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Asymptotic blowup vortex solutions of the NLS

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Contents

Chapter 1

Introduction

In this thesis, we will analyze the cubic nonlinear Schrödinger (NLS) equation

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + \Phi|\Phi|^2 = 0, \tag{1.1}$$

$$\Phi(\mathbf{x}, 0) = \Phi_0(\mathbf{x}), \mathbf{x} \in R^d, 2 < d < 4. \tag{1.2}$$

The NLS is a useful model for many physical processes including nonlinear optics, plasma physics and quantum theory, as well as forming the envelope equation for many other important problems. The NLS is an example of a unitary Hamiltonian partial differential equation, and during the evolution of the solution $\Phi(x, t)$ both the mass

$$M = \int_{R^d} |\Phi|^2 d\mathrm{x}$$

and the Hamiltonian

$$H = \int_{R^d} (|\nabla\Phi|^2 - \frac{1}{2}|\Phi|^4) d\mathrm{x}$$

remain unchanged.

It has been proved that in dimensions $4 > d \geq 2$, and if the Hamiltonian of the initial condition $\Phi_0(x)$ is negative, there exists solutions that become infinite at a single point in a finite time T, forming a growing and increasingly narrow peak [4]. We call this the "blow-up", in plasma physics this is called a collapse and in nonlinear optics it corresponds to an extreme increase of the field amplitude due to self-focusing of a laser beam. [1,6]

It has also been shown [4] that solutions of the NLS remain bounded for $d < 2$, therefore $d = 2$ is so-called "critical dimension", with $d > 2$ called "supercritical dimension". For the critical case $d = 2$, there's no self-similar blowup solutions for NLS.

In 2007, G.Fibich and N.Gavish [3] presented a first study of singular vortex solutions of the two-dimensional NLS.

The word 'vortex' is widely known as a spinning, often turbulent, flow of fluid, the motion of the fluid swirling rapidly around a center. In dynamical systems, a solution which have the form

$$\Phi(t, r, \theta) = A(t, r)e^{im\theta}, \qquad A_r(t, 0) = 0 \tag{1.3}$$

is called 'vortex'.

In the polar coordinate system, Φ satisfies

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{d-1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}. \tag{1.4}$$

Since then the equation (1.1) becomes

$$i\frac{\partial\Phi}{\partial t} + \frac{\partial^2\Phi}{\partial r^2} + \frac{d-1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \Phi|\Phi|^2 = 0, \qquad \Phi_r(t, 0, \theta) = 0. \tag{1.5}$$

In this equation, the dimension d can take non-integer values. In other words, d can now be considered as a bifurcation parameter. The solution Φ can also be considered as a function of d.

Many previous works investigated the solutions for $m = 0$ which are only depend on r (radially symmetric case)[1,2,6,7]. There is numerical evidence that there appears to be a single-bump, self-similar, blowup solution [6,7] (it means that Φ can be written as a function depends on variables that are functions of x and t) for each $2 < d \leq 4$ in terms of a locally unique quantity that we will denote by $a = a(d)$ (see later in this Chapter). Especially, for solutions asymptotic to the blow-up, it is proved that their exist solutions when a and d vary slowly with time where $a \to 0$ and $d \to 2$ as $t \to T$, where T is the blow-up time.

Some former works concentrate in the case that d is close to 2. Most of them discussed the case that $d - 2 = \mathcal{O}(e^{-\pi/a})$ which is exponentially small [1,2], but also the case when d is algebraically close to 2 that $d - 2 = a^l$ is considered [7].

In this paper, we will also concentrate in the critical limit of

$$d \to 2^+ \tag{1.6}$$

but for the case $m \neq 0$. We will compare its behaviour with the radially symmetric case.

We can find that the equation (1.5) is invariant under a change in the scale of r, t and Φ such that

$$t \mapsto \lambda t,\ r \to \lambda^{1/2} r,\ \Phi \to \lambda^{-1/2}\Phi,\ \theta \to \theta \tag{1.7}$$

for some $\lambda > 0$.

Rescale

$$y = \frac{|x|}{L(t)} = \frac{r}{L(t)}, \tau = \int_0^t \frac{1}{L^2(s)} ds, u(y,\theta,\tau) = L(t)\Phi(x,\theta,t) \tag{1.8}$$

with some $L(t)$ to be decided.

Then u satisfies

$$iu_\tau + (u_{yy} + \frac{d-1}{y} u_y + \frac{1}{y^2} u_{\theta\theta}) + |u|^2 u + ia(\tau)(yu)_y = 0. \tag{1.9}$$

Where

$$a = -L\frac{dL}{dt} = -\frac{1}{L}\frac{dL}{d\tau}. \tag{1.10}$$

There is numerical evidence [2] that u has the form

$$u(y, \theta, \tau) = e^{i\omega\tau} A_m(y, \theta). \tag{1.11}$$

And also numerical evidence that we can take a as a positive constant, that is,

$$L(t) = \sqrt{2a(T-t)}. \tag{1.12}$$

Since then the amplitude of u remains constant in time and as a sequence, the rescaled system is no longer singular and can be simplified to something which is independent of τ.

Since we can rescale ω by changing other coefficients it is convenient to consider only $\omega \equiv 1$.

Also τ can be calculated,

$$\tau = \int_0^t \frac{1}{L^2(s)} ds = \int_0^t \frac{1}{2a(T-t)} ds = \frac{1}{2a} \log \frac{T}{T-t}. \tag{1.13}$$

In previous work, C.J.Budd presented a simplified equation when $A_m(y, \theta) = Q(y)$ [1] and $a \ll 1$. The equation is given by

$$Q_{yy} + \frac{(d-1)}{y} Q_y + ia(yQ)_y - Q + Q|Q|^2 = 0. \tag{1.14}$$

It is shown in [9] that there is a global solution $Q(y)$ such that

$$Q(y) \to 0, \qquad \text{as} \qquad y \to \infty. \tag{1.15}$$

Theoretical study of (1.14) is given in [1], C. J. Budd gave an asymptotic description of a doubly countable set of multi-bump solutions $Q(y) \equiv Q_{K,J}(y)$ with $(K, J) \in (0, 1, 2, 3, ..) \times (0, 1, 2, 3, ..)$ and for which $|Q_{K,J}(y)|$ has $K + J$ local maxima.

He showed that if y is fixed and $a \to 0$, $d \to 2$ then $Q_{K,J}(y) \to R_K(y)$, where $\{R_K(y)\}$ is the set of solutions of the ODE

$$R_{yy} + \frac{1}{y}R_y - R + R^3 = 0, \qquad R_y(0) = 0 \ R \to 0 \ as \ y \to \infty \tag{1.16}$$

and $R_K(y)$ has precisely $K-1$ zeros and K turning points [8]. For small a the function $|Q_{K,J}(y)|$ has J bumps with maxima located at the points κ_j/a, with κ_j is to leading order $1 + \mathcal{O}(a \log(a))$.

Now for the vortex case,

$$A_m(y, \theta) = Q(y)e^{im\theta}. \tag{1.17}$$

The equation of Q can be calculated from (1.9),

$$Q_{yy} + \frac{(d-1)}{y}Q_y + ia(yQ)_y - (1 + \frac{m^2}{y^2})Q + Q|Q|^2 = 0. \tag{1.18}$$

Consider that Q is only depend on y, and for every fixed $r = r_0$, $y \to \infty$ when $t \to T$. Then we must have the initial condition

$$Q(y) \to 0, \qquad \text{as} \qquad y \to \infty. \tag{1.19}$$

Otherwise Φ will be infinite at r_0 as $t \to T$ by definition of u. Since r_0 arbitrary, it is contradict with what we assumed.

Also the amplitude of A_m is symmetric in y, therefore $y = 0$ must be a local maximum or minimum so that $A_{m,y}|_{y=0} = 0$. Hence

$$Q_y(0) = 0. \tag{1.20}$$

And observe that Q has a phase invariance symmetry, that is, we can always replace Q by $Qe^{i\varphi}$ for some constant $\varphi \in \Re$. Thus without lost generality we can take

$$Q(0) \in \Re. \tag{1.21}$$

Thus the initial conditions are same with the radially symmetric case.

In this thesis, we will present an asymptotic analysis of the solution of (1.18) when $m \neq 0$ and $d \to 2$ with single peak.

Chapter 2

How we construct the solution

2.1 Basic assumptions

As we mentioned in the previous chapter, there exist both analytical and numerical results that in case for $m = 0$ and a is small, there exists a solution with single peak and the peak lies at $\mathcal{O}(1/a)$ ($Q_{0,1}(y)$). The width of bump region is $\ll \mathcal{O}(1/a)$. We look for the similar solution in case for $m \neq 0$. Thus we assume that the peak lies at $y = \kappa/a$, with the $\mathcal{O}(1)$ parameter κ to be calculated.

Figure 2.1 shows the graph of the solution of (1.18) when $d = 2$ but $m \neq 0$ [3], which has a similar shape as $Q_{0,1}(y)$. Hence we have strong motivation to do the above assumptions and we can estimate the shape of solution. (see Figure (2.2)).

2.2 Multiple Scale and Matching

To analyze the solution of (1.18) and derive the relationship between a, m and d near $d = 2$, it is hard to investigate the solution directly in the whole region. Thus we split it into different regions and approximate every regions use different asymptotic analysis methods, and get different types of asymptotic approximations in different regions.

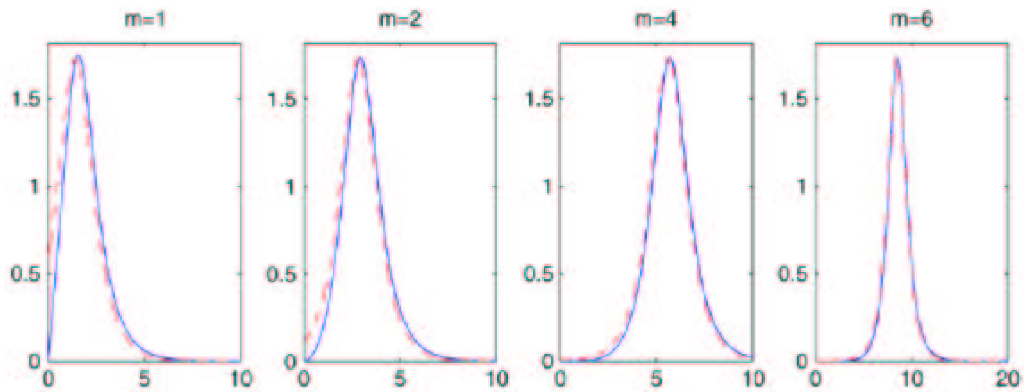

Figure 2.1: The solution of (1.18) when $d = 2$ and for m=1,2,4 and 6 generated by Fibich and Gavish [3]

