Solution Concepts in Cooperative Game Theory

Stolwijk, A.

A. Stolwijk

Solution Concepts in Cooperative Game Theory

Master's Thesis, defended on October 12, 2010

Thesis Advisor: dr. F.M. Spieksma

Mathematisch Instituut, Universiteit Leiden

Contents

Chapter 1

Introduction

1.1 Background and Aims

In Game Theory situations are studied in which multiple people each strive to achieve his or her goal. Thereby it is assumed that all participants behave rationally. An important characteristic of games is that the actions of one person have influence on the outcomes of other people in the game and vice versa. Because of these interdependencies, Robert Aumann, who received a Noble award for his contributions to Game Theory in 2005, suggested that *Interactive Decision Theory* would be a better name for the discipline called 'Game Theory'. From its definition it follows that game theory can be applied to all situations where peoples' actions are both utility maximizing and interdependent. The major applications of game theory are to economics, political science, tactical and strategic military problems, evolutionary biology and computer science. According to Aumann (1987) there are also important connections with accounting, statistics, the foundations of mathematics, social psychology and branches of philosophy.

In this thesis I will investigate a particular set of game theoretic problems: namely those in which people who participate in the game, can cooperate with each other to compel a certain solution. In politics they can form coalitions, in the economy they can form cartels. This branch of game theory is called Cooperative Game Theory. In Cooperative Game Theory we are interested in what players can achieve by cooperation. We look at all feasible outcomes where players can make binding commitments. We assume that there is some mechanism which enforces these commitments.

The method of game theory consists of the construction of models or methodologies that can in principle be applied to a wide variety of abstract interactive situations. In theory most situations of interactive decision making have many outcomes, often infinitely many. For example if, say, three persons have to share a cake or to accomplish a certain task, a great variety of outcomes can be defined. Or, if political parties try to form a majority coalition, the number of potential outcomes, in the form of the set of compromises, is sheer unlimited.

However, not all outcomes are rational or equally plausible. By defining so called solution concepts, cooperative game theory tries to characterize the set of outcomes that are, seen from a viewpoint of rationality, interesting. In this thesis I will describe and discuss the main solution concepts that have, in the course of time, been proposed by different game theorists. In the discussion I will also point at the relations between the different concepts and their limitations.

1.2 Outline

After this introductory chapter, Chapter 2 starts with a general description of the model of a cooperative game. The discussion is mainly limited to the basic concepts, e.g the characteristic function and the solution space, and some general properties of a cooperative game.

The solution space of a model usually consists of more points or payoffs. These are also called imputations. Chapter 3 investigates the relation between different imputations. Starting point is the preference relation as defined by Van Neumann and Morgenstern. This preference relation makes it possible to compare outcomes in a logical way. I also define two other variants on the Von Neumann Morgenstern preference relation.

In Chapter 4, I look into more detail in the most general solution concept of a cooperative game namely the Core. Imputations in the Core have the attractive property that they are not dominated. With the weaker forms of the preference relation, I define a new kind of Core, the Dual Core. The Core and the Dual Core and their relations are also discussed here.

Chapter 5 is devoted to the Nash equilibrium. Although the Nash equilibrium is basically a solution concept of a non-cooperative game, I define a variant to cooperative games.

Stable Sets are the subject of Chapters 6 and 7. Stable Sets were introduced by Von Neumann and Morgenstern. It appears that solutions in a Stable Set have some nice properties.

Chapter 6 starts with a general introduction of the concept. Contrary to the (Dual) Core and Nash Equilibria, Stable Sets are difficult to find. For a three person zero sum game it will be shown how they can be constructed.

Chapter 7 is an extension of chapter 6. Successively properties of Partially Stable Sets, Strictly Stable Sets and Weakly Stable Sets will be investigated. The Chapter ends with a conjecture. Unfortunately, I have not succeeded in proving the conjecture.

The Core, the Nash equilibrium and Stable Sets are solution concepts which have, from a rational point of view, the property to be stable in some sense. The Nucleolus and the Shapley Value, which I discuss in Chapters 8, 9 and 10, are mainly motivated by some sense of fairness.

The idea behind the Nucleolus, the subject of Chapter 8, is to make the least happy coalition of players of a game as happy as possible. In Chapter 9 I slightly adjust the Nucleolus by assuming that coalitions are already given (decided).

Chapter 10 is devoted to the Shapley Value. The basic idea behind the Shapley Value is to share the payoff of a game according to the relative importance of the individual players of the game.

Chapter 11 discusses some special classes of games, respectively: composite games and convex games. The discussion add to a better understanding of the various concepts introduced in the earlier chapters.

The thesis ends with Chapters 12 and 13. In Chapter 12 three existence questions with respect to Stable Sets are discussed and Chapter 13 contains some concluding remarks.

Chapter 2

The Model: Some Basic Concepts

In this thesis we will look at a certain type of games: games in which a mechanism is available that can enforce agreements between players. These games are called cooperative games. A problem is how to characterize these games. If we look for example at the following weighted majority game[1] $[5; 2, 3, 4]$, we see that this game is in a sense the same game as the weighted majority game $[2; 1, 1, 1]$. In both games, any two parties can always have a majority. So the utility of the votes of every player is in both games the same. In this chapter we define a function which gives the utility or the payoff of a coalition. Every cooperative game can be described by such a function.

2.1 Characteristic Function

To characterize a cooperative game with n-players, we use the so called *characteristic* function. The characteristic function tells us for every feasible coalition what the utility or payoff is. In a more formal way this can be defined as follows:

Definition 2.1 An n-*persons* coalitional game is a pair (N, v), with $N = \{1, 2, ..., n\}$ the set of n players and v, the characteristic function, which maps every subset $S \subseteq N$ to a number in $\mathbb{R}$.

We take $v(\emptyset) = 0$.

We call a nonempty subset S a coalition with players $\{i | i \in S\}$. The number of indices in a set S is denoted $|S|$. If $|S| = k$, S is called a k-*persons* coalition. A *one-person* coalition is called a player.

For now we look at models with the property that cooperation leads to a greater or equal gain for every player of the coalition, by coordinating the strategies of its members, no matter what the other players do. In formula: $v(S \bigcup T) \geq v(S) + v(T), \forall S, T \subset N, S \bigcap T = \emptyset$. This property is called *superadditivity.*

[1] In a weighted majority game $[Q; p_1, ..., p_i, ..., p_n]$ each voter i of n voters is assigned a certain nonnegative real number p_i. There is a positive real number quota Q such that a group of voters can pass a resolution if the sum of the weights of the group members is at least as high as the given quota. Thus if $\sum_S p_i \geq Q$ for some $S \subseteq \{1, ..., n\}$, coalition S can pass the resolution.

If the characteristic function says that cooperation of S and T leads to less gain then not cooperation, we always can transform the characteristic function into a superadditive characteristic function with the intuitive argument that *cooperation is by not cooperating*.

Definition 2.2 We can transform a game (N, v) to a game in his superadditive form (N, v_{SA}):

$$v_{SA}(S) = \max\{\sum_i v(T_i) | \bigcup_i T_i = S,\ T_i \bigcap T_j = \emptyset,\ i \neq j\}.$$

Note that a game which is already in superadditive form is transformed to itself.

In models with n players there are 2^n possibilities to make coalitions; the number of subsets of $\{1, ..., n\}$. We will only specify the non-zero values of the characteristic function.

Before we investigate this model in more detail, we look at some examples:

Example 2.3 Three player zero-sum game We have three players in this game. They can form coalitions. If a player cooperates with another player their total gain is 1 and the third player loses 1. If they don't cooperate, they all lose 1. So we have the following cooperative game:

$$v(1) = v(2) = v(3) = -1,$$
$$v(12) = v(13) = v(23) = 1.$$

To make the characteristic function complete for all players in all coalitions we have to define the value $v(S)$ for $S = \{123\}$:

$$v(123) = \max\{v(1) + v(2) + v(3) = -3, v(12) + v(3) = 0, v(13) + v(2) = 0, v(1) + v(23) = 0\} = 0$$

Example 2.4 In Friesland there is a farmer, called F, who has one cow. Every week the farmer gets 1 barrel of milk from this cow, but without good processing this milk is useless. A company B offers its service to process the milk into butter, another company Y offers its service to process the milk into yoghurt. After processing the total payoff for butter or yoghurt, or the profit minus the costs, is 1. We can write this problem as a coalitional game as follows:

$$v(FB) = v(FY) = v(FBY) = 1,$$
$$v(S) = 0 \text{ for all other } S.$$

From the characteristic function it can directly be seen that cooperation leads to extra value. But there is also a value for a coalition S for not cooperation; this is what economists call the *opportunity costs*. This is the value the great coalition N loses if coalition S is not cooperating with the great coalition. We will call this the *dual value* of a coalition.

Definition 2.5 The Dual Value of a coalition S is $v^*(S) = v(N) - v(N \backslash S)$.

Note that the function v^* is not necessarily superadditive.

We explain this by continuing Example 2.4.

Continuation of Example 2.4 Now $v^*(F) = v^*(BY) = 1$ and $v^*(S) = v(S)$ for all other S. The $v^*(S)$ for this game is not superadditive, $v^*(F) + v^*(BY) > v^*(FBY)$.

What is meant by the dual values in this game, is that if coalition F or coalition BY don't cooperate with the other member(s) of the great coalition, they together lose 1. Thus in some way, coalition BY can blackmail the other member F of the great coalition FBY. On the other hand the coalition F can also blackmail the other members BY of the great coalition FBY.

2.2 Solution Space: Transferable and Non-Transferable Utilities

The characteristic function does not contain all information of a game. The possible payoffs of $v(S)$ to its members are not given, or the characteristic function does not tell us anything about the possible sharing of the total payoff among its members. With respect to sharing the profit or the utility, a division can be made by games with *transferable* and games with *non-transferable utilities*.

Example 2.6 Take the following example. Two children, 1 and 2, can get one bar of chocolate. They both love chocolate. But they only get it if they agree that only one of the children may eat it. So they can't share it. We see that $v(12) > v(1) + v(2)$. The possible payoffs x for this total game are $(v(1), v(2))$, $(v(1), v(12))$ and $(v(12), v(2))$. This type of cooperative games are called games with *non-transferable utilities*.

If the bar of chocolate, i.e. the $v(12)$, can be divided in all possible ways, we speak of a game with *transferable utilities*.

In this thesis we mainly restrict ourself to games with transferable utilities. For these games we define the following sets:

Definition 2.7 The set of all payoff possibilities is described as A''.
The set of all payoffs for which the sum of the payoffs is not greater than $v(N)$ is called the set of pre-imputations A'. This set is given by $A' = \{x | \sum_{i \in N} x_i \leq v(N)\}$ The set of individually rational pre-imputations is the set of imputations $A = \{x | \sum x_i \leq v(N), x_i \geq v(i) \forall i\}$.

We see that $A \subset A' \subset A''$.

From Definition 2.7 it follows that a payoff in an n-person game corresponds to a vector in $\mathbb{R}^n$. The set of pre-imputations A' corresponds to a half space. The set of imputations A is a closed and convex set.

To illustrate these definitions, we look at the following Example.

Example 2.8 Suppose we have a game (N, v), with $N = \{1, 2, 3\}$ and characteristic function

- $v(i) = 0$ *for* $i = 1, 2, 3$,
- $v(12) = v(13) = v(23) = 1$.

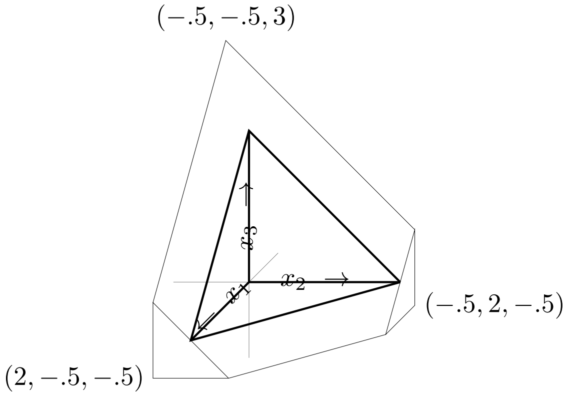

Figure 2.1

Figure 2.1 shows the space $B = A'' \bigcap \{x| - \frac{1}{4} \leq x_1 \leq 1,\ -\frac{1}{4} \leq x_2 \leq 1,\ -\frac{1}{4} \leq x_3 \leq 1\frac{1}{2}\}$. The set of pre-imputations is the half space $A' = \{x| \sum x_i \leq 1\}$. The set of imputations $A = \{x| \sum x_i \leq 1, x_i \geq 0\}$. The borders of $A' \bigcap B$ are the thin lines. The borders of A are given by the thick lines.

Thus we see that the characteristic function of a cooperative game with transferable utility contains all the information about the set of imputations.

2.3 Equivalence between Games

We start with a definition.

Definition 2.9 Two games (N, v) and (N, w) are equivalent if there exists a transformation $s : v \to w$ for which s is a composition of the following maps, $\pi,\ \upsilon,\ \tau$:

(a) A permutation π of players

(b) A linear transformation $\upsilon : v(S) \to \lambda v(S)$ for λ in R_+, and $\upsilon : x_i \to \lambda x_i$ for all $x \in A$, for all $S \subseteq N$.

(c) An affine transformation $\tau : v(S) \to v(S) + \sum_{i \in S} c_i$, where $c_1, c_2, ..., c_n \in \mathbb{R}$ are fixed constants, and $\tau : x_i \to x_i + c_i$ for all $x \in A$, for all $S \subseteq N$.

Thus if (N, v) and (N, w) are equivalent, then for some $\pi,\ \upsilon,\ \tau$ we know that $s(v(S)) = \pi \circ \upsilon \circ \tau(v(S)) = \lambda v(\pi(S)) + \sum_{i \in \pi(S)} c_i = w(S)$. We can see that s is bijective, because $\pi,\ \upsilon,\ \tau$ are bijective.

Thus we see that the characteristic function of two equivalent games contains the same information. We also see that the imputation space is equivalent. From Definition 2.9b, it follows that the dual values of two equivalent games, are also equivalent. The next Definition and Theorem allow us to focus on a smaller group of possible characteristic functions.

Definition 2.10 We say that the characteristic function of a superadditive game is in *(0,1)-normalised* form if $v(i) = 0$ for $i = 1, 2, ..., n$, $v(S) \geq 0$ and $v(N)$ is either 1 or 0. We say that the game is additive if and only if its characteristic function in (0,1)-normalised form gives $v(S) = 0 \ \forall S \subseteq N$.

In [4] the following theorem is given without the proof.

Theorem 2.11 [4] *Every superadditive game* (N, v) *is equivalent with exactly one superadditive game* (N, w) *in (0,1)-normalised form.*

PROOF:

We prove two steps:

- We create an superadditive, (0,1)-normalised game,
- We show that this game is unique.

First let $v_{SA}(N) = \max_{J \in \mathcal{J}} \sum_{i \in J} v(T_i)$ for which $\mathcal{J} = \{J | \bigcup_{i \in J} T_i = N,\ T_j \bigcap T_k = \emptyset \text{ for } j, k \in J\}$. Let $\lambda = \begin{cases} (|v_{SA}(N) - \sum v(i)|)^{-1} \text{ if } v_{SA}(N) - \sum v(i) \neq 0 \\ 0 \text{ else} \end{cases}$. Let $c_i = -\lambda v(i)$. With these λ and $c_1, ..., c_n$ we make the transformation $w(s) = \lambda v(S) + \sum_{i \in S} c_i$. Because s is a combination of the actions in Definition 2.9 it is trivial that $w(S)$ and $v(S)$ are equivalent. Now we will check that the game is in $(0, 1)$-*normalised* form:

$$\begin{aligned} w(i) &= \lambda v(i) + c_i = \lambda v(i) - \lambda v(i) = 0 \\ w(N) &= \lambda v_{SA}(N) + \sum_n c_i = \lambda v(N) - \sum_n \lambda v(i) \\ &= \lambda(v_{SA}(N) - \sum_n v(i)) = \begin{cases} \frac{v_{SA}(N) - \sum_n v(i)}{|v_{SA}(N) - \sum_n v(i)|} = 1 \text{ if } \lambda \neq 0 \\ 0 \text{ else} \end{cases} \end{aligned}$$

Every superadditive game has also one unique equivalent (0,1)-normalised game:

If a game is equivalent with more than one (0,1)-normalised game, there have to be two (0,1)-normalised games, (N, w) and (N, w') which are equivalent. Because $w(i) = 0$ and $w'(i) = \lambda w(i) + c_i$ we cannot add a constant $c_i \neq 0$ to any of the players, or else $w(i) > 0$. And with every $\lambda \neq 1$ we have either:

- if $w(N) = 1$ then $w'(N) = \lambda w(N) \neq 1$ and w' is not a $(0, 1)$-normalised game, or
- if $w(N) = 0$ then $w'(N) = 0$. Thus $w' = w$.

□

If we have a game in which $v(N) = 0$ in (0,1)-normalised form, the formation of a coalition has no use for any of the players, if they are individually rational. These games are called *inessential* games. If we have a player i for which $v(S \bigcup \{i\}) = v(S) + v(i)$ for all $S \subseteq N$, we say $\{i\}$ is a dummy player. In the $(0, 1)$-normalised game a dummy player always gets 0.

2.4 Properties of Solutions

As we pointed out in the introduction, in Game Theory situations are studied in which people try to achieve their goals. A solution $\sigma(N, v)$ of a game is a subset of the pre-imputation space which are in some sense more plausible and attractive than the pre-imputations not in this subset. In this section we look at useful properties a solution $\sigma(N, v)$ can have.

Definition 2.12 [10] Let $\sigma(N, v)$ a solution for (N, v). Suppose (N, w) is equivalent with (N, v), thus we know there exists an $s : v \longrightarrow w$. We say that solution σ is **Covariant under Strategic Equivalence** (COV) if the following holds: if $\sigma(N, v)$ is a solution for (N, v) then $s(\sigma(N, v))$ is a solution for (N, w).

Definition 2.13 Let $\sigma_1(N_1, v_1)$ a solution for (N_1, v_1) and let $\sigma_2(N_2, v_2)$ a solution for disjoint game (N_2, v_2). We say the solutions $\sigma_1(N_1, v_1)$, $\sigma_2(N_2, v_2)$ are **Covariant under Composition** (COCO) if $\sigma_{1\times 2}(N_1 \bigcup N_2, v_1 + v_2) = \sigma_1(N_1, v_1) \times \sigma_2(N_2, v_2)$ is a solution for $(N_1 \bigcup N_2, v_1 + v_2)$.

Definition 2.14 We say a solution $\sigma(N, v)$ has the **Strong Anonymous** (SAN) property if for all pairs of players i, j for whom $v(S \bigcup\{i\}) - v(i) = v(S \bigcup\{j\}) - v(j)$ for all $S \subset N \backslash (\{i\} \bigcup \{j\})$ then for all $\sigma \in \sigma(N, v)$ holds: $\sigma_i = \sigma_j$.

Definition 2.15 [10] We call a solution $\sigma(N, v)$ **Pareto Optimal** (PO) if $\sum \sigma_i = v(N)$ for all $\sigma \in \sigma(N, v)$.

Chapter 3

Comparing Imputations

In optimisation problems we optimise a function in $\mathbb{R}$ for which the preference relations $>$, $\geq$ are transitive. But if we look at two imputations x, y for the game with the two children and the chocolate bar, the situation is fundamentally different. Take for example x the imputation for which child 1 gets the whole chocolate bar, and imputation y where they share it equally. Both imputations are Pareto optimal. Thus if both children try to maximize their own quantity of chocolate, not both children prefer x over y or vice versa. So, if every player wants to maximize his own profit, in a transferable utility game we don't have a transitive relation between Pareto optimal imputations. In this chapter we investigate the relation between different imputations in more detail.

3.1 Strong Domination

Von Neumann and Morgenstern (1944)[8] defined a specific preference relation which they based on the "Behavior of Human". According to this preference relation if all players of a coalition S gain strictly more in imputation x than in imputation y and they are in a situation to change from y to x, then x dominates y over coalition S. More formally this can be described as follows:

Definition 3.1 An imputation x dominates an imputation y (written $x >_S y$) **strongly** over a coalition S if:

- $x_i > y_i \ \forall \ i \in S$,
- $\sum_{i \in S} x_i \leq v(S)$.

See figure 3.1 for an illustration.

All the pre-imputations in the light red area in figure 3.1 are strongly dominated by x. Thus we can see x as the strong Pareto optimum for coalition (12) of all the pre-imputations in the grey area united with x.

Definition 3.2 If there exists a coalition S for which $x >_S y$ then we say x dominates y **strongly** (written as $x \succ y$)

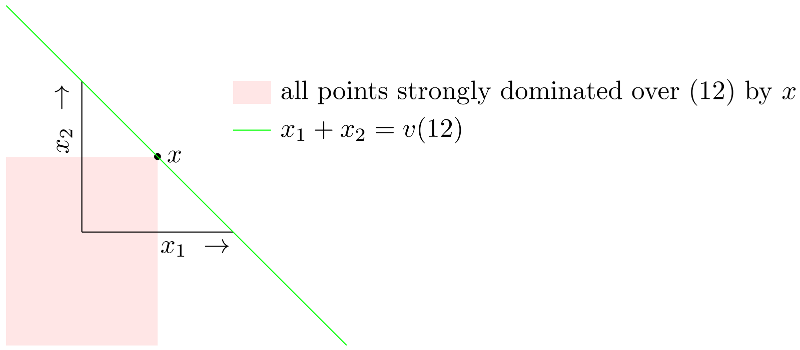

Figure 3.1

Definition 3.3 A set S is called *effective* for x, with respect to v, if $\sum x_i \leq v(S)$. S is called *strictly effective* if the inequality is strict and *strongly effective* if the equality holds. If $\sum x_i > v(S)$ we say S is *ineffective* for x.

With the definition of strong domination we can define the following sets:

$$\mathrm{Dom}_S\ y = \{x \in A : y \succ_S x\},$$

$$\mathrm{Dom}\ y = \{x \in A : y \succ x\},$$

$$\mathrm{Dom}^{-1}\ y = \{x \in A : x \succ y\}.$$

We see that $\bigcup_S \mathrm{Dom}_S\ y = \mathrm{Dom}\ y$. If we compare $\mathrm{Dom}_S\ y$ and $\mathrm{Dom}_T\ y$ for an $S \subset T$ if both are effective, we see that $\mathrm{Dom}_S\ y = \{x | y_i > x_i,\ i \in S\} \supset \mathrm{Dom}_T\ y = \{x | y_i > x_i,\ i \in T\}$. Thus we have to look at the sets at the union of the sets $\bigcup_U \mathrm{Dom}_U\ x$ for which $\sum_{i \in U} x_i \leq v(U)$ and $\sum_{i \in U'} x_i > v(U')$ for every $T' \subsetneq T$.

Continuation of Example 2.8 We've looked at A and at a part of A'. Let's look at imputation $x = (\frac{1}{2}, \frac{1}{2}, 0)$. We see that for $S = \{12\}$, $S = \{13\}$, $S = \{23\}$ or $S = \{3\}$ the equation $\sum_{i \in S} x_i \leq v(S)$ holds. So these coalitions are effective and domination over these coalitions is possible. Because $\{y | y_2 < \frac{1}{2},\ y_3 < 0\} \subset \{y | y_3 < 0\}$ and $\{y | y_1 < \frac{1}{2},\ y_3 < 0\} \subset \{y | y_3 < 0\}$ we have to focus only on domination over $S = \{12\}$ and $S = \{3\}$. The sets dominated by x are sketched in figure 3.2.

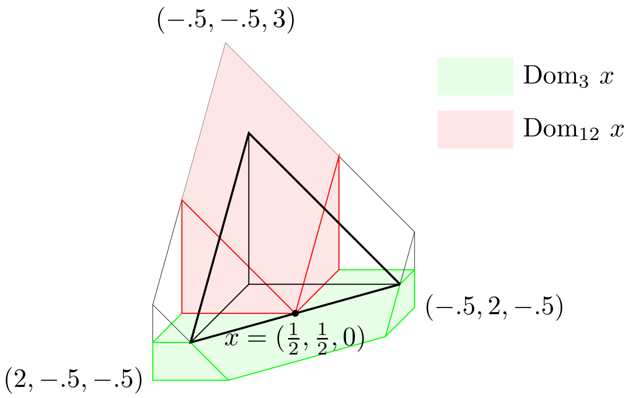

Figure 3.2

Note that in this example the boundary of the dominated areas is shaded to make the Figure more clear. The boundery is not in the set of dominated imputations.

3.1.1 Properties of Strong Domination

We start with a Theorem on the behavior of domination in equivalent games.

Theorem 3.4 [8] *Domination is transformation preserved: If (N,v) and (N,w) are equivalent games with $s: v \to w$ and $x \succ y$ for $x,y \in A(v)$, then $s(x) \succ s(y)$ for $s(x), s(y) \in A(w)$. Here $A(v) = \{x | \sum_{i\in N} x_i \leq v(N)\}$ and $A(w) = \{x | \sum_{i\in N} x_i \leq w(N)\}$.*

PROOF:

We show that there is a coalition T such that $s(x)_i > s(y)_i$ for all $i \in T$ and $\sum_{i\in T} s(x)_i \leq w(T)$ if the assumptions hold.
So, we know that $x \succ y$ over S. Thus $x_i > y_i$ for all $i \in S$. Then we know that $s(x)_{\pi(i)} = \lambda x_{\pi(i)} + c_{\pi(i)} > \lambda y_{\pi(i)} + c_{\pi(i)} = s(y)_{\pi(i)}$ for all $i \in S$.
Because $w(\pi(S)) = \lambda v(\pi(S)) + \sum_{i\in\pi(S)} c_i$ and $\sum_{i\in S} x_i \leq v(S)$ we know that $\sum_{i\in S} s(x)_{\pi(i)} = \sum_{i\in S} \lambda x_{\pi(i)} + c_{\pi(i)} = (\sum_{i\in\pi(S)} \lambda x_i) + \sum_{i\in\pi(S)} c_i \leq w(\pi(S))$. We now see that $s(x) \succ s(y)$ over $\pi(S)$.

□

Now we look in more detail at the preference relations $>_S$ and $\succ$.

Theorem 3.5 [2] *$>_S$ is a strict partial ordering. $\succ$ is not a partial ordering.*

PROOF:

We have to prove that:

- $\neg(a <_S a)$ (Irreflexivity),
- if $a <_S b$ then $\neg(b <_S a)$ (Asymmetry),
- if $a <_S b$ and $b <_S c$ then $a <_S c$ (Transitivity).

- Irreflexivity: We see that $a \not>_S a$, because $a_i = a_i$ for all $i \in S$.
- Asymmetry: If $x >_S y$ then for all $i \in S$ holds $x_i > y_i$. So $y >_S x$ cannot hold between x and y.
- Transitivity: If $x >_S y$ and $y >_S z$ then $x_i > y_i > z_i$. And because S is effective for x, $x >_S z$.

So $>_S$ is a strict partial ordering.

But if we look at the game $v(1) = v(2) = 1$, then $(1,0) \succ (0,1)$ over coalition (1), and $(0,1) \succ (1,0)$ over coalition (2). So $\succ$ is not a partial ordering.

$\square$

Theorem 3.6 [2] *Dom_S and Dom are monotone non-decreasing functions of point sets into open sets in A'.*

PROOF:

$\mathrm{Dom}_S y$ is either the empty set or $\bigcap_{i\in S}\{y_i > x_i\}$ and this is a finite intersection of open sets. A finite intersection of open sets is open and the empty set is open, thus $\mathrm{Dom}_S y$ is open.// If Y is a set, $\mathrm{Dom}_S Y = \bigcup_{y\in Y} \mathrm{Dom}_S y$ is a union of open sets and that is open. // If Y' is any set, then $\mathrm{Dom}_S Y \bigcup Y' = \mathrm{Dom}\ _S Y \bigcup \mathrm{Dom}_S Y' \supseteq \mathrm{Dom}_S Y$. So Dom_S is monotone non-decreasing.// Because Dom is the finite union over all S of Dom_S, thus a finite union of open sets, it is an open set. Because Dom_S is a monotone non-decreasing function of point sets into open sets, and Dom is the finite union of monotone non-decreasing functions of point sets into open sets, Dom is open and monotone non-decreasing also.

$\square$

Definition 3.7 For a given $x \in A'$, if there exists a $y \in A'$ such that $y \succ x$ and if for every $z \in A'$ such that $z \succ y$ it follows that $z \succ x$ we say that y strongly majorizes x, written $y \to x$.

We say an imputation x is majorized, if there exists an y such that $y \to x$.

Theorem 3.8 [2] *$\to$ is a strict partial ordering.*

PROOF:

We have to prove that:

- $\neg(a \to a)$ (irreflexivity),
- if $a \to b$ then $\neg(b \to a)$ (asymmetry),
- if $a \to b$ and $b \to c$ then $a \to c$ (transitivity).

- reflexivity: If $a \to a$, then $a \succ a$ (per definition). But $a_i \not\succ a_i$ for all i. Thus $a \not\succ a$.
- antisymmetry: Suppose $b \to a$ and $a \to b$. Then $a \succ b$ and $b \succ a$. Thus $b \to b$. and thus $b \succ b$, which gives a contradiction.
- transitivity: let $a \to b$ and $b \to c$. Then $a \succ b$. Thus $a \succ c$. Look at a d for which $d \succ a$. Then $d \succ b$ per definition, and also $d \succ c$ per definition. This holds for every d for which $d \succ a$. Thus $a \to c$.

So $\rightarrow$ is a strict partial ordering.

