A H2 well-posedness result for second order quasilinear parabolic PDE's on the real line with an application to a generalisation of the Gray-Scott model

Siero, E.P.J.A.

E.P.J.A. Siero

A H^2 well-posedness result for second order quasilinear parabolic PDE's on the real line with an application to a generalisation of the Gray-Scott model

Master thesis

September 2011

Thesis advisors:
Dr. J.D.M. Rademacher
Prof. Dr. A. Doelman

Mathematisch Instituut
Universiteit Leiden

Contents

Introduction

This thesis has been written as a product of a traineeship at CWI, the National Research Institute for Mathematics and Computer Science in the Netherlands. Research was conducted under supervision of Jens Rademacher, to whom I owe a debt of gratitude.

The aim of this study was to apply existence theory for quasilinear PDE's on $\mathbb{R}$ to a model for vegetation patterns that uses porous medium flow for the water. This model is given by the *Generalised Klausmeier Gray-Scott* model (16):

$$\text{GKGS:} \begin{cases} u_t = & \mathcal{D}\left(u^2\right)_{xx} + \mathcal{C}u_x + \mathcal{A}(1-u) - uv^2 \\ v_t = & v_{xx} - \mathcal{B}v + uv^2 \end{cases}$$

on $\mathbb{R}_{\geq 0} \times \mathbb{R}$, where $\mathcal{A}, \mathcal{B}, \mathcal{D}$ are assumed to be strictly positive constants. For $u > 0$ this is a second order parabolic quasilinear system of PDE's.

In the article [3] by Amann an existence theory is developed for quasilinear systems on bounded domains. In this article, on page 225 he makes the remark that the domain Ω can be chosen to be unbounded, and refers to his own article [2] which only considers semilinear parabolic systems on unbounded domains. In their article [11] Wu and Zhao refer to a third article by Amann [4] to conclude that they have local existence of a solution to a quasilinear system on what seems to be an unbounded domain. An application of [4] to the bounded domain is included in §2.4. But an explicit treatment of existence theory for *quasilinear* parabolic *systems* on the *unbounded domain* has not been done in [4] and was generally not found in the literature. This caused the focus of this thesis to shift to the existence theory itself.[1]

In this thesis a well-posedness result is presented for quasilinear systems of second order PDE's on the unbounded domain $\mathbb{R}$. The function space chosen for this framework is the Sobolev space of twice weakly differentiable functions on $\mathbb{R}$, $H^2(\mathbb{R})$, which is a Banach algebra (section 3.2). The algebra property is convenient for estimating the norm of nonlinear reaction terms and ultimately provides a local Lipschitz property of the Nemytskii operator (see corollary 3.14). A remarkable property of the existence results below is that they are obtained without the use of fractional power spaces: the proofs are similar to the proof of Picard-Lindelöf existence theorem for ODE's, at the prize of requiring smooth initial data.

We first return to GKGS to see how such an existence result could be applied. By substituting $w = u^2$ into GKGS it can be rewritten to a slightly more transparent form (equation (18)):

$$\begin{cases} w_t = & \mathcal{D}\sqrt{2w}w_{xx} + \mathcal{C}w_x + \sqrt{2w}\mathcal{A}(1-\sqrt{w}) - \sqrt{2}wv^2 \\ v_t = & v_{xx} - \mathcal{B}v + \sqrt{w}v^2 \end{cases} .$$

[1] After completing the thesis the attention was drawn to an article by Kato [7], which does treat existence theory for quasilinear systems on the unbounded domain, a comparison is to be made.

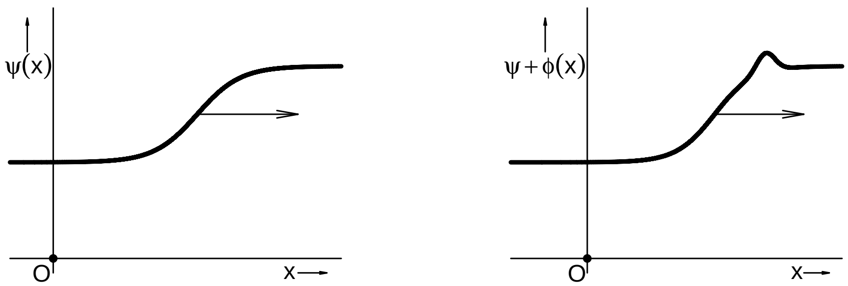


Figure 1: Sketch of travelling wave ψ without and with perturbation ϕ in $H^2(\mathbb{R})$.

From this one directly sees that the coefficient of the highest order derivative depends on the solution itself in such a way that the PDE degenerates as w vanishes. This is unavoidable for w in $H^2(\mathbb{R})$, confer corollary 3.11. Instead of choosing a different function space, this problem is circumvented by looking at $H^2(\mathbb{R})$ perturbations of existing solutions that stay away from 0.

To illustrate this, assume that there exists a travelling wave solution $\psi(t) = \binom{w_\psi}{v_\psi}$ of GKGS with $w_\psi(t, x) \geq \delta > 0$. For a function $\phi = \binom{w}{v}$ it holds that $\psi + \phi$ solves GKGS iff ϕ solves:

$$\phi_t = A_\psi(t, \phi)\phi + f_\psi(t, \phi); \tag{1}$$

$$\text{with} \quad A_\psi(t, \phi)\phi = \begin{pmatrix} \mathcal{D}\sqrt{2(w_\psi + w)}w_{xx} + \mathcal{C}w_x & 0 \\ 0 & v_{xx} \end{pmatrix}; \tag{2}$$

see equations (19) and (20). The advantage of this PDE is that the coefficient of the highest order derivative vanishes nowhere for $w \in H^2(\mathbb{R})$ small. Any solution of GKGS that stays away from 0 suffices, so the travelling wave could for instance be replaced by a homogeneous steady state of GKGS. These states coincide with those found for Gray-Scott in section 2.1.2.

A crucial role is played by the Sobolev imbedding of $H^1(\mathbb{R})$ into the Hölder continuous functions $C^{0,\gamma}(\mathbb{R})$ (theorem 3.9), and the pointwise L^∞-bounds this implies. A complete simple proof of this well-known result is provided next to some additional properties of Hölder continuous functions. By applying our main existence result below to the PDE for the perturbation with $\phi(0)$ small, for some time a solution ϕ exists and thus we obtain short time existence of a solution $\psi + \phi$ to GKGS. This is illustrated by figure 1. In §2.3 it is derived, for illustration, that (periodic) solutions of GKGS which start out with u-component $u \geq \delta > 0$ remain that way as long as the v-component remains bounded.

Let n denote the number of PDE's present in the system. The main existence result can be formulated as follows, but for the full details see §1.3.[2] Let $A(t, \phi)$ be a second order differential operator, suppose we have the following quasilinear n-dimensional system of PDE's:

$$\phi_t = A(t, \phi)\phi + f(\phi), \quad \phi(0) = u \tag{Q}$$

[2] In §1.3, to simplify notation only, the theory is developed for a system containing only a single PDE ($n = 1$). Generalisation to larger n is straightforward.

Theorem (Main existence result). *Suppose that f is locally Lipschitz on bounded subsets of $\left(H^2(\mathbb{R})\right)^n$ and $A(t,\phi)$ generate evolution systems $\{U_\phi(t,s)\}_{0\le s\le t\le T}$ which are Lipschitz in ϕ. If $u \in D(A)$ then a mild solution of (Q) exists.*

We give a sketch of the proof for $n = 1$. The proof is based on the Banach contraction mapping theorem, just like the proof of Picard-Lindelöf for ODE's. The Banach space chosen to define a contraction on is:

$$X_\tau = \left(C\left([0,\tau], H^2(\mathbb{R})\right), ||\cdot||_\infty\right);$$
$$\text{with } ||\phi||_\infty = \sup_{0\le t\le\tau} ||\phi(t)||_{H^2}.$$

Sketch of proof. In the semilinear case with $A(t,\phi) = A(t)$ a single evolution system $U(t,s)_{0\le s\le t\le\tau}$ is generated. Mild solutions are fixed points of the map:

$$J_{u,\tau} : X_\tau \to X_\tau$$
$$\phi \mapsto U(t,0)u + \int_0^t U(t,s) f_N(\phi(s))ds.$$

While showing that for some τ this defines a contraction on some neighbourhood of u (as is done in the proof of theorem 1.8), the following estimate is made, where $v(s)$ is some element of $H^2(\mathbb{R})$:

$$\left|\left|\int_0^t U(t,s)v(s)\, ds\right|\right|_{H^2} \le \tau \cdot \sup_{0\le s\le t\le\tau} ||U(t,s)||_{\mathscr{L}(H^2(\mathbb{R}),H^2(\mathbb{R}))} \cdot \sup_{0\le s\le\tau} ||v(s)||_{H^2}.$$

Note that since $u \in H^2(\mathbb{R})$ such a simple estimate suffices to make the argument; usually (in the parabolic case) the smoothening properties of the evolution system are used.

To prove the theorem in the quasilinear case we first note that for fixed $\bar{\phi}$ in a neighbourhood $B_{\tau,\epsilon}$ of u we are back in the non-autonomous semilinear case $\phi(t) = A(t,\bar{\phi})\phi + f(\phi)$, denote its solution by $\phi(\bar{\phi})$. Then mild solutions of (Q) coincide with fixed points of the map:

$$K_{u,\tau,\epsilon} : B_{\tau,\epsilon}(u) \to B_{\tau,\epsilon}(u)$$
$$\bar{\phi} \mapsto \phi(\bar{\phi});$$

for which we show that this again defines a contraction on a (possibly smaller) neighbourhood of u. □

There are some subtleties involved in defining the map K and showing that it becomes a contraction. This is discussed in section §1.3, which ends the discussion on abstract general well-posedness theory.

The second chapter contains results of a more applied nature. We start by looking into local stability of a model by Klausmeier and the Gray-Scott model, which reveals existence of a Hopf bifurcation. This leads to existence of travelling wave train solutions of GKGS. As discussed previously in this introduction, the abstract framework can be applied to perturbations of solutions of GKGS to obtain a short time existence result. Arguments on how the assumptions of the abstract framework can be met in the case of GKGS are contained in §2.5.

The third chapter, as the title suggests, contains background material. These are either well known results that are included for convenience of the reader or rather lengthy or technical calculations that have been excluded from the main body.

1 Existence result for semilinear and quasilinear second order PDE's on the real line

There is a close relationship between Cauchy problems and evolution operators. Let the *linear Cauchy problem* be given by:

$$\begin{cases} \dot{\phi}(t) = & A\phi(t) \quad \text{if } t \geq 0 \\ \phi(0) = & u \end{cases} ;$$

where A is a linear operator on a Banach space X with domain $D(A) \subset X$. The following relation with semigroups (see section 3.1) is well known and can be found in [6, Proposition II.6.2].

Theorem 1.1. *If $(A, D(A))$ generates a strongly continuous semigroup $S(t)$, and $u \in D(A)$, then $\phi(t) = S(t)u$ is the unique classical solution of the linear Cauchy problem.*

In this section we look at a more general Cauchy problem: we introduce a reaction term and later on have A depend on t and ϕ. Using the Banach contraction mapping principle we obtain a similar result as theorem 1.1 for the linear Cauchy problem.

Although our main interest goes out to an essentially parabolic equation (see (16) below), only *strong continuity* of the generated semigroup is assumed. So regularisation properties of the *analytic* semigroup are not used: the use of interpolation theory for Banach spaces (fractional power spaces) is circumvented at the price of a weaker but more transparant result. As a consequence the framework below is built for regular initial conditions.

1.1 Autonomous semilinear case

Consider the autonomous Cauchy problem with reaction term:

$$\begin{cases} \frac{\partial}{\partial t}\phi(t,x) & = A\phi(t,x) + f(\phi(t,x)) \quad \text{if } t \geq 0 \\ \phi(0,x) & = u(x) \end{cases} ; \tag{3}$$

where $(A, D(A))$ with $D(A) = H^2(\mathbb{R})$ is assumed to generate a strongly continuous semigroup $S(t)$ on $L^2(\mathbb{R})$.

Theorem 1.2 (Variation of constants formula)**.** *Suppose that $\phi : [0,\tau] \times \mathbb{R}$ with $\phi(\cdot, x) \in C^1([0,\tau])$ and $\phi(t,\cdot) \in C^2(\mathbb{R})$ is a classical solution of (3), then it holds:*

$$\phi(t,x) = S(t)u(x) + \int_0^t S(t-s)f(\phi(s,x))ds.$$

Proof. Let ϕ be a classical solution of (3) and fix t. Then the X valued function

$s \mapsto S(t-s)\phi(s)$ is differentiable:

$$\begin{aligned}
\frac{d}{ds}S(t-s)\phi(s) &= \lim_{h\to 0}\frac{S(t-(s+h)\phi(s+h)-S(t-s)\phi(s)}{h}\\
&= \lim_{h\to 0}\frac{S(t-(s+h)\phi(s+h)-S(t-s)\phi(s+h)}{h}\\
&\quad+\lim_{h\to 0}\frac{S(t-s)\phi(s+h)-S(t-s)\phi(s)}{h}\\
&= -S(t-s)A\phi(s)+S(t-s)\frac{d}{ds}\phi(s);
\end{aligned}$$

so it holds that:

$$\begin{aligned}
\phi(t,x)-S(t)\phi(0,x) &= \int_0^t \frac{d}{ds}S(t-s)\phi(s,x)ds\\
&= \int_0^t \left(-S(t-s)A\phi(s,x)+S(t-s)\frac{d}{ds}\phi(s,x)\right)ds\\
&= \int_0^t S(t-s)\left(-A\phi(s,x)+\frac{d}{ds}\phi(s,x)\right)ds\\
&= \int_0^t S(t-s)f(\phi(s,x))ds.
\end{aligned}$$

Replacing $\phi(0,x)$ by $u(x)$ we obtain: $\phi(t,x) = S(t)u(x)+\int_0^t S(t-s)f(\phi(s,x))ds$. □

We wish to interpret (3) as an ordinary differential equation on $H^2(\mathbb{R})$. For this, we introduce the Nemytskii operator f_N, see section 3.5. Equation (3) can thus be rewritten:

$$\begin{cases}\dot\phi(t) = & A\phi(t)+f_N(\phi(t)) \quad \text{if } t\geq 0\\ \phi(0) = & u\end{cases}. \tag{4}$$

We formulate the following assumption:

(A0) It holds that $f_N \in \mathscr{C}(H^2(\mathbb{R}))$ and f_N is Lipschitz on any bounded subset of $D(A) = H^2(\mathbb{R})$.

In light of corollary 3.14, this assumption could be replaced by the assumption that $f_N(0) = 0$ and $f \in \mathscr{C}^3(\mathbb{R})$. Since A is assumed to generate a strongly continuous semigroup, A is a closed operator. In the following, $D(A)$ is assumed to be endowed with the graph norm $||\cdot||_A$, see section 3.2.

Definition For $\tau > 0$ introduce the Banach space :

$$X_\tau := \left(C\left([0,\tau], D(A)\right), ||\cdot||_\infty\right).$$

For $u \in D(A)$ and $\epsilon > 0$ let the closed ball centered around u with radious ϵ be given by:

$$B_\epsilon(u) := \{v \in D(A) \,|\, ||u - v||_A \leq \epsilon\};$$
$$B_{\tau,\epsilon}(u) := \{\phi \in X_\tau \,|\phi(t) \in B_\epsilon(u) \text{ for } 0 \leq t \leq \tau\}.$$

Definition An element $\phi \in X_\tau$ is a *mild solution* of (4) if on $[0, \tau]$ it holds that:

$$\phi(t) = S(t)u + \int_0^t S(t-s)f(\phi(s))ds.$$

By theorem 1.2 every classical solution is a mild solution.

Using the Banach contraction mapping theorem we will prove the existence of a mild solution to (4). To this end we define a map:

$$J_{u,\tau} : X_\tau \to X_\tau$$
$$\phi \mapsto S(t)u + \int_0^t S(t-s)f(\phi(s))ds,$$

and we want to show that for sufficiently small τ, $J_{u,\tau} : X_\tau \to X_\tau$ is a contraction when restricted to some $B_{\tau,\epsilon}(u) \subset X_\tau$.

We first need some preliminary results. For strongly continuous semigroups $\{S(t)\}$ it is well known that $S(t)u \in D(A)$ and $AS(t)u = S(t)Au$ if $u \in D(A)$ [6, Lemma II.1.3(ii)].

