Pricing Derivatives on Multiple Assets

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Recombining Multinomial Trees Based on Pascal's Simplex

Master's Thesis

Defended on March 8, 2013

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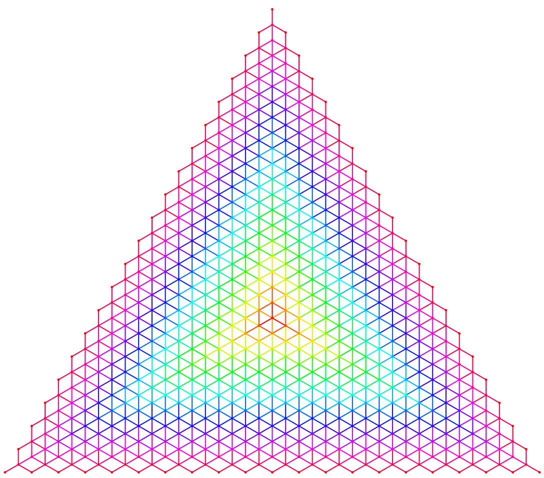

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Abstract

In this thesis[1] a direct generalisation of the recombining binomial model by Cox, Ross, and Rubinstein [16] based on Pascal's simplex is constructed. This discrete method approximates the price of derivatives on multiple assets in a Black-Scholes market environment. It consists of a sequence of recombining multinomial trees based on Pascal's simplex. The generalisation keeps most aspects of the binomial model intact, of which the following are the most important: The direct link to Pascal's simplex; the matching of the moments of the log-transformed process; and the completeness of the model. The goal of this thesis is to privide a theoretical satisfactory solution. However, the recombining multinomial model might also have the potential to provide a practical satisfactory solution.

[1]Hospitality has been provided for D. Sierag for two weeks at University College Cork by B. Hanzon in October 2012, while Leiden University has supported funding for this visit.

Contents

1 Introduction

Thousands of years ago basic forms of economic systems emerged around the world. Throughout the ages the economic system has evolved by the hand of human society to what it is today: A global system of production, distribution, and consumption of goods and services. Unfortunately, millenia of experience has not led by far to the understanding of the economic system to the fullest extent. Problems which seem clear at first sight, take a great deal to completely understand. While in the twentieth century knowledge grew vastly and at the dawn of the twenty-first century is still growing, world leaders and scientists still struggle to manage the economic system to benefit all civilians and to map out the details of the structure of the economic system.

A major player in the economic system is the financial market. On the financial market various assets are traded, such as stocks, bonds, and other financial contracts. Financial contracts which are based on other assets are called *derivatives*. An example of a derivative is a *European option*. The holder of the contract has the right, but not the obligation, to exercise the option at some moment in the future specified in the contract. If the holder exercises the option at that time, he receives a payoff equal to an amount specified in the contract. One of the main questions is what a fair price of the option is today. The statement of the problem is clear, but the solution is not obvious.

Many theories and models have been developed to map out the system and behaviour of financial markets. In 1965 Samuelson [36] proposed a popular model for the behaviour of asset prices. In 1973 Black and Scholes [6] provided an equation called the *Black-Scholes equation* to price derivatives on a single asset in the *Black-Scholes model*, an adjusted Samuelson market environment. In 1985 Cox, Ingersoll, and Ross [15] extended the Black-Scholes equation to the *generalised Black-Scholes equation* to price derivatives on multiple assets. Analytical solutions to these equations have been established for some derivatives, but solving the equation for an arbitrary derivative is often too hard. Therefore discrete methods are needed to approximate the analytical solution.

In 1979 Cox, Ross, and Rubinstein [16] provided a discrete method involving recombining binomial trees based on Pascal's triangle to approximate the price of derivatives on one asset. In this thesis we provide a direct generalisation of this model to a discrete method involving recombining multinomial trees based on Pascal's simplex to approximate the price of derivatives on multiple assets. The path of generalisation that we follow gives new insight in the binomial model as well as the multinomial model. The recombining multinomial trees have a nice structure, which allows them to be used in an efficient way. Moreover, recombining multinomial trees are a useful technique to approximate the value of American options.

The outline of this thesis is as follows. In Section 2 we introduce some basic concepts of financial mathematics. This theory is split up into two parts: First the univariate model is described in Section 2.1; and then the multivariate model is described in Section 2.2. Section 3 provides the main theory of this thesis. In Section 3.1 the recombining binomial tree method is derived in a different way than is found in literature. Along this path the generalisation to recombining multinomial trees is presented in Section 3.2. After the derivation of the model it is reviewed in Section 4. We provide some numerical examples in Section 4.1; we briefly discuss literature related to our model in Section 4.2; and we propose topics for further research in Section 4.3. Finally in Section 5 we conclude this thesis.

2 Basic Concepts

In this section we introduce some basic concepts of financial mathematics to lay a foundation before we start to work on the main topics of this thesis in Section 3. Because the structure of the financial market is too complicated to study directly, we need to use a simplified approach. We use the model introduced by Samuelson in 1965 [36] to model the behaviour of asset prices. Furthermore, to price derivatives we use the popular *Black-Scholes model*, introduced in 1973 by Black and Scholes [6], which is based on the Samuelson market environment. Although the Black-Scholes model is based on several assumptions which are not realistic, the results of pricing derivatives are satisfying and the model is used widely. The Black-Scholes model also serves as a foundation for our theory, on which we will elaborate in Section 3. In this section we first introduce a univariate model of financial markets in Section 2.1. In Section 2.2 we extend the univariate model to a multivariate model.

2.1 Univariate Model

On the financial markets around the world lots of financial products are traded. For instance there are the stock market, the stock derivative market, the bond market, and the fixed income derivative market, where company stocks, derivatives of stocks, bonds, and derivatives of bonds are traded, respectively. *Derivatives* are financial contracts based on one or more underlying assets. These contracts can be quite complicated. Also, it is not clear beforehand that a unique fair price for the contract exists. The fair price for a derivative turns out to be dependent on the price of the underlying assets rather than market forces.

Examples of derivatives are *European call options* and *European put options* on a single stock. A European call option or European put option has three parameters recorded in the contract: The underlying stock, the expiration date, and the strike price. At the *expiration date*, the holder of the European call option has the right, but not the obligation, to buy the underlying stock at the strike price; the holder of the European put option then has the right, but not the obligation, to sell the underlying stock at the strike price. If the holder of the contract makes use of his right at the expiration date, we say that the option is *exercised*. The writer of the option then has the obligation to either sell or buy the stock at the strike price, in the case of a European call option or European put option, respectively. Other related examples are *American call options* and *American put options*. American options can be exercised early, i.e., at any time before the expiration date the holder can choose to exercise the option and receive a payment according to the payoff function.

One on the main questions in financial mathematics is how to value derivatives. It turns out that the answer to this question is not that obvious. Other important problems include hedging derivatives and finding optimal early exercise times for American options. In the Black-Scholes model many methods to find answers to these problems have been proposed, of which we will discuss the most significant.

The Fixed Income Market and Risk-free Bonds

There also exist financial markets where coupon bonds from either companies or governments are traded. A *coupon bond* is a financial contract where the buyer of the contract lends a certain amount of money to the writer of the contract. According to the structure of the contract, the writer pays the buyer an amount of cash, the *coupon*, at certain moments in time specified in the contract until the expiration date, when the loan is repaid. If the only moment in time when the buyer receives a coupon is at the expiration date, the coupon bond is called a *zero coupon bond*. Bonds are assumed to be a more safe investment than other assets. However, there is always the risk that the writer of the contract will not pay his debt, for example if the company goes bankrupt. Credit rating agencies like Standard & Poors, Moody's, and Fitch rate the risk of bonds of companies and authorities. Writers of bonds with the best rating, often addressed as *AAA-rating* or *triple-A rating*, are expected to repay their debts always. Bonds of writers with *AA-rating* or *double-A rating* are more risky, but are assumed to repay their debts almost surely. From there it goes all the way down to *junk-bonds*, for which there is a high probability that the writer of the bond will not repay all of his debts. Often writers with a high rating have to pay a small interest rate and writers with a low rating have to pay a high interest rate. Bonds of governments of the western world have typically high ratings, while bonds of companies in unstable countries have a low rating. Note that in practice a high rating of a company or government does not necessarily mean that your money is safe there. On September 15, 2008, the former AA-rated company Lehman Brothers went bankrupt. The next day the former AAA-rated company AIG had to be bailed out by the United States Federal Reserve Bank to prevent it from going bankrupt. Recently, on February 1, 2013, owners of coupon bonds of SNS REAAL lost their investment when the Dutch financial institution had to be bailed out by the Dutch government.

We make the assumption that there exists a *risk-free bond*, where we can store our cash against a certain interest rate, without any risk. In literature this is also called a *money account*. Although the name suggests it the risk-free bond is not a bond. We assume that we can deposit and withdraw any amount of cash at any time, which is typically not possible with coupon bonds.

Let B be a risk-free bond with interest rate r. Suppose that the writer of the bond pays the interest rate n times per year. Then after a year the value of the bond is

$$B_0(1 + r/n)^n,$$

with B_0 the initial value of the bond. After T years the value of the bond is equal to

$$B_0(1 + r/n)^{nT},$$

If the interest rate is *continuously compounded*, that is, the writer pays the interest constantly, then the value of the bond after T years equals

$$\lim_{n\to\infty} B_0(1 + r/n)^{nT} = B_0 e^{rT}.$$

Arbitrage Opportunities

Often the same asset is traded on different asset markets, with different bid prices and ask prices. The *ask* price of an asset is the price for which you can buy the asset, and the *bid* price of an asset is the price for which you can sell the asset on the financial market. In general, the ask price of an asset is higher than the bid price. If this would not be the case the market provides an unrefusable offer: We could buy the asset and sell it at the same time, which leaves us a profit equal to the difference between the bid price and ask price. If this is the case, that is, if we can make a trade of various financial products without running any risk such that we end up with more cash than we started with, this is called an *arbitrage opportunity*. Note that this is also possible via a combination of multiple financial markets: If one asset is traded on two markets, where the bid price on one market is lower than the ask price on the other, we could buy the asset on the first market and sell it simultanuously on the other, locking in a profit equal to the difference between the bid price and the ask price. An arbitrage opportunity could also exist of more than one financial product. For example, we could buy a stock on a stock market in one currency; sell it on another stock market for an amount of cash in another currency; with that cash we buy some gold bars; and these gold bars we sell for an amount of money in the first currency. If we end up with more cash than we had before, this is an arbitrage opportunity.

It is sometimes also possible to sell assets that we do not have. If we do this, and we have a negative amount of the asset on our balance, we say that we have a *short* position of the asset. If we have a possitive amount of an asset on our balance, we say that we have a *long* position of the asset. When it is possible to take a short position of the products of an arbitrage opportunity, we can trade the assets involved in the arbitrage opportunity simultaneously, that is, we do not have to buy the stocks before we can sell them. This lowers the risk of failing to make use of the arbitrage opportunity.

Derivatives can also be used in an arbitrage opportunity in a more complicated setting. In Example 2.1 we illustrate this problem with European options on one asset. More complicated arbitrage opportunities involving several financial products can occur if the derivatives (or stocks) are mispriced. For the writers or resellers of the derivatives it is therefore necessary to quote the correct price.

Example 2.1. (Put-Call Parity)
Consider a European call option c and a European put option p, both with the same underlying asset Z, expiration date T, and strike price K. At the expiration date, the value of the call option is

$$c(T) = \max\{Z_T - K, 0\}, \tag{1}$$

and the value of the put option is

$$p(T) = \max\{K - Z_T, 0\}. \tag{2}$$

Consider two portfolio's:

1. The portfolio consisting of the European put option and the underlying asset;
2. The portfolio consisting of the European call option and an amount of cash equal to the discounted strike price.

At the time of maturity, the values of both portfolio's is equal to $\max\{Z_T, K\}$. Since it is only possible to exercise the option at the expiration date, the value of both portfolio's are also equal at each time t before the expiration date. Therefore we have the following relation, which is called the *put-call parity*:

$$p(t) + Z(t) = c(t) + Ke^{-r(T-t)},$$

for all $t \in [0, T]$. Suppose that the bid price of the European call option and the ask price of the European put option and underlying asset at time t are such that $p(t) + Z(t) > c(t) + Ke^{-rT}$, then this is an arbitrage opportunity: By selling the asset and European put option and buying the European call option, we lock in a profit of $p(t) + Z(t) - c(t) - Ke^{-rT}$.

Complete Markets

Consider a financial market with one asset and one risk-free bond, which follow the price processes Z and B respectively. Within this market we can write financial derivatives depending on only those two price processes, and introduce them into the market. The concept of financial derivatives is part of the much wider concept of *contingent claims*, where the contract can be dependent on any process, not only a price process. We present the formal definition of a contingent claim and financial derivative in Definition 2.1.

Definition 2.1. A stochastic variable X is called a *contingent claim* if the value of X at time T is determined by the stochastic process $Z = (Z(t))_{t\in[0,T]}$. If the value of X at time T is only dependent on $Z(T)$ and T, then X is called a *simple claim*. If Z is a price process, then X is called a *(financial) derivative* and a *simple (financial) derivative*, respectively.

Under certain conditions it can be shown that at time t the value of a simple financial derivative $X(t)$ is determined by $Z(t)$, for all $t \in [0, T]$. See also Theorem 2.3.

In the last decades the interest in the research on financial derivatives has increased rapidly. The main foci of interest are hedging, the pricing of financial derivatives, and optimal exercise times of American options. The pricing of financial derivatives can be quite complicated, even though the statement in the contract is very clear. A major breakthrough in pricing derivatives was the article by Black and Scholes in 1973 [6], followed by the article of Merton [29] in the same year. Coincidentally, in that same year for the first time standardised option contracts were traded on the newly founded Chicago Board Options Exchange [16]. In their articles Black, Merton, and Scholes showed that under certain model assumptions the theoretical fair price of a derivative is unique and satisfies a partial differential equation, which has become known as the Black-Scholes equation. In fact, under the assumptions of the Black-Scholes model every financial derivative can be priced in a fair way. We say that the Black-Scholes Model is *complete*, as we formally define in Definition 2.2.

Definition 2.2. A *self-financing portfolio* is a portfolio for which the purchase of a new portfolio is financed solely by selling assets already in the portfolio.

A simple claim X is said to be *reachable* or *attainable* is there exists a self-financing dynamic portfolio $\Delta(t) = (\Delta_1(t), \Delta_2(t))$ such that

$$X(T) = \Delta_1(T)Z + \Delta_2(T)B,$$

with $\Delta_1(t)$ the amount invested in the asset price process Z at time t and $\Delta_2(t)$ the amount of cash invested in a risk-free bond process B at time t, and $\Delta(t)$ only depending on information available at time t. This dynamic portfolio $\Delta(t)$ is called a *replicating portfolio*. A market is called *complete* if every simple claim can be reached.

It turns out that under the assumption that there are no arbitrage opportunities, the financial derivative can be hedged by its replicating portfolio, as we propose in Proposition 2.1.

Proposition 2.1. Consider a financial derivative F and a replicating portfolio Δ. Then under the assumption that there are no arbitrage opportunities the only price process $F(t)$ which is consistent is given by

$$F(t) = \Delta_1(t)Z + \Delta_2(t)B.$$

Proof. See Björk [4], Proposition 8.2, page 112. □

In practice the value of the derivative using the Black-Scholes model is often easy to compute if the exact solution is known; otherwise it is very time consuming. To overcome this problem approximations are used, such as finite difference methods. However, we could also approach the pricing of the financial derivatives in another way. In 1979, Cox, Ross, and Rubinstein [16] made a major breakthrough in pricing financial derivatives. They introduced a discrete method involving a recombining binomial tree to approximate the value of financial derivatives. This model is also complete, and converges to the same value of the price of a financial derivative as the value in the Black-Scholes model [16]. We will discuss the recombining binomial tree model by Cox, Ross, and Rubinstein in section 3.1. First we introduce some essential building bricks of financial mathematics.

The Geometric Brownian Motion

One of the assumptions that we make on the behaviour of assets in the financial market is that they follow a geometric Brownian motion. The basic idea is that the price of an asset will grow continuously through time at a certain rate, but is affected by some noise, which causes the price of the asset to fluctuate. A stochastic process that plays a prominent role in the Brownian motion is the Wiener process, which is defined in Definition 2.3.

Definition 2.3. A stochastic process W is called a *Wiener process* if it satisfies the following properties:

1. $W(0) = 0$;
2. The function

$$f\colon \begin{array}{ccc} \mathbb{R}^+ & \to & \mathbb{R} \\ t & \mapsto & W(t) \end{array}$$

   is continuous with probability 1;
3. For two points in time $0 \leq s < t$ the change $\delta W = W(t) - W(s)$ is $N(0, t-s)$ distributed;
4. W has independent increments, i.e., for any two time intervals $[a, b]$ and $[c, d]$, with $0 \leq a < b \leq c < d$, $W(b) - W(a)$ and $W(d) - W(c)$ are independent.

Each Wiener process satisfies the *Markov property*. This follows from the fact that any increment on the interval $[s, t]$ is independent from increments before s, and independent of the value of the Wiener process at time s.

The expected value of the change of the Wiener process is 0 and the variance of the change of the Wiener process is $(t - s)$ on any interval $[s, t]$. We want to generalise the Wiener process to a stochastic process with an expected value of the change of the Wiener process equal to $\mu(t-s)$ and the variance of the change of the Wiener process equal to $\sigma^2(t - s)$ on any interval $[s, t]$ for constants μ and σ. This generalisation is defined in Definition 2.4.

Definition 2.4. Let W be a Wiener process. Let $\mu, \sigma, X_0 \in \mathbb{R}$. Let X be the stochastic process defined by

$$dX = \mu dt + \sigma dW, X(0) = X_0,$$

Then X is called a *generalised Wiener process.*

For $0 \leq s < t$ the increment $X(t) - X(s)$ is $N\big(\mu(t - s), \sigma^2(t - s)\big)$ distributed.

Another generalisation of the Wiener process and the generalised Wiener process is to relax the restriction of μ and σ being constant, as is stated in Definition 2.5.