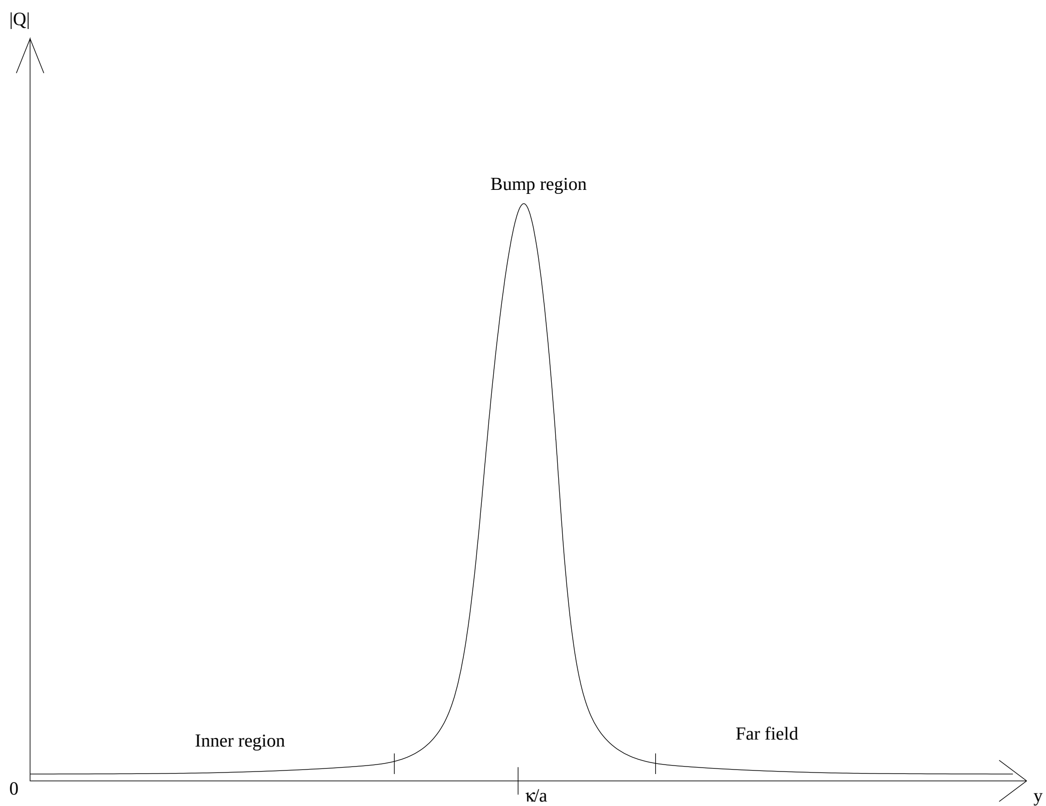

Figure 2.2: Approximate shape of the solution

It is important that we can link solutions together. Therefore, in the transition region between two regions we should expect that the two expansions give the same result. We use the technique of matching developed in asymptotic analysis.

For instance consider Figure (2.2), we may intuitively split the whole region into three parts: Inner region, Bump region and Far field. There are analytical evidences that $|Q|$ is small at inner region in case $m = 0$, here we assume that we also have this in case $m \neq 0$.

We also have to discuss the scale of m in terms of a. For instance, if $am = \mathcal{O}(1)$ then some parts of the second term in the expansion may be as large as the first term. The solution will be very different when $m = \mathcal{O}(1)$ versus when $m = \mathcal{O}(1/a)$. We do not consider $m \gg \mathcal{O}(1/a)$.

Chapter 3

The solution at inner region

In this chapter, we construct the inner region solution of (1.18) for $y = \mathcal{O}(1)$, we have already assumed that $|Q|$ is small in this region. Since Q is complex it is common to use the amplitude and phase decomposition

$$Q(y) = A(y)e^{i\theta(y)}. \tag{3.1}$$

Thus, its derivative can also be calculated in terms of A and θ, we have

$$Q_y = (A_y + iA\theta_y)e^{i\theta(y)}, \tag{3.2}$$

and

$$Q_{yy} = (A_{yy} + 2iA_y\theta_y + iA\theta_{yy} - A\theta_y^2)e^{i\theta(y)}. \tag{3.3}$$

Put into (1.18), then take the imaginary part

$$2A_y\theta_y + A\theta_{yy} + \frac{d-1}{y}A\theta_y + aA + ayA_y = 0. \tag{3.4}$$

Times Ay and put the third term to right-hand side we get

$$(A^2\theta_y + 2AyA_y\theta_y + A^2y\theta_{yy}) + (aA^2y + ay^2AA_y) = -(d-2)A^2\theta_y. \tag{3.5}$$

Observe that the $left{-}hand$ side is equal to the derivative of $\theta_y yA^2 + ay^2A^2/2$. Integrate both sides,

$$\theta_y + \frac{a}{2}y = -\frac{(d-2)}{yA^2}\int_0^y \theta_y A^2 dy'. \tag{3.6}$$

By assumption $d-2$ is small and $Q(0) \in \Re$, that is, $\theta(0)=0$, hence θ can be estimated by

$$\theta(y) \approx \int_0^y -\frac{ay'}{2} dy' = -\frac{ay^2}{4}. \tag{3.7}$$

This motivates using the following decomposition:

$$Q(y) = P(y) e^{-iay^2/4}. \tag{3.8}$$

Combine with (1.18) we have

$$P_{yy} + \frac{d-1}{y} P_y - (1 + \frac{m^2}{y^2}) P + \frac{a^2 y^2}{4} P - ia(\frac{d-2}{2}) P + P|P|^2 = 0. \tag{3.9}$$

By assumption we have that $|P|$ is small in the inner region, thus the term $P|P|^2$ can be ignored. Then (3.9) can be simplified as

$$P_{yy} + \frac{d-1}{y} P_y - (1 + \frac{m^2}{y^2}) P + \frac{a^2 y^2}{4} P - ia(\frac{d-2}{2}) P = 0. \tag{3.10}$$

For y is not large it is treated as a perturbation of Bessel equation

$$P_{yy} + \frac{1}{y} P_y - (1 + \frac{m^2}{y^2}) P = 0 \tag{3.11}$$

which is solved by linear combinations of modified Bessel functions $I_m(y)$ and $K_m(y)$. Since $K_m(y)$ goes to infinity as $y \to 0$ which is in contradiction to the initial condition, we have to leading order

$$P(y) = \alpha \sqrt{2\pi} I_m(y). \tag{3.12}$$

Here we put the $\sqrt{2\pi}$ term together with a coefficient α in order to be more convenient for calculation. We have asymptotically that

$$I_m(0) = 0 (m \neq 0) \text{ and } \sqrt{2\pi} I_m(y) \to \frac{e^y}{\sqrt{y}} \text{ as } y \to \infty (m = \mathcal{O}(1)). \tag{3.13}$$

Chapter 4

Behavior in the bump-region

4.1 The multi-bump solutions for $m = \mathcal{O}(1)$

In this section we analyze the solution of (1.18) at multi-bump region. We use a similar method with previous work to rescale (1.18) by setting

$$y = \kappa/a + s, \tag{4.1}$$

where κ/a is the location of the peak we assumed.

Put into (1.18) then it satisfies

$$Q_{ss} - Q + i\kappa Q_s + Q|Q|^2 + ia(sQ)_s + a\frac{d-1}{\kappa + as}Q_s - \frac{a^2m^2}{(\kappa + as)^2}Q = 0. \tag{4.2}$$

Here we use the asymptotic expansion to solve this problem.

The method of asymptotic analysis is widely used in researching ODEs and PDEs. The basic idea is that if there is a small parameter ε in the equation, then we assume that the solution F can be written as

$$F = F_0 + \varepsilon^{\alpha}F_1 + \varepsilon^{\beta}F_2 + \cdots, \tag{4.3}$$

where the leading term F_0 is much larger than the others and $\beta > \alpha > 0$. If we put the expansion into the initial equation and take only the leading order terms into account, we get an equation for the function F_0 which is not depend on ε. This is called the asymptotic approximation of the solution.

We always have to analyze not only the leading term, but also the second term, third term...etc. And the values of parameters α and β should be decided.

Now a is the small parameter, thus Q can be expressed by

$$Q(s) = Q_0(s) + aQ_1(s) + a^2Q_2(s) + \cdots . \tag{4.4}$$

We hereby consider the leading order of the solution. Let $Q_0(s)$ be the leading term of $Q(s)$, then again we have to pay attention to the range of m.

If $m \ll \mathcal{O}(1/a)$, then $a^2m^2 \ll \mathcal{O}(1)$ will be small. Then

$$Q_{0,ss} - Q_0 + i\kappa Q_{0,s} + Q_0|Q_0|^2 = 0. \tag{4.5}$$

Do the rescaling

$$Q_0(s) = e^{-i\kappa s/2}S_0(s). \tag{4.6}$$

Then the ordinary differential equation of S_0 is

$$S_{0,ss} - (1-\kappa^2/4)S_0 + S_0|S_0|^2 = 0. \tag{4.7}$$

This equation has been solved in [B00], the solution is unique up to a constant of unit modulus, and is given by

$$S_0(s) = \sqrt{2(1-\kappa^2/4)}\ \mathrm{sech}(\sqrt{1-\kappa^2/4}\ s). \tag{4.8}$$

Consider for the shape of the solution, we must have $S_0 \to 0$ as $|s| \to \infty$. In other words, it should be $1-\kappa^2/4 > 0$, otherwise the solution will be oscillating.

Hence we have at peak $s = 0$

$$|S_0(0)| = |Q_0(\kappa/a)| = \sqrt{2(1-\kappa^2/4)} \tag{4.9}$$

to leading order.

4.2 The multi-bump solutions for $m = \mathcal{O}(1/a)$

If $m = \mathcal{O}(1/a)$, then $a^2m^2 = \mathcal{O}(1)$ will not be small anymore. Hence

$$Q_{0,ss} - Q_0 + i\kappa Q_{0,s} + Q_0|Q_0|^2 + \frac{4\rho^2}{\kappa^2}Q_0 = 0 \tag{4.10}$$

where $\rho = am/2$.

Here we do the same rescaling as above, and get

$$S_{0,ss} - (\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)S_0 + S_0|S_0|^2 = 0. \tag{4.11}$$

The solution is again unique similarly given by

$$S_0(s) = \sqrt{2(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)}\ \mathrm{sech}(\sqrt{\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4}\ s). \tag{4.12}$$

Also we must have $\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4 > 0$. And

$$|S_0(0)| = |Q_0(\kappa/a)| = \sqrt{2(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)}. \tag{4.13}$$

Remark: Note that for the case $m = \mathcal{O}(1/\sqrt{a})$, in the leading order is equal to the case $m = \mathcal{O}(1)$. But the solution for this case will be somehow different in later discussion.

Chapter 5

Locating the maxima

In this section we determine the value of κ, this is done by considering the second term of (4.4). Here we must discuss the order of m more precisely. It has been found for equation (4.2) that when $m \ll \mathcal{O}(1/a)$, the term contains m is trivial compare to the leading order of the equation (4.2). But consider if $m = \mathcal{O}(1/\sqrt{a})$, then $a^2m^2 = \mathcal{O}(a)$. Hence it is not small compare to the second term.

5.1 In case $m \ll \mathcal{O}(1/\sqrt{a})$

We begin with the simplest case $m \ll \mathcal{O}(1/\sqrt{a})$. In this case a^2m^2 is small compared to both the first and the second term of equation (4.2). Since then this case is the same as the $m = 0$ case which C.J.Budd done before[1].

