The Riemann-Roch theorem is a special case of the Atiyah-Singer index formula

Raynor, S.C.

S.C. Raynor

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Thesis supervisor: dr. M. Lübke

Mathematisch Instituut, Universiteit Leiden

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Introduction

The Atiyah-Singer index formula equates a purely analytical property of an elliptic differential operator P (resp. elliptic complex E) on a compact manifold called the analytic index $\text{ind}_a(P)$ (resp. $\text{ind}_a(E)$) with a purely topological property, the topological index $\text{ind}_t(P)$(resp. $\text{ind}_t(E)$) and has been one of the most significant single results in late twentieth century pure mathematics. It was announced by Michael Atiyah and Isadore Singer in 1963, with a sketch of a proof using cohomological methods. Between 1968 and 1971, they published a series of papers[1] in which they proved the formula using topological K-theory, as well as filling in the details of the original proof.

The history of the Atiyah-Singer index formula reads as a "Who's Who" in twentieth century topology and analysis. The formula can be seen as the culmination of a project of generalisation of index theorems that began in the mid 1800's with the Riemann-Roch theorem (and the Gauss-Bonnet theorem), and which involved many of the greatest names in topology and analysis of the last 150 years. It is an achievement for which Atiyah and Singer were awarded the Abel Prize in 2004. The significance of their formula reaches beyond the fields of differential topology and functional analysis: it is also fundamental in much contemporary theoretical physics, most notably string theory.

For the purpose of this paper however, the only results which we shall consider are the classical Riemann-Roch theorem (1864), the Hirzebruch-Riemann-Roch theorem (1954), and the Atiyah-Singer index formula (1963). In fact, we will only really look at the latter two in the context of being direct generalisations of the classical Riemann-Roch theorem.

The (classical) Riemann-Roch theorem, proved as an equality in 1864, links analytic properties of certain objects called divisors on compact Riemann surfaces, with topological properties of holomorphic line bundles defined in terms of the divisors. Though the terms involved will only be properly defined later in this paper, it is convenient, nonetheless, to state the theorem here.

Let X be a compact Riemann surface and D a divisor on X, that is, a function $D : X \to \mathbb{Z}$ with discrete support. Then the Riemann-Roch theorem states that

$$h^0(X, \mathcal{O}_D) - h^1(X, \mathcal{O}_D) = 1 - g + \deg(D). \tag{0.1}$$

Here $h^0(X, \mathcal{O}_D)$ is the dimension of the space of meromorphic functions f such that, for all $x \in X$, $\text{ord}_x(f) \geq -D(x)$, where $\text{ord}_x(f) = n$ if f has a zero of order n or a pole of order $-n$ at x, and $h^1(X, \mathcal{O}_D)$ is the dimension of another space of meromorphic functions also with only certain prescribed poles and zeroes (we will discuss this in detail in chapter 3). The degree, $\deg(D)$, of the divisor D is the sum

[1] *The index of elliptic operators: I-V.* (Paper II from 1968 is authored by Atiyah and Segal, rather than Atiyah and Singer.) [**AS1, AS2, AS3, AS4, AS5**].

of its values over X. Since X is compact, the support of D is finite and so $\deg(D)$ is well-defined. Finally g denotes the genus of the surface X. It is clear that these are all integral values.

The left hand side of equation (0.1) can be described in terms which depend on the holomorphic structure of certain line bundles on X, whilst we shall see that the right hand side depends only on the topology of these bundles.

There is a natural equivalence relation on the space of divisors of a Riemann surface X and it will be shown that there is a one to one correspondence between equivalence classes of divisors on X and isomorphism classes of line bundles on X. (This will be described in chapter 3.)

The Riemann-Roch theorem provides the conditions for the existence of meromorphic functions with prescribed zeroes and poles on a compact Riemann surface. Its significance did not go unnoticed and its implications were studied by many of the greatest names in topology and analysis (even including Weierstrass). Interestingly it was initially regarded fundamentally as a theorem of analysis and not of topology.

It was not until 1954, nearly a century after its original discovery, that Hirzebruch found the first succesful generalisation of the Riemann-Roch theorem to holomorphic vector bundles of any rank on compact complex manifolds of any dimension.[2] This came a few months after J.P Serre's 1953 discovery of what is now known as *Serre duality*, which provides a powerful tool for calculation with the Riemann-Roch theorem, but also deep insights into the concepts involved. Serre had applied sheaf theory to the Riemann-Roch theorem and Hirzebruch also used these newly emerging methods of topology to find techniques suitable for the project of generalisation. The so-called Hirzebruch-Riemann-Roch theorem says that the *Euler characteristic* $\chi(E)$ of a holomorphic vector bundle E on a compact complex manifold X is equal to its *T-characteristic* $T(E)$. We will define these terms in chapters 3 and 4. Of significance here is that, in the case that the X has dimension 1 and E rank 1, if D is the divisor that corresponds to E, then the Euler characteristic $\chi(E)$ is equal to the left hand side of equation (0.1) and $T(E)$ is equal to the right hand side of (0.1).

After Hirzebruch's theorem, progress to the Atiyah-Singer index formula was very swift indeed. Grothendieck discovered the *Grothendieck-Riemann-Roch theorem* around 1956[3], and the Atiyah-Singer index formula was published in its complete form in 1964.