Definition 2.5. Let W be a Wiener process. Let X be a stochastic process, and $\mu(X, t)$ and $\sigma(X, T)$ integrable functions of X and time t, such that

$$dX = \mu(X, t)dt + \sigma(X, t)dW.$$

Then X is called an *Itō process.* If $\mu(X, t) = \hat{\mu}X$ and $\sigma(X, t) = \hat{\sigma}X$, with $\hat{\mu}$ and $\hat{\sigma}$ constants, then X is called a *geometric Brownian motion.* We call $\mu(X, t)$ the *drift rate* and $\sigma(X, t)$ the *variance rate.*

Now we are able to examine the behaviour of the asset price process via the geometric Brownian motion. However, the behaviour of contingent claims, or financial derivatives, is also interesting to examine. Itō's Lemma (Theorem 2.1) provides a good solution to this problem.

Theorem 2.1. (Itō's Lemma)
Suppose that X follows an Itō process given by

$$dX = \mu(X, t)dt + \sigma(X, t)dW,$$

where W is a Wiener process and μ and σ are functions of X and t. Let $F \colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $(x, t) \mapsto F(x, t)$ be a twice continuously differentiable function. Then F follows the process

$$dF = \left(\frac{\partial F}{\partial X}\mu + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2\right) dt + \frac{\partial F}{\partial X}\sigma dW.$$

Proof. For a proof of Itō's Lemma, see [26], Theorem 3.3, pages 149 to 153. □

A well known property of the asset price process is that it follows a log-normal distribution. The statement is given in Proposition 2.2. First we define the *log-transformed process* in Definition 2.11.

Definition 2.6. Let Z be an asset price process following an Itō process. Let the price process $\hat{Z}$ be defined by

$$\hat{Z} = \log(Z).$$

Then $\hat{Z}$ is called the *log-transformed process* of Z.

Proposition 2.2. (Log-Normal Property)
Let Z be an asset price process following the Itō process given by

$$dZ = Z\mu dt + Z\sigma dW,$$

where μ and σ are constants and W a Wiener process. Then the asset price process Z follows a log-normal distribution. Moreover, the log-transformed process $\hat{Z}$ is given by

$$d\hat{Z} = \hat{\mu}dt + \sigma dW,$$

where $\hat{\mu} = (\mu - \frac{1}{2}\sigma^2)$.

Proof. See Björk [4], Chapter 5.2, page 65. □

The Black-Scholes Model

In 1973 Black and Scholes used the model of the financial market introduced by Samuelson [36] to value options under certain conditions [6]. Their theory was a breakthrough in theory on pricing European simple financial derivatives. It is set in a complete market environment and has an exact solution. We present their model in Theorem 2.2.

Theorem 2.2. (Black-Scholes Equation)
Consider a financial market consisting of a risk-free bond B and a risky asset Z. Furthermore we make the following assumptions:

1. The price process of the asset follows the geometric Brownian motion

$$dZ = Z\mu dt + Z\sigma dW,$$

with W a Wiener process, and μ and σ constant.

2. Short selling and long selling of assets is permitted for every finite amount of the asset.
3. There are no transaction costs or taxes.
4. It is possible to trade any amount $a \in \mathbb{R}$ of any asset, so there is no restriction to whole numbers or even fractions of the asset.
5. There are no dividends during the life of the derivative.
6. There are no riskless arbitrage opportunities.
7. Trading in the assets happens continuously over time.
8. The risk-free interest rate r is constant and the same for all expiration dates. The borrowing rate equals the lending rate.

A market environment for which these conditions hold is called a *Black-Scholes model*.

Consider a Black-Scholes model with a European simple derivative F with expiration date T. Then the *Black-Scholes equation* holds:

$$rF = \frac{\partial F}{\partial t} + rZ\frac{\partial F}{\partial Z} + \tfrac{1}{2}\sigma^2 Z^2 \frac{\partial^2 F}{\partial Z^2}. \quad (3)$$

Proof. See Björk [4], Theorem 7.7, page 97. □

For European call and put options on one asset the exact solution is known. It is described in Example 2.2.

Example 2.2. Consider a European call option c and a European put option p on an asset Z with strike price K and time to maturity T. Solving the Black-Scholes equation requires some theory on stochastic integrals, which we will not elaborate. The solutions to the Black-Scholes equation are given by

$$c = Z_0 N(d_1) - KN(d_2)e^{-rT},$$
$$p = KN(-d_2) - Z_0 N(-d_1),$$

with $N(\cdot)$ the cumulative distribution function of the standard normal distribution $N(0,1)$, and

$$d_1 = \frac{\ln(Z_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(Z_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

See also [4], Proposition 7.10, page 101, and [23], Section 12.8, page 246.

Note that none of the assumptions we made are realistic. We look back at the assumptions we made and elaborate on them.

1. There is no certainty that the price of an asset follows a geometric Brownian motion. This assumption can be relaxed to the assumption that the asset price process folows a general Itō process, where μ and σ are not constant but depend on the asset price S and the time t (see [4], page 94-97). However, even then there is no certainty that this assumption holds in real life.
2. In practice the quantity one can buy or sell is limited.
3. There are transaction costs and taxes.
4. It is not possible to buy or sell all fractions of assets.
5. Some assets, like most stocks, pay dividend during the lifetime of the asset.
6. Empirical evidence suggests that the financial markets are not arbitrage free.
7. While most of the time there is a financial market open for business, assets are not traded continuously. Most assets cannot be traded any at time of the week. Although at any time during the opening hours of the market the asset can be traded, only a finite number of trades take place. Therefore continuous trading is not feasible.
8. There is no bond that is risk-free. Also, the interest rate of all bonds is variable and the borrowing rates typically are not equal to the lending rate.

An important property of the Black-Scholes model is that it is complete. Therefore every European simple derivative F has a dynamic replicating portfolio, which only depends on the time t and the asset price Z. Hence F is a function of t and Z. The result is given in Theorem 2.3.

Theorem 2.3. Consider a financial market under the Black-Scholes model. Then the following statements hold:

1. The market is complete;
2. Every European simple financial derivative F, with maturity date T, is a function of the price process Z at time t, i.e., $F(Z(t), t)$, for all $t \in [0, T]$.

Proof. See Björk [4], Theorem 8.3, page 112. □

2.2 Multivariate Model

The theory of financial derivatives on one asset described in Section 2.1 can be extended to financial derivatives depending on multiple assets. Consider a financial market with k assets and one risk-free bond which follow the vector price processes Z and price process B, respectively. Though finanical contracts depending on those price processes are not as popular as the financial derivatives depending on one asset, they are traded on a regular basis. As with derivatives on one asset, derivatives depending on multiple assets can lead to an arbitrage opportunity. Market makers therefore need to quote correct prices of their derivatives to avoid being exploited by an arbitrageur. In 1985 the Black-Scholes equation, described in Section 2.1, was extended to multiple dimensions by Cox, Ingersoll, and Ross [15]. We describe this generalised Black-Scholes model in this section. The partial differential equation of the generalised Black-Scholes model is also often difficult to solve. Therefore it is a good idea to also look at discrete models to price derivatives depending on multiple assets. We present a recombining multinomial tree which converges to the solution of the generalised Black-Scholes equation in Section 3.2. This discrete model has the property that it is complete, just like the generalised Black-Scholes model. First the formal definition of *(finincial) derivatives* and *contingent claims* on multiple assets are presented in Definition 2.7.

Definition 2.7. A stochastic vector X is called a *contingent claim* if the value of X at time T is determined by the stochastic process $Z = (Z(t))_{t\in[0,T]}$. If the value of Z at time T is only dependent on $Z(T)$ and T, then X is called a *simple claim.* If Z is a multivariate price process, then X is called a *(financial) derivative* and a *simple (financial) derivative*, respectively.

Analogous to the univariate model the multivariate complete market environment is presented in Definition 2.8.

Definition 2.8. Consider a stochastic price vector Z of length k and a risk-free bond process B. A simple claim X is said to be *reachable* or *attainable* if there exists a self-financing dynamic portfolio $\Delta(t) \in \mathbb{R}^{k+1}$ such that

$$X(T) = \sum_{i=1}^{k} \Delta_i(T)Z_i(T) + \Delta_{k+1}(T)B(T),$$

with $\Delta_i(t)$ the amount of money invested in asset i at time t, $1 \le i \le k$; $\Delta_{k+1}(t)$ the amount of money invested in the risk-free bond at time t; and $\Delta(t)$ only depending on information available at time t. This dynamic portfolio $\Delta(t)$ is called a *replicating portfolio.* A market environment is called *complete* if every simple claim can be reached.

Just as in the univariate model the simple financial derivative can be hedged by its replicating portfolio, as we propose in Proposition 2.3.

Proposition 2.3. Consider a simple financial derivative F and a replicating portfolio Δ. Then under the assumption that there are no arbitrage opportunities the only price process $F(t)$ which is consistent is given by

$$F(t) = \Delta_1(t)Z + \Delta_2(t)B.$$

Proof. See Björk [4], Proposition 8.2, page 112. □

The Multivariate Geometric Brownian Motion

In the univariate model described in Section 2.1 we assumed that the asset price follows a geometric Brownian motion. The generalisation to multiple dimensions takes into account the correlation between the geometric Brownian motions. The assumption we make is that the multivariate price process Z follows a *multivariate geometric Brownian motion*, which we define in Definition 2.9.

Definition 2.9. Let $Z = (Z_1, \dots, Z_k)^\top$ be the multivariate price process defined as

$$dZ_i = Z_i \mu_i dt + Z_i \sigma_i dW_i,$$

with W a vector of correlated Wiener processes such that $E[dW_i dW_j] = \rho_{ij} dt$ for $i \neq j$, where ρ_{ij} is the covariance between Wiener process i and j; $E[dW_i^2] = dt$; and $dt^2 = 0$. Then Z is called a *multivariate geometric Brownian motion*.

In Proposition 2.2 it was shown that the natural logarithm of the geometric Brownian motion follows a univariate normal distribution. Conveniently the multivariate geometric Brownian motion has a similar property, namely that the element wise natural logarithm of the multivariate geometric Brownian motion follows a multivariate normal distribution[2]. This is shown in Proposition 2.4. First the formal definition of this price process and the generalisation of Itō's Lemma (Theorem 2.1) are given in Definition 2.11 and Theorem 2.4.

Definition 2.11. Let Z be a process given by a k-variate geometric Brownian motion and let the k-variate vector price process $\hat{Z}$ be defined by

$$\hat{Z}_i = \log Z_i,$$

for all $1 \leq i \leq k$. Then $\hat{Z}$ is called the *log-transformed price process* of Z.

Theorem 2.4. (Itō's Lemma)
Suppose X follows a multivariate geometric Brownian motion in k dimensions given by

$$dX_i = X_i \mu_i dt + X_i \sigma_i dW_i,$$

with W a vector of correlated Wiener processes such that $E[dW_i dW_j] = \rho_{ij} dt$ for $i \neq j$, $E[dW_i^2] = dt$, and $dt^2 = 0$. Let $F\colon \mathbb{R}^k \times \mathbb{R}^+ \to \mathbb{R}$, $(x,t) \mapsto F(x,t)$ be a twice continuously partially differentiable function. Then F follows the process

$$dF = \frac{\partial F}{\partial t} dt + \sum_{i=1}^{k} \frac{\partial F}{\partial X_i} dX_i + \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 F}{\partial X_i \partial X_j} dX_i dX_j.$$

[2]

Definition 2.10. Let $Z = (Z_1, \dots, Z_n)^\top$ be such that the elements of Z are independent standard normal distributed random variables. We say that a random vector X is *multivariate-normally distributed* with parameters μ and Σ if it has the same distribution as the vector $\mu + LZ$, where L is a matrix such that $\Sigma = LL^\top$. We write $N_k(\mu, \Sigma)$ for the multivariate normal distrubution.

Proof. See Björk [4], Theorem 4.16, page 54. □

Proposition 2.4. Let Z be a multivariate geometric Brownian motion in k dimensions. Define $\hat{Z} = \log Z$. Then the price process $\hat{Z}$ is given by

$$d\hat{Z}_i = \hat{\mu}_i dt + \sigma_i dW_i,$$

for all $1 \leq i \leq k$, with

$$\hat{\mu} = \begin{pmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_k \end{pmatrix} = \begin{pmatrix} \mu_1 - \frac{1}{2}\sigma_1^2 \\ \vdots \\ \mu_k - \frac{1}{2}\sigma_k^2 \end{pmatrix}.$$

Furthermore, Z follows a k-variate log-normal distribution, and $\hat{Z}$ follows the k-variate normal distribution $N_k(\hat{\mu}dt, \Sigma dt)$, where Σ is the $k \times k$ covariance matrix given by

$$\Sigma_{ij} = \sigma_i \sigma_j \rho_{ij},$$

for all $1 \leq i, j \leq k$.

Proof. See Appendix A. □

The Multivariate Black-Scholes Model

The Black-Scholes model for pricing simple derivatives on one asset can be extended to multiple assets. Cox, Ingersoll, and Ross presented this generalisation in 1985 [15]. We present their result in Theorem 2.5.

Theorem 2.5. Let $n \in \mathbb{N}$ and consider a financial market consisting of a risk-free bond B and k risky asset $\{Z_i\}_{i=1}^k$. Furthermore we make the following assumptions:

1. The price vector Z of the assets follows the multivariate geometric Brownian motion

$$dZ_i = Z_i \mu_i dt + Z_i \sigma_i dW_i.$$

2. Short selling and long selling of assets is permitted for every finite amount of the asset.
3. There are no transaction costs or taxes.
4. It is possible to trade any amount $a \in \mathbb{R}$ of any asset, so there is no restriction to whole numbers or even fractions of the asset.
5. There are no dividends during the life of the derivative.
6. There are no riskless arbitrage opportunities.
7. Trading in the assets happens in continuous time.
8. The risk-free rate of interest r is constant and the same for all maturities. The borrowing rate equals the lending rate.

Then this market environment is called the *(generalised) Black-Scholes model.*

Consider a Black-Scholes model with a European simple derivative F with expiration date T. Then the *(generalised) Black-Scholes equation* holds:

$$rF = \frac{\partial F}{\partial t} + rZ^\top \frac{\partial F}{\partial Z} + \tfrac{1}{2} tr\left(\sigma^\top DHD\sigma\right), \tag{4}$$

where $tr(\cdot)$ is the trace of the $k \times k$ matrix and H is the Hessian of F, and D is given by

$$D = \begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z_k \end{pmatrix}.$$

.

Proof. For a proof of this Theorem, see Cox, Ingersoll, and Ross [15]. □

Example 2.3. Consider a Black-Scholes market with two assets $Z = (Z_1, Z_2)$ and a European simple derivative F with time T payoff function equal to

$$\max\{Z_1(T) - Z_2(T), 0\}.$$

In 1978 Fischer [18] and Margrabe [28] independently found the analytical solution F^* to price this option. It is given by

$$F^* = Z_1(0)N(u) - Z_2(0)N(v),$$

where u and v are given by

$$v = \sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)T},$$
$$u = \frac{\log(Z_1(0)/Z_2(0)) + v^2/2}{v}.$$

As is the case in the one dimenional case, the generalised Black-Scholes model is complete and thus every simple derivative can be replicated. The formal statements are given in Theorem 2.6.

Theorem 2.6. Consider a financial market under the generalised Black-Scholes model. Then the following statements hold:

1. The market is complete;
2. Every European simple financial derivative F, with time to maturity T, in the market is a function of the multivariate asset price process $Z(t)$ and time t, i.e., $F(Z(t), t)$, for all $t \in [0, T]$.

Proof. See Björk [4], Theorem 8.3, page 112. □

3 Recombining Multinomial Trees

On financial markets options on one or more assets are traded. To avoid arbitrage opportunities, the market makers want to quote the correct price. However, finding the correct price is not always straightforward. An analytical solution by the Black-Scholes equation can be difficult to find. Therefore discrete methods to approximate the analytical solution are used. In this section we propose a discrete method to approximate the price of simple derivatives on multiple assets. This discrete method consists computing one or more elements of a sequence of recombining multinomial trees based on Pascal's simplex which converges to the continuous time analytical solution. It is a generalisation of the popular binomial tree method, which can provide a good approximation for pricing options on one asset. Before we propose the generalisation we give a new insight in the binomial model in Section 3.1. The derivation of the recombining binomial tree is derived via Pascal's triangle. This view serves as a foundation for the generalisation to recombining multinomial trees based on Pascal's simplex in Section 3.2.

3.1 Univariate Model

In 1979, Cox, Ross and Rubinstein made a major breakthrough by introducing a method to approximate the value of options on a single asset Z via a binomial tree [16]. We will explain this concept in a way that is not known in literature, to our knowledge. It is based on the two-dimensional grid $\mathbb{N}^2$ and Pascal's triangle. The two-dimensional derivation presented in this section is a good foundation to extend to multiple dimensions, as we will show in Section 3.2.

The goal of this section is to construct a recombining binomial tree with N levels to approximate the price process of an asset Z and the price of an option. This tree can be interpreted as follows. In each timestep, the value of the asset can go either up or down (although $0 < d < u$ is all that is required), where the asset price at time 0 equals Z_0. Each node in the tree represents a possible outcome of the asset price and has a certain probability to be reached. However, it is important to realise that we do not assume that the asset price has only two possible outcomes in the next timestep. The main goal is to approximate the price process, and in what follows it turns out that the recombining binomial tree we construct does the trick. We will construct the tree in several steps, starting with the construction of a random walk X on $\mathbb{N}^2$. This lattice already has the looks of the recombining binomial tree, but needs to be adjusted to match the asset price Z. Using some linear algebra we transform the random walk on $\mathbb{N}^2$ to a random walk X' on $\mathbb{R}$ in a specific way. We show in Theorem 3.1 that an associated sequence of random variables converges to the asset price process Z at the expiration date. Furthermore, this tree can be used to price derivatives. The motivation for this construction is linked to Pascal's triangle and the convergence of a sequence of recombining multinomial trees to the asset price process Z.

