The Grothendieck-Riemann-Roch Theorem

Javan Peykar, Ariya.

Ariyan Javanpeykar

The Grothendieck-Riemann-Roch Theorem

With an Application to Covers of Varieties

Master's thesis, defended on June 17, 2010

Thesis advisor: Jaap Murre

Mathematisch Instituut
Universiteit Leiden

Contents

Introduction

The classical Riemann-Roch problem can be stated as follows in modern language. For a compact Riemann surface X of genus g and a divisor D on X, how can we calculate $\dim H^0(X, \mathcal{O}_X(D))$? There is no general answer to this question. Instead, we can show that

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(K - D)) = \deg D + 1 - g,$$

where K is the cotangent bundle of X and $\deg D$ is the degree of D. This is the Riemann-Roch theorem for Riemann surfaces. Invoking Serre duality and writing $\mathcal{L} = \mathcal{O}_X(D)$, we see that the Riemann-Roch theorem is equivalent to

$$\dim H^0(X, \mathcal{L}) - \dim H^1(X, \mathcal{L}) = \int_X \left(\frac{c_1(K^\vee)}{2} + c_1(\mathcal{L}) \right),$$

where c_1 is the first Chern class and $K^\vee$ is the dual of K. The left-hand side of this equation is the Euler characteristic $\chi(X, \mathcal{L})$. Now, one would like to generalize the Riemann-Roch theorem to compact complex manifolds X of any dimension, i.e., to give a formula for $\chi(X, \mathcal{L})$ when $\mathcal{L}$ is a line bundle on X. The general formula was shown by Hirzebruch ([**Hirz**]): for any holomorphic vector bundle $\mathcal{E}$ on a compact complex manifold X, we have that

$$\chi(X, \mathcal{E}) = \int_X \operatorname{ch}(\mathcal{E}) \operatorname{td}(X),$$

where $\operatorname{ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$ and $\operatorname{td}(X)$ is the Todd class of the tangent bundle $\mathcal{T}_X$ of X. Now, the above theorem is known as the Hirzebruch-Riemann-Roch theorem and could also be interpreted as

$$\text{some cohomological invariant of } \mathcal{E} = \int_X \left(\text{some characteristic class of } X \text{ and } \mathcal{E}\right).$$

By now, the importance of the Euler characteristic

$$\chi(X, E) = \sum (-1)^i \dim H^i(X, E)$$

was noticed.

In proving a Riemann-Roch theorem for smooth projective varieties, Grothendieck took on a completely different approach. For starters, the base field $\mathbf{C}$ was replaced by a field of any characteristic. Hirzebruch's analytic methods are thus not applicable. Also, Grothendieck proved a "relativized version" of the Riemann-Roch theorem which is much more powerful than Hirzebruch's theorem. For example, in a review of Grothendieck's work for *Mathematical Reviews*, Bott wrote "Grothendieck has generalized the theorem to the point where not only it is more generally applicable than Hirzebruch's version, but it depends on a simpler and more natural proof". Moreover, while developing "the" right setting for his theorem, he developed many new concepts such as K-theory and λ-rings while providing new perspectives for intersection theory and characteristic classes. By "the" right setting, we mean Grothendieck's idea to consider all coherent sheaves (i.e., not just the locally free ones) and to replace the cohomology ring by the Chow ring.

We can explain Grothendieck's approach by looking a bit closer at Hirzebruch's theorem. Let X be a compact complex variety and let f be a morphism from X to a point. We can rewrite Hirzebruch's theorem as

$$\sum(-1)^i \dim R^i f_*\mathcal{E} = f_*\left(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X)\right), \tag{1}$$

where f_* on the left-hand side is the direct image functor (i.e., global sections) and f_* on the right-hand side is the Gysin homomorphism (i.e., integration). Now, let X be a projective smooth variety over a field k with structure morphism $f : X \longrightarrow \operatorname{Spec} k$. Assuming we have defined "the" right objects, the Riemann-Roch theorem for f should be similar to equation (1). A proof of such a theorem could then be approached as follows. One starts by embedding X into a projective space $\mathbf{P}^n_k$ via a closed immersion $i : X \longrightarrow \mathbf{P}^n_k$. Then, one proves equation (1) with f replaced by i and combines this with some simple facts about projective spaces. Grothendieck actually took a much more general approach and considered morphisms $f : X \longrightarrow Y$ of smooth projective varieties. As it turns out, there is not a big difference between the proof of the above case and this case because f factors into a closed immersion $X \longrightarrow \mathbf{P}^n_Y$ and the projection $\mathbf{P}^n_Y \longrightarrow Y$. Now, the Grothendieck-Riemann-Roch theorem can be summarised in the following statement: if $f : X \longrightarrow Y$ is a proper morphism of smooth quasi-projective varieties over a field k, the following diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(X)} & A^{\cdot}(X)\otimes_{\mathbf{Z}}\mathbf{Q} \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(Y)} & A^{\cdot}(Y)\otimes_{\mathbf{Z}}\mathbf{Q} \end{array}$$

is commutative. The objects and the maps will be explained in Chapter 1 and 2. We give examples and applications of the Grothendieck-Riemann-Roch theorem in Chapter 3. The Grothendieck-Riemann-Roch theorem turns out be of fundamental value in the study of heights for certain covers of varieties fibered over a curve as we shall see in Section 6 of Chapter 3.

A ring will always be unitary, associative and commutative unless stated otherwise.

Ik wil graag Bas Edixhoven bedanken. Ik heb met groot genoegen gewerkt aan deze scriptie waarvan het onderwerp mij werd voorgelegd door hem. Ik voel me ook genoodzaakt hem te bedanken voor de hulp die hij me heeft geboden vanaf de dag dat ik hem vroeg of ik in Parijs kon studeren. Uiteraard ben ik ook veel dank verschuldigd aan meneer Murre. Onze lange gesprekken over wiskunde hebben zeker een grote rol gespeeld in mijn keuze om verder te gaan in de wiskunde. In het bijzonder hebben ze de beslissing naar Parijs te gaan ook een tikkeltje zwaarder gemaakt. Je tiens à remercier Professeur Bost d'avoir accepté de me diriger à Paris. Ces conseils étaient d'une grande aide. Ik wil ook graag Robin de Jong bedanken voor zijn suggesties en het corrigeren van deze scriptie. This thesis would not have been possible if it weren't for the Algant programme. Doing my first year in Paris was one of the most amazing experiences I have had educationwise. Ik dank Arno voor het beantwoorden van mijn vragen.

CHAPTER 1

Grothendieck's K_0-theory

1. Grothendieck groups

Let $\mathcal{C}$ be a full additive subcategory of an abelian category $\mathcal{A}$.

EXAMPLE 1.1. The category of A-modules is abelian for any ring A. The category of finitely generated A-modules is a full additive subcategory. It is abelian if and only if A is noetherian.

Let $\mathrm{Ob}(\mathcal{C})$ denote the class of objects in $\mathcal{C}$ and let $\mathrm{Ob}(\mathcal{C})/\cong$ be the set of isomorphism classes[1]. Let $F(\mathcal{C})$ be the free abelian group on $\mathrm{Ob}(\mathcal{C})/\cong$, i.e., an element $T \in F(\mathcal{C})$ is a finite formal sum

$$\sum n_X[X],$$

where $[X]$ denotes the isomorphism class of $X \in \mathrm{Ob}(\mathcal{C})$ and n_X is an integer.

DEFINITION 1.2. To any sequence

$$(E) \qquad 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\mathcal{C}$, which is exact in $\mathcal{A}$, we associate the element $Q(E) = [M] - [M'] - [M'']$ in $F(\mathcal{C})$. Let $H(\mathcal{C})$ be the subgroup generated by the elements $Q(E)$, where E is a short exact sequence. We define the *Grothendieck group*, denoted by $K(\mathcal{C})$, as the quotient group

$$K(\mathcal{C}) = F(\mathcal{C})/H(\mathcal{C}).$$

- The Grothendieck group $K(\mathcal{C})$ depends on $\mathcal{A}$. Therefore, we will always make explicit what $\mathcal{A}$ is. In case $\mathcal{C}$ itself is abelian, we will always take $\mathcal{A} = \mathcal{C}$.
- The class of an element $\alpha \in F(\mathcal{C})$ in $K(\mathcal{C})$ is denoted by $\mathrm{cl}_{\mathcal{C}}(\alpha)$ or just $\mathrm{cl}(\alpha)$. This gives us a homomorphism $\mathrm{cl} : F(\mathcal{C}) \longrightarrow K(\mathcal{C})$ such that any homomorphism $F(\mathcal{C}) \longrightarrow A$ of abelian groups which is additive on short exact sequences factors uniquely through $K(\mathcal{C})$.
- Since $\mathcal{C} \subset \mathcal{A}$ is an additive category, it has finite direct sums and a zero object. Clearly $\mathrm{cl}(0) = 0$ and $\mathrm{cl}(M) = \mathrm{cl}(M')$ in $K_0(\mathcal{C})$ for any two isomorphic objects M and M' of $\mathcal{C}$. By the fact that the sequence

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0$$

  is exact in $\mathcal{A}$, the addition in $K_0(\mathcal{C})$ is given by $\mathrm{cl}(M \oplus N) = \mathrm{cl}(M) + \mathrm{cl}(N)$.

EXAMPLE 1.3. Let us give some examples.

[1] Here we should restrict ourselves to categories $\mathcal{C}$ for which $\mathrm{Ob}(\mathcal{C})/\cong$ is a set. Such categories are called skeletally small categories.

(1) Let A be a ring and let $\mathcal{C}$ denote the (abelian) category of A-modules. To avoid set-theoretical difficulties, the reader may consider A-modules of bounded cardinality. For any A-module M, it holds that $M \oplus \left(\bigoplus_{n\in\mathbf{N}} M\right) \cong \bigoplus_{n\in\mathbf{N}} M$. Thus

$$\mathrm{cl}(M) + \mathrm{cl}(\bigoplus_{n\in\mathbf{N}} M) = \mathrm{cl}(\bigoplus_{n\in\mathbf{N}} M)$$

and $\mathrm{cl}(M) = 0$ in $K(\mathcal{C})$. We see that $K(\mathcal{C}) = 0$.

(2) More generally, for any additive category $\mathcal{C}$ which admits countable direct sums, we have that $K_0(\mathcal{C}) = 0$. (This is independent of the abelian category $\mathcal{A}$.)

(3) Let A be a principal ideal domain and $\mathcal{C}$ denote the (abelian) category of finitely generated A-modules. By the structure theorem of A-modules, any finitely generated A-module is isomorphic to the direct sum of a free module and a torsion module, where the latter is isomorphic to a direct sum of cyclic modules. The rank of a finitely generated A-module is defined as the rank of its free part. The rank gives us a surjective map $\mathrm{rk} : \mathrm{Ob}(\mathcal{C})/\cong \longrightarrow \mathbf{Z}$ which induces a surjective homomorphism from $F(\mathcal{C})$ to $\mathbf{Z}$. Since the rank is additive on short exact sequences, it induces a homomorphism $K(\mathcal{C}) \longrightarrow \mathbf{Z}$. For any nonzero ideal $I = (x)$, we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\cdot x} A \longrightarrow A/I \longrightarrow 0$$

and therefore that $\mathrm{cl}(A/I) = 0$ in $K(\mathcal{C})$. Thus, since the rank of A equals 1, the rank induces an isomorphism from $K(\mathcal{C})$ to $\mathbf{Z}$.

(4) Let A be a ring and let $\mathcal{C}_m$ be the category of finitely generated free A-modules of rank 0 or rank greater than or equal to some fixed positive integer m. Since it has finite direct sums and the zero object, it is an additive subcategory of the abelian category $\mathcal{C}$ of finitely generated free A-modules. Assuming $A \neq 0$, for $m \geq 2$, the kernel of the natural projection $A^{m+1} \longrightarrow A^m$ is not an object of $\mathcal{C}_m$. Therefore, $\mathcal{C}_m$ is not an abelian subcategory in this case. Assuming A is a principal ideal domain, the reasoning above shows that the rank map induces an isomorphism $K(\mathcal{C}_m) \cong \mathbf{Z}$ with generator $\mathrm{cl}(A^{m+1}) - \mathrm{cl}(A^m)$. In particular, the natural inclusion $\mathcal{C}_m \subset \mathcal{C}$ induces an isomorphism on the level of Grothendieck groups.

(5) Let A be a local ring with residue field k. Let $\mathcal{C}$ denote the category of finitely generated projective A-modules viewed as a full subcategory of the abelian category of A-modules. By Nakayama's Lemma, every finitely generated projective A-module M is isomorphic to a free A-module of rank equal to $\dim_k M \otimes_A k$. We see that the rank induces an isomorphism $K(\mathcal{C}) \cong \mathbf{Z}$.

(6) Let $\mathcal{C}$ be the category of finite abelian p-groups for some prime number p. The length of such a group induces an isomorphism $K(\mathcal{C}) \longrightarrow \mathbf{Z}$.

The above construction of the Grothendieck group coincides with the more general construction of the Grothendieck group associated to an exact category in [**Weibel**, Chapter II.7].

2. The Grothendieck group of coherent sheaves

References for the basics of coherent sheaves are [**Liu**, Chapter 5] and [**Har**, Chapter II.5]. Although we will precise this always, every scheme will be noetherian.

Let X be a noetherian scheme.

Let $\mathrm{Coh}(X)$ denote the category of coherent sheaves on X. It is a full abelian subcategory of the category of $\mathcal{O}_X$-modules. If $X = \operatorname{Spec} A$ is affine, the global sections functor $\Gamma(X, -)$ gives an equivalence of categories from $\mathrm{Coh}(X)$ to the category of finitely generated A-modules. Its quasi-inverse assigns to each finitely generated A-module M the coherent sheaf $\widetilde{M}$.

DEFINITION 1.4. The *Grothendieck group of coherent sheaves* of X, denoted by $K_0(X)$, is defined as

$$K_0(X) := K(\mathrm{Coh}(X)) = F(\mathrm{Coh}(X))/H(\mathrm{Coh}(X)).$$

For a ring A, we write $K_0(A) = K_0(\operatorname{Spec} A)$. By the equivalence of categories, $K_0(A)$ is the Grothendieck group associated to the category of finitely generated A-modules.

EXAMPLE 1.5. Let K be the function field of $\mathbf{P}^1_k$ and let η be its generic point. The map $K_0(\mathbf{P}^1_k) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$ given by $\mathcal{F} \mapsto (\dim_K \mathcal{F}_\eta, \chi(\mathbf{P}^1, \mathcal{F}))$ is an isomorphism.

EXAMPLE 1.6. Let x be a closed point in X. Then $K_0(\{x\}) = K_0(k(x)) \cong \mathbf{Z}$.

For completeness, we state the following well-known Lemma ([**BorSer**, Proposition 1]).

LEMMA 1.7. Let U be an open subset of X and let $\mathcal{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}|_U \cong \mathcal{F}$. Moreover, if there is a coherent sheaf $\mathcal{G}$ on X with $\mathcal{F} \subset \mathcal{G}|_U$, then there is a coherent sheaf $\mathcal{F}'$ on X which extends $\mathcal{F}$ such that $\mathcal{F}' \subset \mathcal{G}$. □

Recall that the *support* of a coherent sheaf $\mathcal{F}$ on X, denoted by $\operatorname{Supp} \mathcal{F}$, is the subset of points $x \in X$ such that $\mathcal{F}_x \neq 0$. Since the stalk $\mathcal{F}_x = 0$ if and only if $\mathcal{F}|_U = 0$ for some open neighborhood U of x, the support of $\mathcal{F}$ is a closed subset of X. In fact, $\operatorname{Supp} \mathcal{F}$ is the closed subscheme defined by the sheaf of ideals $\operatorname{Ann} \mathcal{F}$ and $\mathcal{F}$ is the extension by zero of a coherent sheaf on $V(\operatorname{Ann} \mathcal{F})$.

LEMMA 1.8. Let $\mathcal{F}$ be a coherent sheaf on X with support S. Then there is a filtration

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \ldots \supset \mathcal{F}_n = 0,$$

where $\mathcal{F}_i$ is a coherent sheaf on X with support in S, such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is an $\mathcal{O}_S$-module.

PROOF. Let $\mathcal{I}$ be the ideal sheaf defining S in X. It suffices to show that $\mathcal{I}^n\mathcal{F} = 0$ for some integer $n \in \mathbf{Z}$. Then the filtration

$$\mathcal{F} = \mathcal{I}^0\mathcal{F} \supset \mathcal{I}\mathcal{F} \supset \mathcal{I}^2\mathcal{F} \supset \ldots \supset \mathcal{I}^{n-1}\mathcal{F} \supset 0$$

will be of the desired form. Thus, let $x \in S$ and let $U = \operatorname{Spec} A$ be an affine open subset of X containing x. Let I be the ideal of A defining $U \cap S$ and let $M = \mathcal{F}(U)$. Note that M is a finitely generated A-module. For $f \in I$, let $D(f)$ be the complement of $V(f)$ in $\operatorname{Spec} A$ and note that $M \otimes_A A_f = 0$. That is, all elements of M are annihilated by a power of f. Since M is finitely generated, there is an integer $r \in \mathbf{Z}$ such that $f^r M = 0$. Therefore, since I is also finitely generated (A is noetherian), there is an integer $s \in \mathbf{Z}$ such that $I^s M = 0$. Now, covering X by a finite number of affine open subsets, we see that $\mathcal{I}^n\mathcal{F} = 0$ for some integer $n \in \mathbf{Z}$. □

THEOREM 1.9. **(Localization sequence)** Let $Y \subset X$ be a closed subscheme of (the noetherian scheme) X and $X \backslash Y = U$. There exists a sequence

$$K_0(Y) \longrightarrow K_0(X) \longrightarrow K_0(U) \longrightarrow 0$$

for which the first arrow is induced by extension by zero of sheaves from Y to X and the second arrow is induced by the restriction of sheaves from X to U. This sequence is exact.

PROOF. It is clear that this sequence exists. (The extension by zero is an exact functor in this case and so is the restriction of sheaves.) By Lemma 1.7, the map on the right is surjective. Furthermore, it is clear that the composition of the two maps is zero. Therefore, we have a natural surjective homomorphism $\beta : A \longrightarrow K_0(U)$, where $A = K_0(X)/\mathrm{im}(K_0(Y))$ is the cokernel of the first map. To prove the theorem, it suffices to give an inverse $\gamma : K_0(U) \longrightarrow A$ to β.

Firstly, suppose that $\mathcal{F}$ is a coherent sheaf on U which extends to a coherent sheaf $\mathcal{G}$ on X. We claim that the image of $\mathcal{G}$ in A only depends on $\mathcal{F}$. To prove this we consider another extension $\mathcal{G}'$ of $\mathcal{F}$ and the diagonal embedding

$$\mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} = (\mathcal{G} \times \mathcal{G}')|_U.$$

By definition, the composition with a projection on the first or second factor is given by the identity morphism $\mathcal{F} \longrightarrow \mathcal{F}$. By Lemma 1.7, there exists a coherent subsheaf $\mathcal{G}'' \subset \mathcal{G} \times \mathcal{G}'$ such that $\mathcal{G}''|_U = \mathcal{F}$. Therefore, there is also a morphism $\varphi : \mathcal{G}'' \longrightarrow \mathcal{G}$ which induces the identity on U. The exact sequence corresponding to the morphism φ shows that

$$\mathrm{cl}(\ker \varphi) - \mathrm{cl}(\operatorname{coker} \varphi) = \mathrm{cl}(\mathcal{G}'') - \mathrm{cl}(\mathcal{G}).$$

Since $\operatorname{Supp} \ker \varphi \cap U = \operatorname{Supp} \operatorname{coker} \varphi \cap U = \emptyset$, we see that $\operatorname{Supp} \ker \varphi, \operatorname{Supp} \operatorname{coker} \varphi \subset Y$. By Lemma 1.8, we have that $\mathrm{cl}(\mathcal{G}'') - \mathrm{cl}(\mathcal{G})$ is in the image of the map $K_0(Y) \longrightarrow K_0(X)$. We conclude that $\mathcal{G}'' = \mathcal{G}$ in A. Similarly, one can show that $\mathcal{G}'' = \mathcal{G}'$ in A. Therefore, $\mathcal{G} = \mathcal{G}'$ in A. Thus, for any extension $\mathcal{G}$ of $\mathcal{F}$, we may denote its image in A by $\gamma(\mathcal{F})$. To finish the proof, we shall show that the map $\gamma : K_0(U) \longrightarrow A$ is well-defined (i.e., the assignment γ is additive on short exact sequences). To prove this we let

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of sheaves on U. By Lemma 1.7, we may choose an extension $\mathcal{G}$ of $\mathcal{F}$ to X. Then $\mathcal{F}'$ extends to a subsheaf $\mathcal{G}'$ of $\mathcal{G}$ and $\mathcal{F}''$ extends to the quotient sheaf $\mathcal{G}/\mathcal{G}'$. This shows that the map γ is indeed additive on short exact sequence, by the fact that it is independent of the extension one chooses. □

COROLLARY 1.10. For any noetherian scheme X, it holds that the restriction homomorphism

$$K_0(X \times_{\mathbf{Z}} \operatorname{Spec} \mathbf{Z}[t]) \longrightarrow K_0(X \times_{\mathbf{Z}} \operatorname{Spec} \mathbf{Z}[t, \frac{1}{t}])$$

induced by the open immersion $\operatorname{Spec} \mathbf{Z}[t, \frac{1}{t}] \longrightarrow \operatorname{Spec} \mathbf{Z}[t]$ is an isomorphism. In particular, for any noetherian ring A, we have that $K_0(A[t]) \cong K_0(A[t, \frac{1}{t}])$.

PROOF. The open immersion $\operatorname{Spec} \mathbf{Z}[t, \frac{1}{t}] \longrightarrow \operatorname{Spec} \mathbf{Z}[t]$ induces an open immersion $X \times_{\mathbf{Z}} \operatorname{Spec} \mathbf{Z}[t, \frac{1}{t}] \longrightarrow X \times_{\mathbf{Z}} \operatorname{Spec} \mathbf{Z}[t]$ by base change. Note that the closed subscheme $X \times_{\mathbf{Z}} \operatorname{Spec} \mathbf{Z} = X$ is the complement of $X \times_{\mathbf{Z}} \operatorname{Spec} \mathbf{Z}[t, \frac{1}{t}]$. By the exact sequence

$$K_0(X) \longrightarrow K_0(X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}}) \longrightarrow K_0(X \times_{\mathbf{Z}} \operatorname{Spec} \mathbf{Z}[t, \tfrac{1}{t}]) \longrightarrow 0\ ,$$

it suffices to show that the first homomorphism $K_0(X) \longrightarrow K_0(X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}})$ is zero. To prove this, note that we have a short exact sequence of coherent sheaves

$$0 \longrightarrow p^*\mathcal{F} \xrightarrow{\cdot t} p^*\mathcal{F} \longrightarrow i_*\mathcal{F} \longrightarrow 0\ .$$

Here $p : \mathbf{A}^1_X \longrightarrow X$ is the projection and $i : X \longrightarrow \mathbf{A}^1_X$ is the closed immersion (as above). □

EXAMPLE 1.11. Let $\mathfrak{p}$ be a maximal ideal of a principal ideal domain A and let $n \geq 1$ be an integer. Note that $A/\mathfrak{p}^n A$ is a zero-dimensional local noetherian ring. The length induces an isomorphism $K_0(A/\mathfrak{p}^n) \longrightarrow \mathbf{Z}$ with generator the class of $A/\mathfrak{p}$.

EXAMPLE 1.12. Let $n \geq 1$. Let A be a principal ideal domain and let $Y \subset \operatorname{Spec} A[x_1, \ldots, x_n]$ be the closed subscheme defined by the ideal $I \subset A$ and choosing $x_1 = x_2 = \ldots = x_n = 0$. There is an exact sequence of abelian groups

$$K_0(A/I) \longrightarrow K_0(\mathbf{A}_A^n) \longrightarrow K_0(U) \longrightarrow 0 ,$$

where $U = \mathbf{A}_A^n - Y$. Let us show that the homomorphism $K_0(A/I) \longrightarrow K_0(\mathbf{A}_A^n)$ is the zero map. We distinguish two cases.

(1) Suppose that $I = 0$. We have a short exact sequence of $A[x_1, \ldots, x_n]$-modules

$$0 \longrightarrow A[x_1] \longrightarrow A[x_1] \longrightarrow A \longrightarrow 0 .$$

This shows that the class of A is zero in $K_0(\mathbf{A}_A^n)$. Since $K_0(A) \cong \mathbf{Z}$ with generator (the class of) A, we conclude that $K_0(A) \longrightarrow K_0(\mathbf{A}_A^n)$ is the zero map,

(2) Suppose that $I \neq 0$. For any nonzero ideal $J = xA$, we have a short exact sequence of A-modules

$$0 \longrightarrow A \xrightarrow{\cdot x} A \longrightarrow A/J \longrightarrow 0 .$$

Therefore, the homomorphism $K_0(A/I) \longrightarrow K_0(A)$ is zero. From the functoriality of extension by zero, we can conclude that the composition $K_0(A/I) \longrightarrow K_0(A) \longrightarrow K_0(\mathbf{A}_A^n)$ is zero.

For a morphism $f : X \longrightarrow Y$ of schemes, the direct image of a sheaf $\mathcal{F}$ on X is denoted by $f_*\mathcal{F}$. This defines a functor f_* from the category of sheaves on X to the category of sheaves on Y.

EXAMPLE 1.13. For a closed immersion $f : X \longrightarrow Y$, the direct image coincides with the extension by zero of a sheaf. In particular, the functor f_* is exact in this case.

EXAMPLE 1.14. For a field k and morphism $f : X \longrightarrow \operatorname{Spec} k$, the push-forward coincides with the global sections functor $f_* = \Gamma(X, -)$. In general, this functor is only left exact. Its right derived functors in the category of sheaves on X are the cohomology functors $H^i(X, -)$.

Recall that f_* is right adjoint to the inverse image functor f^{-1}. Therefore, it is left exact. We can form the right derived functors $R^i f_*$ in the category of sheaves on X. These functors are called the higher direct image functors. It is not hard to see that, for any sheaf $\mathcal{F}$ on X, it holds that $R^i f_*(\mathcal{F})$ is the sheaf associated to the presheaf

$$V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

on Y. In particular, for any noetherian and finite-dimensional scheme, we have that $R^i f_* = 0$ when $i > \dim X$.

EXAMPLE 1.15. Let k be a field and let $f : \mathbf{A}_k^1 \longrightarrow \operatorname{Spec} k$ be the projection. Then $f_*\widetilde{k[x]}$ can be identified with the the k-module $k[x]$ which is clearly not finitely generated. Thus, the direct image does not preserve coherence necessarily. Note that f is not proper. (Make a change of basis by taking the product of $\mathbf{A}_k^1$ over k and note that the image of the hyperbola $xy - 1$ is not closed.)

EXAMPLE 1.16. Suppose that f is a closed immersion. Then $R^if_* = 0$ for $i > 0$. Furthermore, since f is a finite morphism, we have that $R^0f_*\mathcal{F}$ is coherent if $\mathcal{F}$ is coherent.

Let $f : X \longrightarrow Y$ be a morphism of noetherian schemes. Recall that the derived functors of f_* in the category of sheaves on X coincide with the derived functors of f_* in the category of $\mathcal{O}_X$-modules.

THEOREM 1.17. Suppose that f is proper. Let $\mathcal{F}$ be a coherent sheaf on X. For any $i \geq 0$, the higher direct image $R^if_*\mathcal{F}$ is a coherent sheaf on Y.

PROOF. Since the question is local on Y, we may assume $Y = \operatorname{Spec} A$ is affine with A a noetherian ring. Now, let us show that

$$R^if_*\mathcal{F} \cong \widetilde{H^i(X,\mathcal{F})} \tag{2}$$

as sheaves on Y. Firstly, note that this holds for $i = 0$ by the fact that $f_*\mathcal{F}$ is quasi-coherent on Y. Secondly, since the "tilde" functor $\sim$ from the category of A-modules to the category of $\mathcal{O}_Y$-modules is exact, we see that both sides of (2) are δ-functors from the category of quasi-coherent sheaves on X to the category of $\mathcal{O}_Y$-modules. But both sides are effaceable for $i > 0$. (Any quasi-coherent sheaf $\mathcal{F}$ on X can be embedded in a flasque, quasi-coherent sheaf.) Thus, there is a unique isomorphism of δ-functors which gives the isomorphism in (2) by the fact that $R^0f_*\mathcal{F} \cong \widetilde{\Gamma(X,\mathcal{F})}$. We conclude that $R^if_*\mathcal{F}$ is quasi-coherent. Since the coherence is a bit more tricky, we will now assume f to be projective. This will suffice for our applications. The general proof uses Chow's Lemma ([**Har**, Chapter II, Exercise 4.10]), which says that proper morphisms are fairly close to projective morphisms.

By the above, we have to show that $H^i(X,\mathcal{F})$ is a finitely generated A-module when $f : X \longrightarrow \operatorname{Spec} A$ is projective. There is a closed immersion $i : X \longrightarrow \mathbf{P}^m_A$ for some integer m. This allows us to reduce to the case $X = \mathbf{P}^m_A$. Explicit computations in Cech cohomology show that $H^i(X,\mathcal{F})$ is finitely generated for sheaves of the form $\mathcal{O}_X(n)$, $n \in \mathbf{Z}$. The same holds for direct sums of such sheaves. Now, for a general coherent sheaf $\mathcal{F}$ on X, we have a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Here $\mathcal{E}$ is a direct sum of sheaves $\mathcal{O}_X(n)$ and $\mathcal{K}$ is coherent. In fact, there exists an integer $n < 0$ such that the twisted sheaf $\mathcal{F}(-n)$ is generated by its global sections. Since X is quasi-compact, we may cover X with a finite number of open affine subsets U_i $(i = 1, \ldots, d)$. On each U_i, we have that $\mathcal{F}(-n)(U_i)$ is generated by a finite number of global sections. Therefore, there exist a finite number of global sections $s_1, \ldots, s_r \in \mathcal{F}(-n)(X)$ which generate $\mathcal{F}(-n)$ on every open U_i. Therefore there is a surjective morphism $\mathcal{O}^r_X \longrightarrow \mathcal{F}(-n)$. Tensoring this with $\mathcal{O}_X(n)$ gives a surjective morphism $\mathcal{O}^r_X(n) \longrightarrow \mathcal{F}$. Its kernel is $\mathcal{K}$ by definition. Now, the long exact sequence of cohomology applied to the above short exact sequence implies the result by descending induction on i. □

From the previous theorem we get the following facts. For any coherent sheaf $\mathcal{F}$ on X, the element $\operatorname{cl}(R^if_*\mathcal{F})$ is well-defined in $K_0(Y)$. Then, assuming X to be also finite-dimensional, the alternating sum $\sum(-1)^i\operatorname{cl}(R^if_*\mathcal{F})$ is well-defined in $K_0(Y)$. Note that the long exact sequence for derived functors shows that the map $[\mathcal{F}] \mapsto \sum(-1)^i\operatorname{cl}(R^if_*\mathcal{F})$ is additive on short exact sequences and therefore induces a homomorphism $K_0(X) \longrightarrow K_0(Y)$. This morphism is denoted by $f_!$. By the Leray spectral sequence, we have that $g_! \circ f_! = (g \circ f)_!$.