Put (4.4) into (4.2) we get

$$Q_{1,ss} - Q_1 + i\kappa Q_{1,s} + 2Q_1|Q_0|^2 + Q_0^2\overline{Q_1} = -i(sQ_0)_s - \frac{d-1}{\kappa}Q_{0,s}. \quad (5.1)$$

Do the similar rescaling as above

$$Q_1(s) = e^{-i\kappa s/2}S_1(s). \quad (5.2)$$

Thus $\overline{Q_1(s)} = e^{i\kappa s/2}\overline{S_1(s)}$. Together with the similar relationship between $\overline{Q_0(s)}$ and $\overline{S_0(s)}$ we get

$$S_{1,ss} - (1-\kappa^2/4)S_1 + 2S_1|S_0|^2 + S_0^2\overline{S_1} = -iS_0 - isS_{0,s} - \frac{\kappa}{2}sS_0 - \frac{d-1}{\kappa}S_{0,s}. \tag{5.3}$$

Let $t(s)$ be the real part of S_1. Since S_0 is real, the absolute value of S_0 can be replaced by itself. We get the ODE of t is

$$t_{ss} - (1-\kappa^2/4)t + 3S_0^2 t = -\frac{\kappa}{2}sS_0 - \frac{d-1}{\kappa}S_{0,s} \equiv f_1(s). \tag{5.4}$$

For the asymptotic expansion to be consistent we require $|t| \to 0$ as $|s| \to \infty$. In order to solve (5.4), first consider the homogeneous equation

$$t_{ss} - (1-\kappa^2/4)t + 3S_0^2 t = 0. \tag{5.5}$$

The behavior of this equation has already been recognized. It has an odd, exponentially decaying solution $\psi_1(s) = S_{0,s}$ and a linearly independent, exponentially growing, even valued solution $\psi_2(s)$. With $\psi_1(s)$ and $\psi_2(s)$ having constant Wronskian W (see[B00]). Hence

$$\psi_2(s) = S_{0,s}\int_0^s \frac{dy}{S_{0,y}^2(y)} \sim \exp(\sqrt{1-\kappa^2/4}s) \text{ for large } s. \tag{5.6}$$

From the variation of constants formula, we can express $t(s)$ by

$$t(s) = A\psi_2(s) + A'\psi_1(s) - \psi_1\int_0^s \frac{\psi_2 f_1}{W}ds' + \psi_2\int_0^s \frac{\psi_1 f_1}{W}ds' \tag{5.7}$$

with some constants A and A'. Also ψ_1 decays exponentially, so we can find

$$t(s) \to (A - I/W)\psi_2(s) \text{ as } s \to \infty \tag{5.8}$$

$$t(s) \to (A + I/W)\psi_2(s) \text{ as } s \to -\infty \tag{5.9}$$

where

$$I = \int_0^\infty f_1\psi_1 ds = -\frac{2}{3\kappa}(1-\kappa^2/4)^{1/2}(1-\kappa^2) \tag{5.10}$$

to leading order.

From the fact that $|t| \to 0$ as $|s| \to \infty$ and $\psi_2(s)$ grows exponentially, we have $A = 0$ and $I = 0$. And also we already have that $1 - \kappa^2/4 > 0$. Thus

$$\kappa = 1 \tag{5.11}$$

is the leading order of maxima.

We do not have to consider the imaginary part of S_1.

5.2 In case $m = \mathcal{O}(1/a)$

For $m = \mathcal{O}(1/a)$, $a^2m^2 = \mathcal{O}(1)$ is significant. Again we put (4.4) into (4.2) and get

$$Q_{1,ss} - Q_1 + i\kappa Q_{1,s} + 2Q_1|Q_0|^2 + Q_0^2\overline{Q_1} = -i(sQ_0)_s - \frac{d-1}{\kappa}Q_{0,s} - \frac{8\rho^2 s}{\kappa^3}Q_0.$$

Define S_1 and t the same to above, hence

$$\begin{aligned} S_{1,ss} - (1 - \kappa^2/4 + 4\rho^2/\kappa^2)S_1 + 2S_1|S_0|^2 + S_0^2\overline{S_1} = \\ -iS_0 - isS_{0,s} - \frac{\kappa}{2}sS_0 - \frac{d-1}{\kappa}S_{0,s}. \end{aligned}$$

We can write the differential equation of t

$$\begin{aligned} t_{ss} - (1 - \kappa^2/4 + 4\rho^2/\kappa^2)t + 3S_0^2 t =& (-\frac{\kappa}{2} - \frac{8\rho^2}{\kappa^3})sS_0 - \frac{d-1}{\kappa}S_{0,s} \\ =& -\frac{2}{\kappa}(\kappa^2/4 + 4\rho^2/\kappa^2)sS_0 - \frac{d-1}{\kappa}S_{0,s} =: f_2(S) \end{aligned} \tag{5.12}$$

with

$$|t| \to 0 \text{ as } |s| \to \infty. \tag{5.13}$$

The solution of homogeneous equation

$$t_{ss} - (1 - \kappa^2/4 + 4\rho^2/\kappa^2)t + 3S_0^2 t = 0 \tag{5.14}$$

is given by $\psi_1(s) = S_{0,s}$ and

$$\psi_2(s) = S_{0,s} \int_0^s \frac{dy}{S_{0,y}^2(y)} \sim \exp(\sqrt{1 - \kappa^2/4 + 4\rho^2/\kappa^2}\, s) \text{ for large } s. \tag{5.15}$$

Also we can express the solution by

$$t(s) = C\psi_2(s) + C'\psi_1(s) - \psi_1 \int_0^s \frac{\psi_2 f_2}{W} ds' + \psi_2 \int_0^s \frac{\psi_1 f_2}{W} ds' \tag{5.16}$$

for some C and C'.

And

$$t(s) \to (C - J/W)\psi_2(s) \text{ as } s \to \infty \tag{5.17}$$

$$t(s) \to (C + J/W)\psi_2(s) \text{ as } s \to -\infty \tag{5.18}$$

where

$$J = \int_0^\infty f_2 \psi_1 ds = -\frac{2}{3\kappa}(1 - \kappa^2/4 + 4\rho^2/\kappa^2)^{1/2}(1 - \kappa^2 - \frac{8\rho^2}{\kappa^2}) \tag{5.19}$$

to leading order.

Again we have $C = 0$ and $J = 0$. And also $1 - \kappa^2/4 + 4\rho^2/\kappa^2 > 0$ as we mentioned before. Thus we have

$$1 - \kappa^2 - \frac{8\rho^2}{\kappa^2} = 0 \tag{5.20}$$

at the peak.

We may rewrite above equation as

$$\kappa^4 - \kappa^2 + 8\rho^2 = 0. \tag{5.21}$$

It has no solutions for $\rho > \frac{1}{4\sqrt{2}}$, and

$$\kappa = \sqrt{\frac{1 \pm \sqrt{1-32\rho^2}}{2}} \tag{5.22}$$

otherwise.

This gives two κ that satisfy the conditions $1-\kappa^2-\frac{8\rho^2}{\kappa^2}=0$ and $1-\kappa^2/4+4\rho^2/\kappa^2 = (1-\kappa^2-\frac{8\rho^2}{\kappa^2}) + 3\kappa^2/4 + 12\rho^2/\kappa^2 > 0$. Thus it maybe true that there exists two possible solutions.

5.3 In case $m = \mathcal{O}(1/\sqrt{a})$

For special case $m = \mathcal{O}(1/\sqrt{a})$, we take $b = am^2 = \mathcal{O}(1)$. Again put (4.4) into (4.2) and hence there will be one term more compare to (5.1)

$$Q_{1,ss} - Q_1 + i\kappa Q_{1,s} + 2Q_1 Q_0 \overline{Q_0} + Q_0^2 \overline{Q_1} = -i(sQ_0)_s - \frac{d-1}{\kappa} Q_{0,s} + \frac{b}{\kappa^2} Q_0. \tag{5.23}$$

The calculation will be totally the similar way as $m \ll \mathcal{O}(1/\sqrt{a})$, the corresponding equation will be

$$t_{ss} - (1-\kappa^2/4)t + 3S_0^2 t = -\frac{\kappa}{2} s S_0 - \frac{d-1}{\kappa} S_{0,s} + \frac{b}{\kappa^2} S_0 \equiv f_3(s). \tag{5.24}$$

And K (which corresponds to I) will be

$$K = \int_0^\infty f_3 S_{0,s} ds = \frac{2}{3\kappa}(1-\kappa^2/4)^{1/2}(1-\kappa^2) - \frac{b}{\kappa^2}(1-\kappa^2/4). \tag{5.25}$$

It is difficult to give an explicit solution of κ given $K = 0$. But we can implicitly give b as a function of κ

$$b = \frac{2\kappa(1-\kappa^2)}{3\sqrt{1-\kappa^2/4}}. \tag{5.26}$$

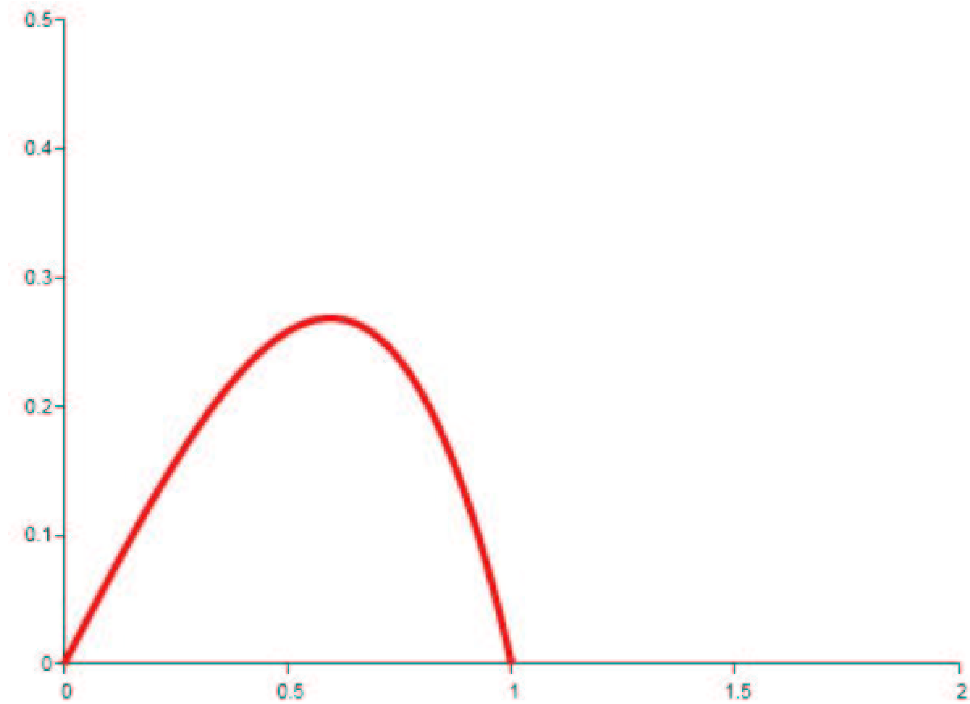

Figure 5.1: The function of b by κ

We plot this function (the horizontal axis is κ), we only take that $b \geq 0$, since when $\kappa > 1$, then $b < 0$, there is no solutions at $\kappa > 1$. Also if b greater than the maximum $\frac{2(3-\sqrt{7})\sqrt{2\sqrt{7}-4}}{3}$ there will be no value for κ. This maximum happens when $\kappa = \sqrt{3-\sqrt{7}}$ and it is approximately between 0.26 and 0.27. For each b between 0 and the maximum there are two solutions of κ.

Chapter 6

The WKB solution at far field

6.1 Rescaling

In this section we consider the behavior of equation (1.18) at far field, where $y - \kappa/a = \mathcal{O}(1/a)$. As the value of $y - \kappa/a$ becomes large, then the solution computed in previous section decays exponentially. When this happens we have simultaneously that $y = \mathcal{O}(1/a)$ and $|Q| \ll 1$.

For this region, the term $Q|Q|^2$ in equation (1.18) is small compared to the other terms. Then (1.18) can be approximated by

$$Q_{yy} + \frac{(d-1)}{y} Q_y + ia(yQ)_y - (1 + \frac{m^2}{y^2})Q = 0. \tag{6.1}$$

In [1], the following rescaling was used,

$$Q(y) = e^{-iay^2/4} y^{(1-d)/2} Z(y). \tag{6.2}$$

The rescaled function $Z(y)$ satisfies the ODE

$$-Z'' + \left(-\frac{a^2y^2}{4} + 1 - ia(\frac{d-2}{2}) - y^{1-d}|Z|^2 + \frac{(d-1)(d-3)}{4y^2} - \frac{m^2}{y^2} \right) Z = 0. \tag{6.3}$$

Because y is large, $d \to 2$ and $|Q|$ is small, the terms $ia(\frac{d-2}{2})$, $y^{1-d}|Z|^2$ and

$\frac{(d-1)(d-3)}{4y^2}$ are small compare to the term '1'. Hence we may drop them out and get

$$Z'' + \left(\frac{a^2y^2}{4} - (1 + \frac{m^2}{y^2})\right) Z = 0. \tag{6.4}$$

6.2 A solution for $m \ll \mathcal{O}(1/a)$

It is also necessary to consider the order of m here. As we see in (6.4), if $m = \mathcal{O}(1/a)$ and $y = \mathcal{O}(1/a)$ then $\frac{a^2y^2}{4}$ and $\frac{m^2}{y^2}$ will both be $\mathcal{O}(1)$.