Two-Dimensional Lattice and Pascal's Triangle

Consider the two-dimensional Cartesian product $\mathbb{N}^2$. On this lattice, consider a random walk

$$X = \{X_n\}_{n=0}^{\infty} \subset \mathbb{N}^2,$$

starting at $X_0 = (0,0) \in \mathbb{N}^2$. In each step of the random walk we are allowed to move one unit in the positive direction in one of the axes. We move with probability p in the positive direction of the first axis of $\mathbb{N}^2$ ($x \mapsto x + e_1$), and we move with probability $1-p$ in the positive direction of the second axis of $\mathbb{N}^2$ ($x \mapsto x + e_2$). The random walk X is thus defined as follows:

$$X_n := \sum_{i=1}^{n} Y_i,$$

where the sequence $Y = \{Y_n\}_{n\in\mathbb{N}}$ is defined by a sequence of i.i.d. random vectors Y_n with

$$P(Y_n = e_1) = p,$$
$$P(Y_n = e_2) = 1 - p.$$

After N steps there is just a finite number of possible values which X_N can take. Say after N steps, there have been i moves in the direction of the first axis and $N - i$ moves in the direction of the second axis, with $0 \leq i \leq N$. Furthermore, there are $\binom{N}{i}$ possible walks which lead to $X_N = (i, N - i)$, of which each walk occurs with probability $p^i(1 - p)^{N-i}$. Therefore, the probability that $X_N = (i, N - i)$ equals

$$P(X_N = (i, N - i)) = \binom{N}{i} p^i (1 - p)^{N-i},$$

for all $0 \leq i \leq N$. We see that at level N the value of X_N has a binomial distribution $B(N, p)$.

If we construct the lattice this way we see that we are actually constructing Pascal's triangle. After N steps we arrive at the N-th level of Pascal's triangle. See Figure 1 for a visualisation of the construction of this two-dimensional lattice.

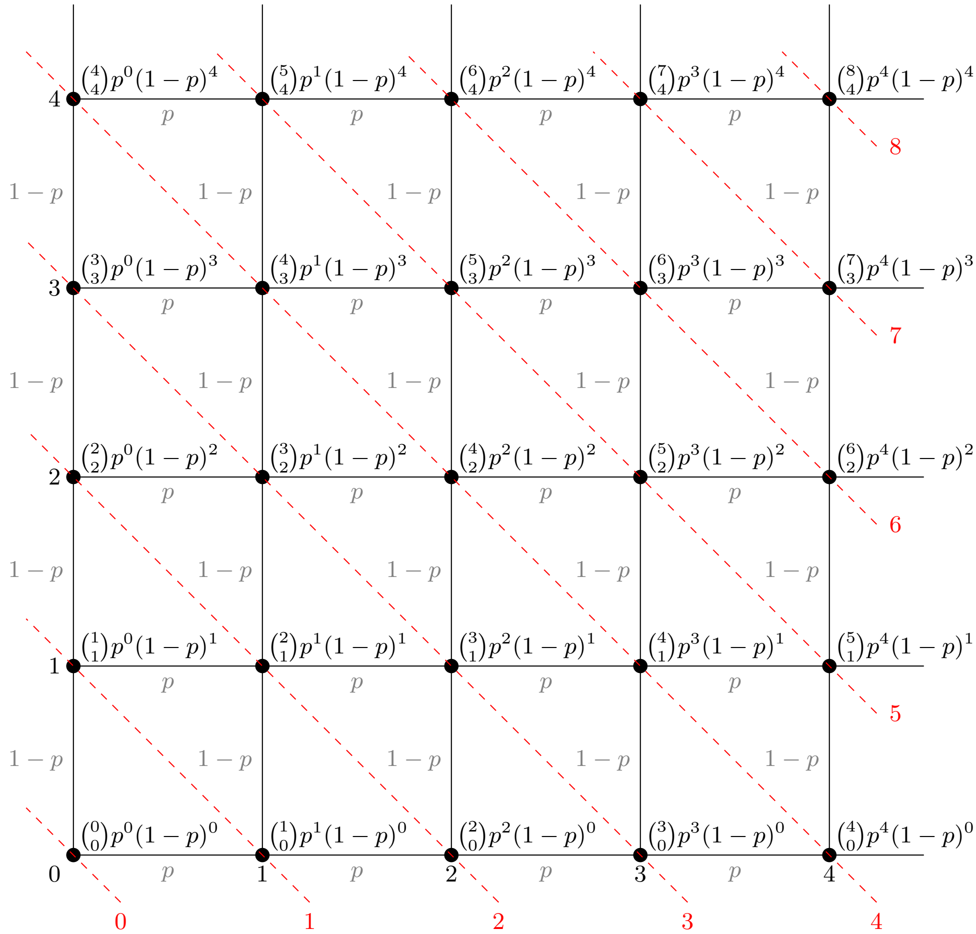

Figure 1: Visualisation of the Pascal-based two-dimensional lattice. Each red dotted line represents the level at which we arive after n steps, where n is the corresponding number in red.

One-Dimensional Representation of the Two-Dimensional Random Walk

Projecting the random walk X on the hyperplane orthogonal to $\iota = (1, 1)$ yields a random walk on the line orthogonal to ι. Now if we rotate the whole system clockwise by 45 degrees such that ι coincides with the unit vector e_1, the projected random walk coincides with a random walk on the second axis of $\mathbb{R}^2$. By eliminating the first axis we find a random walk X' on $\mathbb{R}$.

The random walk X' on $\mathbb{R}$ can also be found in the following way. First the system is rotated such that e_1 coincides with ι. Then the image is reflected on the line through ι. Finally the system is projected on the plane orthogonal to e_1 through $(0, 0)$. This results in a random walk on the hyperplane orthogonal to e_1 through $(0, 0)$. We use this method because it turns out that this method works well in the generalisation to multiple dimensions in Section 3.2. The orthogonal matrix that represents the rotation and reflection is

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The matrix that projects the unit vectors on $\mathbb{R}$ is therefore $M = (1/\sqrt{2}, -1/\sqrt{2})$. By projecting the random walk X on $\mathbb{R}$ via M, we find a new random walk $X' = \{X'_n\}$ on $\mathbb{Z}$ defined by

$$X'_n := MX_n,$$

for all $n \geq 0$. Note that X' starts in the origin, i.e., $X'_0 = 0$. In each step, the probability to move up by one unit is equal to p, and the probability to move down by one unit is equal to $1 - p$. See Figure 2 for a visualisation of the two-dimensional rotation, and Figure 3 for the projection of the two-dimensional random walk to the one-dimensional walk. In this visualisation we use a different oriantation for convencience.

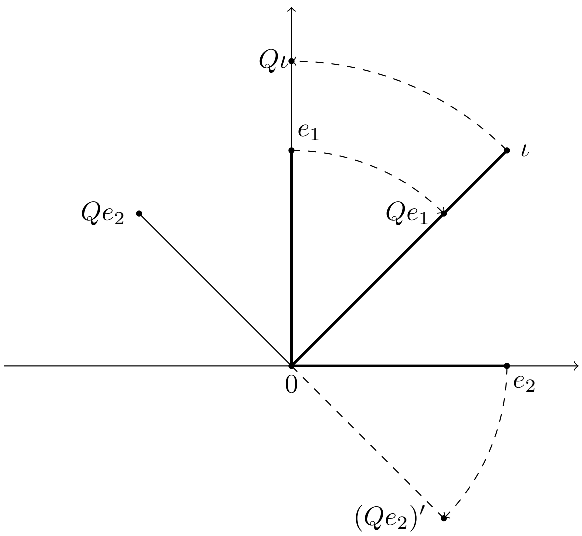

Figure 2: The rotation of the unit vectors and ι by the orthogonal matrix Q. $(Qe_2)'$ is the reflection of Qe_2 in the line through ι.

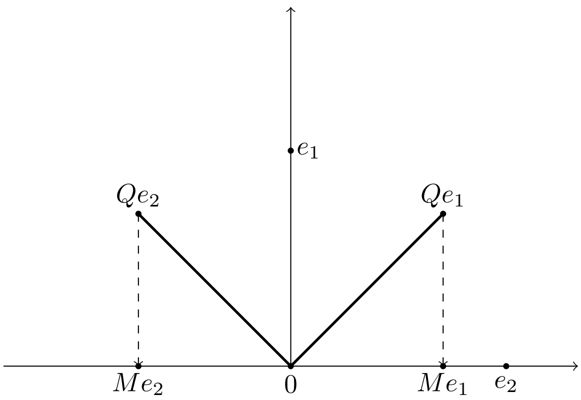

Figure 3: The projection of the rotated unit vectors on $\mathbb{R}$ by M.

Construction of the Recombining Binomial Tree

Let $Y' = \{Y'_n\}_{n\in\mathbb{N}}$ be defined by the projection of Y on $\mathbb{R}$ via M, i.e.,

$$Y'_n := MY_n,$$

for all $n \in \mathbb{N}$. Then Y' consists of a sequence of i.i.d. random variables $\{Y'_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}$, with

$$P(Y'_n = 1/\sqrt{2}) = p,$$
$$P(Y'_n = -1/\sqrt{2}) = 1 - p.$$

Also we find for X' that

$$X'_N = MX_N = M\sum_{n=0}^{N} Y_n = \sum_{n=0}^{N} MY_n = \sum_{n=0}^{N} Y'_n.$$

Theorem 3.1. Suppose that $p = \frac{1}{2}$, and let $\sigma > 0$, $\mu \geq 0$. Then the sequence

$$\{\sigma\sqrt{2/N}X'_N + \mu\}_{N\in\mathbb{N}},$$

converges in distribution to the normal distribution $N(\mu, \sigma^2)$.

Proof. See Appendix A. □

The result of Theorem 3.1 is of significant importance for approximating the price process of assets. Consider a price process Z of an asset that follows the geometric Brownian motion given by

$$dZ = Z\mu dt + Z\sigma dW.$$

By virtue of Proposition 2.2 $\log Z$ has the normal distribution $N(\hat{\mu}T, \sigma^2 T)$ on any interval of length T, with $\hat{\mu} = \mu - \frac{1}{2}\sigma^2$. By applying Theorem 3.1, we see that we can approximate the price process $\log Z$ on any interval of length T by the sequence

$$\{\sigma\sqrt{2\delta t}X'_N + \hat{\mu}T\}_{N\in\mathbb{N}},$$

with $p = \frac{1}{2}$ and $\delta t = T/N$. The elements of this sequence can be rewritten such that for each $N \in \mathbb{N}$ we have that

$$\sigma\sqrt{2\delta t}X'_N + \hat{\mu}T = \sum_{n=1}^{N} \left\{\sigma\sqrt{2\delta t}Y'_n + \hat{\mu}\delta t\right\}. \tag{5}$$

Since Y'_n is a random draw from $\{1/\sqrt{2}, -1/\sqrt{2}\}$, we can interpret equation (5) as the sum of random draws of $\{\sigma\sqrt{\delta t} + \hat{\mu}\delta t, -\sigma\sqrt{\delta t} + \hat{\mu}\delta t\}$. In each timestep δt, we can move in one of these directions. We use this fact to define the *direction vectors* for the log-normal process in Definition 3.1

Definition 3.1. Let d_1 and d_2 be given by

$$d_1 = \exp\left\{\sigma\sqrt{\delta t} + \hat{\mu}\delta t\right\}, \qquad d_2 = \exp\left\{-\sigma\sqrt{\delta t} + \hat{\mu}\delta t\right\}. \tag{6}$$

Then d_1 and d_2 are called the *direction vectors*.

With the direction vectors the log-transformed price process $\hat{Z}$ at time T can thus be approximated by summing the log-transformed asset price at time 0 with $N \in \mathbb{N}$ independent random draws from the distribution χ_N, with χ_N given by

$$P\left(\chi_N = \log d_1\right) = \tfrac{1}{2},$$
$$P\left(\chi_N = \log d_2\right) = \tfrac{1}{2}.$$

The price process Z at time T on the other hand can be approximated by multiplying the asset price at time 0 with $N \in \mathbb{N}$ independent random draws from the distribution $\chi'_N := \exp(y_N)$, with χ'_N given by

$$P\left(\chi'_N = d_1\right) = \tfrac{1}{2},$$
$$P\left(\chi'_N = d_2\right) = \tfrac{1}{2}.$$

Definition 3.2. Let $N \in \mathbb{N}$. Then the graph containing all possible paths of the random walk with N timesteps described above is called a *recombining binomial tree.*

See Figure 4 for a visualisation of a recombining binomial tree with $N = 4$. Note the similarities between the recombining binomial tree in Figure 4 and the two-dimensional lattice in Figure 1. Pascal's triangle

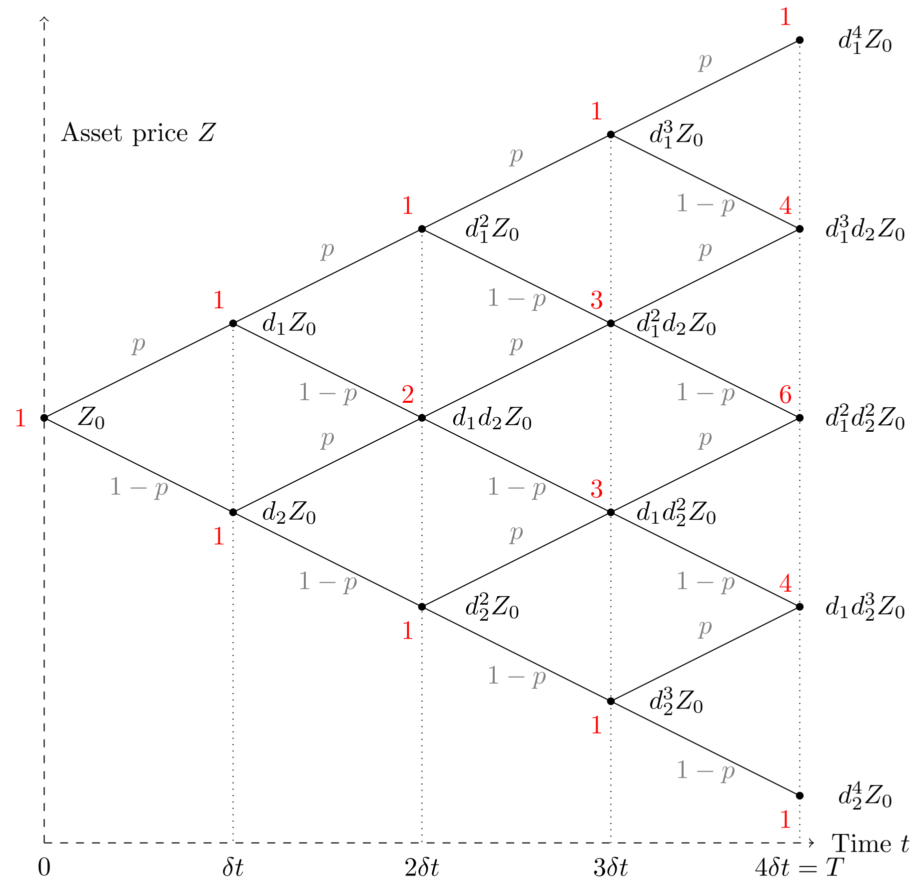

Figure 4: Visualisation of a four-step binomial tree. The red numbers indicate the number of ways to get to the corresponding node.

can be viewed as the log-transformed tree corresponding to the recombining multinomial tree. Therefore the recombining binomial tree can be viewed as a lattice. This nice structure of the recombining binomial trees implies the potential of the trees to be implemented in an efficient algorithm. We elaborate on the conseqequences of this property in Section 4.3.

If the movement is in the first direction d_1, we call this a move *upward*, and if the movement is in the second direction d_2, we call this a move *downward*. In literature the upward move d_1 is often addressed as u, and the downward move d_2 is often addressed as d. The direction vectors d_1 and d_2 coincide in the binomial case with the direction vectors of the generalisation of the recombining binomial tree to a recombining multinomial tree described in Section 3.2.

Completeness of the Model and Pricing Derivatives on One Asset

A nice and well known property of the recombining binomial tree is that it is complete, which we show in Theorem 3.2. Therefore we can find a replicating portfolio for any simple financial derivative, which we can use to price the simple financial derivative.

Theorem 3.2. Let $N \in \mathbb{N}$ and consider a European simple financial derivative F on one asset Z. Let $\sigma \neq 0$ and $\mu \geq 0$. Let d_1 and d_2 be the direction vectors. Then the recombining binomial tree with N levels described by the direction vectors is complete.

Proof. See Appendix A. □

Consider a simple derivative F with the expiration date at time T, and suppose we know the value of the derivative $F(Z)$ at time T for any Z. Consider the recombining binomial tree with N steps of length $\delta t = T/N$ corresponding to the underlying asset Z of F. Then we can calculate $F(\nu)$ for any node ν in the recombining binomial tree at time T. Suppose that the value $F(\nu')$ is know for all nodes at time $m\delta t$, $1 \leq m \leq N$. Let ν be a node at time $(m-1)\delta t$ and let $\nu_1 = \nu + e_1$ and $\nu_2 = \nu + e_2$. By virtue of Theorem 3.2 we can find a replicating portfolio $\Delta = (\Delta_1, \Delta_2)$. For example we can use Equation (8) in Appendix A to find Δ. If F is European, the value of $F(\nu)$ equals

$$F(\nu) = \Delta_1 Z(\nu) + \Delta_2 e^{-r\delta t}.$$

If however F is American, we have to take in account the possibility of an early exercise. The early exercise leads to a payoff of according to the payoff function F_{payoff}. Therefore the value of the American option $F(\nu)$ equals

$$F(\nu) = \max\left\{F_{\text{payoff}}(\nu), \Delta_1 Z(\nu) + \Delta_2 e^{-r\delta t}\right\}.$$

By working backwards through the recombining binomial tree we can find the value of the option $F(0)$ (American or European). For the American option the recombining binomial tree has an advantage over other methods: It also shows what the optimal exercise time is. More theory on recombining binomial trees on American options on one asset can be found in Shreve [37] and Shreve [38].

Illustration of a Recombining Binomial Tree

In this section we derived a discrete method to approximate the value of simple derivatives on one asset via recombining binomial trees. We conclude this section with Example 3.1 for an illustration of this procedure. This example is also used by Cox, Ross, and Rubinstein [16].

