Effective resistance: And other graph measures for network robustness

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Abstract

Robustness is the ability of a network to continue performing well when it is subject to failures or attacks. In this thesis we survey robustness measures on simple, undirected and unweighted graphs, network failures being interpreted as vertex or edge deletions. We study graph measures based on connectivity, distance, betweenness and clustering. Besides these, reliability polynomials and measures based on the Laplacian eigenvalues are considered.

In addition to surveying existing measures, we propose a new robustness measure, the normalized effective resistance, which is derived from the total effective resistance. Total effective resistance is — within the field of electric circuit analysis — defined as the sum of the pairwise effective resistances over all pairs of vertices. The strength of this measure lies in the fact that all (not necessarily disjoint) paths are considered, in other words, the more backup possibilities, the larger the normalized effective resistance and the larger the robustness. A chapter is dedicated to optimizing the normalized effective resistance, first for graphs with a fixed number of vertices and diameter, and second for the addition of an edge to a given graph.

For all of the measures described above we evaluate the effectiveness as a measure of network robustness. The discussion and comparison of robustness measures is illustrated by a number of examples. Where possible we make extensions to weighted graphs and for all statements we provide either an elaboration of the original proof, or — when a rigorous proof is not available — we provide one ourselves.

Keywords: network robustness; graph measures; Laplacian eigenvalues; graph spectrum; effective resistance; optimization

AMS classification: 05C50; 05C81; 05C90; 94C05; 94C15

Preface

To conclude the Master program in Mathematics at the University of Leiden, I have carried out a nine-month research project at TNO Information and Communication Technology. This thesis is a mathematical report of this project and as such supposes some knowledge about graph theory, linear algebra and Markov chains.

I would like to thank my TNO supervisors Almerima Jamaković and Robert Kooij for their guidance and confidence, and my university supervisor Floske Spieksma for the time we spent together solving major and minor mathematical problems. I would also like to say thanks to Piet Van Mieghem, chairman of the Network Architecture and Service section of the Delft University of Technology, for his interest in my project, his spontaneous emails containing research questions, and his collaboration in the paper 'Effective graph resistance' that Piet, Floske, Almerima, Rob and I wrote together.

It has been a pleasure working with all these people as well as working on the interesting and important subject of network robustness. The subject made it possible to review fields of mathematics like graph theory, linear algebra, percolation and Markov chains, but also to get to know new fields such as spectral graph theory and electric circuit analysis.

Contents

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List of Symbols

Graph theory

V, E	set of vertices, set of edges
n, m	number of vertices, number of edges
$N(v)$	neighborhood of vertex v (set of vertices adjacent to v)
δ_v, $\delta_{\min}$	degree of vertex v (number of adjacent vertices), minimum degree
s_v	strength (weighted degree) of vertex v
K_n, C_n, S_n, P_n, O_n	complete, cycle, star, path, empty graph with n vertices

Complex networks

κ, κ_v, κ_e	graph connectivity, vertex connectivity, edge connectivity
d_{ij}, $\bar{d}$	distance between vertices i and j, average vertex distance
$d_{\max}$	diameter (maximum vertex distance)
b_x, $\bar{b}_v$, $\bar{b}_e$	betweenness of vertex/edge x, average vertex, edge betweenness
C	clustering coefficient
Rel	reliability polynomial
ξ	number of spanning trees
τ	network criticality

Linear algebra

$\operatorname{sp} S$	span or linear hull of a set of vectors S
$\langle \boldsymbol{v}, \boldsymbol{w} \rangle$	inner product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{v} \perp \boldsymbol{w}$	$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal ($\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$)
$W^{\perp}$	orthogonal complement of set subspace W
A^T	transposed of matrix A
$\operatorname{tr}(\mathsf{A})$	trace of matrix A
I	identity matrix (diagonal matrix with ones on the diagonal)
J	all-one matrix
$\mathbf{0}$, $\mathbf{1}$	all-zero column vector, all-one column vector
$\boldsymbol{e}_i$	unit vector (column vector with a one at position i and zeros elsewhere)

Spectral graph theory

$\mathsf{\Delta}$	degree matrix
S	matrix of strengths
A, a_{ij}	adjacency matrix with entries a_{ij}, $a_{ij} = 1$ if $(i, j) \in E$, $a_{ij} = 0$ otherwise
W, w_{ij}	matrix of weights with entries w_{ij}, the edge weights
T, t_{ij}	traffic matrix with entries t_{ij}, the traffic between i and j

L, L^W, $\left(\mathsf{L}^W\right)^+$	Laplacian, weighted Laplacian, weighted Laplacian pseudoinverse
λ_i (λ_i^W)	the i-th smallest (weighted) Laplacian eigenvalue
$\boldsymbol{\lambda}$	vector of Laplacian eigenvalues

Electrical circuit analysis

Y	net current through the network
y_{ij}	current between vertices i and j
r_{ij}	resistance of edge (i, j)
$\boldsymbol{v}$, v_i	vector of potentials, potential of vertex i
R_{ij}, R^{tot}	effective resistance between vertices i and j, total effective resistance
R^{norm}	normalized (inverse total) effective resistance

Probability theory

$\mathbf{P}(A)$	the probability that event A occurs
$\mathbf{E}(X)$	the expectation of the random variable X

Random walks on graphs

p_{ij}	transition probability from vertex i to vertex j
π_i	stationary probability of vertex i
T_{ij}	hitting time, number of transitions from i to j
T_{ii}^+	first return time to vertex i
B_{ivj}	number of visits to v in between i (included) and j (excluded)

Chapter 1

Introduction

1.1 The motivation for studying network robustness

As we live in a highly networked world, where vital facilities such as hospitals and fire brigades depend on a large amount of networks of different kinds, it is of highest importance that these networks are robust. Think of the consequences if for example telecommunication systems, power grids, water supplies, or road networks are malfunctioning. But what do we mean by network robustness? Let us start by giving a working definition.

Definition 1.1 *Robustness* is the ability of a network to continue performing well when it is subject to failures or attacks.

In order to decide whether a given network is robust, a way to quantitatively measure network robustness is needed. In other words, we would like to measure the impact of failures on the functionality of the network. During the past years a lot of robustness measures have been proposed [32], but scientists do not agree on which one to use, therefore the quest for robustness measures continues. Intuitively robustness is all about back-up possibilities [30], or alternative paths [42], but it remains a challenge to capture these concepts in a mathematical formula.

In short, network robustness research aims at finding a method for quantifying network robustness. Once such a measure has been established, we will be able to compare networks, to improve existing networks and to design robust networks.

1.2 The field of network robustness research

Network robustness research is carried out by scientists with different backgrounds, like mathematics, physics, computer science and biology [30]. As a result, quite a lot of different approaches to capture the robustness properties of a network have been undertaken [38]. All of these approached are based on the analysis of the underlying *graph* — consisting of a set of *vertices* connected by *edges* — of a network. We will use the words vertices and edges used in graph theory instead of the words nodes and links as these concepts are usually called in network theory.

One of these approaches is the study of *percolation theory* [7, 9, 35], which takes place within the field of *complex networks*. The scope of complex network theory is to analyze the properties of large networks (such as the so-called 'scale-free' and 'small-world' properties) and to define models that are able to describe real-world networks [1, 5, 13, 25]. In percolation theory robustness is interpreted as the ability to keep most of the vertices connected after deletion of some vertices (and the edges adjacent to them) or some edges. Percolation is especially adapted as a tool to determine the robustness of large networks like the Internet, because the results are valid for random graphs with a number of vertices growing to infinity.

We have focused on the study of measures (also called graph metrics) on finite, deterministic graphs. For a review of these measures see for example [5, 13, 12]. The graph measures considered in these review papers are topological measures, indicating that they describe the *network topology* (the geografical design consisting of vertices, which may be connected by edges), neglecting any processes running on top of the network.

Spectral graph theory [11] is another field of research used in network robustness studies [19] and applied in this thesis. It is the study of the eigenvalues of several matrices (like the adjacency matrix and the Laplacian) associated to a graph.

1.3 The aim of this research

In this thesis we are looking for a quantitative measure, that is intuitively clear to measure network robustness, easy to calculate, and applicable to all kinds of *simple* (without loops or multiple edges), undirected and connected graphs. The restriction to simple and connected graphs do not exclude any real-world networks, because we only speak about a network when the underlying graph is connected, and multiple edges can be replaced by one edge with a weight equal to the sum of the original edge weights. Although we focus on topological robustness measures, we consider both weighted and unweighted graphs. This leads to the following research question.

Research question *What is the best way to measure the robustness of simple, undirected, connected and possibly weighted graphs?*

We try to answer this question by reviewing existing graph measures possibly useful for measuring robustness and by comparing these measures. Furthermore, we compare them with the new measure we propose, the normalized effective resistance, a graph measure that is both informative and computationally tractable. A review of the normalized effective resistance and its properties, as well as its optimization for several optimization criteria is part of this thesis. Concretely, we aim at answering the following questions:

Secondary research questions

- Which graph measures have been proposed for measuring network robustness?
- Which other existing graph measures are suited for measuring network robustness?
- Which new measures can be defined for measuring network robustness?
- What are the properties of the above mentioned measures?
  - Which values can be obtained by the measure?
  - How does the measure change when edges are added or deleted?
  - Which graphs are optimal for this measure (for some given optimization criteria)?
- What graph properties do the measures capture, are these important for network robustness?

1.4 Thesis outline

Chapter 2 contains a review and a discussion of some classical graph measures. Sections 2.1 untill 2.4 consider a broad range of classical graph measures from complex network theory. The central question is whether these measures, which are not specifically introduced as network robustness measures, could be used to determine the robustness properties of a graph. The subject of Section 2.5 is the reliability polynomial, which represents a classical method to measure network robustness. Section 2.6 compares the measures treated in this chapter by means of a few example graphs.

Chapter 3 is about spectral theory and measures based on the eigenvalues of matrices associated to graphs (called *spectral measures*). The first section, Section 3.1, gives an introduction to the Laplacian matrix and its eigenvalues. In section 3.2 the second smallest Laplacian eigenvalue (also called *algebraic connectivity*) is discussed, because it has been proposed as a robustness measure. The discussion of this eigenvalue is followed by an exposition of a few other spectral measures, which also have been proposed as measures for robustness.

In Chapter 4 a new robustness measure, based on the notion of effective resistance in electrical circuits, is proposed. Section 4.1 treats the relation between the pairwise effective resistance and the robustness of the connection between two vertices. In Section 4.2 a formal definition of the pairwise effective resistance and the total effective resistance are given, as well as expressions of both in terms of the Laplacian. The basic properties of the pairwise effective resistance and the *total effective resistance* (the sum of the pairwise effective resistances over all pairs of vertices) are described in Section 4.3. In the next section, Section 4.4, an analysis of random walks on graphs is developed in order to derive some alternative expressions for the pairwise and total effective resistance. Section 4.5 discusses some examples and the final section (Section 4.6) contains the arguments for the introduction of the *normalized effective resistance* (which is proportional to the inverse total effective resistance) as a new measure for network robustness.

Chapter 5 is dedicated to the maximization of the normalized effective resistance in order to be able to design robust networks and to improve existing networks. The first optimization criteria to be considered are a fixed number of vertices and a fixed largest distance in the graph. Section 5.1 gives a characterization of the class of graphs containing the optimal graphs for these criteria and discusses the results of exhaustive search on this class. The optimization of an approximation of the normalized effective resistance, by using only a part of the Laplacian eigenvalues, is the subject of Section 5.2. Chapter 5 is concluded by a section (Section 5.3) on the optimal way to add an edge to a graph, which clearly is a relevant problem when the robustness of real-world networks is considered.

Chapter 2

Classical graph measures

In the past decades, numerous measures have been introduced to characterize graphs. In this chapter we treat these classical graph measures that are intuitively relevant for evaluating the robustness of a network. Unless differently stated, by a graph $G = (V, E)$ we mean a simple, undirected, connected and unweighted graph, with $|V| = n$ vertices and $|E| = m$ edges. We also explore the possibilities to adapt these measures in order to measure weighted graphs or to take the traffic of the network into account. Each section describes and discusses a specific graph measure or a class of measures. Section 2.6 contains a comparison and evaluation of the measures treated in this chapter.

2.1 Connectivity

Apart from the classical binary *connectivity* measure κ, which distinguishes *connected* graphs ($\kappa = 1$) having paths between all pairs of vertices and *unconnected* graphs ($\kappa = 0$) for which at least one pair of vertices lacks a connecting path, two more connectivity measures have been defined: vertex and edge connectivity.

The *vertex connectivity* κ_v of an incomplete graph is the minimal number of vertices to be removed in order to disconnect it. The number of edges that need to be removed to disconnect the graph is called the *edge connectivity* κ_e. It is easy to see that $\kappa_v \leq \kappa_e \leq \delta_{\min}$, where $\delta_{\min}$ is the minimum degree of the vertices. For a complete graph K_n the vertex connectivity cannot be determined by the definition above, because it cannot be disconnected by deleting vertices. In order for the inequality $\kappa_v \leq \kappa_e \leq \delta_{\min}$ to hold also in the case of a complete graph, its vertex connectivity is defined to be $\kappa_v = n - 1$.

It seems natural to say that the higher the vertex or edge connectivity of a graph, the more robust it is. However, these connectivity measures do not take into account the importance of the deleted vertices or edges. A graph broken into disconnected components, might still be functioning well, if there is a small amount of traffic among the components. Therefore two graphs with the same vertex or edge connectivity may not be equally robust.

2.2 Distance

Let the *distance* d_{ij} be the length of the shortest path between vertices i and j. The maximum $d_{\max}$ over these distances is called the *diameter* and the average over all pairs is denoted by $\bar{d}$,

$$\bar{d} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij}. \tag{2.1}$$

The length of a path in an unweighted graph is equal to the number of edges it consists of. If the edge weights in a weighted graph denote the length of the edge, then the length of a path is the sum of the weights of its edges. The meaning of the diameter and the average distance as robustness measures follows from the fact that the shorter a path, the less vulnerable it is. Nevertheless, the vulnerability of a path can be compensated by adding back-up paths, which are not considered by the two measures. This clearly is a disadvantage. The average distance is more sensible than the diameter, as the first is strictly decreasing when edges are added, while the latter may remain equal while adding edges.

The presence of frequently used paths that are long and thus vulnerable decreases the robustness of a network. On the other hand, a long path that is less frequently used almost does not negatively affect the overall robustness. If for a network the *traffic matrix* T is given, with entries $\mathsf{T}_{ij} = t_{ij}$ that denote the amount of traffic between vertices i and j, then we define a new measure based on the concept of graph distance as

$$\bar{d}^T = \frac{1}{\sum_{i=1}^{n}\sum_{j=i+1}^{n} t_{ij}} \sum_{i=1}^{n}\sum_{j=i+1}^{n} t_{ij} d_{ij}.$$

In the comparison of robustness measures at the end of this chapter the unweighted variants of the measures are considered, as not all of the measures in this chapter have a weighted counterpart. We will, however, define a weighted variants when possible.

2.3 Betweenness

The *betweenness* denotes the number of shortest paths between pairs of vertices, passing through a vertex or edge x. If there exists more than one shortest path between two vertices, then each of these k paths is counted $1/k$ times. The formal definition of the betweenness of a vertex or an edge x is

$$b_x = \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{n_{ij}(x)}{n_{ij}},$$

where $n_{ij}(x)$ is the number of shortest paths between i and j passing through x and n_{ij} is the total number of shortest paths between i and j. The vertex betweenness is sometimes called betweenness centrality, because it has been introduced to determine the vertices that occupy central positions in the network [16].

Betweenness can be used when a graph is given, but traffic flows are not known. Suppose there is one unit of traffic between all pairs of vertices and traffic travels by shortest paths (dividing the load if there is more than one shortest path), then the load of a vertex/edge is given by its betweenness. Hence, if the actual vertex/edge loads are known, it makes no sense to calculate the betweenness. If the traffic matrix is known, we define a new version of the betweenness for a vertex or node x as follows.

$$b_x = \sum_{i=1}^{n}\sum_{j=i+1}^{n} t_{ij}\frac{n_{ij}(x)}{n_{ij}},$$

where t_{ij}, $n_{ij}(x)$ and n_{ij} are defined as before.

