Decomposing positive representations in Lp-spaces for Polish transformation groups

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Master's thesis
Leiden University, 26 April 2011

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Introduction

In representation theory, one is often interested in decomposing a representation of a group G into a direct sum of irreducible representations. Several results of this kind are known, although most research in this area has focused upon representations of groups on Hilbert spaces. The case of groups acting as bounded operators on a Banach space is less well-known, let alone groups acting on ordered Banach spaces or Banach lattices. In this thesis we study specific representations of groups on a specific class of Banach lattices, spaces of Lebesgue integrable functions.
Sometimes it may not be possible to decompose a representation into a direct sum of irreducible representations, and we would like to consider a 'continuous' direct sum of spaces, a direct integral so to speak. Such a concept exists for Hilbert spaces, and it is defined to be an L^2-space of sections of a family of Hilbert spaces. In this thesis we examine a similar concept in the case of Banach spaces, so-called Banach bundles. We then consider spaces of integrable sections of these bundles, a kind of direct integral for Banach spaces. We construct an isometric lattice isomorphism between L^p-spaces of scalar-valued functions and L^p-spaces of sections of some Banach bundle.
We do this by using results on measure decompositions. These results tell us that, under certain assumptions on the spaces involved, we can decompose a measure μ on some space X which is invariant for the action of some group G into an integral of ergodic measures. Combining these concepts we construct a Banach bundle $\mathfrak{B}$ of L^p-spaces, and show that the L^p-space of p-integrable scalar-valued functions $L^p(X, \mu)$ is isomorphic to a subspace of the space of sections that have finite p-upper integral. We then decompose the representation of G on $L^p(X, \mu)$ induced by the action of G on X into band irreducible representations, by viewing it as a representation on the above subspace.
We assume that the reader has some basic knowledge of functional analysis, topology and measure theory. We will try to explain the basics of most of the objects and properties which we use, but since we do not intend to write a textbook on any of these areas, we will sometimes skip the details. For a thorough exposition on these subjects we refer to such works as [2], [5] and [11].
The first chapter treats some of the necessary background knowledge. A few concepts from the theory of Riesz spaces are presented, as well as several other results which we will use later on. Then we state the precise setting which we consider in this thesis and the relation between band irreducible representations and ergodic measures.
In chapter 2 we state the measure decomposition result that we will use and examine some of its corollaries.
We treat the theory of Banach bundles in chapter 3. First we define these bundles and derive some of their properties, after which we move on to consider integration in Banach bundles.
We present our main results in Chapter 4. Using the ideas of the previous chapters we make the decomposition of an L^p-space and the action of a group on such a space precise. Finally, in the conclusion we make some remarks on possible extensions and generalizations of this research.

Chapter 1

Background and preliminaries

For the reader to be able to understand later chapters in this thesis, he or she must know something about Riesz spaces and transformation groups. In this chapter some elementary facts about these structures are given.

1.1 Riesz spaces and Banach lattices

First we present a short overview of some of the necessary concepts from the theory of ordered vector spaces. This is not meant to be a complete overview of this theory, and a substantial part of the definitions and results can be generalized to a wider class of structures. However, the concepts as presented here will suffice to examine the rest of this thesis. A thorough introduction to this field, including proofs of the results below, can be found in [1] and [16], among others.
All vector spaces are assumed to be real.

Definition 1.1.1. Let E be a vector space and $\leq$ a partial ordering on E. The pair $(E, \leq)$ is said to be an *ordered vector space* if the following properties hold true for all $x, y \in E$:

- $x \geq y$ implies $x + z \geq y + z$ for all $z \in E$.
- $x \geq y$ implies $\alpha x \geq \alpha y$ for all $\alpha \in \mathbb{R}_{\geq 0}$.

Usually, we will not explicitly mention the underlying ordering in a partial ordered vector space $(E, \leq)$ and simply speak of an ordered vector space E.
Once we have the concept of an ordered vector space, it is natural to consider positive elements. An element x in an ordered vector space E is said to be *positive* if $x \geq 0$ holds. We can also define positivity of linear operators on ordered vector spaces. Let $T : E \to F$ be a linear operator between ordered vector spaces E and F. Then T is called a *positive operator* if $T(x) \geq 0$ holds in F for all $x \geq 0$ in E.
Let E be a partially ordered vector space and $F \subset E$ a subset. An element $x \in E$ is said to be an *upper bound* for F if $x \geq y$ holds for all $y \in F$. Similarly, a *lower bound* x for F satisfies $x \leq y$ for all $y \in F$. An $x \in X$ is a *supremum* for F if x is an upper bound for F such that $x \leq z$ for all upper bounds $z \in E$ of F. A largest lower bound for F is called an *infimum* for F.[1]

[1] Clearly, the supremum and infimum of a set, when they exist, are unique.

Definition 1.1.2. A *Riesz space* is an ordered vector space E such that, for all $x, y \in E$, the supremum $x \vee y$ and infimum $x \wedge y$ of $\{x, y\}$ exist in E.

If $x \in E$ is an element in a Riesz space, then we can define the *positive part* x^+ of x as $x^+ := x \vee 0 \in E$. Similarly, the *negative part* x^- of x is $x^- := (-x) \vee 0 \in E$. The *absolute value* $|x|$ of x is $|x| := x \vee (-x) \in E$.
We also have maps between Riesz spaces that respect the lattice structure of these spaces. A linear map $T : E \to F$ between Riesz spaces E and F is called a *lattice homomorphism* if $T(x \vee y) = T(x) \vee T(y)$ for all $x, y \in E$. If T^{-1} is a well-defined lattice homomorphism as well, then T is a *lattice isomorphism.*

Remark 1.1.3. One can easily show that $T(x \wedge y) = T(x) \wedge T(y)$ for all $x, y \in E$ if $T : E \to F$ is a lattice homomorphism. Also note that T is then a positive operator.

Example 1.1.4. Let (X, μ) be a non-empty measure space, $p \in [1, \infty)$, and $\mathcal{L}^p(X, \mu)$ the space of p-integrable real-valued functions on X, that is, the set of all measurable functions $f : X \to \mathbb{R}$ such that $\int_X |f(x)|^p d\mu(x)$ is finite. We define a partial ordering on $\mathcal{L}^p(X, \mu)$ by: $f \leq g$ in $\mathcal{L}^p(X, \mu)$ if $f(x) \leq g(x)$ in $\mathbb{R}$ for all $x \in X$. It is straightforward to check that this turns $\mathcal{L}^p(X, \mu)$ into an ordered vector space.
If $f, g \in \mathcal{L}^p(X, \mu)$ are given, then the function $f \vee g : X \to \mathbb{R}$ given by $(f \vee g)(x) := f(x) \vee g(x)$ for all $x \in X$, is the well-defined supremum of f and g in $\mathcal{L}^p(X, \mu)$. Similarly, $f \wedge g \in \mathcal{L}^p(X, \mu)$ given by $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in X$, is the infimum of $\{f, g\}$. So $\mathcal{L}^p(X, \mu)$ is in fact a Riesz space.
Also, the map $|| \cdot ||_p : \mathcal{L}^p(X, \mu) \to [0, \infty)$ given by

$$||f||_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}$$

for $f \in \mathcal{L}^p(X, \mu)$, is a seminorm on $\mathcal{L}^p(X, \mu)$. Often we wish to divide out the kernel of this seminorm, and this provides us with the well-known space $L^p(X, \mu) = \mathcal{L}^p(X, \mu)/\ker(|| \cdot ||_p)$ of equivalence classes of p-integrable functions. As is common practice, we will view the elements of $L^p(X, \mu)$ as functions on X, identifying two of them if the p-norm of their difference is zero. The latter is the case precisely when two functions are equal μ-almost everywhere on X.
The ordering on $L^p(X, \mu)$ induced by the one on $\mathcal{L}^p(X, \mu)$ is given by $f \leq g$ in $L^p(X, \mu)$ if $f(x) \leq g(x)$ for μ-almost all $x \in X$. $L^p(X, \mu)$ is an ordered vector space, and because the equivalence classes of $f \vee g$ and $f \wedge g$, for $f, g \in \mathcal{L}^p(X, \mu)$, form the supremum respectively infimum of the equivalence classes of f and g in $L^p(X, \mu)$, we see that $L^p(X, \mu)$ is also a Riesz space.
An element $f \in \mathcal{L}^p(X, \mu)$ is positive if $f(x) \geq 0$ for all $x \in X$, and an $f \in L^p(X, \mu)$ is positive if $f(x) \geq 0$ for μ-almost all $x \in X$. The positive part, negative part, and absolute value of an element of $\mathcal{L}^p(X, \mu)$ or $L^p(X, \mu)$ correspond with the usual definitions of these concepts as maps to $\mathbb{R}$. So $f^+(x) = f(x) \vee 0$, $f^-(x) = (-f(x)) \vee 0$ and $|f|(x) = |f(x)|$ for all $x \in X$ and $f \in \mathcal{L}^p(X, \mu)$.
For each $\lambda \in (0, \infty)$, the multiplication operator $f \mapsto \lambda f$ is a lattice isomorphism, both on $\mathcal{L}^p(X, \mu)$ and on $L^p(X, \mu)$. We will encounter other examples of lattice isomorphisms on these spaces later on, when we consider the action of a group G on X.
Similar statements hold for the μ-almost everywhere bounded functions $\mathcal{L}^\infty(X, \mu)$ on X, the set of all measurable functions $f : X \to [0, \infty)$ for which there exists an $M \geq 0$

such that $\mu\{x \in X : |f(x)| > M\} = 0$. The smallest such M will be denoted by $||f||_\infty$, and the map $||\cdot||_\infty : \mathcal{L}^\infty(X,\mu) \to \infty$, $f \mapsto ||f||_\infty$ for $f \in \mathcal{L}^\infty(X,\mu)$, is a seminorm on $\mathcal{L}^\infty(X,\mu)$. We view elements of $L^\infty(X,\mu) = \mathcal{L}^\infty(X,\mu)/\ker(||\cdot||_\infty)$ as functions on X and we identify two of them if their difference has ∞-norm 0, which holds if they are equal μ-almost everywhere. Then $\mathcal{L}^\infty(X,\mu)$ and $L^\infty(X,\mu)$ are Riesz spaces and the supremum respectively infimum of two elements are as in the case $p < \infty$ above. The multiplication operators from above are also lattice isomorphisms on $\mathcal{L}^\infty(X,\mu)$ and $L^\infty(X,\mu)$.

Ideals and bands

Assumption 1.1.5. From here on we will suppose that all Riesz spaces are *Dedekind complete* and have the *countable sup property*. The former means that every non-empty order bounded set has a supremum, and the countable sup property tells us that, for every subset $F \subset E$ of the Riesz space E having a supremum in E, there exists a countable subset of F with the same supremum. These properties imply that we can adjust the definitions we give to the (often simpler) case of sequences. This is justified since the spaces of Lebesgue integrable functions that we will we consider in this thesis are Dedekind complete and have the countable sup property.

Consider a sequence $\{x_n\}_{n=1}^\infty$ in a Riesz space E. It is said to be *increasing* if $n \geq m$ in $\mathbb{N}$ implies $x_n \geq x_m$ in E. We write $x_n \uparrow x$ if $\{x_n\}_{n=1}^\infty$ is increasing and $\sup_{n\in\mathbb{N}} x_n = x \in E$. Decreasing sequences are defined similarly, and $x_n \downarrow x$ means that $\{x_n\}_{n=1}^\infty$ is decreasing and $\inf_{n\in\mathbb{N}} x_n = x$. We are now ready to define order convergence in a Riesz space.

Definition 1.1.6. A sequence $\{x_n\}_{n=1}^\infty$ in a Riesz space E is said to be *order convergent* to an element $x \in E$, notation $x_n \xrightarrow{o} x$, if there exists a sequence $\{y_n\}_{n=1}^\infty \subset E$ such that $y_n \downarrow 0$ and $|x_n - x| \leq y_n$ for each $n \in \mathbb{N}$.

Two special types of subspaces of a Riesz space are ideals and bands. An *ideal* in a Riesz space E is a linear subspace $A \subset E$ such that $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A *band* in E is an ideal which is closed under order convergence, i.e. $x_n \xrightarrow{o} x \in E$ and $\{x_n\}_{n=1}^\infty \subset A$ imply $x \in A$. A linear operator $T : E \to E$ on E is said to be *band irreducible* if $TB \subset B$ for a band $B \subset E$ implies $B = 0$ or $B = E$. A collection Γ of operators on E is called band irreducible if, for every band $B \subset E$, $\Gamma B \subset B$ implies that B is trivial.

Remark 1.1.7. All ideals (and thus all bands as well) are closed under the lattice operations $\vee$ and $\wedge$. In other words, if $x, y \in A$ are elements of an ideal $A \subset E$, then $x \vee y \in A$ and $x \wedge y \in A$.

So far we have considered vector spaces with a partial ordering on them. In a lot of examples, specifically in the ones that we will consider later on, the vector space is also endowed with a norm. If this norm is compatible with the ordering in some sense, then we use the following terminology:

Definition 1.1.8. Let E be a Riesz space with a norm $||\cdot|| : E \to [0,\infty)$. $(E, ||\cdot||)$[2] is called a *normed Riesz space* if $|x| \leq |y|$ in E implies $||x|| \leq ||y||$. If E is furthermore complete with respect to the norm $||\cdot||$, then $(E, ||\cdot||)$ is a *Banach lattice*.

[2]Just as we did for the ordering on a Riesz space, often we will not mention the norm explicitly and simply speak of a normed Riesz space space E.

Remark 1.1.9. In a normed Riesz space, the lattice operations are norm continuous. In other words, if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in a normed Riesz space E such that $x_n \to x$ and $y_n \to y$ in norm for certain $x, y \in E$, then $x_n \vee y_n \to x \vee y$ and $x_n \wedge y_n \to x \wedge y$ in norm.

Remark 1.1.10. We now have two concepts of convergence on a normed Riesz space, convergence in order and in norm. In general, neither will imply the other. However, certain relations between the two concepts do hold, and in fact the L^p-spaces that we examine have the property, for $p < \infty$, that order convergence implies norm convergence, as we will see below.

We can give an alternative characterization of bands in a normed Riesz space using the concept of a disjoint complement. Let $F \subset E$ be a subset of a Riesz space. The *disjoint complement* of F is the subset $F^d := \{x \in E : |x| \wedge |y| = 0 \text{ for all } y \in F\}$.

Proposition 1.1.11. In a normed Riesz space E, a subset $F \subset E$ is a band if and only if $F = F^{dd}$.

From this and Remark 1.1.9 we get

Corollary 1.1.12. Every band in a normed Riesz space is norm closed.

If E is a Riesz space, then we may wish to decompose E into simpler parts, as is done for vector spaces by writing the space as a direct sum of subspaces. For ordered vector spaces we also have an ordering to account for, so we would like to incorporate this ordering into such a decomposition.

Definition 1.1.13. Let E be a Riesz space. We say that E is the *order direct sum* of linear subspaces $F, G \subset E$ if $E = F \oplus G$ as a vector space and if $x = y + z \geq 0$ in E implies $y \geq 0$ and $z \geq 0$, where $y \in F$ and $z \in G$ form the unique decomposition of x in $F \oplus G$.

In this thesis we will attempt to decompose a space of Lebesgue integrable functions in a similar manner. In that light, the following proposition motivates our interest in bands:

Proposition 1.1.14. If a Riesz space $E = F \oplus G$ is the order direct sum of subspaces $F, G \subset E$, then F and G are bands such that $G = F^d$ and $F = G^d$.

Example 1.1.15. Again let a non-empty measure space (X, μ) and a $p \in [1, \infty]$ be given. We have seen in Example 1.1.5 that the spaces $\mathcal{L}^p(X, \mu)$ and $L^p(X, \mu)$ are Riesz spaces, and that $L^p(X, \mu)$ can be endowed with the norm $||\cdot||_p : X \to [0, \infty)$ given by

$$||f||_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}$$

for all $f \in L^p(X, \mu)$ if $p \in [1, \infty)$, and

$$||f||_\infty = \inf \{M \geq 0 : \mu\{x \in X : |f(x)| > M\} = 0\}$$

for all $f \in L^\infty(X, \mu)$. In fact, $L^p(X, \mu)$ is a normed Riesz space, and since it is complete with respect to $||\cdot||_p$, a Banach lattice. Indeed, if $f, g \in L^p(X, \mu)$ are such that $|f| \leq |g|$

almost everywhere, then monotonicity of the integral implies $||f||_p \leq ||g||_p$ for $p < \infty$, and for $p = \infty$ $||f||_\infty \leq ||g||_\infty$ follows immediately.

We now examine order convergence in this space. Suppose $\{f_n\}_{n=1}^\infty \subset L^p(X,\mu)$ converges in order to an $f \in L^p(X,\mu)$. Then there exists a positive decreasing sequence $\{g_n\}_{n=1}^\infty \subset L^p(X,\mu)$ such that $\inf_{n\in\mathbb{N}} g_n = 0$ almost everywhere and $|f_n - f| \leq g_n$ almost everywhere for all $n \in \mathbb{N}$. This implies that $\{f_n\}_{n=1}^\infty$ converges pointwise almost everywhere to f. So order convergence implies almost everywhere convergence.

However, the converse does not hold in general. To see this, set $X := [0,1]$ and let μ be Lebesgue measure on X. For each $n \in \mathbb{N}$, set $X_n := (2^{-n}, 2^{1-n}) \subset X$ and $f_n := 2^{n/p}\mathbf{1}_{X_n}$ if $p \in [1,\infty)$, $f_n := n\mathbf{1}_{X_n}$ if $p = \infty$. Then $\{f_n\}_{n=1}^\infty \subset L^p(X,\mu)$ converges to zero almost everywhere, but it does not converge in order. Indeed, suppose $\{g_n\}_{n=1}^\infty \subset L^p(X,\mu)$ is such that $g_n \downarrow 0$ and $f_n \leq g_n$ (since the sequence $\{f_n\}_{n=1}^\infty$ converges to zero almost everywhere, its order limit, if it exists, must be equal to 0). First assume $p < \infty$. Then

$$||f_n||_p = \left(\int_X f_n^p d\mu\right)^{1/p} = \left(\int_{X_n} 2^n d\mu\right)^{1/p} = 1$$

for each $n \in \mathbb{N}$, and $g_n \geq g_m \geq f_m$ for all $m \geq n$. So $g_n \geq \sup_{m\geq n} f_n$ and, because the X_n are mutually disjoint,

$$\int_X g_n^p d\mu \geq \int_X \sup_{m\geq n} |f_n|^p d\mu = \sum_{m=n}^{\infty} \int_{X_n} 2^n d\mu = \infty$$

for each $n \in \mathbb{N}$, a contradiction. In the case $p = \infty$ we find in a similar manner $||g_n||_\infty \geq ||f_m||_\infty = m$ for all $m \geq n$, which contradicts $\{g_n\}_{n=1}^\infty \subset L^\infty(X)$.

We also examine the relation between order convergence and norm convergence in these spaces. To this end we consider the cases $p \in [1,\infty)$ and $p = \infty$ separately. For $p < \infty$ one need only note that we can apply the dominated convergence theorem to see that order convergence implies norm convergence. The reverse implication need not hold. Indeed, set $X := [0,1]$ and let μ be Lebesgue measure on X. Let $\{X_n\}_{n=1}^\infty$ be the sequence of intervals $[0,1], [0,\frac{1}{2}], [\frac{1}{2},1], [0,\frac{1}{3}], [\frac{1}{3},\frac{2}{3}], [\frac{2}{3},1], \ldots$ in X and let $\{f_n\}_{n=1}^\infty \subset L^p(X,\mu)$ be the sequence of characteristic functions of these intervals. This sequence converges to zero in norm, as the lengths of the intervals decrease to zero, but the sequence does not converge pointwise anywhere. Since order convergence implies pointwise almost everywhere convergence, we conclude that the sequence does not converge in order.

For $p = \infty$ the situation is reversed. Indeed, since $||f - g||_\infty \leq \epsilon$ implies $|f - g| \leq \epsilon\mathbf{1}$ almost everywhere, with $\mathbf{1}$ the constant function on X and $f, g \in L^\infty(X,\mu)$, it is easy to see that norm convergence implies order convergence. However, if we consider $X := [0,1]$ and μ Lebesgue measure on X once again and set $X_n := (0, \frac{1}{n}) \subset X$, $f_n := \mathbf{1}_{X_n}$ for each $n \in \mathbb{N}$, then $\{f_n\}_{n=1}^\infty$ converges in order to 0 but it is not a Cauchy sequence and therefore not norm convergent.

Finally, we determine the bands in $L^p(X,\mu)$ for μ finite. Let $B \subset L^p(X,\mu)$ be a band, so $B^{dd} = B$. A measurable set $Y \subset X$ is called a *null set* for B if every $f \in B$ is zero almost everywhere on Y. Let Γ be the collection of all null sets of X. Since μ is finite, $\gamma := \sup\{\mu(Y) : Y \in \Gamma\}$ is a finite quantity. Therefore there exists a sequence $\{Y_n\}_{n\in\mathbb{N}} \subset \Gamma$ such that $\mu(Y_n) \uparrow \gamma$ as $n \to \infty$. Set $Y := \cup_{n=1}^\infty Y_n$ for such a sequence. Then $Y \in \Gamma$ and there does not exist a subset $Z \subset X \setminus Y$ of positive measure with $Z \in \Gamma$. From this we deduce that $B^d = \{f \in L^p(X,\mu) : f(x) = 0 \text{ for almost all } x \in X \setminus Y\}$ and

$$B = B^{dd} = \{f \in L^p(X) : f(y) = 0 \text{ for almost all } y \in Y\}. \qquad (1.1)$$

On the other hand, for any measurable $Y \subset X$ the set of functions in $L^p(X,\mu)$ which vanish almost everywhere on Y is band. We conclude that any band $B \subset L^p(X,\mu)$ is of the form (1.1) for some measurable $Y \subset X$.
This statement can be generalized to the case where μ is σ-finite, by restricting to subsets on which μ is finite.

1.2 Transformation groups

In this section we aim to describe in detail the setting which will be considered in this thesis. First we recall some concepts and results from topology, functional analysis and measure theory. For more details we refer to textbooks such as [2], [5] and [11].