Example 3.1. Consider a market environment with one asset Z and a risk-free bond with risk-free interest rate $r = 0.05$. At time 0 the asset price equals $Z(0) = 40$. The standard deviation is given by $\sigma = 0.2$. Suppose that we want to consider a European call option c depending on Z with expiration date in one month, so that $T = 1/12$ if we take one year as one unit of time. We will calculate the option price at time 0 using a recombining binomial tree in two steps, i.e., $N = 2$. The direction vectors are thus given by

$$d_1 = 1.043$$
$$d_2 = 0.9611.$$

We can construct a recombining binomial tree with the given direction vectors. The two-step tree consists of the nodes (i,j) with i upward moves and j downward moves, $0 \leq i, j \leq 2$. In each node we can evaluate the asset price $Z(i,j)$. Then we can evaluate value of the option $c(i,j)$ at the three nodes in the recombining binomial tree at time T, i.e., at the nodes (i,j) such that $i + j = 2$, using Equation (1). Using the replicating portfolio we can calculate the value of the option in the other nodes. We find that the approximation of the recombining binomial tree of the option at time 0 equals

$$c(0) = 5.142.$$

We can evaluate the exact value according to the Black-Scholes using the equations from Example 2.2. This gives

$$c(0) = 5.148.$$

A visualisation of the tree with asset prices and option prices is given in Figure 5.

The approximations of call options with strike price 35, 40, and 45 by recombining binomiral trees with 5, 10, 20, and 50 steps are listed in Table 1, accompanied by the analytical solution by Example 2.2.

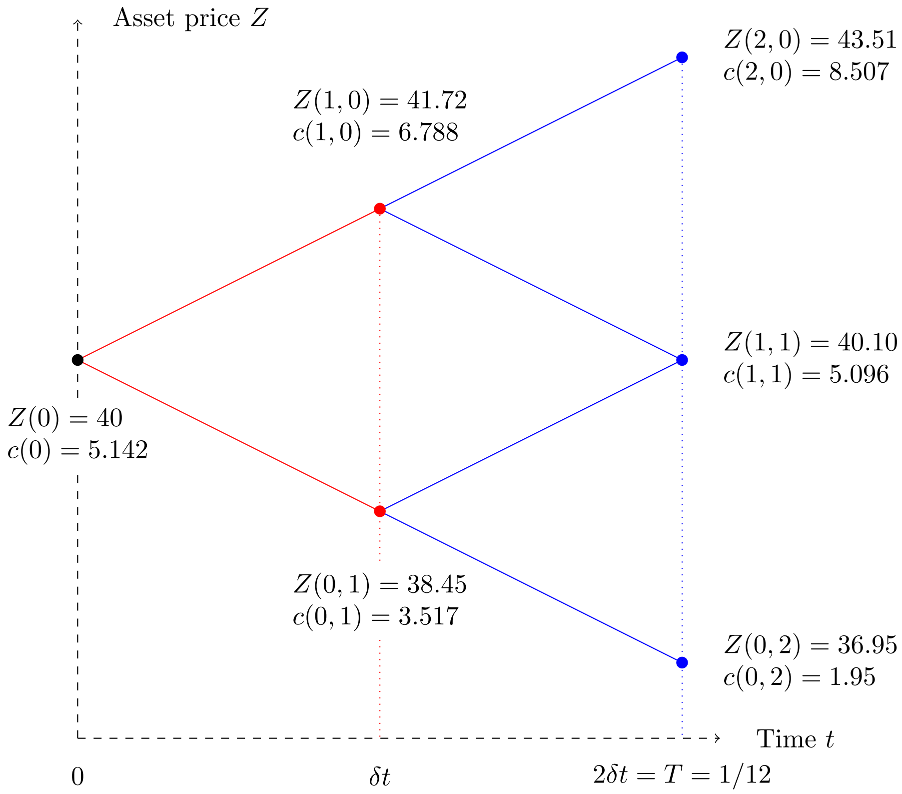

Figure 5: Approximation of the value of a call option c with strike price $K = 35$ and time to maturity $T = 1/12$ on a single asset Z with inital value $Z(0) = 40$, standard deviation $\sigma = 0.2$, and risk-free interest rate $r = 0.05$.

Strike price	Number of timesteps				Analytical solution
	5	10	20	50	
35	5.142	5.147	5.147	5.148	5.148
40	1.049	0.991	0.999	1.003	1.003
45	0.01934	0.01668	0.02168	0.02112	0.02250

Table 1: Approximation of the value of call options with strike price 35, 40, and 45 and time to maturity $T = 1/12$ on a single asset Z with inital value $Z(0) = 40$, standard deviation $\sigma = 0.2$, and risk-free interest rate $r = 0.05$.

3.2 Multivariate Model

Many attempts to generalise the Cox-Ross-Rubinstein model have been made, some more succesful than others. There are two main generalisations. The first generalises the number of branches, but keeps the number of underlying assets constant, namely equal to one. The second generalises the number of onderlying assets from one to many. An example of the first case is the 1986 article by Boyle [8], where he introduced a trinomial tree to price options on one asset. Unfortunately, this model lacked the property of a complete market environment. An example of the second case is the tree-based model to price options depending on two assets Boyle introduced two years later [9]. Again these models are not set up in a complete market environment. In 1990 He introduced a satisfactory discrete model that approximates the price of options depending on multiple assets in a complete market environment [20]. He showed that the derivative prices derived from a discrete time model converge to the derivative price derived from the generalised Black-Scholes model. We will show how we can derive the discrete time multivariate generalisation of the recombining binomial tree using Pascal's simplex, the generalisation of Pascal's triangle. Furthermore, we will give some examples of how to price simple derivatives depending on multiple assets in Section 4.1, using the recombining multinomial tree derived in this Section.

$k+1$-Dimensional Lattice and Pascal's Simplex

The recombining binomial tree method in section 3.1 is based on Pascal's triangle. In the multivariate model the generalisation is based on Pascal's simplex. Pascal's simplex is the generalisation of Pascal's triangle in higher dimensions. The numbers in Pascal's triangle are based on the coefficients of powers of a *binomial*, a polynomial with two terms. The coefficients of the n-th level of Pascal's triangle are the coefficients of the n-th power of the binomial $(x+y)$. The Binomial Theorem states that for all binomials $(x+y)$ and $n \in \mathbb{N}$ the following statement holds:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-1}.$$

The coefficients in the n-th level of Pascal's triangle are therefore given by $\binom{n}{i}$, $0 \le i \le n$.

Pascal's simplex is constructed in a similar way in higher (and lower) dimensions. The elements of Pascals simplex in k dimensions are the coefficients of powers of a *multinomial*, a polynomial with k terms. The Multinomial Theorem states that for any multinomial $(x_1 + \ldots + x_k)$ we have that

$$(x_1 + \ldots + x_k)^n = \sum_{n_1+\ldots+n_k=n} \binom{n}{n_1, \ldots, n_k} \prod_{i=1}^{k} x_i^{n_i}.$$

The elements in the n-th level of Pascal's simplex therefore consist of the multinomial coefficients

$$\binom{n}{n_1, \ldots, n_k},$$

where $n_1 + \ldots + n_k = n$.

In section 3.1 Pascal's triangle was constructed on the two-dimensional lattice $\mathbb{N}^2$. Pascal's simplex can be constructed on a $(k+1)$-dimensional lattice $\mathbb{N}^{k+1}$ in a similar way. Consider a random walk $X = \{X_n\}_{n\ge0}$ on $\mathbb{N}^{k+1}$, such that in each step we move one unit in the positive direction of only one of the axes of $\mathbb{N}^{k+1}$. We move in the direction of the i-th axis with probability p_i, with $p_1 + \ldots + p_{k+1} = 1$. Let $\{Y_n\}_{n\ge1}$ be a sequence of i.i.d. random vectors in $\mathbb{N}^{k+1}$ such that

$$P(Y_n = e_i) = p_i,$$

for all $1 \le i \le k+1$. Then the random walk X is defined as

$$X_N := \begin{cases} \sum_{n=1}^{N} Y_n & \text{if } N > 0, \\ (0, \ldots, 0) & \text{if } N = 0, \end{cases}$$

for all $N \in \mathbb{N}$. Suppose that after N steps, there have been n_i moves in the direction of the i-th axis, for all $1 \le i \le k+1$. There are exactly $\binom{N}{n_1,\ldots,n_{k+1}}$ routes to get to this point, and each route has probability $p_1^{n_1} \cdot \ldots \cdot p_{k+1}^{n_{k+1}}$. The probability to get to $X_N = (n_1, \ldots, n_{k+1})$ therefore equals

$$P\left(X_N = (n_1, \ldots, n_{k+1})\right) = \binom{N}{n_1, \ldots, n_{k+1}} \prod_{i=1}^{k+1} p_i^{n_i}.$$

Derivation of the M-vectors: Method I

The movements in $(k+1)$-dimensions need to be translated into a hyperplane of k-dimensions, such that we can link them to the assets. We therefore project the random walk on the hyperplane orthogonal to the vector $\iota = (1, \ldots, 1)$. Since the random walk is defined by the unit vectors, we only need to look at the unit vectors. After the projection the $(k+1)$ unit vectors are represented in a k-dimensional hyperplane in $\mathbb{R}^{k+1}$, but we prefer a representation in $\mathbb{R}^k$. We can apply a rotation on the whole system such that ι coincides with the first unit vector e_1. The $(k+1)$ rotated unit vectors now lie in a hyperplane orthogonal to e_1. Therefore these vectors can be represented in $\mathbb{R}^k$ by the projection on the plane orthogonal to e_1 (the first entries of all $(k+1)$ projected vectors equal 0). We call the set of $(k+1)$ vectors thus constructed the *M-vectors* (see also Definition 3.3 below).

To find the M-vectors we can also first apply the rotation, and then project the $(k+1)$ rotated vectors on the hyperplane orthogonal to the first unit vector. The difference is subtle, but turns out to be a more efficient calculation. A visualisation of this process for $k=2$ is given in Figure 6 and Figure 7. Again consider the $(k+1)$ unit vectors and ι in $\mathbb{R}^{k+1}$. By applying the correct orthogonal matrix $Q^\top$ to these vectors, we can rotate the system properly. After the rotation the unit vectors form an orthogonal set of $k+1$ vetors in $\mathbb{R}^{k+1}$. These are the columns of an orthogonal matrix $Q^\top$. Because the rotated unit vectors lie in a plane orthogonal to the first unit vector, the first row of $Q^\top$ is a constant vector with length 1 such that all entries are equal to $1/\sqrt{k+1}$. The transpose Q has first column vectors equal to $\iota/\sqrt{k+1}$ and other column vector arbitrary with length 1 such that Q is orthogonal. We can construct this orthogonal matrix Q by first applying the Gram-Schmidt process[3] to ι and k of the $k+1$ unit vectors, of which the last already form an orthonormal basis of $\mathbb{R}^{k+1}$. Note that ι and any k of the $k+1$ unit vectors form a basis of $\mathbb{R}^{k+1}$. Then by normalizing the orthogonal basis constructed by the Gram-Schmidt process we find an orthonormal basis Q. The result is given in Proposition 3.1.

Proposition 3.1. The orthonormal basis constructed by applying the Gram-Schmidt process and normalisation on ι and the unit vectors $\{e_1, \ldots, e_k\}$ in $\mathbb{R}^{k+1}$ is described by the column vectors of the $(k+1)\times(k+1)$ orthogonal matrix Q given by

$$Q(i,j) = \begin{cases} \dfrac{1}{\sqrt{k+1}} & \text{if } j=1, \\ \sqrt{\dfrac{k+2-j}{k+3-j}} & \text{if } j=i+1, \\ -\dfrac{1}{\sqrt{(k+2-j)(k+3-j)}} & \text{if } 1<j<i+1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $1 \le i,j \le k+1$, i.e.,

$$Q = \begin{pmatrix} \frac{1}{\sqrt{k+1}} & \sqrt{\frac{k}{k+1}} & 0 & \cdots & 0 & \cdots & 0 \\ \frac{1}{\sqrt{k+1}} & -\frac{1}{\sqrt{(k+1)k}} & \sqrt{\frac{k-1}{k}} & \cdots & 0 & \cdots & 0 \\ \frac{1}{\sqrt{k+1}} & -\frac{1}{\sqrt{(k+1)k}} & -\frac{1}{\sqrt{k(k-1)}} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{k+1}} & -\frac{1}{\sqrt{(k+1)k}} & -\frac{1}{\sqrt{k(k-1)}} & \cdots & \sqrt{\frac{j}{j+1}} & \cdots & 0 \\ \frac{1}{\sqrt{k+1}} & -\frac{1}{\sqrt{(k+1)k}} & -\frac{1}{\sqrt{k(k-1)}} & \cdots & -\frac{1}{\sqrt{(j+1)j}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{k+1}} & -\frac{1}{\sqrt{(k+1)k}} & -\frac{1}{\sqrt{k(k-1)}} & \cdots & -\frac{1}{\sqrt{(j+1)j}} & \cdots & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{k+1}} & -\frac{1}{\sqrt{(k+1)k}} & -\frac{1}{\sqrt{k(k-1)}} & \cdots & -\frac{1}{\sqrt{(j+1)j}} & \cdots & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Proof. See Appendix A. □

[3] The *Gram-Schmidt process* transforms a basis to an orthogonal basis. If $x=(x_1,\ldots,x_{k+1})$ is a basis of $\mathbb{R}^{k+1}$, then the orthogonal basis $v=(v_1,\ldots,v_{k+1})$ can be found by the following formulae:

$$\begin{aligned} v_1 &= x_1, \\ v_2 &= x_2 - \frac{(v_1,x_2)}{(v_1,v_1)}v_1, \\ v_3 &= x_3 - \frac{(v_1,x_3)}{(v_1,v_1)}v_1 - \frac{(v_2,x_3)}{(v_2,v_2)}v_2, \\ &\vdots \\ v_{k+1} &= x_{k+1} - \sum_{i=1}^{k} \frac{(v_i,x_{k+1})}{(v_i,v_i)}v_i, \end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product. See also [33], Theorem 5.15, page 386.

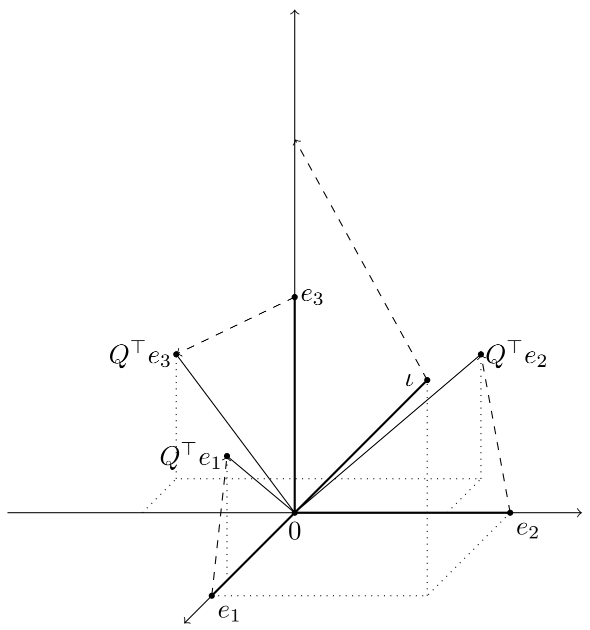

Figure 6: The rotation of the unit vectors and ι by the orthogonal matrix Q.

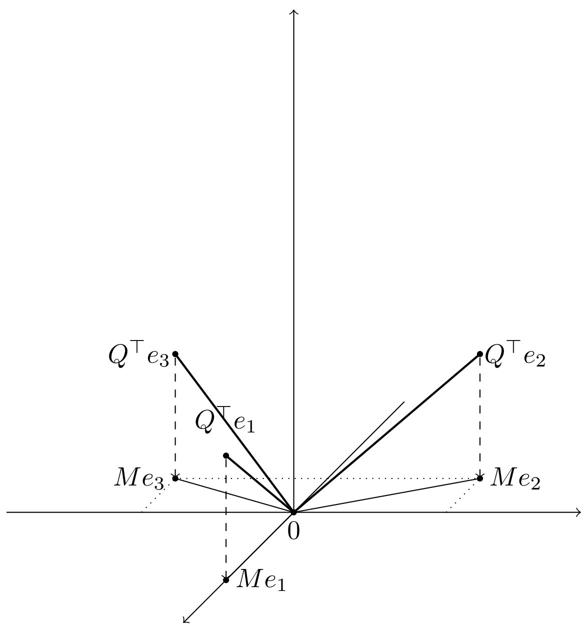

Figure 7: The projection of the rotated unit vectors on $\mathbb{R}$.

Consider the orthogonal matrix Q described in Proposition 3.1 and the unit vectors of $\mathbb{R}^{k+1}$. By applying $Q^\top$ to the unit vectors, we find that the rotation gives an orthonormal basis which consists of the column vectors of $Q^\top$, which are the row vectors of the orthogonal matrix Q described in proposition 3.1. For $k = 2$ $Q^\top$ is given by

$$Q^\top = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

See Figure 6 for a visualisation of the rotation for $k = 2$.

The first row of $Q^\top$ has all entries equal to $1/\sqrt{k+1}$, which is consistent with our prior assumption that $Q^\top$ is a rotation matrix that rotates ι to the unit vector. The $(k+1)$ column vectors of $Q^\top$ therefore lie in a hyperplane orthogonal to the first unit vector. The projection of these vectors on the hyperplane orthogonal to e_1 is realised by eliminating the entries in the first row of $Q^\top$. This leads to a $k \times (k+1)$ matrix M with column vectors in $\mathbb{R}^k$. See Figure 7 for a visualisation of the projection of $Q^\top$ on the plane orthogonal to e_1 for $k = 2$. From the matrix M we can define the M-vectors, as we do in Definition 3.3.