But for $m \ll \mathcal{O}(1/a)$ and $y = \mathcal{O}(1/a)$, the $\frac{m^2}{y^2}$ term is small, and equation (6.4) becomes

$$Z'' + (\frac{a^2y^2}{4} - 1)Z = 0. \tag{6.5}$$

In this section we first consider this case.

Introduce the rescaling parameter x by

$$x = \frac{ay}{2}. \tag{6.6}$$

Change into x scale, then we have

$$Z_{xx} + \frac{4}{a^2}(x^2 - 1)Z = 0. \tag{6.7}$$

Equation (6.7) satisfies the condition to use the WKB method, which is introduced in Appendix 1. For $x < 1$ and not close to 1, the WKB solution is given by

$$Z(x) = \frac{A_+}{(1-x^2)^{1/4}} e^{\frac{2}{a}g(x)} + \frac{A_-}{(1-x^2)^{1/4}} e^{-\frac{2}{a}g(x)} \tag{6.8}$$

where

$$g(x) = \int_0^x \sqrt{1-s^2}ds = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x), \tag{6.9}$$

for some constants A_+ and A_-.

The turning point will be at $x = 1$, which is after the bump region. For $x > 1$ and not close to 1, equation (6.8) becomes elliptic.

6.3 The solution for $m = \mathcal{O}(1/a)$

For $m = \mathcal{O}(1/a)$, there is a parameter we set in section 4

$$\rho = \frac{am}{2}. \tag{6.10}$$

Then this gives

$$Z_{xx} + \frac{4}{a^2}(x^2 - \frac{\rho^2}{x^2} - 1)Z = 0. \tag{6.11}$$

Similarly equation (6.11) satisfies the condition to use the WKB method. The turning point located where $x^2 - \frac{\rho^2}{x^2} - 1 = 0$, hence

$$x^2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \rho^2}. \tag{6.12}$$

There is only one real positive solution for (6.12)

$$x = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \rho^2}}. \tag{6.13}$$

It follows that the turning point will be at $x_0 = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \rho^2}}$. Recall that we have the relation $\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4 > 0$ derived in section 4, this gives that $\kappa < 2x_0$. Thus the turning point lies after the bump region.

For $x < x_0$ and not close to x_0, the WKB approximation is

$$Z(x) = \frac{C_+}{(1 + \frac{\rho^2}{x^2} - x^2)^{1/4}} e^{\frac{2}{a}f(x)} + \frac{C_-}{(1 + \frac{\rho^2}{x^2} - x^2)^{1/4}} e^{-\frac{2}{a}f(x)}, \tag{6.14}$$

where

$$f(x) = \frac{1}{2}\rho\log(x^2) - \frac{1}{2}\rho\log(2\rho^2 + 2\rho\sqrt{x^2+\rho^2-x^4} + x^2)$$
$$+\frac{1}{2}\sqrt{x^2+\rho^2-x^4} - \frac{1}{4}\tan^{-1}(\frac{1-2x^2}{2\sqrt{x^2+\rho^2-x^4}}), \tag{6.15}$$

for some constants C_+ and C_-. Also when $x > x_0$ and not close to x_0, the solution becomes elliptic.

Chapter 7

Solutions near the turning-point

According to Appendix 1, the WKB methods will not be available when x is close to the turning point. In this section we look for the solutions near the turning point.

7.1 Solution when $m \ll \mathcal{O}(1/a)$

For $m \ll \mathcal{O}(1/a)$, the turning point lies at $x = 1$. It can be approximated that $x^2 - 1 \approx 2(x - 1)$ as $x \to 1$. Since then, equation (6.7) can be approximated by

$$Z_{xx} = \frac{8(1-x)}{a^2} Z. \tag{7.1}$$

Let $\eta = 2(1-x)a^{-2/3}$, then we get $Z_{\eta\eta} = \eta Z$. Hence equation (7.1) can be solved using Airy functions according to Appendix 2. The solution is given by

$$Z(x) = \beta_1 \mathrm{Ai}(\eta) + \beta_2 \mathrm{Bi}(\eta). \tag{7.2}$$

Also by L'Hospital rule, here $g(x)$ (which is defined in section 6.2) can be estimated by

$$g(x) = g(1) - \int_x^1 \sqrt{2(1-x')}dx' = \frac{\pi}{4} - \frac{2\sqrt{2}}{3}(1-x)^{3/2}. \tag{7.3}$$

7.2 Solution when $m = \mathcal{O}(1/a)$

For $m = \mathcal{O}(1/a)$, when x is close to x_0, we find that

$$\begin{aligned}
\lim_{x\to x_0} 1 + \frac{\rho^2}{x^2} - x^2 &= \lim_{x\to x_0} \frac{\rho^2 + x^2 - x^4}{x^2} \qquad (7.4)\\
&= \lim_{x\to x_0} -\frac{(x^2 - (\frac{1}{2} + \sqrt{\frac{1}{4}+\rho^2}))(x^2 - (\frac{1}{2} - \sqrt{\frac{1}{4}+\rho^2}))}{x_0^2}\\
&= \lim_{x\to x_0} \frac{(x_0 - x)(x_0 + x)(x_0^2 - (\frac{1}{2} - \sqrt{\frac{1}{4}+\rho^2}))}{x_0^2}\\
&= \lim_{x\to x_0} \frac{2(x_0 - x)\sqrt{1+4\rho^2}}{x_0}.
\end{aligned}$$

Since $f'(x) = \sqrt{1 + \frac{\rho^2}{x^2} - x^2}$, we get from L'Hospital rule

$$f(x) = f(x_0) - \frac{2\sqrt{2}}{3}\frac{(x_0 - x)^{3/2}(1+4\rho^2)^{1/4}}{\sqrt{x_0}} \tag{7.5}$$

for $x \to x_0^-$.

Then equation (6.11) can be written as

$$Z_{xx} = \frac{8(x_0 - x)\sqrt{1+4\rho^2}}{a^2 x_0} Z. \tag{7.6}$$

According to Appendix 2, we let $\zeta = \frac{2(1+4\rho^2)^{1/6}}{x_0^{1/3}}\frac{x_0 - x}{a^{2/3}}$, this gives

$$Z_{\zeta\zeta} = \zeta Z, \tag{7.7}$$

which can also be solved in terms of Airy function that

$$Z(x) = \alpha_1 \mathrm{Ai}(\zeta) + \alpha_2 \mathrm{Bi}(\zeta). \tag{7.8}$$

Chapter 8

Matching

8.1 Intermediate region

At last, it is time to match solutions of different regions together. We introduced how to match solutions in Section 2 briefly. Now we explain it in more details.

First we introduce the intermediate region. As we know, we assumed that $y = \mathcal{O}(1)$ at the inner region and $|y - \kappa/a| \ll \mathcal{O}(1/a)$ at the bump-region, but how about the region with both $y = \mathcal{O}(1/a)$ and $\kappa/a - y = \mathcal{O}(1/a)$ between these two regions? We define it by the 'intermediate region' (see Figure (8.1)). In this region, it is also true that $|Q| \ll \mathcal{O}(1)$, hence the solution in the intermediate region must be the same with the WKB solution at far field.

8.2 Matching the WKB solution with the multi-bump solutions

In this subsection we consider the matching of the multi-bump solution with the WKB solutions at both the intermediate region and far field. Here we

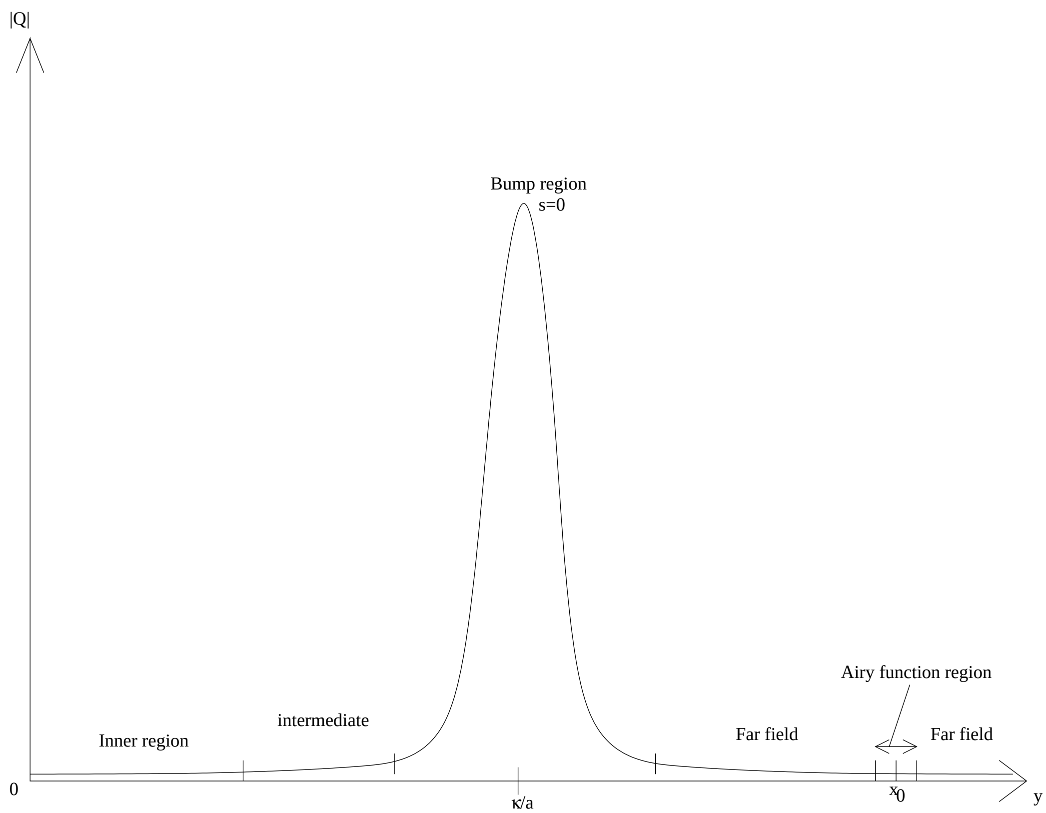

Figure 8.1: Different regions used for our matching

The intermediate region is the region where $|Q|$ is small and
$y = \mathcal{O}(1)$ between the inner region and the bump region;
WKB method is used in both intermediate region and far field;
The Airy function region is the region where x close
to the turning point of the WKB Method at far field.

set

$$t = x - \kappa/2 = as/2. \tag{8.1}$$

8.2.1 If $m \ll \mathcal{O}(1/a)$

In the case of $m \ll \mathcal{O}(1/a)$, rescale the WKB solution near the bump region use the Taylor expansion of small t, equation (6.9) becomes

$$\begin{aligned} g(x) &= g(\kappa/2) + tg'(\kappa/2) \\ &= \frac{1}{4}\kappa\sqrt{1-\kappa^2/4} + \frac{1}{2}\sin^{-1}(\frac{\kappa}{2}) + t\sqrt{1-\kappa^2/4} + \mathcal{O}(t^2). \end{aligned} \tag{8.2}$$

Combine with (6.2) and (6.8), Q can to leading order be expressed in terms of t by

$$\begin{aligned} Q(t) =& \frac{e^{-i\kappa t/a}}{(1-\kappa^2/4)^{1/4}}(\frac{a}{\kappa})^{(d-1)/2}[A_- e^{-\frac{2}{a}(g(\kappa/2)+t\sqrt{1-\kappa^2/4})} \\ &+ A_+ e^{\frac{2}{a}(g(\kappa/2)+t\sqrt{1-\kappa^2/4})}]. \end{aligned} \tag{8.3}$$

In the other hand, recall the leading order Q_0 and S_0 in the bump region which we computed in section 4.1, in terms of t it is given by

$$Q(t) = e^{-i\kappa t/a}\sqrt{2(1-\kappa^2/4)}\ \mathrm{sech}(\frac{2}{a}\sqrt{1-\kappa^2/4}\ t). \tag{8.4}$$

Here we look at the transition region where $0 < t = \mathcal{O}(a^{1/2})$, in this region the leading term of the multi-bump solution is given by

$$Q(t) = e^{-i\kappa t/a} 2\sqrt{2(1-\kappa^2/4)} e^{-\frac{2}{a}\sqrt{1-\kappa^2/4}\ t}. \tag{8.5}$$

Expressions (8.3) and (8.5) can be perfectly matched when $A_+ = 0$ and

$$A_- = 2\sqrt{2}(1-\kappa^2/4)^{3/4}(\frac{\kappa}{a})^{1/2} e^{\frac{2}{a}g(\kappa/2)}. \tag{8.6}$$

We also have to match the multi-bump solution with the WKB solution in the intermediate region. This time we match the WKB solution when $t \to 0_-$ together with the bump solution where $s \to -\infty$.