Background A topological space X is said to be *completely metrizable* if there exists a metric on X which induces the topology of X such that X is complete with respect to this metric. Also, X is *Polish* if it is separable and completely metrizable. Any Polish space is second-countable and any subset of a Polish space is metrizable and separable.
If E is a Banach space, then we denote by $\mathcal{B}(E)$ the set of bounded linear operators on E. Apart from the norm topology there is another topology on $\mathcal{B}(E)$, the *strong operator topology*. In this topology a net $\{T_i\}_{i\in I} \subset \mathcal{B}(E)$ converges to a $T \in \mathcal{B}(E)$ if $T_i(x) \to T(x)$ for all $x \in E$. Clearly, this topology is weaker than the norm topology on $\mathcal{B}(E)$. If X is a topological space and $\rho : X \to \mathcal{B}(E)$ a map, then we say that ρ is *strongly continuous* if it is continuous with respect to the strong operator topology on $\mathcal{B}(E)$.
If X is a topological space, then by the *Borel σ-algebra* we mean the σ-algebra generated by the open sets in X. In what follows all matters of measurability on a topological space X will refer to this Borel structure.
If μ is a measure on a Hausdorff topological space X, then μ is *outer regular* if

$$\mu(Y) = \inf\{\mu(U) : Y \subset U \text{ open}\}$$

for every $Y \subset X$ measurable. Also, μ is said to be *inner regular* if

$$\mu(Y) = \sup\{\mu(F) : F \subset Y \text{ closed}\}$$

for all $Y \subset X$ measurable. The measure μ is *normal* if it is both outer and inner regular. Any finite Borel measure on a metrizable space is normal [2, Theorem 12.5]. Furthermore, μ is *tight* if

$$\mu(Y) = \sup\{\mu(K) : K \subset Y \text{ compact}\}$$

for all $Y \subset X$ measurable. Clearly, a tight measure is inner regular. Finally, μ is said to *regular* if $\mu(K) < \infty$ for all $K \subset X$ compact and if μ is both outer regular and tight. Any finite Borel measure on a Polish space is regular [2, Theorem 12.7].
For any locally compact Hausdorff space X, μ any regular finite measure on X and $p \in [1,\infty)$, the equivalence classes of the compactly supported continuous functions $C_c(X) \subset L^p(X,\mu)$ lie dense in $L^p(X,\mu)$. The compactly supported continuous functions need not be dense in $L^\infty(X,\mu)$.
For any metrizable space X, the set $\mathcal{P}(X)$ of Borel probability measures on X can be

endowed with the *weak* topology*[3]. In this topology a net $\{\mu_i\}_{i\in I} \subset \mathcal{P}(X)$ converges to a $\mu \in \mathcal{P}(X)$ if

$$\int_X f\,d\mu_i \to \int_X f\,d\mu$$

for all $f : X \to \mathbb{R}$ continuous and bounded. A base for this topology is given by the sets

$$\left\{\lambda \in \mathcal{P}(X) : \max_{1\leq i\leq n} \left|\int_X f_i d\mu - \int_X f_i d\lambda\right| < \epsilon\right\} \subset \mathcal{P}(X) \tag{1.2}$$

for $\mu \in \mathcal{P}(X)$, $n \in \mathbb{N}$, $f_1, \ldots, f_n \in C_b(X)$ and $\epsilon > 0$. Moreover, $\mathcal{P}(X)$ is compact if and only if X is compact and $\mathcal{P}(X)$ is Polish if and only if X is Polish.

Groups acting on topological spaces

Definition 1.2.1. Let G be a group and X a set. We say that G *acts on* X if we have a map $G \times X \to X$, which we denote by $(g, x) \mapsto gx$ for all $g \in G$ and $x \in X$, that satisfies the following properties:

- For all $x \in X$ we have $ex = x$, where $e \in G$ is the identity in G.
- For all $g, g' \in G$ and $x \in X$ we have $(gg')x = g(g'x)$.

We then call X a (left) *G-set*.
Moreover, if X is a measurable space, then X is a *measurable G-space* if the map $x \mapsto gx$ on X is measurable for each $g \in G$. We also say that G acts on X in a measurable manner.
If X is a topological space, then X is a *topological G-space* if $x \mapsto gx$ is continuous on X for each $g \in G$, and we say that G acts on X in a continuous manner.
Finally, if G is a topological group and the map $G \times X \to X$ given by $(g, x) \mapsto gx$ for $(g, x) \in G \times X$ is continuous, then the pair (G, X) is called a *transformation group*.

Clearly, if G acts on X in a continuous manner, then each $g \in G$ defines a homeomorphism on X. In that case, $x \mapsto gx$ is a map on X which is measurable with respect to the Borel σ-algebra on X, for each $g \in G$. Also, if (G, X) is a transformation group then X is a topological G-space.
When we ascribe a certain topological property to a transformation group (G, X), then we mean that both G and X have this property. For instance, if (G, X) is a locally compact Polish transformation group, then G is a locally compact Polish group and X is a locally compact Polish space.
From Definition 1.2.1 we see that giving an action of G on a set X is equivalent to giving a *representation* of G on X, a group homomorphism $\rho : G \to \mathrm{Aut}(X)$, with $\mathrm{Aut}(X)$ the group of automorphisms of X. If E is a Banach space then we are interested in representations $\rho : G \to \mathcal{B}(E)$. As noted in the Introduction we would like to decompose such representations, much as natural numbers can be decomposed into prime numbers by factorization. One of the ways to decompose a representation is the following.

[3]In probability theory this is usually called the weak topology. However, we can view the probability measures on X as elements of the dual of $C_b(X)$, the space of continuous bounded functions on X. We would then like to consider the weak* topology on this space, not the weak topology.

Definition 1.2.2. Let $\rho : G \to \mathcal{B}(E)$ be a representation of a group G on a Banach space E. A *direct sum decomposition* of ρ is a set $\{\rho_i\}_{i \in I}$, for I some index set, of representations $\rho_i : G \to \mathcal{B}(E_i)$, with $E_i \subset E$ a closed subspace for each $i \in I$, such that $E = \oplus_{i \in I} E_i$ and $\rho_i(g) = \rho(g)|_{E_i}$ for each $i \in I$ and $g \in G$.

Now let X be a measurable G-space. A measure μ on X is said to be an *invariant measure* (for G) if $\mu(g^{-1}(Y)) = \mu(Y)$ for all $Y \subset X$ measurable and $g \in G$. A measurable subset $Y \subset X$ is said to be an *invariant set* (for G) if $GY := \cup_{g \in G} gY = Y$. Moreover, we say that μ is *ergodic* if it is an invariant probability measure such that $\mu(Y) = 0$ or $\mu(Y) = 1$ for all invariant sets $Y \subset X$. We will study these ergodic measures a bit more later on.
Let $f : X \to \mathbb{R}$ be a function on X. For each $g \in G$ we can then define a function $gf : X \to \mathbb{R}$ by $gf(x) = f(g^{-1}x)$ for all $x \in X$. If f is measurable, then so is gf because $(gf)^{-1}(Y) = g(f^{-1}(Y))$ is measurable in X for each $Y \subset \mathbb{R}$ measurable. If μ is an invariant measure on X, then we can say even more:

Proposition 1.2.3. Suppose μ is a G-invariant measure on X. For each $g \in G$ and $p \in [1, \infty]$, the map $f \mapsto gf$ is an isometric lattice isomorphism on $L^p(X, \mu)$. G acts on $L^p(X, \mu)$ in a continuous manner and the map $\rho : G \to \mathcal{B}(L^p(X, \mu))$, $\rho(g)(f) := gf$ for all $g \in G$ and $f \in L^p(X, \mu)$, to the space of bounded operators on $L^p(X, \mu)$ is a representation of G as a group of isometric lattice isomorphisms on $L^p(X, \mu)$.
If (G, X) is in fact a transformation group, X is locally compact, μ is a regular finite measure and $p \in [1, \infty)$, then ρ is strongly continuous.

Proof:
Let $g \in G$ be given. We have already seen that gf is measurable for f measurable, and it is easy to see that g acts linearly. To show that $||f||_p = ||gf||_p$ for $f \in L^p(X, \mu)$, first assume $p \in [1, \infty)$. We use the standard machine. For f an indicator function the statement follows from the invariance of μ. The linearity of g then extends the result to simple functions. One easily checks that the monotone convergence theorem and splitting into positive and negative parts lead to $||f||_p = ||gf||_p$ for all $f \in L^p(X, \mu)$. On the other hand, for $p = \infty$ the statement follows immediately because

$$\mu \{x \in X : |gf(x)| > M\} = \mu \left\{x \in X : |f(g^{-1}x)| > M\right\}$$

$$= \mu(g \{x \in X : |f(x)| > M\}) = \mu \{x \in X : |f(x)| > M\}$$

for all $f \in L^\infty(X, \mu)$ and $M \geq 0$. So $\rho(g) \in \mathcal{B}(L^p(X, \mu))$ is indeed an isometric operator on $L^p(X, \mu)$ for all $p \in [1, \infty]$.
Now let $f, f' \in L^p(X, \mu)$ be given. Then

$$g(f \vee f')(x) = (f \vee f')(g^{-1}x) = f(g^{-1}x) \vee f'(g^{-1}x) = gf(x) \vee gf'(x) = (gf \vee gf')(x)$$

for almost all $x \in X$. So g acts as a lattice homomorphism. Since the same holds for $g^{-1} \in G$, $\rho(g) \in \mathcal{B}(L^p(X, \mu))$ is an isometric lattice isomorphism.
G acts on $L^p(X, \mu)$ and ρ is a representation because

$$(gg')f(x) = f((g')^{-1}g^{-1}x) = g'f(g^{-1}x) = g(g'f)(x)$$

for all $g, g' \in G$, $f \in L^p(X, \mu)$ and almost all $x \in X$. Clearly $ef = f$ for all $f \in L^p(X, \mu)$, where $e \in G$ is the identity element.
As for the strong continuity of ρ, assume that (G, X) is a transformation group, that

X is locally compact, μ a regular finite measure and that p is finite. Let $\{g_n\}_{n=1}^{\infty} \subset G$ be a sequence converging to some $g_0 \in G$. It suffices to show that $g_n f \to g_0 f$ for all $f \in C_c(X)$. Indeed, the continuous compactly supported functions lie dense in $L^p(X, \mu)$, and it is straightforward to reduce the general case to this dense subset, using that the operators $\rho(g) \in \mathcal{B}(L^p(X, \mu))$ are uniformly bounded by 1.
So let an $f \in C_c(X)$ be given. Since the map $(g, x) \mapsto gx$ is continuous on $G \times X$, we have $g_n^{-1}x \to g_0^{-1}x$ as $n \to \infty$ for each $x \in X$. Because f is continuous, $g_n f \to g_0 f$ pointwise. We also have

$$|g_n f(x)| \leq \sup_{y \in X} |f(y)| < \infty$$

for all $x \in X$ and $n \in \mathbb{N}$. Hence we can apply the dominated convergence theorem to see that $g_n f \to g_0 f$. □

We are now ready to formulate the main question which we will attempt to answer in this thesis. A representation of a group G on a Riesz space E is said to be *band irreducible* if G leaves only the trivial bands 0 and E in E invariant.

Let G be group, X a measurable G-space, μ a G-invariant measure and $p \in [1, \infty]$. Is it possible to decompose the representation ρ on $L^p(X, \mu)$ from Proposition 1.2.3 into band irreducible representations in some manner, and if so, under what hypotheses on the spaces involved? Moreover, if G is a topological group, when are the representations involved strongly continuous?

The answer, as we will see in Theorem 4.1.5, is that this can indeed be done for $p < \infty$. However, we do not use a direct sum decomposition as in Definition 1.2.2, but a type of integral decomposition which will be described in later chapters.
The reason why we ask the representations to be band irreducible lies partly in Proposition 1.1.14. We want to decompose the space $L^p(X, \mu)$ into simpler parts, and if we do this in a way that respects the lattice properties, we can expect bands to be involved. Since we would like to decompose a representation on this space in such a manner that we cannot decompose it any further, it seems natural to require that the representations which we decompose it into only leave trivial bands invariant.

Ergodic measures and band irreducibility In this section we let (G, X) be a locally compact Polish transformation group (These assumptions can be somewhat weakened. For details see [14]). We will now investigate the relationship between ergodic measures and band irreducibility of the action of G on $L^p(X, \mu)$.
We have remarked that the space $\mathcal{P}(X)$ of probability measures on X is a Polish space when endowed with the weak* topology. It is easy to see that the subset $\mathcal{I} \subset \mathcal{P}(X)$ of all G-invariant probability measures is convex. We will show that the subset $\mathcal{E} \subset \mathcal{I}$ of extreme points of $\mathcal{I}$ consists precisely of the ergodic measures on X. For this we need a lemma from [14, pp. 196-197] to help us classify these ergodic measures.
For subsets $Y, Y' \subset X$ we denote by $Y \Delta Y' := (Y \cup Y') \setminus (Y \cap Y')$ the *symmetric difference* of Y and Y'.

Lemma 1.2.4. Let $Y \subset X$ be a measurable subset. There exists a G-invariant set $Y' \subset X$ such that $\mu(Y \Delta Y') = 0$ for any $\mu \in \mathcal{I}$ with the property that $\mu(gY \Delta Y) = 0$ for every $g \in G$.

Note that the set Y' does not depend on the invariant measure μ, only on Y and G.

Corollary 1.2.5. For an invariant probability measure μ on X, the following are equivalent:

1. μ is ergodic.
2. $\mu(Y) = 0$ or $\mu(Y) = 1$ for every measurable $Y \subset X$ such that $\mu(gY\Delta Y) = 0$ for every $g \in G$.
3. $\mu \in \mathcal{E}$.

Proof:
First suppose that μ is ergodic. Let $Y \subset X$ be measurable such that $\mu(gY\Delta Y) = 0$ for all $g \in G$. By the above lemma, there exists an invariant set $Y' \subset X$ such that $\mu(Y\Delta Y') = 0$. Since μ is ergodic, $\mu(Y') = 0$ or $\mu(Y') = 1$. This then implies $\mu(Y) = 0$ or $\mu(Y) = 1$.
Now suppose that condition (2) holds and that we have $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for $\mu_1 \neq \mu_2$ in $\mathcal{I}$ and some $\lambda \in (0, 1]$. Then μ_1 is absolutely continuous with respect to μ. Let $f \geq 0$ be its Radon-Nikodym derivative with respect to μ. Since μ_1 is invariant, it is easy to see that $f(x) = f(g^{-1}x)$ almost everywhere for all $g \in G$. Set $Z := \{x \in X : f(x) \geq 1\}$. Then $\mu(gZ\Delta Z) = 0$ for each $g \in G$, so $\mu(Z) = 0$ or $\mu(Z) = 1$. From $\int_X f d\mu = \mu_1(X) = 1$ we can deduce $\mu(Y) = 1$ and $f(x) = 1$ almost everywhere. Then

$$\mu_1(Y) = \int_Y \mathbf{1} d\mu = \mu(Y)$$

for all $Y \subset X$ measurable and therefore $\mu_1 = \mu$, $\lambda = 1$. So μ is indeed an extreme point of I.
Finally, suppose that $Z \subset X$ is an invariant set such that $c := \mu(Z) \in (0, 1)$. Define measures μ_1 and μ_2 on X by

$$\mu_1(Y) := \frac{1}{c}\mu(Y \cap Z), \quad \mu_2(Y) := \frac{1}{1-c}\mu(Y \cap Z^c)$$

for $Y \subset X$ measurable. Then $\mu_1, \mu_2 \in \mathcal{I}$, $\mu_1 \neq \mu_2$ and $c\mu_1 + (1-c)\mu_2 = \mu$, so $\mu \notin \mathcal{E}$. □

The following proposition gives a hint on where to look for band irreducible decompositions of the representation ρ.

Proposition 1.2.6. Let μ be a G-invariant probability measure on X and $p \in [1, \infty]$. Then the action of G on $L^p(X, \mu)$ is band irreducible if and only if μ is ergodic.

Proof:
First suppose that ρ is band irreducible and let $Y \subset X$ be G-invariant. Consider the band

$$B := \{f \in L^p(X, \mu) : f(y) = 0 \text{ for almost all } y \in Y\} \subset L^p(X, \mu).$$

We have $GB \subset B$, so $B = 0$ or $B = L^p(X, \mu)$. Since $\mathbf{1}_{Y^c} \in B$, it is easy to see that these cases correspond to $\mu(Y) = 1$ respectively $\mu(Y) = 0$. So μ is ergodic.
Conversely, suppose μ is ergodic and let $B \subset L^p(X, \mu)$ be a band such that $GB \subset B$ holds. Write

$$B = \{f \in L^p(X, \mu) : f(y) = 0 \text{ for almost all } y \in Y\}$$

for some measurable $Y \subset X$, as in Example 1.1.15. Then $\mathbf{1}_{Y^c} \in B$ so $g\mathbf{1}_{Y^c} = \mathbf{1}_{gY^c} \in B$ for all $g \in G$. This implies $\mu(Y \cap gY^c) = 0$ and

$$\mu(Y \cap gY) = \mu(Y \setminus gY^c) = \mu(Y) = \mu(gY),$$

so $\mu(gY \Delta Y) = 0$ for all $g \in G$. By Corollary 1.2.5, $\mu(Y) = 0$ or $\mu(Y) = 1$. These cases correspond to $B = L^p(X, \mu)$ respectively $B = 0$. Either way, B is trivial and therefore ρ is band irreducible. $\square$

Chapter 2

Measure disintegration

In the present chapter we examine results from [6] and [14] which, together with Proposition 1.2.6, will be key to decomposing the action of a group on a space of Lebesgue integrable functions.

2.1 Ergodic decomposition

Measurability structures on the ergodic measures In this chapter we let (G, X) be a Polish transformation group, with G locally compact. Loosely speaking, we will see that we can decompose a G-invariant probability measure μ as an integral

$$\mu(Y) = \int_X \beta_x(Y) d\mu(x) \tag{2.1}$$

for each $Y \subset X$ measurable, with the $\beta_x \in \mathcal{E}$ ranging over the ergodic measures on X. However, for this expression to make sense, we need to know that the maps $x \mapsto \beta_x(Y)$ are measurable on X for $Y \subset X$ measurable. One way this could be true is if the map $\beta : X \to \mathcal{P}(X)$ given by $\beta(x) = \beta_x \in \mathcal{E}$ for all $x \in X$, is measurable with respect to the σ-algebra on $\mathcal{P}(X)$ generated by the maps $\lambda \mapsto \lambda(Y)$ on $\mathcal{P}(X)$, for $Y \subset X$ measurable. Indeed, then the composition $x \mapsto \beta_x(Y)$ is measurable on X. Let $\mathcal{A}$ denote this σ-algebra on $\mathcal{P}(X)$, i.e., $\mathcal{A}$ is the smallest σ-algebra for which the maps $\lambda \mapsto \lambda(Y)$ are measurable for each $Y \subset X$ measurable.
Now recall from the previous chapter that we have a weak* topology on the set of probability measures $\mathcal{P}(X)$ which turns $\mathcal{P}(X)$ into a Polish space. So also have a Borel σ-algebra $\mathcal{B}$ on $\mathcal{P}(X)$ with respect to this topology. Fortunately, it turns out these σ-algebras are the same:

Proposition 2.1.1. $\mathcal{A} = \mathcal{B}$.

Proof:
First we show $\mathcal{A} \subset \mathcal{B}$. Let $\mathcal{L}$ be the class of all bounded measurable functions $f : X \to \mathbb{R}$ on X such that the map $\lambda \mapsto \int_X f d\lambda$ is measurable on $\mathcal{P}(X)$ with respect to $\mathcal{B}$. Then it is straightforward to check that $\mathcal{L}$ is a vector space containing $C_b(X)$, the set of continuous bounded functions on X. Also, by applying the dominated convergence theorem one sees that $f \in \mathcal{L}$ when $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}$ is a sequence increasing pointwise to some bounded

$f : X \to \mathbb{R}$. The monotone class theorem now tells us that $\mathcal{L}$ contains all bounded measurable functions on X, and in particular all indicator functions $\mathbf{1}_Y$ of measurable subsets $Y \subset X$. So

$$\lambda \mapsto \int_X \mathbf{1}_Y d\lambda = \lambda(Y)$$

is measurable with respect to $\mathcal{B}$ for all $Y \subset X$ measurable, and $\mathcal{A} \subset \mathcal{B}$ holds true.
For the other inclusion it suffices to show that each open set in $\mathcal{P}(X)$ is an element of $\mathcal{A}$. To this end, remark that for each bounded measurable $f : X \to \mathbb{R}$, the map $\lambda \mapsto \int_X f d\lambda$ is $\mathcal{A}$-measurable on $\mathcal{P}(X)$. Indeed, we know that this is true for characteristic functions, and using linearity and the monotone convergence theorem we can show that it holds for all such f. So for all $\lambda_0 \in \mathcal{P}(X)$, $\epsilon > 0$, $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in C_b(\mathbb{R})$,

$$\left\{\lambda \in \mathcal{P}(X) : \max_{1 \le i \le n} \left| \int_X f_i d\mu - \int_X f_i d\lambda \right| < \epsilon \right\} = \bigcap_{i=1}^n \left\{\lambda \in \mathcal{P}(X) : \left| \int_X f_i d\mu - \int_X f_i d\lambda \right| < \epsilon \right\} \in \mathcal{A}.$$

Since these sets form a basis for the weak* topology on $\mathcal{P}(X)$, any open set in $\mathcal{P}(X)$ is a union of such sets. Furthermore, as $\mathcal{P}(X)$ is Polish and thus second-countable, any open set is a countable union of such elements in $\mathcal{A}$ and is therefore an element of $\mathcal{A}$. □

We also remark that the sets $\mathcal{I}$ and $\mathcal{E}$ in $\mathcal{P}(X)$ are Borel measurable [11, p. 1119]. Applying the above result to the induced weak* topology on the subset $\mathcal{E} \subset \mathcal{P}(X)$ we find:

Corollary 2.1.2. The Borel σ-algebra on $\mathcal{E}$ (with respect to this induced weak* topology) is the σ-algebra generated by the maps $\lambda \mapsto \lambda(Y)$ on $\mathcal{E}$, for $Y \subset X$ measurable.

As a subset of a Polish space, $\mathcal{E}$ is separable and metrizable in the induced weak* topology. In the remainder all matters of topology on $\mathcal{E}$ will refer to this topology, and all matters of measurability to its Borel structure.

Decomposition maps From the discussion in the previous paragraph we conclude that we are looking for a measurable map $\beta : X \to \mathcal{E}$, $x \mapsto \beta_x$, such that (2.1) holds. First we treat an example in which this can be done explicitly.