$\square$

Theorem 3.9 [2] *For a given game (N, v), there exist n quantities, $k_1, ..., k_n$, such that if $a_i > k_i$ for some player i, then either a is in the Core*[1] *or a is majorized.*

PROOF:

Define $k_i = \max_{S \supseteq \{i\}}(v(S) - v(S\backslash\{i\}))$. We can see that $k_i \geq v(i)$, because $v(\emptyset) = 0$. Suppose a is not in the Core and $a_i > k_i$. So there is at least one coalition strictly effective, i.e. one coalition S for which $\sum_S a_i < v(S)$. Now choose an $\epsilon > 0$ so small, that b given by $b_j = \begin{cases} a_j - (n-1)\epsilon \text{ for } j = i \\ a_j + \epsilon \text{ for all other } j \neq i \end{cases}$ has the same strict effective sets. To complete the proof we show that:

1. $b \succ a$,
2. If $c \succ b$, then $c \succ a$.

1. Let S be strictly effective for b. If $i \notin S$ then $b \succ a$ over S. Suppose $i \in S$. Then $v(S\backslash\{i\}) \geq v(S) - k_i \geq \sum_S b_j - b_i = \sum_{S\backslash\{i\}} b_j$. Thus $S\backslash\{i\}$ is an effective set for b and thus $b \succ a$ over $S\backslash\{i\}$.
2. Suppose $c \succ b$ over S. Then $c_j > b_j$ for all $j \in S$. Suppose $i \notin S$. Then $c_j > b_j > a_j$ for all $j \in S$, and S is effective for c, thus $c \succ a$. Suppose $i \in S$. Now $v(S\backslash\{i\}) \geq v(S) - k_i \geq \sum_S c_j - c_i = \sum_{S\backslash\{i\}} c_j$ since $c_i > b_i \geq k_i$. Thus $S\backslash\{i\}$ is effective for c, and thus $c \succ a$ over $S\backslash\{i\}$.

$\square$

3.2 Weak Domination

When Von Neumann and Morgenstern defined strong domination they had a particular 'Behavior of Human' in mind. We can define a weaker version of domination. In case of strong domination, thus $x \succ y$, there is a coalition for which every player prefers x over y. But human can also have as a goal the maximization of the profit by the coalition of which they are member. With this adjusted 'Behavior of Human' in mind I define a weaker form of domination.

[1] The Core is an often used solution concept. It is the set $x \in A$ for which $\sum_{i \in S} x_i \geq v(S) \forall S$. We will look at the Core in more detail in Chapter 4.

Definition 3.10 We say that an imputation x dominates an imputation y (written $x \geq_S y$) **weakly** over a coalition S if:

- $x_i \geq y_i \; \forall \; i \in S$,
- $\sum_{i \in S} y_i < x_i \geq v(S)$.

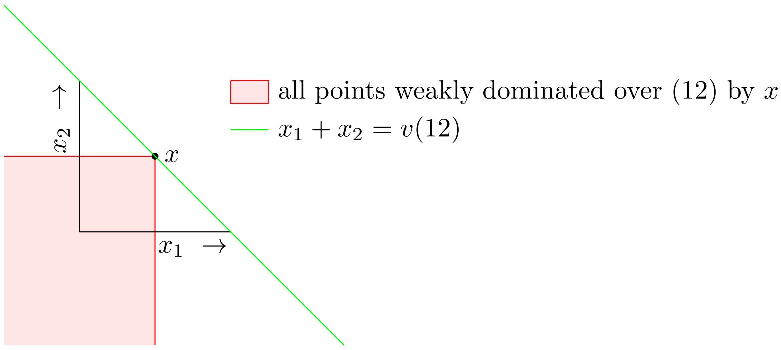

Figure 3.3

Definition 3.11 *If there exists a coalition S for which $x \geq_S y$ we say x dominates y weakly (written $\alpha \succeq \beta$)*

With these definitions in mind we define the next sets:

$$\text{dom}_S \; y = \{x \in A : y \succeq_S x\},$$

$$\text{dom} \; y = \{x \in A : y \succeq x\},$$

$$\text{dom}^{-1} \; y = \{x \in A : x \succeq y\}.$$

If we compare the sets $A = \text{dom}_S \; y \bigcup y$ and $B = \text{Dom}_S \; y \bigcup y$ we see that y is a weak Pareto optimum for coalition S for A and a strict Pareto optimum for B.

We can see that $a \succ b \Rightarrow a \succeq b$. Thus Dom $x \subset$ dom x and $\text{Dom}^{-1} \; y \subset \text{dom}^{-1} \; y$.

Continuation of Example 2.8 We've looked at A and at a part of A'. Let's look at imputation $x = (\frac{1}{2}, \frac{1}{2}, 0)$. We see that for $S = \{12\}$, $S = \{13\}$, $S = \{23\}$ or $S = \{3\}$ the equation $\sum_{i \in S} x_i \leq v(S)$ holds. So these coalitions are effective and domination over these coalitions is possible. Because $\{y | y_2 < \frac{1}{2}, \; y_3 < 0\} \subset \{y | y_3 < 0\}$ we have to focus only on domination over $S = \{12\}$ and $S = \{3\}$. The sets dominated by x are sketched in figure 3.2.

Note that this Figure looks the same as Figure 3.2. But now the imputations on the boundary of the dominated areas are weakly dominated by x.

3.2.1 Properties of Weak Domination

Theorem 3.12 *$\geq_S$ is a strict partial ordering, $\succeq$ not.*

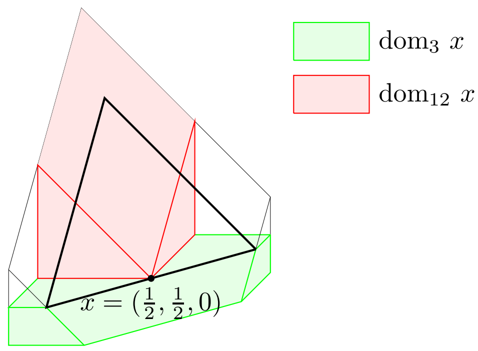

Figure 3.4

PROOF:

We have to prove that:

- $\neg(a \leq_S a)$ (Irreflexivity),
- if $a \leq_S b$ then $\neg(b \leq_S a)$ (Asymmetry),
- if $a \leq_S b$ and $b \leq_S c$ then $a \leq_S c$ (Transitivity).

- Irreflexivity and Asymmetry: If $x \geq_S y$ then for all $i \in S$ holds $x_i \geq y_i$ and $x_j > y_j$ for at least one $j \in S$. So at most one of $x \geq_S y$, $x = y$ or $y \geq_S x$ can hold between x and y.
- Transitivity: If $x \geq_S y$ and $y \geq_S z$ then $x_i \geq y_i \geq z_i$ for all $i \in S$ and $x_j > y_j > z_j$ for at least one $j \in S$. Also S is effective for x. Thus $x \geq_S z$.

So $>_S$ is a partial ordering.

But if we look at the game $v(1) = v(2) = 1$, then $(1, 0) \succeq (0, 1)$ over coalition (1), but also $(0, 1) \succeq (1, 0)$ over coalition (2). So $\succeq$ is not a partial ordering.

$\square$

Theorem 3.13 *For y a single imputation, $\mathrm{dom}_S y \bigcup y$ and $\mathrm{dom}\ y \bigcup y$ are closed sets in A.*

PROOF:

$\mathrm{dom}_S y \bigcup y$ is either y or $\bigcap_{i \in S}(\{y_i \geq x_i \geq v(i)\} \bigcap A)$ and this is a finite intersection of closed sets. A finite intersection of closed sets is closed.
$\mathrm{dom}\ y \bigcup y$ is the finite union of closed sets, and thus it is closed.

$\square$

Theorem 3.14 *Weak Domination is transformation preserved:*
If (N, v) and (N, w) are equivalent games with $s : v \to w$ and $x \succeq y$ for $x, y \in A(v)$, then $s(x) \succeq s(y)$ for $s(x), s(y) \in A(w)$.

PROOF:

We show that if the assumptions hold, then there is an T such that $s(x)_i \geq s(y)_i$ for all $i \in T$, $s(x)_j > s(y)_j$ for a $j \in T$ and $\sum_{i \in T} s(x)_i \leq w(T)$.
Suppose $x \succeq y$ over S, so $x_i \geq y_i$ for all $i \in S$ and $x_j > y_j$ for a $j \in S$. Then we know that $s(x)_{\pi(i)} = \lambda x_{\pi(i)} + c_{\pi(i)} \geq \lambda y_{\pi(i)} + c_{\pi(i)} = s(y)_{\pi(i)}$ for all $i \in S$ and $s(x)_{\pi(j)} = \lambda x_{\pi(j)} + c_{\pi(j)} > \lambda y_{\pi(j)} + c_{\pi(j)} = s(y)_{\pi(j)}$ for $j \in S$.
Because $w(\pi(S)) = \lambda v(\pi(S)) + \sum_{i \in \pi(S)} c_i$ and $\sum_{i \in S} x_i \leq v(S)$ we know that $\sum_{i \in S} s(x)_{\pi(i)} = \sum_{i \in S} \lambda x_{\pi(i)} + c_{\pi(i)} = (\sum_{i \in \pi(S)} \lambda x_i) + \sum_{i \in \pi(S)} c_i \leq w(\pi(S))$. We now see that $s(x) \succeq s(y)$ over $\pi(S)$.

$\square$

3.3 Dual Domination

In the previous sections we have looked at preference relations between imputations. But a coalition can also blackmail another coalition. To look at this, we use the concept of the dual of a coalition.

Definition 3.15 We say that an imputation x dominates an imputation y (written $x \geqq_S y$) **dually** over a coalition S if:

- $x_i \geq y_i \; \forall \; i \in S$
- $\sum_{i \in S} x_i \leq v(S)$
- If $\forall i \in S \; x_i = y_i$ then $\sum_{i \in N \backslash S} y_i > \sum_{i \in N \backslash S} x_i \geq v(N \backslash S)$.

Definition 3.16 If there exist a coalition S such that $x \geqq_S y$ we say x **dominates** y **dually** (written $x \gtrapprox y$).

Theorem 3.17 *If $x \gtrapprox y$, but $x \not\succeq y$, then $\sum_{i \in N} x_i < v(N)$*

PROOF:

Suppose that $x \gtrapprox y$, but $x \not\succeq y$, over coalition S. Than $v(N \backslash S) \leq \sum_{i \in N \backslash S} x_i < \sum_{i \in N \backslash S} y_i$ and $y_i = x_i$ for all $i \in S$. Thus $\sum_{i \in N} x_i = \sum_{i \in N \backslash S} x_i + \sum_{i \in S} x_i < \sum_{i \in N \backslash S} y_i + \sum_{i \in S} y_i \leq v(N)$.

$\square$

We define the following sets:

$\mathrm{dm}_S \; y = \{x \in A : y \gtrapprox_S x\}$,

$\mathrm{dm} \; y = \{x \in A : y \gtrapprox x\}$,

$\mathrm{dm}^{-1} \; y = \{x \in A : x \gtrapprox y\}$.

To illustrate Dual Domination, we proceed with Example 2.4.

Continuation of Example 2.4 We know that:

- $v(i) = 0$
- $v(FY) = v(FB) = 1$.

Lets look at the imputation $x = (x_F, x_Y, x_B) = (1, 0, 0)$. As we will see in chapter 4 this point is in the so called Core, because $\sum_{i\in S} x_i \geq v(S)$ for all S. This imputation is not strongly and not weakly dominated. But for example $y = (0, 0, 0)$ gives $y \gtrapprox x$ over $S = YB$. So x is dually dominated.

Thus a coalition YB can prefer $(0, 0, 0)$ over $(1, 0, 0)$. The reason is because they don't have any chance to get more than 0 if they agreed with a payoff $(1, 0, 0)$.

3.3.1 Properties of Dual Domination

We can state the following theorems.

Theorem 3.18 $\geqq_S$ *is a strict partial ordering,* $\gtrapprox$ *not.*

PROOF:

We have to prove that:

- $\neg(a \leqq_S a)$ (Irreflexivity),
- if $a \leqq_S b$ then $\neg(b \leqq_S a)$ (Asymmetry),
- if $a \leqq_S b$ and $b \leqq_S c$ then $a \leqq_S c$ (Transitivity).

- Irreflexivity and Asymmetry: Suppose $a \leqq_S b$, then $a_i \leq b_i$ for all $i \in S$. Thus if $a \leqq_S b$ and $b \leqq_S a$, then $a_i = b_i$ for all $i \in S$. But then $\sum_{i\in N\backslash S} a_i < \sum_{i\in N\backslash S} b_i < \sum_{i\in N\backslash S} a_i$ which gives a contradiction.
- Transitivity: Note that we have to prove 4 different cases for $\geqq_S$ is a strict partial ordering:
  1. If $x \geqq_S y$, $x \geq_S y$, and $y \geqq_S z$, $y \geq_S z$, then $x \geqq_S y$,
  2. If $x \geqq_S y$, but not $x \geq_S y$, and $y \geqq_S z$, $y \geq_S z$, then $x \geqq_S y$,
  3. If $x \geqq_S y$, $x \geq_S y$, and $y \geqq_S z$, but not $y \geq_S z$, then $x \geqq_S y$,
  4. If $x \geqq_S y$, but not $x \geq_S y$, and $y \geqq_S z$, but not $y \geq_S z$, then $x \geqq_S y$

  Case 1 is equivalent as Theorem 3.12.
  Suppose 2. Then $x_i = y_i \geq z_i$ for all $i \in S$, $\sum_{i\in S} x_i \leq v(S)$, and $x_j = y_j > z_j$ for a $j \in S$. So $x \geq_S z$, so $x \geqq_S z$.
  Suppose 3. Then $x_i \geq y_i = z_i$ for all $i \in S$, $\sum_{i\in S} x_i \leq v(S)$, and $x_j > y_j = z_j$ for a $j \in S$. So $x \geq_S z$, so $x \geqq_S z$.
  Suppose 4. Then $x_i = y_i = z_i$ for all $i \in S$. But $v(N\backslash S) \leq \sum_{i\in N\backslash S} x_i < \sum_{i\in N\backslash S} y_i < \sum_{i\in N\backslash S} z_i$. Thus $x \geqq_S z$.

And thus $\geqq_S$ is a partial ordering.

But if we look again at the game $v(1) = v(2) = 1$, then $(1, 0) \gtrapprox (0, 1)$ over coalition (1), but also $(0, 1) \gtrapprox (1, 0)$ over coalition (2). So $\gtrapprox$ is not a partial ordering.

$\square$

Theorem 3.19 *Dual Domination is transformation preserved: // If (N, v) and (N, w) are equivalent games with $s : v \to w$ and $x \gtrapprox y$ for $x, y \in A(v)$, then $s(x) \gtrapprox s(y)$ for $s(x), s(y) \in A(w)$.*

PROOF:

We can have two cases if $x \gtrapprox y$ over an S:

1. $x \succeq y$ over this S,
2. $x \not\succeq y$ over this S.

In Theorem 3.14 we have proven that there is a T such that $s(x) \succeq s(y)$ over this T, thus also $s(x) \gtrapprox s(y)$ over this T.

We now show that if $x \not\succeq y$ over S then there is a T such that $s(x) \not\succeq y$ over this T. Thus we have to show that $s(x)_i = s(y)_i$ for all $i \in T$, $\sum_{i \in T} s(x)_i \leq w(s)$ and $\sum_{i \in N \setminus T} s(y)_i > \sum_{i \in N \setminus T} s(x)_i \geq w(N \setminus S)$. Because $x \gtrapprox y$ over S and $x \not\succeq y$ over this S. Then $x_i = y_i$ for all $i \in S$, $\sum_{i \in S} x_i \leq v(S)$ and $\sum_{j \in N \setminus S} y_j > \sum_{j \in N \setminus S} x_j \geq v(N \setminus S)$.
Then we see that $s(x)_{\pi(i)} = \lambda x_{\pi(i)} + c_{\pi(i)} = \lambda y_{\pi(i)} + c_{\pi(i)} = s(y)_{\pi(i)}$ for all $i \in S$. Also $w(\pi(N \setminus S)) \leq \sum_{j \in N \setminus S} s(x)_{\pi(j)} = \sum_{j \in N \setminus S} \lambda x_{\pi(j)} + c_{\pi(j)} < \sum_{j \in N \setminus S} \lambda y_{\pi(j)} + c_{\pi(j)} = s(y)_{\pi(j)}$. Because $w(\pi(S)) = \lambda v(\pi(S)) + \sum_{i \in \pi(S)} c_i$ we see that $\sum_{i \in S} s(x)_{\pi(i)} = \sum_{i \in S} \lambda x_{\pi(i)} + c_{\pi(i)} = (\sum_{i \in \pi(S)} \lambda x_i) + \sum_{i \in \pi(S)} c_i \leq w(\pi(S))$. We see that now $s(x) \gtrapprox s(y)$ over $\pi(S)$.

□

Continuation of Example 2.4 We know the game is

- $v(i) = 0$ for $i = 1, 2, 3$,
- $v(12) = v(13) = 1$.

Look at imputation $x = (1, 0, 0)$. Because $\sum_{i \in S} x_i \geq v(S)$ for all S, imputations dually dominated by x are imputations weakly dominated by x. According to theorem 3.13 this set is closed.

Now look at an imputation $y = (0, 0, 0)$. This imputation dominates imputations for which $x_i > 0$ for one player $\{x | x_i > 0,\ x_1 = 0,\ x_j = 0 \text{ for } i, j \in \{2, 3\}\}$, or if $x_2 > 0$ and $x_3 > 0$ $(\{x | x_2 > 0,\ x_3 > 0,\ x_1 = 0\})$. These sets are not closed. So note that $\mathrm{dm}_S\ y$ can be closed in A (if the domination over S is only weak). But for many imputations it is not closed in A nor open in A'.

Theorem 3.20 *Strong domination $\Rightarrow$ Weak domination $\Rightarrow$ Dual Domination.*

PROOF:

It is easily checked by comparing the definitions.

If we compare imputations with the Definition of strong domination, only improvements for every member of a coalition is taken in account. By weak domination we look at the improvements for the coalition as a whole. With dual domination coalitions can, as we have seen in the Continuation of Example 2.4, also blackmail other coalitions.

Chapter 4

The Core

The results of Chapter 3 enable us to look into more detail at specific solution concepts. We start with the most general one: The Core. In his 1959 paper "Solutions to general non-zero-games" [2] Donald B. Gillies introduced the term Core for a set of imputations which are not dominated. Imputations in the Core are in some way stable; there is no coalition which has the desire and the power to change the outcome of the game.

4.1 The Core

Gillies introduced the Core as "always a subset of the Stable Set[1]". In Economic Theory the Core is widely used nowadays.

Definition 4.1 [2] The Core $C(v)$ consists of those points which are not strongly dominated.

With this definition, we can find the Core of a game with the next Theorem.

Theorem 4.2 [4] *Given a game (N, v), the Core $C(v)$ consists of the imputations $\{x \in A | \sum_{i\in S} x_i \geq v(S)\ \forall S\}$.*

PROOF:

Let $B = \{x | \sum_S x_i \geq v(S)\ \forall S\}$. The proof consists of the following two steps:

(1) $B \subset C(v)$,

(2) $C(v) \subset B$.

Proof of (1): Assume $x \in B$ and $y \succ x$ over coalition S. Then because $y_i > x_i$ for all $i \in S$ and thus $\sum x_i < \sum y_i \leq v(S)$ we have that imputations which are not dominated over S have the property that $\sum_{i\in S} x_i \geq v(S)$. This holds for all S. Thus x is not dominated, and $x \in C(v)$.

Proof of (2): Let x be an imputation that is not dominated, but not in B. So there exists an S such that $\sum_{i\in S} x_i < v(S)$. Let y an imputation such that

[1] The Stable Set is a solution concept we define in Chapter 6

$y_j = \begin{cases} v(j) \text{ for all } j \notin S \\ x_j + \frac{v(S) - \sum_{i \in S} x_i}{|S|} \text{ for all other } j \end{cases}$. Then this imputation y is in A and $y \succ x$ over coalition S. So every imputation not in B is dominated.

So $C(v) = B = \{x \in A | \sum_S x_i \geq v(S)\ \forall S\}$.

□

Because the Core is a set which satisfies a set of countable weak linear inequalities, we can conclude that the Core is a closed and convex set.

Theorem 4.3 *The Core $C(v)$ is also the set of all imputations which are not weakly dominated.*

PROOF:

The proof of the Theorem consists of the following two steps:

1 Imputations in $C(v)$ as in Theorem 4.2 are not weakly dominated,

2 For every imputation $x \in A \backslash C(v)$ there is an imputation $y \in A$ for which $y \succeq x$.

Proof of 1: Let's look at the definition of weak domination. An imputation cannot be weakly dominated if $\sum_{i \in S} x_i \geq v(S)$ for all S. So imputations in the Core, as in Theorem 4.2, are not weakly dominated.

Proof of 2: Now suppose there exists an imputation x not weakly dominated and not in the $C(v)$. Thus there is an x with $\sum x_i < v(S)$. Then we can create a y such that $y \succ x$, and thus $y \succeq x$, according to the arguments used in proving Theorem 4.2.

Thus $C(v)$ is also the set of all not weakly dominated imputations.

□

Nowadays the Core is seen as an important solution concept in Economic Theory. It has some nice properties:

Theorem 4.4 *The Core is a solution concept which has the properties COV, COCO, PO.*

PROOF:

COV: Domination is transformation preserved. If an imputation x is not dominated in a game (N, v), its equivalent imputation $s(x)$ in the equivalent game (N, w) is not dominated either. If an imputation x is dominated by x' in (N, v), then $s(x)$ is dominated by $s(x')$. Thus all not dominated imputations $C(v)$ in (N, v) are not dominated in (N, w), and all dominated imputations in (N, v) are dominated in (N, w).

COCO: Assume imputation x is not dominated in (N_1, v_1) and imputation y is not dominated in (N_2, v_2). We have to prove that $(x; y)$ is not dominated in $(N, v) = (N_1 \bigcup N_2, v_1 + v_2)$. For all coalitions S we know that $v(S) = v(N_1 \bigcap S) + v(N_2 \bigcap S)$ in the composed game $(N, v) = (N_1 \bigcup N_2, v_1 + v_2)$. Thus for $z = (x; y)$ we know that $\sum_{i \in S} z_i = \sum_{i \in N_1 \bigcap S} x_i + \sum_{i \in N_2 \bigcap S} y_i \geq v(N_1 \bigcap S) + v(N_2 \bigcap S) = v(S)$. Thus z is not dominated, and thus z is in the Core.

PO: For an imputations $y \in C(v)$ we know that $\sum_{i \in S} y_i \geq v(S)$ for all S, thus $\sum_{i \in N} y_i \geq v(N)$. Thus all imputations in $C(v)$ are Pareto Optimal.

□

The Core is easy to find. Intuitively, the Core is perhaps the clearest solution concept in Cooperative Game Theory. However a problem is that in many games the Core is empty. This can be concluded from the following theorem.

Theorem 4.5 [4] *The Core is empty for all essential constant sum games*

PROOF:

Suppose $C(v) \neq \emptyset$ and suppose $x \in C(v)$. We know that $v(N) = v(S) + v(N\backslash S)$ for all S in a constant sum game. We also have for every $i \in N$ that $x_i \geq v(i)$ and $\sum_{j \in N\backslash\{i\}} x_j \geq v(N\backslash\{i\})$. In a constant sum game this only can as $x_i = v(i)$ and $\sum_{j \in N\backslash\{i\}} x_j = v(N\backslash\{i\})$. But this holds for every player i. So it can only exist if $x_i = v(i)$ for every player i. This can only be an imputation in the core if the game is inessential. This gives an contradiction with the assumption that (N, v) is an essential game.

□

Continuation of Example 2.3 We have the game:

$$v(i) = -1 \text{ for } i = 1, 2, 3$$
$$v(12) = v(13) = v(23) = 1$$

Now suppose x is an imputation in the Core. Then $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$ and $x_2 + x_3 \geq 1$. Thus $2x_1 + 2x_2 + 2x_3 \geq 3$. This gives a contradiction with $x_1 + x_2 + x_3 \leq 0$. Thus the Core of this game is empty.

So a disadvantage of the Core is that in many games it is empty. But, because of the extremely simple definition of the Core it can be considered as a starting point for more sophisticated solution concepts.

4.2 The Dual Core

In the Definition of the Core, strong domination is the key concept. Because every not strongly dominated imputation is also not weakly dominated and every not weakly dominated imputation is not strongly dominated, we see that if we replace strong domination by weak domination the set stays the same. However, if we replace in the definition 'not strongly dominated' by 'not dually dominated', we get a different set imputations.

Definition 4.6 The Dual Core consists of all points not dually dominated.

What is meant by this is the following. Not all members of a coalition S can improve by leaving the grand coalition. In fact if coalition S leaves the grand coalition there are two possibilities:

1. At least one member of S have to pay a price (> 0) for leaving the grand coalition, or
2. No player in S has to pay a price and no player in $N\backslash S$ has to pay a price.

Thus imputations in the Dual Core are in some sense *more stable* than imputations in the Core. A coalition has no desire for leaving the grand coalition because they have to pay a price. Or if they leave the grand coalition and don't have to pay a price, the other members of the grand coalition don't have to pay a price either.

We can find the Dual Core by solving the next linear equations:

Theorem 4.7 *The Dual Core $DC(v)$ contains the imputations*
$B = \{x \in A | \sum_{i \in S} x_i = v(S) \forall S$ *for which* $v^*(S) = v(S)$, $\sum_S x_i > v(S)$ *for all other* $S\}$.

PROOF:

The proof consists of the following two steps:

(1) $B \subset DC(v)$

(2) $DC(v) \subset B$

Proof of (1): Assume $x \in B$ and $y \gtrapprox x$ over coalition S. If $\sum_{i \in S} x_i > v(S)$ than we have a contradiction. So $\sum_{i \in S} x_i = v(S)$. This can only hold if $v(S) = v^*(S)$. Thus $\sum_{i \in S} x_i = v^*(S)$. But now $\sum_{i \in N \backslash S} x_i = v(N \backslash S)$, which gives again that $y \gtrapprox x$ over coalition S cannot hold.

Proof of (2): Let x be an imputation that is not dually dominated, but not in B. So there exists a S such that either

(a) $\sum_{i \in S} x_i < v(S)$,

(b) $v^*(S) > \sum_{i \in S} x_i = v(S)$.

If (a), we can construct with the same argument as in Theorem 4.2 an imputation which strongly dominates x. So x is not in Dual Core.

Assume (b). For x we see that $\sum_{i \in N \backslash S} x_i > v(N \backslash S)$. Now let y an imputation with

$y_i = \begin{cases} x_i \forall i \in S \\ c_i \text{ such that } \sum_{j \in N \backslash S} c_j = v(N \backslash S) \end{cases}$. Now $y \gtrapprox x$ over S. Thus imputation not in B are not in the Dual Core.

Thus $DC(v) = \{x \in A | \sum_S x_i = v(S) \forall S$ if $v^*(S) = v(S)$, $\sum_S x_i > v(S)$ for all other $S\}$.

□

Because $x \succ y$ implies $x \gtrapprox y$ it is not hard to see the following holds:

Corollary 4.8 $DC(v) \subset C(v)$

PROOF:

Follows also directly from Theorems 4.2 and 4.7.

Thus we can see that the Dual Core is a subset of imputations in the Core which are "more stable" (if the players behave rationally). The solution concept Dual Core also have nice properties:

Theorem 4.9 *The Dual Core is a solution concept with the properties* COV, COCO *and* PO.

PROOF:

COV: Follows by the same argumentation as used in Theorem 4.4 directly from the fact that $\gtrapprox$ is transformation preserved:

COCO: Let $(N, v) = (N_1 \bigcup N_2, v_1 + v_2)$. Suppose $DC(v_1)$ is the nonempty Dual Core for (N_1, v_1) and $DC(v_2)$ is the nonempty Dual Core for (N_2, v_2). We have to prove that if $x \in DC(v_1)$ and $y \in DC(v_2)$ then $z = (x; y)$ is an imputation in $DC(v)$. We show that $\sum_{i \in U} z_i > v(U)$ if $v^*(U) > v(U)$ and $\sum_{i \in U} z_i = v(U)$ if $v^*(U) = v(U)$ for all $U \subset N$. We check all possible coalitions:

- If $U = N_1$ then $\sum_{i \in N_1} x_i = v(N_1) = v^*(N_1)$,
- If $U = N_2$ then $\sum_{i \in N_2} y_i = v(N_2) = v^*(N_2)$,
- If $U = S \bigcup T$ for a $S \subset N_1,\ T \subset N_2$ we have 4 possibilities:
  - If $\sum_{i \in S} x_i > v_1(S)$ and $\sum_{i \in T} y_i > v_2(T)$ for $S \subset N_1,\ T \subset N_2$, than $\sum_{i \in S \bigcup T} z_i > v(S \bigcup T)$,
  - If $\sum_{i \in S} x_i = v_1(S) = v_1^*(S)$ and $\sum_{i \in T} y_i > v_2(T)$ for $S \subset N_1,\ T \subset N_2$, than $\sum_{i \in S \bigcup T} z_i > v(S \bigcup T)$,
  - If $\sum_{i \in S} x_i > v_1(S)$ and $\sum_{i \in T} y_i = v_2(T) = v_2^*(T)$ for $S \subset N_1,\ T \subset N_2$, than $\sum_{i \in S \bigcup T} z_i > v(S \bigcup T)$,
  - If $\sum_{i \in S} x_i = v_1(S) = v_1^*(S)$ and $\sum_{i \in T} y_i = v_2(T) = v_2^*(T)$ for $S \subset N_1,\ T \subset N_2$, than $\sum_{i \in S \bigcup T} z_i = v(S \bigcup T)$, but also $\sum_{i \in N_1 \backslash S} x_i = v(N_1 \backslash S)$ and $\sum_{i \in N \backslash T} y_i = v(N_2 \backslash S)$, thus $\sum_{i \in S \bigcup T} z_i = v^*(S \bigcup T)$.

Thus z is not dually dominated and it is in the Dual Core.