Here the asymptotic behavior of WKB solution remains the same as above. The approximation of multi-bump solution for $s \to -\infty$ is given by

$$Q(t) = e^{-i\kappa t/a} 2\sqrt{2(1-\kappa^2/4)} e^{\frac{2}{a}\sqrt{1-\kappa^2/4}\ t}. \tag{8.7}$$

Hence match with equation (8.3) we get $A_- = 0$ and

$$A_+ = 2\sqrt{2}(1-\kappa^2/4)^{3/4}(\frac{\kappa}{a})^{1/2} e^{-\frac{2}{a}g(\kappa/2)} \tag{8.8}$$

where κ is given in section 5.1 and 5.3 for $m = \mathcal{O}(1)$ and $m = \mathcal{O}(1/\sqrt{a})$ respectively.

8.2.2 If $m = \mathcal{O}(1/a)$

For $m = \mathcal{O}(1/a)$, use the Taylor expansion of (6.15) we get

$$\begin{aligned} f(x) =& \frac{1}{2}\rho \log(\kappa^2/4) - \frac{1}{2}\rho \log(2\rho^2 + 2\rho\sqrt{\kappa^2/4 + \rho^2 - \kappa^4/16} + \kappa^2/4) + \\ & \frac{1}{2}\sqrt{\kappa^2/4 + \rho^2 - \kappa^4/16} - \frac{1}{4}\arctan(\frac{1-\kappa^2/2}{2\sqrt{\kappa^2/4 + \rho^2 - \kappa^4/16}}) \\ & + t\sqrt{\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4} & (8.9) \\ =& f(\kappa/2) + t\sqrt{\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4}. & (8.10) \end{aligned}$$

Then the WKB expression for $Q(y)$ becomes

$$\begin{aligned} Q(t) =& \frac{e^{-i\kappa t/a}}{(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)^{1/4}}(\frac{a}{\kappa})^{(d-1)/2}[C_- e^{-\frac{2}{a}(f(\kappa/2) + t\sqrt{\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4})} \\ & + C_+ e^{\frac{2}{a}(f(\kappa/2) + t\sqrt{\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4})}]. & (8.11) \end{aligned}$$

The leading order of the solution in the multi-bump region is

$$Q(y) = e^{-i\kappa(y-\kappa/a)/2}\sqrt{2(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)}\ \mathrm{sech}(\sqrt{\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4}(y - \kappa/a)). \tag{8.12}$$

Again take $0 < t = \mathcal{O}(a^{1/2})$ the multi-bump solution can be estimate by

$$Q(t) = e^{-i\kappa t/a} 2\sqrt{2(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)} e^{-\frac{2}{a}\sqrt{\frac{4\rho^2}{\kappa^2}+1-\kappa^2/4}\ t}. \tag{8.13}$$

Then we get $C_+ = 0$ and

$$C_-(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)^{-1/4}(\frac{a}{\kappa})^{(d-1)/2} e^{-\frac{2}{a}f(\kappa/2)} = \sqrt{2(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)}. \tag{8.14}$$

So

$$C_- = 2\sqrt{2}(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)^{3/4}(\frac{\kappa}{a})^{(d-1)/2} e^{\frac{2}{a}f(\kappa/2)}. \tag{8.15}$$

In the other side $t < 0$, the approximation of multi-bump solution becomes

$$Q(t) = e^{-i\kappa t/a} 2\sqrt{2(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)} e^{-\frac{2}{a}\sqrt{\frac{4\rho^2}{\kappa^2}+1-\kappa^2/4}\ t}. \tag{8.16}$$

Match again, we get $C_- = 0$ and

$$C_+ = 2\sqrt{2}(\frac{4\rho^2}{\kappa^2} + 1 - \kappa^2/4)^{3/4}(\frac{\kappa}{a})^{(d-1)/2} e^{-\frac{2}{a}f(\kappa/2)}. \tag{8.17}$$

Here κ is given in section 5.2.

8.3 Matching the WKB solution with the solution near turning point

We've already derived the solution in a small neighborhood near the turning point by Airy functions in Section 7. Since then we can match the asymptotic behaviors of Airy functions at infinity with asymptotic behaviors of WKB solutions near turning point together.

8.3.1 If $m \ll \mathcal{O}(1/a)$

For $m \ll \mathcal{O}(1/a)$, combine (6.8) and (7.3), also put $A_+ = 0$ and we get that when $x \to 1^-$, the WKB solution $Z(x)$ is approximated by

$$Z(x) = \frac{A_-}{(1-x^2)^{1/4}} e^{-\frac{\pi}{2a}} e^{\frac{4\sqrt{2}}{3a}(1-x)^{3/2}}. \tag{8.18}$$

From (B.5), for $\eta \to +\infty$ the asymptotic solution in terms of Airy functions is given by

$$Z(x) = \frac{1}{\sqrt{\pi}} \frac{a^{1/6}}{2^{1/4}} \frac{1}{(1-x)^{1/4}} (\frac{1}{2}\beta_1 e^{-\frac{4\sqrt{2}}{3a}(1-x)^{3/2}} + \beta_2 e^{\frac{4\sqrt{2}}{3a}(1-x)^{3/2}}). \tag{8.19}$$

Now we can match two solutions together. Then to leading order, the β_1-term gives no contribution and we're not able to determine the value of β_1 here. The constants A_- and β_2 can be related by

$$\frac{a^{1/6}}{\sqrt{\pi}} \beta_2 = e^{-\pi/2a} A_-. \tag{8.20}$$

For the other side $x \to 1^+$, here the WKB solution becomes elliptic and the amplitude monotone decreasing, it is noted in [1] that it must has form

$$Z(x) \sim \frac{B}{(1-x^2)^{1/4}} \exp(\frac{2i}{a} \int_1^x \sqrt{s^2-1} ds) \tag{8.21}$$

with some constant B. By the approximation $\lim_{x \to 1}(x^2-1) \approx 2(x-1)$ and L'Hospital Rule, we have the estimation

$$Z(x) \approx \frac{B}{(2(1-x))^{1/4}} e^{\frac{4\sqrt{2}}{3a}(x-1)^{3/2}}. \tag{8.22}$$

For Airy function we consider that $\eta \to -\infty$. Combine with (B.5) we have

$$Z(x) \sim \frac{1}{\sqrt{\pi}} \frac{1}{(-\eta)^{1/4}} (\beta_1 \sin(\frac{2}{3}(-\eta)^{3/2} + \pi/4) + \beta_2 \cos(\frac{2}{3}(-\eta)^{3/2} + \pi/4)). \tag{8.23}$$

Also since $|Z|$ is monotone decreasing, the only possibility is that $\beta_1 = i\beta_2$ and hence

$$Z(x) \approx \frac{\beta_2}{\sqrt{\pi}} \frac{a^{1/6}}{(2(x-1))^{1/4}} e^{i(\frac{\pi}{4} + \frac{4\sqrt{2}}{3a}(x-1)^{3/2})}. \tag{8.24}$$

Match two solutions for $x \to 1^+$ together, where we use that $e^{i\pi/4} = (e^{i\pi})^{1/4} = (-1)^{1/4}$, they are perfectly matched by

$$B = \frac{\beta_2 a^{1/6}}{\sqrt{\pi}}. \tag{8.25}$$

Combine with (8.20), it gives that

$$B = A_- e^{-\pi/2a}. \tag{8.26}$$

8.3.2 If $m = \mathcal{O}(1/a)$

In case $m = \mathcal{O}(1/a)$, let $x \to x_0^-$, the WKB solutions can be estimated by combining (6.14) and (7.5)

$$Z(x) = \frac{C_-}{(1 + \frac{\rho^2}{x^2} - x^2)^{1/4}} e^{-\frac{2}{a} f(x_0)} e^{\frac{4\sqrt{2}}{3a} \frac{(x_0 - x)^{3/2}(1+4\rho^2)^{1/4}}{\sqrt{x_0}}}. \tag{8.27}$$

Here we have that $\zeta \to +\infty$. From the asymptotic form of Airy function, we get

$$\begin{aligned} Z(x) =& \frac{1}{\sqrt{\pi}} \frac{a^{1/6}}{2^{1/4}} \frac{x_0^{1/12}}{(x_0 - x)^{1/4}(1+4\rho^2)^{1/24}} (\frac{1}{2}\alpha_1 e^{-\frac{4\sqrt{2}}{3a} \frac{(x_0 - x)^{3/2}(1+4\rho^2)^{1/4}}{\sqrt{x_0}}} \\ &+ \alpha_2 e^{\frac{4\sqrt{2}}{3a} \frac{(x_0 - x)^{3/2}(1+4\rho^2)^{1/4}}{\sqrt{x_0}}}). \end{aligned} \tag{8.28}$$

Matching with the WKB term as $x \to x_0$, we can not determine α_1 now but we can get that

$$\frac{a^{1/6}}{\sqrt{\pi}} \alpha_2 = \frac{x_0^{1/6}}{(1+4\rho^2)^{1/12}} e^{-\frac{2}{a} f(x_0)} C_-. \tag{8.29}$$

For $x > x_0$, ζ goes to $-\infty$. The Airy function for large $-\zeta$ is asymptotic to

$$Z(x) \sim \frac{1}{\sqrt{\pi}}\frac{1}{(-\zeta)^{1/4}}(\alpha_1 \sin(\frac{2}{3}(-\zeta)^{3/2} + \pi/4) + \alpha_2 \cos(\frac{2}{3}(-\zeta)^{3/2} + \pi/4)). \tag{8.30}$$

Again $|Z(x)|$ is monotone decreasing. Hence this is consistent with (8.30) only when $\alpha_1 = i\alpha_2$ so that

$$Z(x) \sim \frac{1}{\sqrt{\pi}}\frac{a^{1/6}}{2^{1/4}}\frac{x_0^{1/12}\alpha_2}{(x-x_0)^{1/4}(1+4\rho^2)^{1/24}}e^{i(\frac{\pi}{4}+\frac{4\sqrt{2}}{3a}\frac{(x-x_0)^{3/2}(1+4\rho^2)^{1/4}}{\sqrt{x_0}})}. \tag{8.31}$$