Example 2.1.3. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| \le 1\} \subset \mathbb{C}$ be the closed unit disc and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$ the unit circle. Then $\mathbb{T}$ is a compact Polish group under multiplication and the induced topology of $\mathbb{C}$. Similarly, $\mathbb{D}$ is a compact Polish space. The map $\mathbb{T} \times \mathbb{D} \to \mathbb{D}$ given by $(e^{i\eta}, re^{i\theta}) \mapsto re^{i(\eta+\theta)}$ for $\eta, \theta \in [0, 2\pi)$ and $r \in [0, 1]$, defines a continuous action of $\mathbb{T}$ on $\mathbb{D}$ and $(\mathbb{T}, \mathbb{D})$ is a compact Polish transformation group.
The ergodic measures on $\mathbb{D}$ are precisely the normalized rotation-invariant measures supported on the circles $r\mathbb{T}$ for $r \in [0, 1]$ (one can determine explicitly that the ergodic measures are supported on the orbits of points, but we will also remark later that this follows in general from the compactness of $(\mathbb{T}, \mathbb{D})$). So $\mathcal{E} = \{\lambda_r : r \in [0, 1]\}$, where

$$\lambda_r(Y) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{1}_Y(re^{i\theta}) d\theta$$

for $Y \subset \mathbb{D}$ measurable and $r \in [0, 1]$.
We show that $\mathcal{E}$, when endowed with the weak* topology, is homeomorphic to the unit interval $[0, 1]$. Indeed, consider the map $\phi : [0, 1] \to \mathcal{E}$ given by $\phi(r) = \lambda_r \in \mathcal{E}$ for $r \in [0, 1]$.

Let $\{r_n\}_{n=1}^{\infty} \subset [0,1]$ be a sequence converging to some $r \in [0,1]$ and $f : \mathbb{D} \to \mathbb{R}$ continuous and bounded. Then we have

$$\int_{\mathbb{D}} f(z) d\lambda_r(z) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

and similar expressions for the integral of f with respect to each λ_{r_n}, $n \in \mathbb{N}$. The functions $\theta \mapsto f(r_n e^{i\theta})$ on $[0, 2\pi]$ converge pointwise to $\theta \mapsto f(re^{i\theta})$ because f is continuous. Moreover, as f is bounded we can use the dominated convergence theorem to find

$$\int_{\mathbb{D}} f(z) d\lambda_{r_n} = \frac{1}{2\pi} \int_0^{2\pi} f(r_n e^{i\theta}) d\theta \to \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta = \int_{\mathbb{D}} f(z) d\lambda_r$$

as $n \to \infty$. By the arbitrariness of $f \in C_b(\mathbb{D})$, $\phi(r_n) = \lambda_{r_n} \to \lambda_r = \phi(r)$ in $\mathcal{E}$. So ϕ is continuous as a map from $[0,1]$ to $\mathcal{E}$. It is clearly bijective. Because $[0,1]$ is compact and $\mathcal{E}$ Hausdorff, we can use a well-known lemma from topology which tells us that ϕ is in fact a homeomorphism, and we conclude that $\mathcal{E}$ is indeed homeomorphic to the compact unit interval $[0,1]$.
Now consider the map $\beta : \mathbb{D} \to \mathcal{E}$ given by $\beta(re^{i\theta}) = \phi(r) = \lambda_r \in \mathcal{E}$, for $r \in [0,1]$ and $\theta \in [0, 2\pi)$. Then β is continuous, and thus Borel measurable, because $\phi^{-1} \circ \beta : \mathbb{D} \to [0,1]$ is continuous. Let μ be the normalized Lebesgue measure on $\mathbb{D}$ given by

$$\mu(Y) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r \mathbf{1}_Y(re^{i\theta}) d\theta dr$$

for $Y \subset \mathbb{D}$ measurable. This is a $\mathbb{T}$-invariant probability measure on $\mathbb{D}$. Furthermore, for any $Y \subset \mathbb{D}$ measurable we have

$$\begin{aligned} \mu(Y) &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r \mathbf{1}_Y(re^{i\theta}) d\theta dr = 2 \int_0^1 r \left(\frac{1}{2\pi} \int_0^{2\pi} \mathbf{1}_Y(re^{i\theta}) d\theta \right) dr = 2 \int_0^1 r \lambda_r(Y) dr \\ &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r \lambda_r(Y) dr d\theta = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r \beta(re^{i\theta})(Y) dr d\theta = \int_{\mathbb{D}} \beta_z(Y) d\mu(z). \end{aligned}$$

So the map β is indeed the decomposition map for μ that we were looking for.

Below we will see that we can find such a decomposition map for any invariant probability measure μ on X. Furthermore, it turns out that this map does not depend on the invariant measure μ that we choose, and that it is in some sense unique. To understand what uniqueness we are referring to, we make the following definition:

Definition 2.1.4. A measurable subset $Y \subset X$ is said to be *G-negligible* if $\mu(Y) = 0$ for all invariant probability measures $\mu \in \mathcal{I}$ on X. We denote the family of all G-negligible subsets of X by $\mathcal{N}$.

The G-negligible sets form a *σ-ideal* in the Borel σ-algebra on X. This means that $\emptyset \in \mathcal{N}$, that $Y \in \mathcal{N}$ for all $Y \subset X$ measurable which satisfy $Y \subset Z$ for some $Z \in \mathcal{N}$, and that $\mathcal{N}$ is closed under countable unions. All these properties are straightforward to check.
We are now ready to state the results of [6] and [14] about the existence of a decomposition map. We have taken the theorem itself from [11, p. 1119], where a convenient summary of their work can be found.

Theorem 2.1.5. Suppose there exists an invariant probability measure on X, so $\mathcal{I} \neq \emptyset$. Then $\mathcal{E} \neq \emptyset$ and there exists a Borel measurable surjection $\beta : X \to \mathcal{E}$, $x \mapsto \beta_x$ for $x \in X$, called a *decomposition map*, satisfying the following properties:

1. For all $x \in X$ and $g \in G$, $\beta_{gx} = \beta_x$.
2. For every $\lambda \in \mathcal{E}$, $\lambda(\beta^{-1}\{\lambda\}) = \lambda\{x \in X : \beta(x) = \lambda\} = 1$.
3. For any invariant probability measure μ on X,

$$\mu(Y) = \int_X \beta_x(Y) d\mu(x) \tag{2.2}$$

for all $Y \subset X$ measurable.

Furthermore, if $\beta' : X \to \mathcal{E}$ is another decomposition map with the above properties, then there exists a G-negligible $Y \in \mathcal{N}$ such that $\beta_x = \beta'_x$ for all $x \in Y^c$.

A few remarks are now in order.
In the above theorem we require that $\mathcal{I} \neq \emptyset$, so a question that remains is when there exists an invariant probability measure on X. A sufficient condition is given by the following result, from [11, p. 1118].

Proposition 2.1.6. If (G, X) is a compact Polish transformation group then $\mathcal{I} \neq \emptyset$.

Property (1) tells us that β is constant on G-orbits. In general it need not be true that the ergodic measures are supported on single G-orbits. However, this was the case in Example 2.1.3 and in fact, if (G, X) is a compact Polish transformation group then for any G-orbit $Gx \subset X$ there exists a unique ergodic measure supported on Gx [11, p. 1119]. So in that case the ergodic measures are indeed supported on the G-orbits.
There are generalizations of these decomposition results to quasi-invariant measures. For more details on measure decompositions see [11, pp. 1101-1140]

2.2 Consequences of the ergodic decomposition

Decomposing integrals on X Now that we know how to decompose an invariant probability measure μ on X into an integral of ergodic measures, we take a look at the consequences of this decomposition for the spaces $L^p(X, \mu)$, $p \in [1, \infty)$.

Proposition 2.2.1. For any $f \in L^1(X, \mu)$ the following holds: $f \in L^1(X, \beta_x)$ for μ-almost all $x \in X$, the map $x \mapsto \int_X f d\beta_x$ is a μ-almost everywhere defined map that is integrable with respect to μ, and we can write

$$\int_X f d\mu = \int_X \left(\int_X f d\beta_x \right) d\mu(x). \tag{2.3}$$

Proof:
We use the standard machine. First assume that $f = \mathbf{1}_Y$ for some $Y \subset X$ measurable. Then $\beta_x(Y)$ is finite for all $x \in X$, so $f \in L^1(X, \beta_x)$ for almost all $x \in X$. Because

$\beta : X \to \mathcal{E}$ is measurable, the map $x \mapsto \beta_x(Y) = \int_X f d\beta_x$ is measurable. Furthermore, equation (2.2) implies

$$\int_X f d\mu = \mu(Y) = \int_X \beta_x(Y) d\mu(x) = \int_X \left(\int_X f d\beta_x \right) d\mu(x),$$

and since this quantity is finite, the map $x \mapsto \int_X f d\beta_x$ is μ-integrable.
If $f \in L^1(X, \mu)$ is a simple function, write $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{Y_i}$ for certain $n \in \mathbb{N}$, $\alpha_i \in [0, \infty]$ and $Y_i \subset X$ measurable, $1 \leq i \leq n$. Then the map $x \mapsto \int_X f d\beta_x = \sum_{i=1}^n \alpha_i \cdot \beta_x(Y_i)$, as a linear combination of measurable functions, is measurable on X. Linearity of the integral implies

$$\begin{aligned} \int_X f d\mu &= \sum_{i=1}^n \alpha_i \mu(Y_i) = \sum_{i=1}^n \alpha_i \int_X \beta_x(Y_i) d\mu(x) \\ &= \int_X \sum_{i=1}^n \alpha_i \left(\int_X \mathbf{1}_{Y_i} d\beta_x \right) d\mu(x) = \int_X \left(\int_X f d\beta_x \right) d\mu(x). \end{aligned}$$

Because this quantity is finite, the map $x \mapsto \int_x f d\beta_x$ is μ-integrable and $\int_X f d\beta_x < \infty$ for μ-almost all $x \in X$.
Now suppose $f = \sup_{n \in \mathbb{N}} f_n \in L^1(X, \mu)$ is the supremum of an increasing sequence of simple functions $\{f_n\}_{n=1}^\infty \subset L^1(X, \mu)$. Then $x \mapsto \int_X f d\beta_x = \sup_{n \in \mathbb{N}} \int_X f_n d\beta_x$ is measurable, as the supremum of a sequence of measurable functions. Furthermore,

$$\int_X f d\mu = \sup_{n \in \mathbb{N}} \int_X f_n d\mu = \sup_{n \in \mathbb{N}} \int_X \left(\int_X f_n d\beta_x \right) d\mu(x) = \int_X \left(\int_X f d\beta_x \right) d\mu(x)$$

by the monotone convergence theorem and what we have shown above. Since $\int_X f d\mu$ is finite, the map $x \mapsto \int_X f d\beta_x$ is μ-integrable and $\int_X f d\beta_x < \infty$ for μ-almost all x.
Finally, let $f \in L^1(X, \mu)$ be arbitrary and let $f = f^+ - f^-$ be its decomposition into a positive part f^+ and negative part f^-. Then $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ are finite, so they are elements of $L^1(X, \beta_x)$ for μ-almost all $x \in X$. The same then holds for f. So $x \mapsto \int_X f d\beta_x = \int_X f^+ d\beta_x - \int_X f^- d\beta_x$ is well-defined and measurable almost everywhere, as the difference of two almost everywhere finite measurable functions. Complete its definition in some measurable way to all of X (for instance by setting it equal to zero where the above expression is not defined). Then

$$\begin{aligned} \int_X f d\mu &= \int_X f^+ d\mu - \int_X f^- d\mu = \int_X \left(\int_X f^+ d\beta_x \right) d\mu(x) - \int_X \left(\int_X f^- d\beta_x \right) d\mu(x) \\ &= \int_X \left(\int_X f^+ d\beta_x - \int_X f^- d\beta_x \right) d\mu(x) = \int_X \left(\int_X f d\beta_x \right) d\mu(x), \end{aligned}$$

where the function $x \mapsto \int_X f d\beta_x = \int_X f^+ d\beta_x - \int_X f^- d\beta_x$ is almost everywhere well-defined and μ-integrable (because the quantity above is finite). □

Push-forward measures and integration on $\mathcal{E}$ So far we have considered integration on X, but we also have a Borel structure on $\mathcal{E}$. We would like to integrate over this space, and so we 'transfer' a measure μ on X to $\mathcal{E}$. To be more precise, the *push-forward measure* of μ through β is defined to be the measure ν on $\mathcal{E}$ given by $\nu(A) = \mu(\beta^{-1}(A))$

for all $A \subset \mathcal{E}$ measurable. This is well-defined because $\beta : X \to \mathcal{E}$ is measurable. If μ is a probability measure then so is ν, and in that case ν is normal, as a finite measure on a metrizable space.
A general result about push-forward measures applied to this specific setting is the following.

Lemma 2.2.2. For any measurable $f : \mathcal{E} \to \mathbb{R}$, $f \in L^1(\mathcal{E}, \nu)$ if and only if $f \circ \beta \in L^1(X, \mu)$, in which case we have

$$\int_{\mathcal{E}} f d\nu = \int_X f \circ \beta \, d\mu. \tag{2.4}$$

Proof:
First remark that $f \circ \beta : X \to \mathbb{R}$ is measurable, as a composition of measurable mappings. Again we use the standard machine. First suppose $f = \mathbf{1}_Y$ for some measurable $Y \subset \mathcal{E}$. Then $f \circ \beta = \mathbf{1}_{\beta^{-1}(Y)}$ and

$$\int_{\mathcal{E}} f d\nu = \nu(Y) = \mu(\beta^{-1}(Y)) = \int_X f \circ \beta \, d\mu.$$

If $n \in \mathbb{N}$, $\alpha_i \in [0, \infty]$ and $Y_i \subset \mathcal{E}$ measurable, for $1 \leq i \leq n$, are such that $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{Y_i}$, then $f \circ \beta = \sum_{i=1}^n \alpha_i \mathbf{1}_{\beta^{-1}(Y_i)}$ and

$$\int_{\mathcal{E}} f d\nu = \sum_{i=1}^n \alpha_i \nu(Y_i) = \sum_{i=1}^n \alpha_i \mu(\beta^{-1}(Y_i)) = \int_X f \circ \beta \, d\mu.$$

Now suppose $f = \sup_{n \in \mathbb{N}} f_n \geq 0$ for some increasing sequence of simple functions on $\mathcal{E}$. Then $f \circ \beta = \sup_{n \in \mathbb{N}} f_n \circ \beta$ and

$$\int_{\mathcal{E}} f d\nu = \sup_{n \in \mathbb{N}} \int_{\mathcal{E}} f_n d\nu = \sup_{n \in \mathbb{N}} \int_X f_n \circ \beta d\mu = \int_X f \circ \beta \, d\mu$$

by the monotone convergence theorem.
Finally let $f : \mathcal{E} \to \mathbb{R}$ be an arbitrary measurable function and let f^+, f^- be its positive respectively negative part. Then $f \circ \beta = f^+ \circ \beta - f^- \circ \beta$. If either $f \in L^1(\mathcal{E}, \nu)$ or $f \circ \beta \in L^1(X, \mu)$, then the following chain of equalities makes sense and the quantities are finite:

$$\int_{\mathcal{E}} f \nu = \int_{\mathcal{E}} f^+ d\nu - \int_{\mathcal{E}} f^- d\nu = \int_X f^+ \circ \beta \, d\mu - \int_X f^- \circ \beta \, d\mu = \int_X f \circ \beta \, d\mu.$$

So $f \circ \beta \in L^1(X, \mu)$ if and only if $f \in L^1(\mathcal{E}, \nu)$. $\square$

By combining Proposition 2.2.1 and Lemma 2.2.2 we can express the integral of a μ-integrable function on X as an integral over $\mathcal{E}$.

Corollary 2.2.3. Let $f \in L^1(X, \mu)$ be given. Then $f \in L^1(X, \lambda)$ for ν-almost all $\lambda \in \mathcal{E}$, the map $\lambda \mapsto \int_X f d\lambda$ is a ν-almost everywhere defined element of $L^1(\mathcal{E}, \nu)$ and

$$\int_X f d\mu = \int_{\mathcal{E}} \left(\int_X f d\lambda \right) d\nu(\lambda). \tag{2.5}$$

Proof:
As was hinted upon in the proof of Proposition 2.1.1, we can show that the map $\lambda \mapsto \int_X |f| d\lambda$ is measurable by using the standard machine. This implies that the set $Y := \left\{\lambda \in E(X) : \int_X |f| d\lambda = \infty\right\}$ is measurable in $E(X)$, and this is precisely the set of λ for which $f \notin L^1(X, \lambda)$. We now apply the first statement in Proposition 2.2.1 to conclude that

$$\nu(Y) = \mu(\beta^{-1}(Y)) = \mu\left\{x \in X : f \notin L^1(X, \beta_x)\right\} = 0.$$

Thus $f \in L^1(X, \lambda)$ indeed holds for ν-almost all $\lambda \in \mathcal{E}$ and $\lambda \mapsto \int_X f d\lambda$ is a ν-almost everywhere defined function on $\mathcal{E}$. As remarked above in the case of $|f|$, the standard machine shows that it is measurable where defined. Complete the definition in some manner to a measurable function g on all of $\mathcal{E}$. Now note that the map $x \mapsto \int_X f d\beta_x$ is the composition $g \circ \beta$ where defined. We have seen in Proposition 2.2.1 that this is the case μ-almost everywhere on X and that $g \circ \beta \in L^1(X, \mu)$. Lemma 2.2.2 tells us that $g \in L^1(\mathcal{E}, \nu)$, so $\lambda \mapsto \int_X f d\lambda$ indeed is an almost everywhere defined element of $L^1(X, \nu)$. Finally, combining the previous two results we find

$$\int_X f d\mu = \int_X \left(\int_X f d\beta_x\right) d\mu(x) = \int_X g \circ \beta \, d\mu = \int_{\mathcal{E}} g \, d\nu = \int_{\mathcal{E}} \left(\int_X f d\lambda\right) d\nu(\lambda).$$

□

We can interpret this result in another way. Fix a $p \in [1, \infty)$ and consider the spaces $L^p(X, \mu)$ and $L^p(X, \lambda)$, for $\lambda \in \mathcal{E}$. Write

$$||f||_\mu = \left(\int_X |f|^p d\mu\right)^{1/p}$$

for the p-norm of an $f \in L^p(X, \mu)$ and

$$||f||_\lambda = \left(\int_X |f|^p d\lambda\right)^{1/p}$$

for the p-norm of an $f \in L^p(X, \lambda)$, for any $\lambda \in \mathcal{E}$. Then the previous result can be alternatively phrased as

Corollary 2.2.4. Let $f \in L^p(X, \mu)$ be given. Then $f \in L^p(X, \lambda)$ for ν-almost all $\lambda \in \mathcal{E}$, the map $\lambda \mapsto ||f||_\lambda$ is a ν-almost everywhere defined element of $L^p(\mathcal{E}, \nu)$ and

$$||f||_\mu = \left(\int_{\mathcal{E}} ||f||_\lambda^p d\nu(\lambda)\right)^{1/p} \tag{2.6}$$

holds.

Proof:
Just apply Corollary 2.2.3 to $|f|^p \in L^1(X, \mu)$. □

Now we know that we can view the norm of an $f \in L^p(X, \mu)$ as a p-integral of the norms of f as an element of $L^p(X, \nu)$, for ν-almost all $\lambda \in \mathcal{E}$. This is precisely what makes us think that there might be a way of decomposing $L^p(X, \mu)$ as 'p-integral' of the spaces $L^p(X, \lambda)$, for $\lambda \in \mathcal{E}$. Furthermore, we have seen in Proposition 1.2.6 that ergodic measures are related to band irreducibility of the action of G. We will see in Chapter 4 that we have in fact already done half the work in proving such a decomposition. All that remains is to establish the formalism necessary to make the phrase 'p-integral over the ergodic measures $\lambda \in \mathcal{E}$ of the spaces $L^p(X, \lambda)$' somewhat more precise. This is what we will do in the next chapter.

Chapter 3

The theory of Banach bundles

In this chapter we take a side-track from what we have considered so far and give a short summary of the theory of Banach bundles, as gathered from [7, pp. 10-30], [8, pp. 99-112] and [9, pp. 125-162]. This concept will prove to be a central tool in our decomposition of the action of the group G on the space $L^p(X, \mu)$.

3.1 Banach bundles

Definition and examples First we give the main definitions and examine some elementary examples of Banach bundles. In this chapter we let X be a Hausdorff space, unless explicitly mentioned. Let $\mathbb{F}$ denote either the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

Definition 3.1.1. A *bundle* $\mathfrak{B}$ over X is a pair (B, π), where B is a Hausdorff space and $\pi : B \to X$ is a continuous open surjection.

We call B the *bundle space* of $\mathfrak{B}$, X the *base space* of $\mathfrak{B}$ and π the *bundle projection* of $\mathfrak{B}$. For any $x \in X$, $\pi^{-1}(x) \subset B$ is the *fiber* over x and we denote it by B_x.

Definition 3.1.2. Let $\mathfrak{B} = (B, \pi)$ be a bundle over X. A function $s : X \to B$ is called a *cross-section* (or simply a section) of $\mathfrak{B}$ if $\pi \circ s = \mathrm{id}_X$, i.e. if $s(x) \in B_x$ for all $x \in X$. A *continuous section* is a cross-section which is continuous as a map from X to B. The set of all continuous cross-sections of $\mathfrak{B}$ will be denoted by $C(\mathfrak{B})$.[1] We say that the bundle $\mathfrak{B}$ has *enough continuous sections* if, for each $b \in B$, there exist a continuous section $s \in C(\mathfrak{B})$ and an $x \in X$ such that $s(x) = b$.

If $f : A \to C$ and $g : B \to C$ are maps between sets, then the *fiber product* of (A, f) and (B, g) over C is the subset $A \times_C B := \{(a, b) \in A \times B : f(a) = g(b)\}$ of $A \times B$. When A and B are topological spaces, then $A \times_C B$ carries the induced topology of $A \times B$. For instance, if (B, π) is a bundle over X, then $B \times_X B = \{(b, c) \in B \times B : \pi(b) = \pi(c)\}$.
We now wish to consider bundles in which the fibers themselves carry the structure of a Banach space. This leads us to the following concept:

[1] Do not confuse this with $C(B)$, the set of continuous scalar-valued functions of the Hausdorff space B.

Definition 3.1.3. A *Banach bundle* over X is a bundle $\mathfrak{B} = (B, \pi)$ over X, together with maps

$$+ : B \times_X B \to B,$$
$$\cdot : \mathbb{F} \times B \to B,$$
$$|| \cdot || : B \to [0, \infty),$$

satisfying the following conditions:

1. For each $x \in X$, $B_x \subset B$ is a Banach space over $\mathbb{F}$ under the restrictions of the operations of addition $+$, scalar multiplication $\cdot$ and norm $|| \cdot ||$ to B_x.
2. $|| \cdot ||$ is continuous on B.
3. The addition operator $+$ is continuous on $B \times_X B$.
4. For each $\lambda \in \mathbb{F}$, the map $b \mapsto \lambda \cdot b$ is continuous as a map from B to B.
5. For any $x \in X$ and any net $\{b_i\}_{i \in I} \subset B$ such that $||b_i|| \to 0$ and $\pi(b_i) \to x$ we have $b_i \to 0_x$, where 0_x is the zero element of B_x.