PO: For an imputation $y \in DC(v)$ we know that $\sum_{i \in S} y_i \geq v(S)$ for all S, thus $\sum_{i \in N} y_i \geq v(N)$. Thus all imputations in $DC(v)$ are Pareto Optimal.

□

4.2.1 Comparing the Core with the Dual Core

In this subsection we discuss two examples to illustrate the differences in performance between the Core and the Dual Core:

Example 4.10 Look at the game $v(1) = v(2) = 0,\ v(12) = 1$.

We see that the $C(v) = \text{span}((0, 1), (1, 0))$ and $DC(v) = C(v) \backslash (0, 1) \bigcup (1, 0)$.

Example 4.11 Let's look at the example of the farmer again. We know that:

- $v(i) = 0$,
- $v(FY) = v(FB) = 1$.

And thus we can see that $x = (x_F, x_Y, x_B) = (1, 0, 0)$ is the only imputation in the Core. But $y = (0, 0, 0)$ gives $y \gtrapprox x$ over $S = YB$, and thus it is not in the Dual Core.

Note that these examples show us another disadvantage of the Core, namely some imputations in the core are not rational. Imputations in $C(v) \backslash DC(v)$ have the next property: There is a smaller coalition S, such that cooperation in the grand coalition N of this coalition S for the benefit of $N \backslash S$ is needed. But coalition S does not improve by this cooperation. The Dual Core does not have this disadvantage.

But, as we already have seen, the Core is empty in many games. And because "the Core is empty" implies "the Dual Core is empty", emptiness of the Dual Core is also a property of many games. In fact, as we also have seen in example 4.11, non emptiness of the Core does not imply non emptiness of the Dual Core, the Dual Core is empty in even more games.

Thus the Dual Core is a solution concept that has nicer rational properties than the Core. If it exists imputations in the Dual Core are more plausible than imputations in $C(v) \backslash DC(v)$.

4.2.2 Strong ϵ-Core

If we look at the definition of the Dual Core we see some great similarities with the definition of the so called Strong ϵ-Core [16]. The idea of the strong ϵ-Core is that leaving the grand coalition gives a penalty (or bonus) of ϵ.

Definition 4.12 [16] The Strong ϵ-Core is the set of imputations for which

$$C_\epsilon(v) = \{x | \sum_{i \in S} x_i \geq v(S) + \epsilon, \ \forall S \subsetneq N \ \textit{for some } \epsilon \in \mathbb{R}, \ \sum_{i \in N} x_i = v(N)\}.$$

If we look at the $\bigcup_{\epsilon > 0} C_\epsilon(v)$, intuitively we see this can be a superset of the Dual Core. Also because $C_{\epsilon_1}(v) \subset C_{\epsilon_2}(v)$ if $\epsilon_1 < \epsilon_2$, we see that $\bigcup_{\epsilon | \epsilon > 0} C_\epsilon(v) \subset \bigcup_{\epsilon | \epsilon \geq 0} C_\epsilon(v) = C(v)$. We can prove this.

Theorem 4.13 *If $\bigcup_{\epsilon > 0} C_\epsilon(v) \neq \emptyset$, then $\bigcup_{\epsilon > 0} C_\epsilon(v) = DC(v)$.*

PROOF:

We prove this Theorem in two steps:

(1) $\bigcup_{\epsilon > 0} C_\epsilon(v) \subset DC(v)$.

(2) $DC(v) \subset \bigcup_{\epsilon > 0} C_\epsilon(v)$.

(1) Let $\epsilon > 0$ such that $C_\epsilon(v) \neq \emptyset$. Thus for every $x \in C_\epsilon(v)$ we know that $\sum_{i \in S} x_i \geq v(S) + \epsilon > v(S)$ for all $S \subset N$. Thus $x \in DC(v)$. This holds for all $\epsilon > 0$, thus $C_\epsilon(v) \subset DC(v)$ for all $\epsilon > 0$. Thus $\bigcup_{\epsilon > 0} C_\epsilon(v) \subset DC(v)$

(2) Assume $\bigcup_{\epsilon > 0} C_\epsilon(v) \neq \emptyset$. We show first that $v(S) \neq v^*(S)$ for all $S \subset N$. Thus assume there is a $T \subset N$ for which $v(T) = v^*(T)$. Because $\sum_{i \in T} x_i > v(T) = v^*(T)$ we know now that $\sum_{i \in N \backslash T} x_i = v(N) - \sum_{i \in T} x_i = v(N) - v(T) - \epsilon < v(N) - v(T) = v(N) - v^*(T) = v(N \backslash T)$. So if there exists an $\epsilon > 0$ for which $C_\epsilon \neq \emptyset$ then $v(S) \neq v^*(S) \ \forall S$.

With this we can prove (2). Let $x \in DC(v)$. We can construct an $\epsilon = \frac{\min_S \ \sum_{i \in S} x_i - v(S)}{2}$ such that $x \in C_\epsilon(v)$; because $v(S) \neq v^*(S)$ for all S we know that $\sum_{i \in S} x_i > v(S)$ for all S for $x \in DC(v)$. Thus $\epsilon > 0$. And thus $DC(v) \subset \bigcup_{\epsilon > 0} C_\epsilon(v)$.

□

If we look at the following example we see that the existence of the Dual Core does not imply existence of a Strong ϵ-Core, for an $\epsilon > 0$:

Example 4.14 Suppose we have the following 4 persons game:

$$v(12) = 1, \ v(34) = 1, \ v(1234) = 2,$$

$v(S) = 0$ for all other S.

We can see that $x_1 + x_2 = 1,\ x_3 + x_4 = 1$ for all $x \in C(v)$. So there doesn't exist an $\epsilon > 0$ and $x \in C(v)$ for which $x_1 + x_2 \geq 1 + \epsilon$ and $x_3 + x_4 \geq 1 + \epsilon$. Thus $C_\epsilon(v) = \emptyset$ for every $\epsilon > 0$.

Also it applies that for $x \in C(v)$ that $x_1 + x_2 = 1 = v^*(12)$ and $x_3 + x_4 = 1 = v^*(34)$. So we see that $DC(v) = \{x | x_i > 0,\ x_1 + x_2 = 1,\ x_3 + x_4 = 1\}$.

This example shows us also that in a so called composed game[2] $(N, v) = (N_1 \bigcup N_2, v_1 + v_2)$ for which in both decomposed games $(N_1, v_1),\ (N_2, v_2)$ the Strong ϵ-Core for an $\epsilon > 0$ exists. The Strong ϵ-Core does not exist for this ϵ in the composed game (N, v).

[2] We look at the behavior of solutions in composed games in more detail in Chapter 11

Chapter 5

Nash Equilibria

In non-cooperative game theory, the Nash equilibrium is probably the most famous solution concept. The idea is that each player individually, and independently from each other maximizes his or her utility. Although the Nash equilibrium is basically a non-cooperative concept, it has been applied to cooperative games also. In this chapter I mainly look at a new cooperative interpretation of the Nash equilibrium, but I start with the non-cooperative equilibrium.

5.1 Strict Nash Equilibria

In a strict Nash equilibrium every player is supposed to know the equilibrium strategies of all other players. None of the players can change its strategy to gain an amount that is greater than or equal to the amount in the strict Nash equilibrium. More formally this can be stated as follows:

Definition 5.1 [11]Let (S, f) be a non-cooperative game with n players, where S_i is the strategy set for player i, $S = S_1 \times S_2 \times ... \times S_n$ is the set of strategy profiles and $f = (f_1(x), ..., f_n(x))$ is the payoff function. Let $x_{\setminus i}$ be a strategy profile of all players except for player i. When each player $i \in \{1, ..., n\}$ chooses strategy x_i resulting in strategy profile $x = (x_1, ..., x_n)$ then player i obtains payoff $f_i(x)$. A strategy profile $x^* \in S$ is a strict Nash equilibrium if:

$f_i(x_i^*, x_{\setminus i}^*) > f_i(x_i, x_{\setminus i}^*)$, if $x_i \neq x_i^*$ for all players $i \in N$.

Let's look closer at the property $f_i(x_i^*, x_{\setminus i}^*) > f_i(x_i, x_{\setminus i}^*)$, if $x_i \neq x_i^*$ for all players $i \in N$ of a strict Nash equilibrium in x^*. We notice two things for this strict Nash equilibrium:

1. If we look at the functions f_i, we see that f_i gives a local optimum in x^* with respect to the neighborhood $N(x) = \{(x_i, x_{\setminus i}) | x_i \in S_i\}$,
2. For a Nash equilibrium in x^* does not exist any $x \in N(x^*)$such that $f_i(x_i^*, x_{\setminus i}^*) \leq f_i(x_i, x_{\setminus i}^*)$ for $x \neq x^*$.

We rewrite the Definition of the strict Nash equilibrium:

Definition 5.2 We call x^* a strict Nash equilibrium, if :

$$f_i(x_i^*, x_{\setminus i}^*) > f_i(x_i, x_{\setminus i}^*) \text{ for all } x \neq x^*,\ x \in N(x^*).$$

and thus there is no $x \in N(x^*)$ such that

$$f_i(x_i^*, x_{\setminus i}^*) \leq f_i(x_i, x_{\setminus i}^*) \text{ for } x \neq x^*.$$

Now we define a similar equilibrium imputation for the Cooperative Game:

Suppose a neighborhood structure is given. An imputation x in the neighborhood $N(x)$ is a strict Nash-equilibrium if:

There is no player i who can change his strategy so as to get more. What we mean by a change of strategy is doing a counter bid x', which is in the neighborhood $N(x)$, for which $x'_i \geq x_i$. This x' must have enough support.

By a counter bid x' with enough support we mean that: "there is a coalition S for which every player gets equal or more and they can claim this new distribution x' because $\sum_{i \in S} x'_i \leq v(S)$". If we compare this sentence, we see that this is exactly the definition of domination.

So if we write this in a more formal way we get the following:

Definition 5.3 We call x a strict Nash equilibrium of the neighborhood $N(x)$, if $\forall x' \in N(x) \backslash x$:

either $x \succ x'$,
or if $x \not\succ x' \Rightarrow x' \not\gtrapprox x$.

With this definition, we can define the maximum neighborhood-structure for which x is still a Nash equilibrium.

Theorem 5.4 *The maximum neighborhood of which x is the strict Nash equilibrium is $\mathcal{X}(x) = \{x \bigcup \text{Dom } x \bigcup (A \backslash \text{dm}^{-1} x)\}$.*

PROOF:

It follows directly from the definition that x is a strict Nash equilibrium of $\mathcal{X}$.

Also if we look at a point $y \notin \mathcal{X}(x)$, we see that $y \gtrapprox x$ and $x \not\succ y$. So if we have a neighborhood Y for which an element y is outside $\mathcal{X}(x)$, x is not a strict Nash equilibrium.

$\square$

We can interpret this Theorem as follows. If we have an imputation x and a switch to an imputation $x' \in \mathcal{X} \backslash \{x\}$ is considered, there is always a coalition S which will stop this switch:

$x' \in \text{Dom } x$: There is always an effective coalition against the switch from x' to x.

$x' \in A \backslash \text{dm}^{-1}\ x$: Either in this set there is no effective coalition for which for all the members the switch is an improvement. Or there is an no effective coalition for which the switch doesn't matter and the switch hits the other members of the grand coalition.

Now we can look at the strict Nash equilibria for the set A:

Theorem 5.5 *If x is a strict Nash equilibrium for $\mathcal{X}(x) = A$, then there is no $y \in A$ such that $y \gtrapprox x$. In other words: If $A = x \bigcup \text{Dom } x \bigcup A \backslash dm^{-1}\ x \Rightarrow dm^{-1}\ x = \emptyset$.*

PROOF:

We use the next argumentation:

If $y \in \mathrm{dm}^{-1}\, x$, thus $x \succ y$ and $y \gtrapprox x$, then there exists an $y' \in A$ such that $x \not\succ y'$ and $y' \gtrapprox x$. Thus $y' \notin \mathcal{X}$.

So for every imputation $y \in A$ must hold that $y \not\gtrapprox x$.

Suppose $x >_T y$ over a T and $y \geqq_S x$. We know that $|S| \geq 1,\ |T| > 1$ and $S \bigcap T = \emptyset$. Thus also $y_i \geq x_i \geq v(i)$ for all $i \in S$, and $x_i > y_i \geq v(i)$ for all $i \in T$. Also $\sum_{i \in S} y_i \leq v(S)$. We have two cases.

- $y_i = x_i$ for all $i \in S$. Then $\sum_{i \in N \backslash S} x_i > v(N \backslash S)$.
  $$\text{Let } y'_j = \begin{cases} \max(y_j, x_j - (v(N\backslash S) - \sum_{i \in N\backslash S} x_i)) \text{ for a } j \in T, \text{ set this } j = k \\ \min(y_j, x_j + \frac{(x_k - y'_k)}{|S|}) \text{ for all } j \in S \\ x_j \text{ for all other } j \in N \end{cases}$$
  Because $x_i > y'_i \geq v(i)$ for one $i \in N$, we see that $x \not\succ y'$. We can also see that $\sum_{i \in N\backslash S} x_i > \sum_{i \in N \backslash S} y'_i \geq v(N\backslash S)$. Because $y'_i \geq x_i$ for all $i \in S$ and $\sum_{i \in S} y'_i \leq \sum_{i \in S} y_i \leq v(S)$ we see that $y' \gtrapprox x$.
  Also $y'_i \geq \min(x_i, y_i)$ and $x \in A,\ y \in A$, thus $y' \in A$.
- $y_i > x_i$ for an $i \in S$.
  $$\text{Let } y'_j = \begin{cases} y_j \text{ for a } j \in T, \text{ set this } j = k \\ \min(y_j, x_j + \frac{(x_k - y'_k)}{|S|}) \text{ for all } j \in S \\ x_j \text{ for all other } j \in N \end{cases}$$
  Because $x_i > y'_i \geq v(i)$ for one $i \in N$, we see that $x \not\succ y'$. We also know that $\sum_{i \in S} x_i \leq \sum_{i \in S} y'_i \leq \sum_{i \in S} y_i \leq v(S)$ and $x_i \leq y'_i \leq y_i$ for all $i \in S$. There is one $i \in S$ such that $y'_i > x_i$. Thus $y' \gtrapprox x$.
  Also $y'_i \geq \min(x_i, y_i)$ and $x \in A,\ y \in A$, thus $y' \in A$.

□

Thus we see that if an imputation in A is dually dominated, it is not a strict Nash equilibrium for A.

Theorem 5.6 *If $DC(v) \neq \emptyset$ is nonempty, then any imputation $x \in DC(v)$ is a strict Nash equilibrium of A.*

PROOF:

$x' \not\gtrapprox x$ for every imputation $x' \in A$ and $x \in DC(v)$. Thus $\mathrm{dm}^{-1}\, x = \emptyset$ and thus $A \backslash \mathrm{dm}^{-1}\, x = A$. And thus x is a strict Nash-equilibrium for $X = x \bigcup \mathrm{Dom}\, x \bigcup A \backslash \mathrm{dm}^{-1}\, x = A$.

□

If we combine the above two Theorems and use our Definition of a strict Nash equilibrium in a cooperative game, we have a new Definition of the Dual Core:

Corollary 5.7 *The Dual Core is the set of all strict Nash equilibria.*

PROOF:

From the definition of the Dual Core (the Dual Core is the set of all not dually dominated imputations) and Theorems 5.5 and 5.6, this follows directly.

□

Theorem 5.8 *A strict Nash equilibrium is a solution concept which has the properties* (COV), (COCO) *and* (PO)*.*

PROOF:

Follows directly from Theorem 4.9.

□

5.2 Weak Nash Equilibria

So we have seen that a strict Nash equilibrium as in Definition 5.3 is a solution concept which is closely related to the Dual Core. It has a number of interesting properties. But a strict Nash equilibrium does not exist if the Dual Core of the game is empty. In non-cooperative game theory there is another equilibrium, which always exists: the weak Nash equilibrium. There is a small difference between these two equilibria in the non-cooperative game:

- In a strict Nash equilibrium there is a good reason not to switch strategy for every player.
- In a weak Nash equilibrium there is no good reason to switch strategy.

We write the definition of a weak Nash equilibrium in formal form in the same manner we did for the strict Nash equilibrium in Definition 5.9:

Definition 5.9 We call x^* a weak Nash equilibrium, if :

$$f_i(x_i^*, x_{\backslash i}^*) \geq f_i(x_i, x_{\backslash i}^*) \text{ for all } x \neq x^*,\ x \in N(x^*).$$

and thus there is not an $x \in N(x^*)$such that

$$f_i(x_i^*, x_{\backslash i}^*) < f_i(x_i, x_{\backslash i}^*) \text{ for } x \neq x^*.$$

To go from the formal Definition of the weak Nash equilibrium for a non-cooperative game to a cooperative game, we use similar arguments as we did above. Here we've gone from the formal Definition of the strict Nash equilibrium for a non-cooperative game to that of a cooperative game.

Definition 5.10 We call x a weak Nash equilibrium of the set X, if $\forall x' \in X$:

either $x \gtrapprox x'$,
or if $x \not\gtrapprox x' \Rightarrow x' \not\succ x$.

Theorem 5.11 *The maximum neighborhood of which x is the weak Nash equilibrium is* $\mathcal{X}(x) = \{x \bigcup \mathrm{dm}\ x \bigcup (A \backslash \mathrm{Dom}^{-1}\ x)\}$.

PROOF:

It follows directly from the definition that x is a weak Nash equilibrium of $\mathcal{X}(x)$.

Also if we look at a imputation $y \notin \mathcal{X}(x)$, we see that $y \succ x$ and $x \not\gtrsim y$. So if we have a neighborhood Y in which an imputation y is in, then x is not a weak Nash equilibrium for Y.

□

Theorem 5.12 *Every weak Nash equilibrium x in A is not strongly dominated by an imputation $x' \in A$. Thus if x is a weak Nash equilibrium, then Dom^{-1} $x = \emptyset$.*

PROOF:

We prove that a weak Nash equilibrium for A cannot be strongly dominated. Let x a weak Nash equilibrium for A. Suppose there is an imputation $x' \in A$ such that $x \gtrapprox x'$ over coalition S, and $x' \succ x$ over T. Now we can create a $y \in A$ such that $x \not\gtrsim y$, and $y >_T x$.

Because $x \gtrapprox x'$ over S we know that either:

(1) $x \succeq x'$ over S or

(2) $x_i = x'_i$ for all $i \in S$. Thus $\sum_{i\in S} x_i = \sum_{i \in S} x'_i \leq v^(S)$, $\sum_{i\in N\backslash S} x'_i > v(N\backslash S)$ and $\sum_{i\in N\backslash S} x'_i > \sum_{i\in N\backslash S} x_i \Rightarrow \sum_{i\in N} x_i < v(N)$.

Suppose (1) holds. Thus there is weak domination over a coalition S. We know $x_i > x'_i$ for at least 1 player i in S.

We take $y_j = \begin{cases} x'_j \text{ for } i = j \\ \min(x'_j, x_j + \frac{x_i - x'_i}{|N-1|}) \text{ for all } j \in T \\ x_j + \frac{x_i - x'_i}{|N-1|} \text{ for all other } j \end{cases}$.

Now we have $x_i > y_i = x'_i$ for player i as above and $y_j > x_j$ for all other players. Thus there is not a coalition S for which $x \not\gtrsim_S y$.

Also $y \succ x$ over T: $y_j > x_j$ for all $j \in T$ and $\sum_{i\in T} y_i \leq \sum_{i\in T} x'_i \leq v(T)$.

Because $x, x' \in A$, we know that $y \in A$.

Now assume (2). Take $y_j = x_j + \frac{\sum x_j - v(N)}{|N|} > x_j$. Now $x_i < y_i$ for all $i \in S$. Thus $x \not\gtrsim y$. But $y \succ x$ over the grand coalition.

Thus we can create a $y \in A$ such that $x \not\gtrsim y$, $y \succ x$.

□

Theorem 5.13 *An imputation x in the Core $C(v)$ is a weak Nash equilibrium for A.*

PROOF:

For an imputation $x \in C(v)$ holds that $x' \not\succ x$ for every $x' \in A$. Thus $\mathrm{Dom}^{-1}\ x = \emptyset$, and thus by Theorem 5.11 we know that x is a weak Nash equilibrium for $X = x \bigcup \mathrm{dm}\ x \bigcup A \backslash \mathrm{Dom}^{-1}\ x = A$.

□

With these two Theorems we can state an alternative Definition for the Core:

Corollary 5.14 *The Core consists of all imputations which are weak Nash equilibria in A*

PROOF:

Imputations not in the Core are strongly dominated. By Theorem 5.12 we know that these imputations cannot be a weak Nash equilibrium.

By the Theorem above we know that all imputations in the Core are weak Nash equilibria.

□

Corollary 5.15 *An essential zero-sum game does not have a weak Nash equilibrium.*

Chapter 6

Stable Sets

Suppose we think of imputations in the core as 'stable', because they are not dominated by other imputations. If we look at imputations outside the Core and compare them with an imputation inside the Core, we should not exclude this imputation just because it is dominated by *some* other imputation. We should demand this dominating imputation itself to be 'stable'. If the imputation y is dominated by an imputation x, which is not in the Core, the argument for excluding y is rather weak. Proponents of y can argue that replacing it by x would not lead to a more stable situation, so we may as well stay where we are, i.e. in an imputation outside the Core.

Von Neumann and Morgenstern therefore correlated imputations with an argumentation with similarities as above. They defined the concept Stable Set V, which gives a certain stability, a "Standard of Behavior". Namely that all the imputations not dominated by an imputation in V are the imputations of V itself. So domination of y by x means the exclusion of y in the Stable Set. Thus an imputation $x' \in V$ cannot be upset by another imputation $x \in V$.

6.1 Definition of Stable Sets

In a more formal way, the Von Neumann and Morgenstern (1944) [8] Stable Set is defined as follows. Because we give alternative Stable Sets in the next chapter, we name them Strongly Stable Sets.

Definition 6.1 A Strongly Stable Set $V \subset A$ is a set imputations such that:

- Internal Stability: $\nexists x, x'$ in V such that $x \succ x'$,
- External Stability: $\forall y \in A \backslash V \exists x \in V$ such that $x \succ y$.

According to next Theorem, we see that these Strongly Stable Sets have some nice rational properties:

Theorem 6.2 *The Strongly Stable Set V is a solution concept which has the properties* (COV), (COCO) and (PO).

PROOF:

COV: Let's look at two equivalent games (N, v) and (N, w). We know that domination is transformation preserved. Thus if V is a Strongly Stable Set for (N, v), then:

- Internal Stability: $s(x_1) \not\succ s(x_2)$ in (N, w) for all $x_1,\ x_2 \in V$
- External Stability: Look at an $s(y) \notin s(V)$. Then there is an $x \in V$ such that $x \succ y = s^{-1}(s(y))$ in (N, v). Thus $s(x) \succ s(y)$.

Thus $s(V)$ is a Strongly Stable Set in (N, w).

COCO: Let $(N_1 \bigcup N_2 = N, v_1 + v_2 = v)$ a composite game. If V_1 is a Strongly Stable Set for v_1 and V_2 is a Strong Stable Set for v_2 then we have to show that the cartesian product $V = V_1 \times V_2$ is a Strongly Stable Set for v.

Thus we have to show that V is internally and externally stable:

- Internal stability: If V is not internally stable, then there are $x_1, x_2 \in V$ such that $x_1 \succ x_2$. There has to be that $x_1 \succ_S x_2$ for a coalition for which $S \bigcap N_1 \neq \emptyset,\ S \bigcap N_2 \neq \emptyset$. But because $v(S) = v(S \bigcap N_1) + v(S \bigcap N_2)$ for such an S, there also has to be domination over $S \bigcap N_1$ or $S \bigcap N_2$. But gives a contradiction with V_1, V_2 stable.
- External stability: Every imputation not in the Strongly Stable Set can be written as $(y; z)$ with $y \in A(v_1) \backslash V_1$ with $\sum_{i \in N_1} y_i \leq v(N_1)$ or $z \in A(v_2) \backslash V_2$ with $\sum_{i \in N_2} z_i \leq v(N_2$, because imputations which are not Pareto Optimal are never in a Strongly Stable Set. Suppose this holds for y. Then y is dominated in the game (N_1, v_1) by imputation x. Take $v \in V_2$ random. Now $(x; v) \succ (y; z)$, and $(x; v) \in V$. Thus every imputation outside the Strongly Stable Set is dominated.

PO: Look at an imputation y for which $\sum y_i < v(N)$. There exists an imputation x in $\overline{A}$ with $x_i = y_i + \frac{v(N) - \sum y_i}{n} > y_i$. So y_i is majorized by x. By the definition of the Strongly Stable Set, or this imputation is in the Strongly Stable Set V, which excludes y to be in V, or it is dominated by an imputation z in V. Because $z \succ x$ and $x \rightarrow y$ we know y is never in a Strongly Stable Set. Thus imputations in V are Pareto optimal.

□

6.2 Stability in A'

From the definition of a Strongly Stable Set it follows that it is a solution set which is externally stable for all imputations in the set A. Thus the stability refers to stability of all individual rational payoffs. With next Theorem we see the relation with Strongly Stable Sets which are stable for the whole pre-imputation space A'.

Theorem 6.3 [2]

a. A necessary and sufficient condition for a Strongly Stable Set V for A to be a stable set in A' is that it touches every face of the simplex A, i.e. for each player i there is an $x \in V$ such that $x_i = v(i)$.

b. A necessary and sufficient condition for a Strongly Stable Set V' of A' to be stable in A is that it is contained in A.

PROOF of 6.3a:

$\Leftarrow$ Suppose V is a Strongly Stable Set for A and it touches every face of the simplex A. Then for every imputation $y \in A' \backslash A$ there is a player i for which $y_i < v(i)$. But one of the assumptions is that for every player i there is an $x \in V$ such that $v(i) = x_i > y_i$ and thus this $x \succ y$ over coalition $S = \{i\}$.

⇒ Suppose V is Strongly Stable for A, but there is an i for which there is no $x \in V$ such that $x_i = v(i)$. We create an y in A', such that $y \not\succ x$.

We know that V is bounded, because of Theorem 3.9 and the fact that all majorized imputations are not in the Strongly Stable Set. Since V is bounded, we can take a K so large that, if $y(K)$ with $y(K)_j = \begin{cases} \frac{v(N)}{n} + K \text{ if } i \neq j \\ \frac{v(N)}{n} - (n-1)K \text{ for } i = j \end{cases}$.

Then for each $x \in V$ we have that $y_i > x_i$ for $i \neq j$ and $y_j < x_j$. If V is stable in A', then there is an $x \in V$ such that $x \succ y$. But this can only be over coalition $S = \{j\}$. Because there is no $x \in V$ such that $x_i = v(i)$, this gives a contradiction.

PROOF of 6.3b:

⇒ Suppose $V' \subset A$. If $V' = A' \backslash \text{Dom } V' = A \backslash \text{Dom } V'$, then V' is Strongly Stable for A.

⇐ Suppose $V' = A' \backslash \text{Dom } V'$, thus $V' \bigcap A = A \bigcap A' \backslash \text{Dom } V' =$ $A' \bigcap (A \backslash \text{Dom } V') = A \backslash \text{Dom } V'$. And thus if V' is Strongly Stable for A, then $V' = A \backslash \text{Dom } V'$.

□

6.3 Construction of Strongly Stable Sets

In the previous chapters we have seen that the (Dual) Core, and thus the set of Nash equilibria, is simply found by solving some linear inequalities. A general algorithm is not available for finding a Strongly Stable Set. Finding is often difficult. In this section we will construct Strongly Stable Sets for the three-person zero-sum game. As we have seen, this game has an empty Core.

Example 6.4 The game has the following characteristic function

$$v(1) = v(2) = v(3) = -1,$$
$$v(12) = v(23) = v(13) = 1.$$

Imputations which are not Pareto optimal are majorized and they are not in the Strongly Stable Set. So we are looking for a solution set V in $\overline{A} = \{x_i | \sum x_i = 0, x_i \geq -1\}$. We can graphically present all possible imputations of the solution space in the following two dimensional plane. If we know in an imputation the payoff for two players and that the imputation is Pareto optimal, we know the payoff for the third player. From now on, we call this the unit-triangle. It is shown in Figure 6.1.

Let's look at an imputation x on this plane. Imputations in the unit triangle are only dominated over the 2-person coalitions:

- The grand coalition gets in every imputation a constant payoff.
- We know that imputations y for which two players get more, dominate an imputation x, and if two players get less in an imputation z these imputations are dominated by x.
- If in two imputations in the unit triangle v and w one player gets a constant payoff, neither v dominates w nor w dominates v.

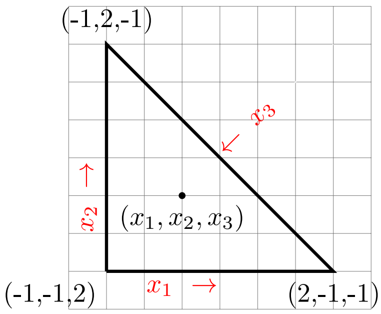

Figure 6.1

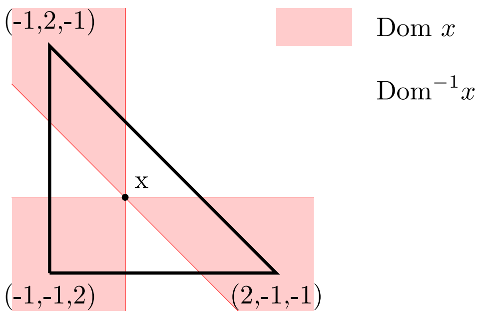

Figure 6.2

We can summarize this in the next picture.

Note that the red lines stand for imputations which neither dominate x, nor are dominated by x.

For a Strongly Stable Set V the next holds by definition:

a There are no $x^1, x^2 \in V$ such that $x^1 \succ x^2$.

b For every $y \in A \backslash V$ $\exists x \in V$ such that $x \succ y$.