Lemma 1.3. *Let $u \in D(A)$. For all $\epsilon > 0$ there exists a $\tau > 0$ such that:*

$$\sup_{0\leq t\leq\tau} ||S(t)u - u||_A \leq \epsilon.$$

Proof. By strong continuity it holds:

$$\lim_{t\downarrow 0} ||S(t)u - u||_A = \lim_{t\downarrow 0} ||S(t)u - u||_{L^2} + \lim_{t\downarrow 0} ||S(t)Au - Au||_{L^2} = 0.$$

□

Lemma 1.4. *Let $\tau > 0$ be given, then there exists a $M(\tau) \geq 1$ such that:*

$$\sup_{0\leq t\leq\tau} ||S(t)||_{\mathscr{L}(D(A),D(A))} = M(\tau).$$

Proof. The restriction of $S(t)$ to $D(A)$, $S(t)|_{D(A)}$, is a strongly continuous semigroup itself (with generator $(A, H^4(\mathbb{R}))$) [6, Proposition II.2.15(ii)]. By strong continuity the orbits are continuous, so it follows that for $u \in D(A)$ the partial orbits $\{S(t)u|\, t \in [0, \tau]\} \subset D(A)$ are bounded. Thus the family $\{S(t)|\, t \in [0, \tau]\} \subset \mathscr{L}(D(A), D(A))$ is pointwise bounded. By the principle of uniform boundedness the family is uniformly bounded. □

Theorem 1.5. *Let $u \in D(A)$ and suppose that (A0) holds. Then for all $\epsilon > 0$ there exists a $\tau > 0$ such that the restriction $J_{u,\tau}|_{B_{\tau,\epsilon}(u)}$ is a contraction.*

Proof. Let $\epsilon > 0$ be given. Let the Lipschitz constant of f_N on $B_\epsilon(u)$ be given by C_{Lip} and let $C_{|f|}$ be given such that $||f_N(v)||_A \leq C_{|f|}$ for $v \in B_\epsilon(u)$. By lemmas 1.3 and 1.4 we can find τ and $M(\tau)$ such that:

$$\sup_{0\leq t\leq\tau} ||(S(t)-1)u||_A \leq \frac{\epsilon}{2};$$
$$\sup_{0\leq t\leq\tau} ||S(t)||_{\mathscr{L}(D(A),D(A))} \leq M(\tau);$$
$$\tau \leq \min\left\{\frac{\epsilon}{2M(\tau)C_{|f|}}, \frac{1}{2M(\tau)C_{Lip}}\right\}.$$

We have for $\phi \in B_{\tau,\epsilon}(u)$:

$$\begin{aligned}
&||J_{u,\tau}(\phi) - u||_{X_\tau} \\
&= \sup_{0\leq t\leq\tau} ||J_{u,\tau}(\phi) - u||_A \\
&\leq \sup_{0\leq t\leq\tau} ||(S(t)-1)u||_A + \sup_{0\leq t\leq\tau} \left|\left|\int_0^t S(t-s)f_N(\phi(s))ds\right|\right|_A \\
&\leq \frac{\epsilon}{2} + \tau \cdot \sup_{0\leq s\leq t\leq\tau} ||S(t-s)f_N(\phi(s))||_A \\
&\leq \frac{\epsilon}{2} + \tau \cdot \sup_{0\leq s\leq t\leq\tau} ||S(t-s)||_{\mathscr{L}(D(A),D(A))} \cdot \sup_{0\leq s\leq\tau} ||f_N(\phi(s))||_A \\
&\leq \frac{\epsilon}{2} + \tau M(\tau) C_{|f|} \\
&\leq \epsilon.
\end{aligned}$$

Thus $B_{\tau,\epsilon}(u)$ is mapped into itself by $J_{u,\tau}$. We also have, for $\phi_1, \phi_2 \in B_{\tau,\epsilon}(u)$:

$$\begin{aligned}
&||J_{u,\tau}(\phi_1) - J_{u,\tau}(\phi_2)||_{X_\tau} \\
&= \sup_{0\leq t\leq\tau} ||J_{u,\tau}(\phi_1) - J_{u,\tau}(\phi_2)||_A \\
&= \sup_{0\leq t\leq\tau} \left|\left|\int_0^t S(t-s)(f_N(\phi_1(s)) - f_N(\phi_2(s)))ds\right|\right|_A \\
&\leq \tau \cdot \sup_{0\leq s\leq t\leq\tau} ||S(t-s)(f_N(\phi_1(s)) - f_N(\phi_2(s)))||_A \\
&\leq \tau \cdot \sup_{0\leq s\leq t\leq\tau} ||S(t-s)||_{\mathscr{L}(D(A),D(A))} \cdot \sup_{0\leq s\leq\tau} ||f_N(\phi_1(s)) - f_N(\phi_2(s))||_A \\
&\leq \tau M(\tau) C_{Lip} \cdot \sup_{0\leq s\leq\tau} ||(\phi_1 - \phi_2)(s)||_A \\
&\leq \tau M(\tau) C_{Lip} \cdot ||\phi_1 - \phi_2||_{X_\tau} \\
&\leq \frac{1}{2} ||\phi_1 - \phi_2||_{X_\tau}.
\end{aligned}$$

So the restriction $J_{u,\tau}|_{B_{\tau,\epsilon}(u)}$ is a contraction. □

Corollary 1.6. *There exists a* $\tau > 0$ *such that* X_τ *contains a unique mild solution of* (4).

Proof. Apply the Banach contraction mapping principle to the mapping $J_{u,\tau}$ from the previous theorem. So there exists a unique fixed point $\phi \in B_{\tau,\epsilon}(u) \subset X_\tau$, i.e.:

$$\phi(t) = S(t)u + \int_0^t S(t-s)f(\phi(s))ds.$$

Thus ϕ is a mild solution of (4). □

1.2 Non-autonomous semilinear case

Now consider the non-autonomous semilinear Cauchy problem with reaction term:

$$\begin{cases} \frac{\partial}{\partial t}\phi(t) & = A(t)\phi(t) + f(\phi(t)) \quad \text{if } t \geq 0 \\ \phi(0) & = u \end{cases} ; \tag{5}$$

where $(A(t), D(A))$ with $D(A) = H^2(\mathbb{R})$ is a time dependent unbounded operator. We use the same approach as in the previous section. In this setting we replace the semigroup by the assumption that the $(A(t), D(A))$ generate an evolution system $U(t,s)$ on $H^2(\mathbb{R})$, see section 3.1. Furthermore, since working with some graph norm is no longer natural, we use the Sobolev norm instead. In section 3.2 equivalence of these norms is proven, so this difference is only minor.

Theorem 1.7 (Variation of constants formula)**.** *Suppose that $\phi : [0,\tau]\times\mathbb{R} \to \mathbb{R}$ with $\phi(\cdot,x) \in C^1([0,\tau])$ and $\phi(t,\cdot) \in C^2(\mathbb{R})$ is a classical solution of* (5)*, then it holds:*

$$\phi(t) = U(t,0)u + \int_0^t U(t,s)f(\phi(s))ds.$$

Proof. Let ϕ be a classical solution of (5) and fix t. Then the X valued function $s \mapsto U(t,s)\phi(s)$ is differentiable:

$$\begin{aligned} \frac{d}{ds}U(t,s)\phi(s) &= \lim_{h\to 0} \frac{U(t,s+h)\phi(s+h) - U(t,s)\phi(s)}{h} \\ &= \lim_{h\to 0} \frac{U(t,s+h)\phi(s+h) - U(t,s)\phi(s+h)}{h} \\ &\quad + \lim_{h\to 0} \frac{U(t,s)\phi(s+h) - U(t,s)\phi(s)}{h} \\ &= -U(t,s)A(t)\phi(s) + U(t,s)\frac{d}{ds}\phi(s); \end{aligned}$$

so it holds that:

$$\begin{aligned} \phi(t) - U(t,0)\phi(0) &= \int_0^t \frac{d}{ds}U(t,s)\phi(s)ds \\ &= \int_0^t \left(-U(t,s)A(t)\phi(s) + U(t,s)\frac{d}{ds}\phi(s)\right) ds \\ &= \int_0^t U(t,s)\left(-A(t)\phi(s) + \frac{d}{ds}\phi(s)\right) ds \\ &= \int_0^t U(t,s)f(\phi(s))ds. \end{aligned}$$

Replacing $\phi(0)$ by u we obtain:

$$\phi(t) = U(t,0)u + \int_0^t U(t,s)f(\phi(s))ds.$$

□

To interpret (5) as an ordinary differential equation on $H^2(\mathbb{R})$, we again introduce the Nemytskii operator f_N. Equation (5) can be rewritten:

$$\begin{cases} \dot{\phi}(t) = & A(t)\phi(t) + f_N(\phi(t)) \quad \text{if } t \geq 0 \\ \phi(0) = & u \end{cases} . \tag{6}$$

In the following definitions the Sobolev norm replaces the graph norm.

Definition For $\tau > 0$ introduce the Banach space :

$$X_\tau := \left(C\left([0,\tau], H^2(\mathbb{R})\right), ||\cdot||_\infty \right).$$

For $u \in H^2(\mathbb{R})$ and $\epsilon > 0$ let the closed ball centered around u with radious ϵ be given by:

$$B_\epsilon(u) := \left\{ v \in H^2(\mathbb{R}) : ||u - v||_{H^2} \leq \epsilon \right\};$$
$$B_{\tau,\epsilon}(u) := \left\{ \phi \in X_\tau : ||u - \phi||_\infty \leq \epsilon \right\}.$$

In the definition above we implicitly used that we can view $H^2(\mathbb{R})$ as the subset of constant functions (in time) in X_τ.

Definition An element $\phi \in X_\tau$ is a *mild solution* of (6) if on $[0,\tau]$ it holds that:

$$\phi(t) = U(t,0)u + \int_0^t U(t,s) f_N(\phi(s)) ds.$$

By theorem 1.7 every classical solution is a mild solution.

Using the Banach contraction mapping theorem we will prove the existence of a mild solution to (6). To this end we define a map:

$$J_{u,\tau} : X_\tau \to X_\tau$$
$$\phi \mapsto U(t,0)u + \int_0^t U(t,s) f_N(\phi(s)) ds,$$

and we want to show that for sufficiently small τ, $J_{u,\tau}$ is a contraction in a neighbourhood of u.

Theorem 1.8. *Let $u \in H^2(\mathbb{R})$ and suppose that (A0) holds. Assume that for some $\tau > 0$ and $0 \leq s \leq t \leq \tau$, $U(t,s)$ is bounded: $||U(t,s)||_{\mathscr{L}(H^2,H^2)} \leq M(\tau)$. Then for all $\epsilon > 0$ there exists a $\tau > 0$ such that the restriction $J_{u,\tau}|_{B_{\tau,\epsilon}(u)}$ is a contraction.*

Proof. Let $\epsilon > 0$ be given, then f_N is Lipschitz on $B_\epsilon(u)$, write $C_{Lip(f)}$ for the Lipschitz constant. There exists a constant $C_{|f|}$ such that for $v \in B_\epsilon(u)$ it holds $||f_N(v)||_{H^2} \leq C_{|f|}$. Since $U(t,s)$ is strongly continuous, we can find a $\tau > 0$ such that:

$$\sup_{0 \leq t \leq \tau} ||(U(t,0) - 1)u||_{H^2} \leq \frac{\epsilon}{2};$$
$$\sup_{0 \leq s \leq t \leq \tau} ||U(t,s)||_{\mathscr{L}(H^2,H^2)} \leq M(\tau);$$
$$\tau \leq \min\left\{ \frac{\epsilon}{2M(\tau)C_{|f|}}, \frac{1}{2M(\tau)C_{Lip(f)}} \right\}.$$

So it holds:

$$
\begin{aligned}
&||J_{u,\tau}(\phi) - u||_{X_\tau} \\
&= \sup_{0\le t\le\tau} ||J_{u,\tau}(\phi)(t) - u||_{H^2} \\
&\le \sup_{0\le t\le\tau} ||(U(t,0)-1)u||_{H^2} + \sup_{0\le t\le\tau} \left|\left| \int_0^t U(t,s) f_N(\phi(s)) ds \right|\right|_{H^2} \\
&\le \frac{\epsilon}{2} + \tau \cdot \sup_{0\le s\le t\le\tau} ||U(t,s) f_N(\phi(s))||_{H^2} \\
&\le \frac{\epsilon}{2} + \tau \cdot \sup_{0\le s\le t\le\tau} ||U(t,s)||_{\mathscr{L}(H^2,H^2)} \cdot \sup_{0\le s\le\tau} ||f_N(\phi(s))||_{H^2} \\
&\le \frac{\epsilon}{2} + \tau M(\tau) C_{|f|} \\
&\le \epsilon;
\end{aligned}
$$

so $J_{u,\tau}$ maps $B_{\tau,\epsilon}(u)$ into itself. We also have:

$$
\begin{aligned}
&||J_{u,\tau}(\phi_1) - J_{u,\tau}(\phi_2)||_{X_\tau} \\
&= \sup_{0\le t\le\tau} ||J_{u,\tau,}(\phi_1) - J_{u,\tau}(\phi_2)||_{H^2} \\
&\le \sup_{0\le t\le\tau} \left|\left| \int_0^t U(t,s)(f_N(\phi_1(s)) - f_N(\phi_2(s))) ds \right|\right|_{H^2} \\
&\le \tau \cdot \sup_{0\le s\le t\le\tau} ||U(t,s)(f_N(\phi_1(s)) - f_N(\phi_2(s)))||_{H^2} \\
&\le \tau \cdot \sup_{0\le s\le t\le\tau} ||U(t,s)||_{\mathscr{L}(H^2,H^2)} \cdot \sup_{0\le s\le\tau} ||f_N(\phi_1(s)) - f_N(\phi_2(s)))||_{H^2} \\
&\le \tau M(\tau) C_{Lip(f)} \cdot \sup_{0\le s\le\tau} ||(\phi_1 - \phi_2)(s)||_{H^2} \\
&\le \tau M(\tau) C_{Lip(f)} \cdot ||\phi_1 - \phi_2||_{X_\tau} \\
&\le \frac{1}{2} ||\phi_1 - \phi_2||_{X_\tau} .
\end{aligned}
$$

So the restriction $J_{u,\tau}|_{B_{\tau,\epsilon}(u)}$ is a contraction. □

Corollary 1.9. *There exists a $\tau > 0$ such that $B_{\tau,\epsilon}(u) \subset X_\tau$ contains a unique mild solution of* (6)*.*

Proof. Apply the Banach contraction mapping principle to the map $J_{u,\tau}|_{B_{\tau,\epsilon}(u)}$ from the previous theorem. □

Remark Let $0 < \tilde{\tau} < \tau$, then $J_{u,\tilde{\tau}}|_{B_{\tilde{\tau},\epsilon}(u)}$ is still a contraction, so $B_{\tilde{\tau},\epsilon}(u) \subset X_\tau$ contains a unique mild solution of (6).

1.3 Quasilinear case

Finally look at the quasilinear Cauchy problem with reaction term:

$$\begin{cases} \dot{\phi}(t) = & A(t,\phi)\phi(t) + f_N(\phi) \quad \text{if } t \geq 0 \\ \phi(0) = & u \end{cases}; \tag{7}$$

where $(A(t,\phi), D(A))$ with $D(A) = H^2(\mathbb{R})$ is an unbounded operator that depends on time and the function ϕ. Let X_τ, $B_\epsilon(u)$ and $B_{\tau,\epsilon}(u)$ be defined as in the previous section. We list some assumptions.

(A1) There exists ϵ and τ_1 such that for $\phi \in B_{\tau_1,\gamma,\epsilon}(u)$ it holds that the $A(t,\phi)$ generate an evolution system $\{U_\phi(t,s)\}_{0\leq s\leq t\leq \tau_1}$ on $H^2(\mathbb{R})$, with $||U_\phi(t,s)||_{\mathscr{L}(H^2,H^2)} \leq M_\phi(\tau_1)$.

We thus obtain a family of evolution systems parametrised by ϕ. With this assumption, similar to the previous section a variation of constants formula holds. Given any classical solution ϕ of (7) we have:

$$\phi(t) = U_\phi(t,0)u + \int_0^t U_\phi(t,s) f_N(\phi(s))ds; \tag{8}$$

which again is the defining equality for a mild solution.

Definition An element $\phi \in X_\tau$ is a *mild solution* if on $[0,\tau]$ equation (8) holds.

To obtain an intermediate result, let $\bar{\phi} \in B_{\tau_1,\epsilon}(u)$ be fixed and consider:

$$\begin{cases} \dot{\phi}(t) = & A(\bar{\phi}(t))\phi(t) + f_N(\phi) \quad \text{if } t \geq 0 \\ \phi(0) = & u \end{cases}. \tag{9}$$

This is a non-autonomous semilinear Cauchy problem, as in the previous section. We continue with another assumption.

