $H\colon I \times I \to D$ such that $H(0,t) = \alpha(t)$, $H(1,t) = \beta(t)$ and for all $s \in I$ the path $H(s,-)$ is smooth, injective and such that $H(s,0) = Q$ and $H(s,t) \in \mathring{D}$ for all $t \neq 0$. A priori the map

$$t \mapsto (1-s)\alpha(t) + s\beta(t)$$

is not injective. However it is enough to determine an $\epsilon \in I$ such that

$$t \mapsto K(s,t) := (1-s)\alpha(\epsilon t) + s\beta(\epsilon t)$$

is injective for all $s \in I$. Indeed for such an ϵ the homotopies

$$\begin{aligned} F\colon I \times I \to D, &\qquad (s,t) \mapsto \alpha\left((1-s(1-\epsilon))t\right) \\ K\colon I \times I \to D, &\qquad (s,t) \mapsto (1-s)\alpha(\epsilon t) + s\beta(\epsilon t) \\ G\colon I \times I \to D, &\qquad (s,t) \mapsto \beta\left((\epsilon + s(1-\epsilon))t\right) \end{aligned}$$

are injective for all $s \in I$ and smooth in t. Moreover the map H given assembling those homotopies is still smooth and realizes the wanted homotopy.

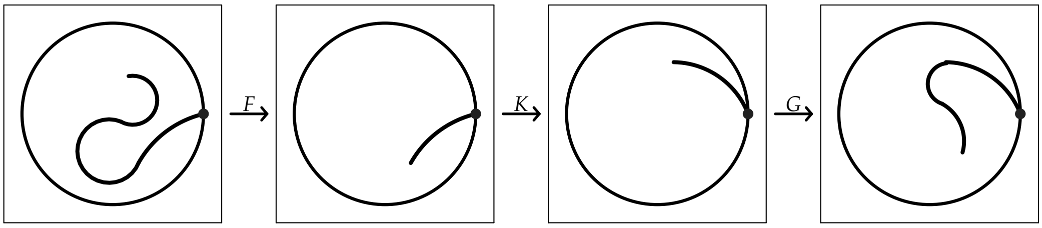


Figure 2.1: The representation of H as composition of A, K and B.

We are left to prove that such ϵ exists. We can assume without loss of generality that $Q = 1$ and we can write $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta(t)$ with α_i and β_j smooth maps of I in $\mathbb{R}$. Moreover since the paths are smooth and the real part has maximum absolute value in $t = 0$, there exists $\epsilon > 0$ such that for all $u \in [0, \epsilon]$ we have

$$\frac{\partial \alpha_1}{\partial t}(u) \leq 0 \qquad \frac{\partial \beta_1}{\partial t}(u) \leq 0$$

These conditions are sufficient to guarantee the injectivity of $K(s, -)$ for all s. $\square$

Using the Isotopy Extension Theorem [10, Chapter 8], Lemma 2.13 allows us to recover the following stronger statement.

Proposition 2.14. *Let α and β be two injective and smooth paths in D such that $\alpha(0) = \beta(0) = Q$ and $\alpha(t), \beta(t) \in \mathring{D}$ for all $t \neq 0$. Then there exists $F \in \mathscr{G}_d$ such that $F\alpha = \beta$.*

We are ready to give the proof of Theorem 2.12.

Proof of Theorem 2.12. Let $\alpha, \beta \in \mathcal{A}_q$ be represented by two ordered $(q+1)$-tuples of non intersecting arcs $(\alpha_0, \dots, \alpha_q)$ and $(\beta_0, \dots, \beta_q)$. Since we work up to isotopy we can assume without loss of generality that there exists an $\epsilon > 0$ such that $\alpha_i|_{[0,\epsilon]}$ and $\beta_i|_{[0,\epsilon]}$ are straight lines. For all $i \in \{0, \dots, q\}$ define the arcs $\widetilde{\alpha}_i(t) := \alpha(\epsilon t)$ and $\widetilde{\beta}_i(t) = \beta_i(\epsilon t)$.

Since D is metric and S is finite we can find closed spaces D_i containing $\alpha_i(I)$ which are homeomorphic to D and such that $D_i \cap D_j = Q$. For example we can define $\delta := \dfrac{1}{3} \min_{\substack{t,s \in [\epsilon, 1] \\ i \neq j \in \{1,\dots,q+1\}}} |\alpha_i(t) - \alpha_j(s)|$. The spaces D_i can be defined as the closure of $\bigcup_{t \in I} B_{\delta \cdot t}(\alpha(t))$. In every D_i we can apply Proposition 2.14 to the paths α_i and $\widetilde{\alpha}_i$ in order to get a diffeomorphism f_i of D_i extending the isotopy between α_i and $\widetilde{\alpha}_i$ and fixing ∂D_i. Moreover, using bump functions we can assume that there is a neighbourhood $U_i \subseteq D_i$ of ∂D_i such that $f_i|_{U_i} = \mathrm{Id}_{U_i}$. In this way the map f defined as the identity on $D \setminus \bigcup D_i$ and as f_i on each D_i is an element of $\mathscr{G}_d$ such that $f\alpha_i = \widetilde{\alpha}_i$ for all $i \in \{0, \dots, q\}$.

Proceeding in a similar way we find $g \in \mathscr{G}_d$ such that $g\widetilde{\beta}_i = \beta_i$ for all $i \in \{0, \dots, q\}$.

We can apply Proposition 2.14 to the paths $\widetilde{\alpha}_0$ and $\widetilde{b}_0$ to find $h_0 \in \mathscr{G}_d$ such that the image of $\widetilde{\alpha}_0$ is $\widetilde{\beta}_0$. Since h_0 preserves the orientation we can ensure that there exists a closed set D_1 containing

$$\widetilde{\beta}_1(I), \dots, \widetilde{\beta}_q(I), h_0(\widetilde{\alpha}_1)(I), \dots, h_0(\widetilde{\alpha}_q)(I)$$

which intersects ∂D and $\widetilde{\beta}_0(I) = h_0(\widetilde{\alpha}_0)(I)$ only in Q and which is homeomorphic to the closed disk. On this closed disk, with the same argument as before, we can find a diffeomorphism h_1' fixing the boundary of D_1 such that the image of $h_0(\widetilde{\alpha}_1)$ is $\widetilde{\beta}_1$. Moreover we can assume, using bump functions, that h_1' is the identity in an open neighbourhood of D_1. It follows that the map $h_1 \colon D \to D$ defined as the identity on the complement of D_1 and as h_1' on D_1 belongs to $\mathscr{G}_d$. Repeating this process we find maps $h_0, \dots, h_q \in \mathscr{G}_d$ such that $h_i(\dots(h_0(\widetilde{\alpha}_i)\dots) = \widetilde{\beta}_i$ and $h_i(\widetilde{\beta}_j) = \widetilde{\beta}_j$ for each $j < i$. The composition $h := h_q \cdots h_0 \in \mathscr{G}_d$ is such that $h(\widetilde{\alpha}_i) = \widetilde{\beta}_i$ for all $i \in \{0, \dots, q\}$.

We can then conclude that the map $\phi := ghf \in \mathscr{G}_d$ is such that $\phi(\alpha_i) = \beta_i$ for all $i \in \{0, \dots, q\}$, and hence maps S to S. Thus ϕ is actually an element of $\mathscr{H}_d$ and hence the action is transitive.

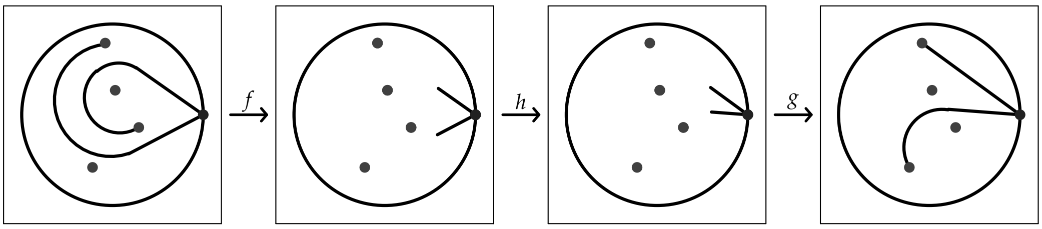


Figure 2.2: The steps in proving the transitivity in the case $n = 4$ and $q = 1$.