We will not distinguish in notation between operations in different fibers, so unless explicitly mentioned we use the same $+$, $\cdot$ and $|| \cdot ||$ for the operations in any fiber. Note that condition 1 implies that $\pi(\lambda b + c) = \pi(b) = \pi(c)$ for all $\lambda \in \mathbb{F}$ and $(b, c) \in B \times_X B$. Also remark that addition of elements in different fibers is in general not defined, hence a Banach bundle is not a vector space but a bundle of vector spaces.
We now give some examples of Banach bundles.

Example 3.1.4. Let A be a Banach space. If we put $B := X \times A$, $\pi(x, a) := x$ for $(x, a) \in X \times A$ and endow B with the product topology, then it is easy to see that $\mathfrak{B} := (B, \pi)$ is a Banach bundle over X when each fiber carries the Banach space structure of A. This bundle is called the *trivial bundle* with constant fiber A. Clearly the trivial bundle has enough continuous sections.
Sometimes the distinction between functions $f : X \to A$ and cross-sections of $\mathfrak{B}$ will be ignored. This is justified since any function $f : X \to A$ gives rise to a cross-section $s_f : X \to B$ via $s(x) = (x, f(x))$. This correspondence is one-to-one, and continuous functions correspond to continuous sections and vice versa.

Example 3.1.5. Let $\mathfrak{B} = (B, \pi)$ be a Banach bundle in which each B_x, $x \in X$, has the structure of a Hilbert space. Then $\mathfrak{B}$ is a called a *Hilbert bundle* over X. In this case the inner product is continuous as a map from $B \times_X B$ to $\mathbb{F}$. Indeed, let $\{(b_i, c_i)\}_{i \in I} \subset B \times_X B$ be a net converging to some $(b, c) \in B \times_X B$, and assume for the moment that $\mathbb{F} = \mathbb{R}$. Then the polarization identity for the inner product and continuity of the Banach bundle operations imply that

$$\langle b_i, c_i \rangle = \frac{1}{4}(||b_i + c_i||^2 - ||b_i - c_i||^2) \to \frac{1}{4}(||b + c||^2 - ||b - c||^2) = \langle b, c \rangle.$$

In the case $\mathbb{F} = \mathbb{C}$ we can use a similar polarization identity to reach the same conclusion.

Example 3.1.6. Let $f : Y \to X$ be a continuous map between Hausdorff spaces and let $\mathfrak{B} = (B, \pi)$ be a Banach bundle over X. Consider the fiber product

$$Y \times_X B = \{(y, b) \in Y \times B : f(y) = \pi(b)\}$$

and the map $\rho : Y \times_X B \to Y$ given by $\rho(y, b) = y$. Then $Y \times_X B$ is Hausdorff and ρ is a continuous surjection. To see that it is open, we use a lemma from topology which will also be useful later on.

Lemma 3.1.7. Let $\rho : Z \to Y$ be a surjection between topological spaces. Then ρ is open if and only if, for any net $\{y_i\}_{i\in I} \subset Y$ converging to some $f(z) \in Y$, there exist a subnet $\{y_{i_j}\}_{j\in J}$ of $\{y_i\}_{i\in I}$ and a net $\{z_j\}_{j\in J} \subset Z$ with the same index set such that $f(z_j) = y_{i_j}$ for all $j \in J$ and $z_j \to z$ in Z.

Proof:
First suppose that the latter condition holds and that f is not open. Let $U \subset Z$ open and $z \in U$ be such that $f(z) \in f(U)$ is not an element of the interior of $f(U)$. Then there exists a net $\{y_i\}_{i\in I} \subset f(U)^c$ such that $y_i \to f(z)$. By assumption, there exist a subnet $\{y_{i_j}\}_{j\in J}$ and a net $\{z_j\}_{j\in J} \subset X$ such that $f(z_j) = y_{i_j}$ for all $j \in J$ and $z_j \to z$. Because U is open, for all large enough $j \in J$, $z_j \in U$. Then $y_{i_j} = f(z_j) \in f(U)$ for all large enough j as well, a contradiction. So f must in fact be open.
Now assume that f is open and let $\{y_i\}_{i\in I}$ and $f(z)$ in Y be as in the statement of the lemma. We form a directed set J in the following manner: let J be the set of all pairs (i, U), where $i \in I$ and $U \subset Z$ is an open neighbourhood of z. Then we define an ordering on J by $(i, U) \le (i', U')$ if $i \le i'$ in I and $U' \subset U$. This indeed makes J a directed set, and for any $j = (i, U)$ we can find an $i_j \ge i$ and a $y_{i_j} \in f(U)$, because f is open and y_i converges to $f(z)$. Choose a $z_j \in U$ such that $f(z_j) = y_{i_j}$. The sequences $\{y_{i_j}\}_{j\in J}$ and $\{z_j\}_{j\in J}$ are as required. □

Note that we have not made use of any assumptions on the spaces Z and Y above or on continuity of the map ρ.
Returning to our example, let $(y, b) \in Y \times_X B$ and a net $\{y_i\}_{i\in I} \subset Y$ such that $y_i \to y = \rho((y, b))$ be given. Then $f(y_i) \to f(y) = \pi(b)$. Since π is an open surjection, we can use the above lemma to find a subnet $\{f(y_{i_j})\}_{j\in J}$ of $\{f(y_i)\}_{i\in I}$ and a net $\{b_j\}_{j\in J} \subset B$ such that $\pi(b_j) = f(y_{i_j})$ for all j and $b_j \to b$. This means that each (y_{i_j}, b_j) is an element of $Y \times_X B$ and that $(y_{i_j}, b_j) \to (y, b)$ in $Y \times_X B$. We can then apply the lemma once again to the sequences $\{(y_{i_j}, b_j)\}_{j\in J}$ and $\{y_{i_j}\}_{j\in J}$ to conclude that ρ is open.
So $\mathcal{C} := (Y \times_X B, \rho)$ is bundle over Y. For each $y \in Y$ give $\rho^{-1}(y) \subset Y \times_X B$ the Banach space structure of $B_{f(y)}$. Then $\mathcal{C}$ is a Banach bundle over Y, called the *bundle retraction* of $\mathfrak{B}$ by f. If $\mathfrak{B}$ has enough continuous sections, then so does $\mathcal{C}$.
An application of this is the following. Let $\mathfrak{B} = (B, \pi)$ be a Banach bundle over X and $Y \subset X$ a subspace of X with the induced topology. Set $B_Y := Y \times_X X = \pi^{-1}(Y)$. Then $\mathfrak{B}_Y := (B_Y, \pi|_{B_Y})$ is a Banach bundle over Y, also called the *reduction* of $\mathfrak{B}$ to Y.

Elementary properties of Banach bundles We now prove some simple properties of a Banach bundle $\mathfrak{B} = (B, \pi)$ over X.

Proposition 3.1.8. The scalar multiplication map $\cdot : \mathbb{F} \times B \to B$, $(\lambda, b) \mapsto \lambda \cdot b$, is continuous.

Proof:
Let $\{(\lambda_i, b_i)\}_{i\in I} \subset \mathbb{F} \times B$ be convergent to some $(\lambda, b) \in \mathbb{F} \times B$. Then $\lambda_i \to \lambda$ in $\mathbb{F}$ and $b_i \to b$ as $i \to \infty$. So $\pi(b_i) \to \pi(b)$. Since $(\lambda_i - \lambda)b_i \in B_{\pi(b_i)}$ for all i, we have $\pi(\lambda_i b_i - \lambda b_i) = \pi(b_i) \to \pi(b)$. Furthermore,

$$||\lambda_i b_i - \lambda b_i|| = |\lambda_i - \lambda| \cdot ||b_i|| \to 0 \cdot ||b|| = 0$$

since the norm is continuous. Now we can apply condition 5 in Definition 3.1.3 to conclude that $(\lambda_i - \lambda)b_i \to 0_{\pi(b)}$. Also, $\lambda b_i \to \lambda b$ by condition 4. Then $\lambda_i b_i = \lambda b_i + (\lambda_i - \lambda)b_i \to \lambda b + 0_{\pi(b)} = \lambda b$ by continuity of addition. □

The next proposition connects the topology of a fiber to the topology it carries as a Banach space.

Proposition 3.1.9. For each $x \in X$ the relative topology of B_x is equal to the topology induced by the norm on B_x.

Proof:
Let $x \in X$ and a net $\{b_i\}_{i\in I} \subset B_x$ which converges to some $b \in B_x$ be given. Then $b_i - b \to b - b = 0_x$ because of continuity of addition, and continuity of the norm implies that $||b_i - b|| \to 0$. Conversely, assume such a net converges in norm to $b \in B_x$. By condition 5 we have $b_i - b \to 0_x$ and therefore $b_i = b + (b_i - b) \to b + 0_x = b$. □

We now consider the cross-sections of $\mathfrak{B}$ and determine some additional structure on them.

Proposition 3.1.10. The set of all cross-sections of $\mathfrak{B}$ is a vector space over $\mathbb{F}$ under pointwise addition and multiplication. The subset $C(\mathfrak{B})$ of all continuous sections is a subspace and a $C(X)$-module. Furthermore, if each fiber of $\mathfrak{B}$ is an ordered vector space then the set of all sections is an ordered vector space under the pointwise ordering: $s \leq t$ in $C(\mathfrak{B})$ if $s(x) \leq t(x)$ in B_x for each $x \in X$.

Proof:
The first statement is clear since each Banach space is closed under addition and scalar multiplication. The zero element of this space is the section $x \mapsto 0_x$, which is continuous by assumption 5. If $s, t \in C(\mathfrak{B})$ and $f \in C(X)$ are given, then $(s + t)(x) = s(x) + t(x)$ defines a continuous section by assumption 3, and $(f \cdot s)(x) = f(x) \cdot s(x)$ is continuous by Proposition 3.1.8. So $C(\mathfrak{B})$ is a $C(X)$-module. By considering the constant functions in $C(X)$ we see that it is a vector space over $\mathbb{F}$. The final statement is straightforward to verify. □

The following proposition gives an alternative condition for convergence in B. We will use it and its corollaries later on.

Proposition 3.1.11. Let $\{b_i\}_{i\in I} \subset B$ be a net such that $\pi(b_i) \to \pi(b)$ in X for some $b \in B$. Suppose that we can find, for each $\epsilon > 0$, a net $\{c_i\}_{i\in I} \subset B$ (having the same index set I) converging to some $c \in B$ such that $\{(b_i, c_i)\}_{i\in I} \subset B \times_X B$, $(b, c) \in B \times_X B$, $||b - c|| < \epsilon$ and $||b_i - c_i|| < \epsilon$ for all large enough $i \in I$. Then $\{b_i\}_{i\in I}$ converges to b in B.

Proof:
Because π is open, we can use Lemma 3.1.7 to find a subnet $\{b_{i_j}\}_{j\in J}$ of $\{b_i\}_{i\in I}$ and a net $\{d_j\}_{j\in J} \subset B$ with $\{(d_j, b_{i_j})\}_{j\in J} \subset B \times_X B$ and $d_j \to b$. Now let $\epsilon > 0$ be

given and choose a net $\{c_i\}_{i\in I} \subset B$ converging to $c \in B$ as in the hypothesis. By continuity of addition and multiplication by -1 we find $d_j - c_{i_j} \to b - c$, and thus $||d_j - c_{i_j}|| \to ||b - c|| < \epsilon$. For large enough $j \in J$ we then have $||d_j - c_{i_j}|| < \epsilon$. So we find $||b_{i_j} - d_j|| \leq ||b_{i_j} - c_{i_j}|| + ||c_{i_j} - d_j|| < 2\epsilon$ for large enough j. Hence $||b_{i_j} - d_j|| \to 0$. By condition 5 in the definition of a Banach bundle, $b_{i_j} - d_j \to 0_{\pi(b)}$. Combining this with $d_j \to b$, we find $b_{i_j} = d_j + (b_{i_j} - d_j) \to b + 0_{\pi(b)} = b$.
So we have shown that for any net $\{b_i\}_{i\in I} \subset B$ as in the hypothesis, a subnet converges to b. Now suppose that the whole net does not converge. Then there exists an open neighbourhood $U \subset B$ of b and a subnet $\{b_{i_j}\}_{j\in J}$ such that $b_{i_j} \notin U$ for all $j \in J$. It is then easy to show that $\{b_{i_j}\}_{j\in J}$ also satisfies the hypothesis of the proposition. So we can find a subnet which converges to b, a contradiction. Therefore $\{b_i\}_{i\in I}$ converges to b. □

Corollary 3.1.12. Let $x \in X$ be given. Define, for each $s \in C(\mathfrak{B})$, $\epsilon > 0$ and each open neighbourhood $U \subset X$ of x,

$$V(s, U, \epsilon) := \{b \in B : \pi(b) \in U, ||b - s(\pi(b))|| < \epsilon\}.$$

Then the $V(s, U, \epsilon)$ are open in B, and the set of all these $V(s, U, \epsilon)$ forms a neighbourhood basis of $s(x) \in B$.

Proof:
That each $V(s, U, \epsilon)$ is open follows from the continuity of s, π and the Banach bundle operations. So the $V(s, U, \epsilon)$ are open neighbourhoods of $s(x)$ in B.
On the other hand, suppose that $\{b_i\}_{i\in I}$ is a net in B such that for each open neighbourhood $U \subset X$ of x and each $\epsilon > 0$, $b_i \in V(s, U, \epsilon)$ for large enough i. Then $\pi(b_i) \to x$, and applying the above proposition to $c := s(x)$ and $c_i := s(\pi(b_i))$ for all $i \in I$, we see that b_i converges to $f(x)$. So convergence with respect to the $V(s, U, \epsilon)$ is the same as convergence in B, and the $V(s, U, \epsilon)$ indeed form a neighbourhood basis for $s(x)$. □

Corollary 3.1.13. Let $s : X \to B$ be a section such that for each $x \in X$ and $\epsilon > 0$ there exists a $t \in C(\mathfrak{B})$ and an open neighbourhood $U \subset X$ of x satisfying $||s(y) - t(y)|| < \epsilon$ for all $y \in U$. Then s is continuous.

Proof:
Choose an $x \in X$ and let $\{x_i\}_{i\in I}$ be a net in X such that $x_i \to x$. Then $\pi(s(x_i)) = x_i \to x = \pi(s(x))$. Now let $\epsilon > 0$ be given and let $t \in C(\mathfrak{B})$ be as in the hypothesis. Define $c_i := t(x_i)$ and $c := t(x)$. Then $\pi(c_i) = x_i = \pi(s(x_i))$ for all $i \in I$, $\pi(c) = x = \pi(s(x))$ and $||c - s(x)|| < \epsilon$. Since x_i converges to x, for large enough i we have $x_i \in U$ and thus $||c_i - s(x_i)|| < \epsilon$. By Proposition 3.1.11, $s(x_i) \to s(x)$ and hence $s \in C(\mathfrak{B})$. □

Corollary 3.1.14. Let $s : X \to B$ be a cross-section such that for each $x \in X$ there exist an open neighbourhood $U \subset X$ of x and a net $\{s_i\}_{i\in I} \subset C(\mathfrak{B})$ converging uniformly to s on U, where the latter statement means that for all $\epsilon > 0$ there exists an $i_0 \in I$ such that $||s_i(y) - s(y)|| < \epsilon$ for all $y \in U$ and $i \geq i_0$. Then $s \in C(\mathfrak{B})$.

We will also want to know when a bundle has enough continuous sections. Here we present a useful criterion for this to be the case.

Proposition 3.1.15. Suppose that we can find, for each $b \in B$ and $\epsilon > 0$, a section $s \in C(\mathfrak{B})$ such that $||s(\pi(b)) - b|| < \epsilon$ holds. Then $\mathfrak{B}$ has enough continuous cross-sections.

Proof:
Let $b \in B$ be arbitrary and set $x := \pi(b) \in X$. We wish to find a section $s \in C(\mathfrak{B})$ such that $s(x) = b$.
To this end, first remark that for each $t \in C(\mathfrak{B})$ we can find a $t' \in C(\mathfrak{B})$ with $t(x) = t'(x)$ and $||t'(y)|| \leq ||t(x)||$ for all $y \in X$. Indeed, this is clear if $t(x) = 0_x \in B_x$, since then we can choose t' to be the zero section. So assume $t(x) \neq 0_x$ and define a function $f : X \to \mathbb{R}$ by $f(y) := 1$ if $t(y) = 0_y \in B_y$ and by $f(y) := \min\left(1, \frac{||t(x)||}{||t(y)||}\right)$ if $t(y) \neq 0_y$, for all $y \in X$. Then $f \in C(X)$ by continuity of the norm, and by Proposition 3.1.10, $t' := f \cdot t \in C(\mathfrak{B})$ is as required.
Now, for each $n \in \mathbb{N}$, let $c_n \in B_x$ and $t_n \in C(\mathfrak{B})$ be such that $t_n(x) = c_n$ and $||c_n - b|| < 2^{-n}$. Set $b_1 := c_1$, $s_1 := t_1$ and $b_n := c_n - c_{n-1}$, $s_n := t_n - t_{n-1}$ for each $n > 1$. Then $\{s_n\}_{n \in \mathbb{N}} \subset C(\mathfrak{B})$ and the series $\sum_{n=1}^{\infty} b_n$ converges absolutely to b, i.e., $\sum_{n=1}^{\infty} b_n = \lim_{n\to\infty} c_n = b$ and

$$\sum_{n=1}^{\infty} ||b_n|| = ||c_1|| + \sum_{n=2}^{\infty} ||c_n - c_{n-1}|| < ||c_1|| + \sum_{n=2}^{\infty} 2^{1-n} < \infty.$$

By the above remarks we can assume that $||s_n(y)|| \leq ||s_n(x)|| = ||b_n||$ holds for each $y \in X$ and $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} ||s_n(y)|| \leq \sum_{n=1}^{\infty} ||b_n|| < \infty$$

for all $y \in X$. Using that absolute convergence implies convergence in a Banach space, we see that $\sum_{n=1}^{\infty} s_n(y) \in B_y$ exists for each $y \in X$. Define a section $s : X \to B$ by $s(y) := \sum_{n=1}^{\infty} s_n(y)$ for all $y \in X$. Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$||s(y) - \sum_{n=1}^{m} b_n|| \leq \sum_{n=m+1}^{\infty} ||s_n(y)|| \leq \sum_{n=m+1}^{\infty} ||b_n|| < \epsilon$$

for all $m \geq N$. Hence the series $\sum_{n=1}^{\infty} s_n$ converges uniformly to s on X. By Corollary 3.1.14, $s \in C(\mathfrak{B})$. Since

$$s(x) = \sum_{n=1}^{\infty} s_n(x) = \sum_{n=1}^{\infty} b_n = b,$$

we are done. □

Although we will not need it for our specific case, we do wish to mention the following result. A proof can be found in Appendix C of [9].

Proposition 3.1.16. Any Banach bundle over a locally compact or paracompact[2] space has enough continuous sections.

Construction of Banach bundles Later on we will consider the situation where we are given a certain set of cross-sectional functions from a Hausdorff space X to a disjoint union (over X) of Banach spaces, and we wish to find a topology on this union such that it becomes a Banach bundle with the obvious projection on X, and such that our set of cross-sectional functions is a set of continuous sections of this bundle. In this light the following proposition will be essential.

[2]A space X is *paracompact* if any open cover admits a locally finite open refinement.

Proposition 3.1.17. Let B be a set and $\pi : B \to X$ a surjection such that each $B_x := \pi^{-1}(x)$, $x \in X$, is a Banach space (again we use the symbols $+$, $\cdot$ and $||\cdot||$ for the operations in all these spaces). Suppose Γ is a vector space (under pointwise addition and scalar multiplication) of maps from X to B satisfying the following conditions:

1. For all $s \in \Gamma$ and $x \in X$ we have $s(x) \in B_x$.
2. For all $s \in \Gamma$ the function $x \mapsto ||s(x)||$ is continuous as a map from X to $[0, \infty)$.
3. For each $x \in X$ the set $\Gamma(x) := \{s(x) : s \in \Gamma\} \subset B_x$ is dense in B_x.

Then there is a unique topology on B such that $\mathfrak{B} := (B, \pi)$ is a Banach bundle over X having enough continuous sections such that $\Gamma \subset C(\mathfrak{B})$.

Proof:
Assume that we have two such topologies $\mathcal{T}$ and $\mathcal{T}'$ on B, and let $\{b_i\}_{i\in I} \subset B$ be a net converging to some $b \in B$ with respect to $\mathcal{T}$. Then $\pi(b_i) \to \pi(b) \in X$ since π is continuous, and $||b_i - s(\pi(b_i))|| \to ||b - s(\pi(b))||$ for all $s \in \Gamma$ by continuity of the Banach bundle operations and each s. Now choose an $\epsilon > 0$ and an $s \in \Gamma$ such that $||s(\pi(b)) - b|| < \epsilon$, which exists since $\Gamma(\pi(b))$ lies dense in $B_{\pi(b)}$. Because $||b_i - s(\pi(b_i))||$ converges to $||b - s(\pi(b))||$, there exists an $i_0 \in I$ such that $||b_i - s(\pi(b_i))|| < \epsilon$ for all $i \geq i_0$. As (B, π) is also a Banach bundle with respect to $\mathcal{T}'$, we can apply Proposition 3.1.11 to $c_i := s(\pi(b_i))$ and $c := s(\pi(b))$ to conclude that b_i converges to b with respect to $\mathcal{T}'$. By reversing the roles of $\mathcal{T}$ and $\mathcal{T}'$ we see that convergence in $(B, \mathcal{T})$ is the same as convergence in $(B, \mathcal{T}')$, and thus the two topologies are equal.
We now construct such a topology for B. Define, for each $s \in \Gamma$, $U \subset X$ open and $\epsilon > 0$, a subset $W(s, U, \epsilon) := \{b \in B | \pi(b) \in U, ||b - s(\pi(b))|| < \epsilon\}$ of B. Let $\mathcal{W}$ be the family of all these sets, with s varying over Γ, U over all open sets in X and ϵ over all positive reals. Let $\mathcal{T}$ be the family of all unions of elements in $\mathcal{W}$ (including the empty union).
We claim that $\mathcal{T}$ is a topology for B. To prove this claim, it suffices to show that the intersection of any two elements of $\mathcal{W}$ is contained in $\mathcal{T}$. So choose $W_1 := W(s, U, \epsilon), W_2 := W(t, V, \delta) \in \mathcal{W}$ and a $b \in W_1 \cap W_2$ (since $\mathcal{T}$ contains the empty union the other case is trivial). Set $x := \pi(b) \in X$ and choose $\epsilon', \delta' > 0$ such that $||b - s(x)|| < \epsilon' < \epsilon$ and $||b - t(x)|| < \delta' < \delta$. Define $\gamma := \frac{1}{2} \min\{\epsilon - \epsilon', \delta - \delta'\} > 0$ and choose an $r \in \Gamma$ such that $||b - r(x)|| < \gamma$, which we can do because of our assumption on $\Gamma(x) \subset B_x$. Then

$$||r(x) - s(x)|| \leq ||r(x) - b|| + ||b - s(x)|| < \gamma + \epsilon',$$

and similarly $||r(x) - t(x)|| < \gamma + \delta'$. Because Γ is a vector space we have $r - s, r - t \in \Gamma$, which implies that $z \mapsto ||r(z) - s(z)||$ and $z \mapsto ||r(z) - t(z)||$ are continuous on X. Combining all this, we see that we can find an open neighbourhood $Z \subset U \cap V$ of x such that $||r(z) - s(z)|| < \gamma + \epsilon'$ and $||r(z) - t(z)|| < \gamma + \delta'$ hold for all $z \in Z$. Because $||b - r(x)|| < \gamma$, $W_3 := W(r, Z, \gamma)$ is an element of $\mathcal{W}$ containing b. Also, if $c \in W_3$, then $\pi(c) \in Z \subset U$ and

$$||c - s(\pi(c))|| \leq ||c - r(\pi(c))|| + ||r(\pi(c)) - s(\pi(c))|| < \gamma + \gamma + \epsilon' \leq \epsilon.$$

So $c \in W_1$ and similarly $c \in W_2$. Therefore W_3 is an element of $\mathcal{W}$ containing b such that $W_3 \subset W_1 \cap W_2$. Since $b \in W_1 \cap W_2$ was arbitrarily chosen, we see that $W_1 \cap W_2 \in \mathcal{T}$ and hence $\mathcal{T}$ is indeed a topology for B.