We conclude that K_0 is a covariant functor from the category of noetherian and finite-dimensional schemes with proper morphisms to the category of abelian groups. To a morphism $f : X \longrightarrow Y$ one assigns the morphism of abelian groups $f_! : K_0(X) \longrightarrow K_0(Y)$ given by $f_! \mathrm{cl}(\mathcal{F}) = \sum(-1)^i \mathrm{cl}(R^i f_* \mathcal{F})$.

The proof of the following Proposition illustrates a technique called *Dévissage*.

PROPOSITION 1.18. The extension by zero $K_0(X_{\mathrm{red}}) \longrightarrow K_0(X)$ is an isomorphism.

PROOF. We treat the affine case $X = \operatorname{Spec} A$, where A is noetherian. Let $I = \sqrt{0}$ be the nilradical of A. Since A is noetherian, there exists a positive integer n such that $I^n = 0$. For any module A-module M, we have a chain of submodules

$$0 = I^n M \subset I^{n-1} M \subset \ldots \subset IM \subset M$$

such that $I^i M / I^{i+1} M = I^i M \otimes_A A/I$ is a module over A/I. We see that

$$\mathrm{cl}(M) = \mathrm{cl}(M/IM) + \mathrm{cl}(IM/I^2 M) + \ldots + \mathrm{cl}(I^{n-1} M)$$

in $K_0(A)$. This implies that the homomorphism $K_0(A/I) \longrightarrow K_0(A)$ is bijective. (In fact, from the above filtration for M, it is clear that the homomorphism $K_0(A/I) \longrightarrow K_0(A)$ is surjective. An inverse to this morphism is given by assigning to the class of each A-module M the element $\sum \mathrm{cl}(I^i M \otimes_A A/I)$ in $K^0(A/I)$. It is easy to see that this is well-defined and inverse to the homomorphism $K_0(A/I) \longrightarrow K_0(A)$.)

In the general case, the reader may verify that the proof is similar to the proof of Lemma 1.8. In fact, for any coherent sheaf $\mathcal{F}$ on X, we have a chain of subsheaves

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \ldots \supset \mathcal{F}_n = 0$$

such that $\mathcal{F}_i / \mathcal{F}_{i-1}$ is an $\mathcal{O}_{X_{\mathrm{red}}}$-module. To prove this, one covers X with a finite number of affine open subsets. □

3. The geometry of $K_0(X)$

Let A be a noetherian ring and M a finitely generated A-module. The support of M is the subset $\operatorname{Supp} M = \operatorname{Supp} \widetilde{M} \subset \operatorname{Spec} A$. We already noted that $\operatorname{Supp} M = V(\operatorname{Ann} M)$, where $\operatorname{Ann} M = \{a \in A \mid aM = 0\}$ is the annihilator of M in A. The following Theorem (which can be found in [**Ser**]) is a bit more precise then Lemma 1.8.

THEOREM 1.19. There exists a chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that $M_i / M_{i-1} \cong A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of A. □

Let X be a noetherian scheme. The class of a coherent sheaf $\mathcal{F}$ in $K_0(X)$ is denoted by $\mathrm{cl}(\mathcal{F})$.

DEFINITION 1.20. A *cycle* on X is an element of the free abelian group $Z(X)$ on the closed integral subschemes of X. That is, an element of $Z(X)$ is a finite formal sum $\sum n_V [V]$, where V is a closed integral subscheme of X and n_V is an integer.

REMARK 1.21. For any open subset $U \subset X$ with complement $Y \subset X$, we have a split exact sequence of abelian groups

$$0 \longrightarrow Z(Y) \longrightarrow Z(X) \longrightarrow Z(U) \longrightarrow 0.$$

The first map is induced by the inclusion $Y \subset X$ and is clearly injective. The second map is induced by the restriction map $[V] \mapsto [V \cap U]$. Its left-inverse is given by assigning to each closed integral subscheme V of U its closure $\overline{V}$ in X. The latter is again an integral subscheme of X. The exactness in the middle is verified easily. We conclude that $Z(X) \cong Z(Y) \oplus Z(U)$. In particular, $Z(X) = Z(X_{\text{red}})$.

The following theorem reveals the geometric nature of $K_0(X)$.

THEOREM 1.22. The homomorphism $Z(X) \longrightarrow K_0(X)$ defined by $[V] \mapsto \text{cl}(\mathcal{O}_V)$ is surjective.

PROOF. The affine case goes as follows. If $X = \text{Spec}\, A$ is an affine scheme, the above map $\gamma : Z(X) \longrightarrow K_0(A)$ is given by $[V(\mathfrak{p})] \mapsto \text{cl}(A/\mathfrak{p})$. Let M be a finitely generated A-module, where A is a noetherian ring. By Theorem 1.19, it has a chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_{\mathrm{i}}$, where $\mathfrak{p}_i$ is a prime ideal of A. This implies that

$$\text{cl}(M) = \text{cl}(A/\mathfrak{p}_n)+\text{cl}(M_{n-1}) = \text{cl}(A/\mathfrak{p}_n)+\text{cl}(A/\mathfrak{p}_{n-1})+\ldots+\text{cl}(A/\mathfrak{p}_1) = \gamma([V(\mathfrak{p}_n)]+\ldots+[V(\mathfrak{p}_1)]).$$

Now, for the general case, let $U = \text{Spec}\, A$ be an open affine in X with complement Y. The groups $Z(Y)$ and $K_0(Y)$ are independent of the closed subscheme structure put on Y. By noetherian induction, we may assume that $Z(Y) \longrightarrow K_0(Y)$ is surjective. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(Y) & \longrightarrow & Z(X) & \longrightarrow & ZU & \longrightarrow & 0\,, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & K_0(Y) & \longrightarrow & K_0(X) & \longrightarrow & K_0(U) & \longrightarrow & 0 \end{array}$$

where the rows are exact. The homomorphism $Z(U) \longrightarrow K_0(U)$ is surjective. By a diagram chase, we conclude that the homomorphism $Z(X) \longrightarrow K_0(X)$ is surjective. □

Let us briefly return to the affine setting. That is, let M be a finitely generated A-module, where A is a noetherian ring. For the convenience of the reader, we include the proof of the following theorem.

THEOREM 1.23. The support of M consists of only maximal ideals if and only if M is of finite length.

PROOF. Suppose that M is of finite length and let $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ be a composition series, i.e., we have that $M_i/M_{i-1} \cong A/\mathfrak{m}_i$ with $\mathfrak{m}_i$ a maximal ideal. Then we have exact sequences

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0\,.$$

By induction, we have that

$$\text{Supp}\, M = \cup_{i=1}^{n} \text{Supp}\, M_i/M_{i-1} = \cup_i \text{Supp}\, A/\mathfrak{m}_i = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}.$$

Conversely, suppose that $\operatorname{Supp} M$ consists of only maximal ideals. We may suppose that $M \neq 0$. Then $\operatorname{Supp} M \neq \emptyset$. Let $(x_1, \dots, x_n)$ be a minimal set of generators for M and consider the proper submodule N generated by $(x_1, \dots, x_{n-1})$. It is clear that $\operatorname{Supp} M = \operatorname{Supp} N \cup \operatorname{Supp} M/N$. Therefore, by induction on n, it suffices to show the theorem for M cyclic. (A composition series for N and M/N gives rise to a composition series for M.) To prove this, let $x \in M$ such that $M = Ax$. Note that $M \cong A/\operatorname{Ann}(x)$. By the assumption that $\operatorname{Supp} M = \operatorname{Supp} A/\operatorname{Ann}(x) = V(\operatorname{Ann}(x))$ consists of only maximal ideals, we have that all prime ideals containing $\operatorname{Ann}(x)$ are maximal. This implies that the noetherian ring $A/\operatorname{Ann}(x)$ is zero-dimensional. In particular, $A/\operatorname{Ann}(x)$ is artinian. Thus M is of finite length over $A/\operatorname{Ann}(x)$ since it is both noetherian and artinian. But since $\operatorname{Ann}(x)M = 0$, we have that M is of finite length over A. □

EXAMPLE 1.24. Let k be a field. Let us show that $K_0(A) \cong \mathbf{Z} \cdot k[x] \oplus \mathbf{Z} \cdot k[y]$, where $A = k[x, y]/(xy)$. We claim that $K_0(A)$ is generated by the classes of the A-modules

$$k[x] = A/(y), \quad k[y] = A/(x), \quad k[x]/(f) = A/(y, f), \quad k[y]/(g) = A/(x, g),$$

where $f \in k[x]$ is an irreducible polynomial and $g \in k[y]$ is an irreducible polynomial. Let us verify this. Take a finitely generated nonzero A-module M. We have precisely two generic points: η_x and η_y. The residue field of η_x is $k(y)$ and the residue field of η_y is $k(x)$. Let $r = \operatorname{rk} M_{(x)}$ be the rank of M at η_x and let $s = \operatorname{rk} M_{(y)}$. Clearly, we have an injective homomorphism

$$k[x]^r \oplus k[y]^s \longrightarrow M$$

whose cokernel N is torsion. Since N is torsion, it has finite support. Therefore, its support must consist of only maximal ideals. (It can't contain a generic point. Else it would be infinite.) Thus, it has a composition series by the above Theorem. As in the proof of Theorem 1.22, this shows that N is a finite sum of the form

$$\sum_{f \text{ irreducible}} n_f \cdot \operatorname{cl}(k[x]/(f)) + \sum_{g \text{ irreducible}} m_g \cdot \operatorname{cl}(k[y]/(g))$$

in the Grothendieck group. (In $K_0(A)$ write N as the sum of the simple quotients that appear in its composition series.) This proves the claim. Now, for any nonzero $f \in k[x]$, the short exact sequence of A-modules

$$0 \longrightarrow k[x] \longrightarrow k[x] \longrightarrow k[x]/(f) \longrightarrow 0$$

shows that the class of $k[x]/(f)$ is zero in $K_0(A)$. Similarly, for any nonzero $g \in k[y]$, the class of $k[y]/(g)$ is zero in $K_0(A)$. Hence $K_0(A)$ is generated by (the classes of) $k[x]$ and $k[y]$. These are linearly independent over $\mathbf{Z}$. In fact, suppose that $a \cdot k[x] + b \cdot k[y] = 0$, where $a, b \in \mathbf{Z}$. Take the rank at (y) to see that $a = 0$. Similarly, take the rank at (x) to see that $b = 0$. Thus, we conclude that $K_0(A) \cong \mathbf{Z} \cdot k[x] \oplus \mathbf{Z} \cdot k[y]$.

We now go back to geometry.

Let X be an algebraic scheme, i.e., a scheme of finite type over a field. In particular, we have that X is noetherian and finite-dimensional. The free abelian group $Z(X) = Z^{\cdot}(X) = \bigoplus_{r \in \mathbf{Z}} Z^r(X)$ is graded by codimension. Here $Z^r(X)$ denotes the free abelian group on the closed integral subschemes of codimension r. For a cycle $\alpha \in Z^{\cdot}(X)$, we let $\alpha_{(r)}$ be its component in $Z^r(X)$. Now, for later use, we shall formulate a "graded" version of Theorem 1.22.

REMARK 1.25. If we let $Z_r(X)$ denote the free abelian group on the closed integral subschemes of dimension r, then $Z_r(X) = Z^{n-r}(X)$ when X is an n-dimensional separated irreducible scheme of finite type over a field. If X is not irreducible and separated, these gradings might not be renumberings of each other.

EXAMPLE 1.26. Let k be an algebraically closed field and let $X = \operatorname{Spec} k[t] = \mathbf{A}^1_k$. Then $Z^0(X) = \mathbf{Z}$ and $Z^1(X) = k(t)^*/k^*$.

Suppose that X is separated and irreducible.

REMARK 1.27. For any irreducible closed subset $Y \subset X$ with complement U, the sequence

$$0 \longrightarrow Z^{r-c}(Y) \longrightarrow Z^r(X) \longrightarrow Z^r(U) \longrightarrow 0$$

is split exact. Here $c = \operatorname{codim}(Y, X)$ and $r \in \mathbf{Z}$.

In general, the Grothendieck group $K_0(X)$ is not naturally graded. Instead, it has a *topological*[2] filtration

$$K_0(X) = F^0X \supset F^1X \supset \ldots\ldots \supset F^{\dim X}X \supset F^{\dim X+1}X = 0,$$

where we define

$$F^iX = \langle \operatorname{cl}(\mathcal{F}) \in K_0(X) \mid \operatorname{codim} \operatorname{Supp} \mathcal{F} \geq i \rangle.$$

Let $\mathcal{F}$ be a coherent sheaf on X and let w be a generic point of S. Since the local ring $\mathcal{O}_{S,w}$ is zero-dimensional, the stalk $\mathcal{F}_w$ is of finite length over $\mathcal{O}_{X,w}$.

DEFINITION 1.28. For a coherent sheaf $\mathcal{F}$ on X, we define the cycle

$$[\mathcal{F}] := \sum_{W \subset \operatorname{Supp} \mathcal{F}} \big(\operatorname{length}_{\mathcal{O}_{X,w}} \mathcal{F}_w\big)[W] \in Z^{\cdot}(X).$$

Here the sum runs through all irreducible components W of $\operatorname{Supp} \mathcal{F}$ with generic point w which are of codimension 0 in $\operatorname{Supp} \mathcal{F}$. For a closed subscheme V of X, we put

$$[V] := [\mathcal{O}_V] = \sum_{W \subset V} \big(\operatorname{length}_{\mathcal{O}_{V,w}} \mathcal{O}_{V,w}\big)[W] \in Z^{\cdot}(X).$$

Also, for any integral subscheme V, this does not conflict with our previous notation for the class of V in $Z(X)$.

EXAMPLE 1.29. Let A be a principal ideal domain and $X = \operatorname{Spec} A$. To give a coherent sheaf on X is to give a finitely generated A-module M. For such an A-module M, there are irreducible $f_1, \ldots, f_r \in A$ such that $M \cong A^{\operatorname{rk} M} \oplus \bigoplus_{i=1}^r M(f_i)$. Here $M(f) = A/(f^{n_1}) \oplus \ldots \oplus A/(f^{n_s})$ for some integers $n_1, \ldots, n_s$. We can show that

$$[M] = \begin{cases} \operatorname{rk} M \cdot [A] & \text{if } M \text{ is not torsion} \\ (n_1^{f_1} + \ldots + n_{s_1}^{f_1}) \cdot [A/f_1] + \ldots + (n_1^{f_r} + \ldots + n_{s_r}^{f_r}) \cdot [A/f_r] & \text{if } M \text{ is torsion} \end{cases}$$

The formula is obvious when M is not torsion. In case M is torsion, the formula is clear since the length of A/f^nA over A is n.

PROPOSITION 1.30. For any coherent sheaf $\mathcal{F}$ with support of codimension r, it holds that the image of $[\mathcal{F}]$ under the morphism $Z^rX \longrightarrow F^rX/F^{r+1}X$ equals the image of $\operatorname{cl}(\mathcal{F})$ in $F^rX/F^{r+1}X$. In particular, the homomorphism $Z^rX \longrightarrow F^rX/F^{r+1}X$ is surjective.

[2]Opposed to having also another filtration which is called the γ-filtration.

PROOF. For any finitely generated A-module M, if $\mathfrak{p}$ is a minimal prime ideal of $\operatorname{Supp} M$, the number of times $A/\mathfrak{p}$ occurs in a filtration for M (as in the proof of Theorem 1.22) is precisely the length of $M_\mathfrak{p}$ over $A_\mathfrak{p}$. □

4. The Grothendieck group of vector bundles

Let X be a noetherian scheme. Let $\mathrm{Vect}(X)$ denote the category of vector bundles on X. By abuse of language, a vector bundle on X will be a coherent sheaf on X which is locally free. A morphism of vector bundles on X is a morphism of $\mathcal{O}_X$-modules. For any noetherian affine scheme $X = \operatorname{Spec} A$, the global sections functor $\Gamma(X, -)$ gives an equivalence of categories from $\mathrm{Vect}(X)$ to the category of finitely generated projective A-modules.

EXAMPLE 1.31. Let S and T be $\mathbf{P}^1_k$, where k is an algebraically closed field. Let $\pi : S \longrightarrow T$ be the morphism given by $[x : y] \mapsto [x^n : y^n]$. Note that π is a finite morphism. Let $m \equiv r \mod n$, where $0 \leq r < n$. We have that

$$\pi_*\mathcal{O}(m) = \mathcal{O}(\lfloor \frac{m+1}{n} \rfloor - 1)^{\oplus(n-r-1)} \bigoplus \mathcal{O}(\lceil \frac{m+1}{n} \rceil - 1)^{\oplus(r+1)}.$$

To prove this formula, cover S by $S_1 = \operatorname{Spec} k[s]$ and $S_2 = \operatorname{Spec} k[s^{-1}]$. Similarly, cover T by $T_1 = \operatorname{Spec} k[t]$ and $T_2 = \operatorname{Spec} k[t^{-1}]$. Now, $\mathcal{O}(m)(S_1)$ is a free $k[s]$-module of rank 1. For any basis (e) of $\mathcal{O}(m)(S_1)$ as a free $k[s]$-module, we have that $(s^{-2m}e)$ is a basis for the free $k[\frac{1}{s}]$-module $\mathcal{O}(m)(S_2)$. By the definition of π_*, we have that $(\pi_*\mathcal{O}(m))(T_1)$ is $\mathcal{O}(m)(S_1)$ considered as a $k[t]$-module. Therefore, it has a basis $(e, se, s^2e, \ldots, s^{n-1}e)$. Similarly, the $k[\frac{1}{t}]$-module $(\pi_*\mathcal{O}(m))(T_2)$ has a basis $(s^{-2m}e, s^{-(2m+1)}e, \ldots, s^{-(2m+n-1)}e)$. We may order these bases such that corresponding elements have exponents of s congruent modulo n. The above formula now follows from some combinatorics. For example, when $m = 0$, we see that we get a transition matrix between our bases which is diagonal with entries $(1, t^{-1}, \ldots, t^{-1})$. The corresponding vector bundle is thus $\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus(n-1)}$. When $m = 1$, we get a transition matrix $(t^{-1}, \ldots, t^{-1}, t, t)$. Therefore $\pi_*\mathcal{O}(1) = \mathcal{O}(-1)^{\oplus(n-2)} \oplus \mathcal{O}(1)^{\oplus 2}$.

Note that $\mathrm{Vect}(X)$ is a full additive subcategory of the abelian category $\mathrm{Coh}(X)$. Therefore, we may define its Grothendieck group via this embedding.

DEFINITION 1.32. We define the *Grothendieck group of vector bundles* on X, denoted by $K^0(X)$, as

$$K^0(X) = K(\mathrm{Vect}(X)) = F(\mathrm{Vect}(X))/H(\mathrm{Vect}(X)).$$

For a ring A, we write $K^0(A) = K^0(\operatorname{Spec} A)$.

The tensor product with respect to $\mathcal{O}_X$ defines a ringstructure on $K^0(X)$ where the identity is given by the class of $\mathcal{O}_X$. In fact, note that any vector bundle is a flat $\mathcal{O}_X$-module. Therefore the subgroup $H(\mathrm{Vect}(X))$ is an ideal of the ring $F(\mathrm{Vect}(X))$. This also shows that $K_0(X)$ becomes a $K^0(X)$-module when multiplication is defined similarly.

REMARK 1.33. For any vector bundle $\mathcal{E}$ on X, there is a locally constant map $\mathrm{rk} : X \longrightarrow \mathbf{Z}$ which sends x to the rank of $\mathcal{E}_x$. One can easily check that this defines a homomorphism $K^0(X) \longrightarrow H^0(X, \mathbf{Z})$. In particular, if X is nonempty, the ring $K^0(X)$ is of characteristic zero. (For a connected scheme, the kernel of the rank morphism $\mathrm{rk} : K^0(X) \longrightarrow \mathbf{Z}$ is the starting point of the so-called γ-filtration for $K^0(X)$. See Chapter 2.)

EXAMPLE 1.34. Let A be a principal ideal domain. Then the rank morphism $K^0(A) \longrightarrow \mathbf{Z}$ is an isomorphism of rings.

EXAMPLE 1.35. Let O_K be the ring of integers for a number field K. Then $K^0(O_K) \cong \mathbf{Z} \oplus \mathrm{Cl}(O_K)$, where $\mathrm{Cl}(O_K)$ is the ideal class group. In fact, for any projective finitely generated A-module M, there is a fractional ideal $\mathfrak{a}$ of A and an integer $n \geq 0$ such that M is isomorphic to $\mathfrak{a} \oplus A^n$. Compare this to Proposition 3.1. The ringstructure on $\mathbf{Z} \oplus \mathrm{Cl}(A)$ is given by $(n, \mathfrak{a})(m, \mathfrak{b}) = (nm, n\mathfrak{b} + m\mathfrak{a})$.

Let $f : X \longrightarrow Y$ be a morphism of noetherian schemes. By the adjointness of f_* and f^{-1}, there is a natural morphism $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$. For a coherent sheaf $\mathcal{F}$ on Y, the inverse image of $\mathcal{F}$ is denoted by $f^*\mathcal{F}$. Recall that it is defined as $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. For a vector bundle $\mathcal{E}$, it holds that $f^*\mathcal{E}$ is a vector bundle of the same rank. Also, for vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on Y, it holds that

$$f^*(\mathcal{E}_1 \otimes_{\mathcal{O}_Y} \mathcal{E}_2) = f^*\mathcal{E}_1 \otimes_{\mathcal{O}_X} f^*\mathcal{E}_2.$$

Moreover, f^* takes short exact sequences of vector bundles into exact sequences. Therefore, f^* induces a ringmorphism $K^0(Y) \longrightarrow K^0(X)$ again denoted by f^*. One easily checks that $g^* \circ f^* = (g \circ f)^*$ for morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$.

We conclude that K^0 defines a contravariant functor from the category of noetherian schemes to the category of rings. To a morphism $f : X \longrightarrow Y$ one assigns the morphism of rings $f^* : K^0(Y) \longrightarrow K^0(X)$ given by $f^*\mathrm{cl}(\mathcal{E}) = \mathrm{cl}(f^*\mathcal{E})$.

EXAMPLE 1.36. Let k be a field and $A = k[x, y]/(xy)$. We have that $K^0(A) \cong \mathbf{Z}$ with generator the class of A. To prove this, let E be a finitely generated projective A-module. Note that $M_{(x)} \cong A^r_{(x)}$ and that $M_{(y)} \cong A^s_{(y)}$, where r and s are the ranks. Localizing M at the origin (x, y), we see that $r = s$. From this it easily follows that $K^0(A)$ is isomorphic to $\mathbf{Z}$ under the rank mapping (at any generic point).

5. The homotopy property for $K_0(X)$

Let X be a noetherian scheme. Then $K_0(X)$ obeys a certain localization sequence (Theorem 1.9) and it has a set of geometric generators (Theorem 1.22). Also, we have seen that the extension by zero $K_0(X_{\mathrm{red}}) \longrightarrow K_0(X)$ is an isomorphism. In this section we shall show that the group $K_0(X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}})$ is naturally isomorphic to $K_0(X)$. In particular, it follows that $K_0(\mathbf{A}^n_A)$ is naturally isomorphic to $K_0(A)$ for any noetherian ring A. This will allow us to compute the Grothendieck group of the projective n-space over a field.

Suppose that $f : X \longrightarrow Y$ is a flat morphism of noetherian schemes. Then the functor $f^* : \mathrm{Coh}(Y) \longrightarrow \mathrm{Coh}(X)$ is exact. Therefore, it induces a natural homomorphism $f^! : K_0(Y) \longrightarrow K_0(X)$. We do not write f^* for this morphism. (See Remark 1.66.)

Let A be a noetherian ring. The inclusion of rings $A \subset A[t]$ is flat and induces by base change a flat morphism $p : X \times_A \mathbf{A}^1_A \longrightarrow X$ for any (noetherian) A-scheme X. This induces a homomorphism $p^! : K_0(X) \longrightarrow K_0(X \times_A \mathbf{A}^1_A)$.

LEMMA 1.37. Suppose that A is reduced. Then the pull-back morphism $K_0(A) \longrightarrow K_0(A[t])$ is surjective.

PROOF. We shall proceed by noetherian induction on Spec A. By the localization theorem, for any $a \in A$, we have a short exact sequence

$$K_0(A/aA) \longrightarrow K_0(A) \longrightarrow K_0(A_a) \longrightarrow 0 .$$

Similarly, we have a short exact sequence

$$K_0(A/aA[t]) \longrightarrow K_0(A[t]) \longrightarrow K_0(A_a[t]) \longrightarrow 0 .$$

It is easy to see that we have a commutative diagram

$$\begin{array}{ccccccc} K_0(A/aA) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A_a) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_a} & & \downarrow & & \downarrow & & \\ K_0(A/aA[t]) & \longrightarrow & K_0(A[t]) & \longrightarrow & K_0(A_a[t]) & \longrightarrow & 0, \end{array}$$

with exact rows. By the induction hypothesis, for any nonzerodivisor $a \in A$, the homomorphism f_a is surjective. Since A is reduced, the nonzerodivisors in A form a directed system S, where $a \leq b$ if and only if $bA \subset aA$. Since the direct limit of abelian groups is an exact functor, we have a commutative diagram

$$\begin{array}{ccccccc} \lim_{a\in S} K_0(A/aA) & \longrightarrow & K_0(A) & \longrightarrow & \lim_{a\in S} K_0(A_a) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \\ \lim_{a\in S} K_0(A/aA[t]) & \longrightarrow & K_0(A[t]) & \longrightarrow & \lim_{a\in S} K_0(A_a[t]) & \longrightarrow & 0, \end{array}$$

where the map f is surjective. Now, the total ring of fractions $K = S^{-1}A$ is a finite product of fields $\prod_i F_i$. Also, for any $a \in S$, the natural inclusion $A_a \subset K$ is flat and induces a homomorphism $K_0(A_a) \longrightarrow K_0(K)$. The latter induces a natural isomorphism of abelian groups $\lim_{a\in S} K_0(A_a) \xrightarrow{\sim} K_0(K)$. Since $K_0(K) = \bigoplus_i K_0(F_i)$, we have a commutative diagram

$$\begin{array}{ccccccc} \lim_{a\in S} K_0(A/aA) & \longrightarrow & K_0(A) & \longrightarrow & \bigoplus_i K_0(F_i) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \\ \lim_{a\in S} K_0(A/aA[t]) & \longrightarrow & K_0(A[t]) & \longrightarrow & \bigoplus_i K_0(F_i[t]) & \longrightarrow & 0, \end{array}$$

where the homomorphism on the right is surjective. By a diagram chase, the homomorphism $K_0(A) \longrightarrow K_0(A[t])$ is also surjective. □

PROPOSITION 1.38. The morphism $K_0(A) \longrightarrow K_0(A[t])$ is surjective.

PROOF. Suppose that the morphism $K_0(A) \longrightarrow K_0(A[t])$ is not surjective. For any ideal $I \subset A$, we have a commutative diagram

$$\begin{array}{ccc} K_0(A/I) & \longrightarrow & K_0(A/I[t]) \\ \downarrow & & \downarrow \\ K_0(A) & \longrightarrow & K_0(A[t]), \end{array}$$

where the vertical maps are induced from the extension by zero. Since A is noetherian, among all ideals $I \subset A$ such that $K_0(A/I) \longrightarrow K_0(A/I[t])$ is not an isomorphism, there is a maximal one $J \subset A$. Then, the ring $B = A/J$ is such that $K_0(B/I) \longrightarrow K_0(B/I[t])$ is an isomorphism for every nonzero ideal $I \subset B$. By Proposition 1.18, the ring B is reduced. Thus, by the above Lemma, the map $K_0(B) \longrightarrow K_0(B[t])$ is an isomorphism. Contradiction. □

THEOREM 1.39. **(Homotopy)** Let X be a noetherian scheme. Then the projection $p : X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}} \longrightarrow X$ induces a bijective homomorphism

$$p^! : K_0(X) \longrightarrow K_0(X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}}).$$

PROOF. Let us show that $p^!$ has a left inverse. This will imply that $p^!$ is injective. The projection $\mathbf{Z}[t] \longrightarrow \mathbf{Z}$ given by $t \mapsto 0$ induces a section $\pi_0 : X \hookrightarrow X \times \mathbf{A}^1_{\mathbf{Z}}$ of p. Dropping the subscripts, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{X\times\mathbf{A}^1} \longrightarrow \mathcal{O}_{X\times\mathbf{A}^1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

of coherent sheaves on $X \times \mathbf{A}^1$. Thus, for any sheaf $\mathcal{F}$ on $X \times \mathbf{A}^1$, it holds that

$$(L^i\pi_0^*\mathcal{F})\mathcal{O}_X = \mathrm{Tor}^i_{\mathcal{O}_{X\times\mathbf{A}^1}}(\mathcal{O}_X, \mathcal{F}) = 0$$

whenever $i \geq 2$. Therefore, the map $\pi_0^! : K_0(X \times \mathbf{A}^1) \longrightarrow K_0(X)$ given by

$$\mathrm{cl}(\mathcal{F}) \mapsto \mathrm{cl}(\mathrm{Tor}^0(\mathcal{O}_X, \mathcal{F})) - \mathrm{cl}(\mathrm{Tor}^1(\mathcal{O}_X, \mathcal{F}))$$

is a well-defined homomorphism. One readily checks that $\pi_0^! \circ p^! = \mathrm{id}$.

Now, let us show that the map $p^!$ is surjective. Let $U = \mathrm{Spec}\, A$ be an affine open subscheme of X. Then $A = \mathcal{O}_X(U)$ is noetherian. By applying the localization sequence to $Y = X - U$, we have a commutative diagram

$$\begin{array}{ccccccc} K_0(Y) & \longrightarrow & K_0(X) & \longrightarrow & K_0(A) & \longrightarrow & 0 \\ \downarrow & & \downarrow p^! & & \downarrow & & \\ K_0(Y \times \mathbf{A}^1) & \longrightarrow & K_0(X \times \mathbf{A}^1) & \longrightarrow & K_0(A[t]) & \longrightarrow & 0 \end{array}$$

with exact rows. Also, the maps on the left and right are surjective by noetherian induction and Proposition 1.38. By a diagram chase, we may conclude that $p^!$ is surjective. □

REMARK 1.40. One can deduce from the above theorem that, for any vector bundle $E \longrightarrow X$, the natural morphism $K_0(X) \longrightarrow K_0(E)$ is an isomorphism. (Here we view E as a scheme.)