Deleting vertices or edges with a higher load can have more impact than deleting others. Betweenness can therefore help to identify bottlenecks and give a tool to improve the robustness of a network. However, the existence of alternative paths for network elements with a high load is not considered. Betweenness is thus again a measure based on shortest paths only.

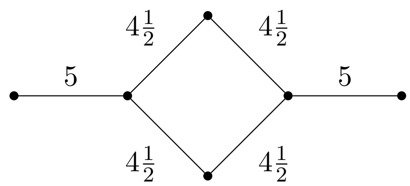
The betweenness can be easily extended to weighted networks where the edge weights give the distances between vertices. Only the shortest paths will change, not the betweenness formula. Newman used the weighted vertex betweenness to identify the most important persons in a social network (a network of collaborating scientists). In his model the edge weight corresponds to the inverse strength of the relation of two scientists instead of the physical distances between them [24]. It seems reasonable to believe that communication in social network does not follow the shortest path in physical distance, but is especially passed by people having strong relations.

It is not very useful to calculate the average vertex $\bar{b}_v$ or edge betweenness $\bar{b}_e$, since they are linear functions of the average vertex distance. The following equalities were found in [36] after we derived them. They can easily be adapted in order to show that $\bar{b}_v^T$ and $\bar{b}_e^T$ are linear functions of $\bar{d}^T$.

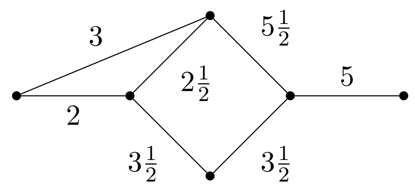
$$\begin{aligned}
\bar{b}_v &= \frac{1}{n}\sum_{v=1}^{n} b_v = \frac{1}{n}\sum_{v=1}^{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{n_{ij}(v)}{n_{ij}} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{1}{n_{ij}}\sum_{v=1}^{n} n_{ij}(v)\\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{1}{n_{ij}}n_{ij}(d_{ij}+1) = \frac{1}{2}(n-1)(\bar{d}+1),\\
\bar{b}_e &= \frac{1}{m}\sum_{e=1}^{m} b_e = \frac{1}{m}\sum_{e=1}^{m}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{n_{ij}(e)}{n_{ij}} = \frac{1}{m}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{1}{n_{ij}}\sum_{e=1}^{m} n_{ij}(e)\\
&= \frac{1}{m}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{1}{n_{ij}}n_{ij}d_{ij} = \frac{n(n-1)}{2m}\bar{d}.
\end{aligned}$$

As a consequence of these linear relations, the average distance and the average vertex betweenness will always indicate the same graph as most robust when comparing the robustness of two graphs, provided the graphs have the same number of vertices. The same holds for the three measures (average distance, average vertex betweenness and average edge betweenness) when the number of vertices and edges of the graphs are equal.

However, we might consider the variance of the vertex or edge betweenness, and investigate whether the network robustness decreases with an increasing variance (and a fixed expectation). As there are no papers known to treat the betweenness variance, it may be considered as a topic for future research.



(a) Graph with maximum edge betweenness of 5



(b) Graph with maximum edge betweenness of $5\frac{1}{2}$

Figure 2.1. The maximum edge betweenness can increase when an edge is added. The betweenness of each edge is given in the graphs.

Sydney et al. have proposed a robustness measure based on the maximum edge betweenness $b_e^{\max}$ and its behavior as vertices are removed, because this maximum determines the bandwidth that can be assigned to each flow [32]. The maximum edge betweenness has a problem though; it can increase while an edge is added. The reason is that flows are assumed to always choose the shortest path. See Figure 2.1 for an example.

2.4 Clustering

The presence of triangles is captured by the *clustering coefficient*, which compares the number of triangles to the number of connected triples. The clustering coefficient gives the portion of vertices j, k sharing a neighbor i that are also neighbors themselves (which means that the edge (j, k) is present, see Figure 2.2). The clustering coefficient c_i of a vertex i is defined as the number of edges among neighbors of i divided by $\delta_i(\delta_i - 1)/2$, the total possible number of edges among its neighbors. The overall clustering coefficient of a graph is the average over the clustering coefficients of the vertices. This definition gives

$$C = \frac{1}{n} \sum_{i \in V; \delta_i > 1} c_i = \frac{1}{n} \sum_{i \in V; \delta_i > 1} \frac{2}{\delta_i(\delta_i - 1)} e_i =$$

$$\frac{1}{n} \sum_{i \in V; \delta_i > 1} \frac{1}{\delta_i(\delta_i - 1)} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} a_{jk} a_{ki} = \frac{1}{n} \sum_{i \in V; \delta_i > 1} \frac{1}{\delta_i(\delta_i - 1)} \left(A^3\right)_{ii},$$

with e_v the number of edges among neighbors of v, and a_{ij} the ij-th element of the adjacency matrix A.

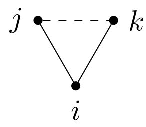


Figure 2.2. Vertices j, k sharing a neighbor i may or may not be neighbors themselves.

Although the clustering coefficient was originally designed for social networks, in which it measures the probability that two friends of a person are friends of each other too, it can also be used to measure robustness in other types of networks. A high clustering coefficient indicates high robustness, because the number of alternative paths grows with the number of triangles.

Several people have given weighted versions of the clustering coefficient, to account for the fact that some edges are more essential than others. The definition of Barrat et al. is easy to work with [4].

$$C^W = \frac{1}{n} \sum_{i \in V; \delta_i > 1} \frac{1}{s_i(\delta_i - 1)} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} a_{jk} a_{ki} \frac{w_{ij} + w_{ki}}{2},$$

where w_{ij} denotes the weight of an edge (i, j) and s_i the so-called *strength* of a vertex i, which equals the sum of the weights of the edges adjacent to it. The advantage of this definition is that the values lie between zero (no triangles) and one (every two adjacent edges are part of a triangle), and it gives back the original definition when all weights are equal to one. Other definitions can be found in [27] and [26].

2.5 Reliability polynomials

Although the reliability polynomial is not part of the standard set of graph measures, we treat it in this chapter, because it is a classical way to quantify network robustness. We start this section by giving the definition of a reliability polynomial as stated by Moore and Shannon in [23].

Definition 2.1 The *reliability polynomial* $\mathrm{Rel}(G)$ of a graph G is equal to the probability that the graph is connected when each edge is (independently of the others) present with probability

$p = 1 - q$, in other words

$$\mathrm{Rel}(G) = \sum_{i=0}^{m} F_i (1-p)^i p^{m-i},$$

when F_i denotes the number of sets of i edges whose removal leaves G connected.

Reliability polynomials are an intuitive way to measure network robustness, although it is difficult to decide what value we should assign to p. The robustness evaluation of graphs depends on the value of p; pairs of graphs for which the reliability polynomial of the first graph is larger for small p, while the reliability polynomial of the second is larger for large p, are known [20]. There even exist pairs of graphs with reliability polynomials that cross twice [10]. It seems reasonable to consider p to be close to one, because in real-world networks edge failures are scarce.

(a) Graph G_1 with $\kappa_e(G_1) = 1$ and $\mathrm{Rel}(G_1) = 1 - q + o(q)$ for $q \to 0$

(b) Graph G_2 with $\kappa_e(G_2) = 2$ and $\mathrm{Rel}(G_2) = 1 - 4q^2 + o\left(q^2\right)$ for $q \to 0$

Figure 2.3. Two graphs for which the reliability polynomial and the edge connectivity give the same evaluation on robustness

In [38] the graphs of Figure 2.3 have been given. The authors have pointed out that the edge connectivity of the second graph is higher and, for p close enough to one, also the reliability polynomial of the second graph is greater than the reliability polynomial of the first graph. We show that the edge connectivity and the reliability polynomial for p close to one always give the same evaluation on robustness. This theorem turned out to be known already to Moore and Shannon in 1956 [23].

Theorem 2.1 (Moore and Shannon, 1956) *The relation between the reliability polynomial* $\mathrm{Rel}(G)$ *of a graph* G *and the edge connectivity* $\kappa_e(G)$ *satisfies the following two properties*

1. *If* $\kappa_e(G_1) < \kappa_e(G_2)$, *then for* p *close enough to one we have* $\mathrm{Rel}(G_1) < \mathrm{Rel}(G_2)$. *This means that the reliability polynomial for* p *close to one and the edge connectivity give the same evaluation on network robustness.*

2. *Let* $s(G)$ *be the number of subsets of* $\kappa_e(G)$ *edges whose removal disconnects* G. *If* $\kappa_e(G_1) = \kappa_e(G_2)$ *and* $s(G_1) > s(G_2)$ *then for* p *close enough to one we have* $\mathrm{Rel}(G_1) < \mathrm{Rel}(G_2)$.

Proof We give a proof of our own. Note that $F_i = \binom{m}{i}$ for $i < \kappa_e(G)$ and $F_{\kappa_e(G)} = \binom{m}{\kappa_e(G)} - s(G)$. Replacing p by $1-q$ in the definition of a reliability polynomial leads to

$$\begin{aligned}\mathrm{Rel}(G) &= \sum_{i=0}^{m} F_i q^i (1-q)^{m-i} \\ &= \sum_{i=0}^{\kappa_e(G)-1} \binom{m}{i} q^i (1-q)^{m-i} + \left(\binom{m}{\kappa_e(G)} - s(G)\right) q^{\kappa_e(G)} (1-q)^{m-\kappa_e(G)} + o\left(q^{\kappa_e(G)}\right) \\ &= 1 - s(G) q^{\kappa_e(G)} + o\left(q^{\kappa_e(G)}\right).\end{aligned}$$

In the last equality we have used the binomial theorem

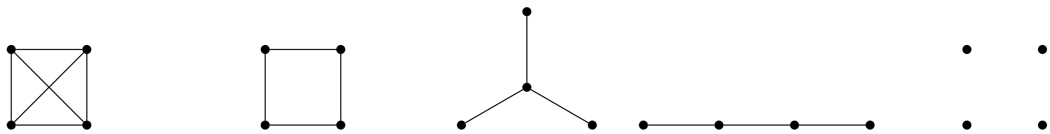
$$\sum_{i=0}^{m} \binom{m}{i} q^i (1-q)^{m-i} = 1.$$

The two statements now follow directly. □

Remark that a reliability polynomial can also be defined for vertex deletion instead of edge deletion. In that case the reliability polynomial for p close to one and the vertex connectivity give the same robustness evaluation.

2.6 Comparison of some classical graph measures

In this section we compare the classical graph measures described above. We have calculated them for the example graphs with four vertices depicted in Figure 2.4. The results are given in Table 2.1 and Figure 2.5.



(a) Complete graph K_4 (b) Cycle graph C_4 (c) Star graph S_4 (d) Path graph P_4 (e) Empty graph O_4

Figure 2.4. Examples of graphs with four vertices

	m	κ	$\delta_{\min}$	κ_v	κ_e	$d_{\max}$	$\bar{d}$	$b_e^{\max}$	$\bar{b}_e$	C	Rel
K_4	6	1	3	3	3	1	1	1	1	1	$-6p^6+24p^5-33p^4+16p^3$
C_4	4	1	2	2	2	2	$1\frac{1}{3}$	2	2	0	$-3p^4+4p^3$
S_4	3	1	1	1	1	2	$1\frac{1}{2}$	3	3	0	p^3
P_4	3	1	1	1	1	3	$1\frac{2}{3}$	4	$3\frac{1}{3}$	0	p^3
O_4	0	0	0	0	0	∞	∞	-	-	0	0

Table 2.1. The values of some graph measures for the five graphs of Figure 2.4

Our intuition says that the graphs are ordered by decreasing robustness. The robustness evaluations of the measures of Table 2.1 correspond to this intuition. Although not all of them can distinguish all graphs, all of the measures would say that the graphs are indeed in order of decreasing robustness.

By analyzing the values in the table we come to the conclusion that the clustering coefficient and the connectedness are poor robustness measures. Also the other two connectivity measures, the diameter and the reliability polynomial cannot distinguish all graphs. The connectivity measures and the reliability polynomial are constant on the set of trees. The maximum edge betweenness performs well in this example, but has been proved to fail in other situations like that of Figure 2.1. The average distance and the average betweenness — which have been shown to always classify the graphs in the same order — seem to be the best robustness measures. Nevertheless, also the average distance and betweenness have a disadvantage, since they consider only the shortest paths in a graph while for the robustness of a network also the longer alternative paths are important.

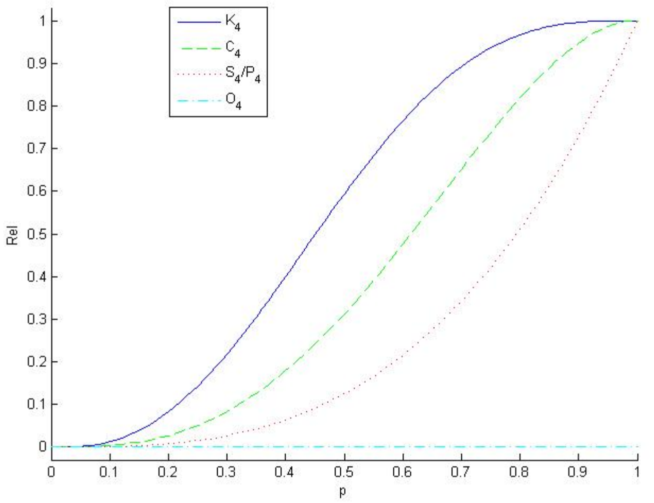


Figure 2.5. Graphs of reliability polynomials for the five graphs of Figure 2.4.

Chapter 3

Spectral graph measures

Networks can be represented by graphs. These graphs can be studied directly, as we have done in the previous chapters, but also by looking at the matrices associated to a graph. Several robustness measures based on the eigenvalues of the adjacency matrix or the Laplacian have been proposed. Section 3.1 gives some properties of the Laplacian and its eigenvalues. Section 3.2 discusses the second smallest Laplacian eigenvalue. In the last section of this chapter some other spectral measures — proposed as a measure for network robustness — are treated.

3.1 The Laplacian and its eigenvalues

The *adjacency matrix* A, which has a one at position i, j when an edge (i, j) is present and is zero elsewhere, gives the most intuitive matrix description of a simple undirected graph $G = (V, E)$. Another matrix associated to a graph is the Laplacian.

Definition 3.1 The *Laplacian* L is the difference $\Delta - \mathsf{A}$ of the *degree matrix* Δ (the diagonal matrix with $\mathsf{D}_{ii} = \delta_i$) and the adjacency matrix, i.e.

$$\mathsf{L}_{ij} = \begin{cases} \delta_i & \text{if } i = j \\ -1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases} .$$

Both the adjacency matrix and the Laplacian fully characterize the graph; when one of these matrices is given, the original graph can be reconstructed. For a graph with non-negative edge weights w_{ij}, the analogue of the adjacency matrix is the *matrix of weights* $\mathsf{W} = (\mathsf{w_{ij}})$, the *weighted Laplacian* is $\mathsf{L}^W = \mathsf{S} - \mathsf{W}$, where S is the diagonal *matrix of strengths*, with elements $\mathsf{S}_{ii} = s_i$. To summarize, the elements of the weighted Laplacian are given by

$$\mathsf{L}^W_{ij} = \begin{cases} s_i = \sum_j w_{ij} & \text{if } i = j \\ -w_{ij} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases} ,$$

which shows that the row sums of L^W are zero, just as for the unweighted Laplacian, such that the all-one vector is an eigenvector for eigenvalue 0. The following theorem is true for the unweighted as well as the weighted version of the Laplacian.

Theorem 3.1 *For the Laplacian of a graph $G = (V, E)$, the multiplicity of the eigenvalue zero corresponds to the number of connected components of G.*

Proof The following proof is our own. Because L and L^W are symmetric matrices, the algebraic multiplicity and the geometric multiplicity are equal. We can therefore use the term 'multiplicity' for both. We only prove the theorem for the weighted Laplacian, as the ordinary Laplacian is a special case of it (where the weights are either zero or one).