The Atiyah-Singer index formula is a direct generalisation of the Hirzebruch-Riemann-Roch theorem since we can assosciate a certain elliptic complex $\overline{\partial}(E)$ with any holomorphic vector bundle E on a compact complex manifold X, and it can be shown that $\chi(E) = \mathrm{ind}_a(\overline{\partial}(E))$ and $T(E) = \mathrm{ind}_t(\overline{\partial}(E))$.

In this paper, we will show how the original Riemann-Roch theorem, formulated for divisors on compact Riemann surfaces, is a special case of the Hirzebruch-Riemann-Roch Theorem and the Atiyah-Singer index formula. The paper does not set out to prove any of these theorems. One of the most striking features of the

[2] These results can be found in [**Hi**], originally published as *Neue topologische Methoden in der algebraischen Geometrie* in 1956.

[3] Grothendieck had originally wished to wait with publishing a proof. With Grothendieck's permission, a proof was first published by Borel and Serre [**BS**] in 1958.

Atiyah-Singer index formula, and a good illustration of the depth and signifcance of the result, is that it admits proofs by many different methods, from the initial cohomology and K-theory proofs, to proofs using the heat equation. We will limit ourselves here to a cohomological formulation of the formula since this is the most natural choice when dealing with the Riemann-Roch theorem. However it is perhaps worth mentioning that the K-theoretic formulation lends itself best to a more general exposition on the Atiyah-Singer index formula.

The paper begins with two purely expository chapters. Chapter 1 sets out the basic definitions and notations concerning vector bundles, sheaves and sheaf cohomology which will be used throughout the paper. Most proofs will not be given. In chapter 2, elliptic differential operators, complexes and the analytic index of an elliptic complex will be defined and a number of examples will be given.

The substantial part of the paper begins in chapter 3. Divisors on a Riemann surface X are defined and the Riemann-Roch theorem is stated in terms of divisors. By constructing a holomorphic line bundle $L = L_D$ on X, associated with the divisor D, it is then shown that the left hand side of the Riemann-Roch equation (0.1) can be interpreted as a special case of the analytic index of an elliptic operator. Finally we show that this also corresponds to the Euler characteristic $\chi(L)$ of L on a Riemann surface.

In chapter 4, we turn to the right hand side of the Riemann-Roch equation (0.1) and show that this can be described in terms of purely topological properties of the surface X and the bundle $L = L_D$. To this end we also define the first Chern classes for the line bundles L_D over X. However, the formulation we obtain for the right hand side of the equation (0.1) is not yet the formulation for the topological index, ind_t, of the Atiyah-Singer index formula or the T-characteristic of the Hirzebruch-Riemann-Roch theorem.

Chapter 5 provides the first step in this further path of generalisation. We show how the Chern classes defined in the previous chapter as topological quantities of holomorphic line bundles over Riemann surfaces, can be generalised to properties of rank r holomorphic bundles over compact complex manifolds of higher dimension n. We then define a number of topological objects on vector bundles which are needed in the description of the T-characteristic and the topological index. Most proofs will be omitted from these expository sections. This information leaves us in a position to show that the right hand side of the Riemann-Roch equality (0.1) is a special case of the T-characteristic of a holomorphic bundle over a compact complex manifold. We will therefore have shown that the classiscal Riemann-Roch theorem is a special case of the Hirzebruch-Riemann-Roch theorem.

In the final chapter 6 it remains to show how, in the case of a holomorphic line bundle L over a compact complex Riemann surface X, the T-characteristic of L is equal to the topological index of L. In doing so we complete the proof that the classical Riemann-Roch theorem is a special case of the Atiyah-Singer index formula.

Unfortunately, there is not space in this paper to show the more general result that the Hirzebruch-Riemann-Roch theorem for higher dimensions is implied by the Atiyah-Singer index formula. However, in the appendix we shall briefly describe some steps that are necessary for doing this.

CHAPTER 1

Review of Basic Material

This chapter serves to review the some of the basic concepts and to establish the notation that we will be using in the rest of the paper. Most proofs of the results will not be included. The books [**We**], [**Fo**], [**Hi**] are excellent sources for this material.

Throughout the paper we will assume that the base manifold X is paracompact and connected.

1. Vector bundles

1.1. Vector bundles, trivialisations, frames and forms. Familiarity with vector bundles is assumed in this paper. The purpose of this section is not to introduce new material but to establish the notation and conventions for the rest of the paper.

In the following, the field K can be $\mathbb{R}$ or $\mathbb{C}$. Let U be an open subset of K^n. We will use the following notation:

- $\mathcal{C}(U)$ refers to the collection of K-valued continuous functions on U.
- $\mathcal{E}(U)$ refers to the collection of K-valued differentiable functions on U.
- $\mathcal{O}(U)$ refers to the collection of $\mathbb{C}$-valued holomorphic functions on U.

In general we will refer to $\mathcal{S}$- functions and $\mathcal{S}$-structures where $\mathcal{S} = \mathcal{C}, \mathcal{E}, \mathcal{O}$.

In this paper we will be dealing with manifolds with real differentiable and complex analytic (holomorphic) structures. That is, manifolds such that the transition (change of chart) functions are real differentiable or holomorphic. We will call these $\mathcal{E}$-, and $\mathcal{O}$- manifolds respectively.