Let us give an application of the homotopy property which is useful in proving the Grothendieck-Riemann-Roch theorem (Theorem 3.6).

Let k be a field and suppose that X and Y are schemes of finite type over k. By base-change, the projection morphisms $X \times_k Y \longrightarrow Y$ and $X \times_k Y \longrightarrow X$ are flat. From the pull-back construction above, these projections induce homomorphisms $K_0(X) \longrightarrow K_0(X \times_k Y)$ and $K_0(Y) \longrightarrow K_0(X \times_k Y)$ which give a natural homomorphism $K_0(X) \otimes K_0(Y) \longrightarrow K_0(X \times_k Y)$.

PROPOSITION 1.41. For any scheme X of finite type over k, the natural homomorphism

$$K_0(X) \otimes K_0(\mathbf{P}^n_k) \longrightarrow K_0(X \times_k \mathbf{P}^n_k)$$

is surjective.

PROOF. We argue by induction on n. For $n = 0$, the statement is trivial. By the localization sequence and the right exactness of $K_0(X) \otimes -$, we have a commutative diagram

$$\begin{array}{ccccccc} K_0(X) \otimes K_0(\mathbf{P}^{n-1}_k) & \longrightarrow & K_0(X) \otimes K_0(\mathbf{P}^n_k) & \longrightarrow & K_0(X) \otimes K_0(\mathbf{A}^n_k) & \longrightarrow & 0 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ K_0(X \times_k \mathbf{P}^{n-1}_k) & \longrightarrow & K_0(X \times_k \mathbf{P}^n_k) & \longrightarrow & K_0(X \times_k \mathbf{A}^n_k) & \longrightarrow & 0 \end{array}$$

with exact rows. By the induction hypothesis, the map f_1 is surjective. By Theorem 1.39, the map f_3 is bijective. By a diagram chase, we conclude that f_2 is surjective. □

6. Algebraic intermezzo: Koszul complexes, complete intersections and syzygy

Let A be a noetherian ring. For elements $x_1, \ldots, x_n$ in A and E the free A-module of rank n with basis $(e_1, \ldots, e_n)$, we define the *Koszul complex* $K^A(x_1, \ldots, x_n)$ associated to the sequence $(x_1, \ldots, x_n)$ to be

$$0 \longrightarrow \Lambda^n E \xrightarrow{d} \Lambda^{n-1} E \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^1 E = E \xrightarrow{d} \Lambda^0 E = A \longrightarrow 0 .$$

Here the boundary map $d : \Lambda^p E \longrightarrow \Lambda^{p-1} E$ is given by

$$d(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} e_{i_1} \wedge \ldots \wedge \hat{e_{i_j}} \wedge \ldots \wedge e_{i_p}.$$

The reader may verify that $d^2 = 0$. Note that for any permutation σ of the set $\{1, \ldots, n\}$, the Koszul complex $K^A(x_1, \ldots, x_n)$ is isomorphic to the Koszul complex $K^A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

EXAMPLE 1.42. The Koszul complex associated to $x_1, x_2 \in A$ is the complex

$$0 \longrightarrow A \xrightarrow{f} A^2 \xrightarrow{g} A \longrightarrow 0 ,$$

where $f : a \mapsto (ax_2, -ax_1)$ and $g : (a, b) \mapsto ax_1 + bx_2$.

DEFINITION 1.43. An element $x \in A$ is called *regular* if the multiplication by x is injective. A sequence $(x_1, \ldots, x_n)$ of elements $x_1, \ldots, x_n \in A$ is said to be a *regular sequence* if x_1 is regular and the image of x_i in $A/(x_1 A + \ldots + x_{i-1} A)$ is regular for all $i = 2, \ldots, n$.

REMARK 1.44. Any sequence of elements in the zero ring is regular. Suppose that A is not the zero ring. Then a nonzero element $x \in A$ is regular if and only if it is a nonzerodivisor. (The zero element is a nonzerodivisor.) Furthermore, a sequence $(x_1, \ldots, x_n)$ is regular if and only if the sequence $(x_1, \ldots, x_n, u)$ is regular for all units $u \in A$.

EXAMPLES 1.45. We give some examples.

(1) Suppose that $A \neq 0$. Then $(0, 1)$ is not a regular sequence in A whereas $(1, 0)$ is. Thus, regular sequences are not invariant under permutation in general.
(2) The sequence $(x_1, \ldots, x_n)$ is regular in $A[x_1, \ldots, x_n]/(1 - x_1 - \ldots - x_n)$.
(3) Linear forms $(f_1, \ldots, f_n)$ in $A = k[X_1, \ldots, X_n]$ define a regular sequence if and only if they form a linearly independent set over k.
(4) Let $A = k[x, y, z]/(xz - y)$. The sequence (x, y, z) is not regular in A.

THEOREM 1.46. Let $(x_1, \dots, x_n)$ be a sequence in A and let I be the ideal generated by it. Assume $I \neq A$.

(1) If $(x_1, \dots, x_n)$ is regular, the augmented Koszul complex

$$0 \longrightarrow \Lambda^n E \xrightarrow{d} \Lambda^{n-1} E \xrightarrow{d} \cdots \xrightarrow{d} E \xrightarrow{d} A \longrightarrow A/I \longrightarrow 0$$

is exact.

(2) If A is local and the augmented Koszul complex

$$0 \longrightarrow \Lambda^n E \xrightarrow{d} \Lambda^{n-1} E \xrightarrow{d} \cdots \xrightarrow{d} E \xrightarrow{d} A \longrightarrow A/I \longrightarrow 0$$

is exact, the sequence $(x_1, \dots, x_n)$ is regular.

PROOF. See [**Lang**, Chapter XXI, Theorem 4.6, pp. 856]. □

EXAMPLE 1.47. Let k be a field. Suppose that $A = k[x,y]/(xy)$ and $\mathfrak{m} = (x,y)A$. Let $k = A/\mathfrak{m}$ be the corresponding residue field. Consider the infinite resolution of free A-modules

$$\cdots \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{f} A \longrightarrow k \longrightarrow 0\,.$$

Here

$$f : (s,t) \mapsto sx + ty,\;\; g : (s,t) \mapsto (sy, tx) \text{ and } h : (s,t) \mapsto (sx, ty).$$

It is easy to see that

$$\mathrm{Tor}_i^A(k,k) = \begin{cases} k & \text{if} \quad i = 0 \\ k^2 & \text{if} \quad i > 0 \end{cases}.$$

To prove this we note that, after tensoring the above resolution with k, the maps become zero. This shows that k does not have a finite projective resolution of A-modules. Else the $\mathrm{Tor}_i^A(k,-)$ functors would be identically zero for $i \gg 0$. In particular, the Koszul complex does not provide us with a resolution in this case.

DEFINITION 1.48. An ideal $I \subset A$ which can be generated by a regular sequence is called a *complete intersection* in A. An ideal $I \subset A$ for which the localization $I_\mathfrak{p}$ at any prime ideal $\mathfrak{p} \subset A$ is a complete intersection in $A_\mathfrak{p}$ is called a *local complete intersection*.

Any complete intersection in A is a local complete intersection. (Localization is exact.)

EXAMPLE 1.49. Let I be a complete intersection in A. Then the class of A/I in $K_0(A)$ equals zero. In fact, consider a Koszul resolution for A/I and use that the alternating sum of the binomial coefficients is zero.

As the following proposition says, the number of equations defining a complete intersection in $\operatorname{Spec} A$ is precisely its codimension.

PROPOSITION 1.50. Suppose that I is an ideal of A which can be generated by a regular sequence $(x_1, \dots, x_r)$. Then $\mathrm{ht}(I) = r$.

PROOF. By Krull's theorem, $\mathrm{ht}(\mathfrak{p}) \leq r$. In particular, it holds that $\mathrm{ht}(I) \leq r$. We shall show by induction on $\dim A$ that $\mathrm{ht}(\mathfrak{p}) \geq r$. Therefore, we may assume that the height of $\mathfrak{p}/x_1 A$ is $r-1$ in $A/x_1 A$. That is, there is a chain of prime ideals

$$\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \ldots \subsetneq \mathfrak{p}_r = \mathfrak{p}$$

with $x_1 A \subset \mathfrak{p}_1$. Now, x_1 does not belong to any minimal prime ideal. For, suppose that x_1 would belong to a minimal prime ideal. Since minimal prime ideals are associated to A, we would have that x_1 is a zero divisor. But that contradicts the fact that $(x_1, \ldots, x_r)$ is a regular sequence. Thus, we have that $\mathfrak{p}_1$ is not a minimal prime ideal. Therefore, there exists a minimal prime ideal $\mathfrak{p}_0$ such that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$. We conclude that $\mathrm{ht}(\mathfrak{p}) \geq r$. □

Suppose that A is a local noetherian ring.

PROPOSITION 1.51. Any minimal set of generators for a complete intersection I in A forms a regular sequence.

PROOF. We may and do assume that $I \neq A$. Let $(x_1, \ldots, x_n)$ be a regular sequence generating I. Note that any minimal set of generators for I has precisely n elements. Thus, suppose that $(y_1, \ldots, y_n)$ is a minimal set of generators for I. Then there exists an invertible $(n \times n)$-matrix $\Lambda \in \mathrm{GL}_n(A)$ such that $a_i = \sum \Lambda_{ij} b_j$. We see that Λ induces an isomorphism of complexes $K^A(x_1, \ldots, x_n) \cong K^A(y_1, \ldots, y_n)$. By Theorem 1.46, it holds that $(y_1, \ldots, y_n)$ is a regular sequence. □

Let $d = \dim A$. Recall that A is called regular if $d = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Equivalently, the maximal ideal $\mathfrak{m}$ can be generated by d elements. Even better, the maximal ideal $\mathfrak{m}$ of A is a complete intersection. (Use that a local noetherian regular ring is an integral domain and Proposition 1.50.) By Proposition 1.51, for any system of parameters $(x_1, \ldots, x_d)$ of A, the Koszul complex $K^A(x_1, \ldots, x_d)$ is exact if and only if A is regular.

LEMMA 1.52. Suppose that A is a local noetherian ring with residue field k. Let M be a finitely generated A-module and suppose that $\mathrm{Tor}_1^A(k, M) = 0$. Then M is free.

PROOF. Let $(m_1, \ldots, m_r)$ be a minimal set of generators for M and consider the exact sequence

$$0 \longrightarrow K \longrightarrow A^r \xrightarrow{\epsilon} M \longrightarrow 0$$

where $\epsilon : (a_1, \ldots, a_m) \mapsto \sum a_i m_i$. The long exact sequence associated to $\mathrm{Tor}_1^A(k, -)$ gives us a short exact sequence

$$0 \longrightarrow k \otimes_A K \longrightarrow \mathrm{Tor}_0^A(k, A^r) \cong k^r \xrightarrow{\epsilon \otimes k} \mathrm{Tor}_0^A(k, M) = k \otimes_A M \longrightarrow 0 \, .$$

By Nakayama's Lemma, $\epsilon \otimes k : k^r \longrightarrow k \otimes_A M = M/\mathfrak{m}M$ is an isomorphism. Thus, $0 = k \otimes_A K = K/\mathfrak{m}K$. Again by Nakayama's Lemma, $K = 0$. We conclude that M is free. □

THEOREM 1.53. **(Syzygy)** Suppose that A is a local noetherian regular ring. Let M be a finitely generated A-module and let

$$0 \longrightarrow N \longrightarrow E_{\dim A - 1} \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence of A-modules where the E_i are free. Then N is free.

PROOF. The long exact sequence for the Tor functor shows that

$$\mathrm{Tor}_1^A(k, N) \cong \mathrm{Tor}_{\dim A + 1}^A(k, M).$$

This is called dimension shifting. Now, the Koszul complex provides us with a resolution for $k = A/\mathfrak{m}$ of length $d = \dim A$. Therefore $\mathrm{Tor}_1^A(k, N) = \mathrm{Tor}_{d+1}^A(k, M) = 0$.Now, apply Lemma 1.52. □

In the next section we will give various applications of the syzygy theorem.

7. The Cartan homomorphism

Let X be a noetherian scheme. The embedding $\mathrm{Vect}(X) \longrightarrow \mathrm{Coh}(X)$ of categories induces a natural homomorphism $K^0(X) \longrightarrow K_0(X)$. This is called the *Cartan homomorphism.*

EXAMPLE 1.54. Let us give some examples of the Cartan homomorphism.

(1) Let p be a prime number, $n \geq 1$ an integer and $A = \mathbf{Z}/p^n\mathbf{Z}$. The Cartan homomorphism $\mathbf{Z} \cong K^0(A) \longrightarrow K_0(A) \cong \mathbf{Z}$ sends 1 to n. (See Examples 1.3 and 1.11).
(2) Let $A = k[x,y]/(xy)$. The Cartan homomorphism $\mathbf{Z} \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$ is given by the diagonal embedding. (See Example 1.24 and 1.36.)

LEMMA 1.55. Suppose that X is regular and finite-dimensional. Let $\mathcal{F}$ be a coherent sheaf and suppose that

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_p \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \tag{3}$$

is an exact sequence of coherent sheaves, where $\mathcal{E}_i$ is a vector bundle for $i = 0, \ldots, p$. Then $\mathcal{G}$ is also a vector bundle whenever $p \geq \dim X - 1$.

PROOF. Firstly, note that $\mathcal{G}$ is coherent. Localize (3) at $x \in X$ and apply Theorem 1.53 to conclude that $\mathcal{G}_x$ is a finitely generated free $\mathcal{O}_{X,x}$-module when $p \geq \dim \mathcal{O}_{X,x} - 1$. Now, we simply observe that $\dim X - 1 \geq \operatorname{codim}(\overline{\{x\}}, X) - 1 = \dim \mathcal{O}_{X,x} - 1$ for all $x \in X$. □

REMARK 1.56. Note that we used the finite dimensionality in an essential way here. Let us give an example of an infinite-dimensional scheme such that the above theorem fails. Let k be a field and let $A = k \times k[t_1] \times k[t_1, t_2] \times \ldots$. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow A \longrightarrow \ldots \longrightarrow A \longrightarrow A \longrightarrow k \longrightarrow 0.$$

The map $A \longrightarrow k$ is the projection. The second map is the projection $A \longrightarrow A/k$ composed with the inclusion $A/k \subset A$. At each stage, the kernel K is of the form $A/(k \times k[t_1] \times k[t_1,t_2] \times \ldots \times k[t_1, \ldots, t_n]$ which is not locally free. Geometrically this corresponds to taking a point, then adding a line, then adding a plane, etc.

EXAMPLE 1.57. Let X be a regular projective scheme over a noetherian ring A and let $\mathcal{F}$ be a coherent sheaf on X. Write $n = \dim X$ and $\mathcal{O} = \mathcal{O}_X$. There exists an integer $m \in \mathbf{Z}$ and a positive integer $r > 0$, such that $\mathcal{F}$ is a quotient sheaf of $\bigoplus^r \mathcal{O}(m) = \mathcal{O}(m)^{\oplus r}$. (See the proof of Theorem 1.17.) Therefore, by Lemma 1.55, we have a resolution of vector bundles

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{E}_i$ is a direct sum of line bundles of the form $\mathcal{O}(m)$. Thus, the Grothendieck group of coherent sheaves $K_0(X)$ is generated by the classes of the line bundles $\mathcal{O}(m)$, where $m \in \mathbf{Z}$. Of course, the same argument applies to $K^0(X)$. We see that the Cartan homomorphism $K^0(X) \longrightarrow K_0(X)$ is surjective.

EXAMPLE 1.58. We apply the above example to compute $K^0(\mathbf{P}^n_k)$, where k is a field. Let $X = \mathbf{P}^n_k$ and $\mathcal{O} = \mathcal{O}_X$. We have an exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{O} \longrightarrow \oplus^{n+1}\mathcal{O}(1) \longrightarrow \oplus^{\binom{n+1}{2}}\mathcal{O}(2) \longrightarrow \cdots \longrightarrow \oplus^{n+1}\mathcal{O}(n) \longrightarrow \mathcal{O}(n+1) \longrightarrow 0\,.$$

In fact, this is just a dualized Koszul complex. From this exact sequence, we get two important facts. Firstly, writing ξ for the class of $\mathcal{O}(1)$ in $K^0(X)$, we see that $(1-\xi)^{n+1}=0$. Here we invoked the ringstructure on $K^0(X)$. Furthermore, we see that $K^0(X)$ is generated by $(1,\xi,\ldots,\xi^n)$. We will show that the surjective homomorphism $\mathbf{Z}^{n+1} \longrightarrow K^0(X)$ given by $(a_0,\ldots,a_n) \mapsto \sum_{i=0}^n a_i\xi^i$ is an isomorphism of abelian groups. To prove this, we proceed in two steps.

Firstly, note that $\xi^m \neq 0$ for $0 \le m \le n$. In fact, for $0 \le i \le n$, the homomorphism $K^0(X) \longrightarrow \mathbf{Z}$ given by $\mathrm{cl}(\mathcal{E}) \mapsto \chi(X,\mathcal{E}(i))$ maps ξ^i to $\dim_k k[T_0,\ldots,T_n]_i \neq 0$. To finish the proof, it suffices to show that the above surjective homomorphism $\mathbf{Z}^{n+1} \longrightarrow K^0(X)$ is injective. In fact, suppose that $\alpha = \sum_{i=0}^n a_i\xi^i = 0$, where $(a_0,\ldots,a_n) \in \mathbf{Z}^{n+1}\backslash\{0\}$. Choose i maximal with $a_i \neq 0$. Then

$$a_i = \chi(X, \alpha\cdot\xi^{-i}) = 0.$$

Contradiction. We conclude that

$$(1,\xi,\ldots,\xi^n)$$

is a $\mathbf{Z}$-basis for the abelian group $K^0(X)$. Also, the map $\mathbf{Z}[x]/(1-x)^n \longrightarrow K^0(X)$ given by $x \mod (1-x)^n \mapsto \xi$ is an isomorphism of rings with inverse given by

$$\mathrm{cl}(\mathcal{E}) \mapsto \sum_{i=0}^n \chi(X,\mathcal{E}(i))x^i \mod (1-x)^n.$$

Definition 1.59. If every coherent sheaf on X is a quotient of a vector bundle, we shall say that $\mathrm{Coh}(X)$ (or just X) has *enough locally frees.*

Example 1.60. Suppose that X is noetherian and has an ample invertible sheaf $\mathcal{L}$. Then, for any coherent sheaf $\mathcal{F}$, there exists an integer m and an epimorphism $\mathcal{O}_X^m \longrightarrow \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes m}$. Tensoring both sides with $(\mathcal{L}^\vee)^{\otimes m}$, we see that any coherent sheaf on X is the quotient of a vector bundle. Thus $\mathrm{Coh}(X)$ has enough locally frees. Since a scheme of finite type over a noetherian ring A is quasi-projective if and only if it has an ample invertible sheaf, a quasi-projective scheme over A has enough locally frees.

It turns out that the schemes we work with have enough locally frees if and only if they are semi-separated. Let us make this more precise.

A scheme is called *semi-separated* if, for every pair of affine open subsets $U, V \subset X$, it holds that $U\cap V$ is affine. Note that separated schemes are semi-separated. (Suppose that X is separated. Then the diagonal morphism $\Delta : X \longrightarrow X\times_{\mathbf{Z}} X$ is a closed immersion. In particular, it is affine. Therefore, for any pair of affine open subsets $U, V \subset X$, the intersection $U\cap V = \Delta^{-1}(U\times V)$ is affine.)

Example 1.61. The affine line over a field with a double origin is semi-separated. This is simply because any open of the affine line is affine. Similarly, the projective line with a double origin is semi-separated.

Example 1.62. The affine plane over a field with a double origin is not semi-separated. This is simply because $\mathbf{A}^2-\{0\}$ is not affine.

Remark 1.63. Suppose that X is a noetherian scheme which has enough locally frees. Then X is semi-separated ([**Tot**, Proposition 1.3]). One can show a converse to this. Suppose that X is a noetherian semi-separated locally $\mathbf{Q}$-factorial scheme. Then X has enough locally frees ([**BrSc**, Proposition 1.3]). In particular, a scheme which is smooth over a field has enough locally frees if and only if it is semi-separated. Totaro actually shows that a scheme which is smooth over a field is semi-separated if and only if the Cartan homomorphism $K^0(X) \longrightarrow K_0(X)$ is surjective. The next theorem settles one implication, whereas Totaro shows the reverse implication ([**Tot**, Proposition 8.1]).

THEOREM 1.64. Let X be a noetherian finite-dimensional regular scheme which is semi-separated. Then the Cartan homomorphism $K^0(X) \longrightarrow K_0(X)$ is an isomorphism of groups.

PROOF. Let $n = \dim X$. Note that X is $\mathbf{Q}$-factorial. Therefore, every coherent sheaf is a quotient of a vector bundle. Thus, we can construct a finite resolution of vector bundles by a standard procedure. In fact, by Lemma 1.55, we have a finite resolution

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

of vector bundles. Thus, the Cartan homomorphism is surjective. Let $\mathcal{E}. = \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_0$ denote this resolution. Consider the element $\gamma(\mathcal{E}.) = \sum(-1)^i \mathrm{cl}(\mathcal{E}_i)$ in $K^0(X)$. Then $\gamma(\mathcal{E}.)$ only depends on $\mathcal{F}$. This induces a homomorphism $\gamma : K_0(X) \longrightarrow K^0(X)$ which is clearly inverse to the Cartan homomorphism ([**BorSer**, Lemme 11,12]). □

EXAMPLE 1.65. Using the structure theorem, we showed that $K_0(A) \cong K^0(A) \cong \mathbf{Z}$ for any principal ideal domain A. Alternatively, the Cartan homomorphism induces an isomorphism $K_0(A) \cong K^0(A)$. Since any finitely generated projective A-module is free, the rank induces an isomorphism $K^0(A) \cong \mathbf{Z}$.

Let us show why one wants to avoid schemes which are not semi-separated. Let X be the projective n-space (over a field k) with a double origin. This is a smooth nonseparated scheme over k. Let 0 be one of the origins of X and let $U \cong \mathbf{P}^n$ be its complement in X.

We have an exact sequence $\mathbf{Z} \longrightarrow K_0(X) \longrightarrow K_0(U) \longrightarrow 0$ associated to the closed point 0. We claim that $K_0(X) \cong \mathbf{Z} \oplus K_0(U) \cong \mathbf{Z}^{n+2}$. To prove this, note that we have a morphism $K_0(U) \longrightarrow \mathbf{Z}$ given by the Euler characteristic. The composition $K_0(X) \longrightarrow K_0(U) \longrightarrow \mathbf{Z}$ determines a left inverse to $\mathbf{Z} \longrightarrow K_0(X)$. Therefore, the homomorphism $K_0(X) \longrightarrow \mathbf{Z} \oplus K_0(U)$ given by $\mathrm{cl}(\mathcal{F}) \mapsto (\chi(U, \mathcal{F}|_U), \mathrm{cl}(\mathcal{F}|_U))$ is an isomorphism of abelian groups. By a similar argument and induction on m, if X is the projective n-space with m origins, we have that

$$K_0(X) \cong \mathbf{Z}^{n+m}.$$

Now, let us determine $K^0(X)$. Firstly, suppose that $n > 1$. Since both origins are of codimension at least 2, we conclude that $K^0(X) \cong K^0(U) \cong \mathbf{Z}^{n+1}$. (In fact, given a vector bundle $\mathcal{E}$ on $U = \mathbf{P}^n$, we can clearly extend it to X by using the same data on the second origin. Now, this extension is unique up to isomorphism. Assume that $\mathcal{F}$ is a vector bundle which extends $\mathcal{E}$. Let $i : U \longrightarrow X$ be the inclusion. Then $i^*\mathcal{F} = \mathcal{E}$. By the adjunction of i_* and i^{-1}, we have a morphism $\mathcal{F} \longrightarrow i_*\mathcal{E}$ of coherent sheaves on X. The kernel of $\mathcal{F} \longrightarrow i_*\mathcal{E}$ is zero on U. Therefore, it is a torsion subsheaf of the vector bundle $\mathcal{F}$. But this implies that it is zero. Thus, we have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow i_*\mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is a coherent sheaf on X. The normality of X implies that $\mathcal{G}^\vee = \mathcal{E}xt^1(\mathcal{O}_X, \mathcal{G}) = 0$. Thus, dualizing the sequence gives us that $\mathcal{F}^\vee \cong (i_*\mathcal{E})^\vee$. Since $\mathcal{F}$ is a vector bundle, we conclude that $\mathcal{F} \cong (\mathcal{F}^\vee)^\vee \cong (i_*\mathcal{E}^\vee)^\vee$. We see that $\mathcal{F}$ is unique up to isomorphism.) Thus, the

Cartan homomorphism $K^0(X) \longrightarrow K_0(X)$ is not an isomorphism. (By Theorem 1.64, this implies Thomason's observation ([**ThTr**, Exercise 8.6]) that X does not have enough locally frees. To see this, one can follow the proof of Theorem 1.64 and arrive at a contradiction or use Totaro's theorem mentioned above.)

Now, suppose that $n = 1$. Then $K^0(X) \cong \mathbf{Z} \oplus K_0(\mathbf{P}^1)$. (Here the factor $\mathbf{Z}$ should be interpreted as $\mathrm{Pic}(\mathbf{P}^1)$.) In fact, any vector bundle on $\mathbf{P}^1$ extends uniquely to X up to the choice of a divisor as the above reasoning shows. The Cartan homomorphism is an isomorphism in this case. Since X is semi-separated, this is consistent with Theorem 1.64.

Let X be a finite-dimensional noetherian semi-separated regular scheme. Since $K^0(X)$ has a ringstructure given by the tensor product, we see that $K_0(X)$ inherits a ringstructure from $K^0(X)$ by the Cartan isomorphism. For $\mathcal{F}$ and $\mathcal{G}$ coherent sheaves on X, this product is given by

$$\mathrm{cl}(\mathcal{F}) \cdot \mathrm{cl}(\mathcal{G}) = \sum_{i=0}^{\dim X} (-1)^i \ \mathrm{cl}(\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

This is a direct consequence of the universality and additivity of the Tor functors.

Remark 1.66. To a morphism $f : X \longrightarrow Y$ of finite-dimensional noetherian semi-separated regular schemes one assigns the composition of ringmorphisms

$$f^! : \ K_0(Y) \xrightarrow{\sim} K^0(Y) \xrightarrow{f^*} K^0(X) \xrightarrow{\sim} K_0(X)$$

which is given by $f^!\mathrm{cl}(\mathcal{F}) = \sum (-1)^i \mathrm{cl}\left(\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})\right)$. Assuming $f : X \longrightarrow Y$ to be proper, one can ask if $f_! : K_0(X) \longrightarrow K_0(Y)$ is a ringmorphism. The following Proposition shows that this is not the case in general.

Proposition 1.67. **(Projection formula)** For $x \in K_0(X)$ and $y \in K_0(Y)$, it holds that

$$f_!(x \cdot f^!(y)) = f_!(x) \cdot y.$$

Proof. It suffices to prove the equality for $x = \mathrm{cl}(\mathcal{F})$ and $y = \mathrm{cl}(\mathcal{E})$, where $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{E}$ is a vector bundle on Y. Firstly, we have a natural isomorphism of coherent sheaves

$$f_*(f^*(\mathcal{E} \otimes \mathcal{F}) = \mathcal{E} \otimes f_*(\mathcal{F}).$$

(In fact, since the statement is local for Y, we may assume that $Y = \mathrm{Spec}\, A$ is affine and $\mathcal{E} = \mathcal{O}_Y^r$. Then, from the definition of f^*, it follows immediately that $f^*\mathcal{F} = \mathcal{O}_X^r$. The equation now becomes obvious.) Thus, for any vector bundle $\mathcal{E}$ on Y, the functor given by

$$\mathcal{G} \mapsto f_*(f^*(\mathcal{E}) \otimes \mathcal{G})$$

is left exact and its right derived functors coincide with those of the functor

$$\mathcal{G} \mapsto \mathcal{E} \otimes f_*\mathcal{G}.$$

Since $\mathcal{E}$ is flat, we get the formula

$$R^i f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E} = R^i f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}).$$

This clearly implies the projection formula. □

Corollary 1.68. Suppose that f is an isomorphism. Then $f^! : K_0(Y) \longrightarrow K_0(X)$ is an isomorphism of rings with inverse $f_!$. □

CHAPTER 2

Intersection theory and characteristic classes

1. Proper intersection

Let k be a field. A variety is a separated integral scheme of finite type over k. A subvariety of a variety is a closed subscheme which is a variety.

Let X be a smooth quasi-projective variety. In particular, we have that X is regular.

Let V (resp. W) be a closed subscheme of X and let $\mathcal{J}_1$ (resp. $\mathcal{J}_2$) be the sheaf of ideals defining V (resp. W). We let $V \cap W$ be the closed subscheme defined by the sheaf of ideals $\mathcal{J}_1 + \mathcal{J}_2$. We call $V \cap W$ the scheme-theoretic intersection of V and W. By definition, $\mathcal{O}_{V \cap W} = \mathcal{O}_X/(\mathcal{J}_1 + \mathcal{J}_2)$. This can be geometrically interpreted as the following.

The set of solutions of the union of two systems of equations is the intersection of the set of solutions of each of them.

By the canonical identity of A-algebras

$$A/I \otimes_A A/J = A/(I + J),$$

where I and J are ideals in the ring A, we have that $\mathcal{O}_{V\cap W} = \mathcal{O}_V \otimes_{\mathcal{O}_X} \mathcal{O}_W$. More directly, the scheme-theoretic intersection of V and W can be defined as the closed subscheme of X given by $V \times_X W$.

The problem in defining "intersection products" of cycles is to associate an "intersection multiplicity" $i_Z(V, W)$ to any irreducible component Z of $V \cap W$ whenever V and W intersect properly. Let us illustrate this by an easy example.