Definition 3.3. Consider the matrix M constructed by the procedure described above:

$$M = \begin{pmatrix}
\sqrt{\frac{k}{k+1}} & \frac{-1}{\sqrt{(k+1)k}} & \frac{-1}{\sqrt{(k+1)k}} & \cdots & \frac{-1}{\sqrt{(k+1)k}} & \frac{-1}{\sqrt{(k+1)k}} & \cdots & \frac{-1}{\sqrt{(k+1)k}} & \frac{-1}{\sqrt{(k+1)k}} \\
0 & \sqrt{\frac{k-1}{k}} & \frac{-1}{\sqrt{k(k-1)}} & \cdots & \frac{-1}{\sqrt{k(k-1)}} & \frac{-1}{\sqrt{k(k-1)}} & \cdots & \frac{-1}{\sqrt{k(k-1)}} & \frac{-1}{\sqrt{k(k-1)}} \\
0 & 0 & \sqrt{\frac{k-2}{k-1}} & \cdots & \frac{-1}{\sqrt{(k-1)(k-2)}} & \frac{-1}{\sqrt{(k-1)(k-2)}} & \cdots & \frac{-1}{\sqrt{(k-1)(k-2)}} & \frac{-1}{\sqrt{(k-1)(k-2)}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \sqrt{\frac{j}{j+1}} & \frac{-1}{\sqrt{j(j+1)}} & \cdots & \frac{-1}{\sqrt{j(j+1)}} & \frac{-1}{\sqrt{j(j+1)}} \\
0 & 0 & 0 & \ddots & 0 & \sqrt{\frac{j-1}{j}} & \cdots & \frac{-1}{\sqrt{j(j-1)}} & \frac{-1}{\sqrt{j(j-1)}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & 0 & 0 & \cdots & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}
\end{pmatrix}.$$

The column vectors of M are called *M-vectors*. The elements of M are given by

$$M(i,j) = \begin{cases} \sqrt{\dfrac{k-i+1}{k-i+2}} & \text{if } i = j, \\ -\dfrac{1}{\sqrt{(k-i+1)(k-i+2)}} & \text{if } i < j \leq k+1, \\ 0 & \text{otherwise.} \end{cases}$$

The M-vectors satisfy certain properties, which are described in Lemma 3.2 below.

Derivation of the M-vectors: Method II

Another method to find the M-vectors is by direct computation. In Section 3.1 we saw that in one dimension there are two vectors in opposite directions and equal length, such that the sum of the two vectors equals 0 and the binomial tree is recombining after two steps. Also the covariance matrix, which is in this case is actually the variance, equals 1. Now consider the problem in two dimensions, that is, for $k = 2$. We then want to find three vectors of equal length for which the sum equals zero, such that the underlying multinomial tree is recombining. Also we want the covariance matrix to equal the identity. A set of vectors which fulfills this restriction is

$$\Big\{ \big(\sqrt{2/3}, 0\big), \big(-1/\sqrt{6}, 1/\sqrt{2}\big), \big(-1/\sqrt{6}, -1/\sqrt{2}\big) \Big\}.$$

This set of vectors has the property that the angle between any two of its vectors is constant. Note that two of the vectors have the same value for x which is equal to half of the x-value of the third vector and that the y-value of one of those two vectors is equal to minus the y-value of the other vector. Note also the link with the Eisenstein integers[4].We will use these properties in the general case. A visualisation of this set of vectors is given in Figure 8.

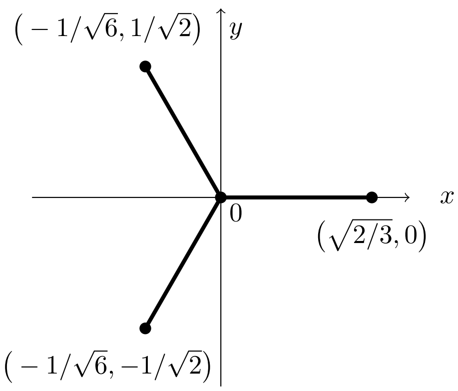

Figure 8: Visualisation of the M-vectors for $k = 2$.

Again consider the general case where $k \in \mathbb{N}$. In each step of the random walk in $\mathbb{R}^k$, we can move in $(k+1)$ directions in $\mathbb{R}^k$. Let M be a $k \times (k+1)$ matrix, of which the column vectors represent the M-vectors in k dimensions. First of all, it makes sense that the M-vectors have equal length. Also we want the process to be recombining after $k+1$ steps, that is, if we make $k+1$ jumps, where each jump is in a different direction, we end up in the point we started in. This means that the sum of the M-vectors equals zero. Finally, we want the covariance matrix $\Sigma = MM^\top$ to equal the identity matrix I, in order to approximate the multivariate standard normal distribution. In summary, we want the M-vector matrix M to satisfy the following properties:

[4]Eisenstein integers are complex numbers of the form

$$z = a + bw,$$

where a and b are integers and $w = 1/2 + i\sqrt{3}/2 = e^{2\pi i/3}$.

1. The length of each two column vectors is the same;
2. Each row sums up to 0;
3. The $k \times k$ covariance matrix $\Sigma = MM^\top$ equals the identity.

To make this set of vectors unique, we construct the M-vectors in the following way. In the two dimensional case we saw that the first vector only had non-zero entries on the first coordinate. We generalise this to the multidimensional case. We also saw that the entries of the first coordinate in the other vectors were equal to each other, but not equal to the first vector. We also make this assumption. Then, since the vectors have equal length, we can calculate the entry of the second coordinate of the second vector. All other entries of the second vector equal zero. Then, again, we choose the entries of the second coordinate of all vectors other than the first two equal. Then we can calculate the entry of the third coordinate of the third vector; and then the entries of the third coordinates of the remaining vectors; etc. This procedure leads to the following. Define the M-vector matrix M by

$$M(i,j) = \begin{cases} \sqrt{\dfrac{k}{k+1} - \displaystyle\sum_{h=1}^{j-1} M(h,j)^2} & \text{if } i = j, \\ -\dfrac{1}{k-i+1} M(i,i) & \text{if } i < j \leq k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1. For all $i \in \{1, \ldots, k\}$ we have that

$$M(i,i) = \sqrt{\frac{k-i+1}{k-i+2}}.$$

Proof. See Appendix A. □

Lemma 3.2. M satisfies properties 1, 2 and 3.

Proof. See Appendix A. □

Remark. Note that this choice for M-vectors is not the only one that does the trick. For every $k \times k$ orthogonal matrix P the column vectors of the matrix PM also satisfy properties 1, 2 and 3 from Lemma 3.2. In what follows we continue to use the M-vectors, but the arguments still hold if we replace M with PM.

Construction of the Recombining Multinomial Tree

The $k \times (k+1)$ matix that contains both the rotation and projection of the unit vectors in $\mathbb{R}^{k+1}$ to $\mathbb{R}^k$ is equal to $Q^\top$ without it's first row, and therefore is equal to M. The random walk X can therefore be projected on $\mathbb{R}^k$ by applying the matrix M. This leads to a random walk $X' = MX$ on $\mathbb{R}^k$ such that in each step we move in the direction of $M(\cdot, i)$ with probability p_i, for all $1 \leq i \leq k+1$. Define the sequence $Y' = \{Y_n\}_{n \geq 1}$ by $Y'_n = MY_n$ for all $n \geq 1$. For all $N \in \mathbb{N}$ we can rewrite X'_N as

$$X'_N = MX_N = M \sum_{n=1}^{N} Y_n = \sum_{n=1}^{N} MY_n = \sum_{n=1}^{N} Y'_n.$$

Theorem 3.3. Let $k \in \mathbb{N}$ and suppose that $p_i = 1/(k+1)$ for all $1 \leq i \leq k+1$. Forthermore, suppose that $\mu \in \mathbb{R}_+^k$ and that Σ is a $k \times k$ symmetric positive definite matrix. Let L be a $k \times k$ matrix such that $\Sigma = LL^\top$. Then the sequence

$$\{L\sqrt{(k+1)/N}X'_N + \mu\}_{N \in \mathbb{N}},$$

converges in distribution to the multivariate normal distribution $N_k(\mu, \Sigma)$.

Proof. See Appendix A. □

The $k \times k$ matrix L in Theorem 3.3 can be computed by applying *Cholesky decomposition*[5]. In fact, every matrix A which satisfies $AA^\top = \Sigma$ is equal to L multiplied by an orthogonal matrix Λ. A more general statement is given in Proposition 3.2.

Proposition 3.2. Let Σ be a symmetric positive definite $k \times k$ matrix. Let L be a real valued $k \times k$ matrix such that $LL^\top = \Sigma$ and let A be a real valued $k \times k$ matrix. Then A satisfies $AA^\top = \Sigma$ if and only if it can be factorised into

$$A = L\Lambda,$$

where Λ is an orthogonal matrix.

Proof. See Appendix A. □

Example 3.2. In this example we consider four matrices L which satisfy $LL^\top = \Sigma$. The reasoning for the choice of these matrices differs per matrix.

1. The first matrix L_{Chol} is derived from the Cholesky decomposition. This is a common method and the evaluation is relatively tractable.
2. The second matrix L_{UDU} is derived via *eigen decomposition*[6] of Σ. The eigen decomposition of Σ gives

   $$\Sigma = UDU^\top,$$

   where U is the orthogonal matrix for which column vectors equal the eigenvectors of Σ and D is the diagonal matrix with the i-th eigenvalue on the i-th diagonal entry. Note that D has only positive entries on the diagonal since Σ is positive definite. Define L_{UDU} by

   $$L_{\text{UDU}} := U\sqrt{D},$$

   where $\sqrt{D}$ is a diagonal $k \times k$ matrix with the i-th entry on the diagonal equal to the root of the i-th entry on the diagonal of D.

[5] Cholesky decomposition is a method to factorise a positive definite matrix Σ in a unique way to a lower triangular matrix L such that $LL^\top$. See Theorem 3.4 for the exact statement. See also Stoer [39], Satz (4.3.3), page 147.

Theorem 3.4. Let A be a positive definite $n \times n$ matrix. Then there exists a unique lower triangular $n \times n$ matrix L, with $l_{ij} = 0$ for $j > i$ and $l_{ii} > 0$ for all i, such that $A = LL^\top$. Furthermore, if A is real valued then L is real valued.

[6] By virtue of the Spectral Theorem we can write a symmetric matrix Σ in the formal

$$\Sigma = UDU^\top,$$

where U is orthogonal and D is diagonal. See also Poole [33], Theorem 5.20, page 400.

3. The third matrix $L_{\sqrt{\Sigma}}$ is the *square root*[7] of the matrix Σ, i.e., $L_{\sqrt{\Sigma}}^2 = \Sigma$ holds. This choice for L is a direct generalisation to the one dimensional case (see Theorem 3.1), where we choose

$$L_{\sqrt{\Sigma}} = \sqrt{\Sigma} = \sigma.$$

Note that $L_{\Sigma} = U\sqrt{D}U^{\top}$.

4. Since the orthogonal matrix Q described in Proposition 3.1 plays a prominent role in Theorem 3.3, the fourth matrix L_Q that we consider is computed by multiplying the Cholesky decomposition L_{Chol} with $Q^{\top}$, i.e.,

$$L_Q := L_{\text{Chol}}Q^{\top}.$$

Here we use the matrix Q that corresponds to model one dimension lower, such that it is a $k \times k$-matrix if L is a $k \times k$ matrix.

Theorem 3.3 makes the connection between the random walk on $\mathbb{R}^k$ and the price process of k assets. Let Z be the price process of k assets that follows a multivariate geometric Brownian motion given by

$$dZ_i = Z_i\mu_i dt + Z_i\sigma_i dW_i,$$

for all $1 \leq i \leq k$. Let $\hat{Z} = \log Z$ be the log-transformed price process of Z. From Proposition 2.4 we know that $\hat{Z}$ follows the k-variate normal distribution $N_k(\hat{\mu}dt, \Sigma dt)$. Let $\delta t = T/N$. From Theorem 3.3 it follows that we can approximate the price process of $\hat{Z}$ by the sequence

$$\{\sqrt{(k+1)\delta t}LX'_N + \hat{\mu}T\}_{N\in\mathbb{N}},$$

where L is a $k \times k$ matrix such that $LL^{\top} = \Sigma$. The elements of this sequence can be rewritten such that

$$\sqrt{(k+1)\delta t}LX'_N + \hat{\mu}T = \sum_{n=1}^{N}\left(\sqrt{(k+1)\delta t}LY'_N + \hat{\mu}\delta t\right).$$

Definition 3.4. Let $d = \{d_i\}_{i=1}^{k+1} \subset \mathbb{R}^k$ be given by

$$d_i(j) := \exp\left\{\sqrt{(k+1)\delta t}(LM(\cdot,i))_j + \hat{\mu}_j\delta t\right\}, \tag{7}$$

for all $1 \leq j \leq k$, for all $1 \leq i \leq k+1$. Then d are called the *direction vectors*.

Using the direction vectors the value of $\hat{Z}$ at time T can thus be approximated by summing the log-transformed asset prices at time 0 and $N \in \mathbb{N}$ independent random draws from the distribution χ_N, with χ_N given by

$$P(\chi_N = \log d_i) = \frac{1}{k+1},$$

for all $1 \leq i \leq k+1$. On the other hand, the value of Z at time T can be approximated by the product of the asset prices at time 0 and $N \in \mathbb{N}$ independent random draws from the distribution $\chi'_N := \exp(\chi_N)$, that is,

$$P(\chi'_N = d_i) = \frac{1}{k+1},$$

for all $1 \leq i \leq k+1$.

[7] The *square root* B of a positive definite matrix Σ is defined as the unique positive definite matrix B such that $\Sigma = B^2$ (see for example Lax [27], pages 115-117). Furthermore, if $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of Σ corresponding to the eigenvectors $v_1, \ldots, v_k$ of Σ, then B is given by

$$B = \sqrt{\lambda_1}v_1v_1^{\top} + \ldots + \sqrt{\lambda_k}v_kv_k^{\top}.$$

Definition 3.5. Let $N \in \mathbb{N}$. Then the graph containing all possible paths of the random walk with N timesteps described above is alled a *recombining multinomial tree*.

The visualisation of a recombining trinomial tree is given in Figure 9.

Completeness of the Model and Pricing Derivatives on Multiple Assets

A nice property of the recombining multinomial tree model is that it is complete. A generalisation of Theorem 3.2 also holds in the multivariate model and is given in Theorem 3.5.

Theorem 3.5. Let $N \in \mathbb{N}$ and consider a simple financial derivative F with time to maturity T on k assets $Z = (Z_1, \dots, Z_k)$ with Σ positive definite and $\mu \geq 0$. Let $d = (d_1, \dots, d_k)$ be the direction vectors. Then the recombining multinomial tree with N levels described by the direction vectors is complete.

Proof. See He [20]. □

Consider a simple derivative F with time to maturity T depending on k assets $Z = (Z_1, \dots, Z_k)$ and suppose we know the value of the derivative $F(Z,T)$ at time T. Consider the recombining multinomial tree with N steps of length $\delta t = T/N$ corresponding to the underlying assets Z. Then we can calculate $F(\nu)$ for any node ν at time T in the recombining multinomial tree. Suppose that the value of the option is known for all nodes at time $m\delta t$, with $1 \leq m \leq N$. Let ν be a node at time $(m-1)\delta t$ and let $\nu_i = \nu + e_i$ for all $1 \leq i \leq k+1$. By virtue of Theorem 3.5 we can find a replicating portfolio $\Delta = (\Delta_1, \dots, \Delta_{k+1})$ of F. This replicating portfolio Δ can be found by solving the following system of equations:

$$\begin{pmatrix} Z_1(\nu_1) & Z_2(\nu_1) & \dots & Z_k(\nu_1) & 1 \\ Z_1(\nu_2) & Z_2(\nu_2) & \dots & Z_k(\nu_2) & 1 \\ \vdots & \vdots & . & \vdots & \vdots \\ Z_1(\nu_{k+1}) & Z_2(\nu_{k+1}) & \dots & Z_k(\nu_{k+1}) & 1 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{k+1} \end{pmatrix} = \begin{pmatrix} F(\nu_1) \\ F(\nu_2) \\ \vdots \\ F(\nu_{k+1}) \end{pmatrix}.$$

If F is a European option, the value of $F(\nu)$ equals

$$F(\nu) = \sum_{i=1}^{k} \Delta_i Z_i(\nu) + \Delta_{k+1} e^{-r\delta t}.$$

By working backwards through the recombining multinomial tree we can find the value of $F(0)$. If however F is an American option, the holder of the option has the choice at node ν to exercise the option and he receives a payoff according to the payoff function $F_{\text{payoff}}(Z(\nu))$. Hence the value of the option at node ν for the American option equals

$$F(\nu) = \max\left\{ F_{\text{payoff}}(Z(\nu)), \sum_{i=1}^{k} \Delta_i Z_i(\nu) + \Delta_{k+1} e^{-r\delta t} \right\}.$$

Convergence of Option Prices

The recombining multinomial tree method described in this section can be a powerful tool to approximate the price of simple derivatives, since the option price derived from the trees converges weakly to the theoretical continuous time option price. In 1990 the convergence of European derivative prices was proved by He [20]. In 1994 the convergence of American derivative prices was proved by Amin and Khanna [2]. However, the proof of the convergence is beyond the scope of this thesis.