Definition 1.1. *Let E, X be Hausdorff spaces and $\pi : E \to X$ be a continuous surjection. $\pi : E \to X$ is called a K-* vector bundle of rank *r over the* base space *X with* total space *E if*

(1) *There exists an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X and, for all $i \in I$, there exists a homeomorphism $\varphi_i : \pi^{-1}(U_i) \to U_i \times K^r$ such that*

$$\varphi_i(E_x) = \{x\} \times K^r, \text{ for all } x \in U_i$$

where $E_x := \pi^{-1}(x)$ is the fibre *of E over x.*

For $x \in U_i$, (U_i, φ_i) is called a local trivialisation *of $\pi : E \to X$ at x. A* local trivialisation of E over X *is a collection $\{(U_i, \varphi_i)_{i\in I}\}$.*

(2) *For all $i, j \in I$ we define the* transition function *$g_{i,j} := \varphi_i \circ \varphi_j^{-1}|_{(U_i\cap U_j)\times K^r}$. Then, for all $x \in U_i \cap U_j$, the map*

$$K^r \cong \{x\} \times K^r \xrightarrow{g_{i,j}} \{x\} \times K^r \cong K^r$$

is a linear isomorphism.

We usually simply say that E is a vector bundle over X and $\operatorname{rk} E = r$.

REMARK 1.2. For $x \in U_i$, identifying the fibre E_x with $K^r \cong \{x\} \times K^r$ via φ_i gives E_x the structure of an r-dimensional K-vector space. By (2), this is independent of the choice of $i \in I$ with $x \in U_i$.

DEFINITION 1.3. *For $\mathcal{S} = \mathcal{E}, \mathcal{O}$, a vector bundle E over X is an $\mathcal{S}$-bundle if E and X are $\mathcal{S}$-manifolds, $\pi : E \to X$ is an $\mathcal{S}$-morphism, and the local trivialisations are $\mathcal{S}$- isomorphisms. It is easily seen that this last condition is equivalent to the transition functions being $\mathcal{S}$-morphisms.*

REMARK 1.4. Note that the definitions imply that, if $\pi : E \to X$ is an $\mathcal{S}$- bundle over X with local trivialisation $\{(U_i, \varphi_i)_i\}$ relative to some cover $\mathcal{U} = \{U_i\}_i$, then, if $\{\phi_i : U_i \to \mathbf{GL}(n, K)\}_i$ are $\mathcal{S}$-maps on U_i, $\{(U_i, \phi_i \cdot \varphi_i)\}_i$ is also a local trivialisation for E.

We calculate the transition functions $\{g'_{i,j}\}_{i,j}$ for E relative to $\{(U_i, \varphi'_i)\}_i$, in terms of the transition functions $\{g_{i,j}\}_{i,j}$ for E relative to $\{(U_i, \varphi_i)\}_i$:

By definition

$$g_{i,j} = \varphi_i \circ \varphi_j^{-1}, \quad \text{on } U_i \cap U_j$$

so

$$g'_{i,j} = \varphi'_i \circ \varphi_j^{'-1} = (\phi_i \circ \varphi_i) \circ (\varphi_j^{'-1} \circ \phi_j^{-1}) = \phi_i g_{i,j} \phi_j^{-1}, \text{ on } U_i \cap U_j.$$

EXAMPLE 1.5. The complex projective line $\mathbb{CP}^1$ is a compact Riemann surface. A point in $\mathbb{CP}^1$ can be specified in homogeneous coordinates $[z_0 : z_1]$ where $z_o, z_1 \in \mathbb{C}$ and z_0 and z_1 are not both zero. Then

$$[z_0 : z_1] = [z'_0 : z'_1] \quad \text{if} \quad [z'_0 : z'_1] = [\lambda z_0 : \lambda z_1],\ \lambda \in \mathbb{C}^*.$$

($\mathbb{C}^*$ denotes the non-zero complex numbers.)

We view $\mathbb{CP}^1$ as the space of complex lines l in $\mathbb{C}^2$ which go through the origin and define $\mathcal{O}_{\mathbb{CP}^1}(-1)$ as the submanifold of $\mathbb{CP}^1 \times \mathbb{C}^2$ given by

$$\mathcal{O}_{\mathbb{CP}^1}(-1) = \{(l, p) : p \in l\} = \{([z_0 : z_1], (\lambda z_0, \lambda z_1)) : \lambda \in \mathbb{C}\} \subset \mathbb{CP}^1 \times \mathbb{C}^2.$$

Now, $\mathbb{CP}^1 = U_0 \cup U_1$ where, for $i = 0, 1$, U_i is the open set given by

$$U_i := \{[z_0 : z_1] \in \mathbb{CP}^1 : z_i \neq 0\}.$$

We wish to show that $\pi : \mathcal{O}_{\mathbb{CP}^1}(-1) \to X$ (where $\pi(l, p) = l$) is a holomorphic line bundle over $\mathbb{CP}^1$:

Local trivialisations $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C},\ i = 0, 1$ are given by

$$\varphi_0 : ([1 : z], (\lambda, \lambda z)) \mapsto ([1 : z], \lambda)$$

and

$$\varphi_1 : ([w : 1], (\mu w, \mu)) \mapsto ([w : 1], \mu)\,.$$

So, on $U_0 \cap U_1$, $[w : 1] = [1 : z]$ and therefore $w = \frac{1}{z}$.