□

Since the action is transitive for every $A, B \in \mathcal{A}_q$ the stabilizers of A and B are conjugate. It follows that in order to have the wanted combinatorial description of $\mathcal{A}_q$ it suffices to compute the stabilizer of only one q-simplex.

Recall that D is the unit disk embedded in the complex plane $\mathbb{C}$. Assume, as in Section 1.3, that $Q = 1$ and

$$P_j = \frac{n+1-2j}{n+1} \cdot i \in \mathbb{C}$$

so that P_k, P_j and Q are not collinear when $k \neq j$. For each $j \in \{0, \dots, n-1\}$ we define the arc $\lambda_j \colon I \to D$ as

$$\lambda_j(t) = (1-t)Q + tP_{j+1}$$

Definition 2.15. The set $\Lambda_q = \{[\lambda_0], \dots, [\lambda_q]\}$ is called the *standard q-simplex* of $\mathcal{A}$.

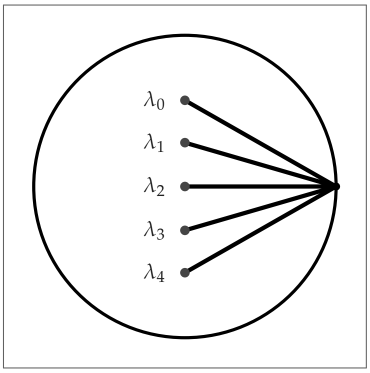


Figure 2.3: Representation of $\lambda_0, \ldots, \lambda_4$ in the case $n = 5$.

Definition 2.16. H_q is the subgroup of B_n generated by $\{\sigma_{q+2}, \ldots, \sigma_{n-1}\}$.

We have the ingredients to state the following:

Theorem 2.17. *The stabilizer of $\Lambda_q \in \mathcal{A}_q$ is $H_q \subseteq B_n$.*

Proposition 2.18. *For all $i \in \{0, \ldots, q\}$ and $j \in \{1, \ldots, n-1\}$ the action of B_n on $\mathcal{A}_0$ satisfies*

$$\sigma_j[\lambda_i] = \begin{cases} [\lambda_i] & \text{if } i \neq j, j-1 \\ [\lambda_{j-1}] & \text{if } i = j \\ \notin \Lambda_q & \text{if } i = j-1 \end{cases}$$

Proof. Since the arcs are linear and the generators permute only two points at the same time we can reduce to the case $n = 2$, as in the proof of Theorem 1.32. It is left to prove that $\sigma_1[\lambda_0] \notin \Lambda_1$ and $\sigma_1[\lambda_1] = [\lambda_0]$. The element σ_1 is represented by the homeomorphism

$$s_1(\alpha e^{i\vartheta}) = \begin{cases} \alpha e^{i(\vartheta+\pi)} = -\alpha e^{i\vartheta} & 0 \leq \alpha \leq 3/4 \\ e^{i\vartheta} \cdot \left(\frac{1}{8} + \frac{7}{8} \cdot e^{4i\pi(1-\alpha)} \right) & 3/4 < \alpha \leq 1 \end{cases}$$

which can be made smooth using bump functions on a neighbourhood of $\alpha = 3/4$ while it is already smooth outside. We can compute, using the explicit formulas, the image of λ_1 via s_1, and since there exists a simply connected neighbourhood of $\lambda_0(I) \cup s_1\lambda_1(I)$ which does not intersect P_2, we can conclude that the two paths are isotopic. If $s_1\lambda_0$ were an element of Λ_1, it should be isotopic to λ_1, since its ending point is P_2. Moreover the choice of P_i's and the definitions of λ_i's implies that $\lambda_0 < \lambda_1$, and since the order is preserved by the action of B_n also $\sigma_1[\lambda_0] < \sigma_1[\lambda_1] = [\lambda_0] < [\lambda_1]$ holds. It is then impossible for $s_1\lambda_0$ to be isotopic to λ_1, thus $\sigma_1[\lambda_0] \notin \Lambda_1$. We refer to Figure 2.4 for a graphical representation of the action of σ_1. $\square$

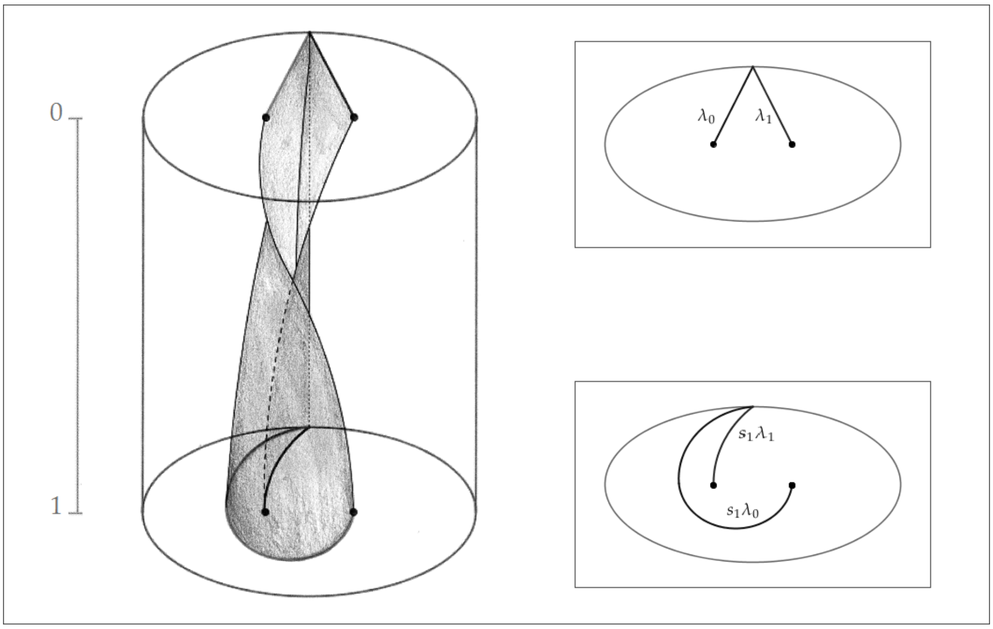


Figure 2.4: The graphical representation of the action of σ_1 on λ_0 and λ_1.

We can then prove that the stabilizer of Λ_q is H_q.

Proof of Theorem 2.17. Define the topological group

$$\mathscr{H}_q := \{h \in \mathscr{H}_d \mid h(\lambda_i(t)) = \lambda_i(t) \text{ for all } i \in \{0, \dots, q\} \text{ and for all } t \in I\}.$$

The class of $h \in \mathscr{H}_d$ belongs to the stabilizer of Λ_q if $h(\Lambda_q)$ is homotopy equivalent to Λ_q. This means that there exists a smooth map $K\colon I \times I \to D^{q+1}$ such that $K(t,0)_i = \lambda_i(t)$ and $K(t,1)_i = h\lambda_i(t)$ for all $i \in \{0, \dots, q\}$ and that for every $s \in I$ the paths $K(-,s)_i$ are arcs such that $K(I,s)_i \cap K(I,s)_j = Q$ if $i \neq j$. Thanks to the isotopy extension theorem this implies the existence of a continuous map $\widehat{K}\colon I \times D \to D$ such that $\widehat{K}(s, \lambda_i(t)) = K(t,s)_i$ for all $i \in \{0, \dots, q\}$. It follows that the class of h is the same as the class of any map which fixes Λ_q point-wise. Hence we deduce that $\mathrm{Stab}(\Lambda_q) = \pi_0(\mathscr{H}_q)$.