The WKB approximation for $Z(x)$ is

$$\begin{aligned} Z(x) &\approx \frac{D}{(\frac{\rho^2}{x^2}+1-x^2)^{1/4}}\exp(\frac{2i}{a}\int_{x_0}^{x}\sqrt{s^2-\frac{\rho^2}{s^2}-1}\mathrm{d}s) \qquad (8.32)\\ &\approx \frac{x_0^{1/4}D}{2^{1/4}(1+4\rho^2)^{1/8}(x_0-x)^{1/4}}\exp(\frac{2i}{a}\int_{x_0}^{x}\sqrt{s^2-\frac{\rho^2}{s^2}-1}\mathrm{d}s) \end{aligned}$$

for some constant D. By similar approximation as (7.4), we have

$$\lim_{x\to x_0}(x^2-\frac{\rho^2}{x^2}-1) \approx \lim_{x\to x_0}\frac{2(x-x_0)\sqrt{1+4\rho^2}}{x_0}. \tag{8.33}$$

Also by L'Hospital Rule, we have

$$Z(x) \approx \frac{x_0^{1/4}D}{2^{1/4}(1+4\rho^2)^{1/8}(x_0-x)^{1/4}}e^{(\frac{4\sqrt{2}}{3a}\frac{(x-x_0)^{3/2}(1+4\rho^2)^{1/4}}{\sqrt{x_0}})i}. \tag{8.34}$$

Thus the two solutions again match perfectly by choosing

$$\frac{a^{1/6}}{\sqrt{\pi}}\alpha_2 = \frac{x_0^{1/6}D}{(1+4\rho^2)^{1/12}}. \tag{8.35}$$

Combining this with the relation of C_- and α_2 gives

$$D = C_- e^{-\frac{2}{a}f(x_0)}. \tag{8.36}$$

8.4 Matching the solutions at infinity

In this section, we consider the solution for $x \gg 1$. Since $x^2 \gg 1$ and $\rho^2/x^2 \ll 1$, the behavior of Z does not depend on the order of m. Hence in this analysis it is not necessary to vary the magnitude.

The equation of $Z(y)$ becomes

$$Z'' + (\frac{a^2y^2}{4} - 1)Z = 0. \tag{8.37}$$

Since the behaviour of $Q(y)$ is the same as the radially symmetric case, it has already been widely investigated in previous work [1,6,7]. Following these discussions, for large y the corresponding $Q(y)$ have the following asymptotic form

$$Q(y) = \mu y^{-1-i/a}[1 + o(1)] = \mu y^{-1} e^{-i\log(y)/a}[1 + o(1)] \tag{8.38}$$

for some μ.

We hereby match (8.38) with (8.21) or (8.32) for $x \to \infty$. We may estimate (8.21) and (8.32) in the same way (B can be replaced by D when m is large)

$$Z(x) \approx Be^{-i\pi/4}x^{-1/2} \exp\left(\frac{2i}{a}\int^x \sqrt{s^2 - 1 + \frac{1}{4s^2}}ds\right). \tag{8.39}$$

Here to leading order the lower bound of the integration is not important since the function is also large for x is large, also the term $\frac{1}{4s^2}$ can be omitted. Thus we can write the approximation by

$$Z(x) \approx Be^{-i\pi/4}x^{-1/2} \exp\left(\frac{2i}{a}\int_1^x \sqrt{s^2 - 1}ds\right) = Bx^{-1/2} \exp\left(\frac{2i}{a}(\frac{x^2}{2} - \frac{1}{2}\log(2x))\right). \tag{8.40}$$

Rescaling (8.38) to $Z(x)$ we get

$$Z(x) \approx \mu e^{i(x^2/a - \log(2x/a)/a)}(\frac{2x}{a})^{-1/2}. \tag{8.41}$$

Hence we get relationship that

$$\mu = (\frac{2}{a})^{1/2}\ B \qquad \mu = (\frac{2}{a})^{1/2} e^{-\frac{2}{a}g(\text{or } f)(x_0)} A_-. \tag{8.42}$$

Here B in the expressions can be replaced by D, and A_- in the expressions can be replaced by C_- while g is replaced by f if $m = \mathcal{O}(1/a)$.

8.5 Matching solutions with inner region

Finally, we match the WKB solution in the intermediate region with the Bessel solution at inner region. In section 3 we derived the leading order solutions of inner region by

$$Q(y) = \alpha\sqrt{2\pi}e^{-iay^2/4} I_m(y). \tag{8.43}$$

And the asymptotic expansion can be given by

$$Q(y) = \alpha\sqrt{2\pi}e^{-iay^2/4} I_m(y)(1 - \frac{a^2y^3}{24} + \cdots). \tag{8.44}$$

For the asymptotic behavior of I_m we have two approximations [5] which we will use:
If $m \ll \mathcal{O}(1/\sqrt{a})$, to leading order we have

$$I_m(y) \sim \frac{e^y}{\sqrt{2\pi y}}. \qquad as \qquad |y| \to \infty \tag{8.45}$$

If $m = \mathcal{O}(1/\sqrt{a})$ or $m = \mathcal{O}(1/a)$, to leading order

$$I_m(y) \sim \frac{1}{\sqrt{2\pi m}} \frac{e^{m\nu}}{(1 + y^2/m^2)^{1/4}}, \tag{8.46}$$

where $\nu = \sqrt{1 + y^2/m^2} + \log \frac{y}{m+\sqrt{m^2+y^2}}$.

8.5.1 If $m \ll \mathcal{O}(1/\sqrt{a})$

First consider if $m \ll \mathcal{O}(1/\sqrt{a})$. In section 8.2 we showed that the WKB solution for this range of m is given by

$$Z(x) = \frac{A_+}{(1-x^2)^{1/4}} e^{\frac{2}{a}g(x)}. \tag{8.47}$$

We hereby match the WKB solution when $x \to 0$ with Bessel solution for $y \to \infty$. For small x we have

$$g(x) \sim \frac{1}{2}x \cdot 1 + \frac{1}{2}x = x. \tag{8.48}$$

Then the approximation of WKB solution can be given by

$$Q \approx e^{-iay^2/4} y^{-1/2} A_+ e^{\frac{2}{a}x} = A_+ e^{-iay^2/4} \frac{e^y}{\sqrt{y}}. \tag{8.49}$$

Use (8.45) to estimate Bessel solution for large y, we have

$$Q \approx \alpha\sqrt{2\pi} e^{-iay^2/4} \frac{e^y}{\sqrt{2\pi y}}. \tag{8.50}$$

We can see that these two solutions match perfectly if

$$\alpha = A_+. \tag{8.51}$$

8.5.2 If $m = \mathcal{O}(1/\sqrt{a})$

In case $m = \mathcal{O}(1/\sqrt{a})$, the approximation of WKB solution is same above. On the other hand in inner region, we use (8.46) for the Bessel solution instead of (8.45).

The expansion (8.44) is only available for $y \ll \mathcal{O}(a^{-2/3})$. Hence we consider the transition region $\mathcal{O}(1/\sqrt{a}) \ll y \ll \mathcal{O}(a^{-2/3})$ and let $z = y/m$. Then z is also large, $(1+y^2/m^2)^{1/4} = (1+z^2)^{1/4} \sim z^{1/2}$ and $\nu = \sqrt{1+z^2} + \log \frac{z}{1+\sqrt{1+z^2}} \sim z + \log \frac{z}{z} = z$. hence the approximation becomes

$$I_m(y) \approx \frac{e^{mz}}{\sqrt{2\pi m z}} = \frac{e^y}{\sqrt{2\pi y}} \tag{8.52}$$

which is also the same as $m \ll \mathcal{O}(1/\sqrt{a})$.

Thus both are the same as in subsection 8.6.1 and we match in a similar way as done there.

8.5.3 If $m = \mathcal{O}(1/a)$

Finally consider case $m = \mathcal{O}(1/a)$ which is more difficult. We must hereby match in an transition region $y = \mathcal{O}(1/\sqrt{a})$, here $x = ay/2 = \mathcal{O}(\sqrt{a})$. By transformation we get $z = y/m = x/\rho$ is also small. Then

$$\nu \approx 1 + \log\frac{z}{2} = 1 + \log x - \log 2\rho = 1 + \log x - \log(am). \tag{8.53}$$

Hence

$$Q = \alpha\sqrt{2\pi}e^{-iay^2/4}I_m(y) \approx \alpha\frac{(\frac{2e}{\rho})^m \cdot e^{m\log x}}{\sqrt{m}}e^{-ix^2/a}. \tag{8.54}$$

Note that the WKB solution in this range is given in terms of $f(x)$ and C_+.

Expression (6.15) gives the definition of $f(x)$. When x is small the first term of $f(x)$ becomes $-\infty$ while others remains $\mathcal{O}(1)$. Hence the first term will dominate the rest. We get

$$\frac{2}{a}f(x) \sim \frac{2}{a}\rho\log(x) = m\log(x). \tag{8.55}$$

Put into the WKB solution

$$Q = e^{-ix^2/a}(\frac{2x}{a})^{-1/2}\frac{C_+}{(1+\frac{\rho^2}{x^2}-x^2)^{1/4}}e^{\frac{2}{a}f(x)}. \tag{8.56}$$

This gives that to leading order

$$Q \approx e^{-iay^2/4}\frac{C_+e^{m\log(x)}}{\sqrt{\rho/x}\sqrt{y}} = e^{-ix^2/a}\frac{C_+e^{m\log(x)}}{\sqrt{m}}. \tag{8.57}$$

Since then, two solutions also match perfectly if

$$\alpha = (\frac{am}{e})^m C_+ = (\frac{2\rho}{e})^{\frac{2\rho}{a}} C_+. \tag{8.58}$$

8.6 Derive the relationship between d, m and a

In this section we relate d to a and m. It is done by estimating μ from the behavior of $Q(y)$ at large y. For further discussion, we use amplitude and phase decomposition of $Q(y)$ again

$$Q(y) = A(y)e^{i\theta(y)}. \tag{8.59}$$

It has already been calculated in Chapter 3 that

$$\theta_y + \frac{a}{2}y = -\frac{(d-2)}{yA^2}\int_0^y \theta_y A^2 dy'. \tag{8.60}$$

For $x \gg 1$ we have

$$A \approx \mu y^{-1} \text{ and } \theta_y \approx -1/(ay). \tag{8.61}$$

Hence θ_y is small and the leading order is

$$\frac{a}{2}y = -\frac{(d-2)y}{\mu^2}\int_0^y \theta_y A^2 dy'. \tag{8.62}$$

Since $\theta_y A^2 = o(y^{-2})$ as y large, the integral converges for $y \to \infty$. Hence we have

$$\frac{a}{2} = -\frac{(d-2)}{\mu^2}\int_0^y \theta_y A^2 dy'. \tag{8.63}$$

So that

$$\mu^2 = -\frac{2(d-2)}{a}\int_0^\infty \theta_y A^2 dy'. \tag{8.64}$$

To estimate this integral, it is necessary to study the different cases of m separately.