Example 2.1. Suppose that $X = \mathbf{A}^2_k = \operatorname{Spec} k[x, y]$, $V = \operatorname{Spec} k[x, y]/(y^2 - x)$ and $W = \operatorname{Spec} k[x, y]/(x - a)$, where $a \in k$. Intuitively, the intersection of V and W should consist of two points. By the above, the intersection is given by

$$V \cap W = \operatorname{Spec} k[x, y]/(y^2 - x, x - a).$$

Clearly, we have that $k[x, y]/(y^2 - x, x - a) \cong k[y]/(y^2 - a)$. If $a \neq 0$ and k is algebraically closed, the Chinese remainder theorem shows that

$$V \cap W = \operatorname{Spec} k[y]/(y - \sqrt{a}) \amalg \operatorname{Spec} k[y]/(y + \sqrt{a}).$$

This clearly coincides with our intuition. Even for $a = 0$, we get that $V \cap W = \operatorname{Spec} k[y]/(y^2)$ which we can interpret as a double point. Of course, when k is not algebraically closed, we also want the multiplicity to be 2. Serre gave a definition for the intersection multiplicity and proved that it is the correct one. See Definition 2.2.

By [**Ser**, Chapter V, Paragraph B.6, Theorem 3], we have that $\operatorname{codim} Z \leq \operatorname{codim} V + \operatorname{codim} W$ for any irreducible component Z of $V \cap W$.

DEFINITION 2.2. We say that V and W *intersect properly* if, for any irreducible component Z of $V \cap W$, we have that

$$\operatorname{codim} Z = \operatorname{codim} V + \operatorname{codim} W.$$

It is clear that this is a local property. Now, suppose that V and W intersect properly and let Z be an irreducible component of $V \cap W$. We define the *(local) intersection multiplicity* of V and W along Z, denoted by $i_Z(V, W)$, as

$$i_Z(V, W) = \sum (-1)^i \operatorname{length}_A(\operatorname{Tor}_i^A(A/I, A/J)).$$

Here A is the local ring $\mathcal{O}_{X,z}$ at the generic point z of Z, I is the ideal defining V and J is the ideal defining W. Moreover, we define the *product cycle* of V and W, denoted by $[V] \cdot [W] \in Z^{\cdot} X$, as

$$[V] \cdot [W] = \sum_Z i_Z(V, W)[Z].$$

Here the sum is over all irreducible components of $V \cap W$. Similarly, we say that cycles α and β on X *meet properly* if their supports intersect properly. In this case, we define the *product cycle* of cycles $\alpha, \beta \in Z^{\cdot}(X)$, denoted by $\alpha \cdot \beta$, by linear extension. Also, we sometimes say that the cycle $\alpha \cdot \beta$ is *defined* if α and β meet properly.

It is clear that, if V and W intersect properly, we have that $[V] \cdot [W] = [W] \cdot [V]$. More precisely, we have that $i_Z(V, W) = i_Z(W, V)$ for any irreducible component Z of $V \cap W$. Also, $[V] \cdot [X] = [X] \cdot [V] = [V]$. Using a spectral sequence argument as explained in [**Ser**, Chapter V, Part C.3, pp. 114], we can show that this "product" is associative. This means that for cycles α, β, γ such that $\alpha \cdot \beta, (\alpha \cdot \beta) \cdot \gamma, \beta \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ are defined, we have that $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$. Moreover, we can show that there is a "exterior product formula" and that in order to calculate the product cycle we may "reduce to the diagonal". For a more precise formulation of these properties see [**Ser**, Chapter V, Paragraph C.3, pp. 114]. We now come to the following important theorem.

THEOREM 2.3. Let Z be an irreducible component of $V \cap W$, where V and W are subvarieties of X.

(1) If V and W do not intersect properly at Z, we have that $i_Z(V, W) = 0$.
(2) If V and W intersect properly at Z, we have that $i_Z(V, W) > 0$ and that $i_Z(V, W)$ coincides with the "classical intersection multiplicity" in the sense of Weil, Chevalley and Samuel.

PROOF. This is [**Ser**, Chapter V, Part C, Theorem 1, pp. 112]. There are two key points in the proof. These are the reduction to the diagonal and the fact that any system of parameters for a regular local ring A determines a finite free resolution for A as an $A \otimes_k A$-algebra by a Koszul complex. □

EXAMPLE 2.4. Let P be the origin of $\mathbf{A}^2 = \operatorname{Spec} k[x, y]$. Let V be the curve in $\mathbf{A}^2$ given by the equation $(x^2 + y^2)^2 + 3x^2y - y^3 = 0$ and let W be the curve given by the equation $(x^2 + y^2)^3 - 4x^2y^2 = 0$. Then V and W do not have any common irreducible component, i.e., they intersect properly. We have that $i_P(V, W) = 14$. (Use the classical intersection multiplicity and its well-known properties. See [**Ful2**, Chapter 3].)

Let V and W be subvarieties of a smooth quasi-projective variety X.

Suppose that V and W are in general position. That is, for every $x \in V \cap W$ there is an affine open subset $U \subset X$ such that $\mathcal{J}_1(U)$ is generated by a regular sequence $(x_1, \ldots, x_r)$ and $\mathcal{J}_2(U)$ is generated by a regular sequence $(g_1, \ldots, g_s)$ with $(f_1, \ldots, f_r, g_1, \ldots, g_s)$ a regular sequence in $\mathcal{O}_X(U)$. Note that V and W intersect properly. In fact, by Proposition 1.50, if $A = \mathcal{O}_{X,c}$ is the local ring at a generic point c of $V \cap W$, we have that $\operatorname{codim} A/(I+J) = r+s$ with $I = (f_1, \ldots, f_r)$ the ideal defining V at x and $J = (g_1, \ldots, g_s)$ the ideal defining W at x.

PROPOSITION 2.5. Let $(x_1, \ldots, x_n)$ be a sequence in a local noetherian ring A and I the ideal generated by it. Suppose that J is an ideal such that $(x_1 \mod J, \ldots, x_n \mod J)$ generates a proper ideal of A/J. Then $(x_1 \mod J, \ldots, x_n \mod J)$ is a regular sequence in A/J if and only if

$$\operatorname{Tor}_i^A(A/I, A/J) = \begin{cases} A/(I+J) & \text{if } \ i = 0 \\ 0 & \text{if } \ i \neq 0 \end{cases}$$

PROOF. Consider the Koszul complex $K^A(x_1, \ldots, x_n)$ associated to the sequence $(x_1, \ldots, x_n)$. Tensoring this with A/J gives us an isomorphism of complexes

$$K^A(x_1, \ldots, x_n) \otimes_A A/J \cong K^{A/J}(x_1 + J, \ldots, x_n + J).$$

The result now follows from Theorem 1.46. □

COROLLARY 2.6. Suppose that V and W are in general position. Then $\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W) = 0$ for all $i \geq 1$.

PROOF. It suffices to show that, for any $x \in X$, the stalk $\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)_x = 0$. We may and do assume that $x \in V \cap W$. Let U be an open affine containing x. Then,

$$\left(\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)\right)_x = \operatorname{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{V,x}, \mathcal{O}_{W,x}) = \operatorname{Tor}_i^A(A/I, A/J),$$

where $A = \mathcal{O}_{X,x} = \mathcal{O}_{U,x}$ and I (resp. J) is the ideal defining V (resp. W) in U. The result now follows from the definition of general position and Proposition 2.5. □

REMARK 2.7. The previous Proposition shows that

$$\operatorname{cl}(\mathcal{O}_V)\operatorname{cl}(\mathcal{O}_W) = \sum (-1)^i \operatorname{cl}\Big(\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)\Big) = \operatorname{cl}(\mathcal{O}_V \otimes_{\mathcal{O}_X} \mathcal{O}_W) = \operatorname{cl}(\mathcal{O}_{V \cap W})$$

in $K_0(X)$ if V and W are in general position. Also, we have that

$$[V] \cdot [W] = \sum_Z i_Z(V,W)[Z] = \sum_Z \operatorname{length} \mathcal{O}_{V \cap W, z}[Z] = [\mathcal{O}_{V \cap W}] = [V \cap W].$$

We see that the product in K_0-theory coincides with the product cycle (under the homomorphism $Z^{\cdot} X \longrightarrow K_0(X)$) and that they both coincide with taking intersections. There is a very deep relation between K-theory and intersection theory. See Section 4 for a modest treatment of this.

EXAMPLE 2.8. Suppose that X is projective over a field k and let $d = \dim X$. Let D be a nonsingular hyperplane section of X. Then $\operatorname{cl}(\mathcal{O}_D)^{d+1} = 0$ in $K_0(X)$. In fact, take d nonsingular hyperplane sections $D_1, \ldots, D_d$ such that

$$D \cap D_1 \cap \ldots \cap D_d = \emptyset,$$

and such that D_i is in general position with the subvariety $D_{i-1} \cap \ldots \cap D_1 \cap D$ $(i = 2, \ldots, d)$. Since all hyperplane sections are linearly equivalent, we have that the ideal sheaves of two hyperplane sections are isomorphic. In particular, we have that $\mathrm{cl}(\mathcal{O}_{D_i}) = \mathcal{O}_D$ for $i = 1, \ldots, d$. By the previous theorem, it holds that $\mathrm{cl}(\mathcal{O}_D)^{d+1} = \mathrm{cl}(\mathcal{O}_{D \cap D_1 \cap \ldots D_d}) = 0$.

Example 2.9. Let H be a hyperplane in $X = \mathbf{P}^n_k$ and let $h = \mathrm{cl}(\mathcal{O}_H)$ in $K_0(X) \cong \mathbf{Z}^{n+1}$. We claim that $(1, h, \ldots, h^n)$ forms a basis for the free abelian group $K_0(X) \cong \mathbf{Z}^{n+1}$. Writing $\xi = \mathrm{cl}(\mathcal{O}_X(1))$, we showed that $(1, \xi, \ldots, \xi^n)$ forms a $\mathbf{Z}$-basis for $K_0(X)$ in Example 1.57. Taking into account the ringstructure on $K_0(X)$, this implies that $(1, \xi^{-1}, \ldots, \xi^{-n})$ is also a $\mathbf{Z}$-basis for $K_0(X)$. The short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

implies that $\xi^{-1} = 1 - h$. Therefore, $h^i \neq 0$ for $0 \leq i \leq n$. Now, suppose that $\sum_{i=0}^n a_i h^i = 0$. Choose j minimal with $a_j \neq 0$. By the previous Example, we have that $0 = h^{n-j} \cdot \sum_{i=0}^n a_i h^i = a_j h^j$. Contradiction. We conclude that $K_0(\mathbf{P}^n_k) \cong \mathbf{Z}[h]/(h^{n+1})$ as a ring.

2. The Chow ring

Let A be a one-dimensional integral domain with fraction field K. For $f \in K^*$, $f = \frac{a}{b}$ with $a, b \in R$, we put

$$\mathrm{ord}_A(f) := \mathrm{length}_A(A/aA) - \mathrm{length}_A(A/bA)$$

and call it the *order of* f. For any $a, b \in A \backslash \{0\}$, we have a short exact sequence

$$0 \longrightarrow A/aA \longrightarrow A/abA \longrightarrow A/bA \longrightarrow 0.$$

Since length_A is additive on short exact sequences, we have that $\mathrm{ord}_A : K^* \longrightarrow \mathbf{Z}$ is a homomorphism.

Example 2.10. For a one-dimensional local noetherian regular ring A, hence a discrete valuation ring, the order of $f \in A$ coincides with the normalized valuation of f.

Let X be an algebraic scheme, i.e., a scheme of finite type over a field k. A prime divisor on X is a closed integral subscheme of codimension 1.

Definition 2.11. Suppose that X is integral. For any prime divisor W with generic point w, we define $\mathrm{ord}_W := \mathrm{ord}_{\mathcal{O}_{X,w}}$ to be the *order along* W.

Let us show that, for any $f \in K(X)^*$, there are only a finite number of prime divisors W with $\mathrm{ord}_W(f) \neq 0$. To prove this, we may assume that $X = \mathrm{Spec}\, A$ is affine and that f is regular on X, i.e., $\mathrm{ord}_W(f) \geq 0$ for all prime divisors W on X. In particular, $\mathrm{ord}_W(f) \neq 0$ if and only if W is contained in $V(f)$. (Here $V(f)$ is the closed subscheme defined by the ideal fA.) But since $f \neq 0$, we have that $V(f)$ is a proper closed subset. Since X is noetherian, we have that $V(f)$ contains only finitely many closed irreducible subsets of codimension 1 of U.

Definition 2.12. For any subvariety V of codimension r in X, and any $f \in K(V)^*$, define a cycle $[\mathrm{div}(f)]$ of codimension $r + 1$ (with support in V) on X by

$$[\mathrm{div}(f)] = \sum_W \mathrm{ord}_W(f)[W] \in Z^{r+1}(X),$$

where the sum is over all prime divisors W of V. A cycle $\alpha \in Z^r(X)$ of codimension r is said to be *rationally equivalent to zero* in $Z^r(X)$, denoted by $\alpha \sim 0$, if there are a finite number of subvarieties Y_i of codimension $r-1$ in X, and $f_i \in K(Y_i)^*$, such that

$$\alpha = \sum [\mathrm{div}(f_i)]$$

in $Z^r(X)$. Since $[\mathrm{div}(f^{-1})] = -[\mathrm{div}(f)]$, the cycles rationally equivalent to zero form a subgroup $\mathrm{Rat}^r(X)$ of $Z^r(X)$. The *group of cycles of codimension r modulo rational equivalence* on X is the factor group

$$A^r(X) = Z^r(X)/\mathrm{Rat}^r(X).$$

We call $A^{\cdot}(X) = \bigoplus A^r(X)$ the *Chow group.*

Just as for $K_0(X)$, we have push-forward and pull-back maps for the Chow group.

Let $f : X \longrightarrow Y$ be a proper morphism of varieties. Let V be a closed subvariety of X with image $W = f(V)$. (Recall that a subvariety is a closed subscheme which is a variety.) If $\dim W < \dim V$, we set $f_*(V) = 0$. If $\dim V = \dim W$, the function field $K(V)$ is a finite extension field of $K(W)$, and we set

$$f_*(V) = [K(V) : K(W)]W.$$

Extending by linearity defines a homomorphism f_* of $Z(X)$ to $Z(Y)$. These homomorphisms are functorial, as follows from the multiplicativity of degrees of field extension.

Let $f : X \longrightarrow Y$ be a flat morphism of relative dimension s. For a closed subscheme V of Y, set

$$f^*[V] = [f^{-1}(V)].$$

Here $f^{-1}(V)$ is the inverse image scheme (a closed subscheme of pure dimension $\dim V + s$) and $[f^{-1}(V)]$ is its cycle. Extending by linearity defines a homomorphism f^* of $Z(X)$ to $Z(Y)$ of degree $-n$. These homomorphisms are clearly functorial. We can also define the pull-back $f^*[V]$ when f is not flat. (Here we use that X is smooth.) See [**Ser**]. These definitions coincide for flat morphisms.

THEOREM 2.13. Let $f : X \longrightarrow Y$ be a morphism of smooth quasi-projective varieties.

(1) Let α be a cycle on X rationally equivalent to zero. Suppose that f is proper. Then the cycle $f_*\alpha$ is rationally equivalent to zero.
(2) Let α be a cycle of codimension r on Y which is rationally equivalent to zero in $Z^r(Y)$. Then the cycle $f^*\alpha$ is rationally equivalent to zero in $Z^{r-s}(X)$.
(3) Let α be a cycle on X and let β be a cycle on Y. Suppose that f is proper. Then $f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta$ whenever both sides are defined.

PROOF. See [**Ful1**, Chapter 1.3, Theorem 1.4], [**Ful1**, Chapter 1.7, Theorem 1.7] and [**Ser**], respectively. □

The following Theorem will allow us to define the product cycle for any two cycles (not necessarily meeting properly). Its proof can be found in [**Chev**].

THEOREM 2.14. **(Chow's moving lemma)** Let V and W be subvarieties of a smooth quasi-projective variety X. Then the cycle $[V]$ is rationally equivalent to a cycle α which meets $[W]$ properly. □

Before we make our definition of the Chow ring we need the following Lemma.

LEMMA 2.15. Given elements α and β in the Chow ring, let α' and β' be cycles representing them which meet properly (these exist by the Moving Lemma). Then the class in $A^{\cdot}(X)$ of $\alpha' \cdot \beta'$ is independent of the choice of representatives $\alpha', \beta' \in Z^{\cdot}(X)$ and depends only on α and β.

PROOF. We sketch the proof. Firstly, using the more classical description of rational equivalence give in [**Ful1**, Proposition 1.6], it suffices to show the following statement. Let $V \subset X \times_k \mathbf{P}^1$ be a $(s+1)$-dimensional subvariety dominating $\mathbf{P}^1$. Fix closed points $a, b \in \mathbf{P}^1$ and let W_a (resp. W_b) be the fibre of $W \longrightarrow \mathbf{P}^1$ above a (resp. b). Now, let V be an r-dimensional subvariety of X such that V intersects both W_a and W_b properly. Then $[V] \cdot [W_a]$ is rationally equivalent to $[V] \cdot [W_b]$.

Let $p : X \times \mathbf{P}^1 \longrightarrow X$ be the projection. Note that $[W_a] = p_*([W] \cdot [X \times \{a\}])$. Similarly, we have that $[W_b] = p_*([W] \cdot [X \times \{b\}])$. Thus, we reduce to showing that

$$[V] \cdot p_*([W] \cdot [X \times \{a\}]) \sim_{rat} [V] \cdot p_*([W] \cdot [X \times \{b\}]).$$

The projection formula implies that

$$[V] \cdot p_*([W] \cdot [X \times \{a\}]) = p_*([V \times \mathbf{P}^1] \cdot ([W] \cdot [X \times \{a\}])),$$

and similar for b. Thus we reduce to showing that

$$p_*([V \times \mathbf{P}^1] \cdot ([W] \cdot [X \times \{a\}])) \sim_{rat} p_*([V \times \mathbf{P}^1] \cdot ([W] \cdot [X \times \{b\}])).$$

Now, we may apply the associativity for the product cycle to conclude that $[V \times \mathbf{P}^1] \cdot ([W] \cdot [X \times \{a\}]) = ([V \times \mathbf{P}^1] \cdot [W]) \cdot [X \times \{a\}]$, and similar for b. Thus we reduce to showing

$$p_*(([V \times \mathbf{P}^1] \cdot [W]) \cdot [X \times \{a\}]) \sim_{rat} p_*(([V \times \mathbf{P}^1] \cdot [W]) \cdot [X \times \{b\}])$$

which is true by [**Ful1**, Proposition 1.6]. □

The previous Theorem tells us that we can define a product on the Chow group by using Chow's Moving Lemma. This product is commutative and associative with unit element $[X]$.

THEOREM 2.16. For every smooth quasi-projective variety X, there is a unique contravariant graded ring structure on $A^{\cdot}(X)$ such that:

(1) It agrees with pull-back of cycles contravariantly.
(2) For any proper morphism $f : X \longrightarrow Y$, we have that $f_* : A^{\cdot}(X) \longrightarrow A^{\cdot}(Y)$ is homomorphism. Also, if $g : Y \longrightarrow Z$ is another proper morphism, then $g_* \circ f_* = (g \circ f)_*$.
(3) If $f : X \longrightarrow Y$ is a proper morphism, and $\alpha \in A^{\cdot}(X)$, $\beta \in A^{\cdot}(Y)$, then we have the projection formula $f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$.
(4) If α and β are cycles on X, and if $\Delta : X \longrightarrow X \times X$ is the diagonal morphism, then we have the exterior product formula $\alpha \cdot \beta = \Delta^*(\alpha \times \beta)$.
(5) For subvarieties V and W of X which intersect properly, we have that the product of $[V]$ and $[W]$ coincides with the product cycle $[V] \cdot [W]$.
(6) It agrees with the product $A^1(X) \times A^r(X) \longrightarrow A^{r+1}(X)$ induced by intersection with Cartier divisors. □

3. Chern classes in the Chow ring

Let X be a smooth quasi-projective variety over a field k. Let $\mathrm{Pic}(X)$ be the group of invertible sheaves on X and let $\mathrm{Cl}(X) = A^1(X)$ be the divisor class group. Every divisor D on X determines up to isomorphism an invertible sheaf $\mathcal{O}_X(D)$ (denoted by $\mathcal{L}(D)$ in Hartshorne) and every invertible sheaf is of this type. This induces an isomorphism $\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X)$. See [**Har**, Chapter II, Proposition 6.16].

Definition 2.17. For any $\mathcal{L} \in \mathrm{Pic}(X)$, we define the *first Chern class* of $\mathcal{L}$ in $\mathrm{Cl}(X)$ by $c_1(\mathcal{L}) = [D]$, where $[D] \in \mathrm{Cl}(X)$ is such that $\mathcal{O}_X(D) = \mathcal{L}$ in $\mathrm{Pic}(X)$. Clearly, the homomorphism $c_1 : \mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$ is inverse to the homomorphism $\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X)$ described above.

Let $\mathcal{E}$ be a vector bundle of rank r and let $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow X$ be the associated projective bundle. Let $\mathcal{O}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ be the canonical invertible sheaf on $\mathbf{P}(\mathcal{E})$. Note that the pull-back $\pi^* : A(X) \longrightarrow A(\mathbf{P}(\mathcal{E}))$ makes $A(\mathbf{P}(\mathcal{E}))$ into an $A(X)$-module.

Example 2.18. Suppose that $\mathcal{E} = \mathcal{O}_X^r$. Then $\mathbf{P}(\mathcal{E}) = \mathbf{P}_X^r = \mathbf{P}_k^r \times_k X$. We have already seen that $K_0(\mathbf{P}(\mathcal{E})) = K_0(X) \otimes_{\mathbf{Z}} K_0(\mathbf{P}_k^r)$ is a free $K_0(X)$-module. If ξ is the class of $\mathcal{O}(1)$ in $K_0(X)$, we have seen that $K_0(\mathbf{P}_k^r)$ is a free abelian group with basis $(1, \xi, \ldots, \xi^r)$. In particular, $K_0(\mathbf{P}(\mathcal{E}))$ is a free $K_0(X)$-module with the same basis.

Theorem 2.19. The Chow ring $A^{\cdot}(\mathbf{P}(\mathcal{E}))$ is a free $A^{\cdot}(X)$-module with basis $(1, \xi, \ldots, \xi^r)$, where $\xi = c_1(\mathcal{O}(1)) \in A^1(\mathbf{P}(\mathcal{E})) \subset A^{\cdot}(\mathbf{P}(\mathcal{E}))$.

Proof. We sketch the proof. We have to show that the map $\oplus_{i=0}^{r-1} A^{\cdot}(X) \longrightarrow A^{\cdot}(\mathbf{P}(\mathcal{E}))$ sending $(x_0, \ldots, x_{r-1})$ to $\sum_{i=0}^{r-1} \pi^*(x_i)\xi^i$ is an isomorphism. Firstly, note that we have a "localization sequence" for the Chow ring ([**Ful1**, Chapter 1, Section 5]). Therefore, by induction on $\dim X$, we may assume X is affine and $\mathcal{E}$ is trivial. Then we have a projective subbundle $i : \mathbf{P}_1 \longrightarrow \mathbf{P}(\mathcal{E})$ of $\mathbf{P}(\mathcal{E})$ of rank $r-1$ such that $\mathbf{P}(\mathcal{E}) \backslash \mathbf{P}_1$ is an affine bundle over X. Then we use a (strong) "homotopy property" for the Chow ring as in [**Ful1**, Chapter 1, Section 8] and the fact that $i_*(A(\mathbf{P}_1)) = A(\mathbf{P}(\mathcal{E}) \cdot \xi$ to conclude that $\oplus_{i=0}^{r-1} A^{\cdot}(X) \longrightarrow A^{\cdot}(\mathbf{P}(\mathcal{E})$ is surjective. Now, the injectivity follows from the fact that $\pi(\pi^*(x) \cdot \xi^r) = x$ and $\pi_*(\pi^*(x) \cdot \xi^i) = 0$ for $0 \leq i < n$. To prove this, we may assume that $x = [X]$ and X is a point. The assertion then follows from the fact that $\xi^n \cdot [\mathbf{P}(\mathcal{E})]$ is represented by any section of $\mathbf{P}(\mathcal{E})$ over X. □

Definition 2.20. There exist unique elements $a_i \in A^i(X)$ $(0 \leq i \leq r)$ such that

$$\xi^r - \pi^*(a_1) \cdot \xi^{r-1} + \pi^*(a_2) \cdot \xi^{r-2} - \ldots + (-1)^r \pi^*(a_r) = 0.$$

We define the *i-th Chern class* of $\mathcal{E}$, denoted by $c_i(\mathcal{E}) \in A(X)$, as $c_i(\mathcal{E}) = a_i$ for $1 \leq i \leq r$. We put $c_0(\mathcal{E}) = 1$. Note that $c_i(\mathcal{E}) = 0$ for $i > r$.

Example 2.21. Let $\mathcal{E} = \mathcal{O}_X(D)$ be an invertible sheaf where D is in $\mathrm{Cl}(X)$. Then $\mathbf{P}(\mathcal{E}) = X$, $\mathcal{O}(1) = \mathcal{O}_X(D)$ and π is the identity map. Therefore, we have that $\xi - \pi^*(a_1) = 0$ showing that $c_1(\mathcal{E}) = [D]$ as one would expect.

Definition 2.22. We define the *Chern polynomial* of $\mathcal{E}$, denoted by $c_t(\mathcal{E})$, as the element

$$c_t(\mathcal{E}) = 1 + c_1(\mathcal{E})t + \ldots + c_r(\mathcal{E})t^r$$

in the ring $A(X)[t]$.

THEOREM 2.23. There is a unique theory of Chern classes for X, which assigns to each vector bundle $\mathcal{E}$ on X an i-th Chern class $c_i(\mathcal{E}) \in A^i(X)$ and satisfies the following properties:

C0: It holds that $c_0(\mathcal{E}) = 1$.
C1: For an invertible sheaf $\mathcal{O}_X(D)$, we have that $c_1(\mathcal{O}_X(D)) = [D]$.
C2: For a morphism of smooth quasi-projective varieties $f : X \longrightarrow Y$ and any positive integer i, we have that $f^*(c_i(\mathcal{E})) = c_i(f^*(\mathcal{E}))$.
C3: If
$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$
is an exact sequence of vector bundles on X, then
$$c_t(\mathcal{E}) = c_t(\mathcal{E}')c_t(\mathcal{E}'')$$
in $A(X)[t]$.
C4: We have that $c_i(\mathcal{E}) = 0$ for $i > \text{rk}\,\mathcal{E}$.
C5: The mapping $\mathcal{E} \mapsto c_t(\mathcal{E})$ can be extended to a homomorphism $c_t : K_0(X) \longrightarrow 1 + \oplus_{i=1}^{\infty} A^i(X) \cdot t^i$.

PROOF. This is a theorem of Grothendieck in [**Groth**, Théorème 1, pp. 144]. Grothendieck shows that the Chern classes constructed above satisfy these properties. (In our case, C2 and C3 are the only nontrivial properties to check.) One of the main ingredients in proving this theorem is the splitting principle stated below. □

EXAMPLE 2.24. Let $f : X \longrightarrow \operatorname{Spec} k$ be the structural morphism and let $\mathcal{E} = \mathcal{O}_X^r$ be free of rank r. By C2, for any $i \geq 1$, we have that $c_i(\mathcal{O}_X^r) = f^* c_i(k^r) = 0$.

The proof of Theorem 2.23 uses in an essential way the so-called Splitting principle for the Chow ring and $\mathcal{E}$.

THEOREM 2.25. **(Splitting Principle)** Fix a vector bundle $\mathcal{E}$ on X of rank r. There exists a smooth quasi-projective variety X' and a morphism $\pi : X' \longrightarrow X$ such that $\pi^* : A(X) \longrightarrow A(X')$ is injective, and $\mathcal{E}' = \pi^*\mathcal{E}$ splits. (That is, it has a filtration $\mathcal{E}' = \mathcal{E}'_0 \supset \mathcal{E}'_1 \supset \ldots \supset \mathcal{E}'_r = 0$ such that $\mathcal{L}_i = \mathcal{E}_{i-1}/\mathcal{E}_i$ is an invertible sheaf for $1 \leq i \leq r$.)

PROOF. There is nothing to prove when $r = 1$. Thus, suppose that $r > 1$. Let $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow X$ be the natural projection. By Theorem 2.19, the ringmorphism π^* is injective. Now, one can check locally that we have a natural surjective morphism $\pi^*\mathcal{E} \longrightarrow \mathcal{O}(1)$. Its kernel $\mathcal{E}'$ is a vector bundle of rank $r - 1$. Now, we repeat this construction with the vector bundle $\mathcal{E}'$ on $\mathbf{P}(\mathcal{E})$ until we reach a line bundle. □

Let $\pi : X' \longrightarrow X$ be as in the Splitting Principle. Then
$$\pi^* c_t(\mathcal{E}) \;=\; c_t(\pi^*\mathcal{E}) = c_t(\mathcal{L}_1)\ldots c_t(\mathcal{L}_r) = \prod_{i=1}^{r}(1 + c_1(\mathcal{L}_i)t) = \prod_{i=1}^{r}(1 + \alpha_i t).$$
Here $\alpha_i = c_1(\mathcal{L}_i)$. Since π^* is injective, this actually determines $c_t(\mathcal{E})$. Thus, let us write
$$c_t(\mathcal{E}) = \prod_{i=1}^{r}(1 + \alpha_i t),$$
where $\alpha_1, \ldots, \alpha_r \in A(X)$. Then $\alpha_1, \ldots, \alpha_r$ are called *Chern roots* of E and this factorization is regarded as purely formal. Note that $c_i(\mathcal{E})$ is the i-th elementary symmetric polynomial in

$\alpha_1, \ldots, \alpha_r$. In particular, $c_1(\mathcal{E}) = \alpha_1 + \ldots + \alpha_r$ and $c_r(\mathcal{E}) = \alpha_1 \ldots \alpha_r$. Also, any symmetric polynomial in Chern roots determines a well-defined polynomial in Chern classes. (By a well-known theorem on symmetric polynomials, any symmetric poynomial can be expressed uniquely as a polynomial in the elementary symmetric polynomials.)

EXAMPLES 2.26. Let $\mathcal{F}$ be another vector bundle of rank s on X and let us write $c_t(\mathcal{F}) = \prod_{i=1}^{s}(1+\beta_i t)$. We apply the splitting principle to determine the Chern polynomial for dual bundles, tensor products and exterior powers.