We now show that B is in fact a Banach bundle with this topology. First note that π is continuous with respect to $\mathcal{T}$. Indeed, if $U \subset X$ is open and $b \in \pi^{-1}(U)$, choose an $s \in \Gamma$ such that $||b - s(\pi(b))|| < 1$. Then $W(s, U, 1)$ is an open neighbourhood of b in B such that $W(s, U, 1) \subset \pi^{-1}(U)$. By the arbitrariness of $b \in \pi^{-1}(U)$, we see that $\pi^{-1}(U)$ is open in B. As U was also arbitrary, π is continuous.
To see that π is open, we wish to use Lemma 3.1.7. So let $\{x_i\}_{i \in I} \subset X$ be a net converging to $x := \pi(b) \in X$ for some $b \in B$. Let J be the set consisting of all pairs $(i, W(s, U, \epsilon))$, where $i \in I$ and $W(s, U, \epsilon) \in \mathcal{W}$ is such that $x \in U$. We define an ordering on J by $(i, W) \leq (i', W')$ if $i \leq i'$ and $W' \subset W$. Then J is a directed set, and for each $j = (i, W(s, U, \epsilon)) \in J$ we can choose an $i_j \geq i$ such that $x_{i_j} \in U$, since $\{x_i\}_{i \in I}$ converges to x. Doing this for all $j \in J$, we find a subnet $\{x_{i_j}\}_{j \in J}$ of our original net and we construct a net $\{b_j\}_{j \in J}$ in B in the following manner: for each $j = (i, W(s, U, \epsilon)) \in J$, define $b_j := s(x_{i_j})$. Then $b_j \in W(s, U, \epsilon)$ and $\pi(b_j) = x_{i_j}$, so $\{b_j\}_{j \in J}$ satisfies the requirements of Lemma 3.1.7. Also remark that $b_j \to b$ as $j \to \infty$. Indeed, for any $W(s, U, \epsilon)$ we have $b_{j_0} \in W(s, U, \epsilon)$, where $j_0 := (i, W(s, U, \epsilon)) \in J$ for some $i \in I$. If $j := (i', W(t, V, \delta)) \in J$ is such that $j \geq j_0$, then $W(t, V, \delta) \subset W(s, U, \epsilon)$, so $b_j \in W(s, U, \epsilon)$. This means that for any $W(s, U, \epsilon) \in \mathcal{W}$ there exists a $j_0 \in J$ such that $b_j \in W(s, U, \epsilon)$ for all $j \geq j_0$, and $\{b_j\}_{j \in J}$ converges to b because the sets $W(s, U, \epsilon) \in \mathcal{W}$ with $\pi(b) = x \in U$ form a neighbourhood basis of b in B. Applying Lemma 3.1.7, we see that π is indeed open.
If $b, c \in B$ are such that $\pi(b) \neq \pi(c)$, then there exist disjoint open neighbourhoods U_b of $\pi(b)$ and U_c of $\pi(c)$, since X is Hausdorff. Choose $s, t \in \Gamma$ such that $||b - s(\pi(b))|| < 1$ and $||c - t(\pi(c))|| < 1$. Then $W(s, U_b, 1)$ and $W(t, U_c, 1)$ are disjoint open neighbourhoods of b respectively c in B. On the other hand, if $b, c \in B$ satisfy $x := \pi(b) = \pi(c)$ but $b \neq c$ then $\alpha := ||b - c|| > 0$ holds. Now let $s, t \in \Gamma$ be such that $||b - s(x)|| < \frac{1}{4}\alpha$ and $||c - t(x)|| < \frac{1}{4}\alpha$ hold. Since $s - t \in \Gamma$, the set $U := \left\{y \in X : ||s(x) - t(x)|| > \frac{1}{2}\alpha\right\}$ is open in X. Define $W_1 := W(s, U, \frac{1}{4}\alpha), W_2 := W(t, U, \frac{1}{4}\alpha) \in \mathcal{W}$. Because

$$\alpha = ||b - c|| \leq ||b - s(x)|| + ||s(x) - t(x)|| + ||t(x) - c|| < \frac{1}{2}\alpha + ||s(x) - t(x)||$$

holds, we have $x \in U$. So W_1 is an open neighbourhood of b, W_2 an open neighbourhood of c, and any $d \in W_1 \cap W_2$ would satisfy

$$||d - t(\pi(d))|| \geq ||s(\pi(d)) - t(\pi(d))|| - ||d - s(\pi(d))|| > \frac{1}{4}\alpha,$$

which contradicts the assumption $d \in W_2$. Therefore such a d cannot exist, and W_1 and W_2 are disjoint open neighbourhoods of b respectively c in B. We conclude that B is Hausdorff, and by combining what we have seen so far we conclude that $\mathfrak{B} = (B, \pi)$ is a bundle over X.
We continue to prove that $\mathfrak{B}$ satisfies the assumptions of a Banach bundle. Since each fiber is a Banach space, the first two conditions are clear. We check condition 2 in 3.1.3. Let $\{b_i\}_{i \in I} \subset B$ be a net converging to some $b \in B$. Let $\epsilon > 0$ be arbitrary and let $s \in \Gamma$ be such that $||b - s(\pi(b))|| < \frac{\epsilon}{3}$. Because we have already seen that π is continuous, $\pi(b_i)$ converges to $\pi(b)$ in X, and since $s \in \Gamma$ we find $||s(\pi(b_i))|| \to ||s(\pi(b))||$. Also, $b_i \in W(s, U, \frac{\epsilon}{3})$ for i large enough because $b \in W(s, U, \frac{\epsilon}{3})$. Choose an $i_0 \in I$ such that $||b_i - s(\pi(b_i))|| < \frac{\epsilon}{3}$ and $|\,||s(\pi(b_i))|| - ||s(\pi(b))||\,| < \frac{\epsilon}{3}$ for $i \geq i_0$. For such i we then have

$$|\,||b_i|| - ||b||\,| \leq \frac{2}{3}\epsilon + |\,||s(\pi(b_i))|| - ||s(\pi(b))||\,| < \epsilon,$$

which can be checked by using the triangle inequality. Hence $||b_i||$ converges to $||b||$.
Now suppose we have nets $\{b_i\}_{i\in I}$ and $\{c_i\}_{i\in I}$ in B such that $x_i := \pi(b_i) = \pi(c_i)$ and $b_i \to b$, $c_i \to c$ for certain $b, c \in B$ such that $x := \pi(b) = \pi(c)$. Let $W(s, U, \epsilon) \in \mathcal{W}$ be an arbitrary basic neighbourhood of $b + c \in B$. Choose $\epsilon' > 0$ such that

$$||b + c - s(x)|| < \epsilon' < \epsilon$$

holds, and put $\delta := \frac{1}{4}(\epsilon - \epsilon') > 0$. Choose $r, t \in \Gamma$ such that $||b - t(x)|| < \delta$ and $||c - r(x)|| < \delta$. We find

$$||t(x) + r(x) - s(x)|| \leq ||t(x) - b|| + ||r(x) - c|| + ||b + c - s(x)|| < 2\delta + \epsilon'.$$

Since $t + r - s \in \Gamma$, there exists an open neighbourhood $V \subset X$ of X such that $V \subset U$ and

$$||t(y) + r(y) - s(y)|| < 2\delta + \epsilon'$$

for all $y \in V$. Now $W(t, V, \delta)$ and $W(r, V, \delta)$ are open neighbourhoods of b respectively c in B. Because $\{b_i\}_{i\in I}$ converges to b and $\{c_i\}_{i\in I}$ to c, we can choose an $i_0 \in I$ such that $b_i \in W(t, V, \delta)$ and $c_i \in W(r, V, \delta)$ for all $i \geq i_0$. From this we then find

$$||b_i + c_i - s(x_i)|| \leq ||b_i - t(x_i)|| + ||c_i - r(x_i)|| + ||t(x_i) + r(x_i) - s(x_i)|| < 4\delta + \epsilon' = \epsilon$$

for all $i \geq i_0$. Since $\pi(b_i + c_i) = x_i \in V \subset U$ for such i, we find $b_i + c_i \in W(s, U, \epsilon)$ for all $i \geq i_0$. Because $W(s, U, \epsilon)$ is an arbitrary basic open neighbourhood of $b + c$ in B, we conclude that $b_i + c_i$ converges to $b + c$ as i tends to infinity, which is the content of condition 3 in 3.1.3.
Let $\{b_i\}_{i\in I}$ be a net in B converging to some $b \in B$, and choose a $\lambda \in \mathbb{F}$. We show that λb_i converges to λb. Remark that the case $\lambda = 0$ follows from condition 5 in 3.1.3, which we will prove after this. So assume $\lambda \neq 0$, and let $W(s, U, \epsilon) \in \mathcal{W}$ be a basic open neighbourhood of λb. Then

$$|\lambda| \cdot ||b - \frac{s}{\lambda}(\pi(b))|| = ||\lambda b - s(\pi(\lambda b))|| < \epsilon,$$

so $b \in W(\frac{s}{\lambda}, U, \frac{\epsilon}{|\lambda|})$. Because $b_i \to b$, there exists an $i_0 \in I$ such that $b_i \in W(\frac{s}{\lambda}, U, \frac{\epsilon}{|\lambda|})$ for all $i \geq i_0$. For such i we then have

$$||\lambda b_i - s(\pi(\lambda b_i))|| = |\lambda| \cdot ||b_i - s(\pi(b_i))|| < \epsilon,$$

which means that $b_i \in W(\frac{s}{\lambda}, U, , \frac{\epsilon}{|\lambda|})$. By the arbitrariness of $W(s, U, \epsilon)$ in $\mathcal{W}$ around λb, we see that λb_i indeed converges to λb.
It remains to prove condition 5. Let $\{b_i\}_{i\in I}$ be a net in B such that $||b_i|| \to 0$ and $\pi(b_i) \to x$ for some $x \in X$ as i tends to infinity, and let $W(s, U, \epsilon) \in \mathcal{W}$ be an arbitrary basic neighbourhood of 0_x in B. Choose an $\epsilon' > 0$ such that $||s(x)|| < \epsilon' < \epsilon$. Since $s \in \Gamma$ and $||b_i||$ converges to zero, we can find an $i_0 \in I$ such that $\pi(b_i) \in U$, $||b_i|| < \epsilon - \epsilon'$ and $||s(\pi(b_i))|| < \epsilon'$ for all $i \geq i_0$. Then

$$||b_i - s(\pi(b_i))|| \leq ||b_i|| + ||s(\pi(b_i))|| < \epsilon$$

and $b_i \in W(s, U, \epsilon)$ for $i \geq i_0$. By the arbitrariness of $W(s, U, \epsilon) \in \mathcal{W}$ around 0_x, we see that $\{b_i\}_{i\in I}$ converges to 0_x.
So we have shown that $\mathfrak{B} = (B, \pi)$ is a Banach bundle over X with respect to this

topology. Now choose an $s \in \Gamma$ and let $\{x_i\}_{i \in I}$ be a net in X converging to some $x \in X$. Choose an arbitrary $W(t, U, \epsilon) \in \mathcal{W}$ around $s(x) \in B$. Then U is an open neighbourhood of x in X, and $||s(x) - t(x)|| < \epsilon$. Since $x_i \to x$ and $s - t \in \Gamma$, $x_i \in U$ and $||s(x_i) - t(x_i)|| < \epsilon$ for large enough i. This in turn means that $s(x_i) \in W(t, U, \epsilon)$ for large enough i, and therefore $s(x_i)$ converges to $s(x)$. Hence we have shown that any $s \in \Gamma$ is a continuous section of $\mathfrak{B}$. Finally, by applying Proposition 3.1.15 to condition 3, we see that $\mathfrak{B}$ has enough continuous sections, and we have completed the proof of this proposition. □

The above proposition also implies that the concept of a Banach bundle generalizes the continuity structures used in [12] and [13]. These continuity structures are used to define a concept of an integral of Banach spaces similar to the one we will consider in the next section. Therefore the concept of integration in Banach bundles that we consider is an extension of that in [12] and [13].

3.2 Integration in Banach bundles

The upper integral Having explored some of the basics of Banach bundles, we now turn to integration in these bundles. For technical reasons we first consider the upper integral concept due to Bourbaki [4].
Let (X, μ) be a measure space.

Definition 3.2.1. Let $f : X \to [0, \infty]$ be a nonnegative function. Define D_f to be the set of all measurable $\phi : X \to [0, \infty]$ such that $\phi(x) \geq f(x)$ for all $x \in X$. The upper integral of f (with respect to μ) is the quantity

$$\overline{\int_X} f \, d\mu := \inf_{\phi \in D_f} \int_X \phi \, d\mu,$$

where the integral on the right is the usual Lebesgue integral. We use the convention $\inf \emptyset = \infty$. If we need to stress the dependence on x we will write $\overline{\int_X} f(x) d\mu(x)$.

Note that this integral is finite if and only if there exists an integrable $\phi \in D_f$. If f is itself measurable then clearly $\overline{\int_X} f \, d\mu = \int_X f \, d\mu$.
Also remark that if $\overline{\int_X} f \, d\mu$ is finite, then there exists an integrable $\phi \in D_f$ such that $\overline{\int_X} f \, d\mu = \int_X \phi \, d\mu$. Indeed, for any $n \in \mathbb{N}$ we can choose a $\phi_n \in D_f$ such that $\int_X \phi_n \, d\mu < \overline{\int_X} f \, d\mu + \frac{1}{n}$. Define $\phi := \inf_{n \in \mathbb{N}} \phi_n$. Then $\int_X \phi \, d\mu \leq \int_X \phi_n \, d\mu$ for all n. Since $\int_X \phi_n \, d\mu$ decreases to $\overline{\int_X} f \, d\mu$, we have $\int_X \phi \, d\mu \leq \overline{\int_X} f \, d\mu$. Because $\phi \in D_f$, by definition $\int_X \phi \, d\mu \geq \overline{\int_X} f \, d\mu$ and the statement holds.
We now prove some simple properties of the upper integral.

Proposition 3.2.2. let f and g be nonnegative functions on X and $\lambda \in [0, \infty)$ a nonnegative scalar. The following hold:

1. (subadditivity) $\overline{\int_X} (f + g) d\mu \leq \overline{\int_X} f \, d\mu + \overline{\int_X} g \, d\mu$.
2. (absolute homogeneity) $\overline{\int_X} (\lambda f) d\mu = \lambda \overline{\int_X} f \, d\mu$.
3. (monotonicity) If $f \leq g$ holds on X, then $\overline{\int_X} f \, d\mu \leq \overline{\int_X} g \, d\mu$.

4. If $\overline{\int_X} f d\mu$ is finite, then f is μ-almost everywhere finite.
5. $\overline{\int_X} f d\mu = 0$ if and only if $f(x) = 0$ for μ-almost all $x \in X$.
6. (monotone convergence theorem) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of increasing nonnegative functions on X. Then $\sup_{n\in\mathbb{N}} \overline{\int_X} f_n d\mu = \overline{\int_X} (\sup_{n\in\mathbb{N}} f_n) d\mu$.

Proof:
(1) For any $\phi \in D_f$ and $\psi \in D_g$ we have $\phi + \psi \in D_{f+g}$, so

$$\overline{\int_X} (f+g)\, d\mu \leq \int_X (\phi+\psi) d\mu = \int_X \phi\, d\mu + \int_X \psi\, d\mu.$$

Taking the infimum over all $\phi \in D_f$ and $\psi \in D_g$ we get the required result.
(2) If $\lambda = 0$, then $\overline{\int_X} \lambda f d\mu = \overline{\int_X} 0\, d\mu = \int_X 0\, d\mu = 0$ because the constant function 0 is measurable. On the other hand, if $\lambda \neq 0$ then $\phi \in D_f \Leftrightarrow \lambda f \in D_{\lambda f}$ holds. This means that

$$\lambda \overline{\int_X} f d\mu = \lambda \inf_{\phi\in D_f} \int_X \phi\, d\mu = \inf_{\phi\in D_f} \int_X \lambda\phi\, d\mu = \inf_{\psi\in D_{\lambda f}} \int_X \psi\, d\mu = \overline{\int_X} \lambda f d\mu.$$

(3) Suppose $f(x) \leq g(x)$ for all $x \in X$. Then any $\phi \in D_g$ is an element of D_f so

$$\overline{\int_X} g d\mu = \inf_{\phi\in D_g} \int_X \phi\, d\mu \geq \inf_{\phi\in D_f} \int_X \phi\, d\mu = \overline{\int_X} f d\mu.$$

(4) Suppose $\overline{\int_X} f d\mu$ is finite. Then there exists an integrable $\phi \in D_f$. From the theory of Lebesgue integration we know that ϕ is finite μ-almost everywhere. Since ϕ dominates f, the same holds for f.
(5) Suppose there exists a subset $A \subset X$ of measure zero such that $f(x) = 0$ for all $x \in A^c$. Define $\phi := \infty \cdot \mathbf{1}_A$. Then $\phi \in D_f$ and

$$0 \leq \overline{\int_X} f d\mu \leq \int_X \phi\, d\mu = 0.$$

Conversely, suppose $\overline{\int_X} f d\mu = 0$. As remarked before, this implies that there exists an integrable $\phi \in D_f$ such that $\overline{\int_X} f d\mu = \int_X \phi\, d\mu$. From the theory of Lebesgue integration we know that $\phi = 0$ almost everywhere on X. Since ϕ dominates f, the same holds for f.
(6) Set $f := \sup_{n\in\mathbb{N}} f_N$. If $\overline{\int_X} f_m d\mu = \infty$ for some $m \in \mathbb{N}$, then $\overline{\int_X} f d\mu \geq \overline{\int_X} f_m d\mu = \infty = \sup_{n\in\mathbb{N}} \overline{\int_X} f_n d\mu$. If this is not the case, then for each $n \in \mathbb{N}$ there exists a $\phi_n \in D_{f_n}$ such that $\overline{\int_X} f_n d\mu = \int_X \phi_n d\mu$. By the monotone convergence theorem for Lebesgue integrals we have

$$\sup_{n\in\mathbb{N}} \overline{\int_X} f_n d\mu = \sup_{n\in\mathbb{N}} \int_X \phi_n d\mu = \int_X \sup_{n\in\mathbb{N}} \phi_n d\mu.$$

Since $\phi := \sup_{n\in\mathbb{N}} \phi_n \in D_f$, we have

$$\sup_{n\in\mathbb{N}} \overline{\int_X} f_n d\mu = \int_X \phi\, d\mu \geq \overline{\int_X} f d\mu.$$

On the other hand, because $f_m \leq f$ for each $m \in \mathbb{N}$, monotonicity of the upper integral implies $\sup_{n\in\mathbb{N}} \overline{\int_X} f_n d\mu \leq \overline{\int_X} f d\mu$. So we indeed conclude that $\sup_{n\in\mathbb{N}} \overline{\int_X} f_n d\mu = \overline{\int_X} f d\mu$. □

The outsized L^p-spaces In this section we let X be a Hausdorff space and μ a Borel measure on X. Let $\mathfrak{B} = (B, \pi)$ a Banach bundle over X and fix a $p \in [1, \infty)$.

Definition 3.2.3. We define the *outsized $\mathcal{L}^p$-space* of $\mathfrak{B}$ with respect to μ to be the set of all sections $s : X \to B$ of $\mathfrak{B}$ such that $\overline{\int_X} ||s(x)||^p d\mu(x) < \infty$, and denote it by $\overline{\mathcal{L}}^p(\mathfrak{B}, \mu)$ (or $\overline{\mathcal{L}}^p(\mathfrak{B})$ if it is clear which measure we are referring to).

It remains to justify calling this set a space. Hereto we use the following lemma, a version of Minkowski's inequality for upper integrals.

Lemma 3.2.4. Let f and g be nonnegative functions on X such that $\overline{\int_X} f^p d\mu$ and $\overline{\int_X} g^p d\mu$ are finite. Then

$$\left(\overline{\int_X} (f+g)^p d\mu\right)^{1/p} \leq \left(\overline{\int_X} f^p d\mu\right)^{1/p} + \left(\overline{\int_X} g^p d\mu\right)^{1/p}$$

holds.