(A2) There exists a $\tau_2 \leq \tau_1$ and $M(\tau_2)$ such that for all $\phi \in B_{\tau_2,\epsilon}(u)$ it holds that:

$$(A2a) \qquad \sup_{0\leq t\leq \tau_2} ||(U_\phi(t,0) - 1)\, u||_{H^2} \leq \frac{\epsilon}{2};$$

$$(A2b) \qquad \sup_{0\leq s\leq t\leq \tau_2} ||U_\phi(t,s)||_{\mathscr{L}(H^2,H^2)} \leq M(\tau_2).$$

Assumptions (A0)-(A2) are sufficient to apply theorem 1.8, so we can find $\tau_3 \leq \tau_2$ and a mild solution $\phi_{\bar{\phi}} = U_{\bar{\phi}}(t,0)u + \int_0^t U_{\bar{\phi}}(t,s)f_N(\phi(s))ds$ of (9) in $B_{\tau_3,\gamma,\epsilon}(u)$. Assumption (A2) also ensures that these τ_3 can be chosen independent of $\bar{\phi} \in B_{\tau_2,\epsilon}(u)$. This enables us to define a map, which sends an arbitrary element $\bar{\phi}$ in $B_{\tau_3,\epsilon}(u)$ to the corresponding unique mild solution of (9):

$$K_{u,\tau_3,\epsilon} : B_{\tau_3,\epsilon}(u) \to B_{\tau_3,\epsilon}(u)$$

$$\bar{\phi} \mapsto \phi_{\bar{\phi}} = U_{\bar{\phi}}(t,0)u + \int_0^t U_{\bar{\phi}}(t,s) f_N(\phi(s))ds.$$

Note that by the remark below corollary 1.9, for $0 < \tau < \tau_3$ the same result holds, so $K_{u,\tau_3,\epsilon}(B_{\tau,\epsilon}(u)) \subset B_{\tau,\epsilon}(u)$. We present a final assumption:

(A3) There exists a $\tau_4 \leq \tau_3$ and $C_{Lip(U)}$ such that for all $\phi_1, \phi_2 \in B_{\tau_4,\epsilon}(u)$:

$$(A3a) \sup_{0\leq s\leq t\leq \tau_4} ||(U_{\phi_1}(t,s) - U_{\phi_2}(t,s))\, u||_{H^2} \leq \frac{1}{2} ||\phi_1 - \phi_2||_{X_\tau};$$

$$(A3b) \sup_{0\leq s\leq t\leq \tau_4} ||U_{\phi_1}(t,s) - U_{\phi_2}(t,s)||_{\mathscr{L}(H^2,H^2)} \leq C_{Lip(U)} ||\phi_1 - \phi_2||_{X_\tau}.$$

Theorem 1.10. *Let $u \in D(A)$ and suppose that assumption (A0)-(A3) hold. Then there exists a $\tau > 0$ such that the restriction $K_{u,\tau_3,\epsilon}|_{B_{\tau,\epsilon}(u)}$ is a contraction.*

Proof. The only thing left to prove is that for some $0 < \tau \leq \tau_3$ the map $K_{u,\tau_3,\epsilon}|_{B_{\tau,\epsilon}(u)}$ is contractive. Let $C_{Lip(f)}$ and $C_{|f|}$ be given as in the proof of theorem 1.8. Choose $\tau < \tau_4$ such that $\tau \leq \min\left\{\frac{1}{8M_\tau C_{Lip(f)}}, \frac{1}{8C_{Lip(U)}C_{|f|}}\right\}$. For $\bar{\phi}_1, \bar{\phi}_2 \in B_{\tau,\epsilon}(u)$ it holds:

$$\begin{aligned}
&\left|\left|K_{u,\tau,\epsilon}(\bar{\phi}_1) - K_{u,\tau,\epsilon}(\bar{\phi}_2)\right|\right|_{X_\tau} \\
&\quad= \sup_{0\leq t\leq\tau} \left|\left|K_{u,\tau,\epsilon}(\bar{\phi}_1) - K_{u,\tau,\epsilon}(\bar{\phi}_2)\right|\right|_{H^2} \\
&\quad= \sup_{0\leq t\leq\tau} \left|\left| U_{\bar{\phi}_1}(t,0)u + \int_0^t U_{\bar{\phi}_1}(t,s) f_N(\phi_1(s))ds \right.\right. \\
&\qquad\qquad \left.\left. - U_{\bar{\phi}_2}(t,0)u - \int_0^t U_{\bar{\phi}_2}(t,s) f_N(\phi_2(s))ds \right|\right|_{H^2} \\
&\quad\leq \sup_{0\leq t\leq\tau} \left|\left|\left(U_{\bar{\phi}_1}(t,0) - U_{\bar{\phi}_2}(t,0)\right) u\right|\right|_{H^2} \\
&\qquad + \tau \cdot \sup_{0\leq s\leq t\leq\tau} \left|\left|U_{\bar{\phi}_1}(t,s) f_N(\phi_1(s)) - U_{\bar{\phi}_2}(t,s) f_N(\phi_2(s))\right|\right|_{H^2} \\
&\quad\leq \frac{1}{2}\left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau} \\
&\qquad + \tau \cdot \sup_{0\leq s\leq t\leq\tau} \left|\left|U_{\bar{\phi}_1}(t,s) f_N(\phi_1(s)) - U_{\bar{\phi}_1}(t,s) f_N(\phi_2(s))\right|\right|_{H^2} \\
&\qquad + \tau \cdot \sup_{0\leq s\leq t\leq\tau} \left|\left|U_{\bar{\phi}_1}(t,s) f_N(\phi_2(s)) - U_{\bar{\phi}_2}(t,s) f_N(\phi_2(s))\right|\right|_{H^2} \\
&\quad\leq \frac{1}{2}\left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau} \\
&\qquad + \tau \cdot \sup_{0\leq s\leq t\leq\tau} \left|\left|U_{\bar{\phi}_1}(t,s)\right|\right|_{\mathscr{L}(H^2,H^2)} \cdot \sup_{0\leq s\leq\tau} ||f_N(\phi_1(s)) - f_N(\phi_2(s))||_{H^2} \\
&\qquad + \tau \cdot \sup_{0\leq s\leq t\leq\tau} \left|\left|U_{\bar{\phi}_1}(t,s) - U_{\bar{\phi}_2}(t,s)\right|\right|_{\mathscr{L}(H^2,H^2)} \cdot \sup_{0\leq s\leq\tau} ||f_N(\phi_2(s))||_{H^2} \\
&\quad\leq \frac{1}{2}\left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau} + \tau M(\tau) C_{Lip(f)} \left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau} \\
&\qquad + \tau C_{Lip(U)} \left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau} C_{|f|}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau} + \frac{1}{8}\left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau} + \frac{1}{8}\left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau} \\
&= \frac{3}{4}\left|\left|\bar{\phi}_1 - \bar{\phi}_2\right|\right|_{X_\tau}.
\end{aligned}
$$

□

Using the Banach contraction mapping theorem, we again obtain existence and uniqueness of a mild solution. A third ingredient for well-posedness is *continuous dependence on initial conditions and parameters*. Though not studied in this thesis, this usually follows from the same contraction mapping theorem with additional smoothness of the Nemytskii operator, as in theorem 3.16.

1.3.1 Some remarks on the assumptions made

A0 This assumption is satisfied when the assumptions of corollary 3.14 are met, which is easy to verify for applications.

A1 The book by Pazy [9] has two different constructions of evolution systems: for the hyperbolic case (§5.3) and the parabolic case (§5.6), see section 3.1.

Since GKGS (equation (16)) is parabolic, it seems natural to apply the construction for the parabolic case. Property (P3) is satisfied if the coefficient of the highest order derivative of $A(t, \phi)$ is Hölder continuous in time, which may be satisfied if ϕ is Hölder continuous in time. Restriction of X_τ to such functions leads to a need for more demanding assumptions then (A2) and (A3).

On the other hand the abstract general well-posedness result was obtained without explicitly demanding the PDE to be parabolic, moreover Pazy also uses the evolution system for the hyperbolic case in a parabolic setting (§6.4).

The boundedness property of the evolution families is a property automatically satisfied in both constructions.

A2 Assumption (A2a) could be described by U uniformly (with respect to ϕ) approximating identity at u.

Assumption (A2b) can be deduced from assumption (A1) together with assumption (A3b). Given (A1) and (A3b) it holds that, for any $\phi \in B_{\tau_4,\epsilon}(u)$:

$$
\begin{aligned}
\sup_{0\le s\le t\le \tau_4} ||U_\phi(t,s)||_{\mathscr{L}(H^2,H^2)} \le & ||U_\phi(t,s) - U_u(t,s)||_{\mathscr{L}(H^2,H^2)} \\
& + \sup_{0\le s\le t\le \tau_4} ||U_u(t,s)||_{\mathscr{L}(H^2,H^2)} \\
\le & C_{Lip(U)} \left|\left|\phi_1 - \phi_2\right|\right|_{X_\tau} + M_u(\tau_4) \\
\le & C_{Lip(U)} \cdot 2\epsilon + M_u(\tau_4).
\end{aligned}
$$

A3 Assumption (A3) describes a property of Lipschitz continuity of U with respect to ϕ. Assumption (A3a) requires the Lipschitz constant to be smaller equal $\frac{1}{2}$ when U is viewed to only map u.

One way to link assumption (A3) to a property of the generator is by making use of the equation:

$$
\begin{aligned}
\left(U_{\phi_1}(t,s)-U_{\phi_2}(t,s)\right)v &= -\int_s^t \frac{\partial}{\partial r} U_{\phi_1}(t,r)U_{\phi_2}(r,s)v\; dr \\
&= \int_s^t U_{\phi_1}(t,r)[A(t,\phi_1(r))-A(t,\phi_2(r))]U_{\phi_2}(r,s)v\; dr;
\end{aligned}
$$

where $\phi_1,\phi_2\in X_\tau$, $v\in H^2(\mathbb{R})$ and $s\le r\le t$. So:

$$
\begin{aligned}
&\sup_{0\le s\le t\le\tau} \left|\left|\left(U_{\phi_1}(t,s)-U_{\phi_2}(t,s)\right)v\right|\right|_{H^2} \\
&\quad = \sup_{0\le s\le t\le\tau}\left|\left|\int_s^t U_{\phi_1}(t,r)[A(r,\phi_1(r))-A(r,\phi_2(r))]U_{\phi_2}(r,s)v\; dr\right|\right|_{H^2} \\
&\quad \le \tau\cdot \sup_{0\le s\le r\le t\le\tau} \left|\left|U_{\phi_1}(t,r)[A(r,\phi_1(r))-A(r,\phi_2(r))]U_{\phi_2}(r,s)v\right|\right|_{H^2}.
\end{aligned}
$$

The next step would be to split the supremum into parts and aim for an estimate:

$$
\sup_{0\le r\le\tau} ||A(r,\phi_1(r))-A(r,\phi_2(r))|| \le C\,||\phi_1(r)-\phi_2(r)||_{X_\tau};
$$

and have the other parts $||U_{\phi_1}(t,r)||$ and $||U_{\phi_2}(r,s)||$ be bounded.

But this is problematic. If v would be an element of $H^4(\mathbb{R})$, then:

$$
\begin{aligned}
&\sup_{0\le s\le r\le t\le\tau} ||U_{\phi_1}(t,r)[A(r,\phi_1(r))-A(r,\phi_2(r))]U_{\phi_2}(r,s)v||_{H^2} \\
&\quad\le \sup_{0\le s\le r\le t\le\tau} ||U_{\phi_1}(t,r)||_{\mathscr{L}(H^2,H^2)}\cdot ||A(r,\phi_1(r))-A(r,\phi_2(r))||_{\mathscr{L}(H^4,H^2)} \\
&\qquad\qquad \cdot ||U_{\phi_2}(r,s)||_{\mathscr{L}(H^4,H^4)}\,||v||_{H^4};
\end{aligned}
$$

in which case $||U_{\phi_1}(t,r)||_{\mathscr{L}(H^2,H^2)}$ and $||U_{\phi_2}(r,s)||_{\mathscr{L}(H^4,H^4)}$ are easily seen to be bounded. Alternatively, one could estimate:

$$
\begin{aligned}
&\sup_{0\le s\le r\le t\le\tau} ||U_{\phi_1}(t,r)[A(r,\phi_1(r))-A(r,\phi_2(r))]U_{\phi_2}(r,s)v||_{H^2} \\
&\quad\le \sup_{0\le s\le r\le t\le\tau} ||U_{\phi_1}(t,r)||_{\mathscr{L}(L^2,H^2)}\cdot ||A(r,\phi_1(r))-A(r,\phi_2(r))||_{\mathscr{L}(H^2,L^2)} \\
&\qquad\qquad \cdot ||U_{\phi_2}(r,s)||_{\mathscr{L}(H^2,H^2)}\,||v||_{H^2};
\end{aligned}
$$

or

$$
\begin{aligned}
&\sup_{0\le s\le r\le t\le\tau} ||U_{\phi_1}(t,r)[A(r,\phi_1(r))-A(r,\phi_2(r))]U_{\phi_2}(r,s)v||_{H^2} \\
&\quad\le \sup_{0\le s\le r\le t\le\tau} ||U_{\phi_1}(t,r)||_{\mathscr{L}(H^2,H^2)}\cdot ||A(r,\phi_1(r))-A(r,\phi_2(r))||_{\mathscr{L}(H^4,H^2)} \\
&\qquad\qquad \cdot ||U_{\phi_2}(r,s)||_{\mathscr{L}(H^2,H^4)}\,||v||_{H^2};
\end{aligned}
$$

in which case there needs to be some uniform smoothening property of $U_{\phi_1}(t, r)$ or $U_{\phi_2}(r, s)$ respectively. But this cannot be expected for a time interval containing zero.

In conclusion, while assumption (A3) is convenient for the abstract approach it is not clear whether it can be verified for specific equations such as GKGS.

2 Generalised Klausmeier Gray-Scott equations

2.1 Comparison of homogeneous steady states of Gray-Scott with Klausmeier

The PDE's of interest will be:

$$\text{Klausmeier:} \begin{cases} u_t = & C_K u_x + A_K - u - uv^2 \\ v_t = & v_{xx} - B_K v + uv^2 \end{cases} \tag{10}$$

$$\text{Gray-Scott:} \begin{cases} u_t = & D_{GS} u_{xx} + A_{GS}(1-u) - uv^2 \\ v_t = & v_{xx} - B_{GS} v + uv^2 \end{cases} \tag{11}$$

on $\mathbb{R}_+ \times \mathbb{R}$, where A_K, B_K, A_{GS}, B_{GS} are assumed to be strictly positive constants. In order to first restrict attention to homogeneous solutions we introduce homogeneous versions of these PDE's:

$$\text{Homogeneous Klausmeier:} \begin{cases} u_t = & A_K - u - uv^2 \\ v_t = & -B_K v + uv^2 \end{cases} \tag{12}$$

$$\text{Homogeneous Gray-Scott:} \begin{cases} u_t = & A_{GS}(1-u) - uv^2 \\ v_t = & -B_{GS} v + uv^2 \end{cases} \tag{13}$$

The system of Klausmeier is used for modelling plant and water dynamics in semiarid regions [8]. The Gray-Scott system models concentrations of chemical reactants. As we shall see, the homogeneous systems of equations exhibit the same qualitative behaviour.

2.1.1 Local bifurcation analysis for Homogeneous Klausmeier

Homogeneous steady state solutions of (12) would have to solve:

$$\begin{cases} 0 = & A_K - u - uv^2 \\ 0 = & -B_K v + uv^2 \end{cases} . \tag{14}$$

In the following we study stability with respect to homogeneous perturbations only. Hence we compute the Jacobian $J = \begin{pmatrix} \frac{\partial u_t}{\partial u} & \frac{\partial u_t}{\partial v} \\ \frac{\partial v_t}{\partial u} & \frac{\partial v_t}{\partial v} \end{pmatrix}$ of the Homogeneous Klausmeier equations:

$$J = \begin{pmatrix} -1 - v^2 & -2uv \\ v^2 & -B_K + 2uv \end{pmatrix}.$$

The eigenvalues λ are given by the characteristic equation:

$$\begin{aligned} 0 =& (-1 - v^2 - \lambda)(-B_K + 2uv - \lambda) - (-2uv)v^2 \\ =& \lambda^2 + \lambda(1 + v^2 + B_K - 2uv) + (1 + v^2)(B_K - 2uv) + 2uv^3. \end{aligned}$$

Desert state One solution of (14) is given by $v = 0$ and $u = A_K$, the so-called *desert state*. The characteristic equation then becomes:

$$\begin{aligned} 0 =& \lambda^2 + \lambda(1 + B_K) + B_K \\ =& (\lambda + 1)(\lambda + B_K); \end{aligned}$$

so $\lambda = -1$ or $\lambda = -B_K$, thus the desert state is stable.

Saddle-node states The other solutions of (14) are given by:

$$uv = B_K \qquad \text{and} \qquad u^2 - A_K u + B_K^2 = 0;$$

so:

$$u_\pm = \frac{A_K}{2} \pm \sqrt{\frac{A_K^2}{4} - B_K^2} \qquad \text{and} \qquad v_\pm = \frac{A_K}{2B_K} \pm \sqrt{\frac{A_K^2}{4B_K^2} - 1};$$

where (u_+, v_-) is one solution and (u_-, v_+) is the other. It is obvious that these solutions only exist if $\frac{A_K^2}{4} > B_K^2$, so if $A_K > 2B_K$, i.e. $B_K \in (0, \frac{A_K}{2})$.

With $uv = B_K$ we obtain:

$$0 = \lambda^2 + \lambda(1 + v^2 - B_K) - B_K + B_K v^2;$$

so $\lambda_\pm = -\frac{1}{2}(1 + v^2 - B_K) \pm \sqrt{B_K(1 - v^2) + \frac{1}{4}(1 + v^2 - B_K)^2}$. Let $\Re(\lambda)$ denote the real part of λ. We can make the following general classification concerning stability of (u, v).

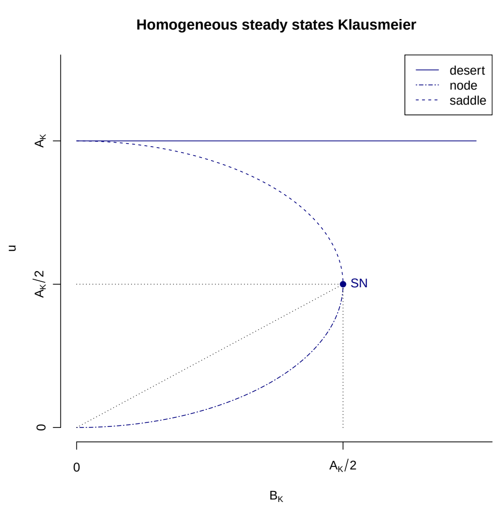
Stability of (u, v)	$1 + v^2 - B_K > 0$	$1 + v^2 - B_K < 0$
$1 - v^2 > 0$	$\Re(\lambda_+) > 0$, $\Re(\lambda_-) < 0$ (saddle, unstable)	$\Re(\lambda_+) > 0$, $\Re(\lambda_-) < 0$ (saddle, unstable)
$1 - v^2 < 0$	$\Re(\lambda_+) < 0$, $\Re(\lambda_-) < 0$ (stable node)	$\Re(\lambda_+) > 0$, $\Re(\lambda_-) > 0$ (unstable node)

We use this table to determine the stability of (u_+, v_-) and (u_-, v_+).