(1) Let us show that $c_t(\mathcal{E}^\vee) = c_{-t}(\mathcal{E})$, where $\mathcal{E}^\vee$ is the dual of $\mathcal{E}$. Firstly, note that $c_t(\mathcal{L}^\vee) = c_{-t}(\mathcal{L})$ for any invertible sheaf $\mathcal{L}$. Also, if $\mathcal{E}$ has a filtration with quotients $\mathcal{L}_i$, then $\mathcal{E}^\vee$ has a filtration with quotients $\mathcal{L}_{r-i}$.
(2) It holds that
$$c_t(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \prod_{i,j}(1 + (\alpha_i + \beta_j)t).$$
This follows from the fact that $c_1(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$ for any invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ combined with the flatness of $\mathcal{E}$ (or $\mathcal{F}$).
(3) We have that
$$c_t(\Lambda^p \mathcal{E}) = \prod_{1 \le i_1 \le \ldots \le i_p \le r} (1 + (\alpha_{i_1} + \ldots + \alpha_{i_p})t).$$
In fact, note that any exact sequence of vector bundles
$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0$$
with $\mathcal{L}$ a line bundle, induces a short exact sequence
$$0 \longrightarrow \Lambda^{p-1}\mathcal{E}' \otimes \mathcal{L} \longrightarrow \Lambda^p \mathcal{E} \longrightarrow \Lambda^p \mathcal{E}' \longrightarrow 0.$$
Note that this also shows that $c_t(\det \mathcal{E}) = 1 + c_1(\mathcal{E})t$. That is, $c_1(\mathcal{E}) = c_1(\det \mathcal{E})$.

Note that, for any Chern root α of $\mathcal{E}$, we have a well-defined element $\exp(\alpha) = 1 + \alpha + \frac{1}{2}\alpha^2 + \ldots$ in $A(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

DEFINITION 2.27. We define the *(exponential) Chern character* of $\mathcal{E}$ as
$$\mathrm{ch}(\mathcal{E}) = \sum_{i=1}^{r} \exp(\alpha_i).$$

Since $\mathrm{ch}(\mathcal{E})$ is a symmetric polynomial in the Chern roots, we have that $\mathrm{ch}(\mathcal{E})$ is a well-defined element of $A(X)_{\mathbf{Q}} := A(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. If $c_i = c_i(\mathcal{E})$, we can show that
$$\mathrm{ch}(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - c_2) + \frac{1}{3!}(c_1^3 - 3c_1c_2 + 3c_3) + \frac{1}{4!}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \text{h.o.t..}$$
The i^{th} term is $\frac{p_i}{i!}$, where p_i is determined inductively by Newton's formula
$$p_i - c_1p_{i-1} + c_2p_{i-2} - \ldots + (-1)^{i-1}c_{i-1}p_1 + (-1)^i i c_i = 0.$$
The following Proposition follows essentially from Theorem 2.23 and the splitting principle.

PROPOSITION 2.28. There is a homomorphism $\mathrm{ch} : K_0(X) \longrightarrow A(X)_{\mathbf{Q}}$ which is uniquely determined by the following properties.

(1) For any morphism $f : Y \longrightarrow X$ of smooth quasi-projective varieties, it holds that
$$f^* \circ \mathrm{ch} = \mathrm{ch} \circ f^*.$$
(2) For any invertible sheaf $\mathcal{L}$ on X with $c_1(\mathcal{L}) = [D]$, it holds that
$$\mathrm{ch}(\mathrm{cl}(\mathcal{L})) = \mathrm{ch}(\mathcal{L}) = \sum_{i\geq 0} \frac{1}{i!}[D]^i.$$
(3) The homomorphism ch is multiplicative, i.e., a ringmorphism. □

We will now consider the power series
$$f(x) = \frac{x}{1-\exp(-x)} \in \mathbf{Q}[[x]].$$
One can show that
$$f(x) = \sum_{j=0}^{\infty} \frac{B_j}{j!}x^j = 1 + \frac{1}{2}x + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}x^{2i},$$
where B_j is the j-th Bernoulli number.

Definition 2.29. The *Todd class* of $\mathcal{E}$, denoted by $\mathrm{td}(\mathcal{E})$, is defined as
$$\mathrm{td}(\mathcal{E}) = \prod_{i=1}^{r} \left(\frac{\alpha_i}{1-\exp(-\alpha_i)}\right) \in A^{\cdot}(X)_{\mathbf{Q}}.$$

Remark 2.30. Note that the Todd class of $\mathcal{E}$ is invertible in $A^{\cdot}(X)_{\mathbf{Q}}$.

Remark 2.31. Let $c_i = c_i(\mathcal{E})$. Then the first few terms of the Todd class of $\mathcal{E}$ are
$$\mathrm{td}(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \dots.$$

Remark 2.32. It holds that
$$\begin{aligned}\sum_{p=0}^{r}(-1)^p \,\mathrm{ch}(\Lambda^p\mathcal{E}^\vee) &= \sum_{p=0}^{r}(-1)^p \sum_{i_1<\ldots<i_p} \exp(-\alpha_{i-1} - \ldots - \alpha_{i_p}) = \prod_{i=1}^{r}(1-\exp(-\alpha_i)) \\ &= \alpha_1 \ldots \alpha_r \prod_{i=1}^{r}\left(\frac{1-\exp(-\alpha_i)}{\alpha_i}\right) = c_r(\mathcal{E})\,\mathrm{td}(\mathcal{E})^{-1}.\end{aligned}$$

Just as with the Chern character, we have a Whitney sum formula. That is, if
$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$
is an exact sequence of vector bundles on X, we have that $\mathrm{td}(\mathcal{E}) = \mathrm{td}(\mathcal{E}')\,\mathrm{td}(\mathcal{E}'')$. In particular, the Todd class induces a homomorphism $\mathrm{td} : K_0(X) \longrightarrow A(X)^*$, where $A(X)^*$ is the multiplicative group of units in the Chow ring.

Definition 2.33. For any smooth quasi-projective variety X, we define the *Todd class of* X, denoted by $\mathrm{td}(X)$, as the Todd class of the tangent sheaf $\mathcal{T}_X$ on X. For a morphism $f : X \longrightarrow Y$, we define the *relative Todd class of* f, denoted by $\mathrm{td}(X/Y)$, as $\mathrm{td}(X/Y) = \mathrm{td}(X) \cdot (\pi^* \,\mathrm{td}(Y))^{-1}$.

4. Notes on the topological filtration

We already mentioned the similarity between the "product" on $Z^{\cdot}(X)$ and $K_0(X)$ in Remark 2.7, where $K_0(X)$ denotes the Grothendieck group of coherent sheaves on X.

Let X be a noetherian scheme. We define the *(topological) graded Grothendieck group* by $\mathrm{Gr}^{\cdot}\, K_0(X) = \bigoplus_{r=0}^{\dim X} F^r X/F^{r+1}X$. Here $F^r X$ denotes the subgroup of $K_0(X)$ generated by the classes of sheaves $\mathcal{F}$ with $\operatorname{codim} \operatorname{Supp} \mathcal{F} \geq r$. By Proposition 1.30, for any $r \in \mathbf{Z}$, we have a surjective homomorphism $Z^r(X) \longrightarrow F^r X/F^{r+1}X$. One can now prove the following theorem which is much easier in dimension 1. See Section 1 of Chapter 3.

THEOREM 2.34. Let X be a smooth quasi-projective variety over an algebraically closed field and let $\gamma : Z(X) \longrightarrow K_0(X)$ be the surjective homomorphism studied in Chapter 1. We will write γ_X for γ if we want to emphasize that this is the homomorphism associated to X.

(1) Suppose that $\alpha \in Z^r(X)$ and $\beta \in Z^s(X)$ meet properly. Then $\gamma(\alpha) \cdot \gamma(\beta) \equiv \gamma(\alpha \cdot \beta) \mod F^{r+s+1}X$.
(2) For any morphism $f : Y \longrightarrow X$ of smooth quasi-projective varieties and any cycle $\alpha \in Z^r(Y)$, we have that $\gamma_X(f^*(\alpha)) \equiv f^*(\gamma_Y(\alpha)) \mod F^{r+1}X$.
(3) Suppose that α and β are cycles on X of codimension r which are rationally equivalent. Then $\gamma(\alpha) \equiv \gamma(\beta) \mod F^{r+1}X$.
(4) Let $V \subset X$ and $W \subset X$ be subvarieties of codimension r and s, respectively. If V and W intersect properly, we have that $\mathrm{cl}(\mathcal{O}_V) \cdot \mathrm{cl}(\mathcal{O}_W) \equiv \mathrm{cl}(\mathcal{O}_{V \cap W}) \mod F^{r+s+1}X$.
(5) For any $r, s \in \mathbf{Z}$, we have that $F^r X \cdot F^s X \subset F^{r+s}X$.
(6) The surjective ringmorphisms $\gamma_X : A^{\cdot}(X) \longrightarrow \mathrm{Gr}^{\cdot}(X)$ define a natural transformation of contravariant functors from the category of smooth quasi-projective varieties to the category of graded rings.

PROOF. We sketch the proof given in [**Mur**]. Firstly, let $V \subset X$ and $W \subset X$ be subvarieties of codimension r and s, respectively. For any $i > 0$, we have that

$$\operatorname{codim} \operatorname{Supp} \operatorname{Tor}_i(\mathcal{O}_V, \mathcal{O}_W) > r + s.$$

This clearly implies (1). To prove (2), one may assume f is a closed immersion by considering the graph of f. Now, by pulling-back, the natural projection $X \times \mathbf{A}^1 \longrightarrow X$ induces an isomorphism of Chow rings $A^{\cdot}(X) \longrightarrow A^{\cdot}(X \times \mathbf{A}^1)$. Combining this with (2), one can show (3). The remaining properties (4), (5) and (6) can be deduced from (1), (2) and (3). □

Later we shall prove, using the Grothendieck-Riemann-Roch theorem, the following theorem.

THEOREM 2.35. The natural transformation $\gamma : A^{\cdot} \longrightarrow \mathrm{Gr}^{\cdot}\, K_0$ becomes a natural isomorphism of contravariant functors after we tensor with $\mathbf{Q}$. (Here we restrict to the category of smooth quasi-projective varieties over an algebraically closed field.) □

Thus, the graded Grothendieck group $\mathrm{Gr}^{\cdot}\, K_0(X)$ may serve as a replacement of the Chow ring if one desires to generalize intersection theory and characteristic classes. (Since we use Chow's Moving Lemma in our definition of the product on the Chow ring, our construction only works for smooth quasi-projective varieties over an algebraically closed field.) Unfortunately, the topological filtration has two disadvantages:

(1) It carries over to $K^0(X)$ only when $K_0(X) = K^0(X)$.

(2) It has been conjectured to be compatible with the ringstructure on $K_0(X)$, but an affirmative answer is only known modulo torsion. (See [**Gil**].) Note that Theorem 2.34 confirms the conjecture for smooth quasi-projective varieties.

We can overcome these difficulties by following Grothendieck and introducing a different filtration on $K^0(X)$. In the following, all schemes will be connected noetherian separated finite-dimensional schemes.

For any scheme X, there are maps $\lambda^i : K^0(X) \longrightarrow K^0(X)$, defined by taking exterior powers: $\lambda^i(\mathrm{cl}(\mathcal{E})) = \mathrm{cl}(\Lambda^i\mathcal{E})$. These are not group homomorphisms, but rather $\lambda^i(x+y) = \sum_{r=0}^{i} \lambda^r(x)\lambda^{i-r}(y)$ for any $x, y \in K^0(X)$. (For a study of this λ-ring structure see [**FuLa**].)

DEFINITION 2.36. The *γ-operations* are defined by:

$$\gamma^i : K^0(X) \longrightarrow K^0(X), \quad x \mapsto \lambda^i(x + (i-1)\mathrm{cl}(\mathcal{O}_X)).$$

Note that $\gamma^0(x) = \lambda^0(x-1) = 1$ and $\gamma^1(x) = \lambda^1(x) = x$, for any $x \in K^0(X)$. Also, we have that $\gamma^i(x+y) = \sum_{r=0}^{i} \gamma^r(x)\gamma^{i-r}(y)$. (In particular, the γ-operations define another λ-ring structure on $K_0(X)$.) Let $\mathrm{rk} : K^0(X) \longrightarrow \mathbf{Z}$ be the rank. We already said that the kernel of this homomorphism is the starting point of the γ-filtration in Remark 1.33.

DEFINITION 2.37. For $i \in \mathbf{Z}_{\geq 0}$, we define abelian groups $F^i_\gamma X$, as opposed to the groups F^iX introduced in the previous Section, as follows. Let $F^0_\gamma X = K^0(X)$ and $F^1_\gamma X = \ker \mathrm{rk}$. Then, we require that if $x \in F^1_\gamma X$, we have $\gamma^i(x) \in F^i_\gamma X$ and that this be multiplicative. That is, we require that

$$F^i_\gamma X \cdot F^j_\gamma X \subset F^{i+j}_\gamma X.$$

The groups $F^i_\gamma X$ define a multiplicative filtration on $K^0(X)$ which we call the *γ-filtration*. The associated graded group $\mathrm{Gr}^\cdot_\gamma K^0(X) = \bigoplus_i F^i_\gamma X / F^{i+1}_\gamma X$ is a graded ring. As opposed to the topological filtration, which we showed to be multiplicative for smooth quasi-projective varieties, we have manually built in the graded ringstructure in $\mathrm{Gr}^\cdot_\gamma K^0(X)$. Unfortunately, it is not clear whether $F^i_\gamma X = 0$ for $i > \dim X$. See [**Man**, Paragraph 9] for a proof of this when X is regular.

Let $\mathcal{E}$ be a vector bundle of rank r on X.

DEFINITION 2.38. For $i \geq 1$, we define the *i-th (generalized) Chern class* of $\mathcal{E}$ by

$$c_i(\mathcal{E}) = \gamma^i(\mathrm{cl}(\mathcal{E}) - r\mathrm{cl}(\mathcal{O}_X)) \mod F^i_\gamma K^0(X) \in \mathrm{Gr}^i_\gamma K^0(X).$$

Thus, for any invertible sheaf $\mathcal{L}$ on X, we have that $c_1(\mathcal{L}) = \mathrm{cl}(\mathcal{L}) - \mathrm{cl}(\mathcal{O}_X) \mod F^2K^0(X)$ and $c_i(\mathcal{L}) = 0$ as long as $i > 2$.

Now, one can define a generalized Chern polynomial as in Section 3 and formulate an analogue of Theorem 2.23. Having defined a generalized Chern character, we can prove that it induces an isomorphism $K_0(X)_{\mathbf{Q}} \longrightarrow \mathrm{Gr}^\cdot_\gamma K_0(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. The proof of the former is less-involved than the proof of its classical analogue (Theorem 2.35) which uses the Grothendieck-Riemann-Roch theorem. In fact one constructs an inverse with the aid of the so-called Adams operations. See [**Man**, Section 11].

CHAPTER 3

The Grothendieck-Riemann-Roch theorem

1. Riemann-Roch for smooth projective curves

A curve is an integral separated 1-dimensional scheme of finite type over a field.

Let X be a smooth projective curve over an algebraically closed field k. Note that $A^{\cdot}(X) = \mathbf{Z} \oplus \mathrm{Cl}(X)$, where $\mathrm{Cl}(X)$ is the divisor class group of X. The group structure on $\mathbf{Z} \oplus \mathrm{Cl}(X)$ is given by $(n, D) + (m, E) = (n + m, D + E)$ whereas the multiplication is given by $(n, D)(m, E) = (nm, mD + nE)$.

We define some natural homomorphisms from $K^0(X)$ to $\mathbf{Z}$ and $\mathrm{Pic}(X)$. The isomorphism $K_0(X) \cong K^0(X)$ will allow us to define these for $K_0(X)$.

There is a unique injective ringmorphism $i : \mathbf{Z} \longrightarrow K^0(X)$ with right inverse the ring morphism $\mathrm{rk} : K^0(X) \longrightarrow \mathbf{Z}$ which assigns to each vector bundle on X its rank.

We have a homomorphism $\det : K^0(X) \longrightarrow \mathrm{Pic}(X)$ which we call the *determinant*. To a vector bundle $\mathcal{E}$ we associate the class of the element $\Lambda^r \mathcal{E}$, where $r = \mathrm{rk}(\mathcal{E})$. As one can easily show, this induces a map from $K^0(X)$ to $\mathrm{Pic}(X)$. (In fact, this works for any noetherian scheme X.)

The map $j : \mathrm{Pic}(X) \longrightarrow K^0(X)$ defined by $[\mathcal{L}] \mapsto \mathrm{cl}(\mathcal{L}) - \mathrm{cl}(\mathcal{O}_X)$ is a homomorphism. To prove this, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be line bundles on X. Then, it suffices to show that

$$\mathrm{cl}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \mathrm{cl}(\mathcal{L}_1) \cdot \mathrm{cl}(\mathcal{L}_2) = \mathrm{cl}(\mathcal{L}_1) + \mathrm{cl}(\mathcal{L}_2) - \mathrm{cl}(\mathcal{O}_X).$$

We proceed in two steps. Firstly, suppose that $\mathcal{L}_1 = \mathcal{O}_X(-D)$ and that $\mathcal{L}_2 = \mathcal{O}_X(-E)$, where D and E are effective divisors on X. Then

$$\begin{aligned}
\mathrm{cl}(\mathcal{O}_X(-D)) \cdot \mathrm{cl}(\mathcal{O}_X(-E)) &= (\mathrm{cl}(\mathcal{O}_X) - \mathrm{cl}(\mathcal{O}_D)) \cdot (\mathrm{cl}(\mathcal{O}_X) - \mathrm{cl}(\mathcal{O}_E)) \\
&= \mathrm{cl}(\mathcal{O}_X) - \mathrm{cl}(\mathcal{O}_D) - \mathrm{cl}(\mathcal{O}_E) + \mathrm{cl}(\mathcal{O}_D)\mathrm{cl}(\mathcal{O}_E) \\
&= \mathrm{cl}(\mathcal{O}_X) - \mathrm{cl}(\mathcal{O}_D) - \mathrm{cl}(\mathcal{O}_E) \\
&= \mathrm{cl}(\mathcal{O}_X(-D)) + \mathrm{cl}(\mathcal{O}_X(-E)) - \mathrm{cl}(\mathcal{O}_X).
\end{aligned}$$

Here we used that $\mathrm{cl}(\mathcal{O}_D) \cdot \mathrm{cl}(\mathcal{O}_E) = 0$ (Example 2.8). Note that, for any effective divisor D on X, we have that

$$j(\mathcal{O}_X(D)) = -j(\mathcal{O}_X(-D)).$$

In fact, note that $\mathrm{cl}(\mathcal{O}_X(D)) \cdot \mathrm{cl}(\mathcal{O}_X(-D)) = 1$ in $K^0(X)$. Therefore, we have that

$$\begin{aligned}
j(\mathcal{O}_X(D)) &= \mathrm{cl}(\mathcal{O}_X(D)) - \mathrm{cl}(\mathcal{O}_X) = (\mathrm{cl}(\mathcal{O}_X(-D)))^{-1} - \mathrm{cl}(\mathcal{O}_X) \\
&= (\mathrm{cl}(\mathcal{O}_X) - \mathrm{cl}(\mathcal{O}_D))^{-1} - \mathrm{cl}(\mathcal{O}_X) = \mathrm{cl}(\mathcal{O}_X) + \mathrm{cl}(\mathcal{O}_D) - \mathrm{cl}(\mathcal{O}_X) \\
&= \mathrm{cl}(\mathcal{O}_D) - \mathrm{cl}(\mathcal{O}_X) + \mathrm{cl}(\mathcal{O}_X) = -(\mathrm{cl}(\mathcal{O}_X) - \mathrm{cl}(\mathcal{O}_D) - \mathrm{cl}(\mathcal{O}_X)) \\
&= -(\mathrm{cl}(\mathcal{O}_X(-D)) - \mathrm{cl}(\mathcal{O}_X)) = -j(\mathcal{O}_X(-D)).
\end{aligned}$$

This shows that, for $\mathcal{L}_1 = \mathcal{O}_X(D)$ and $\mathcal{L}_2 = \mathcal{O}_X(E)$, we have that

$$\mathrm{cl}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \mathrm{cl}(\mathcal{L}_1) \cdot \mathrm{cl}(\mathcal{L}_2) = \mathrm{cl}(\mathcal{L}_1) + \mathrm{cl}(\mathcal{L}_2) - \mathrm{cl}(\mathcal{O}_X).$$

In fact, we may write D as a difference $D_1 - D_2$ of effective divisors and similarly for E. We conclude that j is indeed a homomorphism. (We will see that there is a natural isomorphism $K_0(X) \cong \mathrm{Gr}_\gamma^{\cdot} K_0(X)$ and that j becomes the first generalized Chern class as in Definition 2.38 under this isomorphism.) Note that $\det \circ j = \mathrm{id}_{\mathrm{Pic}(X)}$.

Let us put a graded ringstructure on the abelian group $\mathbf{Z} \oplus \mathrm{Pic}(X)$. We define $(n, \mathcal{L}_1)(m, \mathcal{L}_2) = (nm, n\mathcal{L}_2 + m\mathcal{L}_1)$ for $(n, \mathcal{L}_1), (m, \mathcal{L}_2) \in \mathbf{Z} \oplus \mathrm{Pic}(X)$. (Since $F_\gamma^2 X = 0$, this is just $\mathrm{Gr}_\gamma^{\cdot} K_0(X)$.) Since the first Chern class $c_1 : \mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$, the homomorphism $\mathbf{Z} \oplus \mathrm{Pic}(X) \longrightarrow A^{\cdot}(X)$ given by $(n, \mathcal{L}) \mapsto (n, c_1(\mathcal{L}))$ is an isomorphism of rings.

Proposition 3.1. We have an isomorphism of rings $K_0(X) \longrightarrow \mathbf{Z} \oplus \mathrm{Pic}(X)$ given by $\alpha \mapsto (\mathrm{rk}(\alpha), \det \alpha)$.

Proof. By the above, this is a ringmorphism. Now, we have a complex of abelian groups

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{j} K_0(X) \xrightarrow{\mathrm{rk}} \mathbf{Z} \longrightarrow 0 .$$

Since this complex is left split by det, it suffices to show that this complex is exact. By Theorem 1.22, we have that $\alpha = \mathrm{rk}(\alpha) + \sum_{P \in X \text{ closed}} n_P \cdot \mathrm{cl}(\mathcal{O}_P)$ for any $\alpha \in K_0(X)$. By the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0$$

associated to a point $P \in X$, it holds that $\mathrm{cl}(\mathcal{O}_P) = 1 - \mathrm{cl}(\mathcal{O}_X(-P)) = j(\mathcal{O}_X(-P)^\vee) = j(\mathcal{O}_X(P))$. Thus, it holds that

$$\alpha - \mathrm{rk}(\alpha) = j\left(\bigotimes_{P \in X \text{ closed}} \mathcal{O}_X(P)^{\otimes n_P} \right).$$

This shows that $\ker \mathrm{rk} = \mathrm{im}\, j$, i.e., the above complex is exact. □

The inverse of the above isomorphism is given by $(n, [\mathcal{L}]) \mapsto n \cdot \mathrm{cl}(\mathcal{L})$. Furthermore, the Chern character $\mathrm{ch} : K_0(X) \longrightarrow A(X)$ given by

$$\mathrm{ch}(\alpha) = (\mathrm{rk}(\alpha), c_1(\det(\alpha)) = (\mathrm{rk}(\alpha), c_1(\alpha))$$

is an isomorphism of rings.

Let g be the genus of X. Recall that for every $D \in \mathrm{Cl}(X)$, we have that $\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g$. In fact, since $\chi(X, \mathcal{O}_X) = 1 - g$, it suffices to show that the formula holds for D if and only if it holds for $D + P$, where $P \in \mathrm{Cl}(X)$ is (the class of) some point in X. Now, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0 .$$

Tensoring this with $\mathcal{O}_X(D+P)$ gives us the above formula by the additivity of χ. In particular, combining this formula with Serre duality gives us that $\deg K_X = 2g - 2$.

The Todd class of X is given by $\mathrm{td}(X) = (1, -\frac{1}{2} c_1(\omega_X)) = (1, -\frac{1}{2} K_X)$ in $A(X)_{\mathbf{Q}}$. The *degree* on the Chow ring is the function $\deg : A(X) \longrightarrow \mathbf{Z}$ given by $\deg(n, D) = \deg(D)$, where $\deg D$ denotes the degree of a divisor. (More generally, if $f : X \longrightarrow \mathrm{Spec}\, k$ is the structural morphism, the degree $\deg_X = \deg$ on $A^\cdot(X)$ is f_* composed with the natural isomorphism $A(\operatorname{Spec} k) = \mathbf{Z}$.)

THEOREM 3.2. **(Riemann-Roch)** The following diagram of groups

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}_X} & A(X)_{\mathbf{Q}} \\ \downarrow{\scriptstyle \chi(X,-)} & & \downarrow{\scriptstyle \deg} \\ \mathbf{Z} & \longrightarrow & \mathbf{Q} \end{array}$$

is commutative.

PROOF. It suffices to show that, for any vector bundle $\mathcal{E}$ of rank r, it holds that

$$\chi(X,\mathcal{E}) = \deg(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X)) = \deg(c_1(\mathcal{E})) - \frac{1}{2} r \deg(c_1(\omega_X)).$$

Since $\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g$ for all $D \in \mathrm{Cl}(X)$, the statement holds for (the class of) a line bundle $\mathcal{L}$ in $K_0(X)$. To prove the Riemann-Roch theorem it suffices to do so for the class of a vector bundle $\mathcal{E}$ of rank $r > 1$. Suppose that we have a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0,$$

where $\mathcal{L}$ is a line bundle and $\mathcal{E}'$ is a vector bundle of rank $r-1$. Then, by the additivity of the Euler characteristic and the Riemann-Roch theorem for line bundles, the theorem follows by induction on r. Let us show that we always have such a short exact sequence. Choose $n \gg 0$ such that $\mathcal{E}(n)$ is generated by its global section. Then, since $\dim X = 1$ and $\operatorname{rk} \mathcal{E}(n) > 1$, there is a global section s which is nowhere zero ([**Har**, Exercise II.8.2]). Note that $\mathcal{E}(n)/s\mathcal{O}_X$ is a vector bundle of rank $r-1$ and that we have a short exact sequence

$$0 \longrightarrow s\mathcal{O}_X \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{E}(n)/s\mathcal{O}_X \longrightarrow 0.$$

Now tensor with $\mathcal{O}_X(-n)$ to get the desired short exact sequence. □

REMARK 3.3. As is clear from the definition of the Todd class, we may replace $A^\cdot(X)_{\mathbf{Q}}$ by $A^\cdot(X) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{2}]$ in the above theorem.

We turn to an interesting application of the above result.

A surface is an integral separated 2-dimensional scheme of finite type over a field. Let X be a smooth projective surface over k, where k is algebraically closed. Suppose that there is a flat morphism $f : X \longrightarrow C$, where C is a smooth projective curve over k, such that each fibre is a smooth projective curve. We say that X is a fibered surface over C. Recall that $\mathrm{td}(X/C)$ is defined as $\mathrm{td}(X) \cdot (\forall^* \mathrm{td}(C))^{-1}$ (Definition 2.33).

COROLLARY 3.4. The following diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(X)} & A(X)_{\mathbf{Q}} \\ \downarrow{\scriptstyle \chi(X,-)} & & \downarrow{\scriptstyle \deg} \\ \mathbf{Z} & \longrightarrow & \mathbf{Q} \end{array}$$

is commutative if and only if, for any $\alpha \in K_0(X)$, we have that

$$\deg_C c_1(f_!\alpha) = \deg_X(\mathrm{ch}(\alpha)\,\mathrm{td}(X/C)).$$

PROOF. Firstly, the diagram being commutative amounts to saying that, for any vector bundle $\mathcal{E}$ on X, we have that

$$\chi(X, \mathcal{E}) = \deg_X(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X)).$$

Note that both sides are additive on short exact sequences. Thus, it suffices to show that, for any vector bundle $\mathcal{E}$ on X, we have that $\deg_C c_1(f_!\mathcal{E}) = \deg_X(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X/C))$ if and only if $\chi(X, \mathcal{E}) = \deg_X(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X))$. By the functoriality of K-theoretic push-forward and the above Riemann-Roch theorem for C, we have that

$$\chi(X, \mathcal{E}) \;=\; \chi(C, f_!\mathcal{E}) = \mathrm{rk}(f_!\mathcal{E})\deg(\mathrm{td}(C)) + \deg c_1(f_!\mathcal{E}).$$

If η is the generic point of X and X_η is the generic fibre of f, we have that $\mathrm{rk}\, f_!\mathcal{E} = \chi(X_\eta, \mathcal{E}_\eta)$. Invoking the Riemann-Roch theorem for the smooth projective curve X_η, the Corollary follows from

$$\mathrm{rk}\, f_!\mathcal{E} = \chi(X_\eta, \mathcal{E}_\eta) = r\deg_{X_\eta}(\mathrm{td}(X_\eta)) + \deg_{X_\eta} c_1(\mathcal{E}_\eta). \qquad \square$$

REMARK 3.5. Combining the classical Riemann-Roch theorem for surfaces with Noether's formula, one can show that the above diagram commutes ([**Har**, Example A.4.1.2]). The equality $\deg_C c_1(f_!\alpha) = \deg_X(\mathrm{ch}(\alpha)\,\mathrm{td}(X/C))$ is an expression for the degree of the determinant of cohomology $\det f_!\alpha$. By the functoriality of push-forward in intersection theory, this equality can also be written as

$$\deg_C c_1(f_!\alpha) = \deg_C(f_*(\mathrm{ch}(\alpha)\,\mathrm{td}(X/C)).$$

We can now make three remarks.

(1) One is tempted to conjecture that the stronger equality of cycle classes

$$c_1(f_!\alpha) = f_*(\mathrm{ch}(\alpha)\,\mathrm{td}(X/C))_{(1)}$$

holds in $A^{\cdot}(C)_{\mathbf{Q}}$. This equality holds and follows from the Grothendieck-Riemann-Roch theorem for f given in the next section.

(2) As it appears, the determinant of cohomology plays an important role in Riemann-Roch theorems. In fact, in the above situation, we can define a height over C for covers of X with a fixed branch locus. For a detailed discussion of this see Section 6.

(3) The proof of the above Corollary shows that one can state a similar result for higher-dimensional varieties fibered over a curve, as long as the "Hirzebruch-Riemann-Roch theorem" on the fibres is known. More precisely, let $f : X \longrightarrow C$ be a flat morphism of smooth projective varieties. Suppose that $\dim C = 1$. Then the following diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(X)} & A(X)_{\mathbf{Q}} \\ {\scriptstyle \chi(X,-)}\downarrow & & \downarrow{\scriptstyle \deg} \\ \mathbf{Z} & \longrightarrow & \mathbf{Q} \end{array}$$

is commutative if and only if, for any $\alpha \in K_0(X)$, we have that

$$\deg_C c_1(f_!\alpha) = \deg_X(\mathrm{ch}(\alpha)\,\mathrm{td}(X/C)).$$

2. The Grothendieck-Riemann-Roch theorem and some standard examples

Let $f : X \longrightarrow Y$ be a proper morphism of smooth quasi-projective varieties over a field k. We now come to the main theorem of this thesis.