For every connected component C, the vector $\boldsymbol{y}_C$ — all vectors are column vectors — with $\boldsymbol{y}_i = 1$ if vertex i is part of the component C and $\boldsymbol{y}_i = 0$ otherwise, is an eigenvector for eigenvalue 0, since $\mathsf{L}^W \boldsymbol{y}_C = \mathbf{0}$ (the all-zero vector). Furthermore, the set of these eigenvectors is linearly independent. It is now enough to show that all eigenvectors for eigenvalue 0 are linear combinations of these eigenvectors.

Suppose $\boldsymbol{x}$ is an eigenvector corresponding to eigenvalue 0, then we have for all $i \in V$

$$\boldsymbol{x}_i \sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} \boldsymbol{x}_j w_{ij}. \tag{3.1}$$

This can only hold if $\boldsymbol{x}_i = \boldsymbol{x}_j$ when i, j are part of the same component, for the following reasoning. Let C be an arbitrary component and let i be such that $\boldsymbol{x}_i = \max_{j \in C} \boldsymbol{x}_j$ then $\sum_{j=1}^{n} \boldsymbol{x}_i w_{ij} \geq \sum_{j=1}^{n} \boldsymbol{x}_j w_{ij}$. In order to have the requested equality (3.1), it is necessary that $\boldsymbol{x}_i = \boldsymbol{x}_j$ for all neighbors j of i. Similarly, we see that $\boldsymbol{x}_j = \boldsymbol{x}_k$ for a neighbor k of j. Continuing this argument leads to the conclusion that all eigenvectors $\boldsymbol{x}$ have $\boldsymbol{x}_i = \boldsymbol{x}_j$ when i, j are members of the same connected component, thus all eigenvectors corresponding to eigenvalue 0 are linear combinations of the vectors $\boldsymbol{y}_C$. □

The Laplacian is positive semidefinite, since we can find a B such that $\mathsf{L} = \mathsf{B}^T\mathsf{B}$ as follows. Convert G in a directed graph by choosing an arbitrary direction for each edge, let B be the *edge-vertex incidence matrix*, i.e. for an arc a and a vertex i:

$$\mathsf{B}_{ai} = \begin{cases} 1 & \text{if } a = (i, j) \text{ for some } j \\ -1 & \text{if } a = (j, i) \text{ for some } j \\ 0 & \text{otherwise} \end{cases}.$$

Because the Laplacian is symmetric, positive semidefinite and the rows sum up to 0, its eigenvalues are real, non-negative and the smallest eigenvalue is zero. Hence, we can order the eigenvalues and denote them as λ_i for $i = 1, \ldots, n = |V|$ such that $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The vector with elements λ_i is denoted by $\boldsymbol{\lambda}$.

The weighted Laplacian has the same nice properties of symmetry, positive semidefiniteness, and zero row sums. Hence, we denote the eigenvalues as $\lambda_1^W, \ldots, \lambda_n^W$. To show that the weighted Laplacian is positive semidefinite we convert G again in a directed graph and define B^W as

$$\mathsf{B}_{ai}^W = \begin{cases} \sqrt{w_{ij}} & \text{if } a = (i, j) \text{ for some } j \\ -\sqrt{w_{ij}} & \text{if } a = (j, i) \text{ for some } j \\ 0 & \text{otherwise} \end{cases},$$

which satisfies $\mathsf{L}^W = \left(\mathsf{B}^W\right)^T \mathsf{B}^W$.

We show that the Laplacian eigenvalues do not decrease when an edge is added or an edge weight is increased. Before we come to this theorem we need to state the following characterization of the eigenvalues of a symmetric (or Hermitian) matrix.

Theorem 3.2 (Courant-Fisher or min-max principle) *Let $\mu_1 \leq \ldots \leq \mu_n$ be the eigenvalues of the symmetric matrix M, the next equality holds for the k-th smallest eigenvalue*

$$\mu_k = \min_{S_k} \max_{\substack{\boldsymbol{x} \neq \mathbf{0} \\ \boldsymbol{x} \in S_k}} \frac{\langle \mathsf{M}\boldsymbol{x}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}, \tag{3.2}$$

where S_k is a k-dimensional subset of $\mathbb{R}^n$.

Proof We reproduce the proof of [41]. Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ be an orthogonal set of eigenvectors corresponding to the eigenvalues $\mu_1, \ldots, \mu_n$ respectively. We start by showing that, for all subspaces $S_k \subset \mathbb{R}^n$ with $\dim(S_k) = k$, there is a $\boldsymbol{x} \in S_k$, $\boldsymbol{x} \neq \boldsymbol{0}$ such that

$$\frac{\langle \mathsf{M}\boldsymbol{x}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} \geq \mu_k. \tag{3.3}$$

We have $S_k \cap \mathrm{sp}\{\boldsymbol{u}_k, \ldots, \boldsymbol{u}_n\} \neq \emptyset$, because of the dimensions of both spaces. Choose a vector in the intersection, say $\boldsymbol{x} = \sum_{i=k}^{n} a_i \boldsymbol{u}_i$, it satisfies (3.3):

$$\begin{aligned}
\frac{\langle \mathsf{M}\boldsymbol{x}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} &= \frac{\langle \mathsf{M} \sum_{i=k}^{n} a_i \boldsymbol{u}_i, \sum_{i=k}^{n} a_i \boldsymbol{u}_i \rangle}{\langle \sum_{i=k}^{n} a_i \boldsymbol{u}_i, \sum_{i=k}^{n} a_i \boldsymbol{u}_i \rangle} \\
&= \frac{\sum_{i=k}^{n} \sum_{j=k}^{n} \langle \mathsf{M} a_i \boldsymbol{u}_i, a_j \boldsymbol{u}_j \rangle}{\sum_{i=k}^{n} \sum_{j=k}^{n} \langle a_i \boldsymbol{u}_i, a_j \boldsymbol{u}_j \rangle} \\
&= \frac{\sum_{i=k}^{n} \sum_{j=k}^{n} \mu_i \langle a_i \boldsymbol{u}_i, a_j \boldsymbol{u}_j \rangle}{\sum_{i=k}^{n} \sum_{j=k}^{n} \langle a_i \boldsymbol{u}_i, a_j \boldsymbol{u}_j \rangle} \\
&= \frac{\sum_{i=k}^{n} \mu_i \langle a_i \boldsymbol{u}_i, a_i \boldsymbol{u}_i \rangle}{\sum_{i=k}^{n} \langle a_i \boldsymbol{u}_i, a_i \boldsymbol{u}_i \rangle} \\
&\geq \mu_k \frac{\sum_{i=k}^{n} \langle a_i \boldsymbol{u}_i, a_i \boldsymbol{u}_i \rangle}{\sum_{i=k}^{n} \langle a_i \boldsymbol{u}_i, a_i \boldsymbol{u}_i \rangle} \\
&= \mu_k,
\end{aligned}$$

where we have used that the eigenvectors are orthogonal, such that $\langle a_i \boldsymbol{u}_i, a_j \boldsymbol{u}_j \rangle = 0$ for $i \neq j$. As a consequence of (3.3), it holds that

$$\inf_{S_k} \max_{\substack{\boldsymbol{x} \neq \boldsymbol{0} \\ \boldsymbol{x} \in S_k}} \frac{\langle \mathsf{M}\boldsymbol{x}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} \geq \mu_k.$$

On the other hand, for $S_k = \mathrm{sp}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ we have

$$\max_{\substack{\boldsymbol{x} \neq \boldsymbol{0} \\ \boldsymbol{x} \in S_k}} \frac{\langle \mathsf{M}\boldsymbol{x}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \frac{\langle \mathsf{M}\boldsymbol{u}_k, \boldsymbol{u}_k \rangle}{\langle \boldsymbol{u}_k, \boldsymbol{u}_k \rangle} = \mu_k,$$

which together imply (3.2). □

Inserting L^W for M and λ_k^W for μ_k gives us alternative expressions for the Laplacian eigenvalues. The Courant-Fisher principle allows for developing a monotonicity theorem for Laplacian eigenvalues, which is a consequence of Weyl's theorem [18].

Theorem 3.3 (Weyl) *Let a graph G be given and let G' be obtained by increasing the weight of an edge, the Laplacian eigenvalues of the new graph satisfy*

$$\lambda_k^W(G') \geq \lambda_k^W(G).$$

This means that increasing edge weights does not decrease the Laplacian eigenvalues.

Proof The proof follows from the proof of Weyl's theorem [18]. Suppose weight w_{ij} of edge (i,j) is increased by an amount a. For the matrix $\mathsf{K} = a\,(\boldsymbol{e}_i - \boldsymbol{e}_j)\,(\boldsymbol{e}_i - \boldsymbol{e}_j)^T$, with $\boldsymbol{e}_i$ the i-th unit vector, we find by Theorem 3.2

$$\begin{aligned}
\lambda_k^W(G') &= \min_{S_k} \max_{\substack{\boldsymbol{x}\neq \boldsymbol{0}\\ \boldsymbol{x}\in S_k}} \frac{\langle \mathsf{L}^W(G')\boldsymbol{x},\boldsymbol{x}\rangle}{\langle \boldsymbol{x},\boldsymbol{x}\rangle} \\
&= \min_{S_k} \max_{\substack{\boldsymbol{x}\neq \boldsymbol{0}\\ \boldsymbol{x}\in S_k}} \frac{\langle \left(\mathsf{L}^W(G)+\mathsf{K}\right)\boldsymbol{x},\boldsymbol{x}\rangle}{\langle \boldsymbol{x},\boldsymbol{x}\rangle} \\
&= \min_{S_k} \max_{\substack{\boldsymbol{x}\neq \boldsymbol{0}\\ \boldsymbol{x}\in S_k}} \frac{\langle \mathsf{L}^W(G)\boldsymbol{x},\boldsymbol{x}\rangle + \langle \mathsf{K}\boldsymbol{x},\boldsymbol{x}\rangle}{\langle \boldsymbol{x},\boldsymbol{x}\rangle} \\
&\geq \min_{S_k} \max_{\substack{\boldsymbol{x}\neq \boldsymbol{0}\\ \boldsymbol{x}\in S_k}} \frac{\langle \mathsf{L}^W(G)\boldsymbol{x},\boldsymbol{x}\rangle}{\langle \boldsymbol{x},\boldsymbol{x}\rangle} \\
&= \lambda_k^W(G),
\end{aligned}$$

where we have used that A is positive semidefinite. □

By increasing a weight $w_{ij} = 0$ by $a = 1$ we find the following corollary.

Corollary 3.4 *Adding an edge does not decrease the Laplacian eigenvalues.*

The next theorem gives the Laplacian spectrum of a complement graph.

Theorem 3.5 *Let $\bar{G}$ be the complement of graph G, that is the graph containing the same vertex set as G and all possible edges except for those of G. The Laplacian eigenvalues of $\bar{G}$ are $\lambda_1^W(\bar{G}) = 0$ and*

$$\lambda_i^W(\bar{G}) = n - \lambda_{n-i+2}^W(G), \quad \textit{for } i = 2, \ldots n.$$

Proof We have provided a proof ourselves. Since the Laplacian is symmetric, it has an orthogonal basis of eigenvectors. Let $\boldsymbol{u}_i(G)$ be an eigenvector corresponding to eigenvalue $\lambda_i^W(G)$ $(i > 1)$, then $\boldsymbol{u}_i(G)$ is orthogonal to $\mathbf{1}$ (the all-one vector). Using that $\mathsf{L}^W(G)+\mathsf{L}^W(\bar{G}) = n\mathsf{I} - \mathsf{J}$ (with I the identity matrix and J the all-one matrix), we find for $i = 2, \ldots n$

$$\mathsf{L}^W(\bar{G})\boldsymbol{u}_{n-i+2}(G) = \left(n\mathsf{I} - \mathsf{J} - \mathsf{L}^W(G)\right)\boldsymbol{u}_{n-i+2}(G) = \left(n - \lambda_{n-i+2}^W(G)\right)\boldsymbol{u}_{n-i+2}.$$

□

To conclude this section, we give a theorem characterizing the Laplacian spectrum of a graph consisting of several components.

Theorem 3.6 *Our proof only contains one line. Let G be the disjoint union of graphs G_1 and G_2, then the spectrum of G is the union of the spectra of G_1 and G_2.*

Proof Take eigenvectors of G_1 and G_2 and add zeros at the positions corresponding to the other subgraph, then these vectors are independent eigenvectors of G (corresponding to the same eigenvalue as before). □

3.2 Algebraic connectivity

The second smallest Laplacian eigenvalue is believed to measure the connectivity of a graph; the larger λ_2, the more difficult it is to cut the graph into unconnected components. This belief is suggested by two results (Corollary 3.7 and Theorem 3.9) we encounter in this section. The second smallest Laplacian eigenvalue is also related to a lot of other graph characteristics. For a survey of its properties and its applications see [22].

Definition 3.2 The second smallest eigenvalue λ_2 of the Laplacian is called the *algebraic connectivity*.

The first result is a corollary of Theorem 3.1, it therefore also holds for the second eigenvalue of the weighted Laplacian.

Corollary 3.7 to Theorem 3.1 *The algebraic connectivity is equal to zero if and only if the graph is unconnected.*

Miroslav Fiedler first underlined the importance of λ_2 (in his paper of 1973 [15]) and gave it the name algebraic connectivity, which is probably inspired by the previous corollary. A second reason for Fiedler to call the second (unweighted) Laplacian eigenvalue algebraic connectivity might have been that it is bounded above by the vertex connectivity. To prove this, we need the next lemma.

Lemma 3.8 (Fiedler, 1973) *Removing k vertices, reduces the algebraic connectivity by at most k. More formally, let G be a given graph and G_k a graph obtained by deleting k vertices from G, then*

$$\lambda_2(G_k) \geq \lambda_2(G) - k. \tag{3.4}$$

Proof We present the original proof of [15]. It is enough to prove equation (3.4) for $k = 1$, that is to prove that removing a vertex decreases λ_2 by at most one. The general case is then shown by induction on the number of deleted vertices. Without loss of generality we may say that G_1 has been obtained by deleting vertex n (otherwise we rename the vertices). Let us add to G all missing edges incident from this vertex n and call the resulting graph G'. Now we have

$$\mathsf{L}(G') = \begin{pmatrix} \mathsf{L}(G_1) + \mathsf{I} & -\mathbf{1} \\ -\mathbf{1}^T & n-1 \end{pmatrix}.$$

Let $\boldsymbol{u}_2$ be an eigenvector corresponding to the algebraic connectivity of G_1. Because of the orthogonality of the eigenvectors $\mathbf{1}$ (for eigenvalue $\lambda_1(G_1)$) and $\boldsymbol{u}_2$ (for eigenvalue $\lambda_2(G_1)$), we have

$$\mathsf{L}(G') \begin{pmatrix} \boldsymbol{u}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathsf{L}(G_1)\boldsymbol{u}_2 + \boldsymbol{u}_2 \\ -\mathbf{1}^T \boldsymbol{u}_2 \end{pmatrix} = (\lambda_2(G_1) + 1) \begin{pmatrix} \boldsymbol{u}_2 \\ 0 \end{pmatrix},$$

and $(\lambda_2(G_1) + 1)$ is a non-zero eigenvalue of G'. We can conclude that $\lambda_2(G') \leq \lambda_2(G_1) + 1$. Now, by Theorem 3.4:

$$\lambda_2(G) \leq \lambda_2(G') \leq \lambda_2(G_1) + 1,$$

which completes the proof. □

Fiedler's most important theorem is a direct consequence of this lemma.