The group $\pi_0(\mathscr{H}_q)$ is the mapping class group of $\Sigma \setminus \bigcup_{i=0}^{q} \lambda_i(I)$, which coincides with the mapping class group of the disk with $n - q - 1$ punctures. It follows that $\pi_0(\mathscr{H}_q)$ is the mapping class group of the $n - q - 1$ punctured disk, which thanks to Corollary 1.17 is isomorphic to B_{n-q-1}.

As a consequence of Proposition 2.18 we have that $H_q \subseteq \mathrm{Stab}(\Lambda_q)$. Via the identification $\mathrm{Stab}(\Lambda_q) \cong B_{n-q-1}$ the inclusion corresponds to the morphism

$$\phi\colon H_q \to \mathrm{Stab}(\Lambda_q) \cong B_{n-q-1},\ \sigma_k \mapsto \sigma_{k-q-1}$$

It follows that this homomorphism is also surjective because all the generators belong to the image of ϕ. Thus ϕ is an isomorphism which shows that H_q is the stabilizer of Λ_q. □

Remark 2.19. The proof of Theorem 2.17 shows also that H_q is isomorphic to B_{n-q-1}.

We can now state and prove the combinatorial characterization of the arc complex.

Theorem 2.20. *Let $\mathcal{B}$ be the $(n-1)$-dimensional simplicial complex whose q-simplices are the left cosets of H_q in B_n and such that for every $b \in B_n$ the vertices of bH_q are*

$$bH_0,\ b\sigma_1^{-1}H_0,\ \ldots,\ b\sigma_q^{-1}\cdots\sigma_1^{-1}H_0.$$

Then the maps

$$\phi_q\colon \mathcal{B}_q \to \mathcal{A}_q,\ bH_q \mapsto b\Lambda_q$$

define a B_n-equivariant isomorphism of simplicial complexes.

Proof. The orbit-stabilizer theorem implies that the map

$$\phi_q\colon B_n/H_q = \mathcal{B}_q \to \mathcal{A}_q,\ bH_q \mapsto b\Lambda_q$$

is a bijection which respects the action of B_n. So we only need to check that the simplicial structure is preserved. Proposition 2.18 implies that $[\lambda_k] = \sigma_k^{-1}\cdots\sigma_1^{-1}[\lambda_0]$ for all $k \in \{1,\ldots,n-1\}$, then the q-simplex $b\Lambda_q$ is given by the set

$$\{[b\lambda_0],\ldots,[b\lambda_q]\} = \{b[\lambda_0],\ldots,b\sigma_q^{-1}\cdots\sigma_1^{-1}[\lambda_0]\}$$

It follows that the vertices are then of the form $b\sigma_k^{-1}\cdots\sigma_1^{-1}[\lambda_0]$ for all $k \in \{0,\ldots,q\}$ where $\sigma_0 = 1$. It is clear by definition of $\mathcal{B}$ that they correspond to the vertices of bH_q via ϕ_0. □

Remark 2.21. Using the combinatorial description and the presentation of B_n we can see that for all $b \in B_n$ the faces of bH_q are the $(q-1)$-simplices

$$bH_{q-1},\ b{\sigma_{q-1}}^{-1}H_{q-1},\ \ldots,\ b{\sigma_1}^{-1}\cdots{\sigma_{q-1}}^{-1}H_{q-1}$$

Remark 2.22. Since ϕ_q depends on the choice of the simplex Λ_q the description of $\mathcal{A}$ in terms of B_n is not canonical.

2.3 Some preliminaries about simplicial complexes

This section is devoted to give some definitions and state properties about simplicial complexes which will be used in next section. We use [8], [12] and [11] as main references.

Definition 2.23. Let $\mathcal{C}$ be a simplicial complex. Let $I^{(\mathcal{C}_0)}$ be the space of functions $t\colon \mathcal{C}_0 \to I$ with finite support. The *geometric realization* $|\mathcal{C}|$ of $\mathcal{C}$ is the subspace of maps t such that $\sum_{c\in\mathcal{C}_0} t(c) = 1$ and $\{c \in \mathcal{C}_0 | t(c) > 0\} \in \mathcal{C}$.

By the very definition of $|\mathcal{C}|$ we can identify every point of $|\mathcal{C}|$ with a formal sum $\sum_{c\in C} t_c c$ where $C \in \mathcal{C}$ and t_c are non negative real numbers such that $\sum_{c\in C} t_c = 1$.

Definition 2.24. For every simplex $C \in \mathcal{C}$ we denote by $|C|$ the geometric realization of the complex $\Delta(C) = \{A \subseteq C \mid A \neq \emptyset\}$. Denote by ∂C the complex $\Delta(C) \setminus \{C\}$.

Lemma 2.25. *The space $|C|$ is homeomorphic to the closed ball of dimension* $\dim C$*, denoted* $D^{\dim C}$*, while* $|\partial C| = \partial|C| \cong S^{\dim C - 1}$.

Definition 2.26. The space $|C|^\circ := |C| \setminus \partial|C|$ is called the interior of $|C|$.

Remark 2.27. Fix $n \in \mathbb{N}$. Let $\mathcal{P}(n)$ be the power set of $\{0, \dots, n-1\}$ and define the *standard combinatorial n-simplex* $\Delta(n)$ to be $\mathcal{P}(n) \setminus \{\emptyset\}$. It is a simplicial complex of dimension n. A point $P \in |\Delta(n)|$ is written as $\sum_{i=0}^{n-1} a_i t_i$ with $a_i \in I$ and $\sum a_i = 1$. Its geometric realization is homeomorphic to D^n. The subcomplex $S(n-1) = \Delta(n) \setminus \{0, \dots, n-1\}$ is also denoted $\partial\Delta(n)$. A point $P \in |\partial\Delta(n)|$ is written as $\sum_{i=0}^{n-1} a_i t_i$ where at least one coefficient a_i is zero. Its geometric realization is homeomorphic to S^{n-1}, the boundary of D^n.

Definition 2.28. A *simplicial map* between the complexes $\mathcal{C}$ and $\mathcal{D}$ is a map $\phi_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ such that for every $A \in \mathcal{C}$ we have $\phi(A) \in \mathcal{D}$. This determines a map $\phi\colon \mathcal{C} \to \mathcal{D}$ given by $\phi(A) = \phi_0(A)$ for all $A \in \mathcal{C}$.

Lemma 2.29. *Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial complexes with geometric realizations $|\mathcal{C}|$ and $|\mathcal{D}|$. Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a simplicial map. The map*

$$|\phi|\colon |\mathcal{C}| \to |\mathcal{D}|, \sum_{c\in C} t_c c \mapsto \sum_{c\in C} t_c \phi(c)$$

is continuous.

Remark 2.30. It is not true that every continuous map $f\colon |\mathcal{C}| \to |\mathcal{D}|$ arises from a simplicial map of the simplicial complexes $\mathcal{C}$ and $\mathcal{D}$. If this happens we say, by abuse of language, that f is a *simplicial map* with respect to the complexes $\mathcal{C}$ and $\mathcal{D}$.

Definition 2.31. A *triangulation* of a topological space X is a simplicial complex $\mathcal{C}$ together with a homeomorphism $h\colon |\mathcal{C}| \to X$.

Example 2.32. Let $\mathcal{C}$ be a simplicial complex. Then $(\mathcal{C}, \mathrm{Id})$ is a triangulation of $|\mathcal{C}|$.

Lemma 2.33 ([8, Theorem 2C.1]). *Let X be triangulated by $(\mathcal{C}, h)$ where $\mathcal{C}$ is a finite complex and Y be triangulated by $(\mathcal{D}, g)$. For every continuous map $\phi\colon X \to Y$ there exists a finite triangulation $(\mathcal{C}', h')$ that refines $(\mathcal{C}, h)$, such that ϕ is homotopic to a map $\phi'\colon X \to Y$ which is simplicial with respect to the complexes $\mathcal{C}'$ and $\mathcal{D}$.*

Remark 2.34. The simplicial approximation theorem holds also in a relative way [17, Theorem 2.34]. This means that if there exists a subcomplex $\mathcal{A}$ of $\mathcal{C}$ such that $\phi|_{|\mathcal{A}|}$ is simplicial, then the refinement $\mathcal{C}'$ can be chosen to contain $\mathcal{A}$ and such that the homotopy between ϕ and ϕ' is relative to $|\mathcal{A}|$.