It holds that:

$$\begin{aligned} & u_+ - B_K \\ =& \frac{A_K}{2} - B_K + \sqrt{\frac{A_K^2}{4} - B_K^2} \\ >& 0 \end{aligned}$$

for $B_K \in (0, \frac{A_K}{2})$. So $u_+ > B_k$, thus $v_- = \frac{B_K}{u_+} < 1$. From this it follows that $1 - v^2 > 0$, so (u_+, v_-) is a *saddle*.



As for (u_-, v_+), it holds that $u_- - B_K = \frac{A_K}{2} - B_K - \sqrt{\frac{A_K^2}{4} - B_K^2} < 0$ since for $B_K \in (0, \frac{A_K}{2})$:

$$\begin{aligned}\left(\frac{A_K}{2} - B_k\right)^2 &= \frac{A_K^2}{4} - A_K B_K + B_K^2 \\ &< \frac{A_K^2}{4} - \frac{A_K B_K}{2} \\ &< \frac{A_K^2}{4} - B_K^2 \\ &= \left(\sqrt{\frac{A_K^2}{4} - B_K^2}\right)^2.\end{aligned}$$

So $u_- < B_k$, thus $v_+ > 1$. So (u_-, v_+) is a *node*.

The homogeneous steady states are shown in the figure above. When B_K drops below $\frac{A_K}{2}$ a *saddle-node bifurcation* occurs, depicted by SN in the figure.

To discern between the possibilties of (u_-, v_+) being a stable node (sink) or an unstable node (source) we note that (u_-, v_+) is stable precisely if $v_+^2 > B_K - 1$ (so $B_K \geq 1$). Write:

$$v_+^2 = \frac{A_K^2}{4B_K^2} + \frac{A_K^2}{4B_K^2} - 1 + \frac{A_K}{B_K}\sqrt{\frac{A_K^2}{4B_K^2} - 1};$$

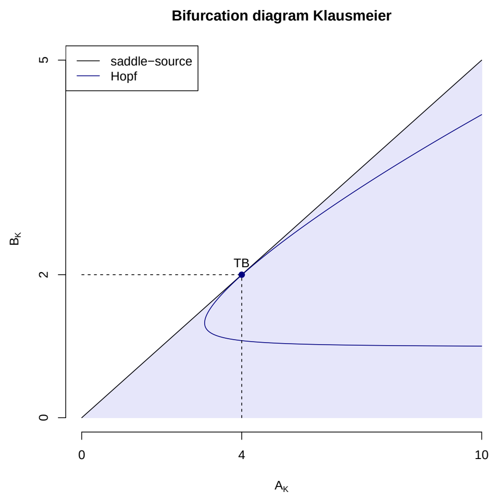
then the following sequence of equivalencies holds.

$$\begin{aligned}(u_-, v_+) \text{ is stable} &\Leftrightarrow \frac{A_K^2}{4B_K^2} + \frac{A_K^2}{4B_K^2} - 1 + \frac{A_K}{B_K}\sqrt{\frac{A_K^2}{4B_K^2} - 1} > B_K - 1 \\ &\Leftrightarrow \sqrt{\frac{A_K^2}{4B_K^2} - 1} > \frac{B_K^2}{A_K} - \frac{A_K}{2B_K} \\ &\Leftrightarrow \frac{A_K^2}{4B_K^2} - 1 > \frac{B_K^4}{A_K^2} + \frac{A_K^2}{4B_K^2} - B_K \\ &\Leftrightarrow -A_K^2 > B_K^4 - B_K A_K^2 \\ &\Leftrightarrow A_K^2(B_K - 1) > B_K^4 \\ &\Leftrightarrow A_K > \frac{B_K^2}{\sqrt{B_K - 1}}.\end{aligned}$$

In the figure on the right, the region where the saddle and node states exist is coloured lavender. The node has a *Hopf instability* precisely when:

$$A_K = \frac{B_K^2}{\sqrt{B_K - 1}}; A_K \neq 4.$$

Only to the right of the Hopf curve the node is stable. Generically, at Hopf instability a *Hopf bifurcation* takes place. Without going into details, at the point depicted by TB we expect a *Takens-Bogdanov bifurcation*.



We proceed by running through the same procedure for the Gray-Scott system.

2.1.2 Local bifurcation analysis for Homogeneous Gray-Scott

The homogeneous steady state solutions of (13) are given by:

$$\begin{cases} 0 = & A_{GS}(1-u) - uv^2 \\ 0 = & -B_{GS}v + uv^2 \end{cases}. \tag{15}$$

The Jacobian of the Homogeneous Gray-Scott equations is:

$$J = \begin{pmatrix} \frac{\partial u_t}{\partial u} & \frac{\partial u_t}{\partial v} \\ \frac{\partial v_t}{\partial u} & \frac{\partial v_t}{\partial v} \end{pmatrix} = \begin{pmatrix} -A_{GS} - v^2 & -2uv \\ v^2 & -B_{GS} + 2uv \end{pmatrix}.$$

The eigenvalues λ are given by the characteristic equation:

$$\begin{aligned} 0 =& (-A_{GS} - v^2 - \lambda)(-B_{GS} + 2uv - \lambda) - (-2uv)v^2 \\ =& \lambda^2 + \lambda(A_{GS} + v^2 + B_{GS} - 2uv) + (A_{GS} + v^2)(B_{GS} - 2uv) + 2uv^3. \end{aligned}$$

Desert state One solution of (15) is given by $v = 0$ and $u = 1$, which we call the *desert state* to emphasise similarities with Klausmeier. The characteristic equation becomes:

$$\begin{aligned} 0 =& \lambda^2 + \lambda(A_{GS} + B_{GS}) + A_{GS}B_{GS} \\ =& (\lambda + A_{GS})(\lambda + B_{GS}); \end{aligned}$$

so $\lambda = -A_{GS}$ or $\lambda = -B_{GS}$, thus the desert state is stable.

Saddle-node states Other solutions of (15) are:

$$uv = B_{GS} \qquad \text{and} \qquad u^2 - u + \frac{B_{GS}^2}{A_{GS}} = 0;$$

so:

$$u_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{B_{GS}^2}{A_{GS}}} \qquad \text{and} \qquad v_{\pm} = \frac{A_{GS}}{2B_{GS}} \pm \sqrt{\frac{A_{GS}^2}{4B_{GS}^2} - A_{GS}};$$

where (u_+, v_-) is one solution and (u_-, v_+) is the other. These solutions only exist if $\frac{B_{GS}^2}{A_{GS}} < \frac{1}{4}$, so if $A_{GS} > 4B_{GS}^2$.

With $uv = B_{GS}$ we obtain:

$$0 = \lambda^2 + \lambda(A_{GS} + v^2 - B_{GS}) + (A_{GS} + v^2) \cdot -B_{GS} + 2B_{GS}v^2.$$

From (15) with $uv = B_{GS}$ it follows that $v^2 = \frac{A_{GS}v}{B_{GS}} - A_{GS}$, so:

$$0 = \lambda^2 + \lambda \frac{A_{GS}v - B_{GS}^2}{B_{GS}} + A_{GS}(v - 2B_{GS});$$

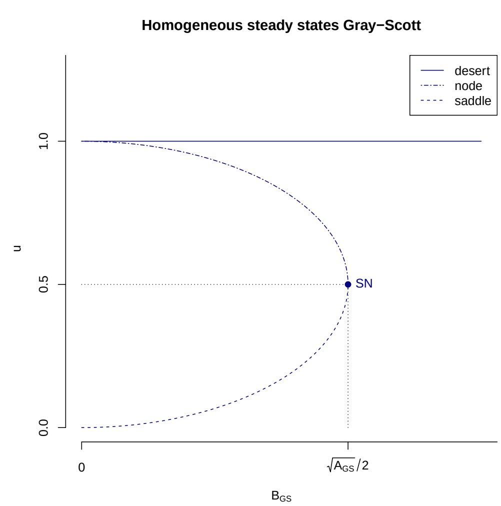
thus $\lambda_{\pm} = -\frac{A_{GS}v - B_{GS}^2}{2B_{GS}} \pm \sqrt{-A_{GS}(v - 2B_{GS}) + \frac{1}{4B_{GS}^2}\left(A_{GS}v - B_{GS}^2\right)^2}$. The following table gives an overview of the dependence of stability on λ.

	$A_{GS}v - B_{GS}^2 > 0$	$A_{GS}v - B_{GS}^2 < 0$
$v - 2B_{GS} > 0$	$\Re(\lambda_+) > 0$, $\Re(\lambda_-) < 0$ (saddle, unstable)	$\Re(\lambda_+) > 0$, $\Re(\lambda_-) < 0$ (saddle, unstable)
$v - 2B_{GS} < 0$	$\Re(\lambda_+) < 0$, $\Re(\lambda_-) < 0$ (stable node)	$\Re(\lambda_+) > 0$, $\Re(\lambda_-) > 0$ (unstable node)

Again we determine stability of (u_+, v_-) and (u_-, v_+), in accordance with the table.
Since $u_- < \frac{1}{2}$ it holds that:

$$v_+ = \frac{B_{GS}}{u_-} > 2B_{GS};$$

so (u_-, v_+) is a *saddle*. On the other hand, we have $u_+ > \frac{1}{2}$ so $v_- < 2B_{GS}$. Thus (u_+, v_-) is a *node*.
The homogeneous steady states are shown in the figure to the right, a *saddle-node bifurcation* (SN) occurs when B_{GS} drops below $\frac{\sqrt{A_{GS}}}{2}$.



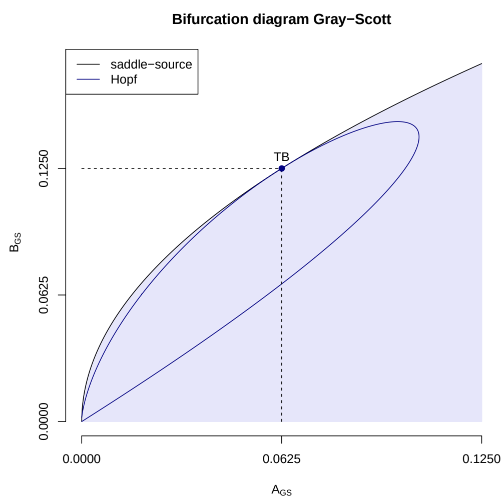
To determine stability of the node, we note that (u_+, v_-) is stable precisely if $v_- > \frac{B_{GS}^2}{A_{GS}}$. Recall that $v_- = \frac{A_{GS}}{2B_{GS}} - \sqrt{\frac{A_{GS}^2}{4B_{GS}^2} - A_{GS}}$. The following list of equalities is true:

$$\begin{aligned}(u_+, v_-) \text{ is stable} &\Leftrightarrow -\sqrt{\frac{A_{GS}^2}{4B_{GS}^2} - A_{GS}} > \frac{B_{GS}^2}{A_{GS}} - \frac{A_{GS}}{2B_{GS}} \\ &\Leftrightarrow \frac{A_{GS}^2}{4B_{GS}^2} - A_{GS} < \frac{B_{GS}^4}{A_{GS}^2} + \frac{A_{GS}^2}{4B_{GS}^2} - B_{GS} \\ &\Leftrightarrow B_{GS}^4 + A_{GS}^3 - A_{GS}^2 B_{GS} > 0.\end{aligned}$$

In the figure, the region of existence of the saddle and node states has a lavender colour. The node has two complex conjugate eigenvalues cross the imaginary axis at:

$$B_{GS}^4 + A_{GS}^3 - A_{GS}^2 B_{GS} = 0.$$

Only within this Hopf curve the node is unstable. At the curve a *Hopf bifurcation* takes place, except for the point depicted by TB, the *Takens-Bogdanov bifurcation*. We refrain from proving this.



2.1.3 Transformation of Homogeneous Klausmeier into Homogeneous Gray-Scott

Let (u_K, v_K) be solutions to the homogeneous Klausmeier system (12). Write:

$$\begin{aligned} u_K &= C_u u; \\ v_K &= C_v v; \\ t &= \sigma\tau. \end{aligned}$$

Substituting this into homogeneous Klausmeier gives:

$$\begin{cases} \sigma C_u u_\tau = A_K - C_u u - C_u C_v^2 u v^2 \\ \sigma C_v v_\tau = -B_K C_v v + C_u C_v^2 u v^2 \end{cases};$$

which yields:

$$\begin{cases} u_\tau = \frac{A_K}{\sigma C_u} - \frac{u}{\sigma} - \frac{C_v^2 u v^2}{\sigma} \\ v_\tau = \frac{-B_K v}{\sigma} + \frac{C_u C_v u v^2}{\sigma} \end{cases}.$$

Now if we assume that (u, v) solves the homogeneous Gray-Scott equations (13) then:

$$\begin{aligned}\frac{A_K}{\sigma C_u} =& A_{GS};\\ \frac{1}{\sigma} =& A_{GS};\\ \frac{C_v^2}{\sigma} =& 1;\\ \frac{B_k}{\sigma} =& B_{GS};\\ \frac{C_u C_v}{\sigma} =& 1;\end{aligned}$$

which after some calculations then yields:

$$\begin{aligned}C_u =& \frac{1}{\sqrt{A_{GS}}};\\ C_v =& \frac{1}{\sqrt{A_{GS}}};\\ \sigma =& \frac{1}{A_{GS}};\\ A_K =& \frac{1}{\sqrt{A_{GS}}};\\ B_K =& \frac{B_{GS}}{A_{GS}}.\end{aligned}$$

Conversely, substituting these values into the homogeneous Klausmeier eqautions:

$$\begin{aligned}u_\tau =& -\frac{1}{\sqrt{A_{GS}}} A_{GS} \sqrt{A_{GS}} - A_{GS} u - \frac{1}{A_{GS}} A_{GS} u v^2\\ =& A_{GS}(1-u) - uv^2;\\ v_\tau =& -\frac{B_{GS}}{A_{GS}} A_{GS} v + \frac{1}{\sqrt{A_{GS}}} \frac{1}{\sqrt{A_{GS}}} A_{GS} u v^2\\ =& -B_{GS} v + uv^2;\end{aligned}$$

which indeed shows that the transformed solution solves the homogeneous Gray-scott equations.

Remark With these transformations it is possible to learn about (the stability of) the homogeneous steady states of Gray-Scott via Klausmeier, without doing the calculations of section 2.1.2.

2.2 A more general system of equations: GKGS

Now we present a new system of equations, that is introduced in [5]. If below $\gamma = 1$ is chosen, it can be viewed as a generalisation of the Klausmeier and the Gray-Scott equations, therefore we call it the *Generalised Klausmeier Gray-Scott* equations, abbreviated by *GKGS*:

$$\text{GKGS:} \begin{cases} u_t = & \mathcal{D}\left(u^\gamma\right)_{xx} + \mathcal{C}u_x + \mathcal{A}(1-u) - uv^2 \\ v_t = & v_{xx} - \mathcal{B}v + uv^2 \end{cases} \tag{16}$$

on $\mathbb{R}_{\geq 0} \times \mathbb{R}$, where $\mathcal{A}, \mathcal{B}, \mathcal{D}$ are assumed to be strictly positive constants and $\gamma = 1$ or $\gamma = 2$. For $\gamma = 1$ the system is semilinear and parabolic.

In the case that GKGS is used to model vegetation in arid regions, where roughly speaking u gives the availability of water and v stands for plant biomass, $\gamma = 2$ is chosen if the water is thought to diffuse as through a porous medium. For $\gamma = 2$ the system is quasilinear, and parabolic for $u > 0$.

Since the reaction terms equal those of the Gray-Scott equations, the homogeneous steady states and stability properties with respect to homogeneous perturbations of these is the same, independent of the choice of γ. Thus for suitable parameters $\mathcal{A}$ and $\mathcal{B}$ a Hopf bifurcation takes place, so we expect a small limit cycle branching from the fixed point. Homogeneous oscillations emerging from the Hopf bifurcation imply existence of wavetrains, confer [10].

In the continuation of this thesis, we only look at the case $\gamma = 2$ as this is viewed as being the harder case of the two.

2.3 Bound below of the u-component of periodic solutions of GKGS

The top equation of (16) for $\gamma = 2$ can be rewritten:

$$\begin{pmatrix} u_t \\ u_x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\mathcal{C} \end{pmatrix} = 2\mathcal{D}uu_{xx} + 2\mathcal{D}u_x^2 + \mathcal{A}(1-u) - uv^2, \tag{17}$$

where $\left(\begin{smallmatrix} u_t \\ u_x \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ -\mathcal{C} \end{smallmatrix}\right)$ is a directional derivative of u.

Denote by S^1 the one-dimensional circle, this corresponds to an interval with periodic boundary conditions.