In terms of the graphical representation above we are looking for the following set points V:

a the imputation set V is not colored light red,

b every imputation not in V is colored light red.

Now we are going to solve this problem. Because none of the imputations v in A dominates all imputations in $A \backslash \{v\}$, we see that a stable set has to consist of more then one imputation. If we want internal stability, we know that for two different imputations in V one player gets a constant payoff. Thus between every two imputations x^i, x^j in a solution V we can span a line parallel to one face of the unit-triangle, otherwise there is internal domination. We have two possibilities for solutions:

1 all imputations of V are on a line parallel to one face of the unit-triangle,

2 there are imputations on more then one line parallel to multiple faces of the unit triangle.

First we assume 2 is the case. If we have less than three imputations, they are on one line. Suppose there are more than three imputations in V. Now there can't be a line between every of these imputations parallel to a face of the unit triangle:

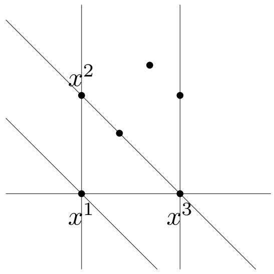

Figure 6.3

Thus if 2 is the case, we have three imputations. If we take the line between each of these two imputations we must make a triangle with the faces parallel to the faces of the unit-triangle. This triangle can lay in two different ways in the unit triangle:

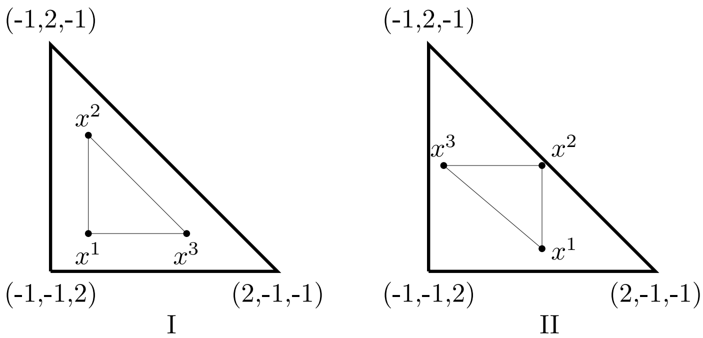

Figure 6.4

If we look at figure 6.4I, we see that imputations within this smaller triangle are never dominated and in this way we never get a Strongly Stable Set:

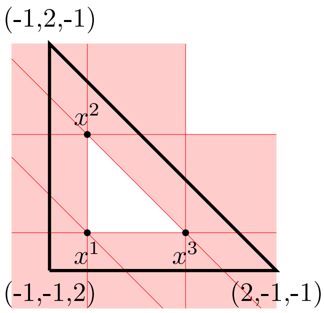

Figure 6.5

So if 2 is the case, then the smaller triangle is like in figure 6.4II. Now suppose at least one of the imputations in V is in the inside of the unit triangle, as in figure 6.6. Then there are imputations in A not dominated by the three imputations of V.

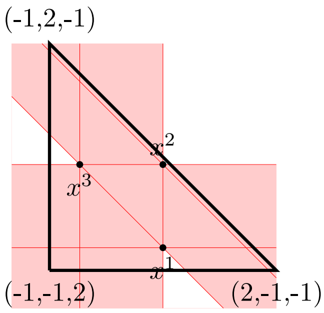

Figure 6.6

So all three imputations have to be on a face of the unit-triangle. If one imputation is not on the half of one of the faces, there must be one other not on one of the faces:

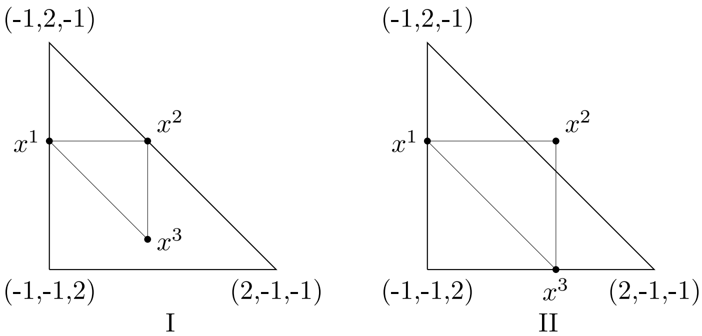

Figure 6.7

So the three imputations $x^2 = (1/2, 1/2, -1)$, $x^1 = (1/2, -1, 1/2)$ and $x^3 = (-1, 1/2, 1/2)$

are the only possibility. If we look at the union of the arced planes by these imputations, we see that every imputation in $\overline{A}\backslash V$ is light red (if an imputation is on one red line red lines, it is dominated):

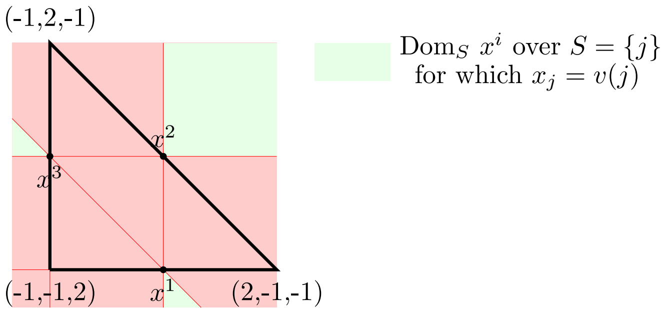

Figure 6.8

And thus every imputation in $A\backslash V$ is dominated. So
$V = \{(\frac{1}{2}, \frac{1}{2}, -1), (\frac{1}{2}, -1, \frac{1}{2}), (-1, \frac{1}{2}, \frac{1}{2})\}$ is a Strongly Stable Set. Note that because for every player there is an imputation in the solution with $x_i = v(i)$ all the pre-imputations in A' are also dominated.

But this is not a unique solution:

Suppose 1 holds. We know that imputations on this line are not dominated by other imputations on this line. Thus the only possibility is that every imputation on this line in the unit-triangle is in the solution set. For these imputations a player gets a constant payoff. If we look at the imputations dominated by such a line we see that:

- every imputation direct above this line, thus every imputation for which the player with constant payoff gets more, is dominated.
- beneath the line is a (pre-)imputation m in A'. This imputation is, because it is on the crossing of two red lines and it dominates every other imputation in V, not dominated over a two person coalition.

For stability in A we want this imputation m to be outside the unit-triangle. We can accomplish this by taking the line such that the player with constant payoff gets less then $\frac{1}{2}$.
And thus the lines, for which $-\frac{1}{2} < x_i + x_j = c \leq 1,\ i,j \in \{123\},\ i \neq j$ is, are Strongly Stable Sets.

Note that the Strongly Stable Sets for which $x_i + x_j \neq 1$, are not stable in the set of pre-imputations A' by Theorem 6.3. If $x_i + x_j = 1$, we know $x_k = -1$, and x_k dominates point m over the one-man coalition $S = \{k\}$. Also every pre-imputation m' 'below' m, for which $m'_k < m_k < v(k)$ is dominated over this one man coalition $S = \{k\}$.

6.3.1 Explanation of the Strongly Stable Set: The Standard of Behavior in the 3-person zero-sum game

In the previous section we saw that there are uncountable many Strongly Stable Sets for the three-person zero-sum game. As we pointed out in section 1.1, we are interested in the performance of solutions in the applications.

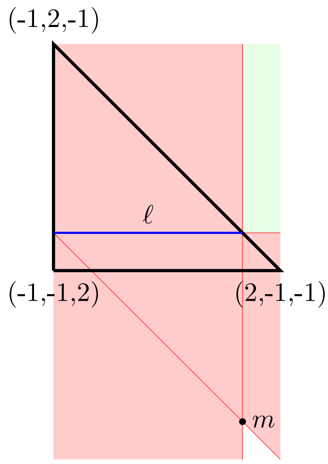

Figure 6.9

Intuitively, the symmetric solution represents a situation in which each of the three players is a possible coalition partner. Which coalition will form is determined by negotiations between the players, based on payoff. The three outcomes in the Strongly Stable Set are stable as a set. In a sense, each is right only because the others are also. The idea is that, since each coalition knows that the other coalitions will divide their $v(S)$ *fifty-fifty* if they form, it too is motivated to divide *fifty-fifty*.

If we look at the Strongly Stable Set where all imputations of the Strongly Stable Set are in one line, we can explain this as follows. One player is excluded from the negotiations. He may be given a certain sum "to keep him quiet"; this sum is fixed and not subject to negotiation. Neither one of the other players considers it a possibility that the excluded player will enter the negotiations. As a consequence, there is no constraint on the two player; the negotiations turn them into a two person game in which there is no internal domination.

In this game we see that the theory of Von Neumann and Morgenstern is in some sense successful. Where there are many examples for which the Core is empty and thus there exist no Nash equilibria, we see that the Strongly Stable Sets give a solution which has also nice properties. If we look at the relation between the Core and the Strongly Stable Set, we see that the Core, and thus the set of Nash-equilibria, is always a subset of the Strongly Stable Set.

Theorem 6.5 *If V is a Strongly Stable Set, then $C(v) \subset V$*

PROOF:

We compare the definitions. For V we know that $A\backslash\text{Dom } V = V$. For a $c \in C(v)$ we know that there is no $x \in A$ such that $c \subset \text{Dom } x$. Thus if $c \notin V$, then $c \notin A\backslash\text{Dom } V$. Thus $c \in \text{Dom } V$. Which gives a contradiction with $c \in C(v)$.

$\square$

Corollary 6.6 *A Nash equilibrium is always in a Strongly Stable Set*

Now we will look at a game with a nonempty Core.

Example 6.7 We will look at the following game, a small variation of the game described by example 2.4:

- $v(F) = 1$, $v(Y) = v(B) = 0$,
- $v(YF) = 2$, $v(BF) = 3$.

For this game, we have the following situation in mind. Again a farmer$F$ has milk from its cow.

- He can sell it directly with a profit of 1
- With cooperation of the yoghurt company Y he can process it into yoghurt, and sells it with a profit of 2
- With cooperation of the butter company B he can process it into butter, and sells it with a profit equal to 3

We know that the imputation space $A = \{x | x_F \geq 1, x_Y \geq 0, x_B \geq 0, \sum x_i \leq 3\}$. Analogue to the construction of the Strongly Stable Set of the 3-person zero-sum game we can represent the not majorized imputations of A in the next triangle:

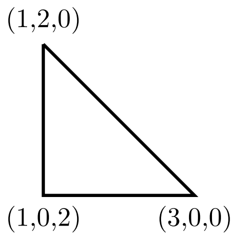

Figure 6.10

If we look at imputations which are dominated over a two-person coalition by an imputation x, we have two possibilities:

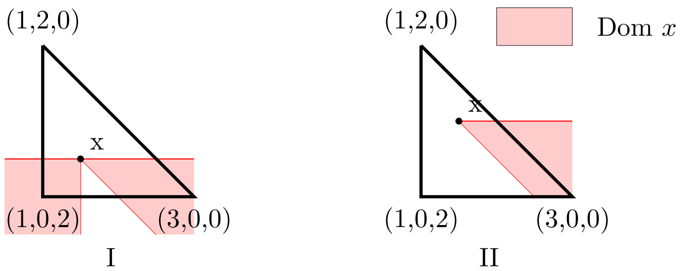

Figure 6.11

We have the case as in figure 6.11I when both the two-person coalitions are effective, thus in imputations $\{x | x_Y + x_M \leq 2,\ x_B + x_M \leq 3\}$. We have the case as in 6.11II when only one of the two-person coalitions is effective, thus $\{x | x_Y + x_M > 2,\ x_B + x_M \leq 3\}$.

But if we construct the Core, we see that it is the span of the imputations $(2, 0, 1)$ and $(3, 0, 0)$. And if we look at all imputations dominated by the Core, we see that Dom $C(v) = \{x | \sum x_B + x_M < 3,\ x_B \leq 1\}$ is. So we have to look for a Strongly Stable Set for $A \backslash \text{Dom } C(v)$.

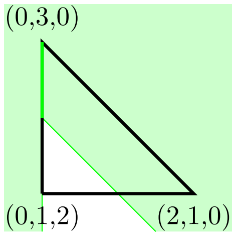

Figure 6.12

If we look at the imputations in A dominated by an imputation in $A\backslash\text{Dom } C(v)$ we see that we have domination as in figure 6.11II. Now we can construct a Strongly Stable Set by taking the whole line $x_Y = 0$. But this Strongly Stable Set is not unique, also taking the line $x_B = 1$ gives a Strongly Stable Set. In fact, every line, for which the slope on every imputation is between the slope of the lines $x_Y = \text{constant}$ and $x_B = \text{constant}$, is a Strongly Stable Set.

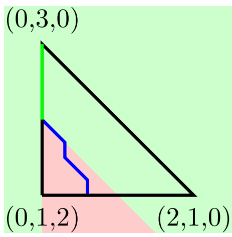

Figure 6.13

Note that all these Strongly Stable Sets are also stable for A'.

Let's look at the performance of these solutions.

If we look at the whole line $x_Y = 0$. We can explain this by the fact that the butter company and the farmer exclude the yoghurt company from the negotiations. Neither of them considers entering negotiation from the yoghurt company an attractive possibility.

The standard of behavior in the combination of the parallel lines (we take the solution as in figure 6.13 as in the example), can be explained as follows. The farmer negotiates with both companies. The companies form a cartel. If we look it from the farmers perspective, and if he sees every penny he gets less than 3 pennies a loss, than every penny the farmer loses 'more' goes to:

> Either the butter company, this is the case if the farmer still gets more than 1.
>
> Or the penny is going to either the yoghurt company or the butter company. This is the case if the farmer already gets less than 1.

We see in these examples that the Strongly Stable Sets perform well. It is a solution concept which expresses the idea of a social organization. E.g. minimal winning coalitions, blocking coalitions and cartel forming as above, systematic discrimination against players or groups of players, etc. One of the great disadvantages is that it is hard to find. Another disadvantage is that the Strongly Stable Set does not always exist. In Chapters 11 and 12 we will investigate the performance and existence of these Strongly Stable Sets in more detail.

6.4 The Union of Stable Sets

Because there is only one eventual payoff and because we don't know the ideas of all players, i.e. we don't know from which strongly stable set the eventual payoff will be an element, we can define a new solution concept.

Definition 6.8 Behavior x is accepted behavior in a cooperative game, if there exists a stable set V with $x \in V$. We say the union over all stable sets $\bigcup V = \mathcal{V}$ are the possible outcomes.

Theorem 6.9 *Accepted behavior, $\{x|x \in \mathcal{V}\}$, is a solution concept which has the properties* COV, COCO *and* PO

PROOF:

COV: Because stable sets are covariant under transformation, the union over stable sets is also.

COCO: Stable sets have the property (COCO), thus the union of these sets also.

PO: Because every imputation in a stable sets is (PO), the imputations in the union of stable sets are also PO.

□

Example 6.10 *If we look at an imputation in $\overline{A} = \{x|x_i \geq -1, \sum x_i = 3$ in the 3-person zero-sum-game we see that this imputation is always in one strong stable set. So every imputation in $\overline{A}$ is accepted behavior. Every imputation not in this set, thus the imputations which are not Pareto Optimal, is majorized, so these are never in a stable set.*

Example 6.11 *If we look at the imputation as in the example 2.4, we see that every imputation which is Pareto optimal and not dominated by the core is accepted behavior. Every imputation which is not in this set, thus the imputations which are not Pareto optimal, are majorized, and thus these are never in a stable set.*

The sets of Accepted Behavior in examples 6.10 and 6.11 are the sets of all imputations that are not majorized and not dominated by the Core.

6.5 Conclusion of Chapter 6

A Strongly Stable Set can be seen as the possible payoffs given a particular "Standard of Behavior". The Union of these Stable Sets can thus be seen as the possible payoffs given all "Standards of Behavior". Some imputations are always in a Strongly Stable Set, as the Core. Some imputations are in one unique Strongly Stable Set[1], and some in more[2] than one, but not in all. We can interpret this as that imputations given a game, not all possible payoffs are equally likely.

[1] Look at imputation $(0, 1, 1)$ in Example 2.4, there is a unique Strongly Stable Set in which this imputation is an element

[2] Look at imputation $(\frac{1}{2}, 1, \frac{1}{2})$ in Example 2.4, which is in uncountable many Strongly Stable Sets. But there are Stable Sets for which this imputation is not an element

Chapter 7

Variants on Strongly Stable Set

In this chapter we investigate some variants on Strongly Stable Sets. Successively we we will define and discuss Partially Stable Sets, Strictly Stable Sets and Weakly Stable sets.

7.1 Interpreting Strongly Stable Sets

By defining the other variants, we interpret the Strongly Stable Set in an alternative way. For this interpretation we use the definition of Nash equilibria[1]. Remind that an imputation x is a strict Nash equilibrium for a neighborhood $N(x)$ if for every $y \in N(x)$:

- either $x \succ y$;
- or if $x \not\succ y$ then $y \not\succsim x$.

And an imputation v is a weak Nash equilibrium for a neighborhood $N(v)$ if for every $w \in N(v)$:

- either $v \succeq w$
- or if $v \not\succeq w$ then $w \not\succ v$

For the Strongly Stable Set V we know the following holds:

> For all $y \notin V$ $\exists x \in V$ such that $x \succ y$: Thus for all $y \notin V$ there is a set $N(x)$ for which $y \in N(x)$ for which x is a strict Nash equilibrium such that $x \succ y$.
>
> For all $x^1, x^2 \in V$ we know that $x^1 \not\succ x^2$: If we look at the set V, we see that all x^i are weak Nash equilibria for V.

We have seen that this Strongly Stable Set can exist of uncountable many imputations.

Continuation of Example 6.4 Let's look at the Strongly Stable Set V for which there are three imputations. We can divide the Pareto optimal imputations of A in 7 areas:

In this Venn-diagram we have 7 areas:

[1]See Chapter 5

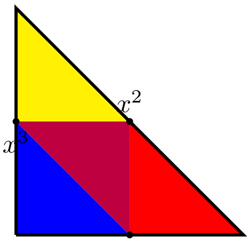

Figure 7.1

- x^1
- x^2
- x^3
- A purple set P'_1. This is the imputation set for which all the members get payoff less than $\frac{1}{2}$. If we look at $P'_1 \bigcup x^i$ we see that x^i, for $i = 1, 2, 3$, is (the unique) strict Nash equilibrium in this sets. Note that the purple set is an open set.
- A yellow set P'_2. If we look at $P'_2 \bigcup x^1$ we see that x^1 is a strict Nash equilibrium in this this set. The yellow set is a closed set minus the imputations x^2 and x^3.
- This also holds for the blue set P'_3. If we look at $P'_3 \bigcup x^2$ we see that only x^2 is a strict Nash equilibrium in this set.
- And for the red set P'_4. If we look at $P'_4 \bigcup x^3$ we see that only x^3 is a strict Nash equilibrium in this set.

So if we take $P_1 = P'_1 \bigcup P'_2 \bigcup x^1$, $P_2 = P'_1 \bigcup P'_3 \bigcup x^2$, $P_3 = P'_1 \bigcup P'_4 \bigcup x^3$ we found three subsets of A, for which $\bigcup P_i \supset A$. If we look at the set V, we see that every x^i is a weak Nash equilibrium for this set.

With this alternative description of the Strongly Stable Set in mind, we look at the following three variants:

1. We are looking for a set V' such that

   External Stability: For all $y \in A \backslash V'$ such that $y \succ x$ for an $x \in V'$ $\exists x' \in V'$ such that $x' \succ y$. Thus if y is an imputation such that there is an $x \in V'$ with $x \in N(y)$ for which y is a strict Nash equilibrium such that $y \succ x$, then there is an $x' \in V'$ and set $N(x')$ for which x' is a strict Nash equilibrium and $y \in N(x')$. If $y \in A \backslash V'$ and for all $x \in V'$ such that $y \not\succ x$, then we know that for all $x \in V'$ that there exists an $N(x)$ such that $y \in N(x)$ and x a strict Nash equilibrium for this set.

   Internal Stability: For all $x^1, x^2 \in V'$ we know that $x^1 \not\succ x^2$. If we look at the set V', we see that all x^i are weak Nash equilibria for V'.

2. We are looking for a set Z such that

   External Stability: For all $y \in A \backslash Z$ there is an $x \in Z$ such that $x \succ y$. Thus if y is an imputation outside V there is an $x \in V$ with a $N(x)$ for which x is a strict Nash equilibrium and $y \in N(x)$ such that $x \succ y$.

Internal Stability: For all $x^1, x^2 \in Z$ we know that $x^1 \not\gtrapprox x^2$. If we look at the set Z, we see that all x^i are strict Nash equilibria for Z.

3. We are looking for a set W such that:

   External Stability: For all $y \in A \backslash W$ there is an $x \in W$ such that $x \gtrapprox y$. Thus if y is an imputation outside V there is an $x \in V$ such that there is a set $N(x)$ for which $y \in N(x)$ for which x is a weak Nash equilibrium such that $x \gtrapprox y$.

   Internal Stability: For all $x^1, x^2 \in W$ we know that $x^1 \not\succ x^2$. If we look at the set W, we see that all x^i are weak Nash equilibria for W.

7.2 Partially Stable Sets

We can state the first variant above in the following definition. This set is defined by Alvin E. Roth in [12]. In this article no name is given to this set, but for clarity we call them Partially Stable Sets.

Definition 7.1 We call a set V' Partially Stable, if

Internal Stability: $x_i \not\succ x_j \ \forall x_i, x_j \in V'$,

External Stability: $\forall y \in A$ such that $y \succ x_i$ for an $x_i \in V'$ $\exists x_j \in V'$ such that $x_j \succ y$

It is not hard to see that the following sets are Partially Stable Sets:

Theorem 7.2

a. A Strongly Stable Set V is a Partially Stable Set,

b. The empty set is a Partially Stable Set,

c. A weak Nash equilibrium for A is a Partially Stable Set,

d. A subset of the Core is a Partially Stable Set.

PROOF of a:

Let V a Strongly Stable Set, then V is internally stable. We also know that for every $y \notin V$ there is an $x \in V$ such that $x \succ V$. Thus for $\forall y \in A$ such that $y \succ x_i$ for an $x_i \in V$ $\exists x_j \in V$ such that $x_j \succ y$, thus V is Partially Stable.

PROOF of b:

Let $V' = \emptyset$. Then there is no internal domination. If we look at an $y \in A$ we see that there is no $x \in V$ such that $y \not\succ x$. Thus the empty set is a Partially Stable Set.

PROOF of c:

Let x be a weak Nash equilibrium for A. By definition there is no internal domination. Also for every $y \in A$ we see that $y \not\succ x$ by definition. Thus a weak Nash equilibrium is a Partially Stable Set.

PROOF of d:

Let $C'(v)$ a subset of the Core. By definition there is no internal domination. Also for every $y \in A$ we see that $y \not\succ x$ for an $x \in C(v)$ by definition. Thus a subset of the Core is a Partially Stable Set.

$\square$

7.2.1 The Existence Proof of Partially Stable Sets

In [12] Alvin E. Roth gives an existence theorem for Partially Stable Sets. With Tarski's fixed point Theorem[2] he proves the existence.

Definition 7.3 A lattice is a partially ordered set $(L, \leq)$ for which every finite nonempty subset S has a unique least upper bound in L denoted as $\vee S$ and a unique maximal lower bound in L denoted as $\wedge S$.

Definition 7.4 A lattice is called a complete lattice for which all subsets H have a unique least upper bound $\vee H$ in L and a unique maximal lower bound $\wedge H$ in L.

We denote elements of L with small letters and subsets of L with capital letters. Let $\phi : L \to L$ with the following property $\mathcal{P}$. For any $A \subseteq L$ we have $\phi(\vee A) = \wedge\phi(A)$, with $\phi(A) = \{\phi(a)|a \in A\}$. From this property it follows that:

Lemma 7.5 *If $a \leq b$ then $\phi(a) \geq \phi(b)$.*

PROOF:

Suppose $a \leq b$, thus $a \vee b = b$. Then $\phi(b) = \phi(a \vee b) = \phi(a) \wedge \phi(b)$ via the Definition of ϕ. This can only hold if $\phi(b) \leq \phi(a)$.

□

Lemma 7.6 *If $a \leq b$ then $\phi^2(a) = \phi(\phi(a)) \leq \phi^2(b)$.*

PROOF:

We can apply Lemma 7.5 twice and we get the above result.

□

Also define $L_D(\phi) = \{a \in L|a \leq \phi(a)\}$, $L_D(\phi^2) = \{a \in L|a \leq \phi^2(a)\}$, property $\mathcal{P}$ gives us:

Lemma 7.7 $\phi^2 : L_D(\phi) \to L_D(\phi)$.

PROOF:

We know that for an element a in $L_D(\phi)$ holds that $a \leq \phi(a)$. It is sufficient to prove that $\phi^2(a) \leq \phi^3(a)$. If $a \leq \phi(a)$, we can conclude from Lemma 7.6 that $\phi^2(a) \leq \phi^3(a)$. Thus $\phi^2(a)$ is an element in $L_D(\phi)$.

□

Lemma 7.8 $\phi^2 : L_D(\phi^2) \to L_D(\phi^2)$.

PROOF:

We know that for an element a in $L_D(\phi^2)$ holds that $a \leq \phi^2(a)$. It is sufficient to prove that $\phi^2(a) \leq \phi^4(a)$. If $a \leq \phi^2(a)$, then via Lemma 7.6 we see that $\phi^2(a) \leq \phi^4(a)$. Thus $\phi^2(A)$ is an element in $L_D(\phi^2)$.

[2]**Tarski's fixed point theorem** Let $(L, \leq)$ be a complete lattice. Suppose $f : L \to L$ is a function for which $A \leq B$ implies $f(A) \leq f(B)$, then the set of all fixed points of f is a complete lattice with respect to $\leq$.

□

Lemma 7.9 *If $A \subset L_D(\phi^2)$ then $\vee A \in L_D(\phi^2)$.*

PROOF:

Suppose $a_i \in A$, then $a_i \in L_D(\phi^2)$. Then via Lemma 7.6 and the definition of ϕ we know that $a_i \leq \phi^2(a_i) \leq \phi^2(\vee A)$. This holds for all $a_i \in A$. Thus $\vee A \leq \vee\phi^2(\vee A) = \vee\phi^2(\vee A)$. Thus $\vee A \in L_D(\phi^2)$.

□

From Tarski's fixed point theorem it follows that ϕ^2 has a fixed point. While it is not generally true that ϕ has a fixed point, the following Theorem holds:

Theorem 7.10 [12] *ϕ^2 has a fixed point $s \in L$ with $s \leq \phi(s)$.*

PROOF:

Let $D = L_D(\phi) \bigcap L_D(\phi^2)$. We know that $\inf L \in D$ we see that $D \neq \emptyset$.

Now let M be a maximal chain in D. Thus $M \subset D$ such that:

- Either $m_i \leq m_j$ or $m_j \leq m_i$ for all $m_i, m_j \in M$,
- $\nexists M' \in D \setminus M$ such that $m_i \leq M'$ for all $m_i \in M$.

Take $s = \vee M$. We are going to prove that $s \in D$:

- $s \in L_D(\phi^2)$ by Lemma 7.9.
- Suppose $s \notin L_D(\phi)$. Thus $s \not\leq \phi(s)$. Thus, by property $\mathcal{P}$, we see that $s \not\leq \wedge\phi(M)$. Thus there is an $m_i \in M$ such that $s \not\leq \phi(m_i)$. Because M is a chain and $s = \vee M$ we know that there is an $m_j \in M$ such that $m_j \not\leq \phi(m_i)$. Because $m_i, m_j \in M$ we know that either $m_i \leq m_j$ or $m_j \leq m_i$:

  If $m_j \leq m_i$, then $m_j \leq m_i \leq \phi(m_i)$ and we have a contradiction.

  If $m_i \leq m_j$, then $m_j \leq \phi(m_j) \leq \phi(m_i)$ and again we have a contradiction.

  Thus $s \in L_D(\phi)$.

Thus we see that $s \in D$.

Because $s \in D$, we know via Lemma 7.7 and Lemma 7.8 that $\phi^2(s) \in D$. Thus suppose $s < \phi^2(s)$, then we have a contradiction with M a maximal chain.

Thus $s \leq \phi(s)$ because $s \in D$ and $s = \phi^2(s)$.

□

Given this Theorem the following is easily proved:

Theorem 7.11 *For every game (N, v) there exists at least one Partially Stable Set V'.*

PROOF:

Take $L = 2^A$ with A the set of imputations. We take the lattice as the set L and the subsets of A ordered by set inclusion. We see that $\phi(X) = A \backslash \text{Dom } X = A \backslash \bigcup_{x \in X} \text{Dom } x = \bigcap_{x \in X} A \backslash \text{Dom } x = \bigcap_{x \in X} \phi(x)$. Thus we can apply Theorem 7.10. So we see that there exists a V' such that $V' = \phi^2(V')$ and $V' \subseteq \phi(V')$.

What this means is that the imputations not dominated by V' is a superset $\phi(V')$ of V'. Also the not-dominated imputations by this $\phi(V')$ is the set of imputations V'. Thus for this V' we have exactly the definition of a Partially Stable Set.

□

If we look closer at Theorem 7.11, it is not hard to see the following holds also:

1. If $C(v) = \emptyset$, then $\emptyset \subset \phi(C(v)) = A$ and $\emptyset = \phi(A) = \emptyset$.

But in [12] there is said that for s as in Theorem 7.11 we can insure that $s > \inf L$. In [13] is said that the existence of such an s, thus a non-empty Partially Stable Set is still a conjecture.

7.3 Strictly Stable Sets

We start with the Definition given in the introduction of this Chapter.

Definition 7.12 Z is a Strictly Stable set if:

External Stability: $\forall y \in A \backslash Z \ \exists x \in Z$ such that $x \succ y$,

Internal Stability: $\forall x, x' \in Z$ holds $x \not\gtrsim x'$.