Lemma 2.35. *Let $\mathcal{C}$ be a simplicial complex, $h\colon |TS| \to S^k$ a finite triangulation of S^k and $f\colon TS \to \mathcal{C}$ a simplicial map. The following are equivalent:*

1. *$|f|$ is null homotopic;*
2. *There exists a finite simplicial complex TD containing TS, a homeomorphism $\widehat{h}\colon |TD| \to D^{k+1}$ extending $h\colon |TS| \to S^k$ and a simplicial map $\widehat{f}\colon TD \to \mathcal{C}$ such that $\widehat{f}|_{TS} = f$.*

Proof. For a continuous map being null homotopic is equivalent to being extendible to the cone, hence the second statement trivially implies the first.

Assume that $|f|\colon |TS| \to |\mathcal{C}|$ is null homotopic. This means that $|f|\colon |TS| \cong S^k \to |\mathcal{C}|$ can be extended to the cone of S^k, that is there exists $\widetilde{f}\colon D^{k+1} \to |\mathcal{C}|$ which restricted to S^k is $|f|$. The finite triangulation (TS, h) of S^k extends to a finite triangulation $(\widetilde{TD}, \widetilde{h})$ of D^{k+1}. Applying the relative version of the simplicial approximation theorem to $\widetilde{f}$ we find a refinement $(TD, \widehat{h})$ of $(\widetilde{TD}, \widetilde{h})$ which extends (TS, h), and a simplicial map $\widehat{f}\colon |TD| \to |\mathcal{C}|$ homotopic to $\widetilde{f}$ relative to $|TS|$. It follows that $|\widehat{f}|$ extends $|f|$. □

We now define some operations with simplicial complexes.

Definition 2.36. Let $\mathcal{C}$ be a simplicial complex and $C \in \mathcal{C}$. The *star* of C, denoted by $St(C)$, is the subcomplex of $\mathcal{C}$ whose simplices are the sets $B \in \mathcal{C}$ such that $B \cup C \in \mathcal{C}$. The *link* of C, denoted by $Lk(C)$, is the subcomplex of $\mathcal{C}$ defined as

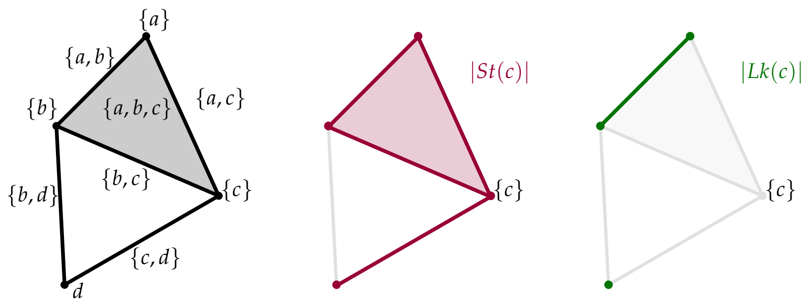
$$Lk(C) = \{B \in \mathcal{C} | B \cap C = \emptyset \text{ and } B \cup C \in \mathcal{C}\}.$$

Note that $St(C)$ and $Lk(C)$ are indeed simplicial complexes.

Example 2.37. Define the simplicial complex $\mathcal{C}$ as the collection of the sets:

$$\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,d\}, \{d,c\}, \{a,b,c\}$$

Figure 2.37 gives a graphical representation of the concepts introduced above. The first picture represents the geometric realization of the complex. In the second one we coloured the subspace $|St(c)|$ and in the third one $|Lk(c)|$.



Definition 2.38. The *join* of two simplicial complexes $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} * \mathcal{D}$, is the simplicial complex

$$(\mathcal{C} * \mathcal{D}) := \mathcal{C} \sqcup \mathcal{D} \sqcup \{C \sqcup D \mid C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}.$$

Definition 2.39. Let X and Y be two topological spaces. The join $X * Y$ is the topological space $X \times Y \times I/R$ where R is the equivalence relation generated by

$$(x, y_1, 0) = (x, y_2, 0) \text{ for all } x \in X \text{ and } y_1, y_2 \in Y$$

$$(x_1, y, 1) = (x_2, y, 1) \text{ for all } x_1, x_2 \in X \text{ and } y \in Y$$

Equivalently every point $P \in X * Y$ can be viewed as a formal sum $(1-t)x + ty$ with $t \in I$ subject to the identifications $0x + 1y = y$ and $1x + 0y = x$ for all $x \in X$ and $y \in Y$.

Lemma 2.40. *The following hold:*

1. *For every vertex $\{c\} \in \mathcal{C}$, the space $|St(\{c\})|$ is contractible.*

2. *The join operators commute with the geometric realization: $|\mathcal{C} * \mathcal{D}| = |\mathcal{C}| * |\mathcal{D}|$.*

3. *For every $C \in \mathcal{C}$ we have $St(C) = C * Lk(C)$.*

Proposition 2.41. *Let $n, m \in \mathbb{N}$. Every homeomorphism $\alpha \colon D^n * S^m \to D^{n+m+1}$ restricts to a homeomorphism between $\partial D^n * S^m$ and ∂D^{n+m+1}.*

Proof. Let α and β be homeomorphisms $D^n * S^m \to D^{n+m+1}$. Then there exists a unique homeomorphism $\gamma \colon D^{n+m+1} \to D^{n+m+1}$ such that $\alpha = \gamma\beta$. Notice also that every homeomorphism $\gamma \colon D^{n+m+1} \to D^{n+m+1}$ restricts to a homeomorphism of the boundary S^{n+m} to S^{n+m}. Combining these two results we conclude that it is enough to prove the statement for only one particular homeomorphism α. We can use Remark 2.27 to substitute $|\Delta(n)|$ and $|S(m)|$ for D^n and S^m. The simplicial map

$$\alpha \colon |\Delta(n)| * |S(m)| \to |\Delta(n+m+1)|$$

defined for every $\sum_{i=0}^{n-1} a_i t_i \in |\Delta(n)|$ and $\sum_{j=0}^{m} b_j t_j \in |S(m)|$ as

$$\sum_{i=0}^{n-1} a_i t_i + \sum_{j=0}^{m} b_j t_j \to \sum_{i=0}^{n-1} a_i t_i + \sum_{j=0}^{m} b_j t_{j+n}$$

is a homeomorphism. The restriction of α to $\partial|\Delta(n)| * |S(m)|$ maps bijectively to $\partial|\Delta(n+m+1)|$, hence the proposition is proved. $\square$

Lemma 2.42. *Let $C = \{t_1, \dots, t_p\} \in \mathcal{C}_{p-1}$ and define $P_C := \sum_{i=1}^{p} p^{-1} t_i \in \mathcal{C}$. Let pt be a one point topological space. Then the map $\phi\colon \partial|C| * pt \to |C|$ defined for all $Q \in \partial|C|$ as $tQ + spt \mapsto tQ + sP_C$ is a homeomorphism which fixes $\partial|C|$ point-wise.*

The following result is a consequence of the Künneth formula

Lemma 2.43 ([14, Lemma 2.3]). *Let $X_0, \dots, X_m$ be topological spaces. Assume that X_j is $(n_j - 1)$-connected for all j. Define $N := \left(m - 1 + \sum_{j=0}^{m} n_j\right)$. Then $X_0 * \cdots * X_m$ is N-connected.*

The following results can be found in [11, Chapter 1].

Definition 2.44. A simplicial complex $\mathcal{C}$ is a *combinatorial n-manifold* if for all $A \in \mathcal{C}$ the space $|Lk(A)|$ is homeomorphic either to $S^{n-\dim(A)-1}$ or to $D^{n-\dim(A)-1}$.