THEOREM 3.6. **(Grothendieck-Riemann-Roch)** The following diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(X)} & A(X)_{\mathbf{Q}} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(Y)} & A(Y)_{\mathbf{Q}} \end{array}$$

is commutative. □

PROOF. We sketch the classical proof ([**BorSer**]). Firstly, since f is the composition of a closed immersion $i : X \longrightarrow \mathbf{P}^n_Y$ and the projection $\pi : \mathbf{P}^n_Y \longrightarrow Y$, we may assume that f is a closed immersion of a smooth subvariety into a smooth projective variety or that f is the projection $\pi : \mathbf{P}^n_Y \longrightarrow Y$. The case of a projection $\pi : \mathbf{P}^n_Y \longrightarrow Y$ is easy. Firstly, assume $Y = \operatorname{Spec} k$ is a point. Then a direct computation shows that the above diagram commutes. In general, we have a diagram

$$\begin{array}{ccc} K_0(\mathbf{P}^n_k) \otimes K_0(Y) & \xrightarrow{(\mathrm{ch}\cdot\mathrm{td}(\mathbf{P}^n_k))\otimes(\mathrm{ch}\cdot\mathrm{td}(Y))} & A^\cdot(\mathbf{P}^n_k)_{\mathbf{Q}} \otimes A^\cdot(Y)_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ K_0(\mathbf{P}^n_Y) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(\mathbf{P}^n_Y)} & A(\mathbf{P}^n_Y)_{\mathbf{Q}} \\ {\scriptstyle \pi_!}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ K_0(Y) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(Y)} & A(Y)_{\mathbf{Q}} \end{array},$$

where the big rectangle is commutative and the upper square is commutative. One simply applies Proposition 1.41 to conclude that the lower square is also commutative. Now, one has reduced to proving the Grothendieck-Riemann-Roch when f is a closed immersion $i : Y \longrightarrow X$ of a smooth subvariety Y into a projective variety X. This is done in four steps. Write $p = \operatorname{codim}(Y, X)$. The first step ([**BorSer**, Proposition 14]) is formal and settles the theorem for $p = 1$ and any element in the image of $i^! : K_0(Y) \longrightarrow K_0(X)$. The second step ([**BorSer**, Corollaire 1]) settles the theorem for $X = \mathbf{P}^n_k$ and Y a closed point. The reasoning is by induction on n and uses that $K_0(Y) \cong \mathbf{Z}$. The main geometric idea is that, given a hyperplane H in $\mathbf{P}^n_k$, one can find another hyperplane Z in $\mathbf{P}^n_k$ and a line D on H, such that D is in general position with $Z \cap H$ on H and $Y = D \cap Z \cap H$. The computation can then be compared to Example 2.8. The third step ([**BorSer**, Corollaire 2]) consists of reducing the problem to $p \geq \dim Y + 2$. The strategy is to consider the composition

$$Y \xrightarrow{i} X \xrightarrow{j} X \times_k \mathbf{P}^n_k \ ,$$

where $j : a \mapsto (a, t_0)$ and $t_0 \in \mathbf{P}^n_k$ is a fixed closed point. Essentially, by choosing n big enough, one can apply the previous steps and the Künneth formula ([**BorSer**, Lemme 16]) to conclude. We come to the fourth and final step. Let $f : \widetilde{X} \longrightarrow X$ be the blow-up[1] of X

[1] See [**Har**, Proposition II.7.16, pp. 166] and [**Har**, Theorem II.8.24, pp. 186].

along Y and let $\mathcal{J}$ be the ideal sheaf defining Y. Let $Y' \subset \widetilde{X}$ be the subscheme defined by the inverse image ideal sheaf $f^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$ and let $j : Y' \longrightarrow \widetilde{X}$ be the natural closed immersion. By construction, we have that $f \circ j = i \circ g$. Now, note that $\widetilde{X}$ and Y' are smooth quasi-projective varieties and that $\operatorname{codim}(\widetilde{Y}, \widetilde{X}) = 1$. The normal sheaf $\mathcal{N}_{Y'/\widetilde{X}}$ is an invertible sheaf on Y' which we shall denote by $\mathcal{L}$. Furthermore, let $\mathcal{F}$ be the quotient sheaf $g^*\mathcal{N}_{Y/X}/\mathcal{N}_{Y'/\widetilde{X}}$. Note that $\mathcal{F}$ is a vector bundle of rank $p-1$. Let $\lambda_{-1}\mathcal{F}^\vee$ denote the element $\sum_{r=0}^{p-1}(-1)^r \operatorname{cl}(\Lambda^r \mathcal{F}^\vee)$ in $K_0(Y')$. In Remark 2.32 we explained where this comes from. Now, the theorem follows from a formal computation ([**BorSer**, Proposition 15]) once we can show the following facts ([**BorSer**, Lemme 19]):

(1) $f_* f^*$ is the identity,
(2) $g_*(c_{p-1}(\mathcal{F})) = 1$,
(3) $f^! i_!(y) = j_!(g^!(y) \cdot \lambda_{-1}\mathcal{F}^\vee)$ for any $y \in K_0(Y)$,
(4) $\lambda_{-1}\mathcal{F}^\vee$ is in the image of $j^!$. □

REMARK 3.7. See [**Ful1**, Theorem 15.2] for a more modern approach to the Grothendieck-Riemann-Roch theorem. The closed immersion $i : Y \longrightarrow X$ is handled differently by a technique called the deformation to the normal bundle.

COROLLARY 3.8. **(Hirzebruch-Riemann-Roch)** For any vector bundle $\mathcal{E}$ on a smooth projective n-dimensional variety X, we have that

$$\chi(X, E) = \deg(\operatorname{ch}(E)\operatorname{td}(X))_{(n)}.$$

In particular, $\chi(X, \mathcal{O}_X) = \deg(\operatorname{td}(X))_{(n)}$. □

EXAMPLE 3.9. Suppose that f is an isomorphism. It is not immediately clear that GRR holds for f, i.e., that the above diagram commutes for f. In fact, by the projection formula, it holds that $f_! = (f^!)^{-1}$. Thus, by the commutativity of $f^!$ with ch, we can conclude that GRR holds for f.

EXAMPLE 3.10. For a closed immersion $i : Y \longrightarrow X$ of a smooth subvariety Y into a smooth projective variety X and a vector bundle $\mathcal{E}$ on Y, we have that

$$\operatorname{ch} i_*\mathcal{E} = i_*(\operatorname{ch}\mathcal{E} \cdot \operatorname{td}(X/Y)^{-1}). \tag{4}$$

Here $\operatorname{td}(X/Y)$ is the Todd class of the normal sheaf $\mathcal{N}_{Y/X}$. It coincides with the inverse of $\operatorname{td}(Y/X)$ (Definition 2.33) by the short exact sequence

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \otimes \mathcal{O}_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0.$$

EXAMPLE 3.11. Suppose that $\dim X = 1$ and $Y = \operatorname{Spec} k$. Then the Grothendieck-Riemann-Roch theorem is just Theorem 3.2.

EXAMPLE 3.12. Suppose that $\dim X = 2$ and $Y = \operatorname{Spec} k$. Then $\operatorname{td}(X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2)$, where we put $c_i = c_i(\mathcal{T}_X)$. In particular, $\chi(X, \mathcal{O}_X) = \frac{1}{12}\deg(c_1^2 + c_2)_2$. If $K = -c_1(\mathcal{T}_X)$ is the class of a canonical divisor, we let $K \cdot K$ denote $\deg(K \cdot K)_2$. If $\chi = \deg(c_2)_2$ is the topological Euler characteristic, the above reads

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(K \cdot K + \chi).$$

By Corollary 3.8, for a vector bundle $\mathcal{E}$ of rank r with $d_i = c_i(\mathcal{E})$, we have that

$$\chi(X, E) = \frac{1}{2}\deg(d_1^2 + 2d_2 + c_1 d_1) + r\chi(X, \mathcal{O}_X).$$

In particular, if D is a divisor on X,

$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{2}(D^2 - K \cdot D) + \chi(X, \mathcal{O}_X).$$

If D is effective, we have that

$$\chi(X, \mathcal{O}_D) = -\frac{1}{2}(D^2 - K \cdot D).$$

Now, if D is an irreducible curve on X, and $p_a(D) = \dim H^1(D, \mathcal{O}_D)$ is its arithmetic genus, then

$$p_a(D) = \frac{1}{2}(D^2 + K \cdot D) + 1.$$

This provides an easy way to compute the arithmetic genus for a curve lying on a surface.

EXAMPLE 3.13. Suppose that $\dim X = 3$ and $Y = \operatorname{Spec} k$. Then

$$\operatorname{td}(X) = 1 + \frac{1}{2}c_2 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2,$$

with $c_i = c_i(\mathcal{T}_X)$. For a vector bundle $\mathcal{E}$ of rank r on X, with Chern classes $d_i = c_i(\mathcal{E})$, it holds that

$$\chi(X, \mathcal{E}) = \deg(\frac{1}{6}(d_1^3 - 3d_1d_2 + 3d_3) + \frac{1}{4}(c_1d_1^2 - 2d_2) + \frac{1}{12}(c_1^2 + c_2)d_1 + \frac{r}{24}c_1c_2)_3.$$

In particular, for a divisor D on X, we have that

$$\chi(X, \mathcal{O}_X(D)) = \deg(\frac{1}{6}D^3 + \frac{1}{4}c_1 \cdot D^2 + \frac{1}{12}(c_1^2 + c_2) \cdot D + \frac{1}{24}c_1c_2)_3.$$

Let us apply this to $X = \mathbf{P}_k^3$. Then $A(X) = \mathbf{Z}[h]/h^4\mathbf{Z}$, where h is the class of a hyperplane. If $\mathcal{E}$ is a vector bundle on X, we write $c_i(\mathcal{E}) = n_ih^i$ where $n_i \in \mathbf{Z}$. Apply the above formula to see that

$$\frac{1}{6}(n_1^3 - 3n_1n_2 + 3n_3 + 11n_1) \in \mathbf{Z}.$$

That is, $n_1^3 - 3n_1n_2 + 3n_3 + 11n_1$ is divisible by 6. For example, if $r = 2$, we have that n_1n_2 must be even.

Let X be a smooth projective variety. In Section 4 we constructed a functorial surjective ringmorphism $A^{\cdot}(X) \longrightarrow \operatorname{Gr}^{\cdot}(X)$ and said that it induces an isomorphism once we tensor with $\mathbf{Q}$. We will now prove this theorem. Firstly, the following example illustrates the role played by the Grothendieck-Riemann-Roch theorem.

EXAMPLE 3.14. Let $P \in X$ be a closed point and let $f : Y \longrightarrow X$ be the inclusion, where $Y = \{P\}$. Note that Y is smooth. Clearly, $\operatorname{td}(Y) = 1 \in A(Y)$. By the Grothendieck-Riemann-Roch theorem, if $[Y] \in A^0(Y)$ denotes the class of Y, it holds that

$$\operatorname{ch}(f_*\mathcal{O}_Y) = f_*([Y] \cdot f^* \operatorname{td}(X)^{-1}).$$

Since $A^j(X) = 0$ for $j > \dim X$ and $f^* \operatorname{td}(X)^{-1} = 1 +$ higher order terms, we conclude that

$$\operatorname{ch}(f_*\mathcal{O}_Y) = f_*[Y].$$

This can be generalized as follows. Let $f : Y \longrightarrow X$ be a closed immersion of a smooth subvariety and let $i = \operatorname{codim}(Y, X)$. Let us show that

$$\operatorname{ch}(f_*\mathcal{O}_Y) = f_*([Y] + \alpha),$$

where $[Y] \in A^0(Y)$ denotes the class of the subvariety Y and α is some element in $\bigoplus_{j>i} A^j(X)_{\mathbf{Q}}$. To prove this, note that $[Y] \in A^i X$ and that $\mathrm{td}(X/Y)$ is of the form $1+$ higher order terms. The same reasoning as above then applies. More precisely, we have that

$$\alpha = [Y]\cdot \mathrm{td}(X/Y)_1 + [Y]\cdot \mathrm{td}(X/Y)_2 + \ldots + [Y]\cdot \mathrm{td}(X/Y)_{\dim Y} \in \bigoplus_{j>i} A^j(Y)_{\mathbf{Q}}.$$

For $i \in \mathbf{Z}$, we define $\varphi^i : A^i(X) \longrightarrow \mathrm{Gr}^i(X)$ to be the homomorphism given by $\varphi([Z]) = \mathrm{cl}(Z)$ mod $F^{i+1}X$ whenever $Z \subset X$ is a closed subscheme of codimension i.

Lemma 3.15. For any closed immersion $f : Y \longrightarrow X$ of codimension i, we have that

$$\mathrm{ch}(\mathrm{cl}(\mathcal{O}_Y)) = f_*([Y] + \alpha),$$

where $\alpha \in \bigoplus_{j>0} A^j(Y)_{\mathbf{Q}}$. In particular, for any $i \in \mathbf{Z}$, we have that $\mathrm{ch}(F^i X) \subset \bigoplus_{j\geq i} A^j(X)_{\mathbf{Q}}$.

Proof. We proceed in two steps.

Step 1: Suppose that Y is smooth. Then this is just Example 3.14.

Step 2: Let $U \subset X$ be an open subset such that $Y \cap U$ is regular and $s := \dim X - U < \dim Y \cap U = \dim Y = d - i$. Put $S = X - U$ and note that $\mathrm{codim}(S, X) = d - s$. For any $j \geq 0$, we have a short exact sequence

$$A^{j-d+s}(S) \longrightarrow A^j(X) \longrightarrow A^j(U) \longrightarrow 0\ .$$

(For the exactness see [**Ful1**, Chapter 1]. We also used this fact in Theorem 2.19.) In the interesting case $j = i$, we see that $A^{i-d+s}(S) = 0$. Therefore, the homomorphism $A^i(X) \longrightarrow A^i(U)$ is an isomorphism. Recall that this isomorphism is induced by the intersection of cycles with U. Since we are only considered with cycles of codimension less or equal to i, we may assume that Y is smooth. Thus, we reduce to Step 1.

The last statement follows from the fact that the morphism φ^i is surjective (Proposition 1.30) and the fact that f_* increases the degree by i. □

Theorem 3.16. The Chern character $\mathrm{ch} : K_0(X) \longrightarrow A^{\cdot}(X)_{\mathbf{Q}}$ induces an isomorphism of rings $\mathrm{ch}_{\mathbf{Q}} : K_0(X)_{\mathbf{Q}} \longrightarrow A^{\cdot}(X)_{\mathbf{Q}}$.

Proof. Lemma 3.15 implies that the Chern character factors through $\mathrm{Gr}^{\cdot} K_0(X)$, i.e., the map $\mathrm{Gr}^{\cdot}(\mathrm{ch}) : \mathrm{Gr}^{\cdot} K_0(X) \longrightarrow A^{\cdot}(X)_{\mathbf{Q}}$ is a well-defined ringmorphism. Since the composition

$$A^i(X) \xrightarrow{\ \varphi^i\ } \mathrm{Gr}^i K_0(X) \xrightarrow{\ \mathrm{Gr}^i(ch)\ } A^i(X)_{\mathbf{Q}}$$

is given by the natural inclusion $A^i(X) \longrightarrow A^i(X)_{\mathbf{Q}}$, we see that both φ^i and $\mathrm{Gr}^i(\mathrm{ch})$ induce bijections after tensoring with $\mathbf{Q}$. This implies that $\mathrm{ch}_{\mathbf{Q}}$ is injective. Since ch is surjective, we have that $\mathrm{ch}_{\mathbf{Q}}$ is surjective. We conclude that it is an isomorphism of rings. □

3. The Riemann-Hurwitz formula

Let $f : X \longrightarrow Y$ be a finite morphism of smooth projective curves over an algebraically closed field k. Note that $f_*\mathcal{O}_X$ is a coherent sheaf by the fact that f is finite. (It is quasi-coherent by the fact that f is affine.) Furthermore, it is locally free by the fact that X is integral and the local rings are discrete valuation rings.[2] Note that $f_*\mathcal{O}_X$ is also a sheaf of rings. Therefore, it is a coherent locally free sheaf of $\mathcal{O}_Y$-algebras. Now, it is easy to see that the rank of $f_*\mathcal{O}_X$, as a vector bundle on Y, is precisely $\deg f$. The Grothendieck-Riemann-Roch theorem for f and $\mathcal{O}_X$ states that

$$\mathrm{ch}(f_*\mathcal{O}_X) = f_*(\mathrm{td}(X/Y)).$$

Comparing terms in degree 0, we immediately get that $\mathrm{rk}\, f_*\mathcal{O}_X[Y] = c_0(f_*\mathcal{O}_X) = f_*(1) = \deg f[Y]$ in $A^{\cdot}(Y)$. That is, we get back that the rank of $f_*\mathcal{O}_X$ is just the degree of f. Outside of the branch locus, this is precisely the number of elements in any fibre of f. Now, what does the Grothendieck-Riemann-Roch theorem give in degree 1? Well, an easy calculation shows that the Grothendieck-Riemann-Roch theorem gives us

$$2c_1(f_*\mathcal{O}_X) = \deg f \cdot K_Y - f_*(K_X). \tag{5}$$

Here K_Y and K_X denote the canonical divisors on Y and X, respectively.

In this section we shall make some remarks on equality (5). We will always assume f to be separable. In particular, the set of ramification points of f is finite.

We show that equality (5) follows from the following well-known theorem ([**Har**, Chapter IV, Prop. 2.3]).

THEOREM 3.17. **(Riemann-Hurwitz)** Let R be the ramification divisor of f. We have that K_X is linearly equivalent to $f^*K_Y + R$ on X. □

In fact, by the projection formula, we have that f_*K_X is linearly equivalent to $\deg f K_Y + f_*R$ on Y. Thus, in order to prove the above statement, it suffices to show that f_*R is linearly equivalent to $-2c_1(f_*\mathcal{O}_X)$ on X. But this follows from a local computation as in [**Ser2**, Chapter 3.6, Proposition 13]. We conclude that the Grothendieck-Riemann-Roch theorem for f and $\mathcal{O}_X$ follows from the Riemann-Hurwitz theorem. We may also conclude that the Grothendieck-Riemann-Roch theorem implies the (less precise) Riemann-Hurwitz theorem ([**Har**, Chapter IV, Prop. 2.4]) which relates the degrees of K_X and K_Y via the degree of the ramification divisor R.

Let us give an example.

Fix an integer $n \geq 1$ and let $\pi : X \longrightarrow Y$ be the morphism given by $(x : y) \mapsto (x^n : y^n)$ from the projective line to itself. In particular, X and Y are isomorphic. Recall that we gave an explicit expression for $\pi_*\mathcal{O}(m)$ in Example 1.31. We now reprove the much weaker equality in K_0-theory to illustrate that we lose a lot of precise information.

Let P be a point in X and let $Q = \pi(P)$. For $m \in \mathbf{Z}$, we have that

$$\mathrm{cl}(\pi_*\mathcal{O}(m)) = n + (m - n + 1)\mathrm{cl}(\mathcal{O}_Q). \tag{6}$$

---

[2] Actually, for any finite flat morphism $f : X \longrightarrow Y$ of noetherian schemes, the coherent sheaf $f_*\mathcal{O}_X$ is locally free. Since a finite morphism of smooth projective curves is automatically flat, this also shows that $f_*\mathcal{O}_X$ is locally free.

In fact, note that $R^i\pi_* = 0$ for $i > 0$. Furthermore, we have that $\pi_!(1) = \mathrm{cl}(\pi_*\mathcal{O}_X) = n + a\mathrm{cl}(\mathcal{O}_Q)$ for some integer $a \in \mathbf{Z}$. This follows from the fact that $\pi_*\mathcal{O}_X$ is locally free of rank n and Example 1.5 (or Proposition 3.1). It is easy to determine a. Namely, we have that $\chi(Y, \pi_*\mathcal{O}_X) = \chi(X, \mathcal{O}_X) = 1$. This implies that $a + n = 1$. Now, by the short exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0$$

and the fact that $\mathrm{cl}(\mathcal{O}_P)^2 = 0$, we have that $\mathrm{cl}(\mathcal{O}(m)) = (1 - \mathrm{cl}(\mathcal{O}_P))^{-m} = 1 + m\mathrm{cl}(\mathcal{O}_P)$. Equation (6) follows from the fact that $\pi_*\mathcal{O}_P = \mathcal{O}_Q$.

Now, we could also easily verify the Grothendieck-Riemann-Roch theorem for π by going the other way around via intersection theory. Note that $\mathrm{cl}(\mathcal{O}(m)) = 1 + m\mathrm{cl}(\mathcal{O}_P)$. Therefore, we have that $\mathrm{ch}(\mathcal{O}(m)) = 1 + m[P]$. It is clear that $\mathrm{td}(Y) = 1 + [Q]$ and $\mathrm{td}(X) = 1 + [P]$. Since $\pi_*[P] = [Q]$, it holds that

$$\mathrm{td}(Y)^{-1}\pi_*(\mathrm{ch}(\mathcal{O}(m)) \cdot \mathrm{td}(X)) = (1 - [Q])(n + (m+1)[Q]) = n + (m - n + 1)[Q].$$

Here we used that $[P]^2 = 0$ and $[Q]^2 = 0$. This equals

$$\mathrm{ch}(\pi_!\mathcal{O}(m)) = n + (m - n + 1)[Q].$$

4. An application to Enriques surfaces

In this section we give an application of the Grothendieck-Riemann-Roch theorem to the study of "classical" Enriques surfaces. It is based on a computation made in [**Pap**]. We shall finish with a short remark on how the result of this Section is applied precisely to the study of the coarse moduli space of Enriques surfaces. A good reference for the study of these surfaces is [**BHPV**].

The base field will be an algebraically closed field k. As usual, a variety is an integral separated scheme of finite type over k.

Definition 3.18. A smooth projective 2-dimensional variety E is called an[3] *Enriques surface* if the following three conditions are satisfied:

(1) $H^1(E, \mathcal{O}_E) = 0$,
(2) $H^2(E, \mathcal{O}_E) = 0$,
(3) for any canonical divisor K on E, $2K$ is linearly equivalent to 0 but K itself it not linearly equivalent to 0.

Furthermore, a *family of Enriques surfaces* is a flat morphism $f : X \longrightarrow Y$ of smooth projective varieties such that each fibre is an Enriques surface.

Let $f : X \longrightarrow Y$ be a family of Enriques surfaces.

Lemma 3.19. The natural map $\mathcal{O}_Y \longrightarrow R^0f_*\mathcal{O}_X$ is an isomorphism.

Proof. Note that f is surjective. This follows from the fact that Y is connected and that f is an open and closed map. Therefore, since X and Y are integral schemes, the natural map $\mathcal{O}_Y \longrightarrow R^0f_*\mathcal{O}_X$ is injective. Now, since $R^0f_*\mathcal{O}_X$ is coherent on Y, we have that $R^0f_*\mathcal{O}_X$ is a finite $\mathcal{O}_Y$-algebra. Apply the Stein factorization theorem, together with the fact that the fibres of f are connected, to conclude that $\mathrm{Spec}\, R^0f_*\mathcal{O}_X = Y$. The statement follows. □

[3] For char $k = 2$ there are "other" Enriques surfaces. Our definition is of the so-called classical Enriques surfaces.

PROPOSITION 3.20. We have that $R^i f_* \mathcal{O}_X = 0$ if $i > 0$.

PROOF. Let $\mathcal{F} = \mathcal{O}_X$. For any $y \in Y$, let $\mathcal{F}_y$ be the coherent sheaf induced on the fibre X_y. It is the structure sheaf on X_y. By Grauert's Theorem ([**Har**, Corollary III.12.9]), for any i, the coherent sheaf $R^i f_* \mathcal{F}$ is a vector bundle on Y and, for every $y \in Y$, the natural map

$$R^i f_*(\mathcal{F}) \otimes k(y) \longrightarrow H^i(X_y, \mathcal{F}_y)$$

is an isomorphism. Since X_y is an Enriques surface, the result follows. □

Note that f is smooth of relative dimension 2. Equivalently, the sheaf of relative differentials $\Omega_{X/Y}$ is a vector bundle of rank 2 on X. In particular, the push-forward $f_* : A(X) \longrightarrow A(Y)$ lowers the degree by 2. Let $\mathcal{T}_f$ be the relative tangent sheaf, i.e., $\mathcal{T}_f$ is the kernel of the surjective morphism $\mathcal{T}_X \longrightarrow f^*\mathcal{T}_Y$. For any fibre of f, the relative tangent sheaf $\mathcal{T}_f$ restricts to the tangent sheaf of the fibre. That is, for any $y \in Y$ with $v : X_y \longrightarrow X$ the natural inclusion, we have that $v^*\mathcal{T}_f = \mathcal{T}_{X_y}$. For $i = 0, 1, 2, \ldots$, let $c_i = c_i(\mathcal{T}_f) \in A(X)_{\mathbf{Q}}$.

LEMMA 3.21. It holds that $f_*(c_2) = 12 \in A^0(Y)_{\mathbf{Q}}$.

PROOF. Since Chern classes commute with pull-back, we have that c_2 is given by $d_2 := c_2(\mathcal{T}_E)$ in the generic fibre $E = X_\eta$ of f. Since all fibres are Enriques surfaces, it holds that $c_2(\mathcal{T}_E) \in A^2(E)_{\mathbf{Q}}$ is of degree 12. In fact, by Example 3.12, it holds that

$$1 = \chi(E, \mathcal{O}_E) = \frac{1}{12}(K_E \cdot K_E + (\deg d_2)_2) = \frac{1}{12}(\deg d_2)_2.$$

Now, we know that $f_*(c_2)$ is a cycle in $A^0(Y) \cong \mathbf{Z}$. That is, $f_*(c_2)$ is given by an integer m. If D is a cycle in the class of c_2, this integer m is given by the intersection number $D \cdot E$. But this is precisely 12. □

Let $\mathcal{L} = \Lambda^2 \Omega_{X/Y} = \det \Omega_{X/Y} = \omega_{X/Y}$. Recall that a line bundle on a scheme X is said to be torsion if its class in $\mathrm{Pic}(X)$ is torsion.

THEOREM 3.22. The coherent sheaf $R^0 f_*(\mathcal{L} \otimes \mathcal{L})$ is a torsion line bundle on Y.

PROOF. Let us apply the Grothendieck-Riemann-Roch theorem to f and $\mathcal{O}_X$. By the above, this gives us that $\mathrm{td}(Y) = f_*(\mathrm{td}(X))$. By the projection formula, we have that $f_*(\mathrm{td}(\mathcal{T}_f)) = 1$. The degree 1 part of this equality reads

$$0 = f_*(\mathrm{td}(\mathcal{T}_f))_{(1)} = f_*(\mathrm{td}(\mathcal{T}_f)_{(3)}) = f_*(\frac{1}{24}c_1c_2) = \frac{1}{24}f_*(c_1 \cdot c_2).$$

We conclude that $f_*(c_1 \cdot c_2) = 0$. Now, we know that $c_1 = -c_1(\Omega_{X/Y}) = -c_1(\det \Omega_{X/Y}) = -c_1(\mathcal{L})$. Also, since the fibres are Enriques surfaces, the line bundle $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}$ is trivial along the fibres of f. This implies that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \cong f^*(\delta)$ for some line bundle δ on Y. By the projection formula and Proposition 3.20, we have that

$$R^0 f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}) = R^0 f_*(f^*(\delta)) = R^0 f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \delta = \delta.$$

This shows that $R^0 f_*(\mathcal{L} \otimes \mathcal{L})$ is a line bundle on Y. Also, it shows that $c_1 = -\frac{1}{2}c_1(\mathcal{L} \otimes \mathcal{L}) = -\frac{1}{2}f^*(c_1(\delta))$. Again, by the projection formula, we have that

$$0 = -2 \cdot f_*(c_1c_2) = f_*(f^*(c_1(\delta)) \cdot c_2) = c_1(\delta) \cdot f_*(c_2) = 12c_1(\delta).$$

We conclude that $c_1(\delta) = 0$ in $A^1(Y)_{\mathbf{Q}}$. Therefore, the line bundle $\delta = R^0 f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L})$ is torsion. $\square$

REMARK 3.23. Writing $\mathcal{K} = R^0 f_*(\mathcal{L} \otimes \mathcal{L})$, we have that $\mathcal{K}^{\otimes 24} \cong \mathcal{O}_Y$. This requires a more precise Grothendieck-Riemann-Roch theorem called the *Grothendieck-Riemann-Roch theorem without denominators*. See [**Pap2**] for a modern proof of the latter(in characteristic zero) based on Hironaka's resolution of singularities ([**Hiro**]) and the weak factorization theorem for birational maps ([**AKMW**]). One can also find a more general proof in [**Ful1**, Chapter 15.3] based on the deformation to the normal bundle.

REMARK 3.24. The above theorem is used in [**Pap**] to prove that the coarse moduli space of complex Enriques surfaces is quasi-affine. Basically, due to the above theorem, a certain line bundle on the coarse moduli space has a nowhere vanishing "invariant" global section after tensoring it with itself a sufficient amount of times. The same line bundle can be shown to be very ample. These facts together can be used to show that the coarse moduli space is quasi-affine.

5. An application to abelian varieties

Let X be an abelian variety over the field k and let $g = \dim X$. A good reference on abelian varieties is [**GeMo**].

LEMMA 3.25. Let $\mathcal{L}$ be a line bundle on X and let $D = c_1(\mathcal{L})$ be its first Chern class in $A^1(X)$. Then $\frac{D^g}{g!}$ is an integer. Furthermore, if $\mathcal{L}$ is ample, we have that $\frac{D^g}{g!}$ is positive.