As we see in this definition, these sets are in some sense internally more stable than Strongly Stable Sets. There is not the smallest rational argument to switch from an imputation x in the Strictly Stable Set to another imputation x' in the Strictly Stable Set.

Theorem 7.13 *Z is a Strictly Stable Set if and only if:*

$\forall y \in A \backslash Z \ \exists x \in Z$ *such that* $x \succ y$,

$\forall x, x' \in Z$ *holds* $x \not\succeq x'$.

PROOF:

$\Leftarrow$ We only have to show that if Z is Strictly Stable then $\forall x, x' \in Z$, if $x \not\succeq x'$ then $x \not\gtrsim x'$. Suppose this does not hold. Then we know that $\sum_{i \in N} x_i < v(N)$. But now this imputation x is majorized by an imputation y. This imputation y is not in Z, because then we have internal domination. But if y is not in Z, then there is a y' such that $y' \succ y$ and thus $y' \succ x$. Thus we have a contradiction.

$\Rightarrow$ Trivial.

□

We can check weak domination easier than dual domination. So from now on we write the Strictly Stable Set as described by Theorem 7.13.

Theorem 7.14 *If Z is a Strictly Stable Set, then Z is Strongly Stable set.*

Corollary 7.15 *The Core is contained in any Strictly Stable Set.*

PROOF:

Follows directly from Theorem 7.13 and Theorem 4.3.

□

Continuation of Example 6.7 In example 6.7 we constructed Strongly Stable Sets for the game (N, v) with characteristic function $v(F) = 1,\ v(FY) = 2,\ v(FB) = 3$. We know that every strictly stable set is strongly stable. We have constructed the set of all the Strongly Stable Sets, so we can look for the Strictly Stable Sets in this set. We also know that the Core is always a subset of a Strongly Stable Set.

Thus we first start by constructing dom $C(v)$:

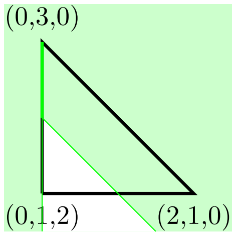

Figure 7.2

and we know that dom x, for an imputation not dominated by the Core is:

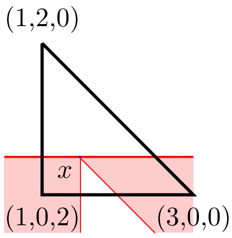

Figure 7.3

In this picture the red lines are also weakly dominated by x.

Thus we see that a Strongly Stable Set is Strictly Stable if the slope of the line of the Strongly Stable Set, except the Core, is never parallel with a face of the triangle. Thus the following set is a Strictly Stable Set:

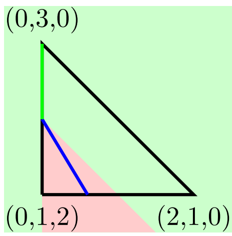

Figure 7.4

We can explain the behavior in this Strictly Stable Set in a similar way as we did for Example 6.7. The butter company can only get a gain greater than 0, if it works together with the yoghurt company:

Assume the farmer starts negotiations with a bid in which he gets all. After all the negotiations, every penny the farmer loses (we assumed the negotiations start by $(3, 0, 0)$) is distributed in the following way:

If the farmer still gets more than 1, the penny goes to the yoghurt company

When the farmer already gets less than 1, a part (> 0) of the penny is going to the yoghurt company and the other part (> 0) is going to the butter company.

If we assume the following: Suppose there is a Strictly Stable Set, this is also a Strongly Stable Set. Then a Strongly Stable Set which is not Strictly Stable is not a "Standard of Behavior". Then we can conclude that in the previous example **a cartel between the two companies will always be formed**.

Continuation of Example 6.4 We have constructed Strongly Stable Sets for the game $v(12) = v(13) = v(23) = 1$, $v(i) = -1$. But if we look for a Pareto optimal imputation $x \in A$ to the set $\text{dom}^{-1}x \bigcup \text{Dom } x \bigcup x$ we see that:

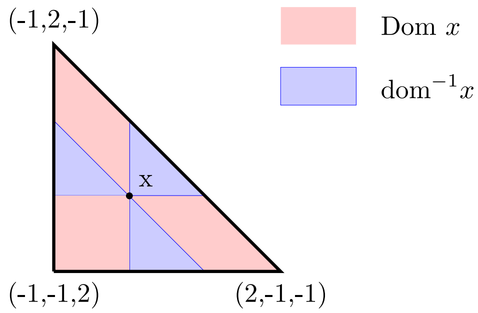

Figure 7.5

We see that the set of all Pareto optimal and individual rational imputations is a subset of the set $\text{dom}^{-1}x \bigcup \text{Dom } x \bigcup x$. Also there is not an $x \in A$ for which $x \succ y$ for all $y \in A$, $y \neq x$. Because imputations which are not Pareto optimal are never in Strictly Stable Sets this game has not a Strictly Stable Set.

Conjecture 7.16 *If (N, v) has an empty Core, then (N, v) has not a Strictly Stable Set*

The existence of a nonempty Core does not imply existence of a Strictly Stable Set. In Section 12.1 we give a game which has a non-convex Strongly Stable Set and a Core, but it does not have a Strictly Stably Set.

So we have seen that if Z is Strictly Stable, then Z Strongly Stable. As we defined, these Strictly Stable Sets have some nicer properties than Strongly Stable Sets which are not Strictly Stable: all imputations in Z are strong Nash equilibria for the set Z.

But as we have also seen, in many games the Strictly Stable Set does not exist.

7.4 Weakly Stable Sets

As pointed out in Section 7.1, we can also look for the following set:

Definition 7.17 We call a set W a Weakly Stable Set if:

External Stability: $\forall y \in A \backslash W \; \exists x \in W$ such that $x \gtrsim y$, Internal Stability: $\forall x, x' \in W$ holds $x \not\succ x'$.

It is not hard to see the following Theorem is true:

Theorem 7.18 *A Strongly Stable Set V is a Weakly Stable Set.*

Theorem 7.19 *W is a Weakly Stable Set if and only if:*

$\forall y \in A \backslash W \ \exists x \in W$ such that $x \succeq y$

$\forall x, x' \in W$ holds $x \not\succ x'$

PROOF:

$\Rightarrow$ Trivial.

$\Leftarrow$ Assume there exists an $y \in A \backslash W$ such that there is a $x \in W$ such that $x \succsim\!\!\!\approx y$ but not $x \succeq y$ over some S. Thus $x_i = y_i$ for all $i \in S$. Also $\sum_{i \in S} x_i < v^*(S)$ and $\sum_{i \in N \backslash S} y_i > \sum_{i \in N \backslash S} x_i v(N \backslash S)$. Thus $\sum x_i < v(N)$. Thus we know that there is a x' in $\overline{A}$ with $x'_i = x_i + \epsilon$ for an $\epsilon > 0$. We see that x' majorizes x. This x' is either:

dually dominated by an $\tilde{x}$ which implies $\tilde{x} \succ y$;

or in the Weakly Stable Set, so $x' \succ y$.

□

Because weak domination is easier checked than dual domination we state the Weakly Stable Sets as in Theorem 7.19.

Theorem 7.20 *The Core is always a subset of a Weakly Stable Set*

PROOF:

If an imputation cannot be weakly dominated, it is in the Weakly Stable Set. By Theorem 4.3 we see thus that the Core is a subset of the the Weakly Stable Set.

□

7.4.1 Differences with Strongly Stable Sets

We construct two Weakly Stable Sets for the three-person zero sum game which are not Strongly Stable.

Continuation of Example 6.4 Remind the game is $v(i) = -1$, for $i = 1, 2, 3$ $v(12) = v(13) = v(23) = 1$ And we know that the Strongly Stable Sets are the sets:

(a) $V = (\frac{1}{2}, \frac{1}{2}, -1), (\frac{1}{2}, -1, \frac{1}{2}), (-1, \frac{1}{2}, \frac{1}{2})$.

(b) $V = \{x \in A | x_i + x_j = a,\ x_k = -a,\ a \in [-1, \frac{1}{2}),\ i \neq j,\ i \neq k,\ j \neq k,\ i, j, k \in \{1, 2, 3\}\}$.

From Theorem 7.18 it follows, that these are also Weakly Stable Sets. But if we look at the set $W = \{x \in A | x_1 + x_2 = -\frac{1}{2},\ x_3 = \frac{1}{2}\}$ we see this is not a Strongly Stable Set. $m = (\frac{1}{2}, \frac{1}{2}, -1)$ is the only imputation not strongly dominated by W or in the Strongly Stable Set W. But m is weakly dominated by $w_1, w_2 \in W$, with $w_1 = (\frac{1}{2}, -1, \frac{1}{2}),\ w_2 = (-1, \frac{1}{2}, \frac{1}{2})$. Thus W is a Weakly Stable Set.

We can also note that every imputation in a Strongly Stable Set is weakly dominated by every other imputation in the Strongly Stable Set. Lets look at what happens if we delete a single imputation of the Strongly Stable Set.

If we delete a single imputation l of the Strongly Stable Set of (b) we see that this single imputation l is weakly dominated. Look at an $x \in \text{Dom } l \bigcap A$. We know that $l_i > x_i,\ l_j > x_j$ and $l_k < x_k$ for a $i, j, k \in \{1, 2, 3\}$. Either $l_i + l_k = c$ or $l_i + l_j = c$ for a constant c and for all imputations in V.

Assume $l_i + l_k = c$, then either $l_k = -1$ or $l_k > -1$. If $l_k > -1$, we take an $l' \in V$ for which $l'_k = 0$, and thus $l'_i > l_i > x_i$ and we have domination over coalition i, j. Suppose $l_k = -1$, then for an $\epsilon = \frac{l_i - x_i}{2} > 0$ we know that there is an $l' \in V$ with $l'_i = l_i - \epsilon$. And we again have domination over coalition ij.

Now assume $l_i + l_j = c$. Then we can take a $l' \in V$ for which $l'_i = l_i - \epsilon$ and $l'_j = l_j + \epsilon$ with $\epsilon = \frac{l_i - x_i}{2}$. We see that $l' \succ x$.

Thus we have seen that all imputations strongly dominated by l are still strongly dominated by another imputation in V. The imputation l is weakly dominated by all other imputations in V.

In fact we can *perforate* a Strongly Stable Set V which is on one line. If we delete countable imputations from V, we still have a Weakly Stable Set.

What can be seen from this example is that a Weakly Stable Set is not necessarily closed.

7.4.2 Questions on Weakly Stable Sets

In their publication [7] Lucas and Rabie noticed in their publication [7][3] is that, if we look at all the possible games as a space, there is a very small line in this space which has no Strongly Stable Set. As we will see for the known games with no Strongly Stable Set, there exists a Weakly Stable Set. From this, we do the following conjecture.

Conjecture 7.21 *Every game has a Weakly Stable Set*

[3] We look at this game in chapter 12.

Chapter 8

The Nucleolus

With the "Standard of Behavior" a non-transitive preference relation (domination) between two different imputations was defined by Von Neumann and Morgenstern. In 1969 D. Schmeidler [14] introduced a transitive preference relation between two different imputations. By taking the optimum with respect to this preference relation, he defined a new solution concept: The Nucleolus. The idea behind the preference relation is that if you compare two imputations, then you look at the least happy coalition. The happiness of a coalition is measured as the difference between $v(S)$ and the total gain of a coalition S: $\sum_{i\in S} x_i$. The least happy coalition, the coalition with the largest difference between $v(S)$ and $\sum_{i\in S} x_i$, is a real valued function on A, so we can minimize this with respect to the set A or A'. The Nucleolus is the minimum of this function; it is the imputation where the least happy coalition is "as happy as possible".

8.1 Defining the Nucleolus

We give the formal definition

Definition 8.1 Let $e(x; S) = v(S) - \sum_{i\in S} x_i$ the excess of a coalition S. For every imputation we order the coalitions $S \subset N$ non-increasing. Put $\theta(x)$ such that $\theta(x)_i \geq \theta(x)_j$ for all $i \leq j$.

Thus we can see $\theta(x)$ as a 2^n-dimensional vector.

Definition 8.2 We can order two imputations x, y lexicographically. $x <_l y$ if $\theta(x)_j = \theta(y)_j$ for $1 \leq j < i$ and $\theta(x)_i < \theta(y)_i$ for a certain i.

With this in mind we can define the following:

Definition 8.3 For a game (N, v) the imputation $x \in A$ for which x attains its minimum with respect to $<_l$ is called the Nucleolus. Thus $N(v) = \{x \in A | x <_l y$ for all $y \in A \backslash N(v)$

By way of illustration, we start with an example.

Example 8.4 Let's look at the three person zero-sum game:

$v(i) = -1,\ v(12) = v(23) = v(13) = 1$

Look at the imputations $x^1 = (\frac{1}{2}, \frac{1}{2}, -1),\ x^2 = (-1, \frac{1}{2}, \frac{1}{2})$ and $x^3 = (0, 0, 0)$. Now we have $\theta(x^1) = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, -\frac{3}{2}, -\frac{3}{2})$ (you can check this with the coalitions $(13, 23, \emptyset, 3, 12, 123, 1, 2)$). We have $\theta(x^2) = (-\frac{3}{2}, -\frac{3}{2}, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})$ (with coalitions $(13, 12, \emptyset, 1, 23, 123, 2, 3)$). And we have $\theta(x^3) = (-1, -1, -1, 0, 0, 0, 0, 0)$ (coalitions $(12, 13, 23, \emptyset, 1, 2, 3, 123)$). We see $x^3 <_l x^1$, $x^3 <_l x^2$ and $x^1 \not<_l x^2$ and $x^2 \not<_l x^1$.

Theorem 8.5 [14] *For every game the Nucleolus consists of a unique imputation*

PROOF:

We prove this in two steps:

(a) $N(v) \neq \emptyset$ and $N(v)$ is compact,

(b) $N(v)$ consists of one imputation.

(a) $v(S_k) - \sum_{j \in S_k} x_j$ is a continuous function on A. Now we show that $\theta(x)_i$ is a continuous function on A.

We use a different notation for $\theta(x)_i = \max\{\min\{v(S) - \sum_{i \in S} x_i | S \in \mathcal{S}\} | \mathcal{S} \subset 2^N, |\mathcal{S}| = i\}$. Because the maximum and minimum of a finite number of continuous functions is continuous, $\theta(x)_i$ is continuous for $i = 1, ..., 2^N$.

Now we define $A^1 = \{x \in A | \theta(x)_1 \leq \theta(y)_1 \forall y \in A\}$ and $A^i = \{x \in A_{i-1} | \theta(x)_i \leq \theta(y)_i \forall y \in A^{i-1}\}$. Because of the continuity of $\theta(x)_i$ and the compactness of A we know a maximum is attained in A. Thus A_1 is nonempty and it is also compact. We can repeat this argument 2^n times, so $N(v) = A^{2^N}$ is nonempty and compact.

(b) Assume $\theta(x) = \theta(y)$ for $x, y \in N(v)$ and $x \neq y$. It is enough to prove that $z <_l x$ for $z = \frac{1}{2}(x + y)$. z is in A because A is convex. Suppose

$\theta(x) = (e(x; S_1), ..., e(x; S_j), e(x; S_{j+1}), ..., e(x; S^{2^N}))$. We reorder this $\theta(x)$ as follows: if $e(x; S_i) = e(x; S_{i+1})$ and $e(y; S_i) \leq e(y; S_{i+1})$ then we take

$\theta'(x) = (e(x; S_1), ..., e(x; S_{i+1}), e(x; S_i), ..., e(x; S^{2^N}))$. We see that the elements of $\theta'(x)$ are still non-increasing.

Now let k the first index such that $e(x; S_k) \neq e(y; S_k)$ for $\theta'(x)$ and $\theta(y)$. So $e(x; S_k) > e(y; S_k)$. Because $z = \frac{1}{2}(x + y)$ we know that $e(z; S_j) = \frac{1}{2}(e(x; S_j) + e(y; S_j))$. Thus $e(z; S_j) = e(x; S_j)$ for every $j < k$.

Now for $j \geq k$ we have that:

- if $e(x; S_j) = e(x; S_k)$ then $e(y; S_j) \leq e(y; S_k)$. Because $e(x; S_k) > e(y; S_k)$ we know that $e(z; S_j) = \frac{1}{2}(e(x; S_j) + e(y; S_j)) < e(x; S_j)$. Thus $e(z; S_j) = \frac{1}{2}(e(x; S_j) + e(y; S_k)) < e(x; S_k)$.
- if $e(x; S_j) < e(x; S_k)$ then $e(y; S_j) \leq e(x; S_j)$. Thus $e(z; S_j) = \frac{1}{2}(e(x; S_j) + e(y; S_j)) < e(x; S_j)$.

Now we have $e(z; S_j) = \frac{1}{2}(e(x; S_j) + e(y; S_j)) < e(x; S_j)$ for all $j \geq k$. Also $e(z; S_i) = e(x; S_i)$ for all $i < j$. So $\theta(z) <_l \theta(x)$. Thus we have a contradiction with $x \in N(v)$. Thus there is at most 1 imputation in $N(v)$.

□

Thus the Nucleolus of a cooperative game is a unique value.

8.2 Computing the Nucleolus

The nucleolus can be found by solving a series of LP-problems:

Algorithm 8.6 [4] We first solve the following LP-problem 1.

$$\min \left\{ x_0 \;\middle|\; \begin{array}{rrcll} & x_i & \geq & v(i) & \forall i \in N \\ x_0 + \sum_{i \in S_j} & x_i & \geq & v(S_j) & \forall S_j \subseteq N \\ \sum_{i \in N} & x_i & = & v(N) & \end{array} \right\}$$

We find a set X^0 for which x^0 attains its minimum. Because $v(\emptyset) = 0$ we have $x_0 \geq 0$. We know that $|X^0| \geq 1$. If $|X^0| = 1$ we have the nucleolus. If not, we take $S^0 = \{S | \sum_{i \in S} x_i = v(S),\ \forall x \in X^0\}$. Now we have to solve the next LP-problem 2 for $k = 0$:

$$\min \left\{ x_0 \;\middle|\; \begin{array}{rrcll} & x_i & \geq & v(i) & \forall i \in N \\ x_0 + \sum_{i \in S_j} & x_i & \geq & v(S_j) & \forall S_j \subseteq N, \quad S_j \notin S^k \\ \sum_{i \in N} & x_i & = & v(N) & \\ & x & \in & X^k & \end{array} \right\}$$

This gives us a set X^{k+1}. If $|X^{k+1}| = 1$ we have the Nucleolus. If not, then we take $S^{k+1} = \{S | \sum_{i \in S} x_i = v(S),\ \forall x \in X^k\}$. Now we use the same Algorithm 2 as above.

Theorem 8.7 [4] *Algorithm 8.6 is finite en finds the Nucleolus*

PROOF:

We use the meaning of Nucleolus in the proof. If we look at what is done in the ith step in the Algorithm, we see that we minimize at least one θ_i. Thus if we compare an imputation $x \in X^i \backslash X^{i+1}$ with an imputation $x' \in X^{i+1}$, we see that $\theta(x)_j = \theta(x')_j$ for $0 \leq j \leq k_i - 1$ and $\theta(x)_{k_i} > \theta(x')_{k_i}$ for a k_i. Thus $x' <_l x$. Because in each step $k_i > k_{i-1}$ and because the Nucleolus is unique, this Algorithm finds the Nucleolus in a finite number of steps.

□

Theorem 8.8

8.8a. [4] *If the Core is nonempty, the Nucleolus is an imputation in the Core.*

8.8b. If the Dual-Core is nonempty, the Nucleolus is an imputation in the Dual-Core

PROOF of 8.8a:

Suppose $x \in C(v)$. Then $e(x; S) \leq 0$ for all S. For any $y \notin C(v)$ we know there exists a T such that $\sum_{i \in T} y_i < v(T)$. Now for this T we have $e(y; T) > 0$, thus $e(y; T) > 0$, and thus $y <_l x$. Thus if the core is nonempty we know that $N(v) \in C(v)$.

PROOF of 8.8b:

Suppose $x \in DC(v)$. Then $e(x; S) = 0$ for all coalitions S for which $v(S) = v^*(S)$ and $e(x; S) < 0$ for all coalitions S for which $v(S) < v^*(S)$. Assume $y \notin DC(v)$. Then, because 8.8a, $y \in C(v)$. Thus there exists an T, such that $v(T) < v^*(T)$ and $e(y; T) = 0$. Now for all S for which $v(S) = v^*(S)$ we know that $e(x; S) = e(y; S) = 0$. $e(x; S') < 0$ for all other $S' \subset N$. But for the T, for which $e(x; T) < 0$, we know that $e(y; T) = 0$. Thus $x <_l y$.

□

Theorem 8.8 gives us a new way to describe the Nucleolus:

Corollary 8.9

8.9a If the game (N, v) has a weak Nash-equilibrium, the Nucleolus is a weak Nash-equilibrium,

8.9b If the game (N, v) has a strict Nash-equilibrium, the Nucleolus is a strict Nash-equilibrium.

8.3 Examples of Nucleoli

We give some examples of how the nucleolus can be found:

Example 8.10 We refer to the three-person zero-sum game:

$$v(i) = -1$$
$$v(12) = v(23) = v(13) = 1.$$

We use Algoritm 8.6.

$$\min\left\{ x_0 \;\middle|\; \begin{array}{rrcl} & x_i & \geq & -1 \quad \text{for } i \in \{1,2,3\} \\ x_0 + x_1 + & x_2 & \geq & 1 \\ x_0 + x_2 + & x_3 & \geq & 1 \\ x_0 + x_1 + & x_3 & \geq & 1 \\ x_1 + x_2 + & x_3 & = & 1 \\ & x_0 & \geq & 0 \end{array} \right\}$$

With this LP-problem we find $X^0 = (1, 0, 0, 0)$. Because X^0 is a unique value, we see that $N(v) = (0, 0, 0)$

Example 8.11 Let's look at the next game:

$$v(1) = v(2) = 0$$
$$v(12) = 1$$

We solve it with Algorithm 8.6.

$$\min\left\{ x_0 \;\middle|\; \begin{array}{rrcl} & x_i & \geq & 0 \quad \text{for } i \in \{1,2\} \\ x_1 + & x_2 & \geq & 1 \\ & x_0 & \geq & 0 \end{array} \right\}$$

With this step we find $X^0 = \{x | x_0 = 0,\ x_1 + x_2 = 1\}$. Thus we go further with the step 2 of the Algorithm:

$$\min\left\{ x_0 \;\middle|\; \begin{array}{rrcl} x_0 + & x_i & \geq & 0 \quad \text{for } i \in \{1,2\} \\ x_1 + & x_2 & = & 1 \end{array} \right\}$$

Now we find $X^1 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Thus $N(v) = (\frac{1}{2}, \frac{1}{2})$.

In this example we see that $N(v)$ is a strict Nash-equilibrium.

8.4 Interpretation of the Nucleolus

As we have seen the Nucleolus is a unique imputation. It can be found with an Algorithm. A question that arises is what the stability of the Nucleolus is? If there is a nonempty Core, the Nucleolus is in a way stable, because it is a weak Nash equilibrium. But what if the Core of the game is empty? What then is the relevance of the sentence "making the least happy coalition as happy as possible". In most situations making "the least happy coalition as happy as possible" is will not be the goal of every player.

Continuation of Example 6.4 If we look at the 3-persons zero-sum game. The Nucleolus is $N(v) = (0, 0, 0)$. If we say now that players 1,2 demand to share their $v(12) = 1$, so $x_1 + x_2 \geq 1$, we see that in terms of dominance every imputation on the line $(a, 1 - a, -1),\ a \in (0, 1)$ dominates $N(v)$ strongly. In fact, every imputation for which two players get more than payoff 0 dominates the nucleolus.

So for the Nucleolus every player has to accept the common goal, they all have to strive for making "the least happy coalition as happy as possible". This will often not be the case in practice.

But we can interpret the Nucleolus differently. We can consider it as an fair allocation based on the coalitions of which players are a member.

Chapter 9

Variants on the Nucleolus

We have just seen that a disadvantage of the Nucleolus is that there is one specific "common goal" involved. In reality, this will often not be the case. As a solution concept, the Nucleolus is often not stable.

Analogue to the Nucleolus we define in this chapter the Weighted Nucleolus: it is a fair allocation based on the coalitions of which players are a member, given a coalition is formed. We do this by minimizing the function $<_l$ over a different area.

Also because the strength of a player is not only dependent on the coalition of which he is a member, I define a alternative version of the Nucleolus. The difference between this alternative version and the Nucleolus is that it is based on excesses not dependent on $v(S)$, but also on $v^*(S)$.

9.1 The Weighted Nucleolus

Definition 9.1 For a game (N, v) the imputation for which x attains its minimum with respect to $<_l$ in the set $A \bigcap \{x| \sum_{i \in S} x_i \geq v(S), \sum_{i \in N \backslash S} x_i \geq v(N \backslash S)\}$ we call the Weighted Nucleolus by coalition S.

Theorem 9.2 *For every game there exists for every coalition S a unique Weighted Nucleolus*

PROOF:

We can use the same arguments as in the proof of Theorem 8.5. Only now we maximize for the nonempty, convex and compact set $A \bigcap \{x| \sum_{i \in S} x_i \geq v(S),\ \sum_{i \in N \backslash S} x_i \geq v(N \backslash S)$.

It is not hard to see that we can find the weighted Nucleolus with Algorithm 8.6. Only we now have to look at a different set for which we maximize x_0:

We replace the first step in the Algorithm by:

$$\min \left\{ x_0 \left| \begin{array}{rrcll} & x_i & \geq & v(i) & \forall i \in N \\ x_0 + \sum_{i \in T} & x_i & \geq & v(T) & \forall T \neq S,\ T \neq N \backslash S \\ \sum_{i \in T} & x_i & \geq & v(T) & \text{for } T = S,\ N \backslash S \\ \sum_{i \in N} & x_i & = & v(N) & \end{array} \right. \right\}$$

With the same arguments as in Theorem 8.7 we see that this Algorithm is finite and finds the unique Weighted nucleolus.

Continuation of Example 6.4 Suppose we want to know the Weighted Nucleolus by coalition $S = (12)$

$$\min \left\{ x_0 \left| \begin{array}{rrcrl} & x_i & \geq & -1 & \text{for } i \in \{1,2,3\} \\ x_i & & \geq & -1 & \text{for } i \in \{1,2,3\} \\ x_1 + & x_2 & \geq & 1 & \\ x_0 + x_2 + & x_3 & \geq & 1 & \\ x_0 + x_1 + & x_3 & \geq & 1 & \\ x_1 + x_2 + & x_3 & = & 1 & \\ & x_0 & \geq & 0 & \end{array} \right. \right\}$$

This LP-problem gives us as $x = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -1)$. Thus the $WN(v) = (\frac{1}{2}, \frac{1}{2}, -1)$ by $S = (12)$ for this problem.

From the algorithm it is also not hard to find that:

Theorem 9.3 *If the Core is nonempty, the Weighted Nucleolus is the Nucleolus*

9.2 Interpretation of the Weighted Nucleolus

The Nucleolus is a particular, not always stable payoff. It is more a way the total payoff can be distributed in a fair way over all the players based on all the coalitions of which he is a member. The Weighted Nucleolus can be interpreted in a similar way, a difference is that it is assumed that the coalitions have already been formed.

Continuation of Example 2.4 We know the game is: $v(F) = 1,\ v(FY) = 2,\ v(FB) = 3$. Now suppose the yoghurt company and the butter company form a cartel. Then, because $v(YB) = 0$, the weighted nucleolus for this game and coalition is the nucleolus which is $(2\frac{1}{2}, \frac{1}{2}, 0)$. It follows that both companies do not benefit if they cooperate.

9.3 Defining the Dual Nucleolus

So the Weighted Nucleolus does not take the possibility of blackmailing into account.

Definition 9.4 Let (N, v) a game. Define (N, v') as

- $v'(S) = v(S) + \frac{v^*(S) - v(S)}{|N|}$ for all S

The imputation for which x attains its minimum with respect to $<_l$ for the game (N, v') in the set A we call the dual nucleolus by coalition S for game (N, v).

Continuation of Example 2.4 We know the game is: $v(F) = 1,\ v(FY) = 2,\ v(FB) = 3$. Again the yoghurt company and the butter company form a cartel. Then, because $v(YB) = 0,\ v^*(BY) = v^*(F) = 2$, we have to solve the next LP-problem:

$$\min \left\{ x_0 \;\middle|\; \begin{array}{rrrcl} x_0+ & x_F+ & x_Y & \geq & 2\frac{1}{3} \\ x_0+ & x_F+ & x_B & \geq & 3 \\ & x_0+ & x_B & \geq & \frac{1}{3} \\ & x_0+ & x_F & \geq & 1\frac{2}{3} \\ x_0+ & x_B+ & x_Y & \geq & \frac{2}{3} \end{array} \right\}$$

Here we see that $(x_F, x_B, x_Y) = (2\frac{1}{3}, \frac{2}{3}, 0)$ is the Dual Nucleolus.

Chapter 10

The Shapley Value

Lloyd S. Shapley[15] proposed a solution-concept with the properties (COCO),(SAN) and (PO). Shapley was looking for a fair allocation of collectively gained profits between cooperating players. The basic criterion is the relative importance of every player in the game and not, as for the Nucleolus, the coalition with the biggest losses. This solution concept is called "The Shapley value" of a cooperative game.