Remark 2.45. Let $\mathcal{C}$ and $\mathcal{C}'$ be simplicial complexes such that $|\mathcal{C}| \cong |\mathcal{C}'|$. Then $\mathcal{C}$ is a combinatorial n-manifold if and only if $\mathcal{C}'$ is a combinatorial n-manifold.

Example 2.46. The simplicial complex $\Delta(n)$ is a combinatorial n-manifold. Since $|\Delta(n)| \cong D^n$ we have that any simplicial complex triangulating D^n is a combinatorial n-manifold.

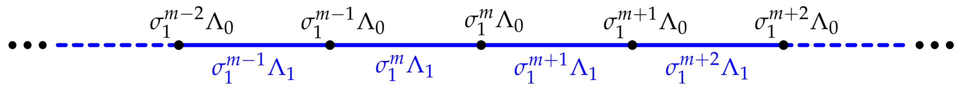
Lemma 2.47 ([11, Lemma 1.18]). *Let D^n be triangulated by $\mathcal{C}$ and let $\mathcal{B}$ be the subcomplex of $\mathcal{C}$ which triangulates the boundary S^{n-1}. Let $A \in \mathcal{C}$. Then $|Lk(A)| \simeq D^{n-\dim(A)-1}$ if and only if $A \in \mathcal{B}$.*

2.4 Contractibility

The main result concerning the topology of the geometric realization of the arc complex is the following theorem.

Theorem 2.48. *$|\mathcal{A}|$ is contractible.*

Example 2.49. We describe the geometric realization in the case $n = 2$. Since $\dim \mathcal{A} = 1$ we have that $\mathcal{A}_q = \emptyset$ for all $q \geq 2$. As described in Theorem 2.20 $\mathcal{A}_i = \{\sigma_1^m \Lambda_i \mid m \in \mathbb{Z}\}$ for $i \in \{0, 1\}$. Moreover $\sigma_1^m \Lambda_i = \sigma_1^n \Lambda_i$ if and only if $n = m$. The combinatorial description of $\mathcal{A}$ implies that for all $m \in \mathbb{Z}$ the vertex $\sigma_1^m \Lambda_0$ is the ending point of exactly two 1-simplices, $\sigma_1^m\Lambda_1$ and $\sigma_1^{m+1}\Lambda_1$. Hence the geometric realization of $\mathcal{A}$ can be depicted as:



It follows that $|\mathcal{A}|$ is contractible.

Even if in the case $n = 2$ the geometric realization of $|\mathcal{A}|$ is very explicit, the situation for $n \geq 3$ is harder to describe.

A partial result about the homotopy type of $|\mathcal{A}|$ is the following statement.

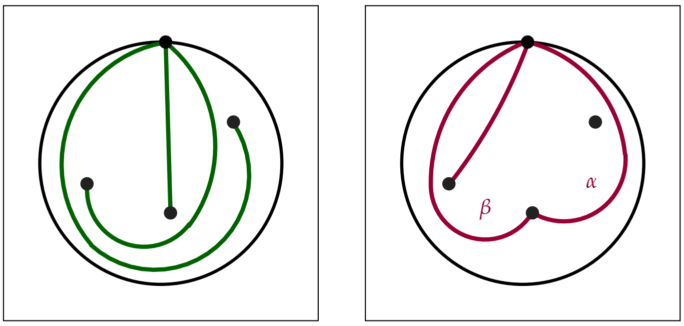
Theorem 2.50 (Hatcher and Wahl)**.** *$|\mathcal{A}|$ is $(n-2)$ connected, i.e. $\pi_j(|\mathcal{A}|) = 0$ for all $j \leq n-2$.*

The theorem is part of the statement of [9, Theorem 7.2]. We give Hatcher and Wahl's proof. For that we introduce a complex $\mathcal{F}$ containing $\mathcal{A}$ whose geometric realization is contractible.

Definition 2.51. The *full arc complex* $\mathcal{F}$ is the simplicial complex whose set of 0-simplices is $\mathcal{A}_0 = \pi_0(Arc)$ and whose q-simplices are subsets $\{a_0, \ldots, a_q\}$ of $\mathcal{A}_0$ of cardinality $q+1$ such that there exist representatives $\alpha_0, \ldots, \alpha_q$ such that $\alpha_i((0,1)) \cap \alpha_j((0,1)) = \emptyset$ for all $i \neq j$.

The difference with the arc complex $\mathcal{A}$ is that arcs in a simplex of $\mathcal{F}$ are allowed to share both their ending points.

Example 2.52. Observe the figures below representing two triples of arcs in the case $n = 3$.



The first one represents a 2-simplex of both $\mathcal{A}$ and $\mathcal{F}$, while the second one represents a 2-simplex of $\mathcal{F}$ which is not a simplex of $\mathcal{A}$ since the ending points of α and β coincide.

It is clear that the definition implies that $\mathcal{A}$ is a subcomplex of $\mathcal{F}$ and that they do not coincide for $n > 1$.

Theorem 2.53. $|\mathcal{F}|$ *is contractible.*

Proof. The proof uses the argument introduced in [7] adapted to the disk.

Let a be a vertex of $\mathcal{F}$. As stated in Lemma 2.40 the space $|St(a)|$ is contractible. It will be enough to show that $|\mathcal{F}|$ can be retracted to $|St(a)|$. Denote by ι the inclusion of $|St(a)|$ in $|\mathcal{F}|$. We will construct a continuous map $\psi\colon |\mathcal{F}| \times I \to |\mathcal{F}|$ such that $\psi(-,0)$ is the identity, $\psi(|\mathcal{F}|,1) \subseteq |St(a)|$ and for all $t \in I$ and $P \in |St(a)|$ we have $\psi(P,t) = P$.

Fix once and for all a representative α of a. Let $P \in |\mathcal{F}|$, then we can write $P = t_0 b_0 + \cdots + t_q b_q$ where the t_i are positive real numbers such that $t_0 + \cdots + t_q = 1$ and $B = \{b_0, \ldots, b_q\} \in \mathcal{F}$.

We say that $(\beta_0, \ldots, \beta_q)$ is a *minimal system* for $(b_0, \ldots, b_q)$ if $[\beta_i] = b_i$ for all $i \in \{0, \ldots, q\}$ and if for all $i \neq j$ we have that $\beta_i((0,1)) \cap \alpha((0,1))$ has minimal cardinality and $\beta_i((0,1)) \cap \beta_j((0,1)) = \emptyset$. For every B there exists a minimal system. To construct it we can first choose arcs which intersect only in their ending points, and then minimize the cardinality of intersection with α. Notice that if $(\beta_0, \ldots, \beta_q)$ is a minimal system of $(b_0, \ldots, b_q)$, then every $(j+1)$-tuple $(\beta_{i_0}, \ldots, \beta_{i_j})$ is a minimal system of $(b_{i_0}, \ldots, b_{i_j})$.

Let β be any minimal system for B. The cardinality $i(b_j)$ of the set

$$\mathcal{I}(b_j) := \beta_j((0,1)) \cap \alpha((0,1))$$

is finite and does not depend on the choice of minimal system β, since every two minimal systems are isotopic via families of minimal systems. We define $i(P)$ the cardinality of

$$\mathcal{I}(P) := \bigcup_{j \in \{0,\ldots,q\}} \beta_j((0,1)) \cap \alpha((0,1))$$

which coincide with the sum $i(b_0) + \cdots + i(b_q)$.

If $i(P) = 0$ then $\{b_0, \ldots, b_q, a\}$ is a simplex of $\mathcal{F}$, and hence $B \in St(a)$. It follows that if $i(P) = 0$ then $P \in |St(a)|$.

Assume that $P \notin |St(a)|$. We show how to associate to P a map $\phi(P)\colon I \to |\mathcal{F}|$ such that $\phi(P)(0) = P$ and $i(\phi(P)(1)) < i(P)$.