Theorem 2.1. *Suppose that (u, v) is a classical solution of (16) on $[0, T] \times S^1$ with $u(0, \cdot) \geq \delta_0 > 0$. If v is bounded, $|v| \leq v_{\max}$, then on $[0, T] \times S^1$ it holds that: $u \geq \min\{\delta_0, \frac{\mathcal{A}}{\mathcal{A}+v_{\max}^2}\} =: \delta$.*

Proof. Let $t_0 \in [0, T]$. Since $u(t_0, \cdot)$ is continuous on S^1, there exists a point $x_{\min}$ such that $(x_{\min}, u(t_0, x_{\min}))$ is a global minimum on S^1. So we have $u_{xx}(t_0, x_{\min}) \geq 0$. Ad absurdum suppose that $0 < u(t_0, x_{\min}) < \delta$. Evaluating (17) in $(t_0, x_{\min})$ gives:

$$\begin{pmatrix} u_t \\ u_x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\mathcal{C} \end{pmatrix} \geq \mathcal{A}(1-u) - uv_{\max}^2.$$

Since $\mathcal{A}(1-u) - uv_{\max}^2$ is a strictly decreasing function of u, and $u < \delta \leq \frac{\mathcal{A}}{\mathcal{A}+v_{\max}^2}$, it holds that:

$$\begin{pmatrix} u_t \\ u_x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\mathcal{C} \end{pmatrix} > \mathcal{A}(1 - \frac{\mathcal{A}}{\mathcal{A}+v_{\max}^2}) - \frac{\mathcal{A}}{\mathcal{A}+v_{\max}^2}v_{\max}^2 = 0.$$

So the directional derivative in $(t_0, x_{\min})$ is positive. Since the directional derivative is a continuous function in x and t, it follows that it is positive in a neighbourhood of $(t_0.x_{\min})$. So in this neighbourhood, in a comoving frame the function will locally increase with time. So the function could never have gotten smaller than δ on $[0, T] \times S^1$. □

Remark Since the coefficients of GKGS are analytic, it is expected that for analytic initial conditions short time existence of a solution is guaranteed by Cauchy-Kovalevskaya theorem. In this thesis the conditions for this theorem are not checked.

The theorem above can be generalised to allow for solutions with isolated irregularities, provided that minima still have a neigbourhood for which the second derivative is positive, via the same proof. Another approach is to look at an interval with Dirichlet or Neumann boundary conditions. At the interior of the interval the same argument works as above and at the boundary the function could be contained by the boundary conditions. For the real line the issue is to control decay to 0 at infinity, which is beyond the scope of this thesis.

2.4 Solutions of GKGS on bounded domains

For bounded domains, well-posedness of GKGS (equation (16), with $\gamma = 2$) is covered by literature. To illustrate this, we show that on an interval, under reasonable assumptions, an existence theorem of Amann [4] can be applied to GKGS. Attention is restricted to solutions with u bounded away from 0, write $u \geq \epsilon > 0$.

For $\gamma = 2$, equation (16) may be rewritten:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} - \frac{\partial}{\partial x}\left(\begin{pmatrix} 2\mathcal{D}u & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} \mathcal{C}u_x + \mathcal{A}(1-u) - uv^2 \\ -\mathcal{B}v + uv^2 \end{pmatrix}.$$

If we put $\Omega = (0,1)$, $n = 1$, $N = 2$, $a_{11} = \left(\begin{smallmatrix} 2\mathcal{D}u & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} \mathcal{C}u_x + \mathcal{A}(1-u) - uv^2 \\ -\mathcal{B}v + uv^2 \end{smallmatrix}\right)$, $G = \{(a,b) \in \mathbb{R}^2 \,|\, a \geq \epsilon\}$, then the following properties hold. Since linear and constant maps are $\mathscr{C}^\infty$ it follows that a_{11} is $\mathscr{C}^\infty$. If we take $f_0 = \left(\begin{smallmatrix} \mathcal{A}(1-u) - uv^2 \\ -\mathcal{B}v + uv^2 \end{smallmatrix}\right)$, $f_1 = \left(\begin{smallmatrix} \mathcal{C} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $f_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then we may write $f(x, w, \eta) = f_0 + f_1\eta_1 + f_2\eta_2$ so f is 'affine in the gradient'. Now GKGS has been rewritten to the form (1) in the article.

Next to check is that assumptions (i), (ii) and (iii) on page 4 of the article hold. Since $\Omega = (0,1)$ it holds that $\partial\Omega = \{0,1\}$. Let δ be a diagonal matrix as in the article, with $\delta_{ii}(x) = 0$ if we impose Dirichlet boundary conditions and $\delta_{ii}(x) = 1$ if we impose Neumann type boundary conditions. If we restrict ourselves to only homogeneous Dirichlet and Neumann boundary conditions, then assumption (iii) holds. We may write, implicitly defining $\mathbb{B}$:

$$\mathbb{B}w := \delta w_x + (1-\delta)w = 0.$$

If we also put:

$$\mathbb{A}(w)w := -\frac{\partial}{\partial x}\left(\begin{pmatrix} 2Dw_1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x}\right) w,$$

then our system can be written in the concise form (6) of the article. Since we have only one matrix a_{jk}, we can define $\mathbf{A}(\cdot, w) = a_{11}(\cdot, w)$ and $\alpha_{11} = 1$, so our equations are trivially of 'separated divergence form' (assumption (i)). The last assumption to check is that the spectrum $\sigma(\mathbf{A}(x,\xi))$ has positive real part for $(x,\xi) \in \bar{\Omega} \times G$. Since $\sigma(\mathbf{A}(x,\xi)) = \{2\mathcal{D}\xi_1, 1\}$, and $\xi_1 \geq \epsilon$ by our definition of G, assumption (ii) holds.

So the theorem on the top of page 5 can be applied, giving existence and uniqueness of a solution of the concise form (6) of the article, so also of GKGS.

Remark Given a solution ψ of GKGS with u away from 0 and a slightly perturbed solution $\psi + \phi$ with $\phi \ll \epsilon$, the perturbation ϕ itself solves a PDE as in theorem 3.17 below.

2.5 Perturbations of solutions of GKGS on the real line

In this section an idea is given on how to use the framework of section 1.3 to find solutions of GKGS (equation (16), with $\gamma = 2$). First problem is that the framework is only set up for a single PDE and not for a system. But generalisation to a system is straightforward: the space $H^2(\mathbb{R})$ has to be replaced by the Cartesian product of copies of $H^2(\mathbb{R})$. Another problem is that from equation (17) it is clear that the coefficient of the highest order derivative of u vanishes as u vanishes. Since u is an element of $H^2(\mathbb{R})$, this can't be avoided (see corollary 3.11).

Instead of choosing a different Banach space for our framework, we take another perspective. Given a solution ψ of GKGS we attempt to find a solution ϕ of a related PDE for perturbations of solutions of GKGS, given by equation (31). By theorem 3.17 this yield a solution $\psi + \phi$ of GKGS. Proving existence of solutions near a given solution can be viewed as a first step towards learning about stability of the given solution. An example of such a given solution would be a wavetrain, whose existence was implied by the presence of a Hopf bifurcation (see section 2.2). We restrict our attention to solutions of GKGS with u positive.

First we rewrite GKGS. Since u is assumed to be positive, define $w = u^2$. Then $u = \sqrt{w}$, so $u_t = \frac{1}{\sqrt{2w}} w_t$ and $u_x = \frac{1}{\sqrt{2w}} w_x$. It holds that:

$$\begin{aligned}\frac{1}{\sqrt{2w}} w_t =& \mathcal{D} w_{xx} + \mathcal{C}\frac{1}{\sqrt{2w}} w_x + \mathcal{A}(1-\sqrt{w}) - \sqrt{w} v^2;\\ w_t =& \mathcal{D}\sqrt{2w} w_{xx} + \mathcal{C} w_x + \sqrt{2w}\mathcal{A}(1-\sqrt{w}) - \sqrt{2} w v^2.\end{aligned}$$

Now a $2u_x^2$ term will not appear as in equation (17), which is important because without this trick it had to be included in the reaction term below. Now the reaction term does not contain any derivatives of u and v. The GKGS equations with substitution $u = \sqrt{w}$ are given by:

$$\left\{\begin{array}{ll} w_t = & \mathcal{D}\sqrt{2w} w_{xx} + \mathcal{C} w_x + \sqrt{2w}\mathcal{A}(1-\sqrt{w}) - \sqrt{2} w v^2 \\ v_t = & v_{xx} - \mathcal{B} v + \sqrt{w} v^2 \end{array}\right. . \tag{18}$$

Derivation of a related PDE for perturbations Let $\psi = \left(\begin{smallmatrix} w_\psi \\ v_\psi \end{smallmatrix}\right)$ be a given solution of this PDE, and let $\phi = \left(\begin{smallmatrix} w \\ v \end{smallmatrix}\right) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. As in section 3.6 below define:

$$\begin{aligned} B(\psi) :=& \begin{pmatrix} \mathcal{D}\sqrt{2w_\psi}\partial_x^2 + \mathcal{C}\partial_x & 0 \\ 0 & \partial_x^2 \end{pmatrix};\\ g(\psi) :=& \begin{pmatrix} \sqrt{2w_\psi}\mathcal{A}(1-\sqrt{w_\psi}) - \sqrt{2} w_\psi v_\psi^2 \\ -\mathcal{B} v_\psi + \sqrt{w_\psi} v_\psi^2 \end{pmatrix};\\ A_\psi(\phi) :=& B(\psi+\phi)\\ =& \begin{pmatrix} \mathcal{D}\sqrt{2(w_\psi + w)}\partial_x^2 + \mathcal{C}\partial_x & 0 \\ 0 & \partial_x^2 \end{pmatrix}\phi; \end{aligned} \tag{19}$$

$$
\begin{aligned}
f_\psi(\phi) :=& B(\psi+\phi)\psi - B(\psi)\psi + g(\psi+\phi) - g(\psi) \\
=& \begin{pmatrix} \mathcal{D}\sqrt{2(w_\psi+w)}(w_\psi)_{xx}+\mathcal{C}(w_\psi)_x \\ (v_\psi)_{xx} \end{pmatrix} - \begin{pmatrix} \mathcal{D}\sqrt{2w_\psi}(w_\psi)_{xx}+\mathcal{C}(w_\psi)_x \\ (v_\psi)_{xx} \end{pmatrix} \\
&+ \begin{pmatrix} \sqrt{2(w_\psi+w)}\mathcal{A}(1-\sqrt{w_\psi+w})-\sqrt{2}(w_\psi+w)(v_\psi+v)^2 \\ -\mathcal{B}(v_\psi+v)+\sqrt{w_\psi+w}(v_\psi+v)^2 \end{pmatrix} \\
&- \begin{pmatrix} \sqrt{2w_\psi}\mathcal{A}(1-\sqrt{w_\psi})-\sqrt{2}w_\psi v_\psi^2 \\ -\mathcal{B}v_\psi+\sqrt{w_\psi}v_\psi^2 \end{pmatrix}.
\end{aligned}
$$

Now the PDE for the perturbation is given by equation (31):

$$
\phi_t = A_\psi(t,\phi)\phi + f_\psi(t,\phi). \tag{20}
$$

To apply theorem 1.10 to this equation we have to look at the requirements (A0)-(A3). The verification of assumptions (A2) and (A3), which are properties of the family of evolution system $U_\phi(t,s)$ uniform with respect to ϕ, is not included.

Nemytskii operator (A0) Note that by lemma 3.18 it holds that $f_\psi(0) = 0$. Also, the only concern for existence of derivatives of f_ψ with respect to w and v is the term $\sqrt{w_\psi + w}$. Now assume that $w_\psi \geq \delta > 0$. Then it is clear that there is no problem differentiating f_ψ as long as ϕ is chosen small, in particular f_ψ is three times continuously differentiable in a neighbourhood of $0 \in R^2$. So by a local two-dimensional version of corollary 3.14, $f_{\psi N}$ is Lipschitz continuous on a neighborhood of $0 \in H^2(\mathbb{R})$. This is a sufficient replacement of (A0).

Generation of an evolution system (A1) Now we shift attention to the operator A_ψ. This is a diagonal operator, so the action of this operator can be split into two separate actions of $A_{\psi 11}$ and $A_{\psi 22}$ on $H^2(\mathbb{R})$. The operator $A_{\psi 22}$ is known to generate an analytic semigroup on $H^2(\mathbb{R})$. For $A_{\psi 11}$, as we already assumed $w_\psi \geq \delta > 0$, the second order coefficient does not cause problems for ϕ small.

Define $\mathcal{L} := A_{\psi 11} - \mathcal{C}\partial_x$, write $\mathcal{L} = \alpha_2 \partial_x^2$ with $\alpha_2 = \mathcal{D}\sqrt{2(w_\psi + w)}$ and domain $D(\mathcal{L}) = H^2(\mathbb{R})$. The convection part of $A_{\psi 11}$ is relatively $\mathcal{L}$-bounded with $\mathcal{L}$-bound equal to 0, confer [6, section III.2]. By [6, Theorem III.2.10] $A_{\psi 11}$ generates an analytic semigroup if $\mathcal{L}$ generates an analytic semigroup, so then $A_\psi(\cdot,\phi)$ generates an analytic semigroup. The rest of this paragraph is devoted to showing that $\mathcal{L}$ generates an analytic semigroup, which is sufficient for property (H1) and (P1) in section 3.1.

Before proving that $\mathcal{L}$ generates an analytic semigroup, we do some preparatory work. Since ϕ is small compared to ψ, let $\underline{\alpha_2}, \overline{\alpha_2} > 0$ be given such that $\underline{\alpha_2} \leq \alpha_2(x) \leq \overline{\alpha_2}$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $L^2(\mathbb{R})$.

Introduce the following alternative inner product on $L^2(\mathbb{R})$:

$$
\begin{aligned}
\langle \cdot, \cdot \rangle_{\alpha_2} : \quad & L^2(\mathbb{R}) \times L^2(\mathbb{R}) && \to \mathbb{R} \\
& (u,v) && \mapsto \int_{\mathbb{R}} \frac{1}{\alpha_2(x)} uv \, dx
\end{aligned} .
$$

For $u, v \in L^2(\mathbb{R})$ it holds that:

$$\frac{1}{\overline{\alpha_2}} \langle \cdot, \cdot \rangle \leq \langle \cdot, \cdot \rangle_{\alpha_2} \leq \frac{1}{\underline{\alpha_2}} \langle \cdot, \cdot \rangle$$

so $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\alpha_2}$ induce equivalent norms. So $L^2(\mathbb{R})$ equipped with $\langle \cdot, \cdot \rangle_{\alpha_2}$ is a Hilbert space.

Definition Let X, Y be Banach spaces with continuous duals X', Y'. Let $D(A) \subset X$ dense and let $(A, D(A))$ with codomain Y be a densely defined operator. The *adjoint* of A, $(A', D(A'))$, is defined on the domain:

$$D(A') = \{y' \in Y' | < A(\cdot), y' >: D(A) \to \mathbb{C} \text{ is continuous}\}$$

by mapping y' to the continous extension of $< A(\cdot), y' >$ in X'.[3]

Note that $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$ is dense, since it holds that the set of infinitely many times continuously differentiable functions with compact support $C_0^\infty(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R})$ [1, Theorem 2.19] and $C_0^\infty(\mathbb{R}) \subset H^2(\mathbb{R})$. So the adjoint of $\mathcal{L}$ is well-defined.