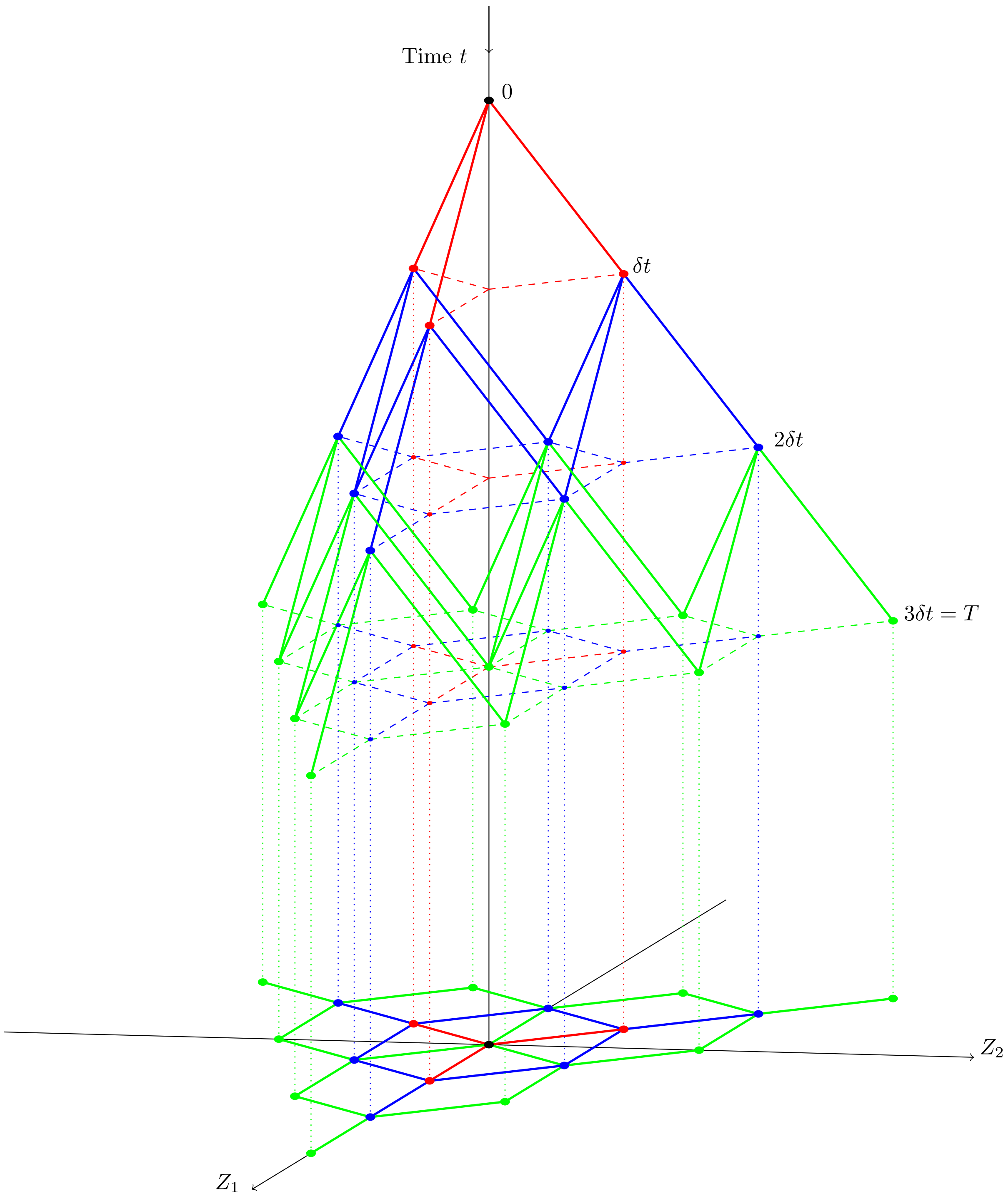

Figure 9: This is a visualisation of a projection of the random walk of the recombining trinomial tree in three dimensions to two dimensions with three timesteps of length δt on two assets Z_1 and Z_2 with mean $\mu = (0, 0)$ and $\Sigma = I$. The random walk starts at the black node on top, which is projected on the black node on the two dimensional plane. From there there are three possible moves to time δt, which are represented by the red nodes and lines. The moves from δt to $2\delta t$ are represented by the blue lines, and the possible outcomes are represented by the blue nodes. Finally, the green lines represent the moves from time $2\delta t$ to time $3\delta t$, and the possible outcomes at time $T = 3\delta t$ are represented by the green nodes and the black node.

Illustration of a Recombining Multinomial Tree

In this section we derived a discrete method to approximate the value of options via recombining multinomial trees. We will conclude this section with Example 3.3 on a recombining trinomial tree on two assets.

This example is also used by Boyle [9] and Chen, Chung, and Yang [14]. In Section 4.1 we continue with Example 3.3 in Example 4.1 and Example 4.2 to compare our results with other tree-based discrete approximations.

Example 3.3. Consider a market environment with two assets $Z = (Z_1, Z_2)$ and a risk-free bond with risk-free interest rate $r = 0.05$. At time 0 the asset price vector is equal to

$$Z(0) = (Z_1(0), Z_2(0)) = (40, 40).$$

The standard deviation vector $\sigma = (\sigma_1, \sigma_2)$ is given by

$$\sigma = (\sigma_1, \sigma_2) = (1/5, 3/10).$$

The correlation coefficient ρ between the first and the second asset is given by

$$\rho = \frac{1}{2}.$$

The 2×2 covariance matrix Σ is therefore given by

$$\Sigma = \begin{pmatrix} 0.04 & 0.03 \\ 0.03 & 0.09 \end{pmatrix}.$$

The 2×2 orthogonal matrix $Q^\top$ described in Proposition 3.1 is given by

$$Q^\top = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

The four matrices described in Example 3.2 are therefore given by

$$L_{\text{Chol}} = \begin{pmatrix} 0.2 & 0 \\ 0.15 & 0.2598 \end{pmatrix}, \qquad L_{UDU} = \begin{pmatrix} 0.1368 & -0.1459 \\ 0.2921 & 0.06833 \end{pmatrix},$$

$$L_{\sqrt{\Sigma}} = \begin{pmatrix} 0.1901 & 0.06203 \\ 0.06203 & 0.2935 \end{pmatrix}, \qquad L_Q = \begin{pmatrix} 0.1414 & 0.1414 \\ 0.2898 & -0.07765 \end{pmatrix}.$$

The 2×3 matrix M containing the M-vectors is given by

$$M = \begin{pmatrix} \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

Suppose that we want to consider the asset prices over a period of seven months, such that $T = 7/12$ if we take one year as one unit of time. Suppose that we divide the timespan into $N = 2$ equal timesteps equal to $\delta t = T/N = 7/24$. For each matrix L from Example 3.2 we can calculate different direction vectors. Let d_{Chol} be the direction vectors corresponding to L_{Chol}; let d_{UDU} be the direction vectors corresponding to L_{UDU}; let $d_{\sqrt{\Sigma}}$ be the direction vectors corresponding to $L_{\sqrt{\Sigma}}$; and let d_Q be the direction vectors corresponding to L_Q. For this example the direction vectors are given by

$$d_{\text{Chol}} = \begin{pmatrix} 1.175 & 0.9346 & 0.9346 \\ 1.123 & 1.123 & 0.7963 \end{pmatrix}, \qquad d_{UDU} = \begin{pmatrix} 1.120 & 0.8694 & 1.054 \\ 1.252 & 0.9372 & 0.8562 \end{pmatrix},$$

$$d_{\text{Chol}} = \begin{pmatrix} 1.166 & 0.9774 & 0.9004 \\ 1.050 & 1.188 & 0.8054 \end{pmatrix}, \qquad d_{\text{Chol}} = \begin{pmatrix} 1.124 & 1.049 & 0.8704 \\ 1.250 & 0.8517 & 0.9438 \end{pmatrix}.$$

With the direction vectors the recombining multinomial tree containing asset prices can be evaluated. The asset price tree for the Cholesky direction vectors d_{Chol} is given in Figure 10.

Consider a European call option c on the maximum of the two assets, i.e.,

$$c(Z, T) = \max\{0, \max\{Z_1(T), Z_2(T)\}\}.$$

We will approximate the value of the option c using a recombining multinomial tree using the Cholesky direction vectors. With the Cholesky asset price tree we are able to calculate the

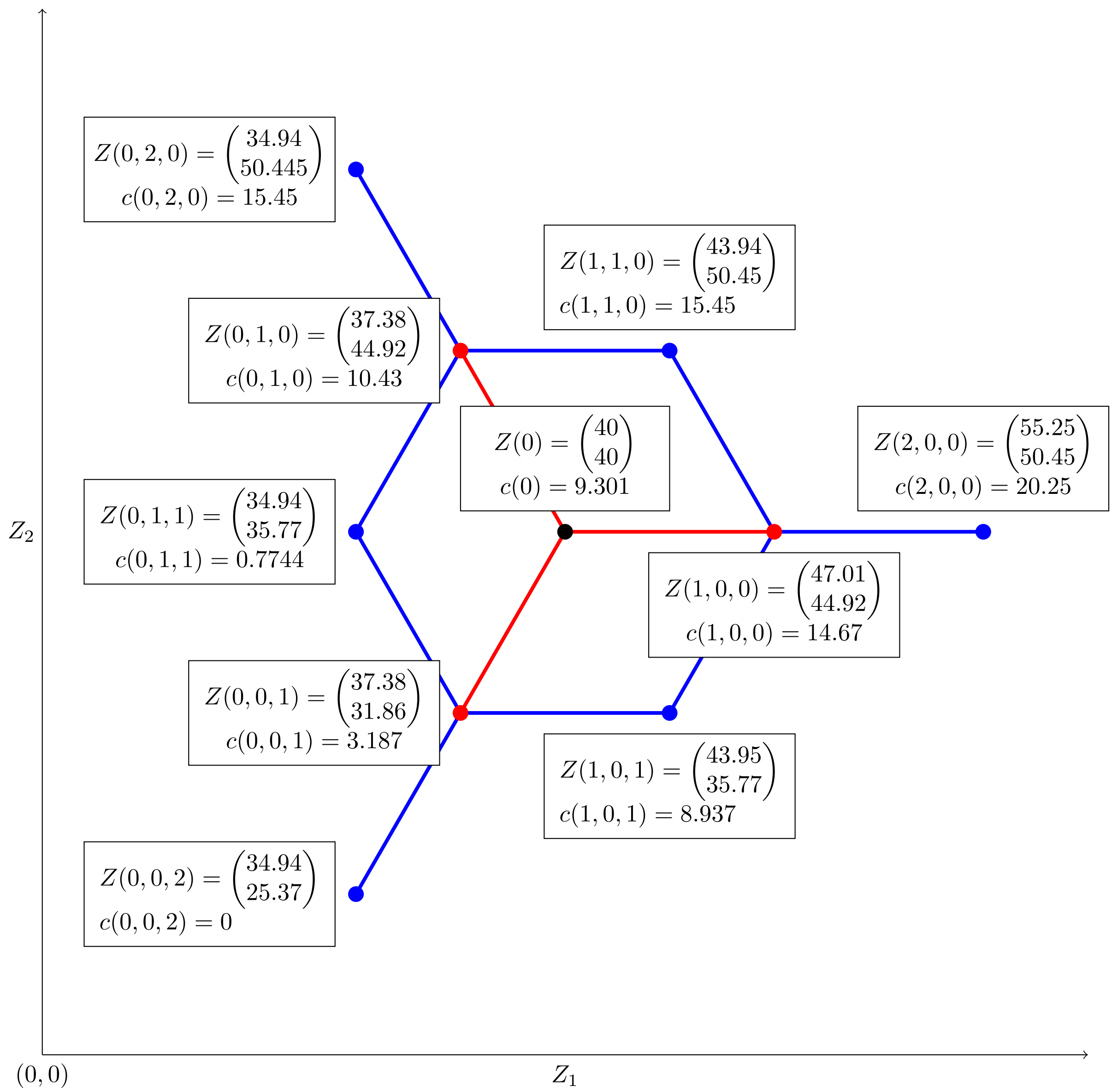

Figure 10

option price tree. The value of the option at all nodes ν at time $T = 2\delta t$ can now be evaluated by using the payoff function of the option c and the approximated asset price at node ν in the asset price tree. For example, the asset price at node $(2,0,0)$ equals $Z(2,0,0) = (55.25, 50.45)$, and therefore the value of the option at node $(2,0,0)$ equals $c(2,0,0) = 20.25$. Now we can work backwards through the tree to find the option prices for the nodes prior to time T. The replicating portfolio can be calculated using Theorem 3.5. Consider for example the option price at node $(1,0,0)$. The system of equations we have to solve is

$$\begin{pmatrix} Z_1(2,0,0) & Z_2(2,0,0) & 1 \\ Z_1(1,1,0) & Z_2(1,1,0) & 1 \\ Z_1(1,0,1) & Z_2(1,0,1) & 1 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} c(2,0,0) \\ c(1,1,0) \\ c(1,0,1) \end{pmatrix}.$$

This gives

$$\Delta = \begin{pmatrix} 0.4246 \\ 0.4437 \\ -25.59 \end{pmatrix}.$$

Hence the approximation of the value of the option at node $(1, 0, 0)$ is equal to

$$c(1, 1, 0) = 14.67.$$

In a similar way we can evaluate the option prices in the other nodes. The approximation of the value of the option at time 0 is equal to

$$c(0) = 9.301.$$

The analytical solution c^* to this problem has been found by Stulz [40]. For this problem it is given by

$$c^* = 9.420$$

The recombinging multinomial tree containing all asset prices and option prices is given in Figure 10.

4 Critical Review

In this Section we review the theory on recombining multinomial trees described in Section 3.2. The goal of this thesis is to provide a theoretical satisfactory solution to the problem of pricing simple derivatives on multiple assets, which is provided in Section 3.2. The tree has a nice structure based on the grid of Pascal's simplex. Therefore the tree has the potential to be incorporated in an efficient algorithm to price derivatives on multiple assets. Unfortunately, the trees also have a downside. While the approximation of the option price derived by the recombining multinomial trees theoretically converges to the analytical option price, in practice it does not converge very fast. However the recombining multinomial trees might have the potential to overcome this problem.

In Section 4.1 we provide some numerical examples to illustrate the slow convergence. The results are compared to similar discrete methods by Boyle [9]; Boyle, Evnine, and Gibbs [10]; and Chen, Chung, and Yang [14]. In Section 4.2 we briefly review other related work, with focus on the two closely related models by Boyle and Chen, Chung, and Yang. In Section 4.3 we use the input from Section 4.1 and 4.2 to substantiate topics for further research to improve the recombining multinomial trees.

4.1 Numerical Examples

In this section we present some numerical examples of pricing simple derivatives on several assets using the theory on recombining multinomial trees discussed in Section 3.2. The analytical values are also provided, and the results are compared with other models on pricing derivatives on multiple assets. The goal of this thesis is to provide a theoretical satisfactory solution rather than a numerical satisfactory solution, and therefore the examples in this section are provided to illustrate the theory. We will see that the estimated option price converges to the analytical value. The convergence is not as fast as the binomial case described in Section 3.1 and shows no preference over the results by Boyle [9]; Boyle, Evnine, and Gibbs [10]; and Chen, Chung, and Yang [14].

In Example 4.1 we will discuss an option on two assets to exchange one asset for another. An analytical solution to this problem is provided independently by Fischer [18] and Margrabe [28]. This solution is described in Example 2.3. In Example 4.2 we discuss call and put options on the maximum and minimum of two assets. The analytical solution to this problem is provided by Stulz [40]. Finally in Example 4.3 we will discuss call and put options on the maximum and minimum of three assets, for which the analytical solution is provided by Johnson [25].

Example 4.1. Options to Exchange One Asset for Another

Consider a European simple derivative F with time to maturity T in a Black-Scholes model with two assets. Let the payoff function of F at time T be given by

$$F(Z,T) = \max\{Z_1(T) - Z_2(T), 0\}.$$

Consider the assets from Example 3.3, where the risk-free interest rate equals $r = 0.05$; initial asset prices $Z(0) = (40, 40)$; and covariance matrix equal to

$$\Sigma = \begin{pmatrix} 0.04 & 0.03 \\ 0.03 & 0.09 \end{pmatrix}.$$

The analytical solution to this problem is given by $F^*(Z.0) = 3.219$ (see Example 2.3). We can approximate the value of the assets using a recombining trinomial tree using the direction vectors d according to Equation 7 in Section 3.2. We calculate the direction vectors using the four different matrices described in Example 3.2, namely: The Cholesky decomposition L_{Chol}; the eigen decomposition L_{UDU}; the square root decomposition $L_{\sqrt{\Sigma}}$; and the decomposition L_Q of the multiplication of the Cholesky decomposition and the orthogonal matrix $Q^\top$ described in Section 3.1. These matrices are also provided in Example 3.2.

We approximate the option price at time 0 via recombining trinomial trees with N timesteps, where we consider the cases $N \in \{10, 20, 50\}$. Chen, Chung, and Yang [14] also provided numerical solutions to this problem, and we compare our results with the results stated in their paper. The results are given in Table 2.

	Number of timesteps (N)			Analytical solution
	10	20	50	
L_{Chol}	3.216	3.202	3.210	3.219
L_{UDU}	3.238	3.240	3.222	
$L_{\sqrt{\Sigma}}$	3.324	3.243	3.224	
L_Q	3.234	3.244	3.223	
AVE	3.253	3.232	3.220	
CCY	3.264	3.235	3.226	

Table 2: CCY denotes the results stated in the paper by Chen, Chung, and Yang [14]. AVE represents the average result of the different choices for L.

The numerical results presented in Table 2 do not imply a significant difference between the different approaches, including the approach by Chen, Chung, and Yang [14]. However, if we consider the trinomial tree with $N = 50$ timesteps, then the Cholesky decomposition gives the worst result. The other methods including the average of our results give almost similar results, but still slightly better than the results stated by Chen, Chung, and Yang [14].

Example 4.2. Options on the Maximum and Mimimum of Two assets

In this section we consider options on the mimumum and maximum of two assets $Z = (Z_1, Z_2)$. The market environment is the same as in Example 4.1, but now we consider options on the maximum and mimimum of the assets. The two options we consider are:

1. Call options on the maximum of the assets. We denote this option by $c(K)$, where K is the strike price of the call option. The payoff function at time T equals

$$\max\{0, \max\{Z_1(T), Z_2(T)\} - K\}.$$

2. Put options on the mimimum of the assets. We denote this option by $p(K)$, where K is the strike price of the put option. The payoff function at time T equals

$$\max\{0, K - \min\{Z_1(T), Z_2(T)\}\}.$$

The analytical solution to this problem is presented by Stulz [40]. A generalisation of the solution to this problem is presented by Johnson [25].

This example has also been used by Boyle [9] and Chen, Chung, and Yang [14], to illustrate their tree-based discrete approximations of European options. We use their results as a benchmark for this example.

We calculate the approximation of the option prices at time zero via recombining multinomial trees with N steps, where $N \in \{10, 20, 50\}$. The direction vectors we use are based on the four matrices described in Example 3.2. The direction vectors are also used in Example 4.1. Furthermore, for both the call and put options we consider all strike prices K with $K \in \{35, 40, 45\}$. The results are given in Table 3.