Proof:
By our assumptions on f and g there exist $\phi \in D_{f^p}$ and $\psi \in D_{g^p}$ such that $\overline{\int_X} f^p d\mu = \int_X \phi \, d\mu$ and $\overline{\int_X} g^p d\mu = \int_X \psi \, d\mu$. We then have $\phi^{1/p} \geq f$ and $\psi^{1/p} \geq g$ on X, so $(f+g)(x) \leq \phi(x)^{1/p} + \psi(x)^{1/p}$ for all $x \in X$. Because $\phi^{1/p} + \psi^{1/p}$ is measurable and vanishes off a σ-bounded set, we have $\phi^{1/p} + \psi^{1/p} \in D_{f+g}$ and $(\phi^{1/p} + \psi^{1/p})^p \in D_{(f+g)^p}$. We now use Minkowski's inequality for Lebesgue integrals to conclude that

$$\left(\overline{\int_X} (f(x)+g(x))^p d\mu\right)^{1/p} \leq \left(\int_X (\phi(x)^{1/p} + \psi(x)^{1/p})^p d\mu\right)^{1/p}$$

$$\leq \left(\int_X \phi(x) d\mu\right)^{1/p} + \left(\int_X \psi(x) d\mu\right)^{1/p} = \left(\overline{\int_X} f^p d\mu\right)^{1/p} + \left(\overline{\int_X} g^p d\mu\right)^{1/p}.$$

□

Proposition 3.2.5. $\overline{\mathcal{L}}^p(\mathfrak{B})$ is a vector space over $\mathbb{F}$ under pointwise addition and scalar multiplication. Furthermore, the map $|| \cdot ||_p : \overline{\mathcal{L}}^p(\mathfrak{B}) \to [0, \infty)$ given by

$$||s||_p = \left(\overline{\int_X} ||s(x)||^p d\mu(x)\right)^{1/p}$$

for all $s \in \overline{\mathcal{L}}^p(\mathfrak{B})$, is a seminorm on $\overline{\mathcal{L}}^p(\mathfrak{B})$. If each fiber in $\mathfrak{B}$ is a normed Riesz space, then $\overline{\mathcal{L}}^p(\mathfrak{B})$ is a Riesz space under the pointwise defined ordering and lattice operations. Furthermore, $|| \cdot ||_p$ is then a Riesz seminorm, i.e. $|s| \leq |t|$ implies $||s||_p \leq ||t||_p$ for all $s, t \in \overline{\mathcal{L}}^p(\mathfrak{B})$.

Proof:
Choose $s, t \in \overline{\mathcal{L}}^p(\mathfrak{B})$. Then $x \mapsto ||s(x)||$ and $x \mapsto ||t(x)||$ satisfy the requirements of Lemma 3.2.4, and combining this with monotonicity of the upper integral and the triangle inequality we find

$$||s+t||_p = \left(\overline{\int_X} ||s(x)+t(x)||^p d\mu\right)^{1/p} \leq \left(\overline{\int_X} (||s(x)|| + ||t(x)||)^p d\mu\right)^{1/p}$$

$$\leq \left(\overline{\int_X} ||s(x)||^p d\mu\right)^{1/p} + \left(\overline{\int_X} ||t(x)||^p d\mu\right)^{1/p} = ||s||_p + ||t||_p < \infty.$$

So $s + t \in \overline{\mathcal{L}}^p(\mathfrak{B})$ and $||\cdot||_p$ is subadditive.
Now choose an $s \in \overline{\mathcal{L}}^p(\mathfrak{B})$ and a $\lambda \in \mathbb{F}$. Proposition 3.2.2 (2) implies that

$$||\lambda s||_p = \left(\overline{\int_X} ||\lambda s(x)||^p d\mu\right)^{1/p} = \left(\overline{\int_X} |\lambda|^p \cdot ||s(x)||^p d\mu\right)^{1/p} = |\lambda| \left(\overline{\int_X} ||\lambda s(x)||^p d\mu\right)^{1/p} < \infty.$$

So $\lambda s \in \overline{\mathcal{L}}^p(\mathfrak{B})$ and $||\cdot||_p$ is absolutely homogeneous. Therefore $\overline{\mathcal{L}}^p(\mathfrak{B})$ is a vector space over $\mathbb{F}$ and $||\cdot||_p$ is a seminorm on $\overline{\mathcal{L}}^p(\mathfrak{B})$.
Finally, suppose each fiber $B_x \subset B$, $x \in X$, is a normed Riesz space. Define an ordering on $\overline{\mathcal{L}}^p(\mathfrak{B})$ by $s \leq t$ for $s, t \in \overline{\mathcal{L}}^p(\mathfrak{B})$ if $s(x) \leq t(x)$ in B_x for all $x \in X$. Then it is easy to check that $\overline{\mathcal{L}}^p(\mathfrak{B})$ is an ordered vector space.
Let $s, t \in \overline{\mathcal{L}}^p(\mathfrak{B})$ be arbitrary and consider the sections $s \vee t : X \to B$ and $s \wedge t : X \to B$ given by $(s \vee t)(x) := s(x) \vee t(x)$, $(s \wedge t)(x) := s(x) \wedge t(x)$ for all $x \in X$. We can use a well-known equality which holds in Riesz spaces to find $(s \vee t)(x) = \frac{1}{2}(s(x) + t(x) + |s(x) - t(x)|)$ for all $x \in X$. Since each fiber is a normed Riesz space, $||\,|s(x) - t(x)|\,|| = ||s(x) - t(x)||$ for all $x \in X$ and hence

$$||\,|s - t|\,||_p = \left(\overline{\int_X} ||\,|s(x) - t(x)|\,||^p d\mu\right)^{1/p} = \left(\overline{\int_X} ||s(x) - t(x)||^p d\mu\right)^{1/p} = ||s - t||_p.$$

Combining all this, we find

$$||s \vee t||_p = ||\frac{1}{2}(s + t + |s - t|)|| = \frac{1}{2}(||s||_p + ||t||_p + ||s - t||_p) < \infty.$$

So $s \vee t \in \overline{\mathcal{L}}^p(\mathfrak{B})$, and similarly for $s \wedge t \in \overline{\mathcal{L}}^p(\mathfrak{B})$. Since $s \vee t$ is clearly the supremum of s and t under the pointwise ordering and $s \wedge t$ the infimum, $\overline{\mathcal{L}}^p(\mathfrak{B})$ is a Riesz space.
Now we can consider the absolute value $|s| = s \vee (-s) \in \overline{\mathcal{L}}^p(\mathfrak{B})$ of any $s \in \overline{\mathcal{L}}^p(\mathfrak{B})$. Suppose $|s| \leq |t|$ for $s, t \in \overline{\mathcal{L}}^p(\mathfrak{B})$. Then $|s(x)| \leq |t(x)|$ for all $x \in X$, and since each fiber is a normed Riesz space, $||s(x)|| \leq ||t(x)||$ for all $x \in X$. Monotonicity of the upper integral implies

$$||s||_p = \left(\overline{\int_X} ||s(x)||^p d\mu\right)^{1/p} \leq \left(\overline{\int_X} ||t(x)||^p d\mu\right)^{1/p} = ||t||_p.$$

□

Having established this we proceed in the expected manner:

Definition 3.2.6. We define $\overline{L}^p(\mathfrak{B}, \mu)$ to be the quotient of $\overline{\mathcal{L}}^p(\mathfrak{B}, \mu)$ and the kernel of the seminorm $||\cdot||_p$:

$$\overline{L}^p(\mathfrak{B}, \mu) := \overline{\mathcal{L}}^p(\mathfrak{B}, \mu)/\ker(||\cdot||_p),$$

and we endow it with the linear operations inherited of $\overline{L}^p(\mathfrak{B}, \mu)$ and the norm $||\cdot||_p$. We then call $\overline{L}^p(\mathfrak{B})$ the *outsized L^p-space* of $\mathfrak{B}$.

We will usually just write $\overline{L}^p(\mathfrak{B})$ and, as is usual practice, we will not distinguish in notation or terminology between sections in $\overline{\mathcal{L}}^p(\mathfrak{B})$ and the equivalence classes to which they belong in $\overline{L}^p(\mathfrak{B})$, and identify two such sections when their difference has p-integral zero. By Proposition 3.2.2 (5) this is the case if and only if two such sections are equal μ-almost everywhere.
$\overline{L}^p(\mathfrak{B})$ is a normed vector space. In fact, even more holds.

Proposition 3.2.7. $\overline{L}^p(\mathfrak{B})$ is a Banach space. If each fiber in B is a Banach lattice, then so is $\overline{L}^p(\mathfrak{B})$, under the ordering and lattice operations inherited from $\overline{\mathcal{L}}^p(\mathfrak{B})$.

Proof:
First recall that a normed vector space is complete if and only if, for any sequence $\{s_n\}_{n\in\mathbb{N}}$ in this space such that $\sum_{n=1}^{\infty} ||s_n|| < \infty$, the series $\sum_{n=1}^{\infty} s_n$ converges. So let $\{s_n\}_{n=1}^{\infty} \subset \overline{L}^p(\mathfrak{B})$ be a sequence such that $\sum_{n=1}^{\infty} ||s_n||_p$ is finite. Then the functions $x \mapsto ||s_n(x)||$, for $n \in \mathbb{N}$, satisfy the requirements of Lemma 3.2.5. Hence, for any $m \in \mathbb{N}$ we have

$$\left(\overline{\int_X} (\sum_{n=1}^{m} ||s_n(x)||)^p d\mu\right)^{1/p} \leq \sum_{n=1}^{m} (\overline{\int_X} ||s_n(x)||^p d\mu)^{1/p} = \sum_{n=1}^{m} ||s_n||_p.$$

Combining this with Proposition 3.2.2 (6), we find

$$\left(\overline{\int_X} (\sum_{n=1}^{\infty} ||s_n(x)||)^p d\mu\right)^{1/p} = \sup_{m\in\mathbb{N}} \left(\overline{\int_X} (\sum_{n=1}^{m} ||s_n(x)||)^p d\mu\right)^{1/p} \leq \sum_{n=1}^{\infty} ||s_n||_p < \infty.$$

By (4) of that same proposition we see that $\sum_{n=1}^{\infty} ||s_n(x)||$ is finite for μ-almost all $x \in X$. Since each fiber is a Banach space, this means that $\sum_{n=1}^{\infty} s_n(x)$ is well-defined for almost all x. Setting it equal to zero otherwise we find a well-defined section, which is an element of $\overline{L}^p(\mathfrak{B})$ because $|| \sum_{n=1}^{\infty} s_n||_p \leq \sum_{n=1}^{\infty} ||s_n||_p < \infty$ holds. Finally, since $\sum_{n=1}^{\infty} ||s_n||_p$ is finite, $\sum_{n=m+1}^{\infty} ||s_n||_p$ converges to zero as m tends to infinity. We find

$$|| \sum_{n=1}^{\infty} s_n - \sum_{n=1}^{m} s_n||_p \leq \sum_{n=m+1}^{\infty} ||s_n||_p \to \infty,$$

so the partial sums $\sum_{n=1}^{m} s_n$ indeed converge in $\overline{L}^p(\mathfrak{B})$ and we conclude that $\overline{L}^p(\mathfrak{B})$ is a Banach space.
Now suppose each fiber is a Banach lattice. It is easy to see that the kernel $\ker(|| \cdot ||_p)$ of $|| \cdot ||_p$ is an ideal in $\overline{\mathcal{L}}^p(\mathfrak{B})$. So $\overline{L}^p(\mathfrak{B}) = \overline{\mathcal{L}}^p(\mathfrak{B}, \mu)/\ker(|| \cdot ||_p)$ is a Riesz space under the ordering inherited from $\overline{\mathcal{L}}^p(\mathfrak{B})$. This ordering corresponds to: $s \leq t$ in $\overline{L}^p(\mathfrak{B})$ if and only if $s(x) \leq t(x)$ for μ-almost all $x \in X$. The lattice operations are the same as those in $\overline{\mathcal{L}}^p(\mathfrak{B})$, $(s \vee t)(x) = s(x) \vee t(x)$ and $(s \wedge t)(x) = s(x) \wedge t(x)$ for all $s, t \in \overline{L}^p(\mathfrak{B})$ and $x \in X$. That $\overline{L}^p(\mathfrak{B})$ is a Banach lattice follows from the last statement in 3.2.5. □

Corollary 3.2.8. If $\{s_n\}_{n\in\mathbb{N}}$ is a sequence in $\overline{L}^p(\mathfrak{B})$ converging to some $s \in \overline{L}^p(\mathfrak{B})$, then there exists a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ such that $s_{n_k}(x) \to s(x)$ as $n \to \infty$ for μ-almost all $x \in X$.

Proof:
Since $\{s_n\}_{n\in\mathbb{N}}$ is Cauchy, we can find a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} ||s_{n_{k+1}} - s_{n_k}||_p < \infty$ holds. Applying the proof of Proposition 3.2.7 to this series, we see that $\sum_{k=1}^{m} (s_{n_{k+1}} - s_{n_k}) = s_{n_{m+1}}$ converges to s almost everywhere. □

The space of p-integrable sections The reason for the term "outsized" is that the spaces $\overline{\mathcal{L}}^p(\mathfrak{B})$ and $\overline{L}^p(\mathfrak{B})$ are often too big to be useful. One example that provides motivation for this statement is the following.

Example 3.2.9. Consider $X = [0,1]$ with Lebesgue measure and the trivial bundle $\mathfrak{B} = (B, \pi)$ with constant fiber $\mathbb{R}$ from Example 3.1.4. So $B = X \times \mathbb{R}$ and $\pi : X \to \mathbb{R}$ is given by $\pi(x, \lambda) = x$ for all $(x, \lambda) \in X \times \mathbb{R}$. As noted before, we can identify sections of this bundle with functions from X to $\mathbb{R}$. So we would expect an L^p-space of sections of this bundle, for $p \in [1, \infty)$, to be equal to the usual $L^p[0,1]$ space of p-integrable functions. However, $\overline{L}^p(\mathfrak{B})$ contains all functions with p-integrable absolute value. This set strictly contains $L^p[0,1]$. Indeed, let $A \subset [0,1]$ be a non-measurable subset and consider the function $f := \mathbf{1}_A - \mathbf{1}_{A^c} : [0,1] \to \mathbb{R}$. It has absolute value $|f| = \mathbf{1} \in L^p[0,1]$, so $f \in \overline{L}^p(\mathfrak{B})$. However, it is clearly not measurable, so $f \notin L^p[0,1]$ and $L^p[0,1] \neq \overline{L}^p(\mathfrak{B})$. It is even true that the set of all functions with integrable absolute value is not closed under addition, as can be seen by considering the function $f + \mathbf{1} = 2\mathbf{1}_A \in \overline{L}^p(\mathfrak{B})$, where f is as above.

To find a more useful space of p-integrable sections one usually passes to a subspace that has many of the properties which we expect of an L^p-space.

Assumption 3.2.10. From here on we assume that $\mathfrak{B}$ is a Banach bundle over a locally compact Hausdorff space X and that μ is an outer regular measure on X such that $\mu(K) < \infty$ holds for all $K \subset X$ compact. We fix a $p \in [1, \infty)$.

Proposition 3.1.16 implies that under this assumption, $\mathfrak{B}$ has enough continuous sections. Let $C_c(\mathfrak{B})$ denote the set of continuous sections of $\mathfrak{B}$ which have compact support. Then $C_c(\mathfrak{B})$ is a linear subspace of $C(\mathfrak{B})$, and for any $s \in C_c(\mathfrak{B})$ the function $x \to ||s(x)||^p$ is continuous and vanishes off a compact set $K \subset X$. Therefore it is bounded and

$$\overline{\int_X} ||s(x)||^p d\mu = \int ||s(x)||^p d\mu \leq \mu(K) \cdot \left(\sup_{x \in X} ||s(x)|| \right)^p < \infty.$$

So $C_c(\mathfrak{B}) \subset \overline{\mathcal{L}}^p(\mathfrak{B}, \mu)$ and the image $L(\mathfrak{B})$ of $C_c(\mathfrak{B})$ under the quotient map from $\overline{\mathcal{L}}^p(\mathfrak{B}, \mu)$ to $\overline{L}^p(\mathfrak{B}, \mu)$, is a sublattice.

Definition 3.2.11. We define the space of p-integrable sections of $\mathfrak{B}$ to be the closure of $L(\mathfrak{B})$ in $\overline{L}^p(\mathfrak{B}, \mu)$, and denote it by $L^p(\mathfrak{B}, \mu)$ ($L^p(\mathfrak{B})$ if no reference to the measure needs to be made).

As a closed subspace of a Banach space, $L^p(\mathfrak{B})$ is a Banach space as well. Just as in 3.2.7, $L^p(\mathfrak{B})$ is a Banach lattice if each fiber is.
Now that we have defined $L^p(\mathfrak{B})$, the first thing to note is that in this space the concept of upper integral is not needed.

Lemma 3.2.12. For any $s \in L^p(\mathfrak{B})$ there exists a $t \in L^p(\mathfrak{B})$, which is almost everywhere equal to s, such that the map $x \mapsto ||t(x)||$ is measurable on X.

Proof:
This is certainly the case if $s \in L(B)$, since we can then choose t to be a continuous section and the map $x \mapsto ||t(x)||$ is a continuous function. If $s \in L^p(\mathfrak{B})$ is arbitrary, let $\{s_n\}_{n \in \mathbb{N}} \subset L(B)$ be a sequence converging in p-norm to s. By Corollary 3.2.8, there exists a subsequence converging almost everywhere to s. Let t be the pointwise limit of this subsequence. Since the norm is continuous on B, $x \mapsto ||t(x)||$ is measurable as a limit of

measurable functions. Since s and t are almost everywhere equal, $||s-t||_p = 0$, $t \in L^p(\mathfrak{B})$, and we are done. $\square$

So for any $s \in L^p(\mathfrak{B})$ we can choose an element $t \in \overline{\mathcal{L}}^p(\mathfrak{B})$ in the equivalence class of s such that $x \mapsto ||t(x)||$ is measurable and

$$\overline{\int_X} ||s(x)||^p d\mu = ||s||_p^p = ||t||_p^p = \int ||t(x)||^p d\mu.$$

From now on we will ignore the distinction between s and t and simply write

$$||s||_p = \left(\int ||s(x)||^p d\mu \right)^{1/p}$$

for the p-norm of any $s \in L^p(\mathfrak{B})$.
This space of p-integrable sections that we have defined will serve as our concept of the "p-integral" of the Banach spaces B_x, for $x \in X$.

Example 3.2.13. Suppose that μ is in fact a regular finite measure and let $\mathfrak{B}$ be the trivial bundle over X with constant fiber $\mathbb{F}$. As noted in Example 3.1.4, continuous sections and continuous functions from X to $\mathbb{F}$ are in one-to-one correspondence, so we can identify $C_c(\mathfrak{B})$ with $C_c(X)$, the space of all continuous compactly supported functions on X. Because $C_c(X)$ lies dense in $L^p(X, \mu)$, we can also identify $L^p(\mathfrak{B}, \mu)$ with $L^p(X, \mu)$ in a natural manner.
In particular, we see that under these assumptions the situation of Example 3.2.9 does not occur for the space $L^p(\mathfrak{B})$.

Example 3.2.14. If $\{B_x\}_{x\in X}$ is a family of Banach spaces, then the direct sum $\oplus_{x\in X} B_x$ of $\{B_x\}_{x\in X}$ is the Banach space of maps $s : x \to \sqcup_{x\in X} B_x$ such that $s(x) \in B_x$ for all $x \in X$ and $\sum_{x\in X} ||s(x)|| < \infty$. In particular, this means that $s(x) = 0_x \in B_x$ for all but countably many $x \in X$. We show that $\oplus_{x\in X} B_x$ can be identified with a space of integrable sections of a Banach bundle over X.
Endow X with the discrete topology and let μ be counting measure on X. Then X is locally compact Hausdorff and μ is regular. Set $B := \sqcup_{x\in X} B_x$ and endow it with the disjoint union topology. Let $\pi : B \to X$ be given by $\pi(b) = x$ if $b \in B_x$ for all $b \in B$. Then it is straightforward to verify that $B := (B, \pi)$ is a Banach bundle over X. As X is discrete, any section of $\mathfrak{B}$ is continuous.
Now consider $L^1(\mathfrak{B})$. Any $s \in L^1(\mathfrak{B})$ can be viewed as an element of $\oplus_{x\in X} B_x$ because

$$\sum_{x\in X} ||s(x)|| = \int_X ||s(x)|| d\mu(x) < \infty.$$

Conversely, if $s \in \oplus_{x\in X} B_x$ then there exists a countable set $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $s(x) = 0$ for all $x \notin \{x_n\}_{n\in N}$. For each $m \in \mathbb{N}$, define a section $s_m : X \to B$ by $s_m(x) = s(x)$ for all $x \in X \setminus \{x_n\}_{n=m+1}^{\infty}$ and $s_m(x_n) = 0_{x_n} \in B_{x_n}$ for all $n > m$. Then $\{s_m\}_{m=1}^{\infty} \subset C_c(\mathfrak{B})$ and

$$||s_m - s|| = \sum_{n=m+1}^{\infty} ||s(x_n)|| \to 0$$

in $\oplus_{x\in X} B_x$ as $m \to \infty$. So $s \in L^1(\mathfrak{B})$ and we can indeed identify $L^1(\mathfrak{B})$ and $\oplus_{x\in X} B_x$.
Therefore the idea of a 1-integral of the spaces B_x, $x \in X$ being a space of integrable sections of a Banach bundle generalizes the notion of a direct sum. Of course, similar statements can be made in the case of a p-direct sum, for an arbitrary $p \in [1, \infty)$.

Example 3.2.15. $L^2(\mathfrak{B})$ is a Hilbert space if $\mathfrak{B}$ is a Hilbert bundle. Indeed, we have seen in Example 3.1.5 that the inner product is continuous on $B \times_X B$. Therefore $x \mapsto \langle s(x), t(x)\rangle$ is continuous if s and t are, and in particular measurable. Using the polarization identities we can extend this to $s, t \in L^2(\mathfrak{B})$. For instance, in the case $\mathbb{F} = \mathbb{R}$ we have

$$\langle s(x), t(x)\rangle = \frac{1}{4}(||s(x) + t(x)||^2 - ||s(x) - t(x)||^2)$$

for all $x \in X$. Let $\{s_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}} \subset L(\mathfrak{B})$ be sequences converging to s respectively t in the 2-norm. By passing to subsequences if necessary, we may assume that $s(x) = \lim_{n\to\infty} s_n(x)$, $t(x) = \lim_{n\to\infty} t_n(x)$ for almost all $x \in X$. By continuity of the Banach bundle operations we find

$$\langle s(x), t(x)\rangle = \frac{1}{4}(||s(x) + t(x)||^2 - ||s(x) - t(x)||^2)$$

$$= \lim_{n\to\infty} \frac{1}{4}(||s_n(x) + t_n(x)||^2 - ||s_n(x) - t_n(x)||^2) = \lim_{n\to\infty} \langle s_n(x), t_n(x)\rangle$$

for almost all $x \in X$. Therefore $x \mapsto \langle s(x), t(x)\rangle$ is almost everywhere equal to a measurable function. We simply choose representatives of s and t such that this map is everywhere measurable. The Cauchy-Schwarz inequality then implies that

$$\int_X |\langle s(x), t(x)\rangle| d\mu \le \int_X (||s(x)|| \cdot ||t(x)||)\, d\mu$$

$$\le \left(\int_X ||s(x)||^2 d\mu\right)^{1/2} \cdot \left(\int_X ||t(x)||^2 d\mu\right)^{1/2} < \infty$$

for $s, t \in L^2(\mathfrak{B})$, so $\langle s, t\rangle := \int_X \langle s(x), t(x)\rangle d\mu$ is well-defined. It is an inner product on $L^2(\mathfrak{B})$ which induces the 2-norm. The complex case is treated similarly.
Because the upper integral is merely subadditive and not additive, the form $\langle s, t\rangle := \overline{\int}_X \langle s(x), t(x)\rangle d\mu$ need not define an inner product on $\overline{L}^2(\mathfrak{B})$ and this space is in general not a Hilbert space.
Also note that, by combining the statements from this example with those in the previous one, we can conclude that our notion of a 2-integral of Hilbert spaces generalizes the direct sum of a family of Hilbert spaces.