We calculate the transition functions relative to U_0 and U_1.

$$g_{0,1} = \varphi_0 \circ \varphi_1^{-1}|_{(U_0 \cap U_1) \times \mathbb{C}} : ([1 : z]), \mu) \mapsto \Big([1 : z], (\frac{\mu}{z}, \mu)\Big) \mapsto \Big([1 : z], \frac{\mu}{z}\Big)\,.$$

Since z is non-zero on $U_0 \cap U_1$, $g_{0,1}|_{(U_0 \cap U_1) \times \mathbb{C})}$ is clearly holomorphic. As a map, $g_{0,1} : U_0 \cap U_1 \to \mathbf{GL}(1, \mathbb{C}) = \mathbb{C}^*$,

$$g_{0,1}([z_0 : z_1]) = \frac{z_0}{z_1} \quad (\text{ so } g_{0,1}([1 : z]) = \frac{1}{z}).$$

It is easy to check that $g_{1,0} = g_{0,1}^{-1} : ([w:1]),\lambda) \mapsto \big([w:1], \frac{\lambda}{w}\big)$, and so

$$g_{1,0}([z_0 : z_1]) = \frac{z_1}{z_0} : U_0 \cap U_1 \to \mathbf{GL}(1,\mathbb{C}).$$

Definition 1.6. *Let E and F be K-vector bundles over X. A map*

$$f : E \to F$$

is a vector bundle homomorphism *if it preserves fibres and $f_x = f|_{E_x}$ is a K-linear map for every $x \in X$. Two $\mathcal{S}$- bundles are* isomorphic *if there is an $\mathcal{S}$-isomorphism*

$$f : E \to F$$

which is a K vector space isomorphism on the fibres of E.

Proposition 1.7. *For $\mathcal{S} = \mathcal{E}, \mathcal{O}$, given a covering $\{U_i\}_i$ of a manifold X and non-vanishing $\mathcal{S}$-functions $g_{i,j} : U_i \cap U_j \to \mathbf{GL}(r,K)$ such that for all i,j,k and for all $x \in U_i \cap U_j \cap U_k$,*

$$g_{i,j}(x) g_{j,k}(x) = g_{i,k}(x)$$

we can construct an $\mathcal{S}$-bundle $\pi : E \to X$ which has transtion functions $\{g_{i,j}\}_{i,j}$ with respect to the covering $\{U_i\}_i$. The bundle E is unique up to isomorphism.

Proof. For an outline of this construction see [**We**, 13-14] .

□

Definition 1.8. *A (local)* section *of a vector bundle $\pi : E \to X$ is a map from X (or an open subset U of X) to E such that $\pi \circ s = \mathrm{id}_X$ (resp. id_U). We denote the $\mathcal{S}$-sections of E over X by $\mathcal{S}(E) := \mathcal{S}(X,E)$. The collection of S-sections of E over an open subspace $U \subset X$ is denoted by $\mathcal{S}(U,E)$. The $\mathcal{S}$-sections of a vector bundle E, defined by $\{U_i\}_i$ and $\{g_{i,j}\}_{i,j}$ are given by $\mathcal{S}$-functions $f_i : U_i \to K^r$ such that*

$$f_i = g_{i,j} f_j, \quad \text{on } U_i \cap U_j.$$

When E is the trivial line bundle $X \times \mathbb{C}$, we write $\mathcal{S} := \mathcal{S}(X \times \mathbb{C})$.

Finally, a meromorphic section *f of a holomorphic line bundle L over a holomorphic manifold X is, relative to a trivialisation $\{U_i\}_i$, a collection of meromorphic functions $f_i : U_i \to \mathbb{C}$ such that*

$$f_i = g_{i,j} f_j, \quad \text{on } U_i \cap U_j.$$

The space of meromorphic sections of a line bunle $L \to X$ is denoted by $\mathcal{M}(L)$.

Definition 1.9. *A* frame *at $x \in X$ for a bundle $E \to X$ is an ordered basis for E_x.*

Since there is a locally trivialising neighbourhood U_x for E, it is clear that we can extend this and define a frame for E above U_x as an ordered set of sections $f = (f_i)_i$ of E over U_x such that, for each $y \in U_x$, $(f_i(y))_i$ is an ordered basis for E_y. A frame over U_x is an $\mathcal{S}$-frame if the sections are $\mathcal{S}$-sections

Remark 1.10. A frame for E on $U \subset X$ defines in a natural way a local trivialisation of $\pi : E \to X$ and vice versa.

Namely, let $f = (f_i)_i$ be a frame for E over U. We wish to construct a local trivialisation $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U \times K^r$. Given $e \in E_x,\ x \in X,\ e = \sum_{i=1}^r \lambda_i(x) f_i(x)$ where $\lambda_i : U \to K$ is an $\mathcal{S}$- function. We define

$$\varphi(e) = (\lambda_i(x), \ldots, \lambda_r(x)).$$

It is easily checked that this is an $\mathcal{S}$-isomorphism.

Conversely, given a trivialisation $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U \times K^r$, we can define an $\mathcal{S}$-frame $f = (f_i)_i$ over U by

$$f_i(x) := \varphi^{-1}(x, e_i)$$

with $(e_1, \ldots, e_r)$ an ordered basis for K^r.

Definition 1.11. *A* vector field V *on* X *is a continuous section of the tangent bundle* TX *of* X.