PROOF. Apply the Hirzebruch-Riemann-Roch theorem to X and $\mathcal{L}$. Since the tangent bundle is trivial, the Todd class of X is trivial. Therefore,

$$\chi(X, \mathcal{L}) = \deg(\mathrm{ch}(\mathcal{L}))_{(g)}.$$

We conclude that $\frac{D^g}{g!} = \chi(X, \mathcal{L})$ is an integer. For the last statement, it suffices to show that D^g is positive for $\mathcal{L}$ very ample. But if $\mathcal{L}$ is very ample, we have that $D^g = \deg(f(X)) > 0$, where $f : X \longrightarrow \mathbf{P}^r$ is the embedding defined by $\mathcal{L}$ such that $f^*\mathcal{O}(1) = \mathcal{L}$. $\square$

Let $p : X \times X \longrightarrow X$ be the projection onto the first coordinate. Similarly, let $q : X \times X \longrightarrow X$ be the projection onto the second coordinate. For any line bundle $\mathcal{L}$ on X, we define its *Mumford line bundle* on $X \times X$, denoted by Λ, as

$$\Lambda := m^*\mathcal{L} \otimes (p^*\mathcal{L})^{-1} \otimes (q^*\mathcal{L})^{-1}.$$

The following theorem is a special case of [**Jah**, Theorem 1.7].

THEOREM 3.26. For any ample line bundle $\mathcal{L}$, we have that

$$(\det q_!(\Lambda \otimes p^*\mathcal{L}))^{-1} = \left(\det q_!(m^*\mathcal{L} \otimes (q^*\mathcal{L})^{-1})\right)^{-1}$$

is an ample line bundle on X.

PROOF. We apply the Grothendieck-Riemann-Roch theorem to the projection $q : X \times X \longrightarrow X$ and the element $\overline{\mathcal{L}} := m^*\mathcal{L} \otimes (q^*\mathcal{L})^{-1}$. Since the Todd class of an abelian variety is trivial, we get that

$$\mathrm{ch}(q_!\overline{\mathcal{L}}) = q_*(\mathrm{ch}(\overline{\mathcal{L}})).$$

Note that q_* lowers the degree by g, where $g = \dim X$. Comparing terms in degree 1, this gives

$$c_1(\det q_!\overline{\mathcal{L}}) = c_1(q_!\overline{\mathcal{L}}) = (q_*(\mathrm{ch}(\overline{\mathcal{L}}))_{(1)} = \left(q_*\left(\sum \frac{c_1(\overline{\mathcal{L}})^i}{i!}\right)\right)_{(1)} = \frac{1}{(g+1)!}q_*(c_1(\overline{\mathcal{L}})^{g+1}).$$

Substituting back $\overline{\mathcal{L}} = m^*\mathcal{L} \otimes (q^*\mathcal{L})^{-1}$, we get

$$c_1(\det q_!(m^*\mathcal{L} \otimes (q^*\mathcal{L})^{-1})^{-1}) = -c_1(\det q_!(m^*\mathcal{L} \otimes (q^*\mathcal{L})^{-1})) = -\frac{q_*\left(c_1(m^*\mathcal{L} \otimes (q^*\mathcal{L})^{-1})^{g+1}\right)}{(g+1)!}.$$

By properties of the first Chern class, the righthandside of the above equation is given by

$$-\frac{q_*\left(c_1(m^*\mathcal{L} \otimes (q^*\mathcal{L})^{-1})^{g+1}\right)}{(g+1)!} = -\frac{q_*\left((m^*c_1(\mathcal{L}) - q^*c_1(\mathcal{L}))^{g+1}\right)}{(g+1)!}.$$

To simplify the computation, we introduce the isomorphism of abelian varieties $s : X \times X \longrightarrow X \times X$ defined as $(x, y) \mapsto (x - y, y)$. We already noted that s_* is an isomorphism with inverse s^*. Now, it is easy to see that $q \circ s = q$ and that $ms = p$. This allows us to rewrite $q_*\left(m^*c_1(\mathcal{L}) - q^*c_1(\mathcal{L}))^{g+1}\right)$ as

$$\begin{aligned} q_*\left((m^*c_1(\mathcal{L}) - q^*c_1(\mathcal{L}))^{g+1}\right) &= q_*s_*s^*\left((m^*c_1(\mathcal{L}) - q^*c_1(\mathcal{L}))^{g+1}\right) \\ &= (qs)_*\left((ms)^*c_1(\mathcal{L}) - (qs)^*(c_1(\mathcal{L}))^{g+1}\right) \\ &= q_*\left((p^*c_1(\mathcal{L}) - q^*c_1(\mathcal{L}))^{g+1}\right). \end{aligned}$$

Writing out the sum $(p^*c_1(\mathcal{L}) - q^*c_1(\mathcal{L}))^{g+1}$ and applying the projection formula, we see that

$$\begin{aligned} q_*\left((p^*c_1(\mathcal{L}) - q^*c_1(\mathcal{L}))^{g+1}\right) &= q_*\sum_{i=0}^{g+1}(-1)^{g+1-i}\binom{g+1}{i}p^*(c_1(\mathcal{L})^i) \cdot q^*(c_1(\mathcal{L})^{g+1-i}) \\ &= \sum_{i=0}^{g+1}(-1)^{g+1-i}\binom{g+1}{i}q_*p^*(c_1(\mathcal{L})^i) \cdot (c_1(\mathcal{L})^{g+1-i}) \\ &= -gq_*\left(p^*(c_1(\mathcal{L}^g)) \cdot c_1(\mathcal{L}) + q_*(c_1(\mathcal{L})^{g+1}\right) \\ &= -gq_*p^*(c_1(\mathcal{L})^g) \cdot c_1(\mathcal{L}). \end{aligned}$$

Here we used that q_* is of degree $-g$ and the graded ring structure on the Chow ring to see that the only summand that could give a nonzero contribution under the image of q_* is the one with $i = d$. Now, let $D = c_1(\mathcal{L})$ and note that D is an ample divisor on X. Also, by Lemma 3.25, we have that $n = \frac{D^g}{g!}$ is a positive integer. We may conclude that

$$c_1(\det q_!(m^*\mathcal{L} \otimes (q^*\mathcal{L})^{-1})^{-1}) = \frac{g}{(g+1)!}(c_1(\mathcal{L})^g) = nD$$

is ample. □

6. Covers of varieties with fixed branch locus

This section is based on the article [**EdJoSc**], where one studies branched covers of surfaces. The goal is to illuminate on how the Grothendieck-Riemann-Roch theorem is applied and extend some of its results to higher-dimensional varieties.

The base field will be the field of complex numbers **C**.

We suppose given the following data.

- A separated smooth integral projective 1-dimensional **C**-scheme C;
- A separated smooth integral projective **C**-scheme X;
- A flat morphism $h : X \longrightarrow C$ of **C**-schemes;
- An effective simple normal crossings divisor D on X such that all components are smooth with multiplicity 1 and intersect transversally;

Let U be the complement of D and let $V \longrightarrow U$ be a finite étale morphism. For simplicity, we will assume V to be connected. By properties of étale morphisms, we have that V is a smooth variety. Let $\pi : Y \longrightarrow X$ be the normalization of X in the function field of V ([**Liu**, Definition 4.1.24]). The variety Y is normal and the morphism $\pi : Y \longrightarrow X$ is finite ([**Liu**, Proposition 4.1.25.]). In particular, $\dim Y = \dim X$. Since C is a Dedekind scheme, we have that Y is projective and flat over C. More precisely, since the local rings of C are discrete valuation rings, Y is flat over C. Since Y is finite over X, it is projective (and affine) over X. Since X is projective over C, we have that Y is also projective over C.

Remark 3.27. Since the topological fundamental group of U is finitely generated, there are only finitely many $V \longrightarrow U$ as above of given degree. See [**SGA7**, Exposé II, Théorème 2.3.1]. In particular, for a fixed degree, the "height over C of the associated covers $Y \longrightarrow X$" is bounded. The aim of [**EdJoSc**] is to give an effective version of this result. We aim at explaining their result and on how the Grothendieck-Riemann-Roch is of fundamental value in the study of this problem.

In order to analyse the singularities of Y we will need the following definition.

Definition 3.28. A point $y \in Y$ is called a *quotient singularity* if there exists a nonsingular variety W and a finite group Γ acting on W such that the quotient W/Γ exists as a variety and is isomorphic to a neighbourhood of y. One says that a quotient singularity is a cyclic quotient singularity if Γ can be taken to be a cyclic group. Note that nonsingular points are quotient singularities.

To understand the situation, we study the local model. The following examples describe the local behaviour of a covering $Y \longrightarrow X$ when X is a smooth projective surface, i.e., $\dim X = \dim Y = 2$. They illustrate some basic properties shared by all associated covers $Y \longrightarrow X$. These properties will be mentioned below.

Example 3.29. Consider the inclusion of **C**-algebras

$$A := \mathbf{C}[x^2, y^2, xy] \subset B := \mathbf{C}[x, y].$$

Note that A is the algebra of invariants of an action by the cyclic group G of order 2 on B, that we describe as follows. Let $G = \{e, a\}$, with e the identity element. We define the action of a on a monomial $x^i y^i$ by

$$a \cdot x^i y^j := (-1)^{i+j} x^i y^j.$$

It is easy to show that A is indeed the algebra of invariants B^G. Since the extension of rings $\mathbf{C}[x^2, y^2] \subset A$ is integral, we have that the Krull dimension of A equals 2. Since A is generated by 3 elements, we may describe it as a quotient of $C := \mathbf{C}[x_1, x_2, x_3]$. Namely, we may describe A as the quotient of C by the ideal

$$I = (x_3^2 - x_1 x_2).$$

Since the Jacobian matrix is given by

$$\begin{pmatrix} -x_2 & -x_1 & 2x_3 \end{pmatrix},$$

it follows that $(x_1, x_2, x_3) \in V(I)$ is singular[4] if and only if $x_1 = x_2 = x_3 = 0$. Consider now the following map

$$\pi : \operatorname{Spec} C/I \longrightarrow X = \operatorname{Spec} B = \mathbf{A}^2_{\mathbf{C}}, \quad (x_1, x_2, x_3) \mapsto (x_1, x_2).$$

Then π is a finite map of degree 2 which is unramified outside $D \subset X$, where the divisor D is the union of the coordinate hyperplanes.

EXAMPLE 3.30. Consider the action $a \cdot x^i y^j = (-1)^i x^i y^j$ in the above example. The normal surface $Y = \operatorname{Spec} \mathbf{C}[x^2, y]$ is smooth in this case.

EXAMPLE 3.31. Let $r > 1$ and consider the inclusion of $\mathbf{C}$-algebras

$$A := \mathbf{C}[x^r, y^r, xy] \subset B := \mathbf{C}[x, y].$$

Note that A is the algebra of invariants of an action by the cyclic group G of order r on B, that we describe as follows. Let a be a generator of G and let ζ_r be a primitive r-th root of unity. We define the action of a on a monomial $x^i y^i$ by

$$a \cdot x^i y^j := (\zeta_r)^{i-j} x^i y^j.$$

It is easy to show that A is indeed the algebra of invariants B^G. (For $r = 2$, this is Example 3.29.) We may describe A as the quotient of $C = \mathbf{C}[x_1, x_2, x_3]$ by the ideal

$$I = (x_3^r - x_1 x_2).$$

It follows that $(x_1, x_2, x_3) \in V(I)$ is singular[5] if and only if $x_1 = x_2 = x_3 = 0$. Consider now the following map

$$\pi : \operatorname{Spec} C/I \longrightarrow X = \operatorname{Spec} B = \mathbf{A}^2_{\mathbf{C}}, \quad (x_1, x_2, x_3) \mapsto (x_1, x_2).$$

Then π is a finite map of degree r which is unramified outside $D \subset X$, where the divisor D is the union of the coordinate hyperplanes.

EXAMPLE 3.32. Let the cyclic group $G = \{e, a, a^2\}$ of order 3 act on $\mathbf{C}[x, y]$ by

$$a \cdot x^i y^j := (\zeta_3)^{i+j} x^i y^j.$$

It is easy to show that $A = \mathbf{C}[x^3, x^2 y, xy^2, y^3]$ is the algebra of invariants. Since A is generated by 4 elements, we can describe it as a quotient of $C := \mathbf{C}[x_1, x_2, x_3, x_4]$. In fact, for $I = (x_1 x_2 - x_3 x_4, x_4^2 - x_1 x_3, x_3^2 - x_2 x_4)$, we have that the morphism $\mathbf{C}[x_1, x_2, x_3, x_4] \longrightarrow A$ given by

$$x_1 \mapsto x^3, \quad x_2 \mapsto y^3, \quad x_3 \mapsto xy^2, \quad x_4 \mapsto x^2 y$$

in surjective with kernel I. We have that $Y = \operatorname{Spec} C/I$ is singular[6] precisely at the point $(0, 0, 0, 0) \in \operatorname{Spec} C/I$. This is easy to see by considering the Jacobian matrix

$$J = \begin{pmatrix} x_2 & x_1 & -x_4 & -x_3 \\ -x_3 & 0 & -x_1 & 2x_4 \\ 0 & -x_4 & 2x_3 & -x_2 \end{pmatrix}.$$

[4]This is called the $\frac{1}{2}(1,1)$ or $A_{2,1}$ cyclic quotient singularity.
[5]This is called the $\frac{1}{r}(1,-1)$ or $A_{r,r-1}$ cyclic quotient singularity.
[6]This is called the $\frac{1}{3}(1,1)$ or $A_{3,1}$ cyclic quotient singularity.

The rank of J equals 0 if and only if $x_1 = x_2 = x_3 = x_4 = 0$. Thus, suppose that the rank of J equals 1. We will show that $x_1 = x_2 = x_3 = x_4 = 0$. Suppose that the bottom row is not equal to zero. Then all the other rows are scalar multiples of this row. It then easily follows that $x_1 = x_2 = x_3 = x_4 = 0$. Thus, we may suppose that the bottom row is zero. Therefore, we have that $x_2 = x_3 = x_4 = 0$. Since the rank of J is 1, this forces $x_1 = 0$. We conclude that Y is singular precisely at the origin. The morphism $\pi : Y \longrightarrow \mathbf{A}^2_{\mathbf{C}}$ given by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2)$ is a finite map of degree 9. If $D \subset \mathbf{A}^2_{\mathbf{C}}$ is the union of the hyperplanes, we see that π is unramified outside D and that the singularities of Y occur in the inverse image of the singularities of D.

Example 3.33. This example is similar to the above examples. Since it is hard to find explicit equations always, we give another approach. Consider the action of the cyclic group $G = \{e, a, \ldots, a^{r-1}\}$ of r elements on $\mathbf{C}[x, y]$ given by $a \cdot x^i y^j = (\zeta_r)^{i+j} x^i y^j$. The algebra of invariants is $A = \mathbf{C}[x^r, x^{r-1}y, \ldots, xy^{r-1}, y^r]$. Note that $\dim A = 2$. In fact, the extension of rings $\mathbf{C}[u, v] \subset A$ is integral, where $u = x^r$ and $v = y^r$. Let $Y = \operatorname{Spec} A$ and denote the morphism associated to the extension of rings $\mathbf{C}[u, v] \subset A$ by

$$\pi : Y \longrightarrow \operatorname{Spec} \mathbf{C}[u, v].$$

Let D be the union of the hyperplanes in $\operatorname{Spec} \mathbf{C}[u, v]$. Let us show that π is unramified outside the divisor D. In fact, for any $0 \leq a \leq n$, we have relations $u^a y^{r-a} = (x^a y^{r-a})^r$. The minimal polynomial of $x^a y^{r-a}$ over $\mathbf{C}[u, v]$ is therefore separable if and only if $u \neq 0$ and $v \neq 0$. Thus, we see that π is unramified outside D. Now, let us show that the singularities of Y occur in the inverse image of the singular locus of D. Suppose that y is a singular point of Y mapping to a nonzero point x on D. Consider the action of the group $\mathbf{C}^* \times \mathbf{C}^*$ on $\mathbf{C}[u, v]$ given by $u \mapsto \lambda_1 u$ and $v \mapsto \lambda_2 v$ for any $(\lambda_1, \lambda_2) \in \mathbf{C}^* \times \mathbf{C}^*$. For any $(\lambda_1, \lambda_2) \in \mathbf{C}^* \times \mathbf{C}^*$, this induces an automorphism of $\operatorname{Spec} \mathbf{C}[u, v]$. Under this action, the orbit of x is infinite. In particular, the orbit of y is infinite. (Here the action on Y is induced by the action on A.) Since every element of the orbit of y is again singular, we see that the singular locus of Y is infinite. But the singular locus of Y is of codimension 2 in X. Therefore, it is finite by Noether's normalization lemma. Contradiction. Moreover, note that Y is singular precisely at the origin. In fact, the singular locus of D is precisely the origin of $\mathbf{A}^2_{\mathbf{C}}$. Apply the defining relations above to see that its fibre under π is precisely the origin[7] in Y.

What do we learn from these examples?

(1) The normal variety Y can be singular.
(2) The singularities of Y are isolated and cyclic quotient in the above examples. This always holds when $\dim X = 2$. We will give an example to show that this is no longer true if $\dim X > 2$.
(3) In the above examples, the singularities of Y occur in the inverse image of the singularities of D. This is always true by Theorem 3.34 below.

Now, we return to our general setting. We write D^{sing} for the singular locus of D and let $n = \dim X$. The following Proposition generalizes [**EdJoSc**, Lemma 2.1]. Its proof invokes the analytic structure on the algebraic variety X. See [**SGA1**, Exposé XII].

Theorem 3.34. We have the following three facts.

[7]This is called the $\frac{1}{r}(1,1)$ or $A_{r,1}$ cyclic quotient singularity.

(1) All singularities of Y are (abelian) quotient.
(2) The singularities of Y occur in the inverse image under π of D^{sing}.
(3) The morphism $\pi^{-1}(D - D^{\mathrm{sing}}) \longrightarrow D - D^{\mathrm{sing}}$ is étale.

Proof. Firstly, if x is a point of X not lying on D, we have that all points in Y mapping to x are regular. Thus, let $x \in X$ be a point lying on D and let y be a point of Y mapping to x. Locally for the analytic topology we identify a neighbourhood W of x in X with the poly-disk

$$\{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid |z_1| < 1, \ldots, |z_n| < 1\},$$

identifying x with the origin and D locally with the zero set of $z_1 \cdot \ldots \cdot z_p$, where $1 \leq p \leq n$ depends on x and D. (The number p is the number of coordinate axes meeting at x. For example, x is a nonsingular point of D if and only if $p = 1$.) Let B be the connected component of $\pi^{-1}(W)$ containing y. Then

$$B - \pi^{-1}(D) \longrightarrow W - D$$

is a connected finite degree topological covering. Therefore, we have that

$$\Gamma_1 := \pi_*(\pi_1(B - \pi^{-1}(D)))$$

is a subgroup of finite index of $\pi_1(W - D)$. Since $W - D$ is homeomorphic to a product of $n - p$ open disks with p punctured open disks, we have that $\pi_1(W - D) \cong \mathbf{Z}^p$. Since this is an abelian group, every covering of $W - D$ is Galois. Let e be the index of the subgroup Γ_1 in $\mathbf{Z}^p$. Note that $\Gamma_2 := e\mathbf{Z}^p$ is a subgroup of index e^{p-1} of Γ_1. The topological covering

$$\zeta : W \longrightarrow W, \quad (z_1, z_2, \ldots, z_n) \mapsto (z_1^e, z_2^e, \ldots, z_p^e, z_{p+1}, \ldots, z_n)$$

is of degree e^p with fundamental group Γ_2. By the Galois correspondence, we have a connected finite degree connected topological covering $\eta : W - D \longrightarrow B - \pi^{-1}(D)$ of degree e^{p-1} and a commutative diagram

$$\begin{array}{ccc} W - D & \xrightarrow{\quad\eta\quad} & B - \pi^{-1}(D) \\ & {\scriptstyle\zeta}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & W - D & \end{array}.$$

By [**SGA1**, Expose XII, Théorème 5.4], there is a unique finite covering $\eta : W \longrightarrow B$ which extends $\eta : W - D \longrightarrow B - \pi^{-1}(D)$ and which fits into a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\quad\eta\quad} & B \\ & {\scriptstyle\zeta}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & W & \end{array}.$$

We see that B is the quotient of W under the action of Γ_2/Γ_1. This shows (1). Now, in order to show (2) and (3), we suppose that x is a closed point of X lying on D but not on D^{sing}. This happens if and only if $p = 1$. Since $\pi_1(W - D)$ is infinite cylic in this case, we have that $\Gamma_1 = \Gamma_2$. Therefore, the covers π and ζ are equivalent and the holomorphic map $\eta : W \longrightarrow B$ is an isomorphism. Thus, we have that B itself is regular. We deduce that Y is regular above $D - D^{\mathrm{sing}}$ and that $\pi^{-1}(D - D^{\mathrm{sing}}) \longrightarrow D - D^{\mathrm{sing}}$ is étale. This proves (2) and (3). $\square$

REMARK 3.35. In the above proof, one can show that if $p = 2$, the point y has an open neighborhood which is the product of a 2-dimensional cyclic quotient singularity and an open linear subspace of $\mathbf{C}^n$ isomorphic to[8] $\mathbf{C}^{n-2}$. See the proof of [**EdJoSc**, Lemma 2.2]. In particular, if $\dim X = 2$, we have that all singularities of Y are cyclic quotient.

The following examples show that Y can have quotient singularities which are not cyclic quotient if $\dim X > 2$.

EXAMPLE 3.36. Consider the inclusion of $\mathbf{C}$-algebras

$$A := \mathbf{C}[x^2, y^2, z^2, xyz] \subset B := \mathbf{C}[x, y, z].$$

Note that A is the algebra of invariants of an action of the noncyclic group G of order four on B, that we describe as follows. Let $G = \{e, a, b, c\}$, with e the identity element. Then we define the action of a, b, c on a monomial $x^i y^j z^k$ by

$$a \cdot x^i y^j z^k = (-1)^{i+j} x^i y^j z^k, \quad b \cdot x^i y^j z^k = (-1)^{j+k} x^i y^j z^k, \quad c \cdot x^i y^j z^k = (-1)^{k+i} x^i y^j z^k.$$

It is easy to see that A is indeed the algebra of invariants B^G. It is clear that we may identify A with $\mathbf{C}[x_1, x_2, x_3, x_4]/I$, where I is the ideal $(x_4^2 - x_1 x_2 x_3)$. The singularities of $\operatorname{Spec} A = \operatorname{Spec} \mathbf{C}[x_1, x_2, x_3, x_4]/I$ consist of the union

$$\{x_1 = x_2 = x_4 = 0\} \cup \{x_1 = x_3 = x_4 = 0\} \cup \{x_2 = x_3 = x_4 = 0\}.$$

Let $Y = \operatorname{Spec} A = \operatorname{Spec} C/I$ and consider the morphism

$$\pi : Y \longrightarrow \mathbf{C}^3, \quad (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3).$$

Then π is a finite morphism of degree 2, unramified outside the divisor $D \subset X$. Here the divisor D is the union of the coordinate hyperplanes. Note that the origin is not cyclic quotient. In view of Remark 3.35, locally around the point $(x, 0, 0, 0)$ with $x \neq 0$, we have that Y is isomorphic to the product of the variety $\{uv = w^2\}$ with a linear space $U \subset \mathbf{C}^3$.

Of course, we can give a similar example in dimension 4.

EXAMPLE 3.37. Consider the inclusion of $\mathbf{C}$-algebras $A := \mathbf{C}[x^2, y^2, z^2, w^2, xyzw] \subset B := \mathbf{C}[x, y, z, w]$. We see that A is the algebra of invariants of an action of the noncyclic group G of order four on B. It is clear that we may identify A with $\mathbf{C}[x_1, x_2, x_3, x_4, x_5]/I$, where $I = (x_5^2 - x_1 x_2 x_3 x_4)$. The singular locus of $\operatorname{Spec} A = \operatorname{Spec} \mathbf{C}[x_1, x_2, x_3, x_4]/I$ is given by the union of $\{x_1 = x_2 = x_5 = 0\}$, $\{x_1 = x_3 = x_5 = 0\}$, $\{x_1 = x_4 = x_5 = 0\}$, $\{x_2 = x_3 = x_5 = 0\}$, $\{x_2 = x_4 = x_5 = 0\}$ and $\{x_3 = x_4 = x_5 = 0\}$. Now, consider the morphism $\pi : Y = \operatorname{Spec} C/I \longrightarrow \mathbf{C}^4$ defined by $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, x_3, x_4)$. Then, we have that π is a finite morphism of degree 2, unramified outside the divisor $D \subset X$. Here the divisor D is again the union of the coordinate hyperplanes. Again, note that the origin is not cyclic quotient.

A proper birational surjective morphism $Y' \longrightarrow Y$ with Y' a smooth variety over $\mathbf{C}$ is called a resolution of singularities for Y. By Hironaka's theorem ([**Hiro**]), we can always find a resolution of singularities for Y. Since C is a Dedekind scheme, we have that Y' is flat over C. If $\dim Y = 2$, this also implies that Y' is projective over C. If $\dim Y > 2$ this is no longer true. In general, we can always arrange Y' to be projective over $\mathbf{C}$, but there are resolutions

[8] Let x and p be as in the above proof. I think one can show that x is a quotient singularity given by a product of $p - 1$ cyclic groups.

$\rho : Y' \longrightarrow Y$ with Y' nonprojective. Let us explain this. Firstly, the algorithms to construct a resolution of singularities $\rho : Y' \longrightarrow Y$ show that Y' can be obtained as a composition of blow-ups. Therefore, in this case Y' is projective. Unfortunately, when $\dim Y > 2$ and $\rho : Y' \longrightarrow Y$ is a resolution of singularities for Y, Hironaka proves that we can find a smooth complete nonprojective variety Y'' which dominates and is birational to Y'. That is, there are always resolutions for Y which are nonprojective.

LEMMA 3.38. For any resolution of singularities $\rho : Y' \longrightarrow Y$, it holds that $R^0\rho_*\mathcal{O}_{Y'} = \mathcal{O}_Y$.

PROOF. Since the question is local on Y, we may assume that $Y = \operatorname{Spec} A$ is affine. Since ρ is proper, we have that $R^0\rho_*\mathcal{O}_{Y'}$ is a coherent sheaf of $\mathcal{O}_Y$-algebras. Therefore, we have that the regular ring $B := \Gamma(Y, R^0\rho_*\mathcal{O}_{Y'})$ is a finitely generated A-module. Note that A and B are integral domains. Therefore, since ρ is surjective, the natural morphism $A \longrightarrow B$ is injective. Since ρ is birational, A and B have the same quotient field. But, since Y is normal, we have that A is integrally closed. Therefore, we have that $A = B$. We conclude that $R^0\rho_*\mathcal{O}_{Y'} = \mathcal{O}_Y$. □

We showed that the singularities of Y are quotient singularities. This is a purely geometric property of the singularity. It turns out that quotient singularities are rational singularities. This is a cohomological property defined below.

DEFINITION 3.39. A singularity $y \in Y$ is called *rational* if there exists a neighbourhood U of y, such that for every resolution of singularities $\rho : Y' \longrightarrow U$ for U we have $R^i\rho_*\mathcal{O}_{Y'} = 0$ for $i > 0$.

Every regular point of Y is a rational singularity. Therefore, by the Leray spectral sequence, it suffices to check the above condition for only one resolution of singularities.

LEMMA 3.40. For any resolution of singularities $\rho : Y' \longrightarrow Y$, we have that

$$R^i\rho_*\mathcal{O}_{Y'} = \begin{cases} \mathcal{O}_Y & \text{for} \quad i = 0 \\ 0 & \text{for} \quad i > 0 \end{cases}.$$

PROOF. Quotient singularities are rational ([**Vie**, Proposition 1]). □

From Lemma 3.40 and the functoriality of push-forward, we get the following result.

COROLLARY 3.41. For any resolution of singularities $\rho : Y' \longrightarrow Y$, we have that $\chi(Y', \mathcal{O}_{Y'}) = \chi(Y, \mathcal{O}_Y)$. □

We already noted the importance of the determinant of cohomology in Remark 3.5.

THEOREM 3.42. Fix a resolution of singularities $\rho : Y' \longrightarrow Y$ and write $f = h \circ \pi \circ \rho$. Then $c_1(f_!\mathcal{O}_{Y'})$ equals

$$f_*(\operatorname{td}(Y')_{(n)}) - h_*(\operatorname{td}(X)_{(n-1)}) \operatorname{td}(C)_{(1)} \deg \pi - \operatorname{td}(C)_{(1)} \cdot h_* \left(\sum_{j=1}^{n-1} \operatorname{ch}(\pi_*\mathcal{O}_Y)_{(j)} \operatorname{td}(X)_{(n-1-j)} \right).$$

PROOF. This is the same computation made in [**EdJoSc**, Section 3]. We apply the Grothendieck-Riemann-Roch theorem to f and $\mathcal{O}_{Y'}$ in degree 1 and combine this with the Grothendieck-Riemann-Roch theorem for h and $\pi_*\mathcal{O}_Y$. □

EXAMPLE 3.43. Suppose that $n = 1$, i.e., X is a smooth projective curve. In particular, the flat morphism $h : X \longrightarrow C$ is finite. Applying the Grothendieck-Riemann-Roch theorem to deduce the above formula seems to be overkill in this case. In fact, we can deduce it more directly as follows. Since Y is normal, it is smooth. Since smooth projective curves are determined by their function fields, any resolution of singularities $\rho : Y' \longrightarrow Y$ is actually an isomorphism. Thus, we have that $\pi : Y \longrightarrow X$ is a finite morphism of smooth projective curves. The composition $f = h \circ \pi$ is also a finite morphism of smooth projective curves. The effective divisor D is just a finite set of points on X and $\deg D$ is just the number of points on D. Note that π is separable and flat. Let R be the ramification divisor of $\pi : Y \longrightarrow X$. Then $R = \sum_{P \in \pi^{-1}D} (e_P - 1)[P]$ and

$$0 \leq \deg R = \sum_{P \in \pi^{-1}D} (e_P - 1) = \deg \pi \deg D - \deg D.$$

Here we used that, for any $Q \in X$, we have that

$$\sum_{P \mapsto Q} e_P = \deg \pi.$$

Now, recall Section 3 and apply the Riemann-Hurwitz theorem to f. This gives us that

$$2c_1(f_*\mathcal{O}_Y) = \deg f \cdot K_C - f_*(K_Y).$$

Apply the Riemann-Hurwitz theorem to $\pi : Y \longrightarrow X$ and rewrite the former as

$$2c_1(f_*\mathcal{O}_Y) = \deg f \cdot K_C - f_*(\pi^* K_X + R).$$

This implies that

$$\deg \det f_*\mathcal{O}_Y \quad = \quad \frac{1}{2}(\deg K_C \deg h - \deg K_X - \deg D) \deg \pi + \frac{1}{2} \deg D.$$

In view of the definition below, we conclude that the "height of π over C" grows linearly in $\deg \pi$ as a function with coefficients depending only on C, X, h and D.