10.1 Definition of the Shapley Value

Lemma 10.1 [4] *There is a unique value $\phi(v)$ which has the properties* COCO, SAN *and* PO

PROOF:

First we look at what these properties mean:

1 SAN: $\phi_i(v) = \phi_j(v)$ if $v(S \bigcup i) = v(S \bigcup j)$ for all $S \subset N \backslash i \bigcup j$ and $i, j \in N$,

2 COCO: $\phi_k(v) = 0$ if $v(S \bigcup k) = v(S)$ for all $S \subset N$ and $k \in N$,

3 COCO: $\phi_i(v) = \phi_i(v_1) + \phi_i(v_2)$ if $v = v_1 + v_2$, we don't ask that in $(N, v) = (N_1 \bigcup N_2, v_1 + v_2)$ that $N_1,\ N_2$ are disjoint.

4 PO: $\sum_{i \in N} \phi_i = v(N)$.

With this in mind we show that there is a unique value which meets these conditions.

First we take a constant $T \subseteq N$ and define the next superadditive characteristic function

$$w_T(S) = \begin{cases} 1 & \text{if} \quad T \subseteq S \\ 0 & \text{else} \end{cases} .$$

For this characteristic function we define:

- $\phi_i(w) = \phi_j(w)$ if $i, j \in T$,
- $\phi_k(w) = 0$ if $k \notin T$,
- $\sum_{i \in T} \phi_i = 1$, thus $\phi_i = \frac{1}{|T|}$ for $i \in T$.

Thus this kind of characteristic function gives a unique ϕ which meets the conditions as in the Theorem.

Now we take the strategic equivalent game cw_T for which

$$cw_T(S) = \begin{cases} c & \text{if} \quad T \subseteq S \\ 0 & \text{else} \end{cases}$$

With the same arguments we take:

- $\phi_i(w) = \phi_j(w)$ if $i, j \in T$,
- $\phi_k(w) = 0$ if $k \notin T$,
- $\sum_{i \in T} \phi_i = c$, thus $\phi_i = \frac{c}{|T|}$ for $i \in T$.

and we have a unique ϕ by a characteristic function which meets the conditions.

We will now show that for every characteristic function v can be written as

$v = \sum_{T \subseteq N} c_T w_T$, for certain constants $c_T,\ T \subseteq N$.

Take $c_\emptyset = 0$. We define, with induction to the number of players $|T|$, $c_T = v(T) - \sum_{S \subset T} v(S)$. Then we have $\{\sum_{T \subseteq N} c_T w_T\}(S) = \sum_{T \subseteq N} c_T w_T(S) = \sum_{T \subseteq S} c_T = c_S + \sum_{T \subset S} c_T = v(S)$.

Because $\phi(v) = \sum_{T \subseteq N} c_T w_T$ and we know that the COCO property holds, we see that $\phi(v)_i = \sum_{T \subseteq N} \phi_i(c_T w_T)$ gives unique values for each player i.

□

Lemma 10.2 $c_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$

PROOF:

We prove it with induction to $|S|$.

For $|S| = 0$ we see $c_S = c_\emptyset = 0$. Thus this is correct.

Suppose it holds for $|S| = n-1$. We know, from Lemma 10.1 that

$c_S = v(S) - \sum_{R \subset S} c_R$. Thus, with the induction hypotheses in mind, we know that

$$\begin{aligned} c_S &= v(S) - \sum_{R \subset S} (\sum_{Q \subseteq R} (-1)^{|R|-|Q|} v(Q)) \\ &= v(S) - \sum_{Q \subset S} (\sum_{Q \subseteq R,\ R \subset S} (-1)^{|R|-|Q|}) v(Q). \end{aligned}$$

If we fix Q, S, we can see that $|Q| \leq |R| < |S|$ and set $|R| = c$ for convenience we know that there can be $\begin{pmatrix} |S| - |Q| \\ c - |Q| \end{pmatrix}$ sets with c players satisfying $R \subset S,\ T \subseteq R$. Thus

$$\begin{aligned} \sum_{Q \subseteq R,\ R \subset S} (-1)^{|R|-|Q|} &= \sum_{c=|Q|}^{|S|-1} \begin{pmatrix} |S| - |Q| \\ c - |Q| \end{pmatrix} (-1)^{c-|Q|} \\ &= \sum_{i=0}^{|S|-|Q|-1} \begin{pmatrix} |S| - |Q| \\ i \end{pmatrix} (-1)^i \\ &= \sum_{i=0}^{|S|-|Q|} \begin{pmatrix} |S| - |Q| \\ i \end{pmatrix} (-1)^i - (-1)^{|S|-|Q|} = (-1)^{|S|-|Q|}. \end{aligned}$$

Here we took $i = c - |Q|$ and we know that the series $\sum_{i=0}^{|S|-|Q|} \binom{|S|-|Q|}{i} (-1)^i = (-1+1)^{|S|-|Q|} = 0$.

Thus we see that $c_S = v(S) + \sum_{Q \subset S} (-1)^{|S|-|Q|} v(Q) = \sum_{Q \subseteq S} (-1)^{|S|-|Q|} v(Q)$ and the induction hypotheses is proven.

□

Theorem 10.3 [4] *For every game v the unique Shapley value for player i is given by* $\phi_i = \sum_{S: i \in S} \frac{(|S|-1)!(n-|S|)!}{n!} [v(S) - v(S \backslash \{i\})]$

PROOF:

From the above Lemma's, we know that the Shapley value is unique, moreover:

$$
\begin{aligned}
\phi_i(v) &= \sum_{T \subseteq N} \phi_i(c_T w_T) = \sum_{T \subseteq N,\ i \in T} \frac{c_T}{|T|} \\
&= \sum_{T \subseteq N,\ i \in T} \frac{\sum_{S \subseteq T} (-1)^{|T|-|S|} v(T)}{|T|} = \sum_{S \subseteq N} \Big(\sum_{S \subseteq T,\ i \in T} \frac{(-1)^{|T|-|S|}}{|T|} \Big) v(S) \\
&= \sum_{S \subseteq N} f_i(S) v(S).
\end{aligned}
$$

Here we have $f_i(S) = \frac{\sum_{S \subseteq T,\ i \in T} (-1)^{|T|-|S|}}{|T|}$.

Choose $i \in N$. Let $T \supset \{i\}$ and choose $S' \subset T \backslash \{i\}$ and $S = S' \bigcup \{i\}$. We see that $f_i(S') = -f_i(S)$, because $1 + |T| - |S| = |T| - |S'|$ and the sets $\{T | S' \subseteq T,\ i \in T\}$ and $\{T | S \subseteq T,\ i \in T\}$ are equal. Thus, by taking the sum over S and $S \backslash \{i\}$, we have that $\phi_i = \sum_{S \subseteq N,\ i \in S} f_i(S)(v(S) - v(S \backslash \{i\}))$.

We have for

$$
\begin{aligned}
f_i(S) &= \frac{\sum_{|T|=|S|}^{n} (-1)^{|T|-|S|} \binom{n-|S|}{|T|-|S|}}{|T|} \\
&= \sum_{|T|=|S|}^{n} (-1)^{|T|-|S|} \binom{n-|S|}{|T|-|S|} \int_0^1 x^{|T|-1} dx \\
&= \int_0^1 \sum_{|T|=|S|}^{n} (-1)^{|T|-|S|} \binom{n-|S|}{|T|-|S|} x^{|T|-1} dx \\
&= \int_0^1 x^{|S|-1} \sum_{|T|=|S|}^{n} (-1)^{|T|-|S|} \binom{n-|S|}{|T|-|S|} x^{|T|-|S|} dx \\
&= \int_0^1 x^{|S|-1} \Big(\sum_{i=0}^{n-|S|} \binom{n-|S|}{i} (-x)^i \Big) dx \\
&= \int_0^1 x^{|S|-1} (1-x)^{n-|S|} dx \\
&= \frac{(|S|-1)!(n-|S|)!}{n!}.
\end{aligned}
$$

This last step can be made because it is the beta-function.

Thus we can see that:

$$\begin{aligned}\phi_i(v) &= \sum_{S\subseteq N,\ i\in S} f_i(S)(v(S) - v(S\backslash\{i\})) \\ &= \sum_{S\subseteq N,\ i\in S} \frac{(|S|-1)!(n-|S|)!}{n!}(v(S) - v(S\backslash\{i\})) \text{ for all } i\end{aligned}$$

.

□

The Shapley Value is a unique value with the combination of properties SAN, COCO and PO. But it does not has the intuitive stability of the imputations in the Core or the Stable Sets.

10.2 Nash Equilibrium vs. Shapley Value

To get a better understanding of the Shapley Value, it can best be compared with a Nash equilibrium. In strict Nash equilibria every coalition is 'happy' with the division of the total gain. A strict Nash equilibrium is therefore a stable outcome. Thus we would like that if there is a strict Nash equilibrium, then a solution is this strict Nash equilibrium or at least we do not want an outcome dominated by it. Now let's look at the next example:

Example 10.4 Take the following game:

1. $v(1) = v(2) = v(3) = 0$
2. $v(12) = 1$, $v(13) = 2$

The Shapley value for this game is:

1. $\phi_1 = \frac{1}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6}$
2. $\phi_2 = \frac{1}{6}$
3. $\phi_3 = \frac{2}{6} + \frac{2}{6}$

But if we look at the core it is $C = \text{span}\ \{(1,0,1),(2,0,0)\}$. If we look at imputation $(\frac{15}{12}, 0, \frac{9}{12})$ we know that this imputation is in the core. Note that it is also in the dual core. So it is a strict Nash equilibrium. Now $x_i > \phi_i$ for the players $i = 1, 3$ and $\sum_{i=1,3} x_i = v(13)$. So the Shapley value in this game is dominated by a strict Nash equilibrium.

From the example follows that the Shapley Value is not a stable outcome.

According to Aumann [1] the image of the Shapley Value that arises from the applications is that it is an index of strength of a player. This strength is based on the strength of the coalitions of which he is a member and of those of which he is not a member. Unlike the Nash equilibrium, the Core and the stable sets, the Shapley Value has no stability on its own.

Chapter 11

Special Classes of Games

In this chapter we look at the behavior of solution concepts in classes for games and special games.

11.1 Composite Games

In real life we know that in a composite game disjoint groups N_1, N_2, in completely different situations, have a choice to cooperate or not to cooperate, i.e. give some kind of utility to the other group. The transfer of utility between disjoint groups is also called *charity*.

Definition 11.1 A game (N, v) is a *composite game* if there are two games (N_1, v_1) and (N_2, v_2) with disjoint player sets N_1, N_2, such that $N_1 \bigcup N_2 = N$, and for which $v(S) = (v_1 + v_2)(S) = v_1(N_1 \bigcap S) + v_2(N_2 \bigcap S)$ for any $S \subset N$.

Because we can find many instances of composite games in the real world, we are interested in the behavior of the solution concepts in these games.

Theorem 11.2 *We can decompose an n-person inessential (0,1)-normalized game in n games with $v_i(j) = 0\ \forall i\ \forall j \in N$, i.e. an n-person inessential game is a composite game of n games.*

PROOF:

Trivially true.

□

Now look at the following example.

Example 11.3 Let $(N \bigcup M, v + w)$ a composite game with $M = \{1, 2\}$ and $N = \{3, 4, 5\}$, and

- $v(i) = 0,\ v(12) = 100$
- $w(i) = 0,\ w(34) = w(35) = w(45) = 80,\ w(345) = 100.$

Lets look at the next set V: $x_1 = 80 - a,\ x_2 = 80 - x_1$ for $0 \leq a \leq 80$ (V_1) and (x_3, x_4, x_5) are union of the three sets $(y, y, 120 - 2y), (y, 120 - 2y, y), (120 - 2y, y, y)$ for $40 \leq y \leq 60$ (V_2). We will show that this set $V = V_1 \times V_2$ is a Strongly Stable Set.

- Internal stability: We see that there is no domination possible over $S = \{345\}$ or $S = \{12\},\ S = \{1\}$ or $S = \{2\}$. So if there is no domination over $S \subset \{345\}$, then V is internally stable. Let x an imputation $x = (120 - 2y, y, y)$ with $40 \leq y \leq 60$. Because $2y > 80$ this imputation can only be dominated over the coalitions $S = \{34\},\ S = \{35\}$. Because an imputation $x' = (120-2y', y', y')$ with $40 \leq y' \leq 60$ has either $x'_1 < x_1,\ x'_2 > x_2,\ x'_3 > x_3$ or $x'_1 > x_1,\ x'_2 < x_2,\ x'_3 < x_3$ there is no internal domination between an x', x as above. So let $\tilde{x} = (\tilde{y}, 120 - 2\tilde{y}, \tilde{y})$ for $40 \leq \tilde{y} \leq 60$. The only possibility is that $x \succ \tilde{x}$ over $S = \{12\}$. Because $\tilde{y} > 120 - 2y$ if and only if $120 - 2\tilde{y} < y$ we have a contradiction. So the set is internal stable.
- External stability: Let x such that $x_1 + x_2 < 80$. This imputation is not in V. And there is a imputation in V with $x'_1 > x_1,\ x'_2 > x_2$ and $x_1 + x_2 = 80$. So there is dominion over $S = \{1, 2\}$.

  Suppose now that for imputation x $x_1 + x_2 > 80$. Again $x \notin V$. Then either $x_3 < 40,\ x_4 < 40$ or $x_5 < 40$. If $x_3 < 40$ and $x_4 < 40$ hold, this imputation is dominated by $(40, 40, 40)$ over the $S = \{34\}$. Because the game is symmetric for the players $1, 2, 3$ this holds for the other possibilities also. Again, because the game is symmetric, we can assume without loss of generality that $x_3 < 40$,$x_4 > 40$ and $x_5 > 40$. Then either $x_3 + x_4 < 80$ or $x_3 + x_5 < 80$. Suppose now $x_3 + x_4 < 80$. Choose an imputation $x' = (120 - 2y', y', y')$ with y' such that $120 - 2y' = x_3 + \epsilon$ for $\epsilon = \frac{x_1+x_2-80}{2} > 0$. Now $x'_4, x'_5 = 60 - \frac{x_3+\epsilon}{2}$. But for imputation x holds that $\min(x_4, x_5) \leq \frac{200-x_1-x_2-x_3}{2} = 60 - \frac{x_3}{2} - 2\epsilon < x'_4,\ x'_5$. So $x' \succ x$ over $S = \{34\}$ or $S = \{35\}$.

  Now lets look at an imputation y with $y_3 + y_4 + y_5 = 120$ but y not in V. We see for $i, j \in \{123\}$ that either

  (1) if y_i, y_j such that $40 \leq y_i, y_j \leq 60$, then $y_i \neq y_j$

  (2) $y_i, y_j < 40$.

  Suppose (2) holds. y is dominated by imputation $(40, 40, 40)$ over $S = \{ij\}$. Now suppose (1) holds. Then $y_k < 40$, for $k \neq j,\ k \neq i$. Without loss of generality we can assume that $k = 3$ and $y_i = \min(y_i, y_j)$ for $i = 4$. Let $y' = (120 - 2u, u, u)$ with $u = \frac{120-y_3-\frac{\epsilon}{2}}{2}$ and $\epsilon = y_5 - y_4 > 0$. Now $y'_4 = \frac{120-y_3-\epsilon}{2} > \frac{120-y_3-\epsilon}{2}$. Thus $y' \succ y$ over $S = \{34\}$.

  Thus we have proven external stability.

Because we saw that in two disjoint games a transfer of money is possible, we can conclude that giving charity is possible in Strongly Stable Sets. Before we prove a Theorem over the possible amount of charity, we first need some definitions. We define the next new game (N, v^q) with:

- $v^q(S) = v(S)$ for $S \subsetneq N$
- $v^q(N) = q$.

Definition 11.4 Let (N, v) and (M, w) games. In the game $(N \bigcup M, v + w)$, a transfer $q - v(N)$, with $q > v(N)$, from M to N is *admissible* if and only if

- $r = v(N) + w(M) - q \geq \sum_{i \in M} v(i)$
- The core of the game v^q has no interior, i.e. there are no imputations x such that $\sum_{i \in S} x_i \geq v(S) + \epsilon$ for $\epsilon > 0$ and all $S \subsetneq N$.

If $q < v(N)$, a transfer of $v(N) - q$ is *admissible* if and only if:

- $q \geq \sum_{i \in N} v(i)$
- the core of w^r, with $r = v(N) + w(M) - q$, has no interior

Theorem 11.5 [9] *Let $q + r = v(N) + w(M)$ as in the definition. Let V, W be Strongly Stable Sets for v^q and w^r. Then $V \times W$ is a Strongly Stable Set for $v+w$ if and only if the associated transfer is admissible.*

PROOF:

We shall assume that $q > v(N)$, otherwise we can use the same argument by looking at $r > w(M)$.

$\Rightarrow$ Assume now that the transfer is not admissible. This is because either:

(a) $r < \sum_{i \in M} w(i)$

(b) the Core of v^q has a non-empty interior

In case (a) we see that w^r has no imputations and so $W, V \times W$ are empty, and that is not stable for $v + w$.

Now assume (b). Thus there is a vector x such that $\sum_{i \in N} x_i = q$ and $\sum_{i \in S} x_i \geq v(S) + \epsilon$ for all $S \subset N$ and some $\epsilon > 0$. Take x' with $x'_i = x_i - \frac{\epsilon}{n}$ and take $y \in W$ with $y'_i = y_i + \frac{\epsilon}{m}$. Now $(x'; y')$ is an imputation of $v + w$. Now $x' \notin V$. Because W is internal stable, also $y' \notin W$. So $(x'; y')$ not in $V \times W$. But this imputation is not dominated:

Let $\tilde{x} \in V$, $\tilde{y} \in W$ such that $(\tilde{x}; \tilde{y}) \succ (x'; y')$. Thus there is an $S \subset M \bigcup N$ such that $\tilde{x}_i > x'_i$, $i \in S \bigcap N$, $\tilde{y}_i > y'_i$, $i \in S \bigcap M$ and $\sum_{S \bigcap N} \tilde{x}_i + \sum_{S \bigcap M} \tilde{y}_i \leq v(S \bigcap N) + w(S \bigcap M)$. But $\sum_{S \bigcap N} \tilde{x}_i \geq v(S \bigcap N)$ for it is in the Core. If $S \bigcap N \neq \emptyset$ it gives $\sum_{S \bigcap N} \tilde{x}_i > v(S \bigcap N)$. Thus $S \bigcap M \neq \emptyset$ and $\sum_{S \bigcap M} \tilde{y}_i \leq w(S \bigcap M)$. But $\tilde{y}_i > y'_i > y_i$ for $i \in S \bigcap M$. And so $\tilde{y} >_S y$ for an $S \bigcap M$, but $\tilde{y}, y \in W$ which gives a contradiction.

$\Leftarrow$ Now assume V, W are Strong Stable for respectively v^q, w^r, and the transfer is admissible. We now have to show that $V \times W$ is stable for $v + w$.

- Internal Stability: Let $(x; y)$, $(x'; y') \in V \times W$. Suppose $(x; y) \succ (x'; y')$. So for an $S \neq \emptyset$ we have that $x_i > x'_i$, $i \in S \bigcap N$, $y_i > y'_i$, $i \in S \bigcap M$ and $\sum_{S \bigcap N} x_i + \sum_{S \bigcap M} y_i \leq v(S \bigcap N) + w(S \bigcap M)$. This means that either $\sum_{S \bigcap N} x_i \leq v(S \bigcap N)$ or $\sum_{S \bigcap M} y_i \leq w(S \bigcap M)$. But this means that either $x \succ x'$ in (N, v) over $S \bigcap N$ or $y \succ y'$ in (M, w) over $S \bigcap M$. Since $\sum_N x_i = \sum_N x'_i$ and $\sum_M y_i = \sum_M y'_i$ we know that $S \bigcup N \neq N$, $S \bigcup M \neq M$. So $x \succ x'$ in v^q or $y \succ y'$ in w^r, which gives us a contradiction.
- External Stability: Suppose now that $(x; y) \in A \backslash V \times W$. There are three possibilities:

  (A) $\sum_{i \in N} x_i = q$, $\sum_{i \in M} y_i = r$

  (B) $\sum_{i \in N} x_i > q$, $\sum_{i \in M} y_i < r$

  (C) $\sum_{i \in N} x_i < q$, $\sum_{i \in M} y_i > r$

  In case (A). Because the assumption we see that either $x \notin V$ or $y \notin W$. Assume $x \notin V$. We know that $\sum_{i \in N} x_i = q$, so there is an imputation $x' \in V$ such that $x' \succ x$ over an $S \subset N$ with $\emptyset \neq S \neq N$. Thus for some $y' \in W$ we have that $(x'; y') \succ (x; y)$ in A through the same coalition S.

  Let's look at (B). Now there exists an $\epsilon > 0$ such that $\sum_{i \in N} x_i = q + \epsilon$, $\sum_{i \in M} y_i = r - \epsilon$. Let y' as follow; $y'_i = y_i + \frac{\epsilon}{m}$, $\forall i \in M$. Either $y' \in W$ or $y' \notin W$:

  - If $y' \in W$ we see that for any $x' \in V$ that $(x'; y') \in V \times W$ and $(x'; y') \succ (x; y)$ over M.

- If $y' \notin W$ we know that there exists an $\tilde{y} \in W$ such that $\tilde{y} \succ y'$ over some $S \subsetneq M$. Over this same S there holds $\tilde{y} \succ y$. With any $\tilde{x} \in V$ we see again that $(\tilde{x}, \tilde{y}) \in V \times W$ and $(\tilde{x}, \tilde{y}) \succ (x, y)$ over this S.

Now for case (C). If $q = v(N)$ we can use the argument of (B), with N and M interchanged. If $q > v(M)$ we define $x' \in v^q$ with $x'_i = x_i + \frac{\epsilon}{n}$, $\forall i \in N$. Because the interior of the core of v^q is empty, there is one $S \subsetneq N$ such that $\sum_{i \in S} x'_i \leq v(S)$. We see that $x' \to x$. Again we know that either $x' \in V$ or $x' \notin V$.

- If $x' \in V$ we see that for any $y' \in W$ that $(x'; y') \in V \times W$. For this $(x'; y')$ we see that $(x'; y') \succ (x; y)$.
- If $x' \notin V$ we know that there is an $\tilde{x} \in V$ such that $\tilde{x} \succ x'$ over some $T \subsetneq M$. Taking random $\tilde{y} \in W$ we see that $(\tilde{x}; \tilde{y}) \in V \times W$. For this $(\tilde{x}; \tilde{y})$ we know $(\tilde{x}; \tilde{y}) \succ (x; y)$.

And thus internal and external stability is proven.

□

We can interpret this Theorem as follows: A coalition S can give charity in a disjoint situation, but this charity does not have to make the disjoint situation in some way *more stable*. Namely, if after the transfer the Core of the disjoint situation gets an interior, in other words, the disjoint game gets a Strict Nash Equilibrium, the transfer is not admissible.

We've seen that Strict Stable Sets act nice on composite games. Because the Nucleolus and the Shapley Value are unique values, it is interesting to look at the differences between these values and the values of the two *sub-games*. We know the Shapley Value[1] doesn't take this into account. Thus, if we again see the Shapley Value as an index of power, this index of power does not change, although the situation is significantly different. This is a disadvantage of the Shapley Value, because in reality the power relation has changed also.

Now look at the following game:

Example 11.6 Let's look at the composite game $(N, v) = (N_1 \bigcup N_2, v_1 + v_2)$, with:

$$
\begin{aligned}
&N_1 = \{1, 2\},\ N_2 = \{3, 4, 5\}, \\
&v_1(1) = v_1(2) = 0,\ v_1(12) = 1, \\
&v_2(3) = v_2(4) = v_2(5) = 0, \\
&v_2(34) = v_2(35) = v_2(45) = v_2(345) = 1.
\end{aligned}
$$

These two games are both symmetric games, so we have as Nucleoli

$N(v_1) = (\frac{1}{2}, \frac{1}{2}),\ N(v_2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for the disjoint games.

We construct the Nucleolus by argumentation. For the Nucleolus of (N, v) we can see that $N(v)_1 = N(v)_2$ and $N(v)_3 = N(v)_4 = N(v)_5$ because the nucleolus is unique and anonymous. We can see that $e(x; i) = 0 - x_i \leq 0$. Also $e(x; v(S)) = 1 - \sum_{i \in S} x_i \geq 0$ for $S = (12), (34), (45), (35)$. We want to minimize the maximum of these, so we see that $x_1 + x_2 = x_3 + x_4 = x_3 + x_5 = x_4 + x_5$.

Thus $N(v) = (\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$.

So we see the Nucleolus can take charity into account. The next theorem gives us when it does and when it does not:

[1] The Shapley Value is defined on the fact that is has property (COCO)

Theorem 11.7 *Let (N_1, v_1), (N_2, v_2) two disjoint games. L If (N_1, v_1) and (N_2, v_2) both have a nonempty Core with Nucleoli $N(v_1)$ and $N(v_2)$, then the Nucleolus of the composed game is $N(v) = (N(v_1); N(v_2))$.*

PROOF:

Suppose x is the Nucleolus and $N(v) = (N(v_1); N(v_2))$. We prove the following two cases:

1. If $\sum_{i \in N_1} x_i > v_1(N_1)$, then $N(v) <_l x$.
2. If $x_i \neq N(v_1)_i$ for an $i \in N_1$ and $\sum_{i \in N_1} x_i = v_1(N_1)$, then $N(v) <_l x$.

Because of Theorem 4.4 we know that (N, v) has a Core. Thus $x \in C(v)$. Because all linear equations stay the same, we see that $\sum_{i \in S} N(v)_i = \sum_{i \in N_1 \bigcap S} N(v_1)_i + \sum_{i \in N_2 \bigcap S} N(v_2)_i \geq v(N_1 \bigcap S) + v(N_2 \bigcap S) = v(S)$. Thus $N(v) \in C(v)$. Thus if $\sum_{i \in N_1} x_i > v_1(N_1)$, then $\sum_{i \in N_2} x_i < v_2(N_2)$. Then $x \notin C(v)$, which gives a contradiction with $x \in C(v)$.

Suppose there is an $x_i \in N_1$ such that $x_i \neq N(v_1)_i$ and $\sum_{i \in N_1} x_i = v_1(N_1)$. Then $\sum_{i \in N_2} x_i = v_2(N_2)$. Thus $x <_l N(v)$, if either $x_1 <_l N(v_1)$ for $x_1 = \{x_i | i \in N_1\}$ or $x_2 <_l N(v_2)$ for $x_2 = \{x_i | i \in N_2\}$. But this is a contradiction with the fact that $N(v_1)$ is the Nucleolus for (N_1, v_1).

□

Theorem 11.8 *Let (N_1, v_1), (N_2, v_2) two disjoint games. Let $N(v_1)$ and $N(v_2)$ the Nucleoli of these games. If (N_1, v_1) has a nonempty Core and (N_2, v_2) has an empty Core, then the Nucleolus $N(v)$ of the composed game $(N_1 \bigcup N_2, v_1 + v_2)$ is $N(v) \neq (N(v_1); N(v_2))$.*

PROOF:

Take $\epsilon = \frac{\min_{S \subset N_2} e(N(v_2); S)}{|N_1| + 1}$. Suppose this minimum is attained in S'. We know $\epsilon < 0$ because $C(v_2) = \emptyset$. Let $x_i = N(v_1)_i - \epsilon$ for all $i \in N_1$. Take $x_i = N(v_2)_i + \frac{|N_1|\epsilon}{|N_2|} > N(v_2)_i$ for all $i \in N_2$.

Then $e((N(v_1); N(v_2)); S') < e((N(v_1); N(v_2)); T)$ for all $T \subset N_1$ and $e((N(v_1); N(v_2)); S') \leq e((N(v_1); N(v_2)); U)$ for all $U \subset N_2$. But $e(x; S') < e((N(v_1); N(v_2)); S')$ and $e(x; U) < e((N(v_1); N(v_2)); U)$ for all $U \subset N_2$. If we look at $e(x; T)$ for a $T \subset N_1$, we see that $e(x; T) < -|T|\epsilon < e(N(v_2); S') < e((N(v_1); N(v_2)); S')$. Thus $x <_l (N(v_1); N(v_2))$. Thus $N(v) \neq (N(v_1); N(v_2))$.

□

In the following example we can see that for a composite game with two empty Cores giving charity and not giving charity can be the case:

Example 11.9 Let (N, v) the game with characteristic function:

$$v(12) = v(13) = v(23) = 1,$$
$$v(45) = v(46) = v(56) = 1,$$
and $v(i) = -1$ for all players.

This game is a composite game of two 3-person zero-sum games, namely $N_1 = \{1,2,3\}$ and $N_2 = \{4,5,6\}$. It has an empty core, because it is a zero-sum game. If we look at the Nucleolus, we see that all players have the same starting position, so $N(v)_i = 0$ for all players i. This is the same as the composition of both the nucleolus of the sub-games. Now look at the game (N, v) with the following characteristic function:

$v(12) = v(13) = v(23) = 1$,
$v(45) = v(46) = v(56) = 1\frac{1}{4}$,
$v(123) = 0,\ v(456) = \frac{1}{4}$,
and $v(i) = -1$ for all players.

Then we see that (N, v) is a composite game of two symmetric games, with $N_1 = \{1,2,3\}$ and $N_2 = \{4,5,6\}$. Thus $N(v_1)_i = 0$ for all players $i \in N_1$ and $N(v_2)_j = \frac{1}{12}$. But $N(v) = (-\frac{1}{48}, -\frac{1}{48}, -\frac{1}{48}, \frac{5}{48}, \frac{5}{48}, \frac{5}{48})$.

Thus we see that the concept of Nucleolus can change. As by Stable sets, the amount of charity is also limited.

11.2 Convex Games

Definition 11.10 We call a game (N, v) convex, if its characteristic function has the following property:
$v(S \bigcup T) + v(S \bigcap T) \geq v(S) + v(T)\ \forall S, T \subseteq N$

This definition means the following. A coalition does not consider to leave a greater coalition, as this new smaller coalition always loses. *The incentives for joining a coalition increases as the coalition grows.*

To see what convexity of a game precisely is, we look at two situations:

Define $m(S,T) = v(S \bigcup T) - v(S) - v(T)$ for disjoint coalitions S, T. Because $m(S,T) = v(S \bigcup T) + v(S \bigcap T) - v(S) - v(T) \geq 0$ for S, T disjoint coalitions, we see that there is not an incentive not to merge all different coalitions.