If E is a vector bundle over X, then $\wedge^p E$ denotes the bundle of *p-vectors* with coefficients in E. That is, for $x \in X$, the fibre $\wedge^p E_x$ of $\wedge^p E$ over x consists of K-linear combinations of elements of the form $v_1 \wedge \cdots \wedge v_p$ with $v_1, \ldots, v_p \in E_x$, where $\wedge$ denotes the exterior product in the exterior algebra $\bigwedge E_x$ of E_x.

For, $\mathcal{S} = \mathcal{C}, \mathcal{E}, \mathcal{O}$, let $\mathcal{S}^k(E)$ denote the $\mathcal{S}$- k-forms of X with coefficients in E. That is

$$\mathcal{S}^k(E) := \mathcal{S}(E \otimes \wedge^k T^*X)$$

where T^*X is the (real) cotangent bundle of X.

(When E, X are complex, $E \otimes \wedge^k T^*X := E \otimes_{\mathbb{C}} \wedge^k T^*X$.)

If X is a complex manifold with basis of local coordinates $(z_1, \ldots, z_n)$, then $(dz_1, \ldots, dz_n)$ is a local frame for $\mathbf{T}$, the *holomorphic cotangent bundle* of X. $\overline{\mathbf{T}}$ is defined as the bundle for which $(d\overline{z_1}, \ldots, d\overline{z_n})$ is a local frame.

We denote by $\mathcal{E}^{p,q}(E)$ the differentiable (p, q)-forms of X with coefficients in E. That is

$$\mathcal{E}^{p,q}(E) := \mathcal{E}(E \otimes \wedge^p \mathbf{T} \otimes \wedge^q \overline{\mathbf{T}}). \tag{1.1}$$

We have

$$\mathcal{E}^p(E) = \bigoplus_{q+r=p} \mathcal{E}^{q,r}(E). \tag{1.2}$$

When $E = X \times \mathbb{C}$, we will often write simply $\mathcal{E}^{q,r} := \mathcal{E}^{q,r}(X \times \mathbb{C})$.

1.2. Metrics on a vector bundle.

Definition 1.12. *Let E be a real differentiable vector bundle over a real differentiable manifold X. A* (bundle) metric *on E is an assignment of an inner product g_x on every fibre E_x such that such that for any open set $U \subset X$ and sections ξ, η of E over U, $g(\xi, \eta)$ is smooth on U.*

Using a trivialisation and a partition of unity, it is easy to see that

Proposition 1.13. *A vector bundle E over a paracompact differentiable manifold X admits a metric.*

Since all base spaces in this paper are paracompact, all bundles will be metrisable (admit a metric).

Definition 1.14. *If $E \to X$ is a complex vector bundle over a manifold X then a* Hermitian metric *on E is the assignment of a Hermitian inner product h_x on every fibre E_x such that for any open set $U \subset X$ and sections ξ, η of E over U, $h(\xi, \eta)$ is smooth on U.*

Given a Hermitian bundle (E, h) of rank r over X and a set of local frames $f = \{f^i\}_i$ where $f^i = (f^i_1, \dots, f^i_r)$ for E, we can define the function matrix

$$h(f^i) := (h_i(f^i_\beta, f^i_\alpha))_{\alpha,\beta}, \quad h(f^i_\beta, f^i_\alpha) : U_i \to \mathbf{GL}(r, \mathbb{C}). \tag{1.3}$$

Then

$$h(f^j) = (h_j(g_{j,i} f^i_\beta, g_{j,i} f^i_\alpha))_{\alpha,\beta} = \overline{g_{i,j}}^t h_i(f^i) g_{j,i}.$$

Remark 1.15. The above implies that for a line bundle L over a Riemann Surface X, defined in terms of a covering $\mathcal{U} = \{U_i\}_i$ and transition functions $\{g_{i,j}\}_{i,j}$, a Hermitian metric h on L is therefore entirely defined by a collection of positive functions $\lambda = \{\lambda_i : U_i \to \mathbb{R}^+\}$. Namely let f_i be a holomorphic frame for L over U_i. Then h_i is completely determined by $\lambda_i := h_i(f_i, f_i) > 0$ which is a continuous positive valued function on U_i.

So a Hermitian metric on L is uniquely determined by a collection of positive functions λ_i on U_i such that $\lambda_j = g_{i,j}\overline{g_{i,j}}\lambda_i$ on $U_i \cap U_j$.

Proposition 1.16. *If $E \to X$ is a smooth complex vector bundle over a complex manifold X, E admits a Hermitian metric.*

Proof. [**We**, 68]. □

Example 1.17. Let $\pi : \mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1$ be as in example 1.1.5. We wish to define a Hermitian metric h on $\mathbb{CP}^1$. If z is a local coordinate on $\mathbb{CP}^1$, the standard Hermitian metric on $\mathbb{CP}^1 \times \mathbb{C}^2 \to \mathbb{CP}^1$ is given by

$$|\,(l, (\alpha, \beta))\,|^2 = |\alpha|^2 + |\beta|^2, \ l \in \mathbb{CP}^1, \ \alpha, \beta \in \mathbb{C}.$$