Example 3.2.16. Another special case is that in which $p = 1$ and $\mathfrak{B}$ is the trivial bundle with constant fiber A, for some Banach space A. We can then compare the Bochner integral for Banach-space valued functions with the space of integrable sections. For details on the relation between these two concepts see [9].

Locally measurable sections We now investigate the space of p-integrable sections some more and give an alternative characterization of the p-integrable sections, in terms of locally measurable sections. Again X is a locally compact Hausdorff space and μ an outer regular Borel measure on X such that $\mu(K) < \infty$ for all $K \subset X$ compact.

Definition 3.2.17. A section $s : X \to B$ is said to μ-*locally measurable* (or simply locally measurable) if, for each compact subset $K \subset X$, there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \subset C(\mathfrak{B})$ of continuous sections such that $s_n(x) \to s(x)$ for μ-almost all $x \in K$.

Clearly any continuous section is locally measurable. Furthermore, the locally measurable sections form a $C(X)$-module which is closed under pointwise μ-almost everywhere convergence on X.
If $s : X \to B$ is a locally measurable section, then for each $K \subset X$ compact, $x \mapsto ||s(x)||$ is μ-almost everywhere on K equal to a measurable function.
To determine the relation between locally measurable sections and p-integrable sections we need the following lemma, a generalized version of Egoroff's theorem.

Lemma 3.2.18. Let $\{s_n\}_{n\in\mathbb{N}}$ and s be sections of $\mathfrak{B}$ and $K \subset X$ compact such that, for each $n \in \mathbb{N}$, $x \mapsto ||s(x) - s_n(x)||$ is almost everywhere equal to a measurable function on K and such that $s_n(x) \to s(x)$ as $n \to \infty$ for μ-almost all $x \in K$. Then for each $\epsilon > 0$ there exists a measurable subset $L \subset K$ of K such that $\mu(L) < \epsilon$ and $||s_n(x) - s(x)|| \to 0$ uniformly on $K \setminus L$.

Proof:
By omitting a set of measure zero from K, we can assume that $s_n(x) \to s(x)$ for all $x \in K$ as $n \to \infty$ and that $x \mapsto ||s(x) - s_n(x)||$ is measurable on K for each $n \in \mathbb{N}$. Let $\epsilon > 0$ be given and define, for all $m, n \in \mathbb{N}$, measurable sets $L_{n,m} \subset K$ by

$$L_{n,m} := \bigcap_{k=n}^{\infty} \left\{ x \in K : ||s(x) - s_k(x)|| < \frac{1}{m} \right\}.$$

For each $m \in \mathbb{N}$, $\{L_{n,m}\}_{n=1}^{\infty}$ is an increasing sequence such that $K = \cup_{n=1}^{\infty} L_{n,m}$. Because Assumption 3.2.10 tells us that $\mu(K) < \infty$ holds, we can use the upper continuity of μ to conclude that $\mu(K \setminus L_{n,m}) \to 0$ as $n \to \infty$. Now choose, for each $m \in \mathbb{N}$, an $n(m) \in \mathbb{N}$ such that $\mu(K \setminus L_{n(m),m}) < 2^{-m}\epsilon$ holds. Set $L := \cup_{m=1}^{\infty} (K \setminus L_{n(m),m})$. Then L is a measurable subset of K such that $\mu(L) \leq \sum_{m=1}^{\infty} \mu(K \setminus L_{n(m),m}) < \epsilon$ and $K \setminus L = K \cap \bigcap_{m=1}^{\infty} L_{n(m),m}$. For any $m \in \mathbb{N}$ and $n \geq n(m)$ we have $K \setminus L \subset L_{n,m}$ and thus

$$||s(x) - s_n(x)|| < \frac{1}{m}$$

for all $x \in K \setminus L$. So $\{s_n\}_{n\in\mathbb{N}}$ indeed converges uniformly to s on $K \setminus L$. □

A subset $Y \subset X$ is said to be *σ-compact* if it is a countable union of compact sets in X. Clearly such a set is measurable. We say that a section $s : X \to B$ *μ-almost vanishes off a σ-compact set* if there exists a σ-compact set $Y \subset X$ with the property that $\mu\{x \in X \setminus Y : s(x) \neq 0_x \in B_x\} = 0$. If $s : X \to B$ is a locally μ-measurable section that almost vanishes off a σ-compact set, then $x \mapsto ||s(x)||$ is almost everywhere on X equal to a measurable function. Indeed, on any compact set it is almost equal to a measurable function, and it almost vanishes off a countable union of such sets.
For any subset $Y \subset X$ and any section $s : X \to B$, the section $s_Y : X \to B$ is given by $s_Y(x) := \mathbf{1}_Y(x)s(x)$ for $x \in X$. If Y and $x \mapsto ||s(x)||$ are measurable, then $x \mapsto ||(\mathbf{1}_Y s)(x)|| = \mathbf{1}_Y(x)||s(x)||$ is measurable. Also, $s \in \overline{L}^p(\mathfrak{B})$ implies $s_Y \in \overline{L}^p(\mathfrak{B})$.
Also, if $Y \subset X$, $M \in [0, \infty)$ and a section $s : X \to B$ are given, then we let the section $s_{Y,M} : X \to B$ be given by $s_{Y,M}(x) := \mathbf{1}_Y(x) \min\left\{1, \frac{M}{||s(x)||}\right\} s(x)$ for $x \in X$ such that $s(x) \neq 0_x$, and by $s_{Y,M}(x) := 0_x$ for $x \in X$ such that $s(x) = 0_x$. Then $s_{Y,M}$ is bounded from above by M and vanishes off Y. If Y and $x \mapsto ||s(x)||$ are measurable, then so is $x \mapsto ||s_{Y,M}(x)||$. If moreover $s \in \overline{L}^p(\mathfrak{B})$, then $s_{Y,M} \in \overline{L}^p(\mathfrak{B})$ as well.
We are now ready to present an alternative way to describe the p-integrable sections.

Proposition 3.2.19. Let $s \in \overline{L}^p(\mathfrak{B})$ be given. Then s is an element of $L^p(\mathfrak{B})$ if and only if it is μ-locally measurable and if it almost vanishes off a σ-compact set.

Proof:
If $s \in L^p(\mathfrak{B}) = \overline{C_c(\mathfrak{B})}$, then Corollary 3.2.8 implies that s is locally μ-measurable and that it almost vanishes off some σ-compact set.
Conversely, suppose s is locally measurable and that $\mu\{x \in X \setminus Y : s(x) \neq 0_x\} = 0$ for some $Y = \cup_{n=1}^{\infty} K_n \subset X$, with each $K_n \subset X$ compact. We can assume that $s(x) = 0_x$ for all $x \in X \setminus Y$ and that $x \mapsto ||s(x)||$ is measurable on X. If we can show that $s_{K,M} \in L^p(\mathfrak{B})$ for all $K \subset X$ compact and $M \in [0, \infty)$, then we can apply the dominated convergence theorem for Lebesgue integrals to the functions $x \mapsto ||s_K(x) - s_{K,M}(x)||$, which converge to zero as $M \to \infty$, to find $s_K \in L^p(\mathfrak{B})$ for all $K \subset X$ compact. Another application of the dominated convergence theorem, this time to $x \mapsto ||s(x) - s_{\cup_{n=1}^{m} K_n}(x)||$ for $m \in \mathbb{N}$, then yields $s \in L^p(\mathfrak{B})$.
So let a compact set $K \subset X$ and an $M \in [1, \infty)$ be given. Let $\{s_n\}_{n \in \mathbb{N}} \subset C(\mathfrak{B})$ be a sequence of continuous sections converging to s pointwise μ-almost everywhere on K. Then $(s_n)_{K,M} \to s_{K,M}$ pointwise almost everywhere on K as well. Now Lemma 3.2.18 tells us that we can find, for each $\epsilon > 0$, a measurable subset $L \subset K$ and an $n \in \mathbb{N}$ such that

$$\mu(L) < \frac{\epsilon}{2(2M)^p} \quad \text{and} \quad ||(s_n)_{K,M} - s_{K,M}|| < \frac{\epsilon}{2\mu(K)}$$

for all $x \in K \setminus L$. Then

$$||(s_n)_{K,M} - s_{K,M}||_p^p \leq \int_L ||(s_n)_{K,M}(x) - s_{K,M}(x)||^p d\mu + \int_{K \setminus L} ||(s_n)_{K,M}(x) - s_{K,M}(x)||^p d\mu$$

$$\leq (2M)^p \mu(L) + \frac{\epsilon}{2\mu(K)} \mu(K \setminus L) < \epsilon.$$

So we can reduce to the case where $s \in C(\mathfrak{B})$. Furthermore, as the section $s_{X,M}$ is continuous if s is, it suffices to show that $s_K \in L^p(\mathfrak{B})$ if $s \in C(\mathfrak{B})$.
Assume that s is continuous and let $U \subset X$ be an open subset such that $K \subset U$ and $\overline{U} \subset X$ is compact. Such a set exists because X is locally compact. Indeed, for any $x \in K$ there exists an open neighbourhood of x with compact closure. These open sets cover K, so there exists a finite subcover. The union of this finite cover is the set U we are looking for. By the outer regularity of μ we can find a decreasing sequence $\{U_n\}_{n=1}^{\infty}$ of open sets which contain K such that $\mu(U_n \setminus K) \downarrow 0$. We can assume that $U_n \subset U$ holds for each $n \in \mathbb{N}$ and that these sets all have compact closure. We can now use Urysohn's lemma to choose a sequence $\{f_n\}_{n \in \mathbb{N}} \subset C(X)$ of continuous functions with $f_n \equiv 1$ on K and $f_n \equiv 0$ on $X \setminus U_n$, for each n. Then $\{f_n s\}_{n \in \mathbb{N}} \subset C_c(\mathfrak{B})$ and

$$||f_n s - s_K||_p^p = \int_X ||f_n(x)s(x) - \mathbf{1}_K(x)s(x)||^p d\mu = \int_{U_n \setminus K} ||f_n(x)s(x) - \mathbf{1}_K(x)s(x)||^p d\mu$$

$$\leq \mu(U_n \setminus K) \sup_{x \in \overline{U}_n} ||s(x)||^p \to 0$$

as $n \to 0$. So $s_K \in L^p(\mathfrak{B})$ and we are done. □

Some remarks So far we have only considered L^p-spaces of sections of a Banach bundle for $p \in [1, \infty)$. There also exists an outsized L^∞-space of sections. For a Banach bundle $\mathfrak{B} = (B, \pi)$ over a Hausdorff space X with a measure μ it is denoted by $\overline{L}^\infty(\mathfrak{B}, \mu)$, and it consists of all sections $s : X \to B$ such that there exists an $M \geq 0$ with $\mu\{x \in X : ||s(x)|| > M\} = 0$. The smallest such $M \in [0, \infty)$ is denoted by $||s||_\infty$, and the map $||\cdot||_\infty : \overline{L}^\infty(\mathfrak{B}, \mu) \to [0, \infty)$, $s \mapsto ||s||_\infty$ for $s \in \overline{L}^\infty(\mathfrak{B}, \mu)$, is a norm on $\overline{L}^\infty(\mathfrak{B}, \mu)$ which makes it a Banach space. If in fact Assumption 3.2.10 holds, then analogous to Proposition 3.2.19 we can define the space of μ-almost everywhere bounded sections to be the subspace $L^\infty(\mathfrak{B}, \mu)$ of $\overline{L}^\infty(\mathfrak{B}, \mu)$ consisting of all $s \in \overline{L}^\infty(\mathfrak{B}, \mu)$ that are locally measurable and vanish μ-almost off a σ-compact set. One can then show this is a closed subspace, and thus a Banach space as well. However, this concept will be less useful to us because, as in the case of an L^∞-space of functions, the compactly supported sections need not lie dense.

The concept of a Banach bundle as we have defined it should not be mistaken with a Banach bundle in differential geometry [10]. There a local triviality condition is imposed on these bundles, something which is too restrictive for our purposes. Instead, we have required condition 5 in Definition 3.1.3 to hold, which can be seen as a "fragment" of this local triviality condition. Moreover, one can show that any Banach bundle over a locally compact Hausdorff space whose fibers are all of the same finite dimension is necessarily locally trivial. In general this need not be the case. See Remark 13.9 in [9, pp.128-129] for more details.

There are also concepts of Banach algebraic bundles and Banach *-algebraic bundles, and these are related in a similar way to Banach algebras and Banach *-algebras as Banach bundles are to Banach spaces. The interested reader is referred to [9].

We have defined the space of p-integrable sections on a locally compact space as the closure of the compactly supported sections in the p-norm, and shown that a section is p-integrable if and only if it has finite p-integral, if it almost vanishes off a σ-compact set and if it is locally measurable. The latter concept can be generalized some more, in terms of local measurability structures. In fact, in [9] these local measurability structures are used to define the p-integrable sections, and then it is shown that this gives the same result as defining them in terms of compactly supported sections. Moreover, for these local measurability structures we can also use other sets besides compact sets. For a locally compact space these are a convenient choice, because there are many of them, but the drawback is that we can only effectively use this method of integration for such spaces. As we will see in the next section, this is a heavy drawback when we do not know whether the space we are dealing with is in fact locally compact.

Chapter 4

Decomposition results

In this chapter we combine the results on measure decompositions and Banach bundles that we have considered so far. We give an isometric lattice isomorphism between a space of p-integrable functions and a space of p-integrable sections of some Banach bundle. Then we use this to decompose the induced action of a group on the space of p-integrable functions into band irreducible representations.

4.1 Decomposing group actions on spaces of integrable functions

We consider a locally compact Polish transformation group (G, X). Fix a $p \in [1, \infty)$ and suppose the set of G-invariant probability measures $\mathcal{I}$ is not empty (Proposition 2.1.6 describes a situation in which this holds). Throughout we take the field of scalars $\mathbb{F}$ equal to $\mathbb{R}$, so all functions are assumed to be real-valued. In Theorem 2.1.5 we found a decomposition map $\beta : X \to \mathcal{E}$, $\beta(x) = \beta_x \in \mathcal{E}$ for $x \in X$, from X to the space $\mathcal{E}$ of ergodic measures on X, that is measurable with respect to the Borel structure on $\mathcal{E}$ given by the weak* topology, with the following properties:

1. For every $g \in G$ and $x \in X$, $\beta_{gx} = \beta_x$[1].
2. For every $\lambda \in \mathcal{E}$, $\lambda(\beta^{-1}\{\lambda\}) = \lambda\{x \in X : \beta(x) = \lambda\} = 1$.
3. For every $\mu \in \mathcal{I}$ and every $Y \subset X$ measurable we have

$$\mu(Y) = \int_X \beta_x(Y) d\mu(x).$$

Choose such a G-invariant probability measure $\mu \in \mathcal{I}$. By Proposition 1.2.3 we have an induced strongly continuous representation $\rho : G \to \mathcal{B}(L^p(X, \mu))$ of G as a group of isometric lattice isomorphisms on $L^p(X, \mu)$.

[1] This property comes with the decomposition, but we will not need it for our results.

Decomposing the space of integrable functions Inspired by Corollary 2.2.4, we would like to construct a Banach bundle over $\mathcal{E}$. Since $\mathcal{E}$ is a subset of a Polish space it is metrizable, hence certainly Hausdorff. For each $\lambda \in \mathcal{E}$, let $B_\lambda := L^p(X, \lambda)$ denote the Banach space of all real-valued functions which are p-integrable with respect to λ. Denote the norm on this space by $||\cdot||_\lambda$, i.e.

$$||f||_\lambda := \left(\int_X |f|^p d\lambda\right)^{1/p}$$

for $f \in L^p(X, \lambda)$, and denote the norm on $L^p(X, \mu)$ by $||\cdot||_\mu$.
Now let $B := \sqcup_{\lambda\in\mathcal{E}} B_\lambda$ be the disjoint union (as a set) of these spaces, and let the surjection $\pi : B \to \mathcal{E}$ be given by $\pi(f) := \lambda$ if $f \in B_\lambda$. We would like to apply Proposition 3.1.17. For this we need to find a suitable set of cross-sections. Let a compactly supported continuous function $f \in C_c(X)$ be given. Then f is bounded, so

$$||f||_\lambda = \left(\int_X |f|^p d\lambda\right)^{1/p} \leq \sup_{x\in X} |f(x)| < \infty$$

and $f \in L^p(X, \lambda) = B_\lambda$ for any $\lambda \in \mathcal{E}$. Define a cross-section $s_f : \mathcal{E} \to B$ by $s_f(\lambda) = f \in B_\lambda$ for $\lambda \in \mathcal{E}$ and let $\Gamma := \{s_f : f \in C_c(X)\}$ be the set of all these cross-sections. Then Γ is a vector space under pointwise addition and scalar multiplication. By definition of the weak* topology on $\mathcal{E}$, for each $f \in C_c(X) \subset C_b(X)$ the mapping

$$\lambda \mapsto ||s_f(\lambda)||_\lambda = ||f||_\lambda = \left(\int_X |f|^p d\lambda\right)^{1/p}$$

is continuous on $\mathcal{E}$. Because X is a Polish space and each $\lambda \in \mathcal{E}$ is a finite Borel measure on X, any $\lambda \in \mathcal{E}$ is regular, as remarked in Section 1.2. As X is locally compact, we know that $\Gamma(\lambda) = C_c(X)$ lies dense in $L^p(X, \lambda) = B_\lambda$ for each $\lambda \in \mathcal{E}$. So (B, π) and Γ satisfy all the assumptions of Proposition 3.1.17, and we conclude:

Proposition 4.1.1. There exists a unique topology on B such that $\mathfrak{B} := (B, \pi)$ is a Banach bundle over $\mathcal{E}$ and such that each $s_f \in \Gamma$, for $f \in C_c(X)$, is a continuous section of $\mathfrak{B}$.

As in Chapter 2, endow $\mathcal{E}$ with the push-forward measure ν of μ though β. So $\nu(A) = \mu(\beta^{-1}(A))$ for all $A \subset \mathcal{E}$ measurable. We now consider integration in the bundle $\mathfrak{B}$. From the previous chapter we know that the outsized L^p-space $\overline{L}^p(\mathfrak{B}, \nu)$ of equivalence classes of sections with finite p-upper integral is a Banach space under the norm

$$||s||_p = \left(\overline{\int_{\mathcal{E}}} ||s(\lambda)||_\lambda^p d\nu(\lambda)\right)^{1/p}$$

for $s \in \overline{L}^p(\mathfrak{B})$.

Theorem 4.1.2. The space $L^p(X, \mu)$ is isometrically lattice isomorphic to a closed sublattice of $\overline{L}^p(\mathfrak{B}, \nu)$.

Proof:
We wish to construct an isometry $S : L^p(X, \mu) \to \overline{L}^p(\mathfrak{B}, \nu)$. To this end, note that

Corollary 2.2.4 tells us that for each $f \in L^p(X,\mu)$ the section $s_f : X \to B$ given by $s_f(\lambda) := f \in B_\lambda$ for all $\lambda \in \mathcal{E}$, is ν-almost everywhere defined. Extend it to all of $\mathcal{E}$, for instance by setting $s_f(\lambda) := 0_\lambda$ for $\lambda \in \mathcal{E}$ such that $f \notin L^p(X,\lambda)$. Then the same corollary gives us

$$||s_f||_p^p = \overline{\int_{\mathcal{E}}} ||s_f(\lambda)||_\lambda^p d\nu(\lambda) = \int_{\mathcal{E}} \left(\int_X |f|^p d\lambda \right) d\nu(\lambda) = \int_X |f|^p d\mu = ||f||_\mu^p < \infty \tag{4.1}$$

for any $f \in L^p(X,\mu)$. This means that the map $S : L^p(X,\mu) \to \overline{L}^p(\mathfrak{B},\nu)$ given by $S(f) := s_f$ for $f \in L^p(X,\mu)$, is a well-defined isometry.
It is linear because, for any $f, g \in L^p(X,\mu)$, $\alpha \in \mathbb{R}$ and ν-almost all $\lambda \in \mathcal{E}$,

$$S(\alpha f + g)(\lambda) = s_{\alpha f+g}(\lambda) = \alpha f + g = \alpha s_f(\lambda) + s_g(\lambda) = \alpha S(f)(\lambda) + S(g)(\lambda).$$

Also, for any $f, g \in L^p(X,\mu)$ and ν-almost all $\lambda \in \mathcal{E}$ we have

$$S(f \vee g)(\lambda) = f \vee g = S(f)(\lambda) \vee S(g)(\lambda).$$

So $S(f \vee g) = S(f) \vee S(g)$ and S is a lattice homomorphism.
We conclude that S is an isometric lattice isomorphism onto its range $S(L^p(X,\mu)) \subset \overline{L}^p(\mathfrak{B},\nu)$, which is closed because S is an isometry. □

Note that property 2 of the decomposition map β tells us that we can in fact write $S(f)(\lambda) = s_f(\lambda) = \mathbf{1}_{\beta^{-1}\{\lambda\}} f$ for any $f \in L^p(X,\mu)$ and ν-almost all $\lambda \in \mathcal{E}$. Therefore, by identifying $L^p(X,\mu)$ with $S(L^p(X,\mu)) \subset \overline{L}^p(\mathfrak{B},\nu)$ via S we can, in some sense, view any $f \in L^p(X,\mu)$ as a "p-integral" of its restrictions $\mathbf{1}_{\beta^{-1}\{\lambda\}} f$ to $\beta^{-1}\{\lambda\}$, for $\lambda \in \mathcal{E}$.