DEFINITION 3.44. Fix a resolution of singularities $\rho : Y' \longrightarrow Y$ and write $f = h \circ \pi \circ \rho$. The *height* of π over C, denoted by $H_C(\pi)$ or just $H(\pi)$, is defined as

$$H(\pi) = |\deg \det(f_! \mathcal{O}_{Y'})|.$$

By Corollary 3.41 and Theorem 3.42, the height is independent of the chosen resolution. In fact, the Hirzebruch-Riemann-Roch theorem shows that

$$\deg_C f_* \operatorname{td}(Y') = \deg \operatorname{td}(Y') = \chi(Y', \mathcal{O}_{Y'}) = \chi(Y, \mathcal{O}_Y).$$

REMARK 3.45. In the process of defining and studying the height, the Grothendieck-Riemann-Roch theorem has shown itself to be of fundamental value. We have applied it to the morphism $h : X \longrightarrow C$, $f : Y' \longrightarrow C$ and the structural morphism $Y' \longrightarrow \operatorname{Spec} \mathbf{C}$ to obtain a well-defined height (i.e., independent of the resolution) and a useful formula (i.e., Theorem 3.42).

In general, as we mentioned in Remark 3.27, for a fixed degree, the height of π over C is bounded from above. Our ultimate goal is actually to give a bound for the height of π over C which is polynomial in $\deg \pi$ and whose coefficients depend only on C, X, h and D. To arrive at such a result, it suffices to bound the following quantities.

Firstly, one should bound

$$\deg \operatorname{td}(Y')_{(n)} = \chi(Y, \mathcal{O}_Y).$$

Then, one should bound the degree of

$$\mathrm{td}(C)_{(1)} \cdot h_* \left(\sum_{j=1}^{n-1} \mathrm{ch}(\pi_* \mathcal{O}_Y)_{(j)} \, \mathrm{td}(X)_{(n-1-j)} \right). \tag{7}$$

For example, let us show how to bound the terms of (7) which involve only $c_1(\pi_*\mathcal{O}_Y)$. That is, let us bound the degree of

$$\mathrm{td}(C)_{(1)} \cdot h_* \left(\sum_{j=1}^{n-1} \frac{(c_1(\pi_*\mathcal{O}_Y))^j}{j!} \, \mathrm{td}(X)_{(n-1-j)} \right) \tag{8}$$

polynomially in $\deg \pi$ with coefficients depending only on C, X, h and D.

Write $D = \sum_{i \in I} D_i$ for the decomposition of D in prime components. Define R to be the Weil divisor, supported on $\pi^{-1}(D)$, given as follows: let D_{ij} be a component of $\pi^{-1}(D)$ mapping onto D_i, then the multiplicity of D_{ij} in R is $e_{ij} - 1$. Here e_{ij} is the ramification index π at the generic point of D_{ij}. Define $B := \pi_* R$. We have that the ideal sheaf of B can be identified with $(\det \pi_*\mathcal{O}_Y)^{\otimes 2}$. To prove this, we invoke the trace pairing $\pi_*\mathcal{O}_Y \otimes \pi_*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$. This trace pairing induces a monomorphism $(\det \pi_*\mathcal{O}_Y)^{\otimes 2} \hookrightarrow \mathcal{O}_X$, identifying $(\det \pi_*\mathcal{O}_Y)^{\otimes 2}$ with the ideal sheaf $\mathcal{O}_X(-B)$ of B as a local computation shows (e.g., [**Ser2**, Chapter 3.6, Proposition 13]). We obtain that

$$c_1(\pi_*\mathcal{O}_Y) = -\frac{1}{2}[B]$$

in $A(X)_{\mathbf{Q}}$. Note that the multiplicity of each D_i in B is $\sum_{j \in J_i} (e_{ij} - 1) f_{ij}$, where f_{ij} is the degree of D_{ij} over D_i. For a fixed $i \in I$, we have

$$\sum_{j \in J_i} e_{ij} f_{ij} = \deg \pi.$$

For each $j = 1, \ldots, n-1$, we conclude that each term

$$\mathrm{td}(C)_{(1)} \cdot h_* \left(\frac{(c_1(\pi_*\mathcal{O}_Y))^j}{j!} \, \mathrm{td}(X)_{(n-1-j)} \right)$$

of the sum in expression (8) is bounded by a polynomial in $\deg \pi$ of degree j with coefficients depending only on C, X, h and D.

The terms in expression (7) involving higher Chern classes are not so easily dealt with. If $\dim X = 2$, we see that the only term that remains to bound is $\mathrm{td}(Y')_{(2)}$.

What about the degree of $\mathrm{td}(Y')_{(n)}$? Well, we can show how to do this when $n = 2$ following [**EdJoSc**]. So from now on, we assume that $n = 2$.

Suppose that $\dim X = 2$. Then the height can be bounded linearly in $\deg \pi$. That is, one can show that there is some positive integer c (which depends only on C, X, h, D) such that $H(\pi) \le c \deg \pi$. Although details can be found in [**EdJoSc**], let us give an idea of how one can bound the degree of $\mathrm{td}(Y')$ from above.

Since the height is independent of the resolution, we can choose Y' to be the minimal resolution of Y. (This always exists by [**Liu**, Proposition 9.3.32] and it is unique up to a unique isomorphism). Write c_i for the degree of $c_i(\mathcal{T}_{Y'})$ and note that

$$\deg \mathrm{td}(Y')_{(2)} = \frac{c_1^2 + c_2}{12}.$$

Thus, it suffices to bound c_1^2 and c_2 from above. Now, by the Bogomolov-Miyaoka-Yau inequality, we have that $c_1^2 \leq 3c_2$, if X is of general type. If X is not of general type, we actually have that $c_1^2 \leq 9$ by Table 10 in Chapter VI of [**BHPV**]. Therefore, in order to bound the degree of $\mathrm{td}(Y')$ from above, it suffices to bound c_2 from above. By the Gauss-Bonnet theorem, we have that c_2 equals the topological Euler characteristic $e(Y')$. In particular, it suffices to bound $e(Y')$. We have that $e(Y') = e_c(Y) + s$, where $e_c(Y)$ is the compactly supported Euler characteristic of Y and s is the total number of exceptional components of $Y' \longrightarrow Y$. Thus, we reduce to bounding $e_c(Y)$ and s. To bound $e_c(Y)$, we use that $\pi^{-1}U \longrightarrow U$ and $\pi^{-1}(D - D^{\mathrm{sing}}) \longrightarrow D - D^{\mathrm{sing}}$ are étale. Then, by [**EdJoSc**, Lemma 2.4],

$$e_c(Y) = e_c(\pi^{-1}U) + e_c(\pi^{-1}D) = e_c(U)\deg\pi + e_c(\pi^{-1}D - D^{\mathrm{sing}}) + e_c(\pi^{-1}D^{\mathrm{sing}}).$$

Now, we have that $e_c(\pi^{-1}D^{\mathrm{sing}}) = \#\pi^{-1}D^{\mathrm{sing}} \leq \#D^{\mathrm{sing}}\deg\pi$. Let us explain how to bound the compactly supported Euler characteristic of $\pi^{-1}(D - D^{\mathrm{sing}})$. Consider the prime decomposition of $D = \sum D_i$ as above and write d_i for the degree of the étale cover $\pi^{-1}(D_i - D^{\mathrm{sing}}) \longrightarrow D_i - D^{\mathrm{sing}}$. Consider the degree f_{ij} of each component D_{ij} of $\pi^{-1}D_i$ over D_i. The bound follows from

$$d_i = \sum_j f_{ij} \leq \sum_j e_{ij} f_{ij} = \deg\pi,$$

where e_{ij} is the ramification index π at the generic point of D_{ij}. Thus, it remains to bound s. This one does by invoking the Hirzebruch-Jung continued fraction associated to a singular point on Y (which we know is cyclic quotient). This finishes our sketch of how to give a linear upper bound for $\deg\mathrm{td}(Y')$.

The lower bound for $\deg\mathrm{td}(Y')$ is harder and requires intersection theory on the normal surface Y.

If $\dim Y = 3$ the above argument to bound $\mathrm{td}(Y')_{(3)}$ from above breaks down for several reasons. For example, let us try and bound the degree of $\mathrm{td}(Y')$ from above. Again we write c_i for the degree of $c_i(\mathcal{T}_{Y'})$. Then, the degree of $\mathrm{td}(Y')$ is $\frac{1}{24}c_1c_2$. One could try to give an upper bound for this by giving an upper bound for c_3 (while hoping for some generalized Bogomolov-Miyaoka-Yau inequality). But this will probably not suffice by ([**LeBrun**, Theorem A and Theorem B]). Thus, we suspect it will not be sufficient to bound c_3 from above in order to bound $\mathrm{td}(Y')$ from above. Furthermore, there is no good notion of minimal resolution when $\dim Y > 2$.

7. Arithmetic curves

We fix a number field K with ring of integers O_K.

DEFINITION 3.46. A connected regular scheme X which is projective and flat over $\mathrm{Spec}\, O_K$ is called an *arithmetic variety* over O_K. A 1-dimensional arithmetic variety over O_K is called an *arithmetic curve* over O_K. A 2-dimensional arithmetic variety over O_K is called an *arithmetic surface* over O_K.

REMARK 3.47. An arithmetic variety over O_K is an arithmetic variety over $\mathbf{Z}$.

Let $X = \mathrm{Spec}\, O_K$. We give an elementary proof of the following Lemma which is usually shown by invoking Serre's criterion for affineness.

LEMMA 3.48. Any open subset of X is affine.

Proof. Clearly, any open subset of X is the complement of a finite set of closed points. Let D be a finite set of closed points of X with complement $U = X - D$ in X. Considering D as a finite set of maximal ideals in O_K, we can define the ring $O_K[\frac{1}{D}]$ to be the elements a in K which have nonnegative valuation at all the primes outside D. Then $\operatorname{Spec} O_K[\frac{1}{D}] \longrightarrow \operatorname{Spec} O_K$ is an open immersion with image U. This can be checked easily on a covering by basic open affines on which the primes in D become principal. □

Let D be a finite set of maximal ideals in O_K. By the above Lemma, we have that $U = X - D$ is affine and we can write $U = \operatorname{Spec} O_K[\frac{1}{D}]$. For any finite étale morphism $V \longrightarrow U$, we have that V is affine and isomorphic to $\operatorname{Spec} O_L[\frac{1}{D}]$ if we assume V to be connected with function field L. The normalization $\pi : Y \longrightarrow X$ of X in L is finite. Since the integral closure of O_K in L is O_L, we have that $Y = \operatorname{Spec} O_L$. Reversely, any arithmetic curve over O_K arises in this matter. (Apply [**Bruin**, Proposition 6.1] and the fact that the set of ramification points of a finite field extension $K \subset L$ is finite.) Of course, the degree of π is the degree of the extension $K \subset L$. As in Section 3, we can interpret this as a Riemann-Roch theorem in degree 0. We now formulate a Riemann-Roch theorem for π in degree 1.

Example 3.49. Consider the field extension $\mathbf{Q} \subset \mathbf{Q}(i)$. The ring of integers of $\mathbf{Q}(i)$ is the ring of Gaussian integers $\mathbf{Z}[i]$. A prime ideal $\mathfrak{P}$ of $\mathbf{Q}(i)$ is ramified over $\mathbf{Q}$ if and only if $\mathfrak{P}$ contains the different $\mathfrak{D}_{\mathbf{Q}(i)/\mathbf{Q}}$. In this case, $\mathfrak{D}_{\mathbf{Q}(i)/\mathbf{Q}} = (2i)\mathbf{Z}[i] = (2)\mathbf{Z}[i]$. We see that $(1+i)\mathbf{Z}[i]$ is the only prime ideal which ramifies. Its ramification index over $\mathbf{Q}$ is 2. There are precisely four covers of $\operatorname{Spec} \mathbf{Z}$ ramified at (2):

A *point at infinity* on X is an embedding $\sigma : K \longrightarrow \mathbf{C}$. For any embedding $\sigma : K \longrightarrow \mathbf{C}$, we define its conjugate $\overline{\sigma} : K \longrightarrow \mathbf{C}$ as the composition of σ with the complex conjugation on $\mathbf{C}$. We say that $\sigma \in X_\infty$ is *real* if $\sigma = \overline{\sigma}$. Clearly, an element $\sigma \in X_\infty$ is real if and only if $\sigma(K) \subset \mathbf{R}$. We let Σ be the set of embeddings $\sigma : K \longrightarrow \mathbf{C}$. Note that $\#\Sigma = [K : \mathbf{Q}]$.

We now give some basic definitions and facts.

The group of *arithmetic cycles of codimension 1* or *Arakelov divisors* on X is defined as

$$\widehat{Z}^1(X) = Z^1(X) \oplus \mathbf{R}^\Sigma.$$

Usually, we will denote elements of $\widehat{Z}^1(X)$ by (D, g), where D is a divisor on X and g is an element of $\mathbf{R}^\Sigma$. We define the map $\widehat{\operatorname{div}} : K^* \longrightarrow \widehat{Z}^1(X)$ by

$$\widehat{\operatorname{div}}(a) = \operatorname{div}(a) \oplus (-\log|\sigma(a)|)_\sigma.$$

Here $\operatorname{div}(a)$ denotes the cycle associated to the fractional ideal $O_K \cdot a$. Note that $\widehat{\operatorname{div}}$ is a homomorphism. We let $\widehat{\operatorname{Rat}}^1(X)$ be its image. We define the *arithmetic Chow group of codimension 1* to be the quotient group

$$\widehat{A}^1(X) = \widehat{Z}^1(X)/\widehat{\operatorname{Rat}}^1(X).$$

Finally, we define the *arithmetic Chow group* as

$$\widehat{A}(X) = \mathbf{Z} \oplus \widehat{A}^1(X).$$

Let (D, g) be an Arakelov divisor on $X = \operatorname{Spec} O_K$. We define its *(arithmetic) degree* in $\mathbf{R}$, denoted by $\widehat{\deg}(D, g)$, as

$$\widehat{\deg}(D, g) = \sum_{\mathfrak{p} \text{ maximal in } O_K} n_{\mathfrak{p}} \log(\# O_K/\mathfrak{p}) + \sum_{\sigma \in \Sigma} \log |\sigma(a)|.$$

This clearly defines a homomorphism $\widehat{\deg} : \widehat{Z}^1(X) \longrightarrow \mathbf{R}$.

LEMMA 3.50. For any $a \in K^*$, the degree of $\widehat{\operatorname{div}}(a)$ equals zero.

PROOF. We may assume that $a \in O_K$ and that $a \neq 0$. Then

$$\widehat{\deg}\widehat{\operatorname{div}}(a) = \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(a) \log(\# O_K/\mathfrak{p}) - \sum_{\sigma \in \Sigma} \log |\sigma(a)|.$$

It is easy to see that

$$\operatorname{ord}_{\mathfrak{p}}(a) \log(\# O_K/\mathfrak{p}) = \log(\# O_{K,\mathfrak{p}}/(a)).$$

Since the O_K-module $O_K/(a)$ has finite support, one has an isomorphism

$$O_K/(a) \cong \bigoplus_{\mathfrak{p}} O_{K,\mathfrak{p}}/(a).$$

This implies that

$$\sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(a) \log(\# O_K/\mathfrak{p}) = \log(\# O_K/(a)) = \log N(a),$$

where $N(a)$ is the norm of the ideal $(a) \subset O_K$. We have a short exact sequence

$$0 \longrightarrow O_K \xrightarrow{m_a} O_K \longrightarrow O_K/(a) \longrightarrow 0\ ,$$

where m_a is the multiplication by a. It is easy to show that $\# O_K/(a) = |\det(m_a)|$. Now, tensor the above short exact sequence with $\mathbf{C}$ over $\mathbf{Z}$. The map induced by m_a from $O_K \otimes_{\mathbf{Z}} \mathbf{C} \cong \oplus_{\sigma \in \Sigma} \mathbf{C}$ to itself is given by the multiplication by $\sigma(a)$ on the coordinate indexed by σ. Therefore, we see that $\det(m_a) = \prod_{\sigma \in \Sigma} \sigma(a)$. This implies the result. □

We conclude that the degree induces a homomorphism $\widehat{\deg} : \widehat{A}^1(X) \longrightarrow \mathbf{R}$.

EXAMPLE 3.51. Suppose that $X = \operatorname{Spec} \mathbf{Z}$. Then $\widehat{\deg} : \widehat{A}^1(X) \longrightarrow \mathbf{R}$ is an isomorphism.

A *metrised O_K-module* on X is the data of a finitely generated O_K-module M together with a family $(h_\sigma)_{\sigma \in \Sigma}$ of hermitian forms h_σ on the complex vector space $\sigma^* M = M \otimes_\sigma \mathbf{C}$ such that $(h_\sigma)_{\sigma \in \Sigma}$ is invariant under conjugation. The former means that, for any $\sigma \in \Sigma$ and $v, w \in \sigma^* M$, we have that

$$\overline{h_\sigma(v, w)} = h_{\overline{\sigma}}(\overline{v}, \overline{w}).$$

A *hermitian vector bundle* on X is a metrised O_K-module $(M, (h_\sigma)_{\sigma \in \Sigma})$, where M is projective. A *hermitian line bundle* on X is a hermitian vector bundle $(M, (h_\sigma)_{\sigma \in \Sigma})$, where M is of rank 1. The projective finitely generated O_K-modules of rank 1 correspond to fractional ideals. We will write just M for a metrised O_K-module when the hermitian form is clear from the context.

For any hermitian line bundle L on X, we can define its degree $\widehat{\deg} L$. One simply takes a section a section s of L and defines $\widehat{\deg} L$ to be the degree of the arithmetic cycle with multiplicity $\mathrm{ord}_{\mathfrak{p}}(s) := \mathrm{ord}_{\mathfrak{p}}(a)$ at a prime ideal $\mathfrak{p}$, where $a \in K^*$ is the unique element such that $s = a \cdot 1$. It is easy to see that $\widehat{\deg} L$ is well-defined, i.e., independent of the section s. (Use the product formula or a similar reasoning as in Lemma 7.)

A morphism of metrised O_K-modules (M, h_σ) and (N, h'_σ) on X is a morphism of O_K-modules $\varphi : M \longrightarrow N$ such that, for any $\sigma \in \Sigma$ and $v \in \sigma^* M$, we have that

$$h'_\sigma(\sigma^*\varphi(v), \sigma^*\varphi(v)) = h_\sigma(v, v).$$

This defines the category $\widehat{\mathrm{Coh}}(X)$ of metrised O_K-modules. An isomorphism in the category of metrised O_K-modules is called an *isometry.* It is not hard to see that morphisms of metrised O_K-modules which are bijective are isometries. The category $\widehat{\mathrm{Vect}}(X)$ of hermitian vector bundles is a full subcategory.

Let us recall some basic constructions. Let $(M, (h_\sigma)_{\sigma\in\Sigma})$ be a metrised O_K-module. Suppose that $N \subset M$ is a submodule and that $(M', (h'_\sigma)_{\sigma\in\Sigma})$ is another metrised O_K-module. In the following σ will be a point in Σ. The restriction of h_σ to σ^*N endows N with a natural structure of a metrised O_K-module. Similarly, the restriction of h_σ to the orthogonal complement of σ^*N makes M/N into a metrised O_K-module if we identity this complement with σ^*M/N. Now, the orthogonal direct sum $h_\sigma \oplus h'_\sigma$ of h_σ and h'_σ makes the direct sum $M \oplus N$ into a metrised O_K-module. Here, for any $v, w \in \sigma^*M$ and $v', w' \in M'$, we define

$$h_\sigma \oplus h'_\sigma(v \oplus v', w \oplus w') = h_\sigma(v, w) + h'(v', w').$$

Also, for any $v, w \in \sigma^*M$ and $v', w' \in M'$, we define

$$h_\sigma \otimes h'_\sigma(v \otimes v', w \otimes w') = h_\sigma(v, w) \cdot h'(v', w').$$

This makes $M \otimes_{O_K} M'$ into a metrised O_K-module. As an application of this, let E be the O_K-module $\mathrm{Hom}_{O_K}(M, M')$ and note that $\sigma^*E = \mathrm{Hom}_{\mathbf{C}}(\sigma^*M, \sigma M')$ is the tensor product of σ^*M and σ^*M'. Therefore, we can make E into a metrised O_K-module by endowing it with this hermitian form on each complex vector space σ^*E. In particular, the dual module $E^\vee = \mathrm{Hom}_{O_K}(M, O_K)$ is endowed with a natural structure of metrised O_K-module. More explicitly, if $v^\vee$ denotes the element $h_\sigma(-, v)$ in $\sigma^*E^\vee$, where $v \in \sigma^*E$, we have that

$$h^\vee_\sigma(v^\vee, w^\vee) = \overline{h_\sigma(v, w)}.$$

Finally, let $p \geq 0$. For any $v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p \in \sigma^*\Lambda^p M$, we define

$$\Lambda^p h_\sigma(v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p) = \det(h_\sigma(v_i, w_j))h'(v', w').$$

This makes $\Lambda^p M$ into a metrised O_K-module. In particular, we can naturally endow the determinant of M with the structure of a metrised O_K-module.

The category $\widehat{\mathrm{Coh}}(X)$ is abelian and $\widehat{\mathrm{Vect}}(X)$ is an additive subcategory of $\widehat{\mathrm{Coh}}(X)$.

Let $\pi : Y \longrightarrow X$ be an arithmetic curve over O_K of degree n. We already said that we can write $Y = \mathrm{Spec}\, O_L$, where L is the function field of Y. Endow O_L with the *trivial* metric. That is, for any $\tau : L \longrightarrow \mathbf{C}$, define the hermitian form h_τ on $\tau^*O_L = \mathbf{C}$ to be the usual hermitian form on $\mathbf{C}$. Thus, for any $v, w \in \mathbf{C}$, we have $h_\tau(v, w) = v\overline{w}$. Now, it is easy to endow the projective finitely generated O_K-module $E := \pi_*O_L$ with the structure of a hermitian vector bundle. In fact, let $\sigma : K \longrightarrow \mathbf{C}$ be an embedding. If $\tau : L \longrightarrow \mathbf{C}$ is an embedding such that $\tau|_K = \sigma$, we will write $\tau|\sigma$. Then, for any $\sigma : K \longrightarrow \mathbf{C}$, we can define a hermitian form $(\pi_* h)_\sigma$ on $\sigma^* E$ as follows. Given $v, w \in \sigma^* E$, we define

$$(\pi_* h)_\sigma(v, w) = \sum_{\tau|\sigma} h_\tau(\tau(v), \tau(w)) = \sum_{i=1}^{n} h_\tau(v_i, w_i) = \sum_{i=1}^{n} v_i \overline{w_i}.$$

Here we use the decomposition $\sigma^* E = \bigoplus_{\tau|\sigma} \tau^* O_L$. To conclude, note that the family of hermitian forms $((\pi_* h)_\sigma)_{\sigma\in\Sigma}$ is invariant under conjugation. Since any fractional ideal $\mathfrak{a}$ of L can be endowed with the trivial metric described above, the above construction shows that $\pi_*\mathfrak{a}$ can be canonically given the structure of a hermitian vector bundle on X.

Example 3.52. The above construction does not always work. Let $n \geq 1$ and let $X \longrightarrow Y$ be the morphism given by $x \mapsto x^n$, where $X = Y = \mathbf{C}$. This is clearly a finite morphism. Note that $\mathcal{E} = \pi_* \mathcal{O}_X$ is a free $\mathcal{O}_Y$-module with basis $(1, w, \ldots, w^{n-1})$. Here w is the coordinate on Y. The trace form $\mathcal{E} \otimes \mathcal{E} \longrightarrow \mathcal{O}_Y$ maps $(f, g) \in \mathcal{E} \times \mathcal{E}$ to $\mathrm{tr}(f \otimes g)$. This is symmetric and $\mathcal{O}_Y$-linear. Let z be the coordinate on X. Then, in the basis $(1, w, \ldots, w^{n-1})$, the matrix of the trace form reads

$$\mathrm{tr}(w^{i-1} w^{j-1}) = \begin{pmatrix} n & 0 & \ldots & 0 \\ 0 & & & nz \\ \vdots & & & 0 \\ & nz & & \end{pmatrix}.$$

The determinant of this matrix is

$$\det \mathrm{tr}(w^{i-1} w^{j-1}) = \epsilon((1 \cdot n - 1)(2n - 1) \ldots (\lfloor \frac{n-2}{2} \rfloor)) n^n (-1)^{\lfloor \frac{n}{2} \rfloor - 1} z^{n-1}.$$

As an application of this we have the following Lemma.

Lemma 3.53. There is a canonical isomorphism

$$(\det \pi_* O_L)^{\otimes 2} \cong \mathfrak{d}_{L/K}$$

of hermitian line bundles. Here $\mathfrak{d}_{L/K}$ denotes the discriminant ([**Neu**, Chapter III.2]).

Proof. Consider the O_K-bilinear map on O_L given by the trace map. It induces a linear map $\tau : \det(\pi_* O_L) \otimes \det(\pi_* O_L) \longrightarrow O_K$ given by $\tau(v_1 \wedge \ldots \wedge v_n, w_1 \wedge \ldots \wedge w_n) = \det(\mathrm{tr}(v_i w_j))$. Its image is, by definition, the discriminant $\mathfrak{d}_{L/K}$. As a local computation shows, it suffices to compare the hermitian metrics on both sides.

Let $\sigma : K \longrightarrow \mathbf{C}$ be an embedding. By construction, the hermitian form on $\sigma^* E$ is given by $h_\sigma(v, w) = \sum_{\tau|\sigma} \overline{\tau(v)} \tau(w)$. The trace form $\sigma^* \mathrm{tr} : \sigma^* E \longrightarrow \sigma^* O_K$ is given by the map $(z_\tau) \mapsto \sum z_\tau$. In particular, the linear map $\sigma^* \tau$ is given by

$$(v_1 \wedge \ldots \wedge v_n) \otimes (w_1 \wedge \ldots \wedge w_n) \mapsto \det \left(\sum_\tau v_{i,\tau} w_{j,\tau} \right) = \det(v_{i,\tau}) \det(w_{j,\sigma}).$$

We conclude that $\sigma^* \tau$ is an isometry. □

Since $\widehat{\deg} \mathfrak{d}_{L/K}$ equals $-\log N(\mathfrak{d}_{L/K})$, we have proven the following arithmetic Hurwitz theorem.

Theorem 3.54. One has $\widehat{\deg} \det \pi_* O_L = -\frac{1}{2} \log N(\mathfrak{d}_{L/K})$. □

Bibliography

[AKMW] D. Abramovich, K. Karu, K. Matsuki, J.W lodarczyk, *Torification and factorization of birational maps.* J. Amer. Math. Soc. 15 (2002), no. 3, 531572.

[BHPV] W. Barth, K.Hulek, C. Peters, A. Van de Ven *Compact complex surfaces.* Springer-Verlag.

[BorSer] A. Borel, J.P. Serre, *Le théorème de Riemann-Roch.* Bull. Soc. math. France, 86, 1985, 97-136.

[BrSc] H. Brenner, S. Schrer, *Ample families, multihomogeneous spectra, and algebraization of formal schemes.* Pacific J. Math. 208 (2003), no. 2, 209-230.

[Bruin] P. Bruin, *Green functions on Riemann surfaces and an application to Arakelov theory.* Master's thesis.

[Chev] *Chevalley exposé 2 et 3.* Sminaire C. Chevalley; 2e anne: 1958. Anneaux de Chow et applications. Secrtariat mathmatique, 11 rue Pierre Curie, Paris, 1958.

[EdJoSc] B. Edixhoven, R. de Jong, J. Schepers, *Covers of surfaces with fixed branched locus.* International Journal of Mathematics.

[Ful1] W. Fulton, *Intersection Theory.* Springer-Verlag.

[Ful2] W. Fulton, *Algebraic curves.* Available online.

[FuLa] W. Fulton, S. Lang, *Riemann-Roch algebra.* Springer-Verlag.

[GeMo] G. van der Geer, B. Moonen, *Abelian varieties.* Available online.

[Gil] H. Gillet, *K-theory and intersection theory.* in the Handbook of K-theory.

[Groth] A. Grothendieck, *La théorie des classes de Chern.* Bulletin de la S.M.F., tome 86 (1958)m, 137-154.

[Har] R. Hartshorne, *Algebraic geometry.* Springer Science 2006.

[Hiro] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero.* I,II, Ann. fo Math. (1) 79 (1964), 109-326.

[Hirz] F. Hirzebruch, *Topological methods in Algebraic Geometry.* Springer-Verlag.

[Jah] J. Jähnel, *Heights for line bundles on arithmetic varieties.* Available online.

[Lang] S. Lang, *Algebra.* Addison-Wesley.

[LeBrun] C. LeBrun *Topology versus Chern Numbers for Complex 3-Folds.* Pac. J. Math. 191 (1999) 123–131.

[Liu] Q. Liu, *Algebraic geometry and arithmetic curves.* Addison-Wesley.

[Man] Y. Manin, *Lecture notes on the K-functor in algebraic geometry.* 1969 Russ. Math. Surv. 24 1.

[Mur] J. Murre, *Algebraic cycles and algebraic aspects of cohomology and K-theory.* Lecture Notes in Mathematics, 1594 Algebraic Cycles and Hodge Theory, Torino, 1993, pp. 93-152.

[Neu] . Neukirch, *Algebraic Number Theory.* Springer.

[Pap] G. Pappas, *Grothendieck-Riemann-Roch and the moduli of Enriques surfaces.* Available online.

[Pap2] G. Pappas, *Integral Grothendieck-Riemann-Roch theorem.* Available online.

[Ser] J.P. Serre, *Algèbre Locale.* Springer-Verlag.

[Ser2] J.P. Serre, *Corps Locaux.* Hermann 1968.

[SGA1] *Séminaire de géométrie algébrique du Bois-Marie 1 (1960-1961).* Revêtement étales et groupe fondamental. Dirigé par A. Grothendieck. Documents Mathématiques 3 (Société Mathématiques de France, 2003).

[SGA7] *Séminaire de géométrie algébrique du Bois-Marie 7-I (1967-1969).* Groupes de monodromie et géométrie algébrique. Dirigé par A. Grothendieck avec la collaboration de M. Raynaud et D.S. Rim. Lecture Notes in Mathematics 288 (Springer-Verlag, 1972).

[ThTr] R. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories, The Grothendieck Festschrift, v. 3, ed. P. Cartier et al..* Birkhaüser (1990), 247436.

[Tot] B. Totaro, *The resolution property for schemes and stacks.* J. Reine Angew. Math. 577 (2004), pp. 1-22

[Vie] E. Viehweg, *Rational singularities of higher dimensional schemes.* Proc. Amer. Math. Soc. **63** (1977), pp. 6-8.

[Weibel] C. Weibel, *The K-book: An introduction to algebraic K-theory.* Available online.