That the incentive to merge increases as the coalition grows, can be seen with the following theorem:

Theorem 11.11 *For two coalitions S, T for which $S \subseteq T \subseteq N\backslash\{i\}$ holds: $v(S \bigcup \{i\}) - v(S) \leq v(T \bigcup \{i\}) - v(T)$.*

PROOF:

Write

$$\begin{aligned} v(T \bigcup \{i\}) + v(S) &= v(S \bigcup (T \bigcup \{i\})) + v(S \bigcap (T \bigcup \{i\})) \\ &= v((S \bigcup \{i\}) \bigcup T) + v(S \bigcap T) \end{aligned}$$

by definition of T, S.

But $v((S \bigcup \{i\}) \bigcup T) + v((S \bigcup \{i\}) \bigcap T) \geq v(S \bigcup \{i\}) + v(T)$.

Thus $v(T \bigcup \{i\}) + v(S) - v(S) - v(T) \geq v(S \bigcup \{i\}) + v(T) - v(T) - v(S)$.

□

As a consequence convex games can be seen as games in which all players want to form the grand coalition. In this section our goal is to prove a Theorem that convex games have unique strongly stable sets, namely the core. Before we can prove this, we first need some terminology.

Definition 11.12 [17] The hyperplane spanned by cooperation of a coalition S we denote as $H_S(v) = \{x | \sum x_i = v(S)\}$. We call the sets $C_S(v) = C(v) \bigcap H_S(v)$ the Core configuration by subset S. We define $C_\emptyset(v) = C(v)$

Definition 11.13 [17] We say a Core configuration is *regular* if $C_N(v) \neq \emptyset$ and $C_S \bigcap C_T \subseteq C_{S \bigcup T} \bigcap C_{S \bigcap T}$ $\forall S, T \subseteq N$.

Theorem 11.14 [17] *In a regular Core configuration C_S we have $C_{S_1} \bigcap C_{S_2} \bigcap ... \bigcap C_{S_m} \neq \emptyset$ for any increasing sequence $S_1 \subset S_2 \subset ... \subset S_m$.*

In the proof of this theorem, we need some lemma's

Lemma 11.15 [17]*Suppose we have a regular Core configuration. Suppose $S \subset T$ and $|T \backslash S| \geq 2$. Let imputation $a \in C_S(v) \bigcap C_T(v)$, then there exists a coalition Q and an imputation b such that $S \subset Q \subset T$ and $b \in C_S \bigcap C_Q \bigcap C_T$. Moreover, we can require that $b_i = a_i$ $\forall i \in S$, and that $j \in Q$, $k \notin Q$ for any two preassigned elements j, k of $T \backslash S$.*

PROOF:

Fix $j, k \in T \backslash S$. Define $b \in C(v)$ by

$$b_i = \begin{cases} a_i - \rho \text{ for } i = j \\ a_i + \rho \text{ for } i = k \\ a_i \text{ for all other } i \end{cases},$$

with ρ maximal such that $b \in C(v)$. Then $\rho \geq 0$. This can be done because $C(v)$ is compact and contains a. It is clear that $\sum_{i \in S} b_i = \sum_{i \in S} a_i$ and $\sum_{i \in T} b_i = \sum_{i \in T} a_i$ and thus $b \in C_S(v) \bigcap C_T(v)$. Since a larger ρ would have taken b out of the Core, there must be a set of players R, with $j \in R$ and $k \notin R$, such that $\sum_{i \in R} b_i = v(R)$ and thus $b \in C_R(v)$. Let $Q = (S \bigcup R) \bigcap T$. Now $b \in C_S \bigcap C_R \bigcap C_T$. Thus because of the regularity of the Core configuration $b \in C_T \bigcap (C_{S \bigcup R} \bigcap C_{S \bigcap R})$. And thus $b \in C_{T \bigcap (S \bigcup R)} \bigcap C_{T \bigcup S \bigcup R}$ and thus $b \in C_Q$. Now Q has all the required properties. Also: $S \subset Q \subset T$, $j \in Q$ and $k \notin Q$.

□

Lemma 11.16 [17] *Let $S \subset N$, $N \backslash S \geq 2$ and $a \in C_S$. Then for any $j \in N \backslash S$ there exists $b \in C_S \bigcap C_{S \bigcup \{j\}}$ such that $b_i = a_i$ $\forall i \in S$*

PROOF:

We set $T = N$ and we use Lemma 11.15 to find a Q and a b such that $S \bigcup \{j\} \subseteq Q \subset N$ and $b \in C_S \bigcap C_Q$. For b holds $b_i = a_i$ for all $i \in S$. If $Q = S \bigcup \{j\}$ the Lemma holds; if not we can set $T = Q$. We can repeat this argument till $Q = S \bigcup \{j\}$.

□

PROOF of theorem 11.14:

We look at the sequence $\{S_k\}$ with maximum length, thus $m = n+1$. We know by definition that $C_{S_1}(v) \neq \emptyset$ for $S_1 = \emptyset$. Let $a_{(1)} \in C_{S_1}(v)$ and apply Lemma 11.16, for $k = 1, ..., n-1$ to find a point $a_{(k+1)} \in C_{S_k}(v) \bigcap C_{S_{k+1}}(v)$, again with $a_{(k),i} = a_{(k+1),i}$ for $i \in S_k$. Because the imputation $a_{(n)}$ is in all of the $C_{S_k}(v)$ for $1 \leq k \leq n$, and also in $C_{S_{n+1}}(v) = C_N(v) = C(v)$ we know $a_{(n)} \in C_{S_1}(v) \bigcap C_{S_2}(v) \bigcap ... \bigcap C_{S_m}$.

□

Definition 11.17 Let ω represent an ordering of the players; ω is one of the $n!$ functions that maps N onto $\{1, 2, ..., n\}$.

Theorem 11.18 [17] *The vertices of the Core in a regular configuration are precisely the points a^ω. a^ω is the imputation with elements $a_i^\omega = v(S_{\omega,\omega(i)}) - v(S_{\omega,\omega(i)-1})$. Here is $S_{\omega,k} = \{i \in N : \omega(i) \leq k\}$ for $k = 0, 1, ..., n$.*

PROOF:

We know $C_{S_{\omega,1}} \subseteq H_{S_{\omega,1}}$ for any ω, so
$a^\omega \in C_{S_{\omega,1}} \bigcap C_{S_{\omega,2}} \bigcap ... \bigcap C_{S_{\omega,n}} \subseteq H_{S_{\omega,1}} \bigcap H_{S_{\omega,2}} \bigcap ... \bigcap H_{S_{\omega,n}}$. By Theorem 11.14 and the assumption that we have a regular Core configuration, we know the left hand-side is not empty, thus a^ω is in the Core.

Suppose a^ω be an imputation in the Core but not a vertex. Then for some non-zero vector d we have $a^\omega \pm d \in C(v)$. But at least one of the hyper planes $H_{S_{\omega,k}}$ must pass strictly between the points $a^\omega + d$ and $a^\omega - d$ excluding one of them being an imputation in the Core.

Now assume there is another vertex in $C(v)$, not belonging to an ω. Let a be a vertex of $C(v)$ and let $\emptyset = S_1 \subset S_2 \subset ... \subset S_m = N$ be a sequence of members of $\{S | a \in C_S\}$, one that maximizes m.

If $m = n+1$, then $a = a^\omega$ for some ω.

Suppose therefore $m < n+1$, so $|S_{k+1} \backslash S_k| \geq 2$ for some k. Let $i, j \in S_{k+1} \backslash S_k$ and $i \neq j$. Since a is a solution for all of the equations $\sum_{i \in S} a_i = v(S)$, $S \in \{T | a \in C_T\}$ it is a unique solution. But $\{S | a \in C_S\}$ is closed under "$\bigcup$" and "$\bigcap$", so there is a $Q \in \{S | a \in C_S\}$, where $Q = (S_k \bigcup R) \bigcap S_{k+1}$. But in order to determine the coordinates a_i, a_j, and not their sum $a_i + a_j$ (otherwise a is not an imputation in a vertex of the core), there must be an equation that separates them, thus $R \in \{S | a \in C_S\}$ that contains either i or j. Again we can take $Q = (S_k \bigcup R) \bigcap S_{k+1}$. But $S_k \subset Q \subset S_{k+1}$ gives a contradiction with the maximality of m.

□

Theorem 11.19 [17] *The Core of a Convex game is non-empty.*

PROOF:

It suffices to show that a^ω is in the core for some ω. Let $T \subset N$ and let j the "first ω-element" of $N \backslash T$, so j is the element for which all ω-predecessors are in T, but j not. We then have $T \bigcup S_{\omega,\omega(j)} = T \bigcup \{j\}$ and $T \bigcap S_{\omega,\omega(j)} = S_{\omega,\omega(j)-1}$.

Because the game is convex we know that $v(T) + v(S_{\omega,\omega(j)}) \leq v(T \bigcup \{j\}) + v(S_{\omega,\omega(j-1)})$. Also $a_j^\omega \leq v(T \bigcup \{j\}) - v(T)$ by definition of a^ω.

Thus we can see that $\sum_{i \in T} a_i^\omega - v(T) \geq \sum_{i \in T \bigcup \{j\}} a_i^\omega - v(T \bigcup \{j\})$.

If we repeat this argument $|N| - |T| - 1$ times we get
$\sum_{i \in T} a_i^\omega - v(T) \geq \sum_{i \in N} a_i^\omega - v(N) = 0$.
Because we have chosen T arbitrary, $\sum_{i \in T} a^\omega \geq v(T)$ for all T and it is thus in the Core.

$\square$

Theorem 11.20 [17] *A game is convex if and only if its Core configuration is regular.*

PROOF:

- $\Rightarrow$ Suppose v is convex. Then $C(v) \neq \emptyset$. Let $S, T \subseteq N$. Suppose a is in $C_S(v) \bigcap C_T(v)$. Then $v(S \bigcup T) + v(S \bigcap T) \geq v(S) + v(T) = \sum_{i \in S} a_i + \sum_{j \in T} a_j = \sum_{i \in S \bigcup T} a_i + \sum_{j \in S \bigcap T} a_j$. But also $\sum_{i \in S \bigcup T} a_i \geq v(S \bigcup T)$ and $\sum_{i \in S \bigcap T} a_i \geq v(S \bigcap T)$. So the equality holds and $a \in C_{S \bigcup T}(v) \bigcap C_{S \bigcap T}(v)$.
- $\Leftarrow$ Suppose the Core configuration is regular. Let $S, T \subseteq N$. Since $S \bigcap T \subseteq S \bigcup T$ there exists an $a \in C_{S \bigcup T}(v) \bigcup C_{S \bigcap T}(v)$ by Theorem 11.14. Then we have $v(S \bigcup T) + v(S \bigcap T) = \sum_{i \in S \bigcup T} a_i + \sum_{j \in S \bigcap T} a_j = \sum_{i \in S} a_i + \sum_{j \in T} a_j \geq v(S) + v(T)$ and thus v is convex.

$\square$

Theorem 11.21 [17] *The Core of a convex game is a Strongly Stable Set.*

PROOF:

Let $\{C_S\}$ be regular and take an imputation b in the imputation set A but not in $C(v)$. Define: $g(S) = 0$ if $S = \emptyset$ and $g(S) = \frac{v(S) - \sum_{i \in S} b_i}{|S|}$ if $S \subseteq N$.

Let g attain its maximum, g^*, at coalition $S = S^*$. Since $b \notin C(v)$ and it is in A we know that $g^* > 0$ and thus $S^* \neq \emptyset$. We also know that $C_{S^*} \neq \emptyset$. Now let $c \in C_{S^*}$, and define $a \in A$ by:

$a_i = \begin{cases} b_i + g^* \text{ if } i \in S^* \\ c_i \text{ if } i \in N \backslash S^* \end{cases}.$

Now we see that $\sum_{i \in N} a_i = v(S^*) + \sum_{i \in N \backslash S^*} c_i = v(N)$. Thus $a \in A$. Because $\sum_{i \in S^*} a_i = v(S^*)$ and $a_i > b_i$ for all $i \in S^*$, we see that $a \succ b$.

Now we have to show that a is in the Core. Let $T \subseteq N$ arbitrary and let $Q = T \bigcap S^*$ and $R = T \backslash S^*$, then $T = Q \bigcup R$. We have:

$$\begin{aligned}
\sum_{i \in T} a_i &= \sum_{i \in Q} a_i + \sum_{j \in R} a_j \\
&= \sum_{i \in Q} b_i + |Q| g^* + \sum_{j \in R} c_j \\
&\geq \sum_{i \in Q} b_i + |Q| g(Q) + \sum_{k \in T \bigcup S^*} c_k - \sum_{l \in S^*} c_l \\
&\geq v(Q) + v(T \bigcup S^*) - v(S^*) \\
&\geq v(T).
\end{aligned}$$

And thus a is in the Core. And thus all imputations outside the Core are dominated by an imputation inside the Core.

□

Theorem 11.22

- a *The Core is the unique Strongly Stable Set* V *in convex games*
- b *The Core is the unique Strictly Stable Set in* S *convex games*
- c *The Core is the unique Weakly Stable Set* W *in convex games*

PROOF:

Look at all imputations outside $C(v)$ of a convex game. These are strongly dominated, thus excluded from any (Weakly/Strongly/Strictly) Stable Set.

Look at all imputations in $C(v)$ of a convex game. The Core is always a subset of a (Weakly/Strongly/Strictly) Stable Set.

Thus in a convex game, all imputations of the Core is the (Weakly/Strongly/Strictly) Stable Set.

□

Thus we see that for a convex game the Core is the unique Stable Set.

Chapter 12

Three Questions

After publication of [8] the following questions arose:

- Does there always exist a Strongly Stable Set?
- For a game (N, v), does there always exist a convex Strongly Stable Set?
- If there is an empty Core, does there always exist a Strongly Stable Set?

The answer to all these questions is negative. In the next sections we will give counterexamples. We will discuss these questions also for Weakly Stable Sets.

12.1 Game with a Unique Non-Convex Strongly Stable Set

All games discussed until now have a Strongly stable set which is convex. If we know that all games have a convex strongly stable set, finding a set is easier than whether this game has not a convex strongly stable set. The next Example illustrates that some games do not have a convex strongly stable set.

Example 12.1

$v(N) = 4,\ v(1467) = 2$

$v(12) = v(34) = v(56) = v(78) = 1,$

and $v(S) = 0$ for all other $S \subset N$.

Theorem 12.2 [6] *The game of Example 12.1 has a unique, non-convex, Strongly Stable Set.*

PROOF:

We see that $A = \{x | \sum x_i \leq 4,\ x_i \geq 0\ \forall i\}$.

The Core are the imputations for which $C(v) = \{x | x_i + x_{i+1} = 1,\ x_1 + x_4 + x_6 + x_7 \geq 2\}$.

Let $B = \{x \in A | x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1\}$. Every imputation in $A \backslash B$ is dominated. Suppose for $x \in A \backslash B$ that $x_1 + x_2 < 1$. This imputation is dominated:

- If either $x_1 < 0$ or $x_2 < 0$ we see it is dominated over a one-person coalition.
- If $x_1, x_2 > 0$ we see that a convex combination of C_1 and C_2 dominates x over the 2-person coalition $S = 12$ for $C_1 = (1,0,0,1,0,1,1,0)$, $C_2 = (0,1,0,1,0,1,1,0)$.

Now we take $F = \{x \in B | \sum_{i=1,4,6,7} x_i < 2,\ x_1, x_4, x_6, x_7 < 1\}$. Every imputation in F is dominated over $S = 1467$ by an imputation in the Core. We will see that for every convex combination $x_1 + x_4 + x_6 + x_7 = 2$, $x_i > 0$ there is an $x \in C(v)$. This is, because all next imputations are imputations in the Core, and the Core is a convex set:

$C_3 = (1,0,0,1,1,0,0,1)$, $C_4 = (1,0,1,0,1,0,1,0)$,
$C_5 = (1,0,1,0,0,1,0,1)$, $C_6 = (0,1,0,1,0,1,0,1)$,
$C_7 = (0,1,0,1,1,0,1,0)$ and $C_8 = (0,1,1,0,0,1,1,0)$.

If we look at an imputation $x \in B \backslash S$ we see that $x_i = 1$ for a player $i \in 1,4,6,7$ and $\sum_{i \in S} x_i \geq v(S)$ and it is not dominated by an imputation in the Core. But there are no two different imputations x, y in $B \backslash F$ for which $y \succ x$: Let for $x \in B \backslash F$ $x_i = 1$ for $i = 1$. Then $y \succ x$ if $y_i > 1$. But then $y \notin B$, so $y \notin B \backslash F$. The same argument holds for $i = 4, 6, 7$.

So the unique Strongly Stable Set for this game is $C \bigcup (B \backslash F)$.

To see that this Stable Set is not convex it is enough to look at the imputations $b^1 = (1,0,1,0,1,0,0,1)$, $b^2 = (0,1,0,1,1,0,0,1)$ and $b^3 = (0,1,1,0,0,1,0,1)$. These imputations are in the solution set, but a linear combination of imputations in the solution set are not. As an example we take all three imputations with weight $\frac{1}{3}$, so $b = \frac{1}{3}b^1 + \frac{1}{3}b^2 + \frac{1}{3}b^3 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, 0, 1)$. Now b is a linear combination of imputations in the solution set, but it is not in the solution set. It is dominated by an imputation in the Core.

□

Theorem 12.3 *The Core is a convex Weakly Stable Set for this game.*

PROOF:

As we can see from the proof of Theorem 12.2, we know that imputations in $x \in B \backslash C(v)$ are not strongly dominated by the Core, because for one player $x_i = 1$. But we can use by the same argument as in the proof of Theorem 12.2, that span$\{C_3, ..., C_8\} \subset C(v)$ for the $c_3, ..., C_8$ as defined in that proof. We see that for every $x \in B \backslash C(v)$ is weakly dominated by an imputation of the Core.

Because every imputation in $A \backslash B$ is strongly dominated by an imputation of the Core, every imputation outside the Core is weakly dominated by the Core.

Because the Core is convex, we see that this game has a convex, Weakly Stable Set.

□

Theorem 12.4 *The game of Example 12.1 does not have a Strictly Stable Set.*

PROOF:

For this game, all imputations not strongly dominated by the Core are in the Strongly Stable Set (see the proof of Theorem 12.2). All imputations in the Strongly Stable Set but outside the Core are weakly dominated by an imputation in the Core (see the proof of Theorem 12.3). So this game has a Core, but not a Strictly Stable Set.

□

12.2 Game With no Strongly Stable Set

After publication of [8] in 1944 an important question for game-theorists was "does a Strongly Stable Set for every game always exist?". In 1964 W.F. Lucas [5] gave the next counterexample which answered this question in the negative.

He gave the following *10-persons* game (N, v) as a counterexample:

Example 12.5

$N = \{1, ..., 9, 0\}$,

$v(N) = 5,\ v(13579) = 4$,

$v(12) = v(34) = v(56) = v(78) = v(90) = 1$,

$v(357) = v(157) = v(137) = 2$,

$v(359) = v(159) = v(139) = 2$,

$v(3579) = v(1579) = v(1379) = 3$,

$v(1479) = v(3679) = v(5279) = 2$,

and $v(S) = 0$ for all other $S \subset N$

Theorem 12.6 [5] *The game of Example 12.5 has no Strongly Stable Set.*

PROOF:

We know that $A = \{x | \sum_{i \in N} x_i \leq 5, x_i \geq 0\ \forall i \in N\}$.

Let's look at the imputations C_1, C_2, C_3, C_4, C_5 and C_6, with

$$\begin{array}{rcl} C_1 & = & (1,0,1,0,1,0,1,0,1,0) \\ C_2 & = & (0,1,1,0,1,0,1,0,1,0) \\ C_3 & = & (1,0,0,1,1,0,1,0,1,0) \\ C_4 & = & (1,0,1,0,0,1,1,0,1,0) \\ C_5 & = & (1,0,1,0,1,0,0,1,1,0) \\ C_6 & = & (1,0,1,0,1,0,1,0,0,1) \end{array} \tag{12.1}$$

. After applying step 1 of Algorithm 8.6 we find that the Core is the convex hull of of $C_k,\ k = 1, 2, 3, 4, 5, 6$.

Now we define the next sets in A.

$$\begin{array}{rl} B = & \{x \in A | x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = x_9 + x_0 = 1\} \\ E_i = & \{x \in B | x_j = x_k = 1, x_i < 1, x_7 + x_9 < 1\} \\ E = & E_1 \bigcup E_3 \bigcup E_5 \\ F_{jk} = & \bigcup_{j,k} \{x \in B | x_j = x_k = 1, x_7 + x_9 \geq 1\} \setminus C(v) \end{array}$$

$$\begin{aligned}
F_p = \quad & \{x \in B | x_p = 1, x_q < 1, x_3 + x_5 + x_q \geq 2, \\
& x_1 + x_5 + x_q \geq 2, x_3 + x_5 + x_q \geq 2\} \setminus C(v) \\
F_{79} = \quad & \{x \in B | x_7 = x_9 = 1\} \setminus C(v) \\
F_{135} = \quad & \{x \in B | x_1 = x_3 = x_5 = 1\} \setminus C(v) \\
F \quad & = F_{15} \bigcup F_{13} \bigcup F_{35} \bigcup F_7 \bigcup F_9 \bigcup F_{79} \bigcup F_{135}
\end{aligned}$$

for $(i, j, k) = (1, 3, 5),\ (3, 5, 1)$, or $(5, 1, 3)$ and $(p, q) = (7, 9),\ (9, 7)$

It is sufficient to show that the following properties hold:

(1) $\{A \setminus B, B \setminus (C(v) \bigcup E \bigcup F), C(v), E, F\}$ is a partition of A

(2) Dom $C(v) \supset [A \setminus B] \bigcup [B \setminus (C(v) \bigcup E \bigcup F)]$

(3) $E \bigcap$ Dom $(C(v) \bigcup F) = \emptyset$

(4) There is no Strongly Stable Set for E.

then we know that this game has no Strongly Stable Set:

If (1) and (2) hold, we see that $V \subset C(v) \bigcup E \bigcup F$. If (3) holds we know that imputations in E are imputations in the Strongly Stable Set, or dominated by an imputation in E. With (4) we see that this does not hold, and we see that this game does not have a Strongly Stable Set.

Lemma 12.7 $\{A \setminus B, B \setminus (C(v) \bigcup E \bigcup F), C(v), E, F\}$ *is a partition of* A.

PROOF:

If we show that the intersections $E \bigcap F = E \bigcap C(v) = F \bigcap C(v) = \emptyset$, we know the Lemma holds.

Note that $C(v) \subset B$ and the intersection $F \bigcap C(v) = \emptyset$ by definition of F.

Now we compare imputations in F and E. For imputations in F we know that either $x_1 + x_3 + x_5 = 3$ or $x_7 + x_9 \geq 1$. For imputations in E we know that $x_1 + x_3 + x_5 < 3$ and $x_7 + x_9 < 1$. So $E \bigcap F = \emptyset$.

For an element of the Core we always have $x_7 + x_9 \geq 1$. So $E \bigcap C(v) = \emptyset$.

Hence (1) holds.

Lemma 12.8 *Dom* $C(v) \supset [A \setminus B] \bigcup B \setminus (C(v) \bigcup E \bigcup F)$:

PROOF:

An imputation x from $[A \setminus B]$ has at least one pair x_i, x_{i+1} with $i \in \{1, 3, 5, 7, 9\}$ for which $x_i + x_{i+1} < 1$. We know for imputations that $x_i, x_{i,i+1} > 0$. Suppose this holds for $x_1 + x_2 < 1$. Then we can find an x' as linear combination of C_1, C_2 as we defined in equations (12.1) for which $x' \succ x$.

Now let $x \in B \setminus (C(v) \bigcup E \bigcup F)$. Then from the previous

(1) $0 \leq x_i \leq 1\ \forall i$.

(2) $x_1 + x_3 + x_5 + x_7 + x_9 < 4$, because otherwise all conditions of the Core are satisfied,

(3) $x_j < 1,\ x_k < 1$ for at least one pair $(jk) = (13),\ (15),\ (35)$, because otherwise it is in E,

(4) $x_q < 1$,

and either for $(p, q) \in \{(7, 9), (9, 7)\}$

(5a) $x_p < 1$

or

(5b) $x_p = 1$ and at least one of the following holds

(5b1) $x_3 + x_5 + x_q < 2$

(5b2) $x_1 + x_5 + x_q < 2$

(5b3) $x_1 + x_3 + x_q < 2$

Now suppose (5a) holds, so $x_7 < 1,\ x_9 < 1$ and $x_1 + x_3 + x_5 + x_7 + x_9 < 4$:

- If $x_1 < 1,\ x_3 < 1$ and $x_5 < 1$ then we can find a $x' \in C(v)$ such that $x'_i > x_i$ for all $i \in \{1, 3, 5, 7, 9\}$ and $\sum x'_i = 4$.
- If $x_1 < 1,\ x_3 < 1$ and $x_5 = 1$ we have $x_1+x_3+x_5+x_7+x_9 = x_1+x_3+1+x_7+x_9 < 4$ so $x_1 + x_3 + x_7 + x_9 < 3$. Now we can find a $x' \in C(v)$ such that $x'_i > x_i$ for $i \in \{1, 3, 7, 9\}$ and $\sum_{i\in\{1,3,7,9\}} x'_i = 3$. So $x' \succ x$.

Now suppose (5b) holds, so $x_j + x_k + x_q < 2$:

Suppose $x_j < 1,\ x_k < 1$ and $x_q < 1$, then we can find a $x' \in C(v)$ such that $x' \succ x$ over $S = \{jkq\}$.

Suppose $x_i < 1,\ x_j < 1$ and $x_k = 1$. Because $x_i < x_k$ we have $x_i + x_j + x_q < 2$ we can find a x' such that $x' \succ x$ over $S = \{ijq\}$.

Lemma 12.9 *$E \bigcap Dom\ (C(v) \bigcup F) = \emptyset$.*

PROOF:

Let's look at an imputation x from E_1. Because $x_3 = x_5 = 1$ and all imputations outside B are dominated by the Core, this imputation can only be dominated over an $S \subset \{1, 2, 4, 6, 7, 8, 9, 0\}$. There can also be no domination over a coalition S with $v(S) = 0$ or over coalitions $S = \{1, 2\},\ S = \{7, 8\}$ or $S = \{9, 0\}$. So x can only be dominated over coalition $S = \{1, 4, 7, 9\}$.

Suppose imputation $x' \in C(v)$ and $x' \succ x$. We must have $x'_1+x'_4+x'_7+x'_9 = 2$. But since $x'_4 > 0$ and $x'_3+x'_4 = 1$ implies $x'_3+x'_5 < 2$, we now have that $x'_1+x'_3+x'_5+x'_7+x'_9 < 4$ which contradicts that $x' \in C(v)$.

Suppose imputation $x' \in F$ and $x' \succ x$. We know that for x' the equations $x'_1 + x'_4 + x'_7 + x'_9 \leq 2$ and $x'_4 > 0$ hold.

If $x' \in F_{35}$ then $x'_4 = 0$ which is a contradiction,

If $x' \in F_{15} \bigcup F_{13}$ we see that if $x'_4 > 0$ then $2 \leq x'_1+x'_7+x'_9 < x'_1+x'_4+x'_7+x'_9 \leq 2$ this is a contradiction,

If $x' \in F_7 \bigcup F_9$ we see that $2 \leq x'_1 + x'_7 + x'_9$ and with same argument as above we again find a contradiction,

If $x' \in F_{79}$ we see again that $2 \leq x'_1 + x'_7 + x'_9$,

If $x' \in F_{135}$ we know that $x'_4 = 0$.

We can use the same arguments for $x \in E_3,\ x \in E_5$.

Lemma 12.10 *There is no Strongly Stable Set for E.*

PROOF:

In the proof of the previous Lemma we have seen that $E_1 \bigcap \mathrm{Dom}_S E = \emptyset$ for all S except $S = \{1, 4, 7, 9\}$. Now lets look at $x \in E_1$. Let $x' \in E_1 \bigcup E_5$, then we see that $x'_4 = 0$ and $x' \not\succ x$, thus (Dom $E_1 \bigcup E_5) \bigcap (E_1 \bigcup E_5) = \emptyset$. So either $x \in V$ or there is an $\tilde{x} \in E_3 \bigcap V$ such that $\tilde{x} \succ x$. Assume $x \in E_1$. We can assume that $x \in V$. Take an $y \in E_3$ such that $y \succ x$. Because we have internal stability $y \notin V$. So there is a $z \in V \bigcap E_5$ such that $z \succ x$. Take a $w \in E_1$ such that $w \succ z$. Internal stability gives us that $w \notin V$. Now there is a $u \in E_3 \bigcap V$ such that $u \succ w$. But we have:

- $u_1 = 1 > x_1$, because $u \in E_3$ and $x \in E_1$
- $u_4 > 0 = x_4$, because $u \in E_3 \subset B$ we know that $u_3 < 1$, this implies $u_4 > 0$. $x \in E_1$ gives $x_3 = 1$ which implies $x_4 = 0$.
- $u_7 > w_7 > z_7 > y_7 > x_7$, because $u \succ w \succ z \succ y \succ x$ over coalitions with players 7 and 9.
- $u_9 > w_9 > z_9 > y_9 > x_9$, because $u \succ w \succ z \succ y \succ x$ over coalitions with players 7 and 9.

and, because $u \succ w$ over $S = \{1, 4, 7, 9\}$ we know that:

$$u_1 + u_4 + u_7 + u_9 \leq 2.$$

Thus we see that $u \succ x$ over $S = \{1, 4, 7, 9\}$. But both $x, u \in V$. This gives a contradiction. Thus $x \notin V$. We can use the same arguments imputations $\tilde{x} \in E_3$ and $\tilde{x}' \in E_5$. Thus none of the imputations in E can be in V. So there is no Strongly Stable Set for E.