$\mathcal{L}$ generates an analytic semigroup

Lemma 2.2. *The operator $(\mathcal{L}, D(\mathcal{L}))$ is a closed operator.*

Proof. Let $\{u_k\} \subset D(\mathcal{L})$ be a sequence such that $u_k \to u \in L^2(\mathbb{R})$ and let $\mathcal{L}u_k \to y \in L^2(\mathbb{R})$. Then it holds:

$$\begin{aligned} ||u_k - u_l||_{\tilde{H}^2} &= ||u_k - u_l||_{L^2} + \left|\left|\partial_x^2 u_k - \partial_x^2 u_l\right|\right|_{L^2} \\ &\leq ||u_k - u_l||_{L^2} + \frac{1}{\underline{\alpha_2}} ||\mathcal{L}u_k - \mathcal{L}u_l||_{L^2} \to 0 \end{aligned}$$

as $k, l \to \infty$. So $\{u_k\}$ is a Cauchy sequence in $H^2(\mathbb{R})$, so $u \in H^2(\mathbb{R}) = D(\mathcal{L})$. Moreover:

$$\begin{aligned} ||\mathcal{L}u - y||_{L^2} &\leq ||\mathcal{L}u - \mathcal{L}u_k||_{L^2} + ||\mathcal{L}u_k - y||_{L^2} \\ &\leq \overline{\alpha_2} \left|\left|\partial_x^2 u - \partial_x^2 u_k\right|\right|_{L^2} + ||\mathcal{L}u_k - y||_{L^2} \\ &\leq \overline{\alpha_2} ||u - u_k||_{\tilde{H}^2} + ||\mathcal{L}u_k - y||_{L^2} \to 0 \end{aligned}$$

as $k, l \to \infty$. So $\mathcal{L}u = y$. $\square$

Lemma 2.3. *The operator $\mathcal{L}$ is a self-adjoint operator on $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle_{\alpha_2})$.*

Proof. Since $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle_{\alpha_2})$ is a Hilbert space, by Riesz representation theorem the continuous dual $L^2(\mathbb{R})'$ can be viewed as the space $L^2(\mathbb{R})$ itself. Using this

[3] This extension exists by Hahn-Banach theorem and is unique because $D(A) \subset X$ is dense.

identification one has, with partial integration and corollary 3.11, for $u, y \in D(\mathcal{L})$:

$$\begin{aligned}\langle \mathcal{L}u, y\rangle_{\alpha_2} &= \int_{\mathbb{R}} \frac{1}{\alpha_2(x)} \alpha_2(x)(\partial_x^2 u(x)) y(x)\, dx \\ &= \int_{\mathbb{R}} (\partial_x^2 u(x)) y(x)\, dx \\ &= \int_{\mathbb{R}} u(x)(\partial_x^2 y(x))\, dx;\end{aligned}$$

so by Hölder's inequality:

$$\begin{aligned}|\langle \mathcal{L}u, y\rangle|_{\alpha_2} &\leq \left|\left|u(\partial_x^2 y)\right|\right|_{L^1} \\ &\leq ||u||_{L^2} \left|\left|\partial_x^2 y\right|\right|_{L^2} \\ &\leq ||u||_{L^2} ||y||_{H^2};\end{aligned}$$

so $y \in D(A')$. Now it holds:

$$\begin{aligned}\langle \mathcal{L}u, y\rangle_{\alpha_2} &= \int_{\mathbb{R}} \frac{1}{\alpha_2(x)} \alpha_2(x)(\partial_x^2 u(x)) y(x)\, dx \\ &= \int_{\mathbb{R}} (\partial_x^2 u(x)) y(x)\, dx \\ &= \int_{\mathbb{R}} u(x)(\partial_x^2 y(x))\, dx \\ &= \int_{\mathbb{R}} \frac{1}{\alpha_2(x)} u(x) \mathcal{L}y\, dx \\ &= \langle u, \mathcal{L}y\rangle_{\alpha_2}.\end{aligned}$$

So it follows that $(\mathcal{L}', D(\mathcal{L}'))$ is an extension of $(\mathcal{L}, D(\mathcal{L}))$. Since $\mathcal{L}$ is closed by lemma 2.2, it holds that $\mathcal{L}$ is self-adjoint. □

Corollary 2.4. *The operator $\mathcal{L}$ generates an analytic semigroup.*

Proof. It holds that:

$$\langle \mathcal{L}u, u\rangle_{\alpha_2} \leq \overline{\alpha_2}\, ||u||_{L^2}^2;$$

so $\mathcal{L}$ is bounded above. By lemma 2.3 it holds that $\mathcal{L}$ is self-adjoint. By [6, Example II.3.27 & Corollary II.4.7] $(\mathcal{L}, D(\mathcal{L}))$ is the generator of an analytic semigroup. □

3 Functional analytic background

3.1 Semigroups and evolution systems

Let X be a Banach space.

Definition A family $(S(t))_{t\geq 0}$ of bounded linear operators on X is called a *semigroup* if:

$$\begin{aligned} &S(t+s) = S(t)S(s) && \text{for all } t,s \geq 0;\\ &S(0) = Id_X. \end{aligned}$$

It is a *strongly continuous semigroup* or C_0*-semigroup* if in addition the orbit maps

$$\mathbb{R}_{\geq 0} \ni t \mapsto S(t)x \in X$$

are continuous.

Lemma 3.1. *Let $S(t)$ be a semigroup. Then $S(t)$ is strongly continuous iff for all $x \in X$:*

$$\lim_{t\downarrow 0} S(t)x = x.$$

Proof. This result is part of [6, Proposition I.1.3] and proven there. □

To every C_0-semigroup one can associate an unbounded operator.

Definition The *generator* $(A, D(A))$ of a C_0-semigroup $S(t)$ on X is the operator given by:

$$\begin{aligned} D(A) &= \left\{ x \in X : \lim_{h\downarrow 0} \frac{S(h)x - x}{h} \text{ exists} \right\}\\ A(x) &= \lim_{h\downarrow 0} \frac{S(h)x - x}{h}. \end{aligned}$$

Necessary and sufficient conditions for an unbounded operator on a Banach space to generate a C_0-semigroup are known, and for instance given by [6, Theorem II.3.8].

Definition A two parameter family $(U(t,s))_{0\leq s\leq t\leq T}$ of bounded linear operators on X is called an *evolution system* if:

$$\begin{aligned} &U(t,s)U(s,r) = U(t,r) && \text{for } 0 \leq r \leq s \leq t \leq T;\\ &U(t,t) = Id_X && \text{for all } t \leq T. \end{aligned}$$

It is *strongly continuous* if moreover

$$\begin{array}{rl} \{(\tau,\sigma) \in \mathbb{R}^2 : \tau \geq \sigma \geq 0\} & \to \mathscr{L}(X) \\ (t,s) & \mapsto U(t,s) \end{array};$$

is continuous.

Remark For $S(t)$ a C_0-semigroup, $U(t,s) := S(t-s)$ defines an evolution system.

The analytic semigroups are a special class of strongly continuous semigroups, related to parabolic PDE's, see for instance [9, section 2.5]. In [9, section 5.3] and [9, section 5.6] two independent constructions of evolution systems are presented, given some sufficient conditions on a set of operators $(A(t), D(A))$. Here we give a summary of these results. We list some properties for the hyperbolic (H) and the parabolic (P) case.

(H1) For $t \in [0,\tau]$ the $A(t)$ generate strongly continuous semigroups $S_t(s)$.
(H2) For $t \in [0,\tau]$ the $A(t)$ form a stable family [9, section 5.2].
(H3) For $t \in [0,\tau]$ the map $t \mapsto A(t) \in \mathscr{L}(Y,X)$ is continuous.

(P1) For $t \in [0,\tau]$ the $A(t)$ generate analytic semigroups $S_t(s)$.
(P2) For $t \in [0,\tau]$ for all λ with $Re(\lambda) \geq 0$ there exists a constant M such that the resolvent $R(\lambda : A(t))$ of $A(t)$ exists and:

$$||R(\lambda : A(t))||_{\mathscr{L}(X,X)} \leq \frac{M}{|\lambda|+1}.$$

(P3) There exist constants L and $0 < \alpha \leq 1$ such that for $s,t,r \in [0,\tau]$:

$$\left|\left|\frac{A(t)-A(s)}{A(r)}\right|\right|_{\mathscr{L}(X,X)} \leq C|t-s|^{\alpha}.$$

If (H1)-(H3) hold, then [9, theorem 5.3.1] yields an evolution system. If (P1)-(P3) hold, then [9, theorem 5.6.1] can be used.

3.2 Norms on Sobolev spaces over $\mathbb{R}$

For convenience some well known definitions are given first. Suppose that $u : \mathbb{R} \to \mathbb{R}$ is locally integrable, so u corresponds to a regular distribution T_u.

Definition Let $m \in \mathbb{N}$ be given. We say that u has a m^{th}*-weak partial derivative* if there exists a locally integrable function $v : \mathbb{R} \to \mathbb{R}$ such that for all infinitely smooth compactly supported functions ϕ in $C_c^\infty(\mathbb{R})$ it holds that:

$$\int_{\mathbb{R}} uD^m\phi \, dx = (-1)^m \int_{\mathbb{R}} v\phi \, dx.$$

Notation: $v = D^m u$.

Remark Alternatively and equivalently, T_u has a m^{th}-weak partial derivative if there exists a regular distribution T_v such that $D^m(T_u) = T_v$.

Definition The subspace $W^{m,p}(\mathbb{R}) \subset L^p(\mathbb{R})$ of functions with weak partial derivatives up to and including order m is called the *Sobolev space of order m in* $L^p(\mathbb{R})$. For the case $p = 2$ we use notation $H^m(\mathbb{R}) := W^{m,2}(\mathbb{R})$.

Definition Let $m \geq 1$ be given.

1. Endow $H^m(\mathbb{R})$ with the *Sobolev norm* given by:

$$||u||_{H^m} = \sum_{j=0}^{m} \left|\left|D^j u\right|\right|_{L^2}.$$

2. Endow $H^m(\mathbb{R})$ with the *simplified Sobolev norm* given by:

$$||u||_{\tilde{H}^m} = ||u||_{L^2} + ||D^m u||_{L^2}.$$

3. Endow $H^m(\mathbb{R})$ with the *algebra norm* given by:

$$||u||_{\text{alg}} = K^* \sqrt{\sum_{j=0}^{m} ||D^j u||_{L^2}^2};$$

with K^* a fixed constant so that $\left(H^m(\mathbb{R}), ||\cdot||_{\text{alg}}\right)$ becomes a Banach algebra, as in [1, Theorem 5.23].

As indicated by the name, the algebra norm has the property that for all $u, v \in H^m(\mathbb{R})$:

$$||uv||_{\text{alg}} \leq ||u||_{\text{alg}} ||v||_{\text{alg}}.$$

A reminiscent property is inherited by norms $||\cdot||_{\text{equiv}}$ equivalent to the algebra norm, write $\frac{1}{C} ||\cdot||_{\text{alg}} \leq ||\cdot||_{\text{equiv}} \leq C ||\cdot||_{\text{alg}}$. Then:

$$||uv||_{\text{equiv}} \leq C ||uv||_{\text{alg}} \leq C ||u||_{\text{alg}} ||v||_{\text{alg}} \leq C^3 ||u||_{\text{equiv}} ||v||_{\text{equiv}}. \tag{21}$$

We now restrict our attention to the case that $m = 1$ or $m = 2$. On $H^1(\mathbb{R})$ the three norms previously defined are easily seen to be equivalent, the same holds for $H^2(\mathbb{R})$ with the help of lemma 3.2 below.

Lemma 3.2. *For $u \in H^2(\mathbb{R})$ it holds:*

$$||u_x||_{L^2} \leq ||u||_{L^2} + ||u_{xx}||_{L^2} \,.$$

Proof. For $a, b \in \mathbb{R}$ it holds that $ab \leq \frac{1}{2}(a^2 + b^2)$ (see the proof of lemma 3.12). Also, elements of $H^1(\mathbb{R})$ vanish at infinity by corollary 3.11 below. Using this it holds:

$$\begin{aligned} ||u_x||_{L^2}^2 &= \int_{\mathbb{R}} u_x^2 dx \\ &= \int_{\mathbb{R}} (-u) u_{xx} dx + [u u_x]_{x=-\infty}^{x=\infty} \\ &\leq \frac{1}{2} \int_{\mathbb{R}} (u^2 + u_{xx}^2) dx \\ &= \frac{1}{2} \left(||u||_{L^2}^2 + ||u_{xx}||_{L^2}^2 \right) \\ &\leq \max \left\{ ||u||_{L^2}^2 \,, ||u_{xx}||_{L^2}^2 \right\} ; \end{aligned}$$

so $||u_x||_{L^2} \leq ||u||_{L^2} + ||u_{xx}||_{L^2}$. □

Remark This is an example of an operator being relatively bounded by another operator, confer [6, section III.2].

Corollary 3.3. *On $H^1(\mathbb{R})$ and $H^2(\mathbb{R})$ the Sobolev norm, simplified Sobolev norm and algebra norm are equivalent.*

Below some simple but valuable properties are presented in a lemma.

Lemma 3.4. *For $u \in H^1(\mathbb{R})$ it holds:*

$$\begin{aligned} ||u||_{L^1} &\leq ||u||_{H^1} \,; \\ ||Du||_{L^1} &\leq ||u||_{H^1} \,. \end{aligned}$$

And for $u \in H^2(\mathbb{R})$:

$$\begin{aligned} ||u||_{H^1} &\leq ||u||_{H^2} \,; \\ ||Du||_{H^1} &\leq ||u||_{H^2} \,. \end{aligned}$$

Graph norm Now let $A = \alpha_2(x)\partial_{xx} + \alpha_1(x)\partial_x + \alpha_0(x)$ be a second order differential operator with domain $D(A) = H^2(\mathbb{R}) \subset L^2(\mathbb{R})$. Assume that A is closed. This operator gives rise to a fourth norm on $H^2(\mathbb{R})$. The rest of this section will be devoted to showing that this norm is equivalent to the previous ones, under light conditions. This result is well known.

Definition Endow $D(A)$ with the *graph norm* given by:

$$||u||_A = ||u||_{L^2} + ||Au||_{L^2} \,.$$

Since A is closed $(D(A), ||\cdot||_A)$ is a Banach space, confer [6, Appendix A.5].

Lemma 3.5. *Suppose for $j = 0, 1, 2$ that $|\alpha_j(x)|$ is a bounded function. Then there exists $C > 0$ such that for all $u \in D(A)$ it holds:*

$$||u||_A \leq C\, ||u||_{H^2}\,.$$

Proof. Write $|\alpha_j| \leq M_j$. Choose $C = \max\{1 + M_0, M_1, M_2\}$. Then:

$$\begin{aligned} ||u||_A =& ||u||_{L^2} + ||\alpha_2(x)u_{xx} + \alpha_1(x)u_x + \alpha_0(x)u||_{L^2} \\ \leq& (1 + M_0)\, ||u||_{L^2} + M_1\, ||u_x||_{L^2} + M_2\, ||u_{xx}||_{L^2} \\ \leq& C\, ||u||_{H^2}\,. \end{aligned}$$

□

The application of the open mapping theorem below needs $\left(H^2(\mathbb{R}), ||\cdot||_A\right)$ to be a Banach space, for which A needs to be closed, as has been assumed.

Corollary 3.6. *If A has bounded coefficients, the Sobolev norm and the graph norm are equivalent on $H^2(\mathbb{R})$.*

Proof. Look at the map Id between Banach spaces:

$$\begin{array}{rcl} Id: & \left(H^2(\mathbb{R}), ||\cdot||_{H^2}\right) & \to \left(H^2(\mathbb{R}), ||\cdot||_A\right) \\ & u & \mapsto u \end{array};$$

this is a bijective map and by lemma 3.5 it is continuous. By the open mapping theorem Id is a homeomorphism. □

3.3 Some properties of Hölder continuous functions in $L^p(\mathbb{R})$

Let $1 \leq p < \infty$ be given. Functions in sequence spaces l^p vanish at infinity. Functions in $L^p(\mathbb{R})$ do not necessarily vanish at infinity, even if they are continuous. For $f \in L^p(\mathbb{R})$ continuous it holds that either $\lim_{x\to\infty} f(x) = 0$ or $\lim_{x\to\infty} f(x)$ does not exist. To illustrate this we construct a continuous function $s_p \in L^p(\mathbb{R})$ such that $||s_p||_{L^\infty} = \infty$ and $\lim_{x\to\infty} f(x)$ does not exist.

In $L^1(\mathbb{R})$. Define $t_n(x) = \max\{0, n^4(\frac{1}{n^3} - |x|)\}$ which is a triangle function with maximum at $x = 0$, $t_n(0) = n$, width $\frac{2}{n^3}$ and slope n^4. It holds that the area under the triangle is given by $||t_n(x)||_{L^1} = \frac{1}{n^3}n = \frac{1}{n^2}$.[4] Now define the function $s_1(x) = \sum_{n=1}^{\infty} t_n(x - 2n)$, a sum of triangles translated over $2n$. Note that the support of the translated triangles is disjoint, so:

$$||s_1||_{L^1} = \sum_{n=1}^{\infty} ||t_n|| = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

So $s_1 \in L^1(\mathbb{R})$ is a continuous function, but $\lim_{x\to\infty} s_1(x)$ does not exist, in fact $||s_1(x)||_{L^\infty} = \infty$.

In $L^p(\mathbb{R})$. We can reach a similar result for $L^p(\mathbb{R})$ by defining $r_n(x) = \sqrt[p]{t_n(x)}$, $s_p(x) = \sum_{n=1}^{\infty} r_n(x-2n)$. Because the summands have disjoint support, it holds:

$$||s_p||_{L^p} = \sqrt[p]{\sum_{n=1}^{\infty} ||r_n||_{L^p}} = \sqrt[p]{\sum_{n=1}^{\infty} ||t_n||_{L^1}} = \sqrt[p]{||s||_{L^1}}.$$

So $s_p \in L^p(\mathbb{R})$ is a continuous function, but $\lim_{x\to\infty} s_p(x)$ does not exist, in fact $||s_p(x)||_{L^\infty} = \infty$ since the maximum of the $r_n(x)$ have height $\sqrt[p]{n}$.

As we have seen, continuous functions f in $L^p(\mathbb{R})$ do not necessarily vanish at infinity, but as we show below they do if they are Hölder continuous.

Definition (Hölder continuity) Let $0 < \gamma \leq 1$. A function $f : \mathbb{R} \to \mathbb{R}$ is *Hölder continuous with exponent* γ if:

$$\exists K > 0 \ \forall x, y \in \mathbb{R} : |f(x) - f(y)| \leq K |x - y|^\gamma .$$

The space of functions that are Hölder continuous with exponent γ is denoted by $C^{0,\gamma}(\mathbb{R})$.