Since $\mathcal{O}_{\mathbb{CP}^1}(-1) \subset \mathbb{CP}^1 \times \mathbb{C}^2$, we can take the restriction of this metric to $\mathcal{O}_{\mathbb{CP}^1}(-1)$. Then, if $\mathbb{CP}^1 = U_0 \cup U_1$ as in example 1.1.5, on U_0 we have

$$|\,([1 : z], (1, z))\,|^2 = 1 + |z|^2,$$

and on U_1 we have

$$|\,([w : 1], (w, 1))\,|^2 = 1 + |w|^2.$$

On $U_0 \cap U_1$, with $w = \frac{1}{z}$,

$$1 + |w|^2 = 1 + \left|\frac{1}{z}\right|^2 = \frac{1}{|z|^2}\left|1 + |z|^2\right| = g_{0,1}\overline{g_{0,1}}\left(1 + |z|^2\right).$$

So, the restriction of the standard metric on $\mathbb{CP}^1 \times \mathbb{C}^2$ is indeed a Hermitian metric for $\mathcal{O}_{\mathbb{CP}^1}(-1)$.

(In the notation of remark 1.1.15, we have $\lambda_0[1 : z] = 1 + |z|^2$ on U_0 and $\lambda_1[w : 1] = 1 + |w|^2$ on U_1.)

Remark 1.18. A Hermitian metric h on a complex bundle $E \to X$ induces a metric g on the underlying real vector bundle. Define

$$g := \operatorname{Re} h = \frac{1}{2}(h + \overline{h}).$$

Then g is positive definite, symmetric, bilinear and real valued.

Definition 1.19. *If $\pi : E \to X$ is a bundle, the* dual bundle $\pi^* : E^* \to X$ *is the bundle with fibres $E^*_x := (E_x)^*$ for all $x \in X$. A choice of metric g on E induces an isomorphism $E \to E^* : \xi \mapsto g(\xi, \cdot)$.*

Proposition 1.20. *If a vector bundle E on X has transition functions $g_{i,j} \in \mathcal{S}(U_i \cap U_j, \mathbf{GL}(n,K))$ with respect to a given covering, the dual bundle E^* has transtion functions $(g_{j,i})^{-1}$.*

Proof. This is a simple exercise in linear algebra. □

Proposition 1.21. *Every complex vector bundle E over a complex manifold X can be described by unitary transition functions.*

Proof. Let h be a Hermitian metric on E and $\{f^i\}_i$ a collection of frames for E. We can apply Gram-Schmidt orthonormalisation to each $h(f^i)$. The transition maps so obtained are then unitary. □

Proposition 1.22. *If $\pi : E \to X$ is a complex bundle with Hermitian metric h, then*

$$E \cong \overline{E}^*.$$

Proof. By the previous proposition, E can be described by unitary transition functions $g_{i,j}$ with respect to a given covering. We have seen, in proposition 1.1.20, that E^* has transition functions $(g_{j,i})^{-1}$, but since $g_{i,j}(x) \in \mathbf{U}(n)$ for all $x \in U_i \cap U_j$, $(g_{j,i})^{-1} = \overline{g_{i,j}}$.

In other words $E^* \cong \overline{E}$ and we are done. □

1.3. Complexification and almost complex structures. We wish to be able to move from complex vector bundles to the underlying real vector bundle and conversely to define (almost) complex structures on even dimensional real bundles.

The map $\psi : \mathbf{GL}(n,\mathbb{C}) \to \mathbf{GL}(2n,\mathbb{R})$ is the embedding obtained by regarding a linear map of $\mathbb{C}^q$ with coordinates $z_1, \ldots, z_q$ as a linear map of $\mathbb{R}^{2q}$ with coordinates $x_1, \ldots, x_{2q}$ where $z_k = x_{2k-1} + ix_{2k}$.

The map $\upsilon : \mathbf{GL}(n,\mathbb{R}) \to \mathbf{GL}(n,\mathbb{C})$ is the *complexification map*, that is the embedding obtained by regarding a matrix of real coefficients as a matrix of complex coefficients.

We have the following commutative diagrams of embeddings:

$$\begin{array}{ccc} \mathbf{U}(n) & \xrightarrow{\psi} & \mathbf{O}(2n) \\ \downarrow & & \downarrow \\ \mathbf{GL}(n,\mathbb{C}) & \xrightarrow[\psi]{} & \mathbf{GL}(2n,\mathbb{R}) \end{array} \tag{1.4}$$

$$\begin{array}{ccc} \mathbf{O}(n) & \xrightarrow{\upsilon} & \mathbf{U}(n) \\ \downarrow & & \downarrow \\ \mathbf{GL}(n,\mathbb{R}) & \xrightarrow[\upsilon]{} & \mathbf{GL}(n,\mathbb{C}) \end{array} \tag{1.5}$$

where in both diagrams the vertical arrows are simply inclusion.

If X is a compact complex manifold, we can extend the maps υ and ψ to maps of vector bundles over X.

LEMMA 1.23. *There is an automorphism* Φ *of* $\mathbf{U}(2q)$ *such that, for* $A \in \mathbf{U}(q)$, $\Phi(\upsilon \circ \psi(A)) \in \mathbf{U}(2q)$ *has the form*

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix},$$

up to a permutation of coordinates. Similarly, if $B \in \mathbf{O}(q)$, *then, up to a permutation of coordinates,* $\psi \circ \upsilon(B) \in \mathbf{O}(2q)$ *has the form*

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

PROOF. (N.B. In this proof, we will not consider the permutations of coordinates. However, this does become relevant when considering the orientation of the spaces.)

$$\psi(A) = \begin{pmatrix} \operatorname{Re} A & -\operatorname{Im} A \\ \operatorname{Im} A & \operatorname{Re} A \end{pmatrix} \in \mathbf{O}(2n).$$

For $M \in \mathbf{U}(2n)$, let

$$\Phi(M) = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} M \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

This is clearly an automorphism and it is easily checked that it is the desired map $\Phi : \mathbf{U}(2n) \to \mathbf{U}(2n)$.