And so there is no Strongly Stable Set for this game.

$\square$

Theorem 12.11 *The Core is a Weakly Stable Set.*

PROOF:

We show that all imputations outside the Core are weakly dominated by an imputation in the Core. From the previous proof we know that all imputations in $A \setminus B$ are strongly dominated by an imputation in $C(v)$. Thus we know that for an imputation $x \in B \setminus C(v)$:

- $0 \leq x_i \leq 1$ for all i,
- $\sum_{i \in S} x_i < v(S)$ for at least one S.

Thus it is sufficient to proof that for every pair x, S for which the above hold, there is an imputation in $x' \in C(v)$ for which $x' \succeq x$.

Remember that the Core is a superset of all imputations which are a convex combination of C_1, C_2, C_3, C_4, C_5 and C_6 as in equations 12.1.

- Look at x and $S = \{1, 2\}$. Then if $\sum_{i \in 1,2} x_i < 1$ we see $x \in A \backslash B$. As we have seen in the previous proof, this x is dominated by an imputation in the Core.
- We can use the same argument for the next coalitions:
  - $S = \{3, 4\}$, $S = \{5, 6\}$,
  - $S = \{7, 8\}$ or $S = \{9, 0\}$.
- Look at the pair x, $S = \{1, 3, 5, 7, 9\}$. Then if $\sum_{i \in 1,3,5,7,9} x_i < 4$, we can find an x' as a convex combination of C_2, C_3, C_4, C_5 and C_6 for which $\sum_{i \in 1,3,5,7,9} x' = 4$ and $x'_i \geq x_i$ for all $i \in S$. For this x' we see that $x' \succeq x$.
- Look at x, $S = \{3, 5, 7\}$. If $\sum_{i \in 3,5,7} x_i < 2$, we can create an x' as a convex combination of C_3, C_4, C_5 and for which $\sum_{i \in 1,3,5,7,9} x' = 4$ and $x'_i \geq x_i$ for all $i \in S$. Thus we see that $x' \succeq x$.
- We can use a similar argument for the next coalitions:
  - $S = \{1, 5, 7\}$, $S = \{1, 3, 7\}$,

  - $S = \{3,5,9\}$, $S = \{1,5,9\}$ or $S = \{1,3,9\}$.
- Look at x and coalition $S = \{3,5,7,9\}$. If $\sum_{i \in 3,5,7,9} x_i < 3$, we can create an x' as a convex combination of C_3, C_4, C_5 and C_6 for which $\sum_{i \in 1,3,5,7,9} x' = 4$ and $x'_i \geq x_i$. Thus for this $x' \succeq x$.
- We can use similar arguments for the next coalitions:
  - $S = \{1,5,7,9\}$ or $S = \{1,3,7,9\}$.
- Look at x, $S = \{1,4,7,9\}$. Then if $\sum_{i \in 1,4,7,9} x_i < 2$ implies, in combination with $0 \leq x_3, x_4 \leq 1$ that $\sum_{i \in 1,3,7,9} x_i < 3 = v(1379)$ and we can use the same argument as we did for this coalition.
- We can use similar argumentation for the next coalitions:
  - $S = \{3,6,7,9\}$ or $S = \{5,2,7,9\}$
- If we look at all other combinations of x and a T, such that $\sum_{i \in T} x_i < v(T)$ we see that either $v(T) = 0$, which implies that $x \notin A$, or it implies that there exists an $S \subset T$ such that $\sum_{i \in S} x_i < v(S)$ for an S as above.

Thus we see that if for an imputation x there is an S such that $\sum_{i \in S} x_i < v(S)$, then there is an $x' \in C(v)$ such that $x' \succeq x$.

Thus every imputation outside the Core is weakly dominated by an imputation inside the Core. Thus the Core is a Convex Weakly Stable Set for this game.

□

12.3 The Game with No Strongly Stable Set and an Empty Core

The Core is a nonempty Partially Stable Set. A Strongly Stable Set is also nonempty and Partially Stable. So in the previous example there is some stability as defined by Von Neumann and Morgenstern. Starting from the game in the previous section Lucas and Rabie [7] constructed a game with an empty Core and no Strongly Stable Set.

Example 12.12 *Take (N_β, v_β), with $N_\beta = \{1, ..., 9, 0\}$ and:*

$$v_\beta(1479) = v_\beta(2579) = v_\beta(3679) = 2 - \gamma,$$
$$v_\beta(N_\beta) = \beta,\ v_\beta(13579) = 0,$$
$$v_\beta(12) = v_\beta(34) = v_\beta(56) = v_\beta(78) = v_\beta(90) = 1,$$
$$v_\beta(1379) = v_\beta(1579) = v_\beta(3579) = 3,$$
$$v_\beta(137) = v_\beta(157) = v_\beta(357) = 2,$$
$$v_\beta(139) = v_\beta(159) = v_\beta(359) = 2,$$
$$v_\beta(S) = 0 \text{ for all other } S \text{ and with } \beta, \gamma \text{ later defined.}$$

Take (N', v') with $N' = \{A, B, C, D\}$ as follows:

$$v'(AB) = v'(AC) = v'(BC) = 2,$$
$$v'(ABCD) = 3,$$

$v'(S) = 0$ for all other S.

From these two games we construct the following game (N, v)

$v(N_\beta \bigcup \{D\}) = 5 + \alpha$,
$v(S) = v_{\beta_0}(N_\beta \bigcap S) + v'(N' \bigcap S)$ for all other $S \subset N$,

with $\alpha < \gamma < \frac{2}{3}$ and $\beta_0 = 5$.

Theorem 12.13 [7] *This game (N, v) has an empty Core and no Strongly Stable Set.*

This game has a Weakly Stable Set.

We prove the first part of Theorem 12.13 by showing that a game satisfying the next conditions does have an empty Core and no Strongly Stable Set:

Look at any three games (N_β, v_β), (N', v') and $(N, v) = (N_\beta \bigcup N', v)$, $N_\beta \bigcap N' = \emptyset$. Only the value $v_\beta(N_\beta)$ depends on β. And those games have the following properties:

1. $C(v_\beta) \neq \emptyset$ if and only if $\beta \geq \beta_0$ for a given β_0
2. $C(v') \neq \emptyset$ and $C(v') \bigcap \{x | x \in A,\ x_k > v(k)\} = \emptyset$ for some player $k \in N'$.
3. $v'(N') - v'(N' - \{k\}) - v'(k) > \alpha > 0$ for this one player k.
4. The characteristic function of (N, v) is given by
   $v(N_\beta \bigcup \{k\}) = \beta_0 + \alpha + v'(k)$
   $v(S) = v_{\beta_0}(S \bigcap N_\beta) + v'(S \bigcap N')$ for all other $S \subset N$

We prove this in the next theorem.

Theorem 12.14 [7] *The composed game (N, v) satisfying the above conditions has an empty Core.*

If (N_β, v_β) has no (N_β, v_β) has no Strongly Stable Set for every $\beta \in [\beta_0, \beta_0 + \alpha]$, then (N, v) has no Strongly Stable Set.

PROOF:

We will first prove this game has an empty Core.

Assume that the Core is nonempty. Let $x \in C(v)$. Note that $\sum_{i \in N_\beta} x_i \geq \beta_0$ and $\sum_{i \in N'} x_i \geq v(N')$ imply that $\sum_{i \in N_\beta} x_i = \beta_0$ and $\sum_{i \in N'} x_i = v(N')$. This implies that also $x_k = v(k)$. But for an imputation $x \in C(v)$ we know that $\sum_{i \in N_\beta \bigcup \{k\}} x_i \geq v(N_\beta \bigcup \{k\}) > v(N_\beta) + v(k)$ must hold, thus we have a contradiction. Thus this game has an empty Core.

Now we prove that this game has no Strongly Stable Set, given that each game (N_β, v_β) has no Strongly Stable Set V_β when $\beta_0 \leq \beta \leq \beta_0 + \alpha$.

It is sufficient to show that if V is a Strongly Stable Set for (N, v), then V_β is a Strongly Stable Set for (N_β, v_β).

So assume V is a stable set for (N, v).

First we define $m = \max\{\sum_{i \in N_\beta} x_i | x \in V\}$. Because V is a closed and bounded set, this value exists. We will show later that $\beta_0 \leq m \leq \beta_0 + \alpha$.

Define the induced Strongly Stable Set

$V_m = A(v_m)\bigcap\{(x_{1_\beta},...,x_{n_\beta})|x_{1_\beta},...,x_{n_\beta},x_{1'},...,x_{n'}) \in V\}$ with

$A(v_m)$ the set of imputations for the game (N_β, v_m) with

$v_m(S) = \begin{cases} v_\beta(S) \text{ if } S \subset N \\ m \text{ if } S = N \end{cases}$. We now show if V is a Strongly Stable Set, then V_m is a Strongly Stable Set for the game (N_β, v_m) with imputation set $A(v_m)$.

We first prove that if V is a Strongly Stable Set for (N, v), then V_m is a Strongly Stable Set for (N_β, v_m). This is the same as $V_m \bigcup \text{Dom} V_m = A(v_m)$. Assume this is not the case. Then there exist an

$x^0 \in A(v_m)\backslash(V_m \bigcup \text{ Dom}V_m)$ for which there is no $z \in V_m$ such that $z \succ x^0$ over an $S \subset N_\beta$ and $\sum_{i\in N_\beta} z_i = m$. Now extend x^0 to an $x \in A(v)$, with $x_i = \begin{cases} x_i^0 \text{ for all } i \in N_\beta \\ y_i \text{ for all } i \in N' \end{cases}$ for some $y \in V$.

By definition of V_m we see for this x that $x \notin V$. Thus there is a $z \in V$ such that $z \succ x$ over some $S \subset N$. If $N_\beta \subset S$ we see that $\sum_{N_\beta} z_i > \sum_{N_\beta} x_i = \sum_{N_\beta} x_i^0 = m$ which gives a contradiction with the definition of m. Thus also $S \neq N_\beta$.

If $S \bigcap N' \neq \emptyset$, then if $\sum_{S\bigcap N'} z_i \leq v(S \bigcap N')$ we see that $z \succ y$ over $S \bigcap N'$, which contradicts $y, z \in V$. Thus $\sum_{S\bigcap N'} z_i > v(S\bigcap N')$, which implies $\sum_{S\bigcap N_\beta} z_i < v(S \bigcap N_\beta)$. Thus $z \succ x$ over an $S \subset N_\beta$ for which $S \neq N_\beta$.

So we can choose some $l \in N_\beta \backslash S$ and take $w \in A(v)$ with

$$w_i = \begin{cases} x_i \text{ for all } i \in N', \\ z_i \text{ for all } i \in N_\beta\backslash\{l\}, \\ z_i + m - \sum_{i\in N_\beta} z_i \text{ for } i = l. \end{cases}.$$

Then $\sum_{i\in N_\beta} w_i = m$ and $w \succ x$ over $S \subset N_\beta$. Because $x^0 \in \text{Dom } w$ we see that $w \notin V$. Thus there is a $u \in V$ such that $u \succ w$ over some $T \subset N$. We can show that if $T \bigcap N' \neq \emptyset$ and $T \bigcap N_\beta \neq \emptyset$ then there exists an $T' \subset N_\beta$ or a $\tilde{T} \subset N'$ such that $u \succ w$ over T' or $\tilde{T}$:

- If $T \bigcap N' \neq \emptyset$, $T \bigcap N_\beta \neq \emptyset$ and $T \neq N_\beta \bigcup\{k\}$ we see that $v(S) = v_\beta(S \bigcap N_\beta) + v'(S \bigcap N')$. If $\sum_{i\in S\bigcap N'} u_i \geq v(S \bigcap N')$, then we must have $\sum_{i\in S\bigcap N_\beta} u_i \leq v(S \bigcap N_\beta)$ and so $u \succ w$ over $S \bigcap N_\beta$. If $\sum_{i\in S\bigcap N'} u_i < v(S \bigcap N')$ then $u \succ w$ over $S \bigcap N'$.
- If $T = N \bigcup\{k\}$ we know that $u_i > w_i$ for all $i \in N_\beta$. So $\sum_{i\in N_\beta} u_i > \sum_{i\in N_\beta} w_i = m$ which gives a contradiction with the definition of m.

So we can consider the following two cases:

1. If $T \subset N_\beta$, then $u \succ z$, which is a contradiction with the fact that $u, z \in V$,
2. If $T \subset N'$, then $u \succ y$, which is a contradiction with the fact that $u, y \in V$.

Thus does not exists an $x^0 \in A(v_m)\backslash(V_m \bigcup \text{Dom } V_m)$. Thus V_m has to be stable for $A(v_m)$.

Now we prove that $\beta_0 \leq m \leq \beta_0 + \alpha$.

Suppose $m > \beta_0 + \alpha$. Take an $x \in V$. Thus $\sum_{i\in N_\beta} x_i = m > \beta_0 + \alpha$. Let $y^0 \in C(v_{\beta_0+\alpha})$. Take $y \in A$ such that $y_i = \begin{cases} y_i^0 \text{ for all } i \in N_\beta \\ x_i + \frac{\epsilon}{|N'|} \text{ for all } i \in N' \end{cases}$ Here we take $\epsilon = m - \beta_0 - \alpha > 0$. We see that $y \succ x$ over N'. Thus there is a $w \in V$ such that $w \succ y$.

Thus $w \succ y$ over an S, we have four possibilities:

1. If $S \subset N_\beta$, then $\sum_S y_i \geq v(S)$ by assumption and domination is impossible.

2. Suppose $S = N_\beta \bigcup \{k\}$. Then $\sum_S y_i = \sum_{N_\beta} y_i^0 + y_k = \beta_0 + \alpha + y_k \geq \beta_0 + \alpha + v(k) = v(N_\beta \bigcup \{k\})$ and thus domination over this subset is not possible.
3. If $S \subset N'$, then also $w \succ x$, which gives a contradiction with $w, x \in V$. Thus $m \leq \beta_0 + \alpha$.
4. Suppose $S \neq N_\beta \bigcup \{k\}$, $S \bigcap N' \neq \emptyset$ and $S \bigcap N_\beta \neq \emptyset$. Then $v(S) = v_{\beta_0}(N_{\beta_0} \bigcap S) + v'(N' \bigcap S)$. But $\sum_{i \in S \bigcap N_\beta} w_i > \sum_{i \in S \bigcap N_\beta} y_i = \sum_{i \in S \bigcap N} y_i^0 \geq v_{\beta_0}(S \bigcap N) = v(S \bigcap N)$. This implies $\sum_{S \bigcap N'} w_i < v'(S \bigcap N') = v(S \bigcap N')$, and thus $w \succ x$ over $S \bigcap N'$. This is a contradiction with $x, w \in V$.

Now suppose $m < \beta_0$. Again take $x \in V$ with $\sum_{i \in N_\beta} x_i = m$. Now take $y' \in C(v')$ and an $y \in A$ such that $y_i = \begin{cases} y_i^0 \text{ for all } i \in N' \\ x_i + \epsilon \text{ for all } i \in N_\beta \end{cases}$. We take $\epsilon = \frac{\beta_0 - m}{|N_\beta|} > 0$. Thus $y \succ x$ over N_β. Thus there is a $w \in V$ such that $w \succ y$ over an S. We have four possibilities:

1. If $S = N_\beta \bigcup \{k\}$. Then $\sum_{i \in N_\beta} w_i > \sum_{i \in N_\beta} y_i > \sum_{i \in N_\beta} x_i = m$ which gives a contradiction with the definition of m.
2. If $S \subset N_\beta$ we see that $w \succ y \succ x$ over this S, which gives a contradiction with $w, x \in V$.
3. If $S \subset N'$, then $\sum_{i \in S} w_i > \sum_{i \in S} y_i = \sum_{i \in S} y_i' \geq v'(S)$. Thus $w \not\succ y$ over this S.
4. If $S \neq N_\beta \bigcup \{k\}$, $S \bigcap N' \neq \emptyset$ and $S \bigcap N \neq \emptyset$. Then $v(S) = v_{\beta_0}(S \bigcap N_\beta) + v'(S \bigcap N')$. Thus $\sum_{i \in S \bigcap N'} w_i > \sum_{i \in S \bigcap N'} y_i = \sum_{i \in S \bigcap N'} y_i' \geq v'(S \bigcap N') = v(S \bigcap N)$. This implies that $\sum_{S \bigcap N} w_i < v_{\beta_0}(S \bigcap N_\beta) = v(S \bigcap N)$. And so $w \succ x$ over $S \bigcap N_\beta$. This gives a contradiction with $w, x \in V$.

Thus $m \not> \beta_0 + \alpha$ and we can conclude that $\beta_0 \leq m \leq \beta_0 + \alpha$.

Thus V_m is a Strongly Stable Set for the induced solution with $\beta_0 \leq m \leq \beta_0 + \alpha$. This contradicts the assumption.

□

PROOF of Theorem 12.13:

$$v_\beta(1479) = v_\beta(2579) = v_\beta(3679) = 2 - \gamma,$$
$$v_\beta(N_\beta) = 5 + \alpha,\ v_\beta(13579) = 0,$$
$$v_\beta(12) = v_\beta(34) = v_\beta(56) = v_\beta(78) = v_\beta(90) = 1,$$
$$v_\beta(1379) = v_\beta(1579) = v_\beta(3579) = 3,$$
$$v_\beta(137) = v_\beta(157) = v_\beta(357) = 2,$$
$$v_\beta(139) = v_\beta(159) = v_\beta(359) = 2,$$
$$v_\beta(S) = 0 \text{ for all other } S.$$

With a proof similar to that of theorem 12.6 we can show that this game has no solution for $0 < \gamma < \frac{2}{3}$ and $0 \leq \alpha \leq \gamma$.

Now take (N', v') as follows:

$$v'(AB) = v'(AC) = v'(BC) = 2$$
$$v'(ABCD) = 3$$
$$v'(S) = 0 \text{ for all other } S.$$

We see that $C(v') = (1,1,1,0)$ and is contained on one face of $A(v')$. Now look at the composed game (N, v) given by:

$$v(N_\beta \bigcup \{D\}) = 5 + \alpha$$
$$v(S) = v_{\beta_0}(N_\beta \bigcap S) + v'(N' \bigcap S) \text{ for all other } S \subset N.$$

Now we can use Theorem 12.14 and thus we see that this game has no solution and an empty core.

□

Now we show that this game has a Weakly Stable Set. It is sufficient to prove the next steps:

1. The games $(N_{\beta_0}, v_{\beta_0})$, (N', v') have Weakly Stable Sets $W_{\beta_0},\ W'$.
2. If $W_{\beta_0},\ W'$ are Weakly Stable Sets for $(N_{\beta_0}, v_{\beta_0})$, (N', v'), then $W_{\beta_0} \times W'$ is a Weakly Stable Set for (N, v).

(1). In a proof similar to that of Theorem 12.6 we can show that $C(v_{\beta_0})$ is a Weakly Stable Set for $(N_{\beta_0}, v_{\beta_0})$.

The set $\text{span}\{(1,2,0,0),(1,1,1,0)\} \bigcup \text{span}\{(0,1,2,0),(1,1,1,0)\} \bigcup \text{span}\{(2,0,1,0), v(1,1,1,0)\}$ is a Weakly Stable Set for (N', v').

- Internal Stability: Look at two imputations $x, y \in W'$. Suppose $y \succ x$ over S. Thus $\sum_{i \in S} x_i < \sum_{i \in S} y_i < v(S)$, $x_j = 1$ and $x_{j\text{mod } 3} < 1$ for a $S = \{j, j \text{ mod } 3\}$. Thus $y_j > 1$. But now $y_{j\text{mod } 3} = 1$ and $\sum_{i \in S} y_i > v(S)$. Thus for $x, y \in W'$ we see that $x \not\succ y$.
- External Stability: Look at an imputations $x \in A(v') \backslash W'$. Then one of the following holds:
  - $x_i, x_j \leq 1$ and $x_i + x_j < 2$ for $i, j \in \{1,2,3\}$, or
  - $x_i, x_j > 1$ for $i, j \in \{1,2,3\}$.
  - If $x_i, x_j \leq 1$ and $x_i + x_j < 2$, then we see the imputation $x' = (1,1,1,0)$ dominates x weakly.
  - If $x_i, x_j > 1$ for $i, j \in \{1,2,3\}$ we see that $x_i, x_j < 2$, otherwise $x \notin A$. We also know that $x_k + x_i < 2$ and $x_k + x_j < 2$ for $k = \{1,2,3\} \setminus \{i, j\}$. Because for one of the sets $S = \{i, k\}$ or $S = \{j, k\}$ there is an imputation $x' \in W'$ such that $x'_v > x_v$ for all $v \in S$ and $\sum_{v \in S} x_v = 2$ we see that this imputation is dominated.

  Thus every imputation in $A(v') \backslash W'$ is weakly dominated by an imputation in W'. Thus W' is externally stable.

Thus W' is a Weakly Stable Set for (N', v').

Now we show that $W = W_{\beta_0} \times W'$ is a Weakly Stable Set for (N, v).

- Internal Stability: Look at $x, y \in W$. Suppose $x \succ y$ over S. Because $W_\beta,\ W'$ are internally stable, we know that $S = N_\beta \bigcup \{k\}$. But $\sum_{N_\beta} x_i = \sum_{N_\beta} y_i$ implies that $x_i > y_i$ for all $i \in N_\beta$ cannot hold and $x \not\succ y$ for all $x, y \in W$.
- External Stability: Has a proof analogue to the proof of COCO in Theorem 6.2.

□

Thus this game has a Weakly Stable Set.

12.4 A Final Remark on Special Games

As we pointed out at the beginning of this chapter, some questions can be asked about the existence of (convex) Strongly Stable Sets. We saw that there are games for which there do not exist a Strongly Stable Set. And there are games, for which there do exist a Strongly Stable Set, but not a convex Strongly Stable Set. The questions whether (convex) Weakly Stable Sets do exist for all games, could unfortunately not be answered.

Chapter 13

Concluding Remarks

In this thesis I have studied the main solution concepts of cooperative game theory.

A solution concept relates an outcome or a set of outcomes to a game. Behind each of these concept is some rationality of the players assumed. According to Aumann, solution concepts should be interpreted more as indicators than as predictors of the outcome of a game. He takes distributions as an analogue. Like concepts as the median and the average are used to characterize a probability distribution, solution concepts of a game tell us something about the game.

After presenting some basic concepts of a cooperative game, I started with a discussion about the way imputations can be compared. Thereafter I investigated a number of important solution concepts.

The first concept was the Core. Imputations in the Core are not strongly dominated by imputations outside the Core. Therefore one may say that imputations in the Core are in some sense stable. However, the Core has also two big disadvantages. First, some imputations in the Core are not very rational. Second, the Core is sometimes empty. To accommodate for the first disadvantage, I defined the Dual Core. Imputations in the Dual Core form a subset of the imputations in the Core. They have the advantage of being more rational. On the other hand, since the Dual Core is a subset of the Core, thus the Dual Core is empty in more games than the Core.

In non-cooperative game theory the Nash equilibrium is probably the most famous solution concept. I transformed the non-cooperative Nash Equilibrium into a solution concept for a cooperative game. It appears that the set of all strict Nash equilibria is equal to the Dual Core. Because the Dual Core does not always exist, it is evident that a strict Nash equilibrium does not always exist either.

I tried to overcome this disadvantage by defining a weaker variant of the strict Nash equilibrium. It was proven that all imputations in the Core are weak Nash equilibria. This means that a weak Nash equilibrium does not always exist, although more cooperative games have a weak Nash equilibrium than a strict Nash Equilibrium. So the problem of emptiness has only partly been resolved.

From the investigation of the Core, the Dual Core and the strict and weak Nash eqilibria, it followed that these solution concepts are intimately related to each other. Because imputations in the (Dual) Core and, consequently, strict and weak Nash equilibria are not dominated by imputations outside the Core, they are in some sense stable. Coalitions of rational players do not have an incentive to look for solutions outside the imputations of these solution concepts.

But this does not mean that imputations outside the Core are always instable. Von Neumann and Morgenstern defined the solution concept Stable Set. For every imputation outside the Strongly

Stable Set there exists an imputation in the Stable Set that is more preferred. Moreover, none of the imputations in the Strongly Stable Set is preferred to an another imputation in the Strongly Stable Set. From the definitions it follows that the (Dual) Core and the Nash equilibrium are always subsets of the Strongly Stable Set. However, as was shown by Lucas in 1964, the Strongly Stable Set does not always exist neither.

So it can be concluded that we need a wider solution concept. Wider in the sense that it exists for all games. In the course of time this route has been followed by some game theorists. Roth introduced the idea of a Partially Stable Set. He proved that this set always exists, however it cannot be proven that it is always non empty.

As a contribution to the discussion, I defined and investigated some variants on the Strongly Stable Set. I first defined the Strictly Stable Set. It appeared that the Strictly Stable Set is internally more stable than the Strongly Stable Set. However, a Strictly Stable Set does not always exist.

Then I introduced the Weakly Stable Set. I proved that the weakly stable set is externally less stable than the Strongly Stable Set. Unfortunately I could not prove whether a Weakly Stable Set always exists.

Rationality and stability are the central criteria in the justification of the imputations in the Core, the Nash equilibrium, the stable set and the variants on these solution concepts. However, there is also another approach in cooperative game theory to solutions. In that approach game theorists lay the emphasis on some form of fairness of the distribution of the total payoff among the players, given the power each (coalition of) players has. In this respect the two main solution concepts that can be found in literature are the Nucleolus and the Shapley Value.

The idea behind the Nucleolus is finding an imputation that makes the least happy coalition as happy as possible. Every game has an unique Nucleolis which can be found by solving a series of LP-problems. However, the nucleolus is not always stable. But if the game has a Core, the Nucleolus is in the core and the distribution of the payoffs over the coalition is stable.

I have formulated two variants on the Nucleolus which I called the weighted Nucleolus and the dual Nucleolus. The difference between the weighted Nucleolus and the Nucleolus is that in the weighted Nucleolus it is assumed that the coalitions have already been formed at the outset of the game.

The Shapley Value is the second widely used solution concept where the attention is on a fair distribution of the total payoffs. The distribution is based on the contribution of a player or a possible coalition to the total payoffs. So it can be seen as a measure of power. Although the Shapley Value is not a stable solution, it has some nice other properties: It is intuitively attractive for all players to cooperate and it can easily be found. Because of this it is, according to Aumann, the most successful of all cooperative solution concepts.

Finally, what have I learned from my journey in the world of cooperative games? First of all of course knowledge of a number of game theoretic concepts and the way these concepts are analysed by mathematicians and game theorists. Secondly, although my thesis is mainly written in mathematics, I became increasingly aware of the relations with the real world. Through the eyes of game theory all interactions between people or groups of people, e.g. political parties, firms, all kind of social groups etc., can be interpreted as a game. Outcomes of these interactions (negotiations) will only last if they are stable or if they have some degree of fairness. What exactly is fair and what is stable depends mainly on the initial strength of the coalitions. So game theory is a very general theory about how (coalitions of) humans interact. At the same time the generality of the theory is also a drawback. It is difficult to apply it to a real situation in a way that an outcome can more or less precisely be predicted.

Bibliography

[1] Aumann, R.J., What is Game theory trying to accomplish, *Frontiers of Economic*, Basil Blackwell, Oxford, 1985.

[2] Gillies, D.B., "Solutions to general non-zero-sum games," *Annals of Mathematics Studies*, No. 40, A. W. Tucker and R. D. Luce (eds.), Princeton University Press, Princeton, N. J., 1959, pp. 47-85.

[3] Grtzer, G., *General Lattice Theory*, Birkhuser Verlag, Basel, Second edition, 2003.

[4] Kallenberg, L.C.M., *Besliskunde 2 dictaat*, 2008.

[5] Lucas, W.F., "A Game with No Solution", *Bull. Amer. Math. Soc.* Volume 74, Number 2, 1968, 237-239.

[6] Lucas, W.F., "Games with unique solutions that are non convex", *Pacific Journal of Mathematics*, Volume 28, Number 3, 599-602, 1969.

[7] Lucas, W.F., Rabie, M., "Games With No Solutions and Empty Cores", *Mathematics of Operations Research*, Volume 7, Number 4, 1982.

[8] Morgenstern, O., Von Neumann, J., *The Theory of Games and Economic Behavior*, Princeton University Press, Princeton, N.J., 1944.

[9] Owen, G., *Game Theory* (3rd ed.), San Diego Academic Press, San Diego, 1995.

[10] Peleg, B., Sudhlter, P., *Introduction to Cooperative Game Theory*, Springer, Berlin Heidelberg, 2nd edition, 2004.

[11] Rubinstein, A., Osborne, Martin J., *A course in game theory,A modern introduction at the graduate level.*, 1994.

[12] Roth, A.E., A Lattice Fixed Point Theorem with constraints, *Bulletin of the American Mathematical Society*, Volume 81, No. 1, 1975.

[13] Roth, A.E., "Subsolutions and the Supercore of Cooperative Games", *Mathematics of Operations Research*, Volume 1, No. 1, February 1976.

[14] Schmeidler, D., "The Nucleolus of a Characteristic Function Game", *Journal on Applied Mathematics*, Volume 17, No. 6, 1163-1170, 1969.

[15] Shapley, Lloyd S., "A value for n-person games", *Annals of Mathematical Studies*, No. 28, 307-317, 1953.

[16] Shapley, Lloyd S., Shubik, M., "Quasi-cores in a monetary economy with non-convex preferences", *Econometrica*, No. 34, 805827, 1966.

[17] Shapley, L.S., "Cores of convex games", *International Journal of Game Theory*, Volume 1, 1971 1126.