[4] Or by direct calculations:

$$||t_n||_{L^1} = \int_{\frac{-1}{n^3}}^{\frac{1}{n^3}} n^4 \left(\frac{1}{n^3} - |x|\right) dx = \int_{\frac{-1}{n^3}}^{0} n + n^4 x dx + \int_0^{\frac{1}{n^3}} n - n^4 x dx$$

$$= n\frac{1}{n^3} - \frac{n^4}{2}\frac{1}{n^6} + n\frac{1}{n^3} - \frac{n^4}{2}\frac{1}{n^6} = \frac{1}{n^2} - \frac{1}{2n^2} + \frac{1}{n^2} - \frac{1}{2n^2} = \frac{1}{n^2}.$$

Lemma 3.7. *If $f \in L^p(\mathbb{R})$ is Hölder continuous on $\mathbb{R}$, then $\lim_{x\to\pm\infty} f(x) = 0$.*

Proof. Suppose that f is Hölder continuous with exponent γ and constant K and that $\lim_{x\to\infty} f(x) \neq 0$, but f is not necessarily an element of $L^p(\mathbb{R})$. Then it holds that:

$$\exists \epsilon > 0 \ \forall N \ \exists x \geq N : |f(x)| \geq \epsilon.$$

So we can find an increasing sequence $\{x_j\}_{j=1}^{\infty}$ with $x_{j+1} - x_j > 2\left(\frac{\epsilon}{K}\right)^{\frac{1}{\gamma}}$ and $|f(x_j)| \geq \epsilon$. Define a function $g_{x_j} \in L^p(\mathbb{R})$ by:

$$g_{x_j}(x) = \begin{cases} \epsilon - K\,|x - x_j|^{\gamma} & \text{if } |x - x_j| \leq \left(\frac{\epsilon}{K}\right)^{\frac{1}{\gamma}} \\ 0 & \text{if } |x - x_j| > \left(\frac{\epsilon}{K}\right)^{\frac{1}{\gamma}} \end{cases}.$$

It holds that:

$$\begin{aligned} \left|\left|g_{x_j}\right|\right|_{L^p} &= \sqrt[p]{\int_{x_j-\left(\frac{\epsilon}{K}\right)^{\frac{1}{\gamma}}}^{x_j+\left(\frac{\epsilon}{K}\right)^{\frac{1}{\gamma}}} (\epsilon - K\,|x - x_j|)^p} \\ &= \sqrt[p]{\int_{-\left(\frac{\epsilon}{K}\right)^{\frac{1}{\gamma}}}^{\left(\frac{\epsilon}{K}\right)^{\frac{1}{\gamma}}} (\epsilon - K\,|x|)^p} \\ &=: M(\gamma, K, \epsilon). \end{aligned}$$

Now define $g : \mathbb{R} \to \mathbb{R}$ by:

$$g(x) = \sum_{j\in\mathbb{N}} g_{x_j}(x).$$

By Hölder continuity it holds that $|f(x)| \geq g(x)$. Since the g_{x_j} have disjoint support, it holds that:

$$||f||_{L^p} \geq ||g||_{L^p} = \sum_{j\in\mathbb{N}} \left|\left|g_{x_j}\right|\right|_{L^p} = \sum_{j\in\mathbb{N}} M(\gamma, K, \epsilon) = \infty.$$

So $f \notin L^p(\mathbb{R})$. □

We proceed with a technical lemma. By showing in $L^p(\mathbb{R}) \cap C^{0,\gamma}(\mathbb{R})$ a uniform lower bound estimate of the L^p-norm in terms of the L^∞-norm we obtain a uniform upper bound estimate of the L^∞-norm in terms of the L^p-norm. We restrict our attention to the case that $p = 2$, $\gamma = \frac{1}{2}$ and K is fixed, see the definition of Hölder continuity above.

Lemma 3.8. *For $u \in C^{0,\frac{1}{2}}(\mathbb{R})$ it holds that $||u||_{L^\infty} \leq \sqrt[4]{2^3} \cdot \sqrt{K\,||u||_{L^2}}$.*

Proof. Let $u \in C^{0,\frac{1}{2}}(\mathbb{R})$ be given. Define for every $\bar{x} \in \mathbb{R}$:

$$g_{\bar{x}}(x) = \begin{cases} 0 & \text{if } |x - \bar{x}| > \left(\frac{u(\bar{x})}{K}\right)^2 \\ |u(\bar{x})| - K\,|x - \bar{x}|^{\frac{1}{2}} & \text{if } |x - \bar{x}| \leq \left(\frac{u(\bar{x})}{K}\right)^2 \end{cases}.$$

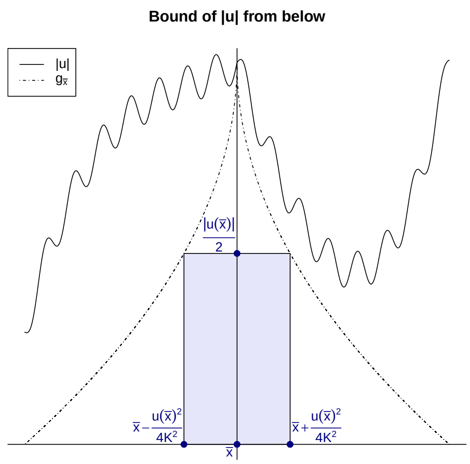
From the Hölder condition it follows that $|u| \geq g_{\bar{x}}$. Since:

$$g_{\bar{x}}\left(\bar{x} - \frac{u(\bar{x})^2}{4K^2}\right) = \frac{|u(\bar{x})|}{2};$$
$$g_{\bar{x}}\left(\bar{x} + \frac{u(\bar{x})^2}{4K^2}\right) = \frac{|u(\bar{x})|}{2};$$

it holds that:

$$g_{\bar{x}} \geq \frac{|u(x)|}{2}\chi_{\left[\bar{x}-\frac{u(\bar{x})^2}{4K^2},\bar{x}+\frac{u(\bar{x})^2}{4K^2}\right]};$$

where χ denotes the characteristic function of the interval. This is illustrated by the figure on the right.



Thus it holds that:

$$\begin{aligned}||u||_{L^2}^2 &\geq ||g_{\bar{x}}||_{L^2}^2 \\ &\geq \left|\left|\frac{|u(x)|}{2}\chi_{\left[\bar{x}-\frac{u(\bar{x})^2}{4K^2},\bar{x}+\frac{u(\bar{x})^2}{4K^2}\right]}\right|\right|_{L^2}^2 \\ &=2\left(\frac{u(\bar{x})^2}{4K^2}\right)\cdot\left(\frac{|u(\bar{x})|}{2}\right)^2 \\ &=\frac{u(\bar{x})^4}{2^3K^2}.\end{aligned}$$

This identity holds for every $\bar{x} \in \mathbb{R}$, thus $2^3K^2\,||u||_{L^2}^2 \geq ||u||_{L^\infty}^4$. Thus it holds that: $||u||_{L^\infty} \leq \sqrt[4]{2^3} \cdot \sqrt{K\,||u||_{L^2}}$. □

Remark Calculating the L^2-norm of $g_{\bar{x}}$ directly gives: $||g_{\bar{x}}||_{L^2} = \frac{|u(\bar{x})|^4}{3K^2}$, so this only improves the estimate by a constant.

3.4 Sobolev embedding theorem

In the previous section Hölder continuity and the space $C^{0,\gamma}(\mathbb{R})$ have been defined. First we define a space of functions whose derivatives up to order m are Hölder continuous.

Definition Let $m \in \mathbb{N}$ and $0 < \gamma \leq 1$. Define:

$$C^{0,\gamma}(\overline{\mathbb{R}}) := \left\{ u : \mathbb{R} \to \mathbb{R} \,\middle|\, \sup_{x,y\in\mathbb{R}} \frac{|u(x) - u(y)|}{|x-y|^{\gamma}} < \infty \quad \& \quad ||u||_{L^\infty} < \infty \right\};$$
$$C^{m,\gamma}(\overline{\mathbb{R}}) := \left\{ u \in C^m(\mathbb{R}) \,\middle|\, \text{for } 0 \leq j \leq m \text{ it holds } D^j u \in C^{0,\gamma}(\overline{\mathbb{R}}) \right\}$$

with norm:

$$||u||_{m,\gamma} := \sum_{j=0}^{m} \left|\left|D^j u\right|\right|_{L^\infty} + \sum_{j=0}^{m} \sup_{x,y\in\mathbb{R}} \frac{|D^j u(x) - D^j u(y)|}{|x-y|^{\gamma}}.$$

We now state a result on continuously embedding $H^1(\mathbb{R})$ and $H^2(\mathbb{R})$. Actually only the first embedding is used in this thesis.

Theorem 3.9. *There exist continuous embeddings:*

$$H^1(\mathbb{R}) \hookrightarrow C^{0,\frac{1}{2}}(\overline{\mathbb{R}})$$
$$H^2(\mathbb{R}) \hookrightarrow C^{1,\frac{1}{2}}(\overline{\mathbb{R}})$$

where each equivalence class in $H^1(\mathbb{R})$ respectively $H^2(\mathbb{R})$ is mapped to one of its representatives. The constant K in the Hölder condition can be chosen uniformly for bounded subsets of $H^1(\mathbb{R})$ respectively $H^2(\mathbb{R})$.

Proof. Let $u \in H^1(\mathbb{R})$, then:

$$u(x) - u(y) = \int_x^y u'(z)dz;$$

using the Hölder inequality (or equivalently, Cauchy-Schwarz) it holds:

$$\begin{aligned} |u(x) - u(y)| &\leq \int_x^y |u'(z)|\, dz \\ &= ||u'||_{L^1[x,y]} \\ &= ||1 \cdot u'||_{L^1[x,y]} \\ &\leq ||1||_{L^2[x,y]} \cdot ||u'||_{L^2[x,y]} \\ &\leq \sqrt{|x-y|} \cdot ||u||_{H^1}. \end{aligned}$$

So it holds that:

$$\sup_{x,y\in\mathbb{R}} \frac{|u(x) - u(y)|}{|x-y|^{\frac{1}{2}}} \leq ||u||_{H^1}.$$

So u is Hölder continuous with exponent $\frac{1}{2}$ and constant $K = ||u||_{H^1}$. So K can be chosen uniformly for bounded subsets of $H^1(\mathbb{R})$.

By lemma 3.8 it holds that:

$$\begin{aligned} ||u||_{L^\infty} \leq & \sqrt[4]{2^3 K^2} \cdot \sqrt{||u||_{L^2}} \\ \leq & \sqrt[4]{2^3} \cdot ||u||_{L^2} \\ \leq & \sqrt[4]{2^3} \cdot ||u||_{H^1} . \end{aligned} \tag{22}$$

so:

$$\begin{aligned} ||u||_{0,\frac{1}{2}} \leq & \sqrt[4]{2^3} \cdot ||u||_{H^1} + ||u||_{H^1} \\ = & \left(\sqrt[4]{2^3} + 1 \right) \cdot ||u||_{H^1} . \end{aligned}$$

So the embedding $H^1(\mathbb{R}) \hookrightarrow C^{0,\frac{1}{2}}(\overline{\mathbb{R}})$ is well-defined and continuous.

Let $u \in H^2(\mathbb{R})$, then u and u' are Holder continuous with exponent $\frac{1}{2}$ and constant $K = ||u||_{H^1}$ and $K = ||u'||_{H^1}$ respectively. By lemma 3.8 it holds that $||u||_{L^\infty} \leq \sqrt[4]{2^3} \cdot ||u||_{H^1}$ and $||u'||_{L^\infty} \leq \sqrt[4]{2^3} \cdot ||u'||_{H^1}$ respectively, so:

$$\begin{aligned} ||u||_{1,\frac{1}{2}} \leq & \sqrt[4]{2^3} \cdot (||u||_{H^1} + ||u'||_{H^1}) + ||u||_{H^1} + ||u'||_{H^1} \\ \leq & \sqrt[4]{2^3} \cdot 2\, ||u||_{H^2} + 2\, ||u||_{H^2} \\ = & 2 \left(\sqrt[4]{2^3} + 1 \right) \cdot ||u||_{H^2} . \end{aligned}$$

So the embedding $H^2(\mathbb{R}) \hookrightarrow C^{1,\frac{1}{2}}(\overline{\mathbb{R}})$ is well-defined and continuous. It is also clear that K can be chosen uniformly for bounded subsets of $H^2(\mathbb{R})$.

□

Corollary 3.10. *For all $u \in H^1(\mathbb{R})$ with $||u||_{H^1} \leq \epsilon$ it holds that:*

$$||u||_{L^\infty} \leq \sqrt[4]{2^3}\epsilon.$$

From this it follows that, for $v \in H^2(\mathbb{R})$ with $||v||_{H^2} \leq \epsilon$:

$$\begin{aligned} ||v||_{L^\infty} \leq & \sqrt[4]{2^3}\epsilon; \\ ||v'||_{L^\infty} \leq & \sqrt[4]{2^3}\epsilon. \end{aligned}$$

Proof. See equation (22) and lemma 3.4. □

Corollary 3.11. *Elements of $H^1(\mathbb{R})$ or $H^2(\mathbb{R})$ vanish at infinity.*

Proof. Apply theorem 3.9 together with lemma 3.7. □

3.5 Nemytskii operators on $H^2(\mathbb{R})$

By interpretating a partial differential equation as an ordinary differential equation on an infinite dimensional Banach space, any reaction term has to be replaced by an operator on this Banach space. Since in this thesis $H^2(\mathbb{R})$ is chosen to be the Banach space, we look at the induced operator on $H^2(\mathbb{R})$.

Definition Given a function $f : \mathbb{R} \to \mathbb{R}$ define the *superposition* or *Nemytskii operator* f_N to be given by:

$$f_N : \begin{array}{ll} H^2(\mathbb{R}) & \to [\mathbb{R} \to \mathbb{R}] \\ u & \mapsto [x \mapsto f(u(x))] \end{array},$$

i.e. $f_N(u)(x) := f(u(x))$.

By straightforward calculations we derive smoothness properties of f_N from smoothness properties of f. The results presented in this section may not be optimal, but they are sufficient for use in this thesis. First we have the following easy lemma.

Lemma 3.12. *Let $x_j \geq 0$, $j = 1, ..., l$ be given, then:*

$$\left(\sum_{j=1}^{l} x_j \right)^2 \leq l \sum_{j=1}^{l} x_j^2.$$

Proof. It holds that:

$$2x_j x_k = x_j^2 + x_k^2 - (x_j - x_k)^2 \leq x_j^2 + x_k^2$$

□

Let $B_\epsilon(0) = \left\{ u \in H^2(\mathbb{R}) \middle| \, ||u||_{H^2} \leq \epsilon \right\}$. By corollary 3.10 it holds that:

$$\sup_{||u||_{H^2} \leq \epsilon} ||u||_{L^\infty} < \sqrt[4]{2^3}\epsilon.$$

So the set $\{ u(x) \in \mathbb{R} | \, u \in B_\epsilon(0), x \in \mathbb{R} \}$ is bounded in $\mathbb{R}$. So the image of this set under a continous map is also bounded. Frequent use of this kind of reasoning is made in the proofs of lemma 3.13 and 3.15 below. Since exact knowledge of constants is not of importance, a generic constant C is introduced that may vary per instance.

Lemma 3.13. *Suppose that $f \in \mathscr{C}^3(\mathbb{R})$, then:*

$$\forall \epsilon > 0 \ \exists C_\epsilon > 0 \ \forall u, v \in B_\epsilon(0) : \ ||f_N(u + v) - f_N(u)||_{H^2} \leq C_\epsilon \, ||v||_{H^2}.$$

Proof. It holds that:

$$\begin{aligned}(f_N(u+v)-f_N(u))(x) &= f(u(x)+v(x))-f(u(x)) \\ &= \int_0^1 f'(u(x)+sv(x))v(x)ds.\end{aligned}$$

For the first part of the Sobolev norm this yields:

$$\begin{aligned}||f_N(u+v)-f_N(u)||_{L^2}^2 &= \int_{\mathbb{R}} \left| \int_0^1 f'(u(x)+sv(x))v(x)ds \right|^2 dx \\ &\leq \int_{\mathbb{R}} |v(x)|^2 \int_0^1 |f'(u(x)+sv(x))|^2 dsdx.\end{aligned}$$

Because f' is continuous and $||u+sv||_{L^\infty} \leq 2\sqrt[4]{2^3}\epsilon$ uniformly, it follows that:

$$||f_N(u+v)-f_N(u)||_{L^2}^2 \leq \int_{\mathbb{R}} |v(x)|^2 \int_0^1 Cdsdx = C\,||v||_{L^2}^2 \leq C\,||v||_{H^2}^2. \quad (23)$$

For the second part of the Sobolev norm it holds that:

$$\begin{aligned}&D_x^2\left[(f_N(u+v)-f_N(u))(x)\right] \\ &\quad = D_x^2\left[\int_0^1 f'(u(x)+sv(x))v(x)ds\right] \\ &\quad = \int_0^1 D_x^2\left[f'(u(x)+sv(x))v(x)\right]ds \\ &\quad \leq \max_{s\in[0,1]} D_x^2\left[f'(u(x)+sv(x))v(x)\right] \\ &\quad = \max_{s\in[0,1]} D_x\left[f''(u(x)+sv(x))(u'(x)+sv'(x))v(x)\right] \\ &\qquad + \max_{s\in[0,1]} D_x\left[f'(u(x)+sv(x))v'(x)\right] \\ &\quad = \max_{s\in[0,1]} f'''(u(x)+sv(x))(u'(x)+sv'(x))^2 v(x) && (\mathfrak{A}) \\ &\qquad + \max_{s\in[0,1]} f''(u(x)+sv(x))D_x\left[(u'(x)+sv'(x))v(x)\right] && (\mathfrak{B}) \\ &\qquad + \max_{s\in[0,1]} f''(u(x)+sv(x))(u'(x)+sv'(x))v'(x) && (\mathfrak{C}) \\ &\qquad + \max_{s\in[0,1]} f'(u(x)+sv(x))v''(x). && (\mathfrak{D})\end{aligned}$$