From these examples it seems that the choice for the direction vectors is of utmost importance. While for the call option $c(35)$ the Cholesky decomposition L_{Chol} is the closest to the analytical solution after 50 steps, for $c(45)$ the eigen decomposition is the closest to the analytical solution. From these examples it is not clear if one of the sets of direction vectors is preferred above all others. There is not one set of direction vectors which always leads to the solution closest to the analytical solution. Also in this example the average of our results seems to be better than an arbitrary choice for L. Furthermore, the solutions of Boyle [9] and Chen, Chung, and Yang [14] give more accurate solutions than our model. If we choose one of our proposed methods, then the approximation of some options lead to better solutions than those of Boyle, while the approximation of other options lead to worse solutions than those of Boyle.

		Number of timesteps (N) 10	20	50	Analytical solution
$c(35)$	L_{Chol}	9.378	9.383	9.403	9.420
	L_{UDU}	9.437	9.427	9.418	
	$L_{\sqrt{\Sigma}}$	9.549	9.464	9.439	
	L_Q	9.422	9.427	9.415	
	AVE	9.447	9.426	9.419	
	B	9.404	9.414	9.419	
	CCY	9.448	9.422	9.419	
$c(40)$	L_{Chol}	5.469	5.478	5.471	5.488
	L_{UDU}	5.497	5.493	5.480	
	$L_{\sqrt{\Sigma}}$	5.650	5.555	5.525	
	L_Q	5.477	5.484	5.475	
	AVE	5.523	5.503	5.488	
	B	5.466	5.477	5.483	
$c(45)$	L_{Chol}	2.792	2.752	2.747	2.795
	L_{UDU}	2.795	2.797	2.792	
	$L_{\sqrt{\Sigma}}$	2.900	2.844	2.821	
	L_Q	2.780	2.789	2.788	
	AVE	2.817	2.796	2.787	
	B	2.817	2.790	2.792	
$p(35)$	L_{Chol}	1.405	1.398	1.409	1.387
	L_{UDU}	1.413	1.409	1.396	
	$L_{\sqrt{\Sigma}}$	1.382	1.368	1.373	
	L_Q	1.418	1.412	1.400	
	AVE	1.405	1.397	1.394	
	B	1.425	1.394	1.392	
$p(40)$	L_{Chol}	3.868	3.839	3.817	3.798
	L_{UDU}	3.885	3.849	3.826	
	$L_{\sqrt{\Sigma}}$	3.769	3.762	3.767	
	L_Q	3.892	3.861	3.831	
	AVE	3.854	3.828	3.810	
	B	3.778	3.790	3.795	
$p(45)$	L_{Chol}	7.572	7.517	7.499	7.500
	L_{UDU}	7.563	7.545	7.521	
	$L_{\sqrt{\Sigma}}$	7.501	7.470	7.475	
	L_Q	7.572	7.556	7.525	
	AVE	7.552	7.522	7.505	
	B	7.475	7.493	7.499	

Table 3: In this table the discrete approximations to price options $c(K)$ and $p(K)$ via recombining trinomial trees are stated. $c(K)$ denotes a European call option with strike price K and $p(K)$ denotes a European put option with strike price K. The benchmarks are the results stated by Boyle [9], which are denoted by B; and the results stated by Chen, Chung, and Yang [14], which are denoted by CCY. AVE represents the average over our results (the different choices for L).

Therefore from these examples it follows that our model does not show preference over the other models.

Example 4.3. Options on the Maximum and Minimum of Three assets

In this example we consider options on the maximum and the mimimum of three assets

$Z = (Z_1, Z_2, Z_3)$. The initial asset prices are given by

$$Z(0) = (Z_1(0), Z_2(0), Z_3(0)) = (100, 100, 100).$$

The interest rate is given by $r = 0.1$. The correlation coefficients are given by $\rho_{12} = \rho_{13} = \rho_{23} = 0.5$ and the standard deviations $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are given by $\sigma = (0.2, 0.2, 0.2)$. Hence the covariance matrix is given by

$$\Sigma = \begin{pmatrix} 0.04 & 0.02 & 0.02 \\ 0.02 & 0.04 & 0.02 \\ 0.02 & 0.02 & 0.04 \end{pmatrix}.$$

The options that we consider have time to maturity equal to $T = 1$ year and an exercise price of $K = 100$. We will consider call options and put options with strike price 100 on the maximum or mimimum of the three assets. We denote $c(\max)$ as the call option on the maximum of the three assets; we denote $c(\min)$ as the call option on the minimum of the three assets; we denote $p(\max)$ as the put option on the maximum of the three assets; we denote $p(\min)$ as the put option on the minimum of the three assets.

		Number of timesteps (N)				Analytical solution
		20	40	60	80	
$c(\max)$	L_{Chol}	22.806	22.761	22.747	22.736	22.672
	L_{UDU}	22.344	22.438	22.480	22.505	
	$L_{\sqrt{\Sigma}}$	23.208	23.049	22.976	22.934	
	L_Q	22.376	22.469	22.506	22.521	
	AVE	22.684	22.679	22.677	22.674	
	BEG	22.281	22.479	22.544	22.576	
	CCY	22.643	22.660	22.664	22.668	
$c(\min)$	L_{Chol}	5.348	5.279	5.297	5.292	5.249
	L_{UDU}	5.658	5.521	5.468	5.437	
	$L_{\sqrt{\Sigma}}$	5.078	5.119	5.142	5.156	
	L_Q	5.386	5.314	5.309	5.301	
	AVE	5.367	5.308	5.304	5.296	
	BEG	5.226	5.237	5.241	5.243	
	CCY	5.293	5.263	5.258	5.259	
$p(\max)$	L_{Chol}	0.906	0.914	0.923	0.925	0.936
	L_{UDU}	0.731	0.790	0.815	0.830	
	$L_{\sqrt{\Sigma}}$	1.061	1.027	1.008	0.998	
	L_Q	0.851	0.887	0.897	0.896	
	AVE	0.887	0.904	0.911	0.912	
	BEG	0.919	0.925	0.928	0.929	
	CCY	0.945	0.940	0.937	0.937	
$p(\min)$	L_{Chol}	7.435	7.384	7.410	7.409	7.403
	L_{UDU}	7.700	7.596	7.558	7.536	
	$L_{\sqrt{\Sigma}}$	7.148	7.211	7.245	7.265	
	L_Q	7.656	7.546	7.526	7.508	
	AVE	7.485	7.434	7.435	7.430	
	BEG	7.240	7.323	7.350	7.364	
	CCY	7.421	7.407	7.406	7.410	

Table 4: In this table the approximations of options on three assets are presented. $c(\max)$ and $c(\min)$ denote European call options with strike price 100 on the maximum and minimum of the three assets, respectively. $p(\max)$ and $p(\min)$ denote European put options with strike price 100 on the maximum and minimum of the three assets, respectively. The results stated by Boyle, Evnine, and Gibbs [10] are denoted by BEG. The results stated by Chen, Chung, and Yang [14] are denoted by CCY. AVE denotes the average of our results.

We approximate the option prices at time zero by using recombining quadrinomial trees. The four methods we use are described in Example 3.2 and are also used in Example 4.1 and Example 4.2. This example has also been used by Boyle, Evnine, and Gibbs [10] and Chen, Chung, and Yang [14]. We compare our results with their results. We also give the analytical solution to the problems using the theory by Johnson [25]. The results are stated in Table 4.

The results stated in Table 4 provide the same pattern as the results in Table 3. The choice for the direction vectors give rise to very different approximations, even after 80 timesteps. However, here we see that the error depends on the choice of the direction vectors. The results derived via the direction vectors calculated with the Cholesky decomposition L_{Chol} are in most cases closer to the results stated by Boyle, Evnine, and Gibbs [10], while the results derived via the direction vectors calculated with $\sqrt{\Sigma}$ are much worse than the results by Boyle, Evnine, and Gibbs [10]. The average of our results is better than our individual models. Here also it may or may not be better than the results from other literature.

4.2 Theoretical Context

In this section we put the theory desribed in this thesis in historical context. A decent amount of research has been published on pricing derivatives via discrete approximations. Some of these approximations are tree-based, while others are based on finite-difference methods or Monte-Carlo simulation. Since the focus of this thesis is on trees, we will only discuss some works related to tree-based methods.

In 1979 Cox, Ross, and Rubinstein presented a method to price derivatives on one asset via a recombining binomial tree [16]. A parallel can be made with Pascal's triangle, as is stated in Section 3.1. In the recombining binomial tree the first and second moment of the continuous time log-transformed process are matched with the first and second moment in the tree corresponding to the discrete time log-transformed process. Moreover, the model is complete and the solution to the binomial model includes a factor of $\sqrt{\sigma^2}$ (see Equation (6) and Theorem 3.1).

There are several aspects for which the binomial model can be generalised. First of all, the number of assets can be generalised from one to many. This is the generalisation derived in this thesis. Other suggestions to generalise the binomial method to multiple assets include Boyle [9]; Boyle, Evnine, and Gibbs [10]; He [20]; Ho, Stapleton, and Subrahmanyam [22]; and Chen, Chung, and Yang [14]; Except for He and Chen, Chung, and Yang, the models are not complete. Boyle and Chen, Chung, and Yang violate other essential properties of the binomial model, which we will discuss later on.

Second, the model could be generalised to models with stochastic parameters instead of the static parameters, such as the standard deviation and the interest rate. The model including stochastic parameters is more realistic and therefore potentially better. Suggestions to generalise the binomial model to include stochastic standard deviation and stochastic interest rate have been made (see for example Black, Derman, and Toy [5]; Nelson and Ramaswamy [31]; Amin and Jarrow [1]; Amin and Ng [3]; Peterson and Stapleton [32]; Ho, Stapleton, and Subrahmanyam [22]; and Ritchken and Trevor [35]). However, the suggested multivariate models are not complete.

Other suggested generalisations include changing the lattice to improve computational efficiency (see for example Boyle [9]; Breen [30]; Broadie and Detemple [13]; and Heston and Zhou [21]) or make the model applicable to path-dependent options such as Asian options (see for example Hull and White [24]; Dai and Lyuu [17]; Boyle and Lau [11]; Ritchken [34]; and Boyle and Tian [12]). Unfortunately, these generalisations often violate essential properties of the binomial model such as the completeness of the model.

The theory on recombining binomial trees described in Section 3.1 gives a detailed new insight on the structure of the binomial method. The direct link to Pascal's triangle and the Eisenstein integers gives a base for the generalisation according to this property in Section 3.2. The implicit incorporation of Pascal's simplex emphasises the nice structure of the grid of the trees and its potential. This consideration can lead to an efficient calculation of option prices and other generalisation, including the incorporation of stochastic volatility and stochastic interest rates.

Over the years articles have been published on pricing derivatives on multiple assets and thusly generalising the recombining binomial model. The two articles most related to our research are He [20] and Chen, Chung, and Yang [14], in chronological order. He showed in 1990 that discrete time models converge to their continuous time counter part. Furthermore, He presented a multidimensional tree as a generalisation of the binomial model. However, in his tree He matched the first and second moment of the log-normal process with the first and second moment of his tree, instead of matching the moments of the log-transformed process with the moments of the tree. Chen, Chung, and Yang do satisfy this property. However their method does not generalise all aspects of the binomial model and matches the moments of the log-transformed processes in a very different way than Cox, Ross, and Rubinstein. We stress that the direct generalisation of the binomial model by Cox, Ross, and Rubinstein should satisfy the following properties:

1. The model should be based on Pascal's simplex, the generalisation of Pascal's triangle;
2. The first and second moment of the continuous time log-transformed process should match the first and second moment of the discrete time log-transformed process;
3. A solution to the generalised model should be based on $\sqrt{\Sigma}$, the square root of the covariance matrix, the generalisation of the varance σ.
4. The model is complete.

While Chen, Chung, and Yang do satisfy properties 1, 2 and 4 and He satisfies properies 1, 3 and 4, both do not present the generalisation we present in this thesis. It should be noted that He doesn't mention the fact that his model satisfies property 3. Moreover, both don't mention Pascal's simplex or stress the fact that they satisfy property 1. However, while the generalisation of He violates property 2, this does not necessarily lead to a worse model, as has been shown in Section 4.1.

4.3 Further Research

The theory on recombining multinomial trees described in Section 3.2 can be an efficient tool to price options on multiple assets. However, more research has to be done to improve the recombining multinomial trees. In this Section we stress the importance and potential of recombining multinomial trees and offer suggestions for further research.

The recombining multinomial tree described in Section 3.2 can be a powerful tool to price options on multiple assets. The recombining multinomial tree has a nice structure, such that an efficient program could lead to efficient calculations by using computational algebra (see Hanzon and Hazewinkel [19], pages 1-6; and Bleylevens [7]). In these calculations we make use of the grid of the M-vectors instead of the usual grids such as $\mathbb{N}^k$. In a recombining multinomial tree for which the moments of the log-normal process are matched the approximation of the option price takes a large number of simple operations. Therefore locating the variables in the physical memory of the computer needed to do the calculation is very time consuming compared to the calculation itself. By allocating the data in the memory in an efficient way the computation time can be reduced. This efficient computation can potentially be achieved because of the grid of the tree and the simple calculations.

The results by He [20] and Amin and Khanna [2] for the convergence of option prices does not provide any information about the error of the estimates. However it is nice to know some bounds on how close the results by the recombining multinomial trees are to the actual values. More knowledge on this subject would improve the recombining multinomial tree method.

Another method to improve the theory on recombining multinomial trees is to generalise the model to a market environment with stochastic standard deviations and interest rate. This market environment is more realistic and therefore leads to more accurate results. However, this brings complications to the model. The extent to which this model can be generalised to a market environment with stochastic standard deviations and interest rates has to be investigated.

A good choice of the direction vectors leads to better results. However, some adjustments have to be made to make sure that the results are satisfying. In Section 4.1 we gave some numerical examples which did not give satisfying results compared to He and Chen, Chung, and Yang. The fact that the models by He and Chen, Chung, and Yang don't satisfy all properties 1, 2, 3, and 4 and hence are not direct generalisations of the binomial model by Cox, Ross, and Rubinstein does not imply that the models are inferior. In fact, a flaw of matching the moments of the log-transformed process is that the moments of the log-normal process are probably not matched. A small error in the log-transformed process leads to a potentially large error in the log-normal process, because the error is enlarged exponentially. Therefore the approximation of the asset price tree is not optimal, and hence the option price derived from the recombining multinomial trees is not optimal. This brings us to the following topic of further research: Expanding the theory on the recombining multinomial trees such that the moments of the log-normal process are matched. There are several ways to approach this problem. First, we can try to solve a system of equations such that the moments are matched. For the binomial model this is a possibility and rather straightforward. However, we have to deal with solving a system of quadratic equations because of the second moment. Moreover, the number of unknowns grows faster than the number of equations as the number of assets increases. Hence in higher dimensions solving the system of equations is not straightforward and a solution to the dimensionality problem has to be found. Chen, Chung, and Yang solve the problem caused by increasing the dimension by considering the angle between the direction vectors. However, they do not apply it to match the moments of the log-normal process. Another approach is to start with the direction vectors calculated by Equation (7), and then improving those vectors such that the moments of the log-normal process are matched. Yet another method is to find another way of defining the direction vectors. He uses an insightful method to match the moments of the log-normal process, which is related to Pascal's simplex. However, the full extent of this relation has not been established. Further research should give more insight in good choices of direction vectors.

5 Conclusion

In this thesis we developed a discrete method to approximate the price of derivatives on multiple assets in a Black-Scholes model. First we introduced some basic concepts of financial mathematics in Section 2. These concepts include the geometric Brownian motion, Itō's Lemma (Theorem 2.1), and the Black-Scholes model. These concepts are provided in a single asset environment and a multiple asset environment. The black-Scholes equation discussed in Section 2.1 and Section 2.2 provide an analytical solution to price derivatives on one or many assets.

We are not always able to solve the Black-Scholes equation. To solve this problem we considered a discrete method to approximate the analytical solution in Section 3. For the univariate case we reviewed the binomial model by Cox, Ross, and Rubinstein in Section 3.1. We derived the binomial model in a different way than is found in literature. This new insight in the derivation of the binomial model served as a foundation to generalise the model to multiple dimensions in Section 3.2. The goal was to keep most aspects of the binomial model instact, such that the difference between the binomial model and the multinomial model would almost only be the different number of underlying assets. Therefore we kept intact the properties of the model concerning the direct link to Pascal's simplex; the matching of the moments of the log-transformed process (opposed to matching the moments of the log-normal process); and the completeness of the model. The link to Pascal's simplex proved to be very useful. Moreover, on account of this property the recombining multinomial trees have the potential to be used in an efficient algorithm using computer algebra. The fact that the recombining multinomial trees are setup in a complete market environment makes the model applicable to all derivatives. Also it can be used to provide hedging strategies and it is setup in an arbitrage free system.

In Section 4 we reviewed the theory on recombining multinomial trees by considering some numerical examples, comparing the theory with other related literature, and offering suggestions for further research. The results of the numerical examples described in Section 4.1 were compared with other related discrete methods. In the examples we considered our model does not show a preference over the other models. The advantage of our model is the relatively low number of nodes in the tree and the simple computations necessary. However, the slow convergence brings the value of matching the moments of the log-transformed price process into question. On this topic we elaborated in Section 4.2 by putting our model in context with other related literature.

In Section 4.3 we provided some topics for further research concerning recombining multinomial trees. First, we argued that recombining multinomial trees have the potential to be used in an efficient algorithm using computer algebra. The fact that the model is based on the grid of Pascal's simplex plays a major role. However, more research needs to be done to work out the details of this problem and to find out the extent to which the recombining multinomial trees can be exploited. Another topic for further research is extending the model to a market environment with stochastic standard deviations and interest rate. This model is more realistic and an efficient generalisation is therefore desirable. Finally we suggested to improve the recombining multinomial tree by matching the moments of the log-normal process instead of the log-transformed process. Several paths to solve this problem have been suggested, as well as several obstacles along these paths have been pointed out.

Altogether the recombining multinomial trees provide a theoretical satisfactory solution. It is a direct generalisation of the recombining binomial trees and provides an alternative to solving the generalised Black-Scholes equation. However, more research needs to be done to transform the recombining multinomial trees to make them applicable in practice.