8.6.1 If $m \ll \mathcal{O}(1/a)$

For $m \ll \mathcal{O}(1/a)$, first we look at the solution (8.4) in the bump region. Rescale it into y and in terms of amplitude and phase we get

$$A^2(y) = 2(1-\kappa^2/4)\text{sech}^2(\sqrt{1-\kappa^2/4}(y-\kappa/a)) \text{ and } \theta_y = -\kappa/2. \quad (8.65)$$

Then $\theta_y A^2 = o(y^{-2})$ if both $|y|$ and $|y-\kappa/a|$ are large, hence the contribution to the integral for large y is small. Also the amplitude at inner region is small. Hence to leading order we have

$$\int_0^\infty \theta_y A^2 dy' = -\int_{-\infty}^\infty \kappa(1-\kappa^2/4)\text{sech}^2(\sqrt{1-\kappa^2/4}\; s)ds = -2\kappa\sqrt{1-\kappa^2/4}. \quad (8.66)$$

Hence

$$\mu^2 = \frac{4(d-2)}{a}\kappa\sqrt{1-\kappa^2/4}. \quad (8.67)$$

Combine with

$$A_- = 2\sqrt{2}(1-\kappa^2/4)^{3/4}(\frac{\kappa}{a})^{1/2}e^{\frac{2}{a}g(\kappa/2)}, \quad (8.68)$$

we get

$$8(1-\kappa^2/4)^{3/2}(\frac{\kappa}{a})^{d-1}e^{\frac{4}{a}(g(\kappa/2)-\frac{\pi}{4})} = 2(d-2)\kappa\sqrt{1-\kappa^2/4}. \quad (8.69)$$

Thus

$$d-2 = \frac{\beta}{a^{d-1}}e^{-\lambda/a} \quad (8.70)$$

where

$$\lambda = 4(g(1)-g(\kappa/2)) = \pi - 2\sin^{-1}(\kappa/2) - \kappa\sqrt{1-\kappa^2/4} \quad (8.71)$$

and

$$\beta = 4\kappa^{d-2}(1-\kappa^2/4). \quad (8.72)$$

If $m \ll \mathcal{O}(1/\sqrt{a})$, then $\kappa = 1$. Hence we have $\beta = 3$ and

$$\lambda = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}. \quad (8.73)$$

8.6.2 If $m = \mathcal{O}(1/a)$

For $m = \mathcal{O}(1/a)$. The leading order of integral (8.64) is

$$\int_0^\infty \theta_y A^2 dy' = -\int_{-\infty}^\infty \kappa(1-\kappa^2/4+4\rho^2/\kappa^2)\text{sech}^2(\sqrt{1-\kappa^2/4+4\rho^2/\kappa^2}\ s)ds \tag{8.74}$$
$$= -2\kappa\sqrt{1-\kappa^2/4+4\rho^2/\kappa^2}.$$

Thus

$$\mu^2 = \frac{4(d-2)}{a}\kappa\sqrt{1-\kappa^2/4+4\rho^2/\kappa^2}. \tag{8.75}$$

This time combine with

$$C_-(\frac{4\rho^2}{\kappa^2}+1-\kappa^2/4)^{-1/4}(\frac{a}{\kappa})^{(d-1)/2}e^{-\frac{2}{a}f(\kappa/2)} = \sqrt{2(\frac{4\rho^2}{\kappa^2}+1-\kappa^2/4)}, \tag{8.76}$$

then

$$8(\frac{4\rho^2}{\kappa^2}+1-\kappa^2/4)^{3/2}(\frac{\kappa}{a})^{d-1}e^{\frac{4}{a}(f(\kappa/2)-f(x_0))} = 2(d-2)\kappa\sqrt{1-\kappa^2/4+4\rho^2/\kappa^2}. \tag{8.77}$$

Thus

$$d-2 = \frac{\gamma}{a^{d-1}}e^{-\delta/a} \tag{8.78}$$

Where

$$\delta = 4(f(x_0)-f(\kappa/2)) = 4\int_{\kappa/2}^{x_0}\sqrt{1-x^2+\rho^2/x^2}dx \tag{8.79}$$

and

$$\gamma = 4\kappa^{d-2}(1-\kappa^2/4+4\rho^2/\kappa^2). \tag{8.80}$$

Both of them must be positive.

It is quite complicate to compute γ and δ precisely. For instance, if we let $\kappa = \sqrt{\frac{1+\sqrt{1-32\rho^2}}{2}}$, combine with the expression of γ, we get

$$\gamma = 4(\frac{1+\sqrt{1-32\rho^2}}{2})^{\frac{d-2}{2}}(\frac{9}{8} - \frac{3}{8}\sqrt{1-32\rho^2}) \tag{8.81}$$

and

$$\begin{aligned}\delta =& 2\rho(\log(1+\sqrt{1+4\rho^2}) - \log(4\rho^2+1+\sqrt{1+4\rho^2})) - \frac{\pi}{2} - 2\rho(\log(1+\sqrt{1-32\rho^2}) \\ &- \log(16\rho^2+1+\sqrt{1-32\rho^2}+\sqrt{6}\rho\sqrt{1+\sqrt{1-32\rho^2}})) - \frac{\sqrt{6}}{4}\sqrt{1+\sqrt{1-32\rho^2}} \\ &+ \arctan(\frac{3-\sqrt{1-32\rho^2}}{\sqrt{6+6\sqrt{1-32\rho^2}}}),\end{aligned} \tag{8.82}$$

which only exist when

$$\rho \leq \frac{\sqrt{2}}{8}. \tag{8.83}$$

Note that we have that $d-2$ is exponentially small in a for every order of m.

Chapter 9

Statement of Results

We hereby state the solutions of (1.18) in terms of Q and y.

9.1 Conditions of the solutions

We do not get the solution in every circumstance. Due to our work, if $m = \mathcal{O}(1/\sqrt{a})$ then solutions only exist when $am^2 \leq \frac{2(3-\sqrt{7})\sqrt{2\sqrt{7}-4}}{3}$, if $m = \mathcal{O}(1/a)$ then solutions only exist when $am \leq \sqrt{2}/4$.

9.2 For $m = \mathcal{O}(1)$

The leading order of location of maxima is at $1/a$.

At inner region, the leading order of solution is given by

$$Q(y) = A_+\sqrt{2\pi}I_m(y)e^{-iay^2/4}. \tag{9.1}$$

At bump region, the leading order of solution is given by

$$Q(y) = \sqrt{3/2}\ \text{sech}(\sqrt{3/4}\ (y - 1/a))e^{-i(y-1/a)/2}. \tag{9.2}$$

At far field, when $0 < 2/a - y \gg \mathcal{O}(1)$, combine (6.2), (6.8) and (6.9), we get

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}\frac{A_-}{(1-a^2y^2/4)^{1/4}}e^{-(\frac{y}{2}\sqrt{1-a^2y^2/4}+\frac{1}{a}\sin^{-1}(\frac{ay}{2}))}. \tag{9.3}$$

When $0 < y - 2/a \gg \mathcal{O}(1)$, combine (6.2), (8.21) and (8.26) we get

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}\frac{A_-e^{-\frac{\pi}{2a}+\frac{i\pi}{4}}}{(a^2y^2/4-1)^{1/4}}e^{i(\frac{y}{2}\sqrt{1-a^2y^2/4}-\frac{1}{a}\log(ay/2+\sqrt{1-a^2y^2/4}))}. \tag{9.4}$$

When $|2/a - y| = \mathcal{O}(1)$, from the solution of Airy function we have

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}\frac{\sqrt{\pi}e^{-\pi/2a}A_-}{a^{1/6}}(i\text{Ai}((2-ay)a^{-2/3}) + \text{Bi}((2-ay)a^{-2/3})). \tag{9.5}$$

For the intermediate region between inner region and bump region, combine (6.2), (6.8) and (6.9), the solution is given by

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}\frac{A_+}{(1-a^2y^2/4)^{1/4}}e^{\frac{y}{2}\sqrt{1-a^2y^2/4}+\frac{1}{a}\sin^{-1}(\frac{ay}{2})}. \tag{9.6}$$

Here we have two parameters A_+ and A_-, from chapter 8 their values are given by

$$A_- = 2\sqrt{2}(3/4)^{3/4}\sqrt{\frac{1}{a}}e^{\frac{1}{a}(\frac{\pi}{6}+\frac{\sqrt{3}}{4})}$$

and

$$A_+ = 2\sqrt{2}(3/4)^{3/4}\sqrt{\frac{1}{a}}e^{-\frac{1}{a}(\frac{\pi}{6}+\frac{\sqrt{3}}{4})}.$$

The relationship between d and a is

$$d - 2 = \frac{3}{a^{d-1}}e^{-(2\pi/3-\sqrt{3}/2)/a}. \tag{9.7}$$

9.3 For $m = \mathcal{O}(1/a)$

The first order of location of maxima is at κ/a, with κ satisfies

$$\kappa^4 - \kappa^2 + 2a^2m^2 = 0. \tag{9.8}$$

It is true that we can only find κ when $am \leq \frac{1}{2\sqrt{2}}$.

Since then we get $a^2m^2 = \frac{1}{2}(\kappa^2 - \kappa^4)$.

At inner region, the first order of solution is given by

$$Q(y) = (\frac{am}{e})^m C_+ \sqrt{2\pi} e^{-iay^2/4} I_m(y). \tag{9.9}$$

At bump region, the first order of solution is given by

$$Q(y) = \sqrt{3 - \frac{3\kappa^2}{2}} \operatorname{sech}(\sqrt{\frac{3}{2} - \frac{3\kappa^2}{4}}\ (y - \kappa/a)) e^{-i(y-\kappa/a)/2}. \tag{9.10}$$

At far field, the turning point happens at $y = 2x_0/a$, where

$$x_0 = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + a^2m^2}}. \tag{9.11}$$

When $0 < 2x_0/a - y \gg \mathcal{O}(1)$, combine (6.2) and (6.14) we have

$$Q(y) = e^{-iay^2/4} y^{(1-d)/2} \frac{C_-}{(1 + \frac{m^2}{y^2} - \frac{a^2y^2}{4})^{1/4}} e^{-\frac{2}{a} f(\frac{ay}{2})} \tag{9.12}$$

where

$$f(x) = \int_0^x \sqrt{1 + \frac{a^2m^2}{4s^2} - s^2 ds}. \tag{9.13}$$

When $0 < y - 2x_0/a \gg \mathcal{O}(1)$, combine (6.2) and (8.32), we have

$$Q(y) = e^{-iay^2/4} y^{(1-d)/2} \frac{C_- e^{-\frac{2}{a} f(x_0) + \frac{i\pi}{4}}}{(\frac{a^2y^2}{4} - \frac{m^2}{y^2} - 1)^{1/4}} \exp(\frac{2i}{a} \int_{x_0}^{ay/2} \sqrt{s^2 - \frac{a^2m^2}{4s^2} - 1 \mathrm{d}s}). \tag{9.14}$$

When $|2x_0/a - y| = \mathcal{O}(1)$, the solution is given in terms of Airy function,

$$Q(y) = e^{-iay^2/4} y^{(1-d)/2} \frac{\sqrt{\pi} x_0^{1/6} e^{-\frac{2}{a} f(x_0)} C_-}{a^{1/6}(1+a^2m^2)^{1/12}} (i\mathrm{Ai}(\frac{(1+a^2m^2)^{1/6}(2x_0 - ay)}{x_0^{1/3} a^{2/3}}) + \mathrm{Bi}(\frac{(1+a^2m^2)^{1/6}(2x_0 - ay)}{x_0^{1/3} a^{2/3}})). \quad (9.15)$$

For the intermediate region between inner region and bump region, combine (6.2), (6.14), the solution is given by

$$Q(y) = e^{-iay^2/4} y^{(1-d)/2} \frac{C_+}{(1 + \frac{m^2}{y^2} - \frac{a^2y^2}{4})^{1/4}} e^{\frac{2}{a} f(\frac{ay}{2})}. \quad (9.16)$$

The values of C_+ and C_- are

$$C_- = \frac{3\sqrt{2}}{2}(2 - kappa^2)^{3/4} (\frac{\kappa}{a})^{(d-1)/2} e^{\frac{2}{a} f(\kappa/2)}$$

and

$$C_+ = \frac{3\sqrt{2}}{2}(2 - kappa^2)^{3/4} (\frac{\kappa}{a})^{(d-1)/2} e^{-\frac{2}{a} f(\kappa/2)}.$$

At last, the relationship between d and a is

$$d - 2 = \frac{3\kappa^{d-2}(2 - kappa^2)}{a^{d-1}} e^{-4(f(x_0) - f(\kappa/2))/a}. \quad (9.17)$$

9.4 For $m = \mathcal{O}(1/\sqrt{a})$

The first order of location of maxima is at $\tilde{\kappa}/a$ (here we use $\tilde{\kappa}$ to distinguish with κ used before), with $\tilde{\kappa}$ satisfies

$$am^2 = \frac{2\tilde{\kappa}(1 - \tilde{\kappa}^2)}{3\sqrt{1 - \tilde{\kappa}^2/4}}. \quad (9.18)$$

This equation have solution only if $am^2 \le \frac{2(3-\sqrt{7})\sqrt{2\sqrt{7}-4}}{3}$.