This yields, using lemma 3.12:

$$\begin{aligned}\left|\left|D_x^2\left[f_N(u+v)-f_N(u)\right]\right|\right|_{L^2}^2 &= \int_{\mathbb{R}} |\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}|^2 dx \\ &\leq \int_{\mathbb{R}} (|\mathfrak{A}|+|\mathfrak{B}|+|\mathfrak{C}|+|\mathfrak{D}|)^2 dx\end{aligned}$$

$$
\begin{aligned}
&\leq 4\int_{\mathbb{R}} |\mathfrak{A}|^2+|\mathfrak{B}|^2+|\mathfrak{C}|^2+|\mathfrak{D}|^2 dx \\
&=4\left(||\mathfrak{A}||_{L^2}^2+||\mathfrak{B}||_{L^2}^2+||\mathfrak{C}||_{L^2}^2+||\mathfrak{D}||_{L^2}^2\right). \quad (24)
\end{aligned}
$$

Still $||u+sv||_{L^\infty} \leq \sqrt[4]{2^3}\,||u+sv||_{H^2} \leq 2\sqrt[4]{2^3}\epsilon$ uniformly, and f', f'', f''' are continuous. This yields, together with the algebra property of $H^1(\mathbb{R})$ (equation (21)):

$$
\begin{aligned}
||\mathfrak{A}||_{L^2} =& \left|\left|\max_{s\in[0,1]} f'''(u+sv)(u'+sv')^2 v\right|\right|_{L^2} \\
\leq& C\max_{s\in[0,1]} \left|\left|(u'+sv')^2 v\right|\right|_{H^1} \\
\leq& C\max_{s\in[0,1]} ||u'+sv'||_{H^1}^2 \,||v||_{H^1} \\
\leq& C\,||v||_{H^2}\,; \\
||\mathfrak{B}||_{L^2} =& \left|\left|\max_{s\in[0,1]} f''(u+sv)D_x\left[(u'(x)+sv'(x))v(x)\right]\right|\right|_{L^2} \\
\leq& C\max_{s\in[0,1]} ||(u'+sv')v||_{H^1} \\
\leq& C\max_{s\in[0,1]} ||u'+sv'||_{H^1}\,||v||_{H^1} \\
\leq& C\,||v||_{H^2}\,; \\
||\mathfrak{C}||_{L^2} =& \left|\left|\max_{s\in[0,1]} f''(u+sv)(u'+sv')v'\right|\right|_{L^2} \\
\leq& C\max_{s\in[0,1]} ||(u'+sv')v'||_{H^1} \\
\leq& C\max_{s\in[0,1]} ||u'+sv'||_{H^1}\,||v'||_{H^1} \\
\leq& C\,||v||_{H^2}\,; \\
||\mathfrak{D}||_{L^2} =& \left|\left|\max_{s\in[0,1]} f'(u+sv)v''\right|\right|_{L^2} \\
\leq& C\,||v''||_{L^2} \\
\leq& C\,||v||_{H^2}\,.
\end{aligned}
$$

From these calculations it follows that:

$$
\left|\left|D_x^2\left[f_N(u+v)-f_N(u)\right]\right|\right|_{L^2}^2 \leq C\,||v||_{H^2}^2\,. \quad (25)
$$

Equivalence of the various Sobolev norms (see section 3.2) together with (23) and (25) yield:

$$
||f_N(u+v)-f_N(u)||_{H^2} \leq C\,||f_N(u+v)-f_N(u)||_{\tilde{H}^2} \leq C_\epsilon\,||v||_{H^2}^2\,.
$$

□

Corollary 3.14. *Let $f \in \mathscr{C}^3(\mathbb{R})$ and suppose that $f_N(0) = 0$. Then it holds that $f_N \in \mathscr{C}(H^2(\mathbb{R}))$ and f_N is Lipschitz on any bounded subset of $H^2(\mathbb{R})$.*

Proof. Let $v \in H^2(\mathbb{R})$ be given. Choose ϵ such that $v \in B_\epsilon(0)$. By lemma 3.13, it holds that:

$$||f_N(v)||_{H^2} = ||f_N(0+v) - f_N(0)||_{H^2} \leq C_\epsilon \, ||v||_{H^2} \, ;$$

so f_N maps into $H^2(\mathbb{R})$.

Let $\epsilon > 0$ and let $u, v \in B_{\frac{\epsilon}{2}}(0)$ be given, then $v - u \in B_\epsilon(0)$. By lemma 3.13 it holds:

$$||f_N(u) - f_N(v)||_{H^2} = ||f_N(u) - f_N(u + (v-u))||_{H^2} \leq C_\epsilon \, ||v-u||_{H^2} \, ;$$

so f_N is Lipschitz on $B_{\frac{\epsilon}{2}}(0)$ with Lipschitz constant C_ϵ. So f_N is Lipschitz on any bounded subset of $H^2(\mathbb{R})$. □

In this thesis corollary 3.14 suffices, lemma 3.15 and theorem 3.16 are not used. A typical application of theorem 3.16 would be to prove continuous dependence on initial conditions and parameters, after a solution has been found by applying the Banach contraction mapping theorem. The proof of the following result is structured the same way as the proof of lemma 3.13. Recall that C is a generic constant that may vary per instance.

Lemma 3.15. *Suppose that $f \in \mathscr{C}^4(\mathbb{R})$, then $\forall \epsilon > 0 \;\; \exists C_\epsilon > 0 \;\; \forall u, v \in B_\epsilon(0)$:*

$$||f_N(u+v) - f_N(u) - (Df)_N(u) \cdot v||_{H^2} \leq C \, ||v||_{H^2}^2 \, .$$

So $f_N \in C^1(H^2(\mathbb{R}))$ and $D(f_N) = (Df)_N$.

Proof. It holds that:

$$\begin{aligned}
&(f_N(u+v) - f_N(u) - (Df)_N(u) \cdot v)(x) \\
&\qquad = f(u(x) + v(x)) - f(u(x)) - f'(u(x))v(x) \\
&\qquad = \int_0^1 f'(u(x) + sv(x))v(x) - f'(u(x))v(x) ds \\
&\qquad = \int_0^1 \int_0^1 f''(u(x) + srv(x)) s (v(x))^2 ds dr.
\end{aligned}$$

Substituting this into the first part of the Sobolev norm yields:

$$\begin{aligned}
&||f_N(u+v) - f_N(u) - (Df)_N(u) \cdot v||_{L^2}^2 \\
&\qquad = \int_{\mathbb{R}} \left| \int_0^1 \int_0^1 f''(u(x) + srv(x)) s (v(x))^2 ds dr \right|^2 dx \\
&\qquad \leq \int_{\mathbb{R}} \left| (v(x))^2 \right|^2 \int_0^1 \int_0^1 \left| f''(u(x) + srv(x)) \right|^2 ds dr dx.
\end{aligned}$$

It holds that $||u+srv||_{L^\infty} \leq 2\sqrt[4]{2^3}\epsilon$ and because f'' is continuous, the following estimate holds:

$$\begin{aligned} ||f_N(u+v) - f_N(u) - (Df)_N(u)\cdot v||_{L^2}^2 \leq & \int_{\mathbb{R}} \left|(v(x))^2\right|^2 \int_0^1 \int_0^1 C dsdrdx \\ \leq & C\left|\left|v^2\right|\right|_{H^2}^2 \\ \leq & C\,||v||_{H^2}^4 . \end{aligned} \tag{26}$$

Substitution into the second part of the Sobolev norm gives:

$$\begin{aligned} & D_x^2\left[(f_N(u+v) - f_N(u) - (Df)_N(u)\cdot v)(x)\right] \\ & \quad = D_x^2\left[\int_0^1\int_0^1 f''(u(x)+srv(x))s(v(x))^2 dsdr\right] \\ & \quad = \int_0^1\int_0^1 D_x^2\left[f''(u(x)+srv(x))s(v(x))^2\right] dsdr \\ & \quad \leq \max_{s,r\in[0,1]} D_x^2\left[f''(u(x)+srv(x))(v(x))^2\right] \\ & \quad = \max_{s,r\in[0,1]} D_x\left[f'''(u(x)+srv(x))(u'(x)+srv'(x))(v(x))^2\right] \\ & \qquad + \max_{s,r\in[0,1]} D_x\left[f''(u(x)+srv(x))\cdot 2v(x)v'(x)\right] \\ & \quad = \max_{s,r\in[0,1]} f''''(u(x)+srv(x))(u'(x)+srv'(x))^2(v(x))^2 && (\mathfrak{A}) \\ & \qquad + \max_{s,r\in[0,1]} f'''(u(x)+srv(x))D_x\left[(u'(x)+srv'(x))(v(x))^2\right] && (\mathfrak{B}) \\ & \qquad + \max_{s,r\in[0,1]} 2f'''(u(x)+srv(x))(u'(x)+srv'(x))v(x)v'(x) && (\mathfrak{C}) \\ & \qquad + \max_{s,r\in[0,1]} 2f''(u(x)+srv(x))D_x\left[v(x)v'(x)\right] && (\mathfrak{D}) \end{aligned}$$

With equation (24), but now for different $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, it suffices to estimate the separate terms $||\mathfrak{A}||_{L^2}$, $||\mathfrak{B}||_{L^2}$, $||\mathfrak{C}||_{L^2}$, $||\mathfrak{D}||_{L^2}$ independently. Again, $||u+srv||_{L^\infty} \leq \sqrt[4]{2^3}\,||u+srv||_{H^2} \leq 2\sqrt[4]{2^3}\epsilon$ uniformly, so:

$$\begin{aligned} ||\mathfrak{A}||_{L^2} = & \left|\left|\max_{s,r\in[0,1]} f''''(u+srv)(u'+srv')^2 v^2\right|\right|_{L^2} \\ \leq & C\max_{s,r\in[0,1]} \left|\left|(u'+srv')^2v^2\right|\right|_{H^1} \\ \leq & C\max_{s,r\in[0,1]} ||u'+srv'||_{H^1}^2\,||v||_{H^1}^2 \\ \leq & C\,||v||_{H^2}^2 \\ ||\mathfrak{B}||_{L^2} = & \left|\left|\max_{s,r\in[0,1]} f'''(u+srv)D_x\left[(u'(x)+srv'(x))(v(x))^2\right]\right|\right|_{L^2} \\ \leq & C\max_{s,r\in[0,1]} \left|\left|(u'+srv')v^2\right|\right|_{H^1} \\ \leq & C\max_{s,r\in[0,1]} ||u'+srv'||_{H^1}\,||v||_{H^1}^2 \end{aligned}$$

$$
\begin{aligned}
&\leq C \left|\left|v\right|\right|_{H^2}^2 \\
\left|\left|\mathfrak{C}\right|\right|_{L^2} =& \left|\left| \max_{s,r\in[0,1]} 2f'''(u+srv)(u'+srv')vv' \right|\right|_{L^2} \\
&\leq C \max_{s,r\in[0,1]} \left|\left|(u'+srv')vv'\right|\right|_{H^1} \\
&\leq C \max_{s,r\in[0,1]} \left|\left|u'+srv'\right|\right|_{H^1} \left|\left|v\right|\right|_{H^1} \left|\left|v'\right|\right|_{H^1} \\
&\leq C \left|\left|v\right|\right|_{H^2}^2 \\
\left|\left|\mathfrak{D}\right|\right|_{L^2} =& \left|\left| \max_{s,r\in[0,1]} 2f''(u+srv)D_x\left[v(x)v'(x)\right] \right|\right|_{L^2} \\
&\leq C \left|\left|vv'\right|\right|_{H^1} \\
&\leq C \left|\left|v\right|\right|_{H^1} \left|\left|v'\right|\right|_{H^1} \\
&\leq C \left|\left|v\right|\right|_{H^2}^2 .
\end{aligned}
$$

From the equations above it follows that:

$$
\left|\left|D_x^2\left[f_N(u+v) - f_N(u) - (Df)_N(u)\cdot v\right]\right|\right|_{L^2}^2 \leq C \left|\left|v\right|\right|_{H^2}^4 . \tag{27}
$$

The result is now proven by combining (26) and (27):

$$
\begin{aligned}
&\left|\left|f_N(u+v) - f_N(u) - (Df)_N(u)\cdot v\right|\right|_{H^2} \\
&\qquad \leq C \left|\left|f_N(u+v) - f_N(u) - (Df)_N(u)\cdot v\right|\right|_{\tilde{H}^2} \\
&\qquad \leq C_\epsilon \left|\left|v\right|\right|_{H^2}^2 .
\end{aligned}
$$

□

Theorem 3.16. *Let $k \geq 3$. If $f \in \mathscr{C}^k(\mathbb{R})$ then $f_N \in \mathscr{C}^{k-3}(H^2(\mathbb{R}))$.*

Proof. For $k = 3$ the result has been shown by corollary 3.14. Assume that the result holds for some $k \geq 3$. Let $f \in \mathscr{C}^{k+1}(\mathbb{R})$ be given arbitrarily, by assumption $(Df)_N \in \mathscr{C}^{k-3}(H^2(\mathbb{R}))$. By the previous lemma it holds that $D(f_N) = (Df)_N$, so $f_N \in \mathscr{C}^{k-2}(H^2(\mathbb{R}))$. This ends the proof by induction. □

3.6 A related PDE for perturbations of solutions

Given a solution and a perturbed solution of a PDE, we derive that the perturbation itself solves a related PDE. Let $(B(t,\psi), D(B))$ be an unbounded operator and let g be a reaction term. Suppose that ψ is a solution of:

$$\psi_t = B(t,\psi)\psi + g(\psi). \tag{28}$$

Now introduce:

$$A_\psi(t,\phi) := B(t,\psi+\phi); \tag{29}$$

$$f_\psi(t,\phi) := B(t,\psi+\phi)\psi - B(t,\psi)\psi + g(\psi+\phi) - g(\psi). \tag{30}$$

Using this, we introduce a related partial differential equation:

$$\phi_t = A_\psi(t,\phi)\phi + f_\psi(t,\phi). \tag{31}$$

Theorem 3.17. *Suppose that ψ is a solutions of* (28)*. Then $\psi+\phi$ is a solution of* (28) *iff ϕ is a solution of* (31)*.*

Proof. Suppose that $\psi+\phi$ is a solution of (28), then it holds:

$$\begin{aligned}
\phi_t =& (\psi+\phi)_t - \psi_t \\
=& B(t,\psi+\phi)(\psi+\phi) + g(\psi+\phi) - B(t,\psi)\psi - g(\psi) \\
=& A_\psi(t,\phi)\phi + B(t,\psi+\phi)\psi - B(t,\psi)\psi + g(\psi+\phi) - g(\psi) \\
=& A_\psi(t,\phi)\phi + f_\psi(t,\phi);
\end{aligned}$$

so ϕ is a solution of (31).

Conversely, suppose that ϕ is a solution of (31), then:

$$\begin{aligned}
(\psi+\phi)_t =& \psi_t + \phi_t \\
=& B(\psi)\psi + g(\psi) + A_\psi(\phi,t,x)\phi + f_\psi(\phi,t,x) \\
=& B(\psi)\psi + g(\psi) + A_\psi(\phi,t,x)\phi + A_\psi(\phi,t,x)\psi \\
& - B(\psi)\psi + g(\psi+\phi) - g(\psi) \\
=& B(\psi+\phi)(\psi+\phi) + g(\psi+\phi);
\end{aligned}$$

so $\psi+\phi$ is a solution of (28). □

Solutions of (31) can be interpreted as perturbations of solutions of (28). Thus they may give information on stability of solutions of (28). Another important property is that f_ψ vanishes at $\phi = 0$.

Lemma 3.18. *For the operator $f_\psi(t,\phi)$ defined above it holds that $f_\psi(t,0) = 0$.*

Proof. It holds that:

$$f_\psi(t,0) = B(t,\psi+0)\psi - B(t,\psi)\psi + g(\psi+0) - g(\psi) = 0.$$

□

References

[1] Robert A. Adams, *Sobolev Spaces.* Academic Press, New York/San Francisco/Londen, 1975.

[2] Herbert Amann, *Existence and regularity for semilinear parabolic evolution equations.* Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 1984: 4e serie, tome 11, no. 4, pp. 593-676

[3] Herbert Amann, *Quasilinear Evolution Equations and Parabolic Systems.* Transactions of the American Mathematical Society, January 1986: vol. 293, no. 1, pp. 191-227

[4] Herbert Amann, *Dynamic Theory of Quasilinear Parabolic Equations II. Reaction-Diffusion Systems.* Differential and Integral Equations, January 1990: vol. 3, no. 1, pp. 13-75

[5] A. Doelman, J.D.M. Rademacher, S. van der Stelt, *Hopf dances near the tips of Busse balloons* Accepted at Discr. Cont. Dyn. Sys. (preprint)

[6] Klaus-Jochen Engel, Rainer Nagel, *One-Parameter Semigroups for Linear Evolution Equations.* Springer-Verlag, New York, 2000

[7] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations.* Springer-Verlag Lecture Notes in Mathematics, 1975: vol. 448, pp. 25-70

[8] Christopher A. Klausmeier, *Regular and Irregular Patterns in Semiarid Vegetation.* Science, 11 June 1999: vol. 284 no. 5421 pp. 1826-1828

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, New York, 1983

[10] Jens D.M. Rademacher, Arnd Scheel, *The saddle-node of nearly homogeneous wave trains in reaction-diffusion systems* Journal of Dynamics and Differential Equations, 2007: vol. 19, no. 2, pp. 479-496

[11] Yaping Wu, Xuezhi Zhao, *The existence and stability of travelling waves with transition layers for some singular cross-diffusion systems.* Physica D, 2005: 200, pp. 325-358