Theorem 3.9 (Fiedler, 1973) *The algebraic connectivity of an incomplete graph is not greater than the vertex connectivity:*

$$\lambda_2 \leq \kappa_v \leq \kappa_e \leq \delta_{\min}.$$

Proof The proof of Fiedler [15] is given. For an incomplete graph G let G_{κ_v} be the graph obtained by deleting the κ_v vertices needed to disconnect G (for a complete graph such vertices do not exist). Lemma 3.8 and Corollary 3.7 give $\lambda_2(G) - \kappa_v \leq \lambda_2(G_{\kappa_v}) = 0$, which directly proves the theorem. □

For a complete graph K_n, we have $\lambda_1 = 0$, $\lambda_2 = \cdots \lambda_n = n$, because the vectors $\boldsymbol{u}_j = \boldsymbol{e}_1 - \boldsymbol{e}_j$ for $j \neq 1$, are $n-1$ linearly independent eigenvectors corresponding to the eigenvalue n. As a consequence, the last theorem does not hold for a complete graph K_n, its algebraic connectivity is n, while the vertex connectivity is defined to be $n-1$. The algebraic connectivity of the example graphs of Section 2.6 are

$$\lambda_2(K_4) = 4, \qquad \lambda_2(C_4) = 2, \qquad \lambda_2(S_4) = 1, \qquad \lambda_2(P_4) \approx 0.59, \qquad \lambda_2(O_4) = 0.$$

Jamaković describes in [19] the relation between algebraic connectivity and graph robustness. She states that 'the algebraic connectivity increases with the increasing node and the link connectivity. This means that the larger the algebraic connectivity, the larger the number of node- or link-disjoint paths. The algebraic connectivity measures the extent to which it is difficult to cut the network into independent components and is therefore a quantifier of the robustness in complex networks.' A lot of research is being done on the algebraic connectivity and many researchers believe in its importance for network connectivity. Bollobás even writes in [6]: 'The second smallest eigenvalue of the Laplacian is far from trivial: in fact, it is difficult to overemphasize its importance.'

Nevertheless, some criticisms can be found in the literature. Baras and Hovareshti [3] remark that the algebraic connectivity is not always strictly increasing when an edge is added. Figure 3.1 shows their example. Another problem of the algebraic connectivity is that the arguments for its proposal as a measure for network robustness are rather weak, since they are only based on the fact that it is bounded above by the vertex connectivity (Theorem 3.9).

(a) $\boldsymbol{\lambda} = (0, 2, 2, 4)$ (b) $\boldsymbol{\lambda} = (0, 2, 4, 4)$

Figure 3.1. Two graphs with identical algebraic connectivity

3.3 Other spectral measures

Several authors argue to use measures based on all Laplacian eigenvalues instead of using the second smallest eigenvalue only. Considering the problem of the strict increasingness, we show that it is not enough to consider the first k (with k a fixed number) Laplacian eigenvalues, because the star graph (Figure 3.2) gives an example of a graph where the first $n-2$ eigenvalues stay equal when adding an edge.

(a) $\boldsymbol{\lambda} = 0, 1, 1, 1, 1, 1, 7$ (b) $\boldsymbol{\lambda} = 0, 1, 1, 1, 1, 3, 7$

Figure 3.2. Only the second last Laplacian eigenvalue changes when an edge is added to a star graph.

Theorem 3.10 *Adding an edge does not necessarily affect the first $n-2$ Laplacian eigenvalues, because:*

1. *The star graph S_n with n vertices has Laplacian spectrum $\boldsymbol{\lambda} = (0, 1, \ldots, 1, 1, n)$;*
2. *Adding an edge gives the graph S_n^* with Laplacian spectrum $\boldsymbol{\lambda} = (0, 1, \ldots, 1, 3, n)$.*

Proof

1. We consider the complement of S_n, it consists of a clique of size $n-1$ and an isolated vertex and hence has spectrum $\boldsymbol{\lambda} = (0, 0, n-1, \ldots, n-1)$. We have used Theorem 3.1, Theorem 3.6 and the spectrum of the complete graph determined in the discussion following Theorem 3.9. Now, using Theorem 3.5 gives the desired result.
2. The complement of S_n^* consists a clique of size $n-1$ missing an edge and an isolated vertex. The complement of the clique minus the edge has one edge and $n-3$ isolated vertices; its spectrum is $(0, \ldots, 0, 2)$. The clique minus the edge thus has spectrum $(0, n-3, n-1, \ldots, n-1)$. It follows that the spectrum of the complement of S_n^* is $(0, 0, n-3, n-1, \ldots, n-1)$ and that of S_n^* is $(0, 1, \ldots, 1, 3, n)$. □

Baras and Hovareshti suggest the number of spanning trees (a *spanning tree* is a subgraph containing $n-1$ edges and no cycles) as an indicator of network robustness [3]. It is a consequence of Kirchhoff's matrix-tree theorem that the number of spanning trees can be written as a function of the unweighted Laplacian eigenvalues.

Theorem 3.11 *The number of spanning trees $\xi(G)$ in a graph G with Laplacian L and Laplacian eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is*

$$\xi(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i.$$

Proof We work out the proof outline of [37]. The matrix-tree theorem of Kirchhoff states that all cofactors of the Laplacian of a graph G are equal to the number of spanning trees of G. The characteristic polynomial $\det(\mathsf{L} - x\mathsf{I}) = c_0 + c_1 x + c_2 x^2 \cdots + c_{n-1} x^{n-1} + (-x)^n$ of the Laplacian satisfies

$$\begin{aligned}
\det(\mathsf{L} - x\mathsf{I}) &= c_0 + c_1 x + c_2 x^2 \cdots + c_{n-1} x^{n-1} + (-x)^n \\
&= (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x) \\
&= \det \begin{pmatrix} \delta_1 - x & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \delta_2 - x & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \delta_n - x \end{pmatrix},
\end{aligned}$$

with λ_i the $i-th$ Laplacian eigenvalue. Using this and the fact that $\lambda_1 = 0$ we find two equivalent expressions for the second coefficient of the characteristic polynomial

$$\begin{aligned}
c_1 &= -\lambda_2 \cdots \lambda_n \\
&= -\det \begin{pmatrix} \delta_2 & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & \delta_3 & \cdots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n2} & -a_{n3} & \cdots & \delta_n \end{pmatrix} - \det \begin{pmatrix} \delta_1 & -a_{13} & \cdots & -a_{1n} \\ -a_{31} & \delta_3 & \cdots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \cdots & \delta_n \end{pmatrix} - \cdots \\
&\quad - \det \begin{pmatrix} \delta_1 & -a_{12} & \cdots & -a_{1\,n-1} \\ -a_{21} & \delta_2 & \cdots & -a_{2\,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1\,1} & -a_{n-1\,2} & \cdots & \delta_{n-1} \end{pmatrix}.
\end{aligned}$$

All n determinants in the last expression are cofactors of L, thus $c_1 = -\lambda_2 \cdots \lambda_n$ equals $-n$ times $\xi(G)$, which completes the proof. □

According to the next theorem [10] the number of spanning trees gives the same judgment about the robustness of a network as the reliability polynomial gives when p goes to zero.

Theorem 3.12 *The reliability polynomial of a graph G satisfies*

$$\mathrm{Rel}(G) = \xi(G)p^{n-1} + o\left(p^{n-1}\right),$$

for $p \to 0$.

Proof The next proof is our own. Let F_i be defined as in Section 2.5. Since connected graphs have at least $n-1$ edges we have $F_i = 0$ for $i > m-n+1$ and since a connected graph with $n-1$ edges is a tree we have $F_{m-n+1} = \xi(G)$. As a consequence the following relation holds.

$$\begin{aligned}\mathrm{Rel}(G) &= \sum_{i=0}^{m} F_i(1-p)^i p^{m-i} \\ &= \sum_{i=0}^{m-n} \left(F_i(1-p)^i p^{m-i}\right) + \xi(G)(1-p)^{m-n+1}p^{n-1} = \xi(G)p^{n-1} + o\left(p^{n-1}\right).\end{aligned}$$

□

Since in real-world networks failures are scarce, it is reasonable to assume p to be close to one. As the number of spanning trees corresponds to the reliability polynomial for 'p close to zero', it does not seem to be a good robustness measure.

For the sake of completeness we mention that Tizghadam and Leon-Garcia have proposed a robustness measure [33]— called *network criticality* — based on the *random walk betweenness*, which they define as the number of visits to a vertex k in a random walk starting in an arbitrary vertex i and ending in an arbitrary vertex j. We further explore this measure in Section 4.4 and show that the network criticality is also a function of the Laplacian eigenvalues.

Chapter 4

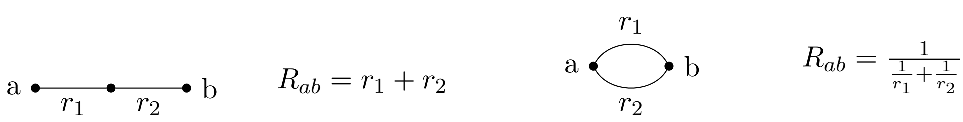
Effective resistance

In this chapter, we argue that the normalized effective resistance is a good measure for network robustness. The normalized total effective resistance is proportional to the inverse total effective resistance, which is defined as the sum of the pairwise effective resistances over all pairs of vertices. The choice of this measure is inspired by the excellent paper of Klein and Randić [21]. The main contribution of their paper is the proof that the total effective resistance can be written in terms of the Laplacian eigenvalues.

We start — in Section 4.1 — with an informal discussion of the pairwise effective resistance and the relation with network robustness. Section 4.2 gives a formal definition of the pairwise and total effective resistance and concentrates on the expressions in terms of the Laplacian. In Section 4.3 some additional properties of the (total) effective resistance are explored. The analysis of random walks on graphs in Section 4.4 allows us to derive more expressions for the (total) effective resistance. This chapter is concluded by a section containing some examples giving an idea of the values the effective resistance can obtain, Section 4.5, and a discussion on the normalized effective resistance and network robustness, Section 4.6.

4.1 Pairwise effective resistance and network robustness

To determine the effective resistance, the (simple, undirected and connected) graph is seen as an electrical circuit, where a edge (i, j) corresponds to a resistor of $r_{ij} = 1$ Ohm. For each pair of vertices the pairwise *effective resistance* between these vertices — the resistance of the total system when a voltage source is connected across them — can be calculated. The calculation of the effective resistance between two vertices of an electrical circuit can easily be done by the well-known series and parallel manipulations. Two resistors with resistances r_1, r_2 in series can be replaced by one resistor with resistance $r_1 + r_2$. If the two resistors are connected in parallel, then they can be replaced by a resistor with resistance $\left(r_1^{-1} + r_2^{-1}\right)^{-1}$. The method is illustrated in Figure 4.1.



(a) Resistors in series (b) Resistors in parallel

Figure 4.1. Calculating the effective resistance between a and b

In Figure 4.2 we have calculated the effective resistance between two vertices a and b in a few small networks. We see that every new path diminishes the effective resistance, but short paths cause a larger decrease than long paths, thus both the number of paths between two vertices and their lengths (number of edges in the path) are taken into account. Having n parallel paths of length l leads to an effective resistance of l/n. It follows that adding the first extra edge has a big impact (it reduces the effective resistance by a factor two), but the more edges we add, the smaller the difference between the old and the new effective resistance. This agrees with our intuition that having a few backup paths improves the connectivity, but adding more backup paths when there already are many, does not lead to any significant further improvement.

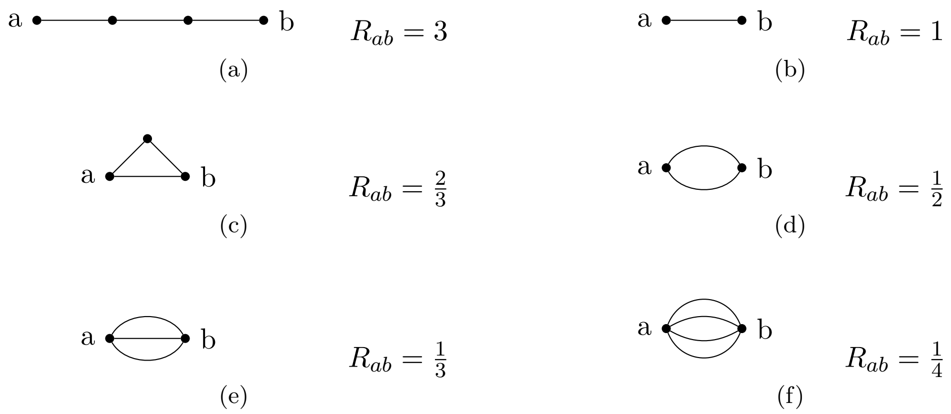


Figure 4.2. Some networks ordered by increasing robustness of the connection between a and b, and the effective resistance between a and b

When the edge weights represent a distance, then the corresponding resistor has to be assigned a resistance of the same value, in order for short paths to still cause a smaller effective resistance than long paths. When the edge weights represent their conductance, then the resistance of each resistor is given by the inverse edge weight, such that edges in parallel increase the effective conductance, while edges in series decrease it. The effective resistance has been proposed to measure the distance in social networks too, that is, the difficulty a message — sent by a particular person in the network — has in reaching another person [21]. In this case the edge weights correspond to the strength of a relation, the stronger this relation the easier the communication, thus the resistances again need to be equal to the inverse edge weight.

4.2 Calculating the effective resistance by the Laplacian

The series and parallel manipulations mentioned in the previous section follow from two important laws of electrical circuit analysis, Kirchhoff's circuit laws — which is indeed the Kirchhoff from the matrix-tree theorem. Let a voltage source be connected between vertices a and b and let $Y > 0$ be the net current out of *source* a and into *sink* b. Then *Kirchhoff's current law* states that the *current* y_{ij} between vertices i and j (where $y_{ij} = -y_{ji}$) must satisfy

$$\sum_{j \in N(i)} y_{ij} = \begin{cases} Y & \text{if } i = a \\ -Y & \text{if } i = b \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

with $N(i)$ the neighborhood of i, that is, the set of vertices adjacent to vertex i. This first law expresses that the total flow into a vertex equals the total flow out of it. The second law, *Kirchhoff's circuital law*, says that for every cycle C in the network

$$\sum_{(i,j) \in C} y_{ij} r_{ij} = 0,$$

where the edges (i, j) are ordered around the cycle. This law is equivalent to saying that to any vertex i there may be associated a *potential* v_i, such that for all edges (i, j)

$$y_{ij} r_{ij} = v_i - v_j, \tag{4.2}$$

which is called *Ohm's law*. Note that y and v depend on the net current Y and on the pair (a, b) which specifies the position of the voltage source.

Definition 4.1 The *effective resistance* is defined as

$$R_{ab} = \frac{v_a - v_b}{Y}.$$

In the next theorem we see by construction that given a (simple, undirected and connected) graph G with edge resistances r_{ij} and given a source-sink pair (a, b), the effective resistance R_{ab} exists and is uniquely defined. It is shown that R_{ab} does not depend on the net current Y. The proof of the theorem shows that given the graph, edge resistances and net current, there always exist currents y and potentials v satisfying Kirchhoff's law (4.1) and Ohm's law (4.2), the vector of potentials $\boldsymbol{v}$ is uniquely defined up to a constant vector by these equations and the currents y are unique.

When r_{ij} is defined as $1/w_{ij}$, then the effective resistance can be calculated by making use of a pseudoinverse of the (weighted) Laplacian. The Laplacian cannot be inverted directly, because it has a zero discriminant, since one of the eigenvalues is equal to zero. However, if we restrict the linear transformation to the subspace orthogonal to the null space $\mathrm{sp}\{\mathbf{1}\}$, the matrix can be inverted, because the image of $(\mathrm{sp}\{\mathbf{1}\})^\perp$ (the subspace perpendicular to $\mathrm{sp}\{\mathbf{1}\}$) is the subspace $(\mathrm{sp}\{\mathbf{1}\})^\perp$ itself. This is a consequence of the fact that symmetric matrices have an orthogonal basis of eigenvectors. Let $\left(\mathsf{L}^W\right)^+$ be the matrix that on $(\mathrm{sp}\{\mathbf{1}\})^\perp$ corresponds to this inverse and on $\mathrm{sp}\{\mathbf{1}\}$ to the zero map. In other words, we have:

Definition 4.2 The *Laplacian pseudoinverse* $\left(\mathsf{L}^W\right)^+$ is defined as the unique matrix satisfying

$$\left(\mathsf{L}^W\right)^+ \mathbf{1} = \mathbf{0}$$

and for every $\boldsymbol{w} \perp \mathbf{1}$

$$\left(\mathsf{L}^W\right)^+ \boldsymbol{w} = \boldsymbol{v} \quad \text{such that } \mathsf{L}^W \boldsymbol{v} = \boldsymbol{w} \text{ and } \boldsymbol{v} \perp \mathbf{1}$$

The definition above is a specific case of the Moore-Penrose pseudoinverse for general $m \times n$-matrices (see for example [31]). It is not known who first proved the following important theorem, but the result has been known at least since the sixties of last century, because it appears in [29].