Let $T = \alpha(z) \in \mathcal{I}(P)$ with minimal z. Then there is a $k \in \{0, \ldots, q\}$ such that $T \in \beta_k(I)$ and without loss of generality we can assume that $T = \beta_k(z)$. Notice that k does not depend on the choice of minimal system. Consider the path $p\colon I \to D$ defined as

$$t \mapsto \begin{cases} \alpha(t) & \text{if } t \leq z \\ \beta_k(t) & \text{if } t \geq z \end{cases}$$

It is homotopy equivalent to a smooth arc $\widetilde{\beta}_k$ by modifying the definition in a neighbourhood of $[0,z]$ so that

$$\widetilde{\beta}_k(I) \cap \alpha(I) = \beta_k(I) \cap \alpha(I) \setminus \{T\} \quad \text{and} \quad \widetilde{\beta}_k((0,1)) \cap \beta_i((0,1)) = \emptyset \text{ for all } i \neq k.$$

In this way the cardinality $\widetilde{i(P)}$ of the set

$$\widetilde{\mathcal{I}(P)} := \widetilde{\beta}_k((0,1)) \cap \alpha((0,1)) \cup \bigcup_{j \neq k} \beta_j((0,1)) \cap \alpha((0,1))$$

is lower than $i(P)$.

Denote by $\widetilde{b}_k$ the isotopy class of $\widetilde{\beta}_k$. Notice moreover that there exists a representative γ of $\widetilde{b}_k$ such that $\gamma((0,1)) \cap \beta_i((0,1)) = \emptyset$ for all $i \in \{0,\dots,q\}$. It follows that $B_1 := \{b_0,\dots,b_q,\widetilde{b}_k\}$ is an element of $\mathcal{F}$ belonging to $\mathcal{F}_q$ or $\mathcal{F}_{q+1}$. We refer to Figure 2.5.

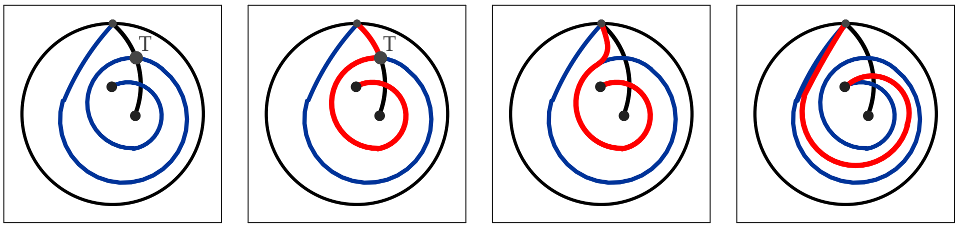


Figure 2.5: Consider the case $B = \{[\beta]\}$ where β is the blue path and α the black one, both represented in the first picture. The second picture represents in red the path p. The third one represents in red the path $\widetilde{\beta}_k$ and the fourth one γ.

Recall that b_k was defined to be the number of intersections between $\beta_k((0,1))$ and $\alpha((0,1))$. We define the map $\phi(P)\colon [0, i(b_k)t_k] \to |\mathcal{F}|$ as

$$s \mapsto P - \frac{s}{i(b_k)t_k} \cdot t_k b_k + \frac{s}{i(b_k)t_k} \cdot t_k \widetilde{b}_k \in |B_1| \subseteq |\mathcal{F}|$$

Since the map is linear in s it is also continuous. Moreover we its value at $i(b_k)t_k$ is

$$\phi(P)(i(b_k)t_k) = P - t_k b_k + t_k c = \sum_{i \neq k} t_i b_i + t_k \widetilde{b}_k$$

We can then conclude that $i(\phi(P)(i(b_k)t_k)) \leq \widetilde{i(P)}$. Indeed $i(\phi(P)(i(b_k)t_k))$ is the minimal number of intersections between α and representatives of $b_0,\dots,c_k,\dots b_q$, while $\widetilde{i(P)}$ is the number of intersections between α and the representatives $\beta_0,\dots,\widetilde{\beta}_k,\dots,\beta_q$. Given that $\widetilde{i(P)} < i(P)$ we conclude that $i(\phi(P)(1)) < i(P)$.

Repeating this argument finitely many times we define a map $\psi(P)_\theta\colon [0, \sum t_j i(b_j)] \to |\mathcal{F}|$ such that $\psi(P)(\theta) \in |St(a)|$. We rescale linearly the interval $[0, \sum t_j i(b_j)]$ to I obtaining $\psi(P)\colon I \to |\mathcal{F}|$ such that $\psi(P)(1) \in |St(a)|$. For every simplex $B \in \mathcal{F}$ the map $\psi(B)\colon |\overset{\circ}{B}| \times I \to |\mathcal{F}|$ sending $(P,s) \to \psi(P)(s)$ is continuous.

Let $P = \sum a_i t_i \in |\mathring{B}|$ and C be a simplex of ∂B such that $P_k = \sum_{i \neq k} a_i t_i \in |\mathring{C}|$. The definitions of the maps $\psi(B)$ and $\psi(C)$ imply that

$$\lim_{t_k \to 0} \psi(B) \left(\sum a_i t_i, s\right) = \psi(C) \left(\sum_{i \neq k} a_i t_i, s\right)$$

for all $s \in I$. It follows that the collection of all maps $\psi(B)$ glue to form a global continuous map $\phi \colon |\mathcal{F}| \times I \to |St(a)|$. By construction this satisfies the properties to be a deformation retraction, so $|\mathcal{F}|$ is contractible. □

Example 2.54. Let $n = 2$ and fix $\alpha \in Arc$. Let $B = \{[\beta_0], [\beta_1]\} \in \mathcal{F}_1$ satisfying the following properties: $\beta_0((0,1)) \cap \beta_1((0,1)) = \emptyset$ and $(\beta_0((0,1)) \cup \beta_1((0,1))) \cap \alpha((0,1))$ is a unique point $T \in \Sigma$. Let $P = t_0[\beta_0] + t_1[\beta_1] \in |B|$. We can represent the point P via the graphs of β_0 and β_1 where the thickness of the lines depends on the coefficients t_i. The Figure 2.6 represents, using this "thickness trick", the images of the map $\phi(P)$ at the times $s = 0$, $s = 0.5$ and $s = 1$. As we can see in the last picture $\widetilde{\beta}_0((0,1)) \cap \alpha((0,1)) = \emptyset = \beta_1((0,1)) \cap \alpha((0,1))$, hence $\{\widetilde{\beta}_0, \beta_1\}$ belongs to $St(\alpha)$.

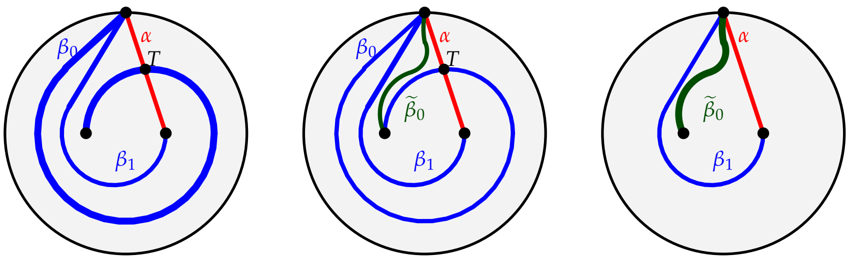


Figure 2.6: A graphical representation of the deformation retraction.

Using the argument of [9, Proposition 7.2] we can prove the $(n-2)$-connectedness of $|\mathcal{A}|$.

Proof of Theorem 2.50. The proof is by induction on n. For $n = 1$ the statement is empty.

Assume $n \geq 2$ and let $k \leq n-2$. By the induction hypothesis we have that $\pi_j(|\mathcal{A}|) = 0$ for all $j < k$. Let $f \colon S^k \to |\mathcal{A}|$ be a continuous map. We want to prove that f is homotopic to a map which can be extended to the cone S^k. Since S^k has a finite triangulation, without loss of generality we can suppose that f is simplicial with respect to a finite triangulation TS of S^k. Since $|\mathcal{A}| \subseteq |\mathcal{F}|$ and the full arc complex is contractible we know that $f \colon S^k \to |\mathcal{F}|$ is null homotopic. Hence by applying Lemma 2.35 there exists a finite triangulation TD of D^{k+1} that extends TS on S^k, and a simplicial map $\widetilde{f} \colon D^{k+1} \to |\mathcal{F}|$ such that $\widehat{f}|_{S^k} = f$.