We regard $B \in \mathbf{O}(n)$ as an element $B = \upsilon B$ of $\mathbf{U}(n)$. For $M \in \mathbf{U}(n)$,

$$\psi(M) = \begin{pmatrix} \operatorname{Re} M & -\operatorname{Im} M \\ \operatorname{Im} M & \operatorname{Re} M \end{pmatrix} \in \mathbf{O}(2n).$$

Since B is real

$$\psi \circ \upsilon(B) = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

□

PROPOSITION 1.24. *If* $E \to X$ *is a complex bundle described by unitary transition functions,* $(\psi \circ \upsilon)(E) \cong E \oplus \overline{E} \cong E \oplus E^*$.

Similarly, if $W \to X$ *is a real bundle,* $(\upsilon \circ \psi)(W) \cong W \oplus W$.

In this case the orientations of $(\upsilon \circ \psi)(W)$ *and* $W \oplus W$ *differ by a factor* $(-1)^{\frac{q}{2}(q-1)}$.

PROOF. By proposition 1.1.22, E is described by unitary transition functions so $E^* \cong \overline{E}$ and the isomorphism follows from the above lemma 1.1.23. Similarly for the second statement. As for the orientations, $(\upsilon \circ \psi)(W)$ is represented by transition matrices $g_{i,j} \in \mathbf{GL}(\mathbf{2q}, \mathbb{R})$ with coordinates $x_1, y_1, \ldots, x_q, y_q$ and the transition matrices of $W \oplus W$ have coordinates $x_1, \ldots x_q, y_1, \ldots, y_q$. □

DEFINITION 1.25. *Let* V *be a real* r*-dimensional vector space. The* complexification $V^{\mathbb{C}}$ *of* V *is given by*

$$V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}.$$

This is equivalent to $V^{\mathbb{C}} = V \oplus iV$ *and therefore there is a natural isomorphism of* $\mathbb{R}$*-vector spaces*

$$V^{\mathbb{C}} \cong V \oplus V.$$

$V^{\mathbb{C}}$ is a complex r-dimensional vector space with complex multiplication given by

$$\lambda(v \otimes \alpha) = v \otimes \lambda\alpha, \quad v \in V, \lambda, \alpha \in \mathbb{C}.$$

There is a canonical conjugation map on $V^{\mathbb{C}}$ defined by

$$\overline{v \otimes \alpha} = v \otimes \overline{\alpha}.$$

If $W \to X$ is a real vector bundle, the complexification *$W^{\mathbb{C}}$ of W over X is the bundle with fibres $(W_x)^{\mathbb{C}}$. If W is given by transtion functions $\{g_{i,j}\}_{i,j}$ with $g_{i,j} \in \mathbf{GL}(r, \mathbb{R})$, then $W^{\mathbb{C}}$ is given by the same transition functions $\{g_{i,j}\}_{i,j}$ but now with the functions $g_{i,j} (= \upsilon(g_{i,j}))$ regarded as elements of $\mathbf{GL}(r, \mathbb{C})$.*

Furthermore $W^{\mathbb{C}} \cong W \oplus W$, although the orientation differs by a factor $(-1)^{\frac{r}{2}(r-1)}$.

Definition 1.26. *Given a $2n$-dimensional real vector space V, there exists a linear map $J \in \mathrm{End}(V)$ such that $J^2 = \mathrm{id}_V$. Then J is called a* complex structure *for V. J gives V the structure of a complex vector space with complex scalar multiplication defined by*

$$(a + ib)v = av + bJv, \quad a, b \in \mathbb{R},\ v \in V.$$

J can be extended to $V^{\mathbb{C}}$ by $J(v \otimes \alpha) = Jv \otimes \alpha$.

Definition 1.27. *An* almost complex structure *θ on a smooth $2n$-dimensional real manifold X is a complex structure on each fibre T_xX of the tangent space TX of X which varies smoothly with $x \in X$.*

Equivalently, given a trivialisation for the tangent bundle of X with transition functions $\{g_{i,j}\}_{i,j}$ with $g_{i,j} \in \mathbf{GL}(2n, \mathbb{R})$, an almost complex structure for X is a bundle E over X with transition functions $\{t_{i,j}\}_{i,j}$, $t_{i,j} \in \mathbf{GL}(n, \mathbb{C})$ relative to the same trivialisation and such that $\psi(t_{i,j}) = g_{i,j}$ (for all i, j).

In particular, if X is a complex manifold then the complex tangent bundle $\mathcal{T} = \mathcal{T}(X)$ is an almost complex structure for X.

Henceforth we shall use the following notation: If X is a complex manifold then

- TX denotes the real tangent bundle of X, and T^*X its dual, the real cotangent bundle of X.
- $\mathcal{T}$ denotes the complex (holomorphic) tangent bundle of X and $\mathbf{T}$ its dual, the complex cotangent bundle of X.