Theorem 4.1 *When the resistance of edge (i, j) is defined as the inverse edge weight (i.e. edge weights refer to conductances), then the effective resistance R_{ab} between vertices a and b satisfies*

$$R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^T \left(\mathsf{L}^W\right)^+ (\boldsymbol{e}_a - \boldsymbol{e}_b) = (\mathsf{L}^W)^+_{aa} - 2(\mathsf{L}^W)^+_{ab} + (\mathsf{L}^W)^+_{bb}.$$

Proof We give the proof that can be found in [21]. Substituting equation (4.2) into equation (4.1) gives for all vertices i

$$\sum_{j \in N(i)} w_{ij} (v_i - v_j) = \sum_{j \in N(i)} \frac{v_i - v_j}{r_{ij}} = \begin{cases} Y & \text{if } i = a \\ -Y & \text{if } i = b \\ 0 & \text{otherwise,} \end{cases}$$

or, equivalently,

$$s_i v_i - \sum_{j=1}^{n} w_{ij} v_j = \begin{cases} Y & \text{if } i = a \\ -Y & \text{if } i = b \\ 0 & \text{otherwise.} \end{cases}$$

In vector notation this equation can be written as

$$\mathsf{L}^{\mathsf{W}} \boldsymbol{v} = (\mathsf{S} - \mathsf{W}) \boldsymbol{v} = Y(\boldsymbol{e}_a - \boldsymbol{e}_b). \tag{4.3}$$

Since the vector at the right hand side is perpendicular to $\mathbf{1}$, this equation can be inverted by the pseudoinverse, finding the vector $\boldsymbol{v}$ in $(\text{sp}\{\mathbf{1}\})^{\perp}$ that satisfies it. The set of all solutions to equation (4.3) is

$$\left\{ \boldsymbol{v} = Y \left(\left(\mathsf{L}^W \right)^+ (\boldsymbol{e}_a - \boldsymbol{e}_b) + c\mathbf{1} \right), c \in \mathbb{R} \right\},$$

thus the vector of potentials is uniquely defined up to a constant vector. The potential difference between an arbitrary pair (i, j) is now unique and given by

$$v_i - v_j = (\boldsymbol{e}_i - \boldsymbol{e}_j)^T \boldsymbol{v} = Y (\boldsymbol{e}_i - \boldsymbol{e}_j)^T \left(\mathsf{L}^W \right)^+ (\boldsymbol{e}_a - \boldsymbol{e}_b).$$

The current y_{ij} for an edge (i, j) is uniquely defined as $v_i - v_j$ and the effective resistance between (a, b) is unique and satisfies

$$R_{ab} = \frac{v_a - v_b}{Y} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^T \left(\mathsf{L}^W \right)^+ (\boldsymbol{e}_a - \boldsymbol{e}_b) = (\mathsf{L}^W)^+_{aa} - 2(\mathsf{L}^W)^+_{ab} + (\mathsf{L}^W)^+_{bb},$$

which does not depend on the net current Y. The last step follows from the symmetry of $(\mathsf{L}^W)^+$, which is a consequence of the fact that $(\mathsf{L}^W)^+$ is orthogonally diagonizable, i.e. there exist an orthogonal matrix U (with $\mathsf{U}^T = \mathsf{U}^{-1}$) and a diagonal matrix D such that $(\mathsf{L}^W)^+ = \mathsf{U}\mathsf{D}\mathsf{U}^T$. □

Now we have seen that the effective resistance is well defined, we can give a definition of the total effective resistance.

Definition 4.3 The *total effective resistance* R^{tot} is the sum of the effective resistances over all pairs of vertices:

$$R^{\text{tot}} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} R_{ij}.$$

In the literature the total effective resistance is also called Kirchhoff index. It can be written as a function of the non-zero Laplacian eigenvalues. This is a result of Klein and Randić ([21]).

Theorem 4.2 (Klein and Randić, 1993) *The total effective resistance R^{tot} satisfies*

$$R^{\text{tot}} = n \sum_{i=2}^{n} \frac{1}{\lambda_i^W}.$$

Proof We reproduce the original proof. Using that $R_{ii} = 0$ and $R_{ij} = R_{ji}$ (which follow from 4.1), the total effective resistance can be written as

$$\begin{aligned}
R^{\text{tot}} &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} R_{ij} \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} R_{ij} \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left((\mathsf{L}^W)^+_{ii} - 2(\mathsf{L}^W)^+_{ij} + (\mathsf{L}^W)^+_{jj} \right) \\
&= n \sum_{i=1}^{n} (\mathsf{L}^W)^+_{ii} - \mathbf{1}^T (\mathsf{L}^W)^+ \mathbf{1} \\
&= n \operatorname{tr} \left((\mathsf{L}^W)^+ \right),
\end{aligned}$$

The last equality follows from the fact that $(\mathsf{L}^W)^+$ on $\text{sp}\{\mathbf{1}\}$ corresponds to the zero map. Since the Laplacian is symmetric, it has an orthonormal basis of eigenvectors. Let U be the matrix that has these eigenvectors as its columns (the i-th column being the eigenvector corresponding to eigenvalue λ_i^W) and let D be given by

$$\mathsf{D} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^W & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^W \end{pmatrix},$$

then the Laplacian satisfies $\mathsf{L}^W = \mathsf{U}\mathsf{D}\mathsf{U}^{-1} = \mathsf{U}\mathsf{D}\mathsf{U}^T$. In other words L^W is given by D when all vectors are written with respect to the orthogonal basis of eigenvectors. The pseudoinverse of L^W with respect to this basis is

$$\mathsf{D}^+ = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \left(\lambda_2^W\right)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\lambda_n^W\right)^{-1} \end{pmatrix}$$

and satisfies $(\mathsf{L}^W)^+ = \mathsf{U}\mathsf{D}^+\mathsf{U}^{-1} = \mathsf{U}\mathsf{D}^+\mathsf{U}^T$. Because similar matrices have the same eigenvalues, the eigenvalues of $(\mathsf{L}^W)^+$ are $0, \left(\lambda_2^W\right)^{-1}, \ldots, \left(\lambda_n^W\right)^{-1}$ and we can conclude that

$$R^{\text{tot}} = n \operatorname{tr} \left(\left(\mathsf{L}^W\right)^+ \right) = n \operatorname{tr} \left(D^+ \right) = n \sum_{i=2}^{n} \frac{1}{\lambda_i^W},$$

because similar matrices also have the same trace. □

The former theorem allows us to derive bounds on R^{tot} in terms of λ_2^W.

Corollary 4.3 *The total effective resistance R^{tot} can be bounded by functions of λ_2^W in the following way*

$$\frac{n}{\lambda_2^W} \leq R^{\text{tot}} \leq \frac{n(n-1)}{\lambda_2^W}.$$

With this bounds we conclude this section on the relation between effective resistance and the Laplacian. In the next section we discover some properties of the effective resistance.

4.3 Properties of the effective resistance

The effective resistance has been called resistance distance by Klein and Randić [21]. It is not difficult to see that the effective resistance R is indeed a distance function.

Theorem 4.4 (Klein and Randić, 1993) *The effective resistance R is a metric (distance function).*

Proof The proof is of Klein and Randić [21].

1. We use the result of Theorem 4.1. We have $R_{ij} = 0$ if and only if $i = j$, because $\boldsymbol{e}_i - \boldsymbol{e}_j$ is in the null space of $\left(\mathsf{L}^W\right)^+$ — which corresponds to sp$\{\mathbf{1}\}$ — if and only if $i = j$. The fact that the pseudoinverse of the Laplacian has eigenvalues $0, \left(\lambda_2^W\right)^{-1}, \ldots, \left(\lambda_n^W\right)^{-1}$ (see the proof of Theorem 4.2 for more explanation) and thus is positive semidefinite, leads to $R_{ij} \geq 0$.

2. Since the pseudoinverse is symmetric, we have $R_{ij} = R_{ji}$.

3. We show that the triangle inequality holds as well. Consider first a situation where a voltage source is connected between vertices a and b. Suppose the current y^{ab} and the potential v^{ab} satisfy Kirchhoff's circuital law (4.1) and Ohm's law (4.2). Then consider the situation in which vertex b is the source and vertex c the sink. Assume that the current y^{bc} and the potential v^{bc} satisfy (4.1) and (4.2) in this situation. Let the net current into the network be Y in both cases. Now, define a current $y^{ac} = y^{ab} + y^{bc}$ and a potential $v^{ac} = v^{ab} + v^{bc}$. It is clear that (4.1) and (4.2) hold for the current y^{ac} and the potential v^{ac} in the case that a voltage source is connected between a and c, with net current Y.

   We have

   $$YR_{ac} = v_a^{ac} - v_c^{ac} = (v_a^{ab} - v_c^{ab}) + (v_a^{bc} - v_c^{bc}) \leq (v_a^{ab} - v_b^{ab}) + (v_b^{bc} - v_c^{bc}) = YR_{ab} + YR_{bc},$$

   where the inequalities $v_c^{ab} \geq v_b^{ab}$ and $v_a^{bc} \leq v_b^{bc}$ are a consequence of the following. We will show that for a source a and a potential v it holds that $v_a \geq v_i$ for all vertices i. It follows from Ohm's law that the vertex with the largest potential can only have outgoing currents (we say vertex v has an outgoing current to vertex w if the current y_{vw} is positive). Now it is clear that no vertex can have a higher potential than the source, because there would be a net current out such a vertex (by the connectivity of the network), which is only possible for the source. Likewise for a sink b we have for the potential $v_b \leq v_i$ for all i.

   The triangle inequality is now a direct consequence:

   $$R_{ac} \leq R_{ab} + R_{bc}.$$

□

In the rest of this section we give some more (probably previously known) results, which are described by Klein and Randić [21]. Proof were not provided in [21], so we have provided them ourselves.

Theorem 4.5 *The effective resistance R is a non-increasing function of the edge weights.*

Proof Since by Weyl's theorem (Theorem 3.3) the Laplacian eigenvalues do not decrease when edge weights are increased, the expression

$$\sum_{i=2}^{n} \frac{\boldsymbol{x}_i^2}{\lambda_i^W}$$

is non-increasing in the edge weights for any $\boldsymbol{x} \in \mathbb{R}^n$. By choosing $\boldsymbol{x} = \mathsf{U}^T(\boldsymbol{e}_a - \boldsymbol{e}_b)$, with U the orthogonal matrix of eigenvectors corresponding to the Laplacian eigenvalues, we can now conclude that

$$\begin{aligned}
R_{ab} &= (\boldsymbol{e}_a - \boldsymbol{e}_b)^T \left(\mathsf{L}^W\right)^+ (\boldsymbol{e}_a - \boldsymbol{e}_b) \\
&= (\boldsymbol{e}_a - \boldsymbol{e}_b)^T \mathsf{U} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \left(\lambda_2^W\right)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\lambda_n^W\right)^{-1} \end{pmatrix} \mathsf{U}^T (\boldsymbol{e}_a - \boldsymbol{e}_b) \\
&= \boldsymbol{x}^T \begin{pmatrix} 0 & 0 \cdots & 0 & \\ 0 & \left(\lambda_2^W\right)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\lambda_n^W\right)^{-1} \end{pmatrix} \boldsymbol{x} \\
&= \sum_{i=2}^{n} \frac{\boldsymbol{x}_i^2}{\lambda_i^W}
\end{aligned}$$

is non-increasing in the edge weights. □

The following corollary follows directly from the last theorem.

Corollary 4.6 *The effective resistance R does not increase when edges are added.*

Before we continue, we state and prove a useful lemma, which is our own formalization of the well-known rule for resistors in parallel.

Lemma 4.7 *Suppose a graph G that does not contain the edge (a, b) is given, and suppose the effective resistance between a and b is R_{ab}. Adding edge (a, b) with resistance r_{ab} gives a effective resistance of*

$$R'_{ab} = \frac{1}{\frac{1}{r_{ab}} + \frac{1}{R_{ab}}}$$

in the new graph G'.

Proof Let the net current Y, potentials v and currents y be given such that they satisfy Kirchhoff's law (4.1) and Ohm's law (4.2) for the original graph G with source a and sink b. Now, for graph G' with source a and sink b define $y_{ab} = (v_a - v_b)/r_{ab}$, $Y' = y_{ab} + Y$ and let the other potentials and currents be as before, then the net current Y', potentials v and currents y satisfy Kirchhoff's and Ohm's laws. The effective resistance R'_{ab} in G' is

$$R'_{ab} = \frac{v_a - v_b}{Y'} = \frac{v_a - v_b}{y_{ab} + Y} = \frac{v_a - v_b}{\frac{v_a - v_b}{r_{ab}} + \frac{v_a - v_b}{R_{ab}}} = \frac{1}{\frac{1}{r_{ab}} + \frac{1}{R_{ab}}}.$$

□

The next theorem was not described by Klein and Randić [21], but is an interesting consequence of Theorem 4.5.

Theorem 4.8 *The total effective resistance strictly decreases when edges are added or weights are increased.*

Proof Suppose edge weight w_{ij} is increased or edge (i, j) is added. It is enough to show that R_{ij} strictly decreases, since effective resistances between other pairs do not increase because of Theorem 4.5. The strict decreasingness of R_{ij} is a direct consequence of Lemma 4.7. □

Theorem 4.9 *For the effective resistance R and the ordinary (shortest-path) distance d (where the length of an edge (i,j) corresponds to the edge resistance r_{ij}) we have for a pair of vertices a, b:*

1. *if there is only one path between a and b, then $R_{ab} = d_{ab}$,*
2. *otherwise, $R_{ab} < d_{ab}$.*

Proof

1. If there is only one path (say P) between a and b, then $y_{ij} = Y$ for all edges (i,j) on this path (due to (4.1)) and $v_i - v_j = Y r_{ij}$. It follows that

$$R_{ab} = \frac{v_a - v_b}{Y} = \sum_{(i,j)\in P} \frac{v_i - v_j}{Y} = \sum_{(i,j)\in P} r_{ij} = d_{ab}.$$

2. Suppose a path between a and b is added, we show that now $R_{ab} < d_{ab}$. The added path may use a part of P, call this part P_1. Consider the rest of P and the rest of the new path, call the shortest of these P_2 and the other P_3 (see Figure 4.3). Now, the current through each edge of P_1 is Y and the current through P_2 and P_3 is equal for every edge of the same path, say Y_2 and Y_3 respectively. Because of (4.1) and (4.2) we have that $Y_2 + Y_3 = Y$ and $Y_2, Y_3 > 0$, which gives $Y_2 < Y$. Finally, we find

$$\begin{aligned} R_{ab} = \frac{v_a - v_b}{Y} &= \sum_{(i,j)\in P_1} \frac{v_i - v_j}{Y} + \sum_{(i,j)\in P_2} \frac{v_i - v_j}{Y} \\ &< \sum_{(i,j)\in P_1} \frac{v_i - v_j}{Y} + \sum_{(i,j)\in P_2} \frac{v_i - v_j}{Y_2} = \sum_{(i,j)\in P_1} r_{ij} + \sum_{(i,j)\in P_2} r_{ij} = d_{ab}. \quad (4.4) \end{aligned}$$

The rest follows from Corollary 4.6.

□

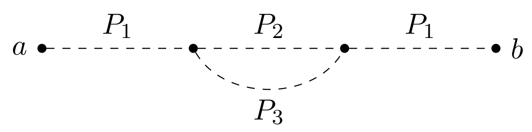


Figure 4.3. The effective resistance R_{ab} is smaller than the distance d_{ab}, when there is more than one path between a and b.

In a tree there is a unique path between every pair of vertices. Therefore the following corollary holds.