At inner region, the first order of solution is given by

$$Q(y) = A'_+\sqrt{2\pi}I_m(y)e^{-iay^2/4}. \tag{9.19}$$

At bump region, the first order of solution is given by

$$Q(y) = \sqrt{2(1-\tilde{\kappa}^2/4)}\ \mathrm{sech}(\sqrt{1-\tilde{\kappa}^2/4}\ (y-\tilde{\kappa}/a))e^{-i\tilde{\kappa}(y-\tilde{\kappa}/a)/2}. \tag{9.20}$$

At far field, when $0 < 2/a - y \gg \mathcal{O}(1)$, combine (6.2), (6.8) and (6.9). Similarly with $m = \mathcal{O}(1)$ case we get

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}\frac{A'_-}{(1-a^2y^2/4)^{1/4}}e^{-(\frac{y}{2}\sqrt{1-a^2y^2/4}+\frac{1}{a}\sin^{-1}(\frac{ay}{2}))}. \tag{9.21}$$

When $0 < y - 2/a \gg \mathcal{O}(1)$, combine (6.2), (8.21) and (8.26) we get

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}\frac{A'_- e^{-\frac{\pi}{2a}+\frac{i\pi}{4}}}{(a^2y^2/4-1)^{1/4}}e^{i(\frac{y}{2}\sqrt{1-a^2y^2/4}-\frac{1}{a}\log(ay/2+\sqrt{1-a^2y^2/4}))}. \tag{9.22}$$

When $|2/a - y| = \mathcal{O}(1)$, from the solution of Airy function we have

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}\frac{\sqrt{\pi}e^{-\pi/2a}A'_-}{a^{1/6}}(i\mathrm{Ai}((2-ay)a^{-2/3}) + \mathrm{Bi}((2-ay)a^{-2/3})). \tag{9.23}$$

For the intermediate region between inner region and bump region (for which we have $y \gg \mathcal{O}(1)$ and $|Q(y)| \ll \mathcal{O}(1)$), the solution given by

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}\frac{A'_+}{(1-a^2y^2/4)^{1/4}}e^{\frac{y}{2}\sqrt{1-a^2y^2/4}+\frac{1}{a}\sin^{-1}(\frac{ay}{2})}. \tag{9.24}$$

Here A'_+ and A'_- are given by

$$A'_- = 2\sqrt{2}(1-\tilde{\kappa}^2/4)^{3/4}\sqrt{\frac{\tilde{\kappa}}{a}}e^{\frac{1}{a}(\frac{\tilde{\kappa}}{2}\sqrt{1-\tilde{\kappa}^2/4}+\sin^{-1}(\tilde{\kappa}/2))}$$

and

$$A'_+ = 2\sqrt{2}(1-\tilde{\kappa}^2/4)^{3/4}\sqrt{\frac{\tilde{\kappa}}{a}}e^{-\frac{1}{a}(\frac{\tilde{\kappa}}{2}\sqrt{1-\tilde{\kappa}^2/4}+\sin^{-1}(\tilde{\kappa}/2))}.$$

The relationship between d and a is

$$d-2 = \frac{4\tilde{\kappa}^{d-2}(1-\tilde{\kappa}^2/4)}{a^{d-1}}e^{-(\pi-2\sin^{-1}(\tilde{\kappa}/2)-\tilde{\kappa}\sqrt{1-\tilde{\kappa}^2/4})/a}. \tag{9.25}$$

Chapter 10

Conclusions

In this thesis, we constructed the vortex solutions of the NLS that become infinite in finite time under some conditions. The idea of solving the equation (1.18) is come from Chris Budd's previous work[1]. His work can also be considered as the special case $m = 0$.

In our work, we discovered that if the parameter m is of order 1, then the location of the peak of the solution is to leading order $1/a$ which is almost the same place with case $m = 0$.
In the range $1 \ll y < 2/a$ we have to leading order

$$|Q(y)| \sim \sqrt{3/2}\, \mathrm{sech}(\sqrt{3/4}\ (y - 1/a)). \tag{10.1}$$

If $0 \leq y \leq a^{-2/3}$, then to leading order

$$Q(y) = e^{-iay^2/4} 4(3/4)^{3/4} \sqrt{\frac{\pi}{a}} e^{-\frac{1}{a}(\frac{\pi}{6}+\frac{\sqrt{3}}{4})} I_m(y). \tag{10.2}$$

And to leading order we have the relationship

$$d - 2 = \frac{3}{a^{d-1}} e^{-(2\pi/3-\sqrt{3}/2)/a}. \tag{10.3}$$

So the first order of solution is almost the same as case $m = 0$, only the solution at inner region is a little different. But if we put $m = 0$ into our result, it just corresponds to what Chris Budd did. This could somehow verify our result. Also the relationship between d and a is the same with previous work.

When we enlarge the scale of m, things become different and more interesting, especially when we consider m to be of order $1/a$, the location of maxima have the following relationship

$$\kappa^4 - \kappa^2 + 2a^2m^2 = 0. \tag{10.4}$$

So the location of maxima can have two possible results if $am < \sqrt{2}/4$, or no results if $am > \sqrt{2}/4$. Here 'two results' doesn't mean the solution have to peaks, but there are two possible solutions where each has one peak.

The relation between d and a is somehow depend on m when m is of order $1/a$ which is different from the case $m = \mathcal{O}(1)$. Despite this however, we can still get the conclusion from our result that $d - 2$ is exponentially small in a (What we assume is only $d - 2$ is small, but not exponentially small), which is similar to results previously found [1].

At last, note again that we didn't find solutions for all m. For $m = \mathcal{O}(1/\sqrt{a})$, we can't find solutions if $am^2 > \sqrt{3 - \sqrt{7}}$. For $m = \mathcal{O}(1/a)$, we can't find solutions if $am > \sqrt{2}/4$. And we didn't investigate non-vortex cases either. Hence there are still many further researches to be made.

Appendix A

WKB Methods

There are two standard methods that we used in our calculation, we hereby put them in the appendix [10].

In order to solve (6.7) and (6.11), we consider the differential equation for general case

$$\varepsilon^2 y'' - q(x)y = 0 \tag{A.1}$$

with ε a small constant.

We first assume that the coefficient q is constant. Then the solution is very easy to give

$$y(x) = a_0 e^{-\frac{1}{\varepsilon} x\sqrt{q}} + b_0 e^{\frac{1}{\varepsilon} x\sqrt{q}} \tag{A.2}$$

as $x \to \infty$. The solution goes exponentially large if $q > 0$ and remains oscilliating otherwise. Since then we discover that the behaviour is totally different for both sides of $q = 0$, which we called the 'turning-point'.

For a more general case, we make the hypothesis that the form of asymptotic expansion of the solution is

$$y \sim e^{\theta(x)/\varepsilon}[y_0(x) + \varepsilon y_1(x) + \cdots]. \tag{A.3}$$

One of the distinctive features of the WKB method is that it is fairly specific on how the solution depends on the fast variation, namely, the dependence is assumed to be exponential.

From (A.3) we get

$$y' \sim (\varepsilon^{-1}\theta_x y_0 + y_0' + \theta_x y_1 + \cdots)e^{\theta/\varepsilon} \tag{A.4}$$

and

$$y'' \sim [\varepsilon^{-2}\theta_x^2 y_0 + \varepsilon^{-1}(\theta_{xx}y_0 + 2\theta_x y_0' + \theta_x^2 y_1) + \cdots]e^{\theta/\varepsilon}. \tag{A.5}$$

Combine with (A.1) leads to the following equations:
$O(1)$ $(\theta_x)^2 = q(x)$ (eikonal equation),
$O(\varepsilon)$ $\theta_{xx}y_0 + 2\theta_x y_0' + \theta_x^2 y_1 = q(x)y_1$ (transport equation).
It follows that $\theta_{xx}y_0 + 2\theta_x y_0' = 0$. Since then, we may find the solution of these two equations are

$$\theta(x) = \pm \int^x \sqrt{q(s)}ds \tag{A.6}$$

and

$$y_0(x) = \frac{c}{\sqrt{\theta_x}} \tag{A.7}$$

where c is an arbitrary constant.

Then the first-term approximation of (A.1) is

$$y \sim q(x)^{-1/4}(a_0 e^{-\frac{1}{\varepsilon}\int^x \sqrt{q(s)}ds} + b_0 e^{\frac{1}{\varepsilon}\int^x \sqrt{q(s)}ds}) \tag{A.8}$$

where a_0 and b_0 are arbitrary, possibly complex, constants. Similarly, we call $q(x) = 0$ the "turning-points".

The WKB solutions become deficient close to the turning points since that $|q(x)|$ is small here and the asymptotic expansion could not use again.

Appendix B

Airy function

In order to solve (7.1), in this part we introduce the solution of the equation

$$y'' = xy. \tag{B.1}$$

The solution is given by the linear combination of Airy functions

$$y(x) = \alpha_0 Ai(x) + \beta_0 Bi(x) \tag{B.2}$$

where

$$Ai(x) \equiv \frac{1}{3^{2/3}\pi} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{k+1}{3}) \sin[\frac{2\pi}{3}(k+1)]}{k!} (3^{1/3}x)^k \tag{B.3}$$

and

$$Bi(x) \equiv e^{\pi i/6} Ai(xe^{2\pi i/3}) + e^{-\pi i/6} Ai(xe^{-2\pi i/3}). \tag{B.4}$$

The Airy functions have asymptotic approximations that

$$Ai(x) \sim \frac{1}{\sqrt{\pi}|x|^{1/4}} [\cos(z - \frac{\pi}{4}) + w \sin(z - \frac{\pi}{4})] \text{ as } x \to -\infty,$$

$$Ai(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-z}(1 - w) \text{ as } x \to +\infty,$$

$$Bi(x) \sim \frac{1}{\sqrt{\pi}|x|^{1/4}}[\cos(z + \frac{\pi}{4}) + w \sin(z + \frac{\pi}{4})] \text{ as } x \to -\infty,$$

$$Bi(x) \sim \frac{1}{\sqrt{\pi}x^{1/4}} e^{z}(1 + w) \text{ as } x \to +\infty, \tag{B.5}$$

where $z = \frac{2}{3}|x|^{3/2}$ and $w = \frac{5}{72z}$.

Consider for more general case

$$y'' = c(x_0 \pm x)y \tag{B.6}$$

where c and x_0 arbitrary constant.

Let $\zeta = c^{1/3}(x_0 \pm x)$, then $y_{xx} = c^{2/3}y_{\zeta\zeta}$. Hence we get

$$c^{2/3}y_{\zeta\zeta} = c \cdot c^{-1/3}\zeta y = c^{2/3}\zeta y. \tag{B.7}$$

Then the solution of (B.6) can be given

$$y(x) = \alpha_0 Ai(\zeta) + \beta_0 Bi(\zeta). \tag{B.8}$$

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