By abuse of notation we still denote by $\widehat{f}$ the maps $TD \to \mathcal{F}$ and $|TD| \to |\mathcal{F}|$ induced by $\widehat{f}$. A simplex $\sigma \in TD$ is *bad* if for every $a \in \widehat{f}(\sigma)$ there exists $b \neq a \in \widehat{f}$ such that $a(1) = b(1)$.

Let σ be a bad simplex of maximal dimension p.

Claim 1. *The map $\widehat{f}|_{|Lk(\sigma)|}$ is null homotopic.*

Let $\{a_0, \dots, a_r\} = \widehat{f}(\sigma)$, and for all i fix a representative α_i of a_i such that $\alpha_k((0,1)) \cap \alpha_j((0,1)) = \emptyset$ whenever $k \neq j$. The maximality of σ means that for every simplex $\tau \in Lk(\sigma)$ the image of τ is a simplex of $\mathcal{A}$ such that $b(1) \neq a(1)$ for all $a \in \widehat{f}(\sigma)$ and $b \in \widehat{f}(\tau)$. In other words every simplex B belonging to the image of $Lk(\sigma)$ satisfies the following properties: the union $\widehat{f}(\sigma) \cup B$ belongs to $\mathcal{F}$ and for every $b \in B$ there exists a representative β such that $\alpha(I) \cap \beta(I) = Q$. In particular $\beta((0,1])$ lies in $\widehat{D} = D \setminus \bigcup_{i=0}^{p} \alpha_i(I)$.

Let $U_1, \dots, U_c$ be the path components of $\widehat{D}$ which have a non-empty intersection with S, and for all $i \in \{1, \dots, c\}$ denote by D_i the closure of the component U_i. For each i call S_i the intersection $S \cap U_i$, and denote its cardinality by n_i. Note that $n_i \geq 1$. Define an arc of $D_i \setminus S_i$ as an injective smooth path α with starting point Q, ending point belonging to S_i and $\alpha((0,1)) \in \overset{\circ}{D}_i \setminus S_i$. Denote by $\mathcal{A}(D_i, S_i)$ the simplicial complex whose q-simplices are $(q+1)$-tuples of homotopy classes of arcs of $D_i \setminus S_i$. Since D_i is homeomorphic to a disk, $\mathcal{A}(D_i, S_i)$ is isomorphic to the arc complex of a disk with n_i punctures.

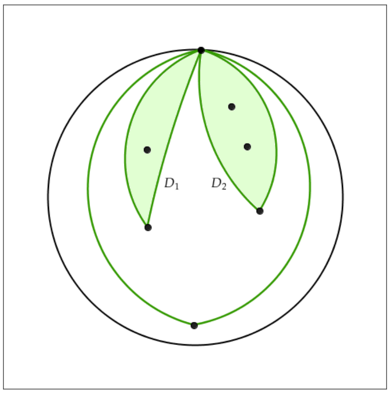


Figure 2.7: The disks D_1 and D_2 obtained cutting along the paths belonging to the image of σ. Note that $n_1 = 1$ and $n_2 = 2$.

The maximality of σ implies that $\widehat{f}(Lk(\sigma))$ is contained in the subcomplex $\mathcal{A}(\sigma)$ of $\mathcal{A}$ defined as the join complex

$$\mathcal{A}(\sigma) := \mathcal{A}(D_1, S_1) * \cdots * \mathcal{A}(D_c, S_c).$$

Notice moreover that since the simplex σ is bad, it cannot belong to TS, the subcomplex of TD triangulating the boundary S^k of D^{k+1}. Thanks to Lemma 2.47 we can conclude that $|Lk(\sigma)|$ is homeomorphic to S^{k-p}.

We can conclude that the restriction of $\widehat{f}$ to $|Lk(\sigma)|$ can be viewed as a simplicial map

$$\widehat{f}|_{|Lk(\sigma)|} \colon |Lk(\sigma)| \cong S^{k-p} \to |\mathcal{A}(\sigma)|$$

Notice that $n_i < n$, hence by induction hypothesis $|\mathcal{A}(D_i, S_i)|$ is $(n_i - 2)$ connected. Furthermore from Remark 2.40 we have that

$$|\mathcal{A}(\sigma)| = |\mathcal{A}(D_1, S_1) * \cdots * \mathcal{A}(D_c, S_c)| \cong |\mathcal{A}(D_1, S_1)| * \cdots * |\mathcal{A}(D_c, S_c)|$$

We can apply then Lemma 2.43 to $|\mathcal{A}(\sigma)|$ obtaining that $|\mathcal{A}(\sigma)|$ is N-connected with

$$N = (c-1) - 1 + \sum_{i=1}^{c}(n_i - 1) - 1 = \sum_{i=1}^{c} n_i - 2.$$

Let d be the cardinality of $\{a(1) \mid a \in \widehat{f}(\sigma)\}$, so that $\sum_{i=1}^{c} n_i = n - d$. Since $\widehat{f}$ is simplicial $d \leq \dim \phi(\sigma) \leq \dim(\sigma) = p$, and together with $k \leq n-2$ we get that

$$k - p \leq n - 2 - p = \sum_{i=1}^{c} n_i + d - 2 - p \leq \sum_{i=1}^{c} n_i - 2$$

This implies that $\widehat{f}|_{|Lk(\sigma)|}$ is null homotopic.

Claim 2. *There exists a continuous map $f_\sigma \colon D^{k+1} \to |\mathcal{F}|$ such that*

$$\begin{cases} f_\sigma(P) = \widehat{f}(P) & \text{if } P \notin |St(\sigma)|^\circ \\ f_\sigma(P) \in |\mathcal{A}| & \text{if } P \in |St(\sigma)|^\circ \end{cases}$$

Since $\widehat{f}|_{|Lk(\sigma)|}$ is null homotopic it can be extended to a map $g \colon \mathrm{pt} * S^{k-p} \simeq P * |Lk(\sigma)| \to |\mathcal{A}(\sigma)|$. The complex $TD_g := \{\mathrm{pt}\} * Lk(\sigma)$ triangulates $\mathrm{pt} * |Lk(\sigma)|$.

Choose a homeomorphism $\phi \colon \partial|\sigma| * P \to |\sigma|$ such that $\phi|_{\partial|\sigma|} = \mathrm{Id}_{\partial|\sigma|}$ (Lemma 2.42). Using Lemma 2.40 we can find the following homeomorphism:

$$\partial|\sigma| * |\{P\} * Lk(\sigma)| = \partial|\sigma| * P * |Lk(\sigma)| \xrightarrow{\cong} |\sigma| * |Lk(\sigma)| = |\sigma * Lk(\sigma)| = |St(\sigma)|$$

where we wrote the equality symbol when the homeomorphism is induced by canonical identifications of complexes. The unique non canonical homeomorphism is induced by ϕ.

It follows that the space $|St(\sigma)|$ is triangulated by the complex $TD_\sigma := \partial\sigma * (P * Lk(\sigma))$ via the homeomorphism Φ induced by ϕ. Notice that the homeomorphism Φ which defines this new triangulation coincides with the canonical one on $|Lk(\sigma)|$ and on $\partial|\sigma|$.

Let $\tau := \partial\sigma * Lk(\sigma)$. Note that $\tau \in St(\sigma)$ and also $\tau \in TD_\sigma$. Note that $\Phi|_{|\partial\sigma|} = \mathrm{Id}_{|\partial\sigma|}$ and that $\Phi|_{|Lk(\sigma)|} = \mathrm{Id}_{|Lk(\sigma)|}$. It follows that $\Phi|_{|\tau|} = \mathrm{Id}_{|\tau|}$. Using Lemma 2.25

we deduce that $|\sigma| \cong D^p$ and as we already noticed $|Lk(\sigma)| \cong S^{k-p}$. It follows that $|St(\sigma)| \cong D^p * S^{k-p}$ which is homeomorphic to D^{k+1}. We can apply Proposition 2.41 to deduce that $\partial|\sigma| * |Lk(\sigma)|$ triangulates $\partial|St(\sigma)|$.