Example 4.1.3. A special case of the above theorem occurs when the invariant measure μ is ergodic, so $\mu \in \mathcal{E}$. Then $\mu(\beta^{-1}\{\mu\}) = 1$ and the push-forward ν of μ through β on $\mathcal{E}$ is given by

$$\nu(A) = \mu(\beta^{-1}(A)) = \mu(\beta^{-1}(A) \cap \beta^{-1}\{\mu\}) = \begin{cases} 1 & \text{if } \mu \in A \\ 0 & \text{if } \mu \notin A \end{cases}$$

for $A \subset \mathcal{E}$ measurable. So ν is the Dirac measure at $\mu \in \mathcal{E}$.
Let $S : L^p(X,\mu) \to \overline{L}^p(\mathfrak{B},\nu)$ be the isometric lattice homomorphism from Theorem 4.1.2 and let $s \in \overline{L}^p(\mathfrak{B},\nu)$ be arbitrary. Set $f := s(\mu) \in L^p(X,\mu)$. Then

$$||S(f) - s||_p^p = \int_{\mathcal{E}} ||s_f(\lambda) - s(\lambda)||_\lambda^p d\nu(\lambda) = ||f - s(\mu)||_\mu^p = 0,$$

Hence $S(f) = s$ and S is surjective. In this case we can identify $\overline{L}^p(\mathfrak{B},\nu)$ and $L^p(X,\mu)$ via S and $L^p(X,\mu)$ is decomposed in a trivial manner.

Unfortunately, in general we do not know what the image of the map S in $\overline{L}^p(\mathfrak{B})$ will be. However, more can be said if $\mathcal{E}$ is locally compact. Since ν is a finite Borel measure on the metrizable space $\mathcal{E}$, it is normal (as remarked in section 1.2). So we can consider the subspace $L^p(\mathfrak{B},\nu) \subset \overline{L}^p(\mathfrak{B},\nu)$ of p-integrable sections from Definition 3.2.11. If there exists a σ-compact set $A \subset \mathcal{E}$ such that $\nu(\mathcal{E} \setminus A) = 0$, then we can describe the image of S explicitly.

Theorem 4.1.4. If $\mathcal{E}$ is locally compact and ν is zero off a σ-compact set, then the map $S : L^p(X,\mu) \to \overline{L}^p(\mathfrak{B},\nu)$ from Theorem 4.1.2 is an isometric lattice isomorphism onto $L^p(\mathfrak{B},\nu) \subset \overline{L}^p(\mathfrak{B},\nu)$.

Proof:
We already know that S is an isometric lattice homomorphism to $\overline{L}^p(\mathfrak{B},\nu)$, so it remains to show that $S(L^p(X,\mu)) = L^p(\mathfrak{B},\nu)$ holds.
For one inclusion, note that μ is regular, as a finite Borel measure on the metrizable space X. As X is locally compact, $C_c(X)$ lies dense in $L^p(X,\mu)$. Because S is an isometry and $L^p(\mathfrak{B},\nu)$ is closed in $\overline{L}^p(\mathfrak{B},\nu)$, we can conclude that $S(L^p(X,\mu) \subset L^p(\mathfrak{B},\nu)$ holds if we know that $S(C_c(X)) \subset L^p(\mathfrak{B},\nu)$.
To prove the latter remark that, since ν is zero off a σ-compact set, any section of $\mathfrak{B}$ almost vanishes off a σ-compact set. For any $f \in C_c(X)$ the section $s_f \in \overline{L}^p(\mathfrak{B},\nu)$ is continuous by Proposition 4.1.1, hence locally measurable. By Proposition 3.2.19, $s_f \in L^p(\mathfrak{B},\nu)$ for any $f \in C_c(X)$. So indeed $S(L^p(X,\mu) \subset L^p(\mathfrak{B},\nu)$.
To prove $S(L^p(X,\mu)) = L^p(\mathfrak{B},\nu)$, it suffices to show that the range lies dense in $L^p(\mathfrak{B},\nu)$. Indeed, since S is an isometry, its image is closed. As $C_c(\mathfrak{B})$ lies dense in $L^p(\mathfrak{B},\nu)$, we merely need to show that we can approximate any compactly supported continuous section by elements of $S(L^p(X,\mu))$. So let $s \in C_c(\mathfrak{B})$ with compact support $K \subset \mathcal{E}$ and an $\epsilon > 0$ be given. Choose, for each $\kappa \in K$, an $f_\kappa \in C_c(X)$ such that

$$||s(\kappa) - S(f_\kappa)(\kappa)||_\kappa = ||s(\kappa) - f_\kappa||_\kappa = \left(\int_X |s(\kappa)(x) - f_\kappa(x)|^p d\kappa(x)\right)^{1/p} < \epsilon,$$

which we can do because $C_c(X)$ lies dense in $B_\kappa = L^p(X,\kappa)$. Because $S(f_\kappa)$ is a continuous section (Proposition 4.1.1), so is $s - S(f_\kappa)$ and the map $\lambda \mapsto ||s(\lambda) - S(f_\kappa)(\lambda)||_\lambda$ is continuous on $\mathcal{E}$. Hence there exists an open neighbourhood $U_\kappa \subset \mathcal{E}$ of κ such that

$$||s(\lambda) - S(f_\kappa)(\lambda)||_\lambda = ||s(\lambda) - f_\kappa||_\lambda < \epsilon$$

holds for ν-almost all $\lambda \in U_\kappa$. In this manner we get an open covering of K, and the compactness of K implies that we can choose an $n \in \mathbb{N}$, $U_1, \ldots, U_n \subset \mathcal{E}$ open and $f_1, \ldots, f_n \in C_c(X)$ such that $K \subset \cup_{i=1}^n U_i$ and $||s(\lambda) - f_i||_\lambda < \epsilon$ for all $\lambda \in U_i$.
Now define sets $A_1, \ldots, A_n \subset X$ by $A_1 := \beta^{-1}(U_1 \cap K)$, $A_2 := \beta^{-1}((U_2 \setminus U_1) \cap K)$ up to $A_n := \beta^{-1}((U_n \setminus \cup_{i=1}^{n-1} U_i) \cap K)$. Since β is measurable, so are the A_i. Furthermore, they are disjoint and $\beta^{-1}(K) = \cup_{i=1}^n A_i$.
Define $f \in L^p(X,\mu)$ by $f := \sum_{i=1}^n \mathbf{1}_{A_i} f_i$. Remark that it follows from property 2 of the decomposition map β that $\int_X g d\lambda = \int_{\beta^{-1}\{\lambda\}} g d\lambda$ for any integrable function $g : X \to \mathbb{R}$ and any $\lambda \in \mathcal{E}$. If $\lambda \in \mathcal{E} \setminus K$, then $s(\lambda) = 0_\lambda$. So for ν-almost all $\lambda \in \mathcal{E} \setminus K$

$$||s(\lambda) - S(f)(\lambda)||_\lambda^p = ||f||_\lambda^p = \int_X |f(x)|^p d\lambda(x) = \int_{\beta^{-1}\{\lambda\}} \left|\sum_{i=1}^n \mathbf{1}_{A_i}(x) f_i(x)\right|^p d\lambda(x) = 0,$$

using that $\beta^{-1}\{\lambda\} \cap (\cup_{i=1}^n A_i) = \beta^{-1}\{\lambda\} \cap \beta^{-1}(K) = \emptyset$.
On the other hand, for any $\lambda \in K$ there exists a $j(\lambda) \in \{1, \ldots, n\}$ such that $\beta^{-1}\{\lambda\} \subset A_{j(\lambda)}$. From property 2 of the decomposition map it follows that $\lambda(A_i) = 0$ for all $i \neq j$ in $\{1, \ldots, n\}$ and $\lambda \in \mathcal{E}$. Hence

$$||s(\lambda) - S(f)(\lambda)||_\lambda^p = ||s(\lambda) - f||_\lambda^p = \int_X \left|s(\lambda)(x) - \sum_{i=1}^n \mathbf{1}_{A_i}(x) f_i(x)\right|^p d\lambda(x)$$

$$= \int_{\beta^{-1}\{\lambda\}} |s(\lambda)(x) - f_{j(\lambda)}(x)|^p d\lambda(x) = \int_X |s(\lambda)(x) - f_{j(\lambda)}(x)|^p d\lambda(x) = ||s(\lambda) - f_{j(\lambda)}||_\lambda^p < \epsilon^p$$

for ν-almost all $\lambda \in K$, by choice of $\{f_1, \ldots, f_n\} \subset C_c(X)$. We conclude that $||s(\lambda) - S(f)(\lambda)||_\lambda < \epsilon$ holds for ν-almost all $\lambda \in \mathcal{E}$, which means that

$$||s - S(f)||_p = \left(\int_{\mathcal{E}} ||s(\lambda) - S(f)(\lambda)||_\lambda^p d\nu(\lambda) \right)^{1/p} < \epsilon.$$

By the arbitrariness of $\epsilon > 0$ and $s \in C_c(\mathfrak{B})$, we see that $S(L^p(X, \mu))$ lies dense in $L^p(\mathfrak{B}, \nu)$ and that S is an isometric lattice isomorphism between $L^p(X, \mu)$ and $L^p(\mathfrak{B}, \nu)$. □

In some sense, Theorem 4.1.2 tells us that we can decompose $L^p(X, \mu)$ as an integral of $L^p(X, \lambda)$-spaces, for λ ranging over the ergodic measures $\mathcal{E}$. Theorem 4.1.4 then explicitly describes the manner in which this is done, if these measures form a locally compact space and if the push-forward measure ν on $\mathcal{E}$ is zero off a σ-compact set. In particular, the latter holds if $\mathcal{E}$ is *σ-locally compact*, which means that it is locally compact and σ-compact. Unfortunately, the author is not aware of any non-trivial conditions on the transformation group (G, X) and the invariant measure μ that guarantee that these requirements are met.

Since any element of $L^p(\mathfrak{B}, \nu)$ is a pointwise ν-almost everywhere limit of a sequence of continuous compactly supported sections, it is clear that ν must indeed vanish off some σ-compact set for the image of S to be contained in $L^p(\mathfrak{B}, \nu)$ (consider the section $S(\mathbf{1}) \in \overline{L}^p(\mathfrak{B}, \nu)$). Similarly, if $\mathcal{E}$ is not locally compact it may have few compact sets, and therefore few compactly supported continuous sections. In that case one might expect $L^p(\mathfrak{B}, \nu)$ to be too small to contain the image of $L^p(X, \mu)$.

Decomposing the group action So far, we have not yet taken the action of the group G on X into account explicitly. From Proposition 1.2.3 we know that G acts on $L^p(X, \lambda)$ for each $\lambda \in \mathcal{E}$. Hence we get an induced action of G on the set of sections $s : X \to B$ of $\mathfrak{B}$ by $(gs)(\lambda) := g(s(\lambda))$ for all $\lambda \in \mathcal{E}$. Because the action of G on $L^p(X, \lambda)$ is isometric for each $\lambda \in \mathcal{E}$, we have

$$||gs||_p^p = \int_{\mathcal{E}} ||g(s(\lambda))||_\lambda^p d\nu(\lambda) = \int_{\mathcal{E}} ||s(\lambda)||_\lambda^p d\nu(\lambda) = ||s||_p^p < \infty \tag{4.2}$$

for each $s \in \overline{L}^p(\mathfrak{B}, \nu)$. As each $g \in G$ defines a lattice isomorphism on each $L^p(X, \lambda)$, G acts as a group of isometric lattice isomorphisms on $\overline{L}^p(\mathfrak{B}, \nu)$. Using the results above, we can say even more.

Theorem 4.1.5. Let $S : L^p(X, \mu) \to S(L^p(X, \mu)) \subset \overline{L}^p(\mathfrak{B}, \nu)$ be the isometric lattice isomorphism from Theorem 4.1.2. Then G acts on $S(L^p(X, \mu))$ and the map $\rho' : G \to \mathcal{B}(S(L^p(X, \mu)))$ given by $\rho'(g)s = gs$ for all $g \in G$ and $s \in S(L^p(X, \mu))$, is a strongly continuous representation of G as a group of isometric lattice isomorphisms on $L^p(\mathfrak{B}, \nu)$ that is fiberwise band irreducible and strongly continuous. For each $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} L^p(X, \mu) & \xrightarrow{\rho(g)} & L^p(X, \mu) \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle S^{-1}} \\ S(L^p(X, \mu)) & \xrightarrow[\rho'(g)]{} & S(L^p(X, \mu)) \end{array} \tag{4.3}$$

Proof:
To see that the diagram commutes and is well-defined, let a $g \in G$ and an $f \in L^p(X, \mu)$ be given. For ν-almost all $\lambda \in \mathcal{E}$ we have

$$g s_f(\lambda) = g(s_f(\lambda)) = gf = s_{gf}(\lambda),$$

so $g s_f = s_{gf} \in L^p(\mathfrak{B}, \nu)$. Hence

$$S^{-1}\rho'(g)S(f) = S^{-1}(g s_f) = S^{-1}(s_{gf}) = gf = \rho(g)f,$$

which means that the diagram is indeed commutative for each $g \in G$. Most properties of ρ' now follow from the corresponding ones of ρ, using that S is an isometric lattice isomorphism.
As for the fiberwise band irreducibility, this follows from Proposition 1.2.6.
Applying Proposition 1.2.3 to the action of G on $L^p(X, \lambda)$ for each $\lambda \in \mathcal{E}$, we see that ρ' is fiberwise strongly continuous. □

The above theorem tells us that, if we identify $L^p(X, \mu)$ with $S(L^p(X, \mu)) \subset \overline{L}^p(\mathfrak{B}, \nu)$ via S, then we can identify ρ and ρ'. Since the latter representation is fiberwise band irreducible we have, in some sense, decomposed ρ into band irreducible representations. Hence Theorem 4.1.5 accomplishes the goal we had set for ourselves in this thesis. Moreover, if the conditions of Theorem 4.1.3 are satisfied, then we can describe the image of S in $\overline{L}^p(\mathfrak{B}, \nu)$ in a direct manner, as the space $L^p(\mathfrak{B}, \nu)$ of p-integrable sections.

Example 4.1.6. We return to the setting of Example 2.1.3 and let $\mathbb{D}$ be the closed unit disc in $\mathbb{C}$, with $\mathbb{T}$ its boundary. Then $\mathbb{T}$ acts on $\mathbb{D}$ by complex multiplication and $(\mathbb{T}, \mathbb{D})$ is a compact transformation group. We have remarked that the ergodic measures are of the form

$$\lambda_r(Y) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}_Y(re^{i\theta})d\theta$$

for $Y \subset \mathbb{D}$ measurable and $r \in [0, 1]$. Also, the map $\phi : \mathcal{E} = \{\lambda_r : r \in [0, 1]\} \to [0, 1]$ given by $\phi(\lambda_r) = r$ for $r \in [0, 1]$ is a homeomorphism, so $\mathcal{E}$ is a compact space. Let μ be the normalized Lebesgue measure on $\mathbb{D}$:

$$\mu(Y) = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} r\mathbf{1}_Y(re^{i\theta})d\theta dr$$

for $Y \subset \mathbb{D}$ measurable. Then μ is a $\mathbb{T}$-invariant probability measure, and the map $\beta : \mathbb{D} \to \mathcal{E}$ given by $\beta(re^{i\theta}) = \lambda_r \in \mathcal{E}$, for $r \in [0, 1]$ and $\theta \in [0, 2\pi)$, is a decomposition map of μ as in part 3 of Theorem 2.1.5 (note that we have not shown that β decomposes all invariant measures in this manner, so we do not know whether β is in fact the map from this theorem, but this is irrelevant since we have fixed μ). It is straightforward to check that β also satisfies the other properties of a decomposition map.
From Proposition 4.1.1 we know that we can construct a Banach bundle $\mathfrak{B} = (B, \pi)$ over $\mathcal{E}$, where $B = \sqcup_{r\in[0,1]} L^p(X, \lambda_r)$ as a set and $\pi(f) = r$ for $f \in L^p(X, \lambda_r)$, such that any $f \in C_c(X)$ gives rise to a continuous section s_f of $\mathfrak{B}$, $s_f(\lambda) = f \in L^p(X, \lambda)$ for all $\lambda \in \mathcal{E}$. Let ν be the push-forward measure of μ by β and let $A \subset \mathcal{E}$ be measurable. Then $A = \phi^{-1}(\phi(A)) = \{\lambda_r \in \mathcal{E} | r \in \phi(A)\}$ and

$$\nu(A) = \mu(\beta^{-1}(A)) = \mu\left\{re^{i\theta} \in \mathbb{D} | r \in \phi(A), \theta \in [0, 2\pi)\right\}$$

$$= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r \mathbf{1}_Y(re^{i\theta}) d\theta dr = 2 \int_{\phi(A)} r dr,$$

with $Y := \{re^{i\theta} \in \mathbb{D} | r \in \phi(A), \theta \in [0, 2\pi)\} \subset \mathbb{D}$.
Since $\mathcal{E}$ is compact, Theorem 4.1.4 tells us that the map $S : L^p(X, \mu) \to L^p(\mathfrak{B}, \nu)$ given by $S(f)(\lambda) = \mathbf{1}_{\beta^{-1}\{\lambda_r\}} f \in L^p(X, \lambda_r)$ for $f \in L^p(X, \mu)$ and $\lambda_r \in \mathcal{E}$, is an isometric lattice isomorphism. As $\beta^{-1}(\lambda_r) = \{z \in \mathbb{D} : |z| = r\}$ for $\lambda_r \in \mathcal{E}$, we can view any $f \in L^p(X, \mu)$ via S as an "integral" of its restrictions to the circles of radius r around 0. The action of $\mathbb{T}$ on $\mathbb{D}$ can then be decomposed into the band irreducible actions of $\mathbb{T}$ on these circles.

4.2 Conclusion and further research

There remain several directions in which the results of this thesis can be generalized and several possible areas for future research to focus upon. Here we discuss a few.
As the reader may have noticed, we have only decomposed $L^p(X, \mu)$ for $p < \infty$, ignoring the space of almost everywhere bounded functions $L^\infty(X, \mu)$. A reason for this is that the action of G on $L^\infty(X, \mu)$ is less interesting because it is generally not strongly continuous. Also, these functions have somewhat different properties and that the approach we used for $p < \infty$ does not transfer directly to the case $p = \infty$. For instance, the compactly supported functions generally do not lie dense in $L^\infty(X, \mu)$. We used these compactly supported functions on crucial points in Proposition 1.2.3 to prove strong continuity of the group action and in Proposition 4.1.1 to find a suitable Banach bundle over the ergodic measures. For that proposition we also used that the weak* topology on $\mathcal{E}$ guaranteed that the continuous bounded functions on X gave rise to continuous sections on such a bundle, but this does not work for the infinity-norm. Finally, as the compactly supported continuous sections need not lie dense in $L^\infty(\mathfrak{B}, \nu)$, one cannot use the same technique of proof as we did in Theorem 4.1.4.
We have proved the theorems in this chapter for G-invariant probability measures, but they also hold for arbitrary finite measures. For the zero measure this is trivial, and if μ is a finite nonzero G-invariant measure, then applying our main theorems to the normalization of μ yields the same result. The question remains whether these results can be extended to the case of σ-finite measures.
So far we have only considered real-valued functions. One reason for this is that these form a Riesz space in a natural manner. With complex-valued functions this is not the case, and therefore we need to consider complex Banach lattices. Adjusting the definitions to this case, the results of Chapter 1 pass on almost verbatim. The measure decomposition results in Chapter 2 are independent of the choice of scalar field, as is Chapter 3. The results of Chapter 4 can also be extended, by splitting complex functions into their real and imaginary parts. So our main theorems also hold in the complex case, altering certain terms from real-valued Banach lattices to their complex counterparts.
A fundamental theorem in [3] states that, for a Polish group G, any standard Borel G-space X is Borel isomorphic to a Polish G-space Y. So (G, Y) is a Polish transformation group and the σ-algebra on X can be identified with that of Y. This provides us with a possibility to generalize our results to a far wider class of structures. See also [11, p. 1115].
Other efforts could go into determining conditions on the spaces involved that guarantee that the space of ergodic measures is σ-locally compact, so that we can use Theorem 4.1.4 to give a nice description of the decomposition. The ergodic measures are the extreme

points of the invariant measures, a convex set in a Polish space. In general sets of extreme points need not inherit much structure from the space in which they lie. For instance, they are quite often not closed. However, it may be that there are useful criteria which guarantee that they are locally compact and σ-compact.

Another way to go could be to use other local measurability structures to construct spaces of p-integrable sections. We have followed [9] in their approach to defining Banach bundles, but in their work they also describe more general local measurability structures, not built around compact sets as in our case. Local measurability structures with compact sets are convenient when the spaces involved are locally compact, and then we can prove results as we have, but another approach could be to work with subsets which are naturally more abundant in the case of the ergodic measures. And yet another possible approach could be to use other concepts than Banach bundles to consider integration.

In this thesis we have considered group actions. However, one of the few points at which we used that each element of G has an inverse, was in defining the action of G on $L^p(X,\mu)$ as $gf(x) = f(g^{-1}x)$ for $g \in G$, $f \in L^p(X,\mu)$ and $x \in X$. That was necessary to ensure that we indeed get a representation of G on $L^p(X,\mu)$. However, if G is in fact a commutative semigroup then $gf(x) := f(gx)$, for $g \in G$, $f \in L^p(X,\mu)$ and $x \in X$, also defines a representation of G on $L^p(X,\mu)$ for all $p \in [1,\infty]$. The author suspects that all results of Chapter 1 can then be extended to the case of commutative semigroups, and we do not need the group assumption until Chapter 2, where it is used to prove the existence of a decomposition map. However, recent work in [15] extends such decomposition results to certain commutative semigroups, and we can use this to prove our main theorems for such semigroups as well.

As mentioned in Chapter 2, there are generalizations of the measure decomposition results to quasi-invariant measures. These do not give rise to induced group actions as in Proposition 1.2.3, but for Theorems 4.1.2 and 4.1.4 we did not use this induced group action directly. So it might be possible to extend these results to quasi-invariant measures, if the decomposition of these measures has the same properties as in the invariant case.

One must also wonder about further properties of the decompositions we have given. Do these L^p-spaces of sections of a Banach bundle have certain universal properties that justify speaking of them as 'integrals' of Banach spaces? Does our decomposition have some sort of uniqueness property? These and more questions remain.

As with all research, one of the main issues that needs to be addressed is that of its use. From a philosophical standpoint, a central principle in mathematics is to decompose something complicated into simpler components. In that light it seems natural to search for decomposition results as we have done, hoping to better understand group actions on spaces of integrable functions. Moreover, research focusing on representations on Hilbert spaces has earned its merits in the past decades. Much less is known about representations on Banach spaces or Banach lattices. We hope that this thesis can help shed some light on representations of (semi)groups on such structures.

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