Corollary 4.10 *The effective resistance and the ordinary distance correspond on a tree, that is, in a tree we have for every pair of vertices i, j:*

$$R_{ij} = d_{ij}.$$

4.4 A random walk analogy

Let a random walk on the undirected graph with edge weights $w_{ji} = w_{ij} = 1/r_{ij}$ be given by the transition probabilities $p_{ij} = w_{ij}/s_i$. The *stationary probability* of vertex i — that is, the probability the random walk is in i in the long run — is $\pi_i = s_i / \sum_{i=1}^{n} s_i$, because this π satisfies the global balance equations $\pi_j = \sum_{i=1}^{n} \pi_i p_{ij}$. We consider the expected *commute time* between two vertices a and b in this random walk. This is the expected number of transitions needed to go from a to b and back. A theorem from Chandra et al. [8] gives a relation between the average commute time and the effective resistance in the same graph. Before we come to this theorem, we state a lemma.

Lemma 4.11 *Let a graph G with edge weights w_{ij} be given. Define a random walk on G by the transition probabilities $p_{ij} = w_{ij}/s_i$. Let T_{ab} be the* hitting time *from vertex a to vertex b (number of transitions to reach vertex b starting in a). The following relation holds*

$$\mathbf{P}\left(T_{ab} < T_{aa}^{+}\right) = \frac{1}{\pi_a \Big(\mathbf{E}\left(T_{ab}\right) + \mathbf{E}\left(T_{ba}\right)\Big)} \quad \textit{for } a \neq b, \tag{4.5}$$

where T_{aa}^{+} denotes the first return time *(number of transition needed to return) to a.*

Proof We give the proof of Aldous and Fill [2]. The renewal theorem (see e.g. [28]) with cycle length S gives

$$\pi_a = \frac{\mathbf{E}(\text{time in } a \text{ during one cycle})}{\mathbf{E}(S)}. \tag{4.6}$$

If we take the cycle length to be 'the time of the first return to b after visiting a', equation (4.6) becomes

$$\pi_a = \frac{\mathbf{E}(B_{baa}) + \mathbf{E}(B_{aab})}{\mathbf{E}(T_{ab}) + \mathbf{E}(T_{ba})} = \frac{\mathbf{E}(B_{aab})}{\mathbf{E}(T_{ab}) + \mathbf{E}(T_{ba})}, \tag{4.7}$$

with B_{avb} the number of visits to vertex v in between the start of the random walk in a and the stop in b, including vertex a but excluding vertex b.

For the random walk starting in a, the number of visits B_{aab} to a (including the start in a) before arriving in b has a geometric distribution with probability of success $p = \mathbf{P}(T_{aa}^{+} < T_{ab})$. The expectation of this geometric distribution is

$$\mathbf{E}(B_{aab}) = \frac{1}{p} = \frac{1}{\mathbf{P}(T_{aa}^{+} < T_{ab})}.$$

Together with (4.7) this gives the desired result. □

Theorem 4.12 (Chandra et al., 1989) *Let a graph G with edge weights w_{ij} be given. First, define an electrical circuit by setting $r_{ij} = 1/w_{ij}$. Second, define a random walk on G by the transition probabilities $p_{ij} = w_{ij}/s_i$. It holds that*

$$R_{ab} = \frac{1}{\sum_{i=1}^{n} s_i}\left(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{ba})\right) \quad \textit{for all } a, b \in V.$$

Proof The proof given is inspired by Chapter 3, Section 3 of the book in preparation by Aldous and Fill [2]. The theorem is clearly true for $a = b$. Suppose now that $a \neq b$. It follows from Lemma 4.11 that

$$\frac{1}{\sum_{i=1}^{n} s_i}\left(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{ba})\right) = \frac{1}{s_a \mathbf{P}(T_{ab} < T_{aa}^{+})},$$

it thus suffices to show that

$$R_{ab} = \frac{1}{s_a \mathbf{P}(T_{ab} < T_{aa}^+)}.$$

Let $v_i = \mathbf{P}(T_{ia} < T_{ib})$, $y_{ij} = w_{ij}(v_i - v_j)$ and $Y = s_a \mathbf{P}(T_{ab} < T_{aa}^+)$. We prove that v, y and Y satisfy Kirchhoff's current law (equation (4.1)). Ohm's law (equation (4.2)) has clearly been fulfilled. We get the following three equations:

$$\begin{aligned}
\sum_{j \in N(a)} y_{aj} &= \sum_{j \in N(a)} w_{aj}(v_a - v_j) = \sum_{j \in N(a)} w_{aj} \Big(\mathbf{P}(T_{aa} < T_{ab}) - \mathbf{P}(T_{ja} < T_{jb}) \Big) \\
&= \sum_{j \in N(a)} w_{aj} \mathbf{P}(T_{jb} < T_{ja}) = s_a \sum_{j \in V} p_{aj} \mathbf{P}(T_{jb} < T_{ja}) = s_a \mathbf{P}(T_{ab} < T_{aa}^+) = Y, \\
\sum_{j \in N(i)} y_{ij} &= \sum_{j \in N(i)} w_{ij}(v_i - v_j) = \sum_{j \in N(i)} w_{ij} \Big(\mathbf{P}(T_{ia} < T_{ib}) - \mathbf{P}(T_{ja} < T_{jb}) \Big) \\
&= s_i \mathbf{P}(T_{ia} < T_{ib}) - \sum_{j \in N(i)} w_{ij} \mathbf{P}(T_{ja} < T_{jb}) = s_i \mathbf{P}(T_{ia} < T_{ib}) - s_i \sum_{j \in V} p_{ij} \mathbf{P}(T_{ja} < T_{jb}) \\
&= s_i \mathbf{P}(T_{ia} < T_{ib}) - s_i \mathbf{P}(T_{ia} < T_{ib}) = 0, \\
\sum_{j \in N(b)} y_{bj} &= \sum_{j \in N(b)} w_{bj}(v_b - v_j) = \sum_{j \in N(b)} w_{bj} \Big(\mathbf{P}(T_{ba} < T_{bb}) - \mathbf{P}(T_{ja} < T_{jb}) \Big) \\
&= - \sum_{j \in N(b)} w_{bj} \mathbf{P}(T_{ja} < T_{jb}) = -s_b \sum_{j \in V} p_{bj} \mathbf{P}(T_{ja} < T_{jb}) = -s_b \mathbf{P}(T_{ba} < T_{bb}^+) \\
&= -s_a \mathbf{P}(T_{ab} < T_{aa}^+) = -Y,
\end{aligned}$$

where we have used the law of total probability three times. The second last equality follows from (4.5). As a result we have

$$R_{ab} = \frac{v_a - v_b}{Y} = \frac{\mathbf{P}(T_{aa} < T_{ab}) - \mathbf{P}(T_{ba} < T_{bb})}{Y} = \frac{1}{Y} = \frac{1}{s_a \mathbf{P}(T_{ab} < T_{aa}^+)}.$$

□

The effective resistance is proportional to the expected commute time, which implies that the total effective resistance is proportional to the expected hitting time averaged over all pairs of vertices.

Corollary 4.13 *We have*

$$R^{\text{tot}} = \frac{1}{\sum_{i=1}^n s_i} \sum_{i=1}^n \sum_{j=1}^n \mathbf{E}(T_{ij}).$$

The number of visits to vertex v in a random walk starting in a, going to b, and back to a, is also related to the expected commute time. This relation is given in a theorem that is easy to prove, but has not been found in the literature. For the proof we need the following lemma.

Lemma 4.14 *Let B_{avb} be the number of visits to vertex v in between the start of the random walk in a and the stop in b, including vertex a but excluding vertex b. Then B_{avb} is given by*

$$\mathbf{E}(B_{avb}) = \pi_v \Big(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{bv}) - \mathbf{E}(T_{av}) \Big) \quad \textit{for } a \neq b.$$

Proof The proof can be found in [2]. The case $a \neq v = b$ is trivial and the case $a = v \neq b$ corresponds to equation (4.7). For the case that a, b, v are distinct, let us use the renewal theorem (4.6) with cycle length S equal to 'the time of the first return to a after visiting b and v (in that order)'. It gives

$$\pi_v = \frac{\mathbf{E}(B_{avb}) + \mathbf{E}(B_{bvv}) + \mathbf{E}(B_{vva})}{\mathbf{E}(T_{ab}) + \mathbf{E}(T_{bv}) + \mathbf{E}(T_{va})} = \frac{\mathbf{E}(B_{avb}) + \mathbf{E}(B_{vva})}{\mathbf{E}(T_{ab}) + \mathbf{E}(T_{bv}) + \mathbf{E}(T_{va})}$$

or, equivalently

$$\pi_v \Big(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{bv}) + \mathbf{E}(T_{va}) \Big) = \mathbf{E}(B_{avb}) + \mathbf{E}(B_{vva}) \tag{4.8}$$

Equation (4.7) becomes

$$\pi_v = \frac{\mathbf{E}(B_{vva})}{\mathbf{E}(T_{va}) + \mathbf{E}(T_{av})},$$

when a is substituted by v, and b by a. If we now subtract

$$\pi_v \Big(\mathbf{E}(T_{va}) + \mathbf{E}(T_{av}) \Big) = \mathbf{E}(B_{vva})$$

from (4.8) we find

$$\pi_v \Big(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{bv}) - \mathbf{E}(T_{av}) \Big) = \mathbf{E}(B_{avb}).$$

□

Theorem 4.15 *Let B_{avb} be as before. The expression*

$$\mathbf{E}(B_{avb}) + \mathbf{E}(B_{bva}) = \pi_v \Big(\mathbf{E}(T_{ab}) + \mathbf{E}(T_{ba}) \Big)$$

holds true.

Proof The theorem is clearly true for $a = b$. Suppose now that $a \neq b$. Adding the expressions for $\mathbf{E}(B_{avb})$ and $\mathbf{E}(B_{bva})$ in Lemma 4.14, directly leads to the desired result. □

This theorem gives us an easy alternative way to prove that the network criticality — proposed as a robustness measure by Tizghadam and Leon-Garcia [33] — is equal to two times the total effective resistance. Tizghadam and Leon-Garcia define the *random walk betweenness* B_v of vertex v as

$$B_v = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{E}(B_{ivj})$$

and the *network criticality* as

$$\tau = 2 \frac{B_v}{s_v},$$

which turns out to be independent of the vertex v.

Theorem 4.16 *The network criticality τ satisfies*

$$\tau = 2R^{\text{tot}}.$$

Proof We use Corollary 4.13 and Theorem 4.15 to find

$$\frac{1}{2}\tau = \frac{B_v}{s_v} = \frac{1}{s_v} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \Big(\mathbf{E}(B_{ivj}) + \mathbf{E}(B_{jvi}) \Big) = \frac{1}{s_v} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \pi_v \left(\mathbf{E}(T_{ij}) + \mathbf{E}(T_{ji}) \right)$$

$$= \frac{1}{\sum_{i=1}^{n} s_i} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(\mathbf{E}(T_{ij}) + \mathbf{E}(T_{ji}) \right) = R^{\text{tot}}.$$

□

Summing the relation

$$s_v R^{\text{tot}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{E}(B_{ivj})$$

over all vertices v gives an alternative proof of Corollary 4.13.

The random walk analysis discussed in this section allows us to derive new expressions for the total effective resistance, which lead to a new method for approximating the total effective resistance [14].

4.5 Examples of graphs and their total effective resistance

If the network is not connected, the effective resistance can be calculated between vertices within the same component, but not between vertices in different components, like the ordinary (shortest path) distance. The total effective resistance is said to be equal to infinity for unconnected graphs.

As a consequence of Corollary 4.8, the minimum total effective resistance is reached by the complete graph K_n. By Theorem 4.2 and the eigenvalues of K_n given in the discussion following Theorem 3.9 we have

$$R^{\text{tot}}(K_n) = n - 1.$$

Among the connected graphs, the path graph has maximum total effective resistance. Theorem 4.9 shows that the connected graph with maximum total effective resistance must be the tree with maximum average distance. The path graph P_n clearly has maximum average distance of all trees with n vertices. We have

$$\begin{aligned} R^{\text{tot}}(P_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij} &= (1 + \cdots + n - 1) + (1 + \cdots + n - 2) + \cdots + (1 + 2) + 1 \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} j = \sum_{i=1}^{n-1} \frac{i(i+1)}{2} = \frac{1}{6}(n-1)n(n+1), \end{aligned}$$

where we have used that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ and $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

The tree with minimum total effective resistance, that is with minimum average distance, is the star graph S_n. Its total effective resistance is

$$R^{\text{tot}}(S_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij} = (n-1) \cdot 1 + \frac{1}{2}(n-1)(n-2) \cdot 2 = (n-1)^2.$$

See Figure 2.4 for examples of the complete, path and star graphs mentioned in this section. The values of the total effective resistance for the small examples in this figure (Section 2.6) are

$$R^{\text{tot}}(K_4) = 3, \qquad R^{\text{tot}}(C_4) = 5, \qquad R^{\text{tot}}(S_4) = 9, \qquad R^{\text{tot}}(P_4) = 10, \qquad R^{\text{tot}}(O_4) = \infty.$$

Figure 4.4 shows two different graphs with the same number of vertices and edges, having the same value for the total effective resistance [40]. As none of the graphs is intuitively more robust than the other, it is no problem that the total effective resistance is not able to discriminate these graphs. It is interesting to note that these graphs also have the same reliability polynomial and the same average distance.

(a) Graph G_1 with $R^{\text{tot}}(G_1) = 64$

(b) Graph G_2 with $R^{\text{tot}}(G_2) = 64$

Figure 4.4. Two graphs with the same total effective resistance

Let us carry out a comparison between the total effective resistance, the algebraic connectivity and the reliability polynomial (for 'p close to one') as robustness measures. We show that they capture different graph properties, because for each pair of these measures there is a pair of graphs such that the first measure indicates one graph as the most robust, while the other measure chooses the other graph.

Recall the example of [38] shown in Figure 2.3. The authors have pointed out that the algebraic connectivity and the reliability polynomial for 'p close to one' (or '$q = 1 - p$ close to zero') give different robustness evaluations. The values of the three measures for these two graphs are

$$\begin{aligned} R^{\text{tot}}(G_1) &\approx 21.67 & R^{\text{tot}}(G_2) &= 21.5 \\ \lambda_2(G_1) &\approx 0.63 & \lambda_2(G_2) &\approx 0.59 \\ \text{Rel}(G_1) &= 1 - q + o(q) \text{ for } q \to 0 & \text{Rel}(G_2) &= 1 - 4q^2 + o\left(q^2\right) \text{ for } q \to 0. \end{aligned}$$

The total effective resistance and the reliability polynomial point at the first graph, while the algebraic connectivity indicates that the second graph is more robust.

(a) Graph G_1 with $R^{\text{tot}}(G_1) = 10$ and $\text{Rel}(G_1) = 1 + o(q)$ for $q \to 0$

(b) Graph G_2 with $R^{\text{tot}}(G_2) = 8.5$ and $\text{Rel}(G_2) = 1 - q + o(q)$ for $q \to 0$

Figure 4.5. Two graphs for which the reliability polynomial and the total effective resistance give a different evaluation of robustness

Figure 4.5 shows two graphs for which the total effective resistance and the reliability polynomial give different robustness evaluations; the reliability polynomial would indicate graph G_1 as most robust, but the total effective resistance is lower for G_2.

4.6 Normalized effective resistance

We believe that the inverse total effective resistance is a good measure for network robustness; the smaller the total effective resistance the more robust the network. We therefore define the normalized (inverse total) effective resistance, which we propose as a measure for robustness.

Definition 4.4 The *normalized effective resistance* is defined as

$$R^{\text{norm}} = \frac{n-1}{R^{\text{tot}}} = \frac{n-1}{n\sum_{i=2}^{n}\frac{1}{\lambda_i}} \in [0,1].$$

The advantage of the normalized effective resistance over the total effective resistance is that a large value indicates a robust network. Furthermore the values lie in the interval $[0,1]$. It is minimal (zero) for unconnected graphs and maximal (one) for complete graphs. Therefore a normalized effective resistance close to one indicates a robust network. We now present the arguments for our proposal of the normalized effective resistance as a measure for network robustness.