It follows that the triangulations $(TD \setminus St(\sigma), \mathrm{Id})$ of $|TD| \setminus |St(\sigma)|$ and (TD_σ, Φ) on $|St(\sigma)|$ glue to a new triangulation $(\widehat{TD}_\sigma, \Psi)$ of $|TD|$. Moreover notice that $TS \subseteq \widehat{TD}_\sigma$ and that $\Psi|_{|TS|} = \mathrm{Id}_{|TS|}$.

We define the map $F\colon \partial|\sigma| * |\{P\} * Lk(\sigma)| \to |\mathcal{F}|$ as

$$tP + sQ \mapsto t\widehat{f}(P) + sg(Q)$$

for every $P \in \partial|\sigma|$ and $Q \in |\{P\} * Lk(\sigma)|$. Notice that the image lies in $|\mathcal{F}|$ since, by maximality of σ we have that $|\mathcal{A}(\sigma)|$ and $|\widehat{f}(\sigma)|$ are disjoint.

We can then deduce that, since $g|_{|Lk(\sigma)|} = \widehat{f}|_{|Lk(\sigma)|}$ we have that on the boundary of $|St(\sigma)|$ the map F coincides with $\widehat{f}$. So the map $f_\sigma\colon D^{k+1} \to |\mathcal{F}|$,

$$f_\sigma(P) = \begin{cases} \widehat{f}(P) & \text{if } P \notin |St(\sigma)| \\ F(P) & \text{if } P \in |St(\sigma)| \end{cases}$$

is continuous. By definition the image of the interior of $|St(\sigma)|$ via f_σ lies in $|\mathcal{A}(\sigma)|$ while it coincides with $\widehat{f}$ otherwise.

Since the triangulation TD is finite the number of bad simplices of maximal dimension is finite too. Notice that given two bad simplices σ and σ' of maximal dimension, we have that $|St(\sigma)|^\circ \cap |St(\sigma')|^\circ = \emptyset$. We can conclude that we can apply the argument above to any bad simplex of maximal dimension p. In this way we find a triangulation $TD(p)$ and a continuous map $f_p\colon D^{k+1} \to \mathcal{F}$ extending f and such that

$$\begin{cases} f_p(P) = \widehat{f}(P) & \text{if } P \notin St(p) \\ f_p(P) \in |\mathcal{A}| & \text{if } P \in St(p) \end{cases}$$

where $St(p)$ denotes the union $\bigcup_\sigma |St(\sigma)|^\circ$ where σ ranges in the set of bad simplices of maximal dimension p. It follows that the bad simplices of $f_p|_{|TD(p)| \setminus St(p)}$ have dimension strictly less than p.

We can repeat the process by decreasing induction on the dimension of the bad simplices. In this way we obtain a triangulation $TD(0)$ of D^{k+1} and a continuous map $f_0\colon D^{k+1} \to |\mathcal{F}|$ with the following properties:

1. The map f_0 extends f;

2. Let $\omega \in TD(0)$ such that $f_0(|\omega|) \subseteq |\mathcal{F}| \setminus |\mathcal{A}|$. Then $f_0|_{|\omega|}$ is simplicial.

These conditions are enough to guarantee that the image of f_0 lies then in $|\mathcal{A}|$. Indeed suppose by contradiction that there exists $\tau \in TD_0$ such that $f_0(|\tau|) \notin |\mathcal{A}|$. Since the map $f_0|_{|\tau|}$ is simplicial that means that there exist $\{a, b\} \subseteq f_0(\tau)$ such that $a(1) = b(1)$ but $a \neq b$. It follows that the simplex $\{a, b\}$ lies in the image of f_0, hence there exists a subset $\tau' \subseteq \tau$ such that $f_0(\tau') = \{a, b\}$. But this means that τ' is a bad simplex. Contradiction!

The map f_0 realizes an extension of f to the cone of S^k and so we can conclude that f is null homotopic. By applying the same argument to every continuous map $f : S^k \to \mathcal{A}$ we deduce that $\pi_k(|\mathcal{A}|) = 0$. By induction we can conclude that $\pi_j(|\mathcal{A}|) = 0$ for all $j \leq n - 2$. □

We can conclude by proving the contractibility of $|\mathcal{A}|$.

Proof of Theorem 2.48. When $n = 1$ the space $|\mathcal{A}|$ is already reduced to a point. Moreover Example 2.49 showed the contractibility in the case $n = 2$, hence we can assume $n \geq 3$.

Since the space $|\mathcal{A}|$ admits a triangulation, in order to get its contractibility it suffices to prove the triviality of all homotopy groups (J.H.C. Whitehead, [16, Theorem 6.32]). We can relate the homotopy groups to the homology groups thanks to the Hurewicz Isomorphism Theorem ([16, Theorem 10.25]):

If X is an $(n-2)$-connected space, with $n \geq 3$, then the Hurewicz homomorphism $h_q : \pi_q(X, P) \to H_q(X, \mathbb{Z})$ is an isomorphism for $q \in \{1, \dots, n-1\}$.

Thanks to Proposition 2.50 we can apply the stated theorem to $|\mathcal{A}|$ and conclude that $H_q(|\mathcal{A}|, \mathbb{Z}) = 0$ for all $q < n - 1$. Since the dimension of the complex is $n - 1$, we can deduce that $H_q(|\mathcal{A}|, \mathbb{Z}) = 0$ for all $q > n - 1$. Combining the two results we have that $H_q(|\mathcal{A}|, \mathbb{Z}) = 0$ for all $q \neq n - 1$. If we prove that $H_{n-1}(|\mathcal{A}|, \mathbb{Z}) = 0$, then Hurewicz's Theorem implies that $\pi_{n-1}(|\mathcal{A}|, P) = 0$ and consequently that $\pi_q(|\mathcal{A}|, P) = 0$ for all q.

In this way we are left to prove the triviality of

$$H_{n-1}(|\mathcal{A}|, \mathbb{Z}) = \ker\left(\mathbb{Z}[\mathcal{A}_{n-1}] \xrightarrow{\partial_{n-2}} \mathbb{Z}[\mathcal{A}_{n-2}]\right)$$

Let $\delta \in \mathbb{Z}[B_n]$ be defined as

$$\delta := 1 - \sigma_{n-1}{}^{-1} + \sigma_{n-2}{}^{-1}\sigma_{n-1}{}^{-1} - \dots + (-1)^{n-1}\sigma_1{}^{-1} \cdots \sigma_{n-1}{}^{-1}$$

The combinatorial description of $\mathcal{A}$ asserted in Theorem 2.20 gives the commutative diagram:

$$\begin{array}{ccc}
\mathbb{Z}[\mathcal{A}_{n-1}] & \xrightarrow{\partial_{n-2}} & \mathbb{Z}[\mathcal{A}_{n-2}] \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{Z}[B_n] & \xrightarrow{\cdot \delta} & \mathbb{Z}[B_n]
\end{array}$$

We hence need to prove that the right multiplication by δ is injective.

Let deg be the group homomorphism

$$\deg\colon B_n \to \mathbb{Z},\, \sigma_i \mapsto 1$$

for all $i \in \{1, \ldots, n-1\}$. Since the relations between the elements of B_n are generated by $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all i and j such that $|i-j| \geq 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $i \in \{1, \ldots, n-2\}$ the map deg is well defined.

Since deg is a homomorphism it induces a graded structure on the ring $\mathbb{Z}[B_n]$.

The homogeneous component of maximal degree of δ is 1. Since 1 is invertible in $\mathbb{Z}[B_n]$ the multiplication on the right by δ is injective.

We conclude that $H_{n-1}(|\mathcal{A}|, \mathbb{Z}) = 0$ and hence that $|\mathcal{A}|$ is contractible. $\square$

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