A List of Proofs

Proof of Proposition 2.4. Using Itō's Lemma (Theorem 2.4) we find that the price process $\hat{Z} = \log(Z) = (\log(Z_1), \ldots, \log(Z_n))^T$ is given by

$$d\hat{Z}_i = \hat{\mu}_i dt + \sigma_i dW_i.$$

For each i, $d\hat{Z}_i = \hat{\mu}_i dt + \sigma_i dW_i$ follows a univariate normal distribution with mean $\hat{\mu} dt$ and variance $\sigma_i^2 dt$. The covariance of $d\hat{Z}_i$ and $d\hat{Z}_j$ equals

$$Cov\left(d\hat{Z}_i, d\hat{Z}_j\right) = E\left[(\hat{\mu}_i dt + \sigma_i dW_i) \cdot (\hat{\mu}_j dt + \sigma_j dW_j)\right] = \sigma_i \sigma_j \rho_{ij} dt,$$

for all i, j. Define Σ by

$$\Sigma_{ij} := \frac{Cov\left(d\hat{Z}_i, d\hat{Z}_j\right)}{dt} = \sigma_i \sigma_j \rho_{ij},$$

for all i, j. Then $\hat{Z}$ follows a k-variate normal distribution $N_k(\hat{\mu} dt, \Sigma dt)$. □

Proof of Theorem 3.1. For $p = \frac{1}{2}$, the expected value of $\sqrt{2}\sigma Y_n'$ is equal to

$$E[\sqrt{2}\sigma Y_n'] = \sqrt{2}\sigma E[M Y_n] = \sqrt{2}\sigma M E[Y_n] = \sqrt{2}\sigma(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}})^\top(\tfrac{1}{2}, \tfrac{1}{2}) = 0,$$

for all $n \geq 0$. On the other hand, the variance of $\sqrt{2}\sigma Y_n'$ is equal to

$$Var[\sqrt{2}\sigma Y_n'] = 2\sigma^2 Var[Y_n'] = 2\sigma^2(p\tfrac{1}{2} + (1-p)\tfrac{1}{2}) = \sigma^2,$$

for each $n \geq 0$. Hence the sequence $\{\sqrt{2}\sigma Y_n'\}_{n \geq 0}$ is a sequence of i.i.d. random variables for which the expected value is equal to 0 and the variance is equal to σ^2. Also, for each $N \in \mathbb{N}$ we have that

$$\sigma\sqrt{2/N} X_N' = \sigma\sqrt{2/N} \sum_{n=0}^{N} Y_n' = \sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N} \sqrt{2}\sigma Y_n'\right).$$

If we apply the central limit theorem[8] to the sequence $\{\sqrt{2}\sigma Y_n'\}_{n \in \mathbb{N}}$ of i.i.d. random variables, we see that the sequence $\{\sigma\sqrt{2/N} X_N'\}_{N \in \mathbb{N}}$ converges in distribution to the normal distribution $N(0, \sigma^2)$. By adding the mean μ it follows that the sequence $\{\sigma\sqrt{2/N} X_N' + \mu\}_{N \in \mathbb{N}}$ converges in distribution to the normal distribution $N(\mu, \sigma^2)$. □

Proof of Theorem 3.2. We only need to show that the model is complete for each timestep. Let $\nu = (i, j)$ be a node in the tree with i upward moves and j downward moves, and let $\nu_1 = \nu + e_1$ and $\nu_2 = \nu + e_2$ be the nodes following node ν by an upward move and a downward move, respectively. Let $F(\nu_1)$ be the price of the derivative at node ν_1 and let $F(\nu_2)$ be the

[8]

Theorem A.1. (Central Limit Theorem)
Let $\{U_n\}_{n \geq 0}$ be a sequence of i.i.d. random variables in $\mathbb{R}$ with finite mean μ and finite variance σ^2. Then the sequence

$$\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N} U_n - \mu\right)$$

converges in distribution to the normal distribution $N(0, \sigma^2)$.

Proof. For a proof of the Central Limit Theorem, see for example Willams [42], Theorem 18.4, page 189; or van der Vaart [41], Example 2.17, page 16. □

price of the derivative at node ν_2. We are looking for a replicating portfolio $\Delta = (\Delta_1, \Delta_2)$ such that if we take a position of Δ_1 of the asset and a position of Δ_2 of the risk-free bond at node ν, the portfolio is worth $F(\nu_1)$ in node ν_1 and $F(\nu_2)$ in node ν_2. Hence the system of equations we have to solve is

$$\begin{pmatrix} Z(\nu_1) & 1 \\ Z(\nu_2) & 1 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} F(\nu_1) \\ F(\nu_2) \end{pmatrix}. \tag{8}$$

We have that $Z(\nu_1) = Z(\nu_2)$ only if $\sigma = 0$. Under the no-arbitrage assumption we have that $F(\nu_1) \neq F(\nu_2)$, so this system of equations has a solution. Therefore every fnancial derivative can be reached and the system is complete. □

Proof of Proposition 3.1.

We apply the Gram-Schmidt process to the column vectors of the $(k+1) \times (k+1)$ matrix

$$U = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

First we calculate a set of $(k+1)$ orthogonal vectors V via the calculation

$$V(\cdot, j) = U(\cdot, j) - \sum_{i=1}^{j-1} \frac{(V(\cdot, i), U(\cdot, j))}{(V(\cdot, i), V(\cdot, i))} V(\cdot, i),$$

for all $1 \leq j \leq k+1$, where (a, b) denotes the inner product of the vectors a and b. Note that the first vector $V(\cdot, 1)$ is given by

$$V(\cdot, 1) = U(\cdot, 1) = \iota.$$

Claim. V satisfies

$$V(i,j) = \begin{cases} \frac{k+2-j}{k+3-j} & \text{if } i+1 = j, \\ -\frac{1}{k+3-j} & \text{if } i+1 > j, \\ 0 & \text{otherwise.} \end{cases}$$

for all $1 < j \leq k+1$ and for all $1 \leq i \leq k+1$.

Proof. We will proof the Claim by induction to j. For $j = 2$ we find that

$$\begin{aligned} V(\cdot, 2) &= U(\cdot, 2) - \frac{(V(\cdot, 1), U(\cdot, 2))}{(V(\cdot, 1), V(\cdot, 1))} V(\cdot, 1) \\ &= e_1 - \frac{(\iota, e_1)}{(\iota, \iota)} \iota \\ &= e_1 - \frac{1}{k+1} \iota \\ &= \left(\frac{k}{k+1}, -\frac{1}{k+1}, \ldots, -\frac{1}{k+1} \right). \end{aligned}$$

Now suppose that $2 < j \leq n$ and suppose that that the statement holds for all values smaller than j. We then have that

$$V(\cdot, j) = e_{j-1} - \sum_{i=1}^{j-1} \frac{(V(\cdot, i), e_{j-1})}{(V(\cdot, i), V(\cdot, i))} V(\cdot, i).$$

If we apply the induction hypothesis we can write the inner products as

$$(V(\cdot,i),e_{j-1}) = -\frac{1}{k+3-i},$$
$$(V(\cdot,i),V(\cdot,i)) = \frac{k+2-i}{k+3-i},$$

for all $2 \le i \le j-1$. For $i=1$ we have that

$$\frac{(V(\cdot,i),e_{j-1})}{(V(\cdot,i),V(\cdot,i))}V(\cdot,i) = \frac{(\iota,e_{j-1})}{(\iota,\iota)}\iota = \frac{1}{k+1}\iota.$$

This gives

$$V(\cdot,j) = e_{j-1} - \frac{1}{k+1}\iota + \sum_{i=2}^{j-1}\frac{1}{k+2-i}V(\cdot,i).$$

Note that

$$-\frac{1}{k+1} - \sum_{i=2}^{j-1}\frac{1}{(k+2-i)(k+3-i)} = -\frac{1}{k+3-j},$$

for all $3 \le j \le k+1$. For $l = j-1$ we then find

$$\begin{aligned} V(l,j) &= e_{j-1}(l) - \frac{1}{k+1} + \sum_{i=2}^{j-1}\frac{1}{k+2-i}V(l,i) \\ &= 1 - \frac{1}{k+1} - \sum_{i=2}^{j-1}\frac{1}{(k+2-i)(k+3-i)} \\ &= 1 - \frac{1}{k+3-j} \\ &= \frac{k+2-j}{k+3-j}. \end{aligned}$$

For $l > j-1$ we find

$$\begin{aligned} V(l,j) &= e_{j-1}(l) - \frac{1}{k+1} + \sum_{i=2}^{j-1}\frac{1}{k+2-i}V(l,i) \\ &= -\frac{1}{k+3-j}. \end{aligned}$$

For $l < j-1$ note that

$$V(l,i) = \begin{cases} \frac{k+2-i}{k+3-i} & \text{if } l = i-1, \\ -\frac{1}{k+3-i} & \text{if } l > i-1, \\ 0 & \text{if } l < i-1, \end{cases}$$

for all $2 \le i \le j-1$. This gives

$$\begin{aligned} V(l,j) &= -\frac{1}{k+1} - \sum_{i=2}^{l}\frac{1}{(k+3-i)(k+2-i)} + \frac{1}{k+2-l} \\ &= -\frac{1}{k+2-l} + \frac{1}{k+2-l} = 0. \end{aligned}$$

◇

We only need to normalise the column vectors of V to complete the Gram-Schmidt process. The norm of $V(\cdot,1)$ is given by $\|V(\cdot,1)\| = \|\iota\| = \sqrt{k+1}$. The norm of $V(\cdot,j)$ for $2 \le j \le k+1$

is given by

$$\begin{aligned}\|V(\cdot,j)\| &= \sqrt{\frac{(k+2-j)^2}{(k+3-j)^2} + \frac{k+2-j}{(k+3-j)^2}} \\ &= \sqrt{\frac{k+2-j}{k+3-j}}.\end{aligned}$$

Defining $Q(\cdot,j) = V(\cdot,j)/\|V(\cdot,j)\|$ for all $1 \leq j \leq k+1$ completes the proof. □

Proof of Lemma 3.1.

We will prove this statement by induction to i. For $i = 1$, we have

$$M(1,1) = \sqrt{\frac{k}{k+1}}.$$

M Let $i > 1$ and suppose the induction hypothesis holds for all values smaller than i. Then we have

$$\begin{aligned}M(i,i) &= \sqrt{\frac{k}{k+1} - \sum_{h=1}^{i-1} M(h,i)^2} \\ &= \sqrt{\frac{k}{k+1} - \sum_{h=1}^{i-2} M(h,i)^2 - M(i-1,i)^2} \\ &= \sqrt{\frac{k}{k+1} - \sum_{h=1}^{i-2} M(h,i-1)^2 - M(i-1,i)^2} \\ &= \sqrt{M(i-1,i-1) - M(i-1,i)^2}.\end{aligned}$$

By applying the induction hypothesis we find

$$\begin{aligned}M(i,i) &= \sqrt{\frac{k-i+2}{k-i+3} - \frac{1}{(k-i+2)^2}\frac{k-i+2}{k-i+3}} \\ &= \sqrt{\frac{(k-i+2)^2-1}{(k-i+3)(k-i+2)}} \\ &= \sqrt{\frac{(k-i+1)(k-i+3)}{(k-i+3)(k-i+2)}} \\ &= \sqrt{\frac{(k-i+1)}{(k-i+2)}}.\end{aligned}$$

□

Proof of Lemma 3.2.

1. Let $j \in \{1, \dots, k+1\}$. Then the length of the square of the vector $M(\cdot, j)$ equals

$$\begin{aligned}
\|M(\cdot, j)\|^2 &= \sum_{i=1}^{k} M(i,j)^2 \\
&= \sum_{i=1}^{j} M(i,j)^2 \\
&= M(j,j)^2 + \sum_{i=1}^{j-1} M(i,j)^2 \\
&= \frac{k}{k+1} - \sum_{i=1}^{j-1} M(i,j)^2 + \sum_{i=1}^{j-1} M(i,j)^2 = \frac{k}{k+1}.
\end{aligned}$$

2. Let $i \in \{1, \dots, k\}$. Then the sum of the elements of row i equals

$$\begin{aligned}
\sum_{j=1}^{k+1} M(i,j) = \sum_{j=i}^{k+1} M(i,j) \\
&= M(i,i) + \sum_{j=i+1}^{k+1} M(i,j) \\
&= M(i,i)(k-i+1)\frac{1}{k-i+1} M(i,i) = 0.
\end{aligned}$$

3. Note that $M(i,j) = M(i,h)$ for all $j, h > i$. Let $i \in \{1, \dots, k\}$. Then we find

$$\begin{aligned}
\Sigma_{ii} &= M(i,\cdot)M(i,\cdot)^\top \\
&= \sum_{j=1}^{k+1} M(i,j)^2 \\
&= M(i,i)^2 + \sum_{j=i+1}^{k+1} M(i,j)^2 \\
&= M(i,i)^2 + (k-i+1)M(i,i+1)^2 \\
&= \frac{k-i+1}{k-i+2} + \frac{(k-i+1)^2}{(k-i+2)(k-i+1)^2} \\
&= \frac{k-i+2}{k-i+2} = 1.
\end{aligned}$$

   Let $i \neq j$ be two rows ($i, j \in \{1, \dots, k\}$). Without loss of generality we assume that $i < j$. Then we have

$$\begin{aligned}
\Sigma_{ij} &= M(i,\cdot)M(j,\cdot)^\top \\
&= \sum_{h=1}^{k+1} M(i,h)M(j,h) \\
&= \sum_{h=j}^{k+1} M(i,h)M(j,h) \\
&= \sum_{h=j}^{k+1} M(i,i+1)M(j,h) \\
&= M(i,i+1)\sum_{h=j}^{k+1} M(j,h) = 0,
\end{aligned}$$

   where the last equation holds because of property 2.

□

Proof of Theorem 3.3.

For $n \in \mathbb{N}$ the expected value of $\sqrt{k+1}LY'_n$ is equal to

$$E[\sqrt{k+1}LY'_n] = \sqrt{k+1}LE[MY_n] = \sqrt{k+1}LME[Y_n] = \sqrt{k+1}LM\iota/(k+1) = (0, \dots, 0),$$

where the last equality holds because each row of M sums up to 0. On the other hand, the covariance matrix of $\sqrt{k+1}LY'_n$ is equal to

$$\begin{aligned} Cov(\sqrt{k+1}LY'_n, \sqrt{k+1}LY'_n) &= E[(\sqrt{k+1}LY'_n)(\sqrt{k+1}LY'_n)^\top] - E[\sqrt{k+1}LY'_n]E[\sqrt{k+1}LY'_n]^\top \\ &= (k+1)LE[MY_n(MY_n)^\top]L^\top \\ &= (k+1)LME[Y_nY_n^\top]M^\top L^\top, . \end{aligned}$$

for all $n \in \mathbb{N}$. The expected value of $Y_nY_n^\top$ is equal to

$$E[Y_nY_n^\top] = \sum_{i=1}^{k+1} p_i e_i e_i^\top = \frac{1}{k+1}I,$$

so that

$$Cov(\sqrt{k+1}LY'_n, \sqrt{k+1}LY'_n) = (k+1)LM\frac{1}{k+1}IM^\top L^\top = \Sigma,$$

for all $n \in \mathbb{N}$. Hence the sequence $\{\sqrt{k+1}LY'_n\}_{n\geq 0}$ is a sequence of i.i.d. random vectors with mean 0 and covariance matrix Σ. Also, for each $N \in \mathbb{N}$ we have that

$$L\sqrt{(k+1)/N}X'_N = L\sqrt{(k+1)/N}\sum_{n=0}^{N} Y'_n = \sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N}\sqrt{k+1}LY'_n\right).$$

If we apply the Multivariate Central Limit Theorem[9] to the sequence $\{\sqrt{k+1}LY'_n\}_{n\geq 0}$ of i.i.d. random vectors, we see that the sequence $\{L\sqrt{(k+1)/N}X'_N\}_{N\in\mathbb{N}}$ converges in distribution to the multivariate normal distribution $N_k(0, \Sigma)$. By adding the mean μ it follows that the sequence $\{L\sqrt{(k+1)/N}X'_N + \mu\}_{N\in\mathbb{N}}$ converges in distribution to the multivariate normal distribution $N_k(\mu, \Sigma)$. □

Proof of Proposition 3.2.

Suppose that Λ is an orthogonal matrix and define A by

$$A := L\Lambda.$$

Then we have that

$$AA^\top = L\Lambda(L\Lambda)^\top = L\Lambda\Lambda^\top L = LL^\top = \Sigma.$$

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Theorem A.2. (Multivariate Central Limit Theorem)
Let $\{U_n\}_{n\geq 0}$ be a sequence of i.i.d. random vectors in $\mathbb{R}^k$ with finite mean μ and finite covariance matrix Σ. Then the sequence

$$\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N} U_n - \mu\right)$$

converges in distribution to the multivariate normal distribution $N_k(0, \Sigma)$.

Proof. For a proof of the Multivariate Central Limit Theorem, see for example van der Vaart [41], Example 2.18, page 16. □

Now suppose that A satisfies $AA^\top = \Sigma$. Σ is positive definite, so all it's eigenvalues are positive and $\det(\Sigma) > 0$. Therefore $\det(AA^\top) = \det(\Sigma) > 0$ holds and A and $A^\top$ are invertible since $\det(A), \det(A^\top) \neq 0$. For the same reason L and $L^\top$ are invertible. By multiplying the equation $\Sigma = LL^\top$ by A^{-1} from the left and by $(A^\top)^{-1}$ from the right we find

$$I = A^{-1}LL^\top(A^\top)^{-1} = A^{-1}LL^\top(A^{-1})^\top = A^{-1}L(A^{-1}L)^\top.$$

By multiplying this equation with $L^{-1}A$ from the left we find

$$(A^{-1}L)^\top = L^{-1}A = (A^{-1}L)^{-1},$$

so $A^{-1}L$ and $(A^{-1}L)^\top = L^{-1}A$ are orthogonal. Define the orthogonal matrix Λ by

$$\Lambda := L^{-1}A.$$

Hence we have that

$$L\Lambda(L\Lambda)^\top = L(L^{-1}A)[L(L^{-1}A]^\top = AA^\top = \Sigma.$$

□

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