First, the total effective resistance is the sum of pairwise effective resistances, which in Section 4.1 we have argued to measure the robustness of the connection between two vertices. The key notion is that pairwise effective resistance takes both the number of (not necessarily disjoint) paths between two vertices and their length into account, therefore the normalized effective resistance considers both the number of back-up paths as well as their quality.

Second, it is a consequence of Theorem 4.8 that the normalized effective resistance strictly increases when edges are added or edge weights are increased. Algebraic connectivity for example does not show this strict increasingness. Moreover, for the simple examples in Section 4.5 the normalized effective resistance gives the same evaluation of robustness as does our intuition. Complete graphs are most robust, unconnected graphs least, trees are the least robust connected graphs, star graphs are the most robust trees, and path graphs the least robust trees.

The third reason is the analogy with random walks; the smaller the effective resistance between vertices a and b, the greater the normalized effective resistance and the smaller the expected duration of a random walk from a to b and back (see Theorem 4.12). Short random walks suffer little from vertex or edge failures, and thus indicate a robust network. In addition, the random walk analogy shows that the robustness measure defined in [33] is equal to two times the total effective resistance (Theorem 4.16). Since both measures have been proposed independently and by different reasonings, it gives a strong indication that the normalized effective resistance is indeed a useful robustness measure.

For the normalized effective resistance, the bounds of Corollary 4.3 become

$$\frac{1}{n}\lambda_2^W \leq R^{\text{norm}} \leq \frac{n-1}{n}\lambda_2^W < \lambda_2^W.$$

The algebraic connectivity can thus be used to approximate the robustness of a network.

Chapter 5

Optimization of the normalized effective resistance

In this chapter graphs are optimized in order to maximize the normalized effective resistance. In each section different conditions have been chosen. Section 5.1 treats the maximization of the normalized effective resistance for graphs with a given number of vertices and diameter. In this section we first characterize the class of graphs, wherein the optimal graph must lie. Afterwards we present some results found by exhaustive search on that class of graphs. Section 5.2 considers the question how many eigenvalues are needed in order to find the same optimal graph as for the effective graph resistance. The topic of the last Section 5.3 is the optimal addition of an edge.

5.1 Graphs with a fixed number of vertices and diameter

Definition 5.1 The *clique chain* $G(n_1, n_2, \ldots, n_{d_{\max}}, n_{d_{\max}+1})$ is a graph obtained from the path graph $P_{d_{\max}+1}$ by replacing the i-th vertex by a clique (subset of vertices which are fully interconnected by edges) of size n_i, such that vertices in distinct cliques are adjacent if and only if the corresponding original vertices in the path graph are adjacent.

For examples of such graphs see Figure 5.2, where the graphs $G(1, 2, 2, 1, 1)$ and $G(1, 1, 3, 1, 1)$ have been drawn.

In [34] Van Dam has shown that the class of graphs $G(n_1 = 1, n_2, \ldots, n_{d_{\max}}, n_{d_{\max}+1} = 1)$ with $\sum_{i=1}^{d_{\max}+1} n_i = n$ contains a graph with maximum *spectral radius* (largest eigenvalue of the adjacency matrix) for fixed number of vertices n and a fixed diameter $d_{\max}$. In [39] it has been shown that also graphs with largest algebraic connectivity, maximum number of links and largest average distance, for fixed n and $d_{\max}$, are obtained within this class. It is easy to see that the same class contains graphs with maximum vertex or edge connectivity and average vertex or edge betweenness for fixed n and $d_{\max}$ as well. We show that the same holds for the normalized effective resistance.

The following theorem is the key to the proof of the statements above. It is easy to verify, see [39].

Theorem 5.1 *Any graph with n vertices and diameter $d_{\max}$ is a subgraph of at least one graph in the class $G(n_1 = 1, n_2, \ldots, n_{d_{\max}}, n_{d_{\max}+1} = 1)$ with $\sum_{i=1}^{d_{\max}+1} n_i = n$.*

Using this theorem and Theorem 4.8 we find the next corollary.

Corollary 5.2 *The maximum normalized effective resistance for fixed n and $d_{\max}$ is equal to the maximum normalized effective resistance achieved within the class of the graphs $G(n_1 = 1, n_2, \ldots, n_{d_{\max}}, n_{d_{\max}+1} = 1)$ with $\sum_{i=1}^{d_{\max}+1} n_i = n$.*

In [14] we have derived a general formula for the total effective resistance of clique chains. This formula allowed us to analytically compute optimal graphs for a given number of vertices and given diameter $d_{\max} \leq 3$. For larger diameters we have written a Matlab program (see Section B.1) to find graphs that maximize the normalized effective resistance for some fixed values of n and $d_{\max}$. The goal of the optimization is to be able to design robust networks when the size of the network (in terms of number of vertices and diameter) is given. The results — which can be found in Table A.1 and Figure 5.1 — are obtained by exhaustive search on the class of clique chains. The same results for the algebraic connectivity are given in [39].

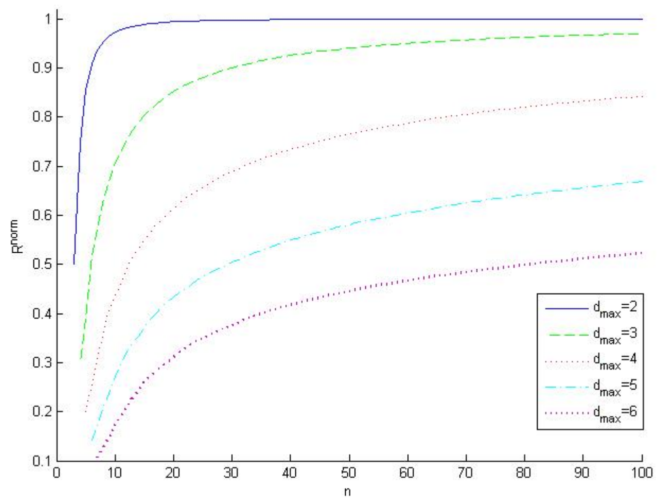
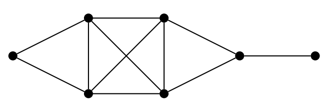
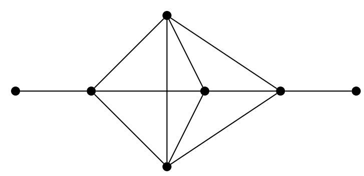


Figure 5.1. The optimal value of the normalized effective resistance for some values of n and $d_{\max}$

We see that for the algebraic connectivity and the normalized effective resistance there exist different optimal graphs. For example for $n = 7$ and $d_{\max} = 4$ the graph with cliques of sizes $(1, 2, 2, 1, 1)$ maximizes the normalized effective resistance, while the graph with clique sizes $(1, 1, 3, 1, 1)$ maximizes the algebraic connectivity. See Figure 5.2.



(a) Graph with $\lambda_2 \approx 0.68, R^{\text{norm}} \approx 0.30$



(b) Graph with $\lambda_2 \approx 0.70, R^{\text{norm}} \approx 0.28$

Figure 5.2. For $n = 7$ and $d_{\max} = 4$, λ_2 and R^{norm} give different optimal graphs.

For both the algebraic connectivity and the normalized effective resistance the optimum is generally achieved for clique graphs with a symmetric sequence of clique sizes. Surprisingly there are a few counterexamples. Regarding the normalized effective resistance, for $n = 100$, $d_{\max} = 7$ we found the optimal graph with clique sizes $(1, 6, 17, 28, 27, 15, 5, 1)$. While optimizing the algebraic connectivity for $n = 122$ and $d_{\max} = 7$ we found that the graph with clique sizes $(1, 11, 20, 29, 28, 21, 11, 1)$ is optimal.

The optimization has shown that in general the clique sizes of the optimal graphs for both measures are larger for cliques closer to the middle. However for the algebraic connectivity there is an example that does not have this structure; the graph with clique sizes $(1, 2, 3, 5, 4, 5, 3, 2, 1)$

is optimal for $n = 26$ and $d_{\max} = 8$.

5.2 An approximation of the normalized effective resistance

The algebraic connectivity has been shown to provide bounds for the normalized effective resistance. In this section we try to answer the question how many Laplacian eigenvalues are needed in order to find the same optimal graph as for the normalized effective resistance.

The sum
$$\frac{k-1}{n\sum_{i=2}^{k}\frac{1}{\lambda_i}} \in [0,1]$$
with $k < n$ is an approximation of the normalized effective resistance by using $k-1$ non-zero eigenvalues instead of all $n-1$ of them. We have optimized this value to find out how many eigenvalues are needed in order to find the same optimal graph as for the normalized effective resistance. For the results see Table 5.1 and for the program Section B.2.

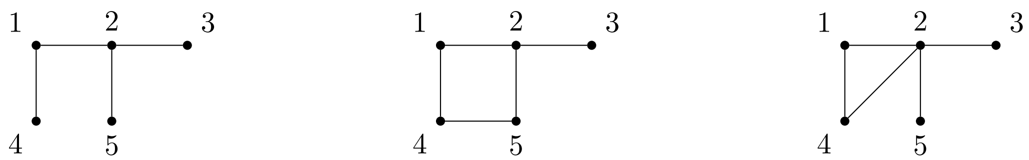
	$n = 26$	$n = 50$	$n = 100$	$n = 122$
$d_{\max} = 2$	$k = 2$	$k = 2$	$k = 2$	$k = 2$
$d_{\max} = 3$	$k = 2$	$k = 2$	$k = 2$	$k = 2$
$d_{\max} = 4$	$k = 15$	$k = 23$	$k = 36$	$k = 42$
$d_{\max} = 5$	$k = 14$	$k = 32$	$k = 93$	$k = 107$

Table 5.1. Minimal number k such that the graph that maximizes $(k-1)/(n\sum_{i=2}^{k}\frac{1}{\lambda_i})$ maximizes R^{norm} as well

In general, for increasing k, the optimal graphs have an increasing number of vertices in the cliques in the middle, but a few surprising counterexamples have been found. For example, for $n = 26$, and $d_{\max} = 4$ (see Table A.2, Appendix A) we have that for $k = 2$ (which corresponds to λ_2/n) the graph with cliques sizes $(1, 7, 10, 7, 1)$ is optimal. For $k = 3, 4, 5, 6$ is the graph with clique sizes $(1, 8, 8, 8, 1)$ optimal. The graph with clique sizes $(1, 7, 10, 7, 1)$ is again optimal for $k = 7, \ldots, 12$.

5.3 Expanding the graph with an edge

For application purposes it is interesting to know which edge has to be added in order to optimize the effective resistance, because it can help improving existing networks. The following example demonstrates that the edge that causes the largest increase in the normal effective resistance, may not causes the largest increase in the algebraic connectivity.



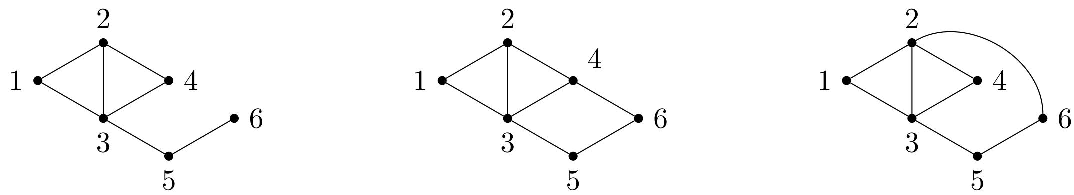
(a) $R^{\text{norm}} \approx 0.22$ and $\lambda_2 \approx 0.52$ (b) $R^{\text{norm}} \approx 0.35$ and $\lambda_2 \approx 0.83$ (c) $R^{\text{norm}} \approx 0.32$ and $\lambda_2 = 1$

Figure 5.3. Adding the edge $(4, 5)$ is optimal for R^{norm}, adding $(2, 4)$ is optimal for λ_2.

A first hypothesis is that it is optimal to add the edge (i, j) for which R_{ij} is maximal. Unfortunately, the example in Figure 5.4 shows that this is not always the case. The corresponding matrix of effective resistances is:

$$\mathsf{R} = \begin{pmatrix} 0 & \frac{5}{8} & \frac{5}{8} & 1 & \frac{13}{8} & \frac{21}{8} \\ \frac{5}{8} & 0 & \frac{1}{2} & \frac{5}{8} & \frac{3}{2} & \frac{5}{2} \\ \frac{5}{8} & \frac{1}{2} & 0 & \frac{5}{8} & 1 & 2 \\ 1 & \frac{5}{8} & \frac{5}{8} & 0 & \frac{13}{8} & \frac{21}{8} \\ \frac{13}{8} & \frac{3}{2} & 1 & \frac{13}{8} & 0 & 1 \\ \frac{21}{8} & \frac{5}{2} & 2 & \frac{21}{8} & 1 & 0 \end{pmatrix}.$$

We see that the pairs $(1, 6)$ and $(4, 6)$ have the largest effective resistance. Nevertheless, edge $(2, 6)$ is the best edge to add.



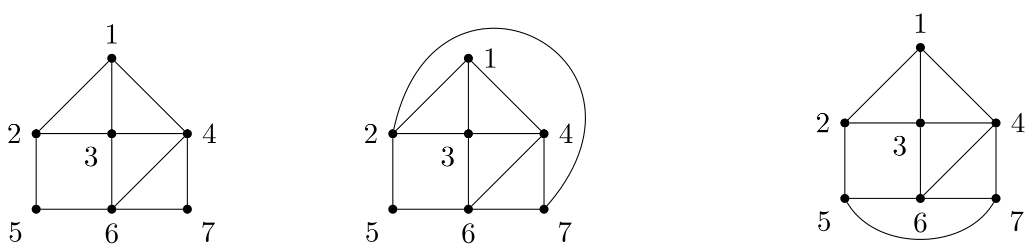
(a) Graph with $R^{\text{norm}} \approx 0.244$ (b) Graph with $R^{\text{norm}} \approx 0.398$ (c) Graph with $R^{\text{norm}} \approx 0.402$

Figure 5.4. Adding the edge $(4, 6)$ is not optimal, although R_{46} gives the maximum effective resistance.

In this counterexample the best edge to add is not the one with maximum pairwise effective resistance, but the one between vertices that lay furthest apart. However, it can neither been demonstrated that the edge (i, j) for which the distance d_{ij} is maximal always is the best edge to add, because the graph in Figure 5.5 gives again a counterexample. The distance matrix corresponding to the graph in Figure 5.5(a) is:

$$\mathsf{D} = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 0 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 0 & 1 \\ 2 & 3 & 2 & 1 & 2 & 1 & 0 \end{pmatrix}.$$

Although the distance is maximal between vertices 2 and 7, it is optimal to add edge $(5, 7)$.



(a) Graph with $R^{\text{norm}} \approx 0.403$ (b) Graph with $R^{\text{norm}} \approx 0.478$ (c) Graph with $R^{\text{norm}} \approx 0.482$

Figure 5.5. Adding the edge $(2, 7)$ is not optimal, although d_{27} gives the maximum distance.

The question which edge to add in order to maximize the normalized effective resistance, is still open.

Chapter 6

Conclusion

6.1 Discussion of the results

The goal of our research was to find a way to quantitatively measure network robustness. More specifically we aimed at answering the following question.

Research question *What is the best way to measure the robustness of simple, undirected, connected and possibly weighted graphs?*

Let us answer this question by discussing the other questions stated in the introduction and repeated here.

- Which graph measures have been proposed for measuring network robustness?
- Which other existing graph measures are suited for measuring network robustness?
- Which new measures can be defined for measuring network robustness?
- What are the properties of the above mentioned measures?
  - Which values can be obtained by the measure?
  - How does the measure change when edges are added or deleted?
  - Which graphs are optimal for this measure (for some given optimization criteria)?
- What graph properties do the measures capture, are these important for network robustness?

We started by reviewing some traditional graph measures from complex network theory: connectivity, vertex connectivity, edge connectivity, average distance, diameter, average vertex betweenness, average edge betweenness, maximum edge betweenness and the clustering coefficient. All connectivity measures, the diameter and the clustering coefficient may stay equal when an edge is added, while the network becomes more robust. The maximum edge betweenness may even increase — and thus indicate that the network becomes less robust — when adding an edge. The remaining measures (average distance, average vertex betweenness, average edge betweenness) qualify better as a measure for robustness, but have been shown to be linear functions of each other (for a given number of vertices).

We continued our research by the inspection of a number of measures specifically proposed for measuring network robustness: the reliability polynomial, algebraic connectivity and the number of spanning trees. Among these, the reliability polynomial most clearly captures the robustness properties of the graph, while the meaning of the algebraic connectivity as a measure for network robustness is the least intuitive. Beside the fact that it is not clear which properties of the graph the algebraic connectivity expresses, it is also a disadvantage that it does not always strictly increase when adding an edge. The reliability polynomial is a function of p, the failure probability of an edge being $1-p$. The robustness evaluation of this measure depends on the choice of p (robustness polynomials of different graphs may intersect each other). The case 'p close to zero' corresponds to measuring the number of spanning trees, which as a consequence is not a good measure since in real-world networks failures are supposed to be scarce. Therefore the reliability polynomial 'for p close to one' seems a better measure. Nevertheless this case corresponds to the edge connectivity, which has been qualified as a poor robustness measure.

To summarize, the requirement that a robustness measure must be strictly increasing when an edge is added, excludes most of the measures mentioned above, except for the average distance (and average betweenness), the reliability polynomial for 'p close to zero' and the number of spanning trees. The latter two measures are related and give the probability that the graph is connected when an edge is removed with large probability, which is a graph property that is not compatible with the fact that failures in real-world networks are scarce. Since the average distance measures the length of the average shortest path between a pair of vertices and does not take the number and length of alternative paths into account, we proposed a new measure for network robustness.

This new measure for network robustness, the normalized effective resistance, has been shown to be zero for unconnected graphs and one for complete graphs. We consider the normalized effective resistance to be a good measure for network robustness for a few reasons. First, the normalized effective resistance considers all paths between two vertices as it increases with the number of paths and decreases when the path lengths are increasing. Second, it strictly increases with the addition of an edge. Third, the larger the normalized effective resistance, the shorter the average random walk between two vertices and the less impact failures have.

The normalized effective resistance can be approximated by the algebraic connectivity, but the optimization in Chapter 5 show that using the algebraic connectivity instead of the normalized effective resistance may lead to different optimal graphs when the number of vertices and the diameter are fixed, and also to the addition of another optimal edge. In Section 5.2 we have shown that it is not possible to use a smaller number than the $n-1$ non-zero Laplacian eigenvalues and still find the right optimal graphs.

6.2 The contribution of this thesis

With this thesis we have contributed to the study of network robustness with as a main goal finding a measure for network robustness — such that the robustness of real-world networks can be evaluated, compared and improved. The contribution of this thesis consists of an extensive survey on robustness measures, the proposal of an functional robustness measure and the optimization of this measure.

The survey of measures (Chapter 2 and Chapter 3) includes a review of more than ten graph measures, their properties and a discussion of their functionality as a robustness measure. Where possible variants of these measures have been proposed where edge weights or traffic matrices are taken into account. Moreover, a complete overview of the characteristics of the Laplacian and its eigenvalues has been given as an introduction to spectral measures. The survey on robustness measures includes a substantial amount of examples and rigorous proofs of all results. For Theorem 2.1, Theorem 3.1, Theorem 3.5, Theorem 3.6 and Theorem 3.12 no earlier proofs were found and other proofs have been elaborated into more detail. New results are: the fact that the maximum edge betweenness may increase when adding an edge (Section 2.3) and Theorem 3.10 on the behavior of the Laplacian spectrum when an edge is added to the star graph.

We have argued that the normalized effective resistance is a highly valuable measure for robustness in Chapter 4. The proposal of this measure is preceded by an introduction of the pairwise and the total effective resistance. In addition, their relation with the Laplacian and random walks has been explained. Some new results (Theorem 4.8 on the decreasingness of the total effective resistance when edges are added and Theorem 4.15 on the relation between the number of visits in a random walk and the pairwise effective resistance) have been proved, proofs of Theorem 4.5 and Theorem 4.9 have been provided, and a new proof of Theorem 4.16 has been developed.

Chapter 5 forms a first step towards the optimization of networks with respect to their robustness. Section 5.1 and Section 5.2 concentrated on the design of robust networks, while Section 5.3 aimed at improving the robustness of existing networks. An interesting result is the finding of asymmetric optimal graphs when maximizing the normalized effective resistance for graphs with a given number of vertices and a given diameter.

6.3 Further research

We wrote a survey on robustness measures on weighted and unweighted graphs. A logical next step is to look for robustness measures on directed graphs or to include traffic matrices and *edge capacities* (bounds on the load of an edge). As mentioned in Section 2.3 the betweenness variance may be considered as a new graph measure.

Further research on the normalized effective resistance may also focus on the generalization to directed graphs and graphs with traffic matrices and capacities. It may as well concentrate on the comparison of the computation time of algorithms for determining the normalized effective resistance.

Concerning the maximization of the normalized effective resistance we would like to analytically compute the optimal graphs of Section 5.1 or at least finding an explanation for the presence of asymmetric optimal graphs. The problem of finding an algorithm for determining the edge that increases the normalized effective resistance most, without having to try all possible edges is still open. Optimization for other optimization criteria, for example a fixed number of vertices and edges, might be interesting as well. Furthermore, research should be done to verify whether real-world graphs exist that have the same structure as the optimal graphs in Section 5.1.

Appendix A

Tables with optimization results

The tables referred to in Section 5.1 and Section 5.2 can be found in this appendix.

$n = 26$	R^{norm}	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
$d_{\max} = 2$	0.9967	1	24	1					
$d_{\max} = 3$	0.8859	1	12	12	1				
$d_{\max} = 4$	0.6644	1	6	12	6	1			
$d_{\max} = 5$	0.4817	1	4	8	8	4	1		
$d_{\max} = 6$	0.3557	1	3	6	6	6	3	1	
$d_{\max} = 7$	0.2678	1	3	4	5	5	4	3	1

$n = 50$	R^{norm}	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
$d_{\max} = 2$	0.9992	1	48	1					
$d_{\max} = 3$	0.9404	1	24	24	1				
$d_{\max} = 4$	0.7653	1	9	29	10	1			
$d_{\max} = 5$	0.5812	1	6	18	18	6	1		
$d_{\max} = 6$	0.4454	1	5	11	15	12	5	1	
$d_{\max} = 7$	0.3516	1	4	9	11	11	9	4	1

$n = 100$	R^{norm}	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
$d_{\max} = 2$	0.9998	1	98	1					
$d_{\max} = 3$	0.9701	1	49	49	1				
$d_{\max} = 4$	0.8425	1	16	67	15	1			
$d_{\max} = 5$	0.6684	1	8	41	41	8	1		
$d_{\max} = 6$	0.5226	1	6	22	41	23	6	1	
$d_{\max} = 7$	0.4175	1	6	17	28	27	15	5	1

$n = 122$	R^{norm}	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
$d_{\max} = 2$	0.9999	1	120	1					
$d_{\max} = 3$	0.9755	1	60	60	1				
$d_{\max} = 4$	0.8601	1	18	84	18	1			
$d_{\max} = 5$	0.6910	1	9	51	51	9	1		
$d_{\max} = 6$	0.5430	1	7	27	51	28	7	1	
$d_{\max} = 7$	0.4347	1	6	19	35	35	19	6	1

Table A.1. Graphs that maximize the normalized total effective resistance for given n, $d_{\max}$

$n = 26$, $d_{\max} = 4$	$n \sum_{i=2}^{k} \frac{1}{\lambda_i}$	n_1	n_2	n_3	n_4	n_5
$k = 2$	4.57	1	7	10	7	1
$k = 3$	7.68	1	8	8	8	1
$k = 4$	9.97	1	8	8	8	1
$k = 5$	11.50	1	8	8	8	1
$k = 6$	13.03	1	8	8	8	1
$k = 7$	14.53	1	7	10	7	1
$k = 8$	15.98	1	7	10	7	1
$k = 9$	17.42	1	7	10	7	1
$k = 10$	18.87	1	7	10	7	1
$k = 11$	20.31	1	7	10	7	1
$k = 12$	21.76	1	7	10	7	1
$k = 13$	23.20	1	7	10	7	1
$k = 14$	24.63	1	7	11	6	1
$k = 15$	25.74	1	6	12	6	1
$k = 16$	26.82	1	6	12	6	1
$k = 17$	27.91	1	6	12	6	1
$k = 18$	28.99	1	6	12	6	1
$k = 19$	30.08	1	6	12	6	1
$k = 20$	31.16	1	6	12	6	1
$k = 21$	32.24	1	6	12	6	1
$k = 22$	33.32	1	6	12	6	1
$k = 23$	34.41	1	6	12	6	1
$k = 24$	35.49	1	6	12	6	1
$k = 25$	36.57	1	6	12	6	1
$k = 26$	37.63	1	6	12	6	1

Table A.2. Graphs that maximize $(k-1)/(n \sum_{i=2}^{k} \frac{1}{\lambda_i})$ for $n = 26$, $d_{\max} = 4$

Appendix B

Matlab codes for the optimization of the normalized effective resistance

This appendix contains the Matlab M-files used for the optimization of the normalized effective resistance in Chapter 5.

B.1 Graphs with a fixed number of vertices and diameter

We start by giving the M-files used for Section 5.1. The following function calculates the normalized effective resistance and the algebraic connectivity, given the adjacency matrix of a graph.

```
function [R,A]=resistance_and_connectivity(G)
%calculates the normalized effective resistance R
%and the algebraic connectivity A, given an adjacency matrix G

n=length(G);        %number of vertices
d=sum(G,1);         %vector of vertex degrees
L=-G+diag(d);       %Laplacian
eigen=sort(eig(L)); %vector of Laplacian eigenvalues

R=0;
A=0;

if (n~=1)&&(eigen(2)>0)
    A=eigen(2);
    for i=2:n
        R=R+n/eigen(i);
    end
    R=(n-1)/R;
end
```

The following function constructs all possible clique chains with a given number of vertices and diameter, it returns the optimal values and graphs for the normalized effective resistance and the algebraic connectivity. It uses the former function.

```
function[x,eff_opt,m_opt,alg_opt,m_opt2]
=increasing_numbers(max,total_number,number,x,eff_opt,m_opt,alg_opt,m_opt2)
%gives sets of ‘total_number’ increasing integers
```

```
%starting with an amount of 'number' integers given in 'x'
%the last integer is 'max'

%first part of increasing numbers
if number==total_number-1
    x(total_number)=max;

    %gives a partition m of 'max'+1 of size 'total_number'+1
    %where the first and last integers are 1
        x2=[0,x,max+1];
        for j=1:length(x2)-1
            m(j)=x2(j+1)-x2(j);
        end

    %gives the adjacency matrix of a clique chain with cliques sizes
    %given by the partition m
    %cliques sizes given by the partition m
        for j=1:total_number+1
            if j>1
                    A(x2(j)+1:x2(j+1),x2(j-1)+1:x2(j))=ones(m(j),m(j-1));
                    A(x2(j-1)+1:x2(j),x2(j)+1:x2(j+1))=ones(m(j-1),m(j));
            end
            A(x2(j)+1:x2(j+1),x2(j)+1:x2(j+1))
             =triu(ones(m(j)),1)+triu(ones(m(j)),1)';
        end

    %calculates the algebraic connectivity,
    %the effective resistance and finds the optima
        [eff,alg]=resistance_and_connectivity(A);
        if eff>eff_opt
            eff_opt=eff;
            m_opt=m;
        end
        if alg>alg_opt
            alg_opt=alg;
            m_opt2=m;
        end

%final part of increasing_numbers
else
    number=number+1;
    for i=x(number-1)+1:max-total_number+number
        x(number)=i;
        [x,eff_opt,m_opt,alg_opt,m_opt2]=increasing_numbers
         (max,total_number,number,x,eff_opt,m_opt,alg_opt,m_opt2);
    end
end
end
```

The following M-file produces the output used in Table A.1, making use of the last function.

```
%finds the graphs with given number of vertices and diameter that optimize
```

```
%the algebraic connectivity or the total ER (effective resistance)

clc;
clear all;

%open output file
fid = fopen('optimize_eff_resist.txt', 'wt');
fprintf(fid, 'The graphs that optimize the algebraic connectivity and
the total effective resistance for given number of nodes and diameter');
fclose(fid);

for N=[26,50,100,122]
    for D=2:7

        eff_opt=0; %optimal value of the total ER
        m_opt=0; %optimal graph for the total ER
        alg_opt=0;  %optimal value of the alg. con.
        m_opt2=0; %optimal graph for the alg. con.

        [x,eff_opt,m_opt,alg_opt,m_opt2]
         =increasing_numbers(N-1,D,1,1,eff_opt,m_opt,alg_opt,m_opt2);

        %print optimal values and graphs in output file
        fid = fopen('optimize_eff_resist.txt', 'at');
        fprintf(fid, '\n\nN: %d, D: %d', N, D);
        fprintf(fid, '\nalg_opt: %6.4f, G:', alg_opt);
        fprintf(fid, ' %d', m_opt2);
        fprintf(fid, '\neff_opt: %6.4f, G:', eff_opt);
        fprintf(fid, ' %d', m_opt);
        fclose(fid);

    end
end
```

B.2 An approximation of the normalized effective resistance

The M-files used in Section 5.2 are very similar to those of Section 5.1. We only give the function that calculates the approximations of the normalized effective resistance. The other two M-files are adaptions of the last two files of the former section, where the normalized effective resistance is replaced by a vector of approximations.

```
function [until_eigenvalue]=all_eigenvalues(G)
%calculates 1 over the sum of k inverse eigenvalues (for k<=n) and
%normalizes
%until_eigenvalue(k-1) gives 1 over the sum of inverse eigenvalue 2
%till eigenvalue k multiplied by (k-1)/n

n=length(G);
d=degree(G);
L=-G+diag(d);
eigen=sort(eig(L));
```

```
until_eigenvalue=zeros(1,n);

if eigen(2)>0
    for i=2:n
        until_eigenvalue(i)=until_eigenvalue(i-1)+1./eigen(i);
    end
    until_eigenvalue(1)=[];
    for i=1:n-1
        until_eigenvalue(i)=i/(n*until_eigenvalue(i));
    end
end

end
```

B.3 Expanding the graph with an edge

For Section 5.3 we have used the following function, which determines which edge has to be added in order to maximize the normalized effective resistance.

```
function [e_opt,e_opt2]=add_best_edge(G)
%gives the edge that optimizes the total eff resistance when added
%and the same for the algebraic connectivity

eff_opt=0;
e_opt=[];
alg_opt=0;
e_opt2=[];

n=length(G);
for i=1:n
    for j=i+1:n
        if G(i,j)==0
            G2=G;
            G2(i,j)=1;
            G2(j,i)=1;
            [eff,alg]=resistance_and_connectivity(G2);
            if eff>eff_opt
                eff_opt=eff;
                e_opt=[i,j];
            end
            if alg>alg_opt
                alg_opt=alg;
                e_opt2=[i,j];
            end
        end
    end
end

end
```

The following program determines whether the best edge to add is the edge with largest distance. The other programs used in Section 5.3 are variants of this progam. It uses the functions 'components' and 'all_shortest_paths' which can be found in [17] and the function 'ER' given below.

```
clc
n=7;
for i=1:100
    G=ER(n,.5); %generation of a random graph with n vertices
    if (sum(sum(G))~=(n^2-n)) %we cannot add an edge to a complete graph
        if (components(sparse(G))==ones(n,1)) %the graph must be connected
            [e_opt_eff,e_opt_alg]=add_best_edge(G);
            D=all_shortest_paths(sparse(G));
            if D(e_opt_eff(1),e_opt_eff(2))~=max(max(D))
                Gfout=G %a counterexample
                e_opt_eff %best edge to add
                D %distance matrix
            end
        end
    end
end

%Generates a random graph with n vertices and edge prob. p

function G=ER(n,p)
    G=zeros(n);
    for i=1:n
        for j=i+1:n
            if rand<p
                G(i,j)=1;
                G(j,i)=1;
            end
        end
    end
end
```

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