If $\mathcal{T}$ is given by transition functions $f = \{f_{i,j}\}_{i,j}$, we can define a bundle $\overline{\mathcal{T}}$ given by $\overline{f} = \{\overline{f_{i,j}}\}_{i,j}$. $\overline{\mathbf{T}}$ is the bundle dual to $\overline{\mathcal{T}}$.

The maps ψ and υ imply the following:

Proposition 1.28. *The following identities hold:*

$$TX^{\mathbb{C}} = \mathcal{T} \oplus \overline{\mathcal{T}}, \tag{1.6}$$

$$(TX^{\mathbb{C}})^* = T^*X^{\mathbb{C}} = \mathbf{T} \oplus \overline{\mathbf{T}}. \tag{1.7}$$

And the r-th exterior power of $T^*X^{\mathbb{C}}$, $\wedge^r T^*X^{\mathbb{C}}$ is given by

$$\wedge^r\, T^*X^{\mathbb{C}} = \bigoplus_{p+q=r} (\wedge^p \mathbf{T}) \wedge (\wedge^q \overline{\mathbf{T}}). \tag{1.8}$$

Proof. This follows directly from the maps υ and ψ and proposition 1.1.24. □

Corollary 1.29. *The isomorphism $T^*X^{\mathbb{C}} \cong T^*X \oplus iT^*X$, together with the projection $p : T^*X^{\mathbb{C}} \cong \mathbf{T} \oplus \overline{\mathbf{T}} \to \overline{\mathbf{T}}$ induces an isomorphism $T^*X \to \overline{\mathbf{T}}$.*

1.4. Connections.

Definition 1.30. *A* connection ∇ *on a differentiable K-vector bundle $E \to X$ with X paracompact, is a collection of K-linear maps*

$$\nabla_U : \mathcal{E}(U,E) \to \mathcal{E}^1(U,E),\ U \subset X \text{ open},$$

such that, if $U' \subset U$ is open and $\xi \in \mathcal{E}(U,E)$, then

$$(\nabla_U \xi)|_V = \nabla_V(\xi|_V)$$

and which satisfies the Leibniz formula

$$\nabla_U(s\xi) = ds \otimes \xi + s\nabla_U(\xi)$$

for any $s \in \mathcal{E}$ and any $\xi \in \mathcal{E}(U,E)$.

If $E \to X$ is a complex bundle, a connection ∇ on E can be written as

$$\nabla = \nabla^{1,0} + \nabla^{0,1}, \quad \text{with } \nabla^{1,0} : \mathcal{E}(E) \to \mathcal{E}^{1,0}(E),\ \nabla^{0,1} : \mathcal{E}(E) \to \mathcal{E}^{0,1}(E).$$

Essentially a connection provides a rule for 'differentiating' a section with respect to a vector field.

Definition 1.31. *If $f = (f_\alpha)_{\alpha=1}^r$ is a frame for E on an open set U, then we can define an $r \times r$ matrix $A = A(\nabla, f)$ of differentials on U, called the* connection matrix of ∇ with respect to f *such that*

$$A_{\beta,\alpha}(\nabla, f) \in \mathcal{E}^1(U), \quad \nabla f_\alpha = \sum_{\beta=1}^r A_{\beta,\alpha}(\nabla, f) \otimes f_\beta. \tag{1.9}$$

A differentiable section of ξ of E over U can be written as $\xi_i = \sum_i \lambda_i f_i$ where $\lambda_i \in \mathcal{E}(U,K)$. Let $\xi(f) := (\lambda_1, \ldots, \lambda_r)$. Then, by the defining properties of the connection ∇,

$$\nabla(\xi|_U) = \sum_{\alpha=1}^r \left(d\lambda_\alpha \otimes f_\alpha + \lambda_\alpha \sum_{\beta+1}^r A_{\beta,\alpha} \otimes f_\beta) \right) = A(\nabla, f)(\xi(f)) + d(\xi, f) \tag{1.10}$$

where $d(\xi, f) := \sum_\alpha (d\lambda_\alpha \otimes f_\alpha)$.

Proposition 1.32. *Every differentiable vector bundle over a paracompact manifold X admits a connection.*

Proof. [**We**, 67]. □

Example 1.33. Let (X, g) be a Riemannian manifold (g is a metric on the tangent bundle TX of X) with tangent bundle TX. The *Levi-Civita* connection ∇ on X is the unique connection on TX which satisfies:

(1) For vector fields V_1, V_2, V_3 on X

$$V_1(g(V_2, V_3)) = g(\nabla_{V_1}(V_2), V_3) + g(V_2, \nabla_{V_1}(V_3)).$$

It is then said that ∇ *preserves the metric.*

(2) For vector fields V_1, V_2 on X

$$\nabla_{V_1}(V_2) - \nabla_{V_2}(V_1) = [V_1, V_2]$$

where $[V_1, V_2]$ is the Lie bracket of TX. ∇ is then said to be *torsion free.*

The following proposition says that, with respect to a given Hermitian metric h on a holomorphic bundle $E \to X$, a unique special connection with very convenient properties called the *canonical connection* exists. If E is taken to be $\mathcal{T}$, the complex tangent bundle of X, this is a Hermitian analogue of the Levi-Civita connection.

PROPOSITION 1.34. *Let X be a complex manifold and E a holomorphic bundle over X with Hermitian metric h. There exists a unique connection $\nabla_{(E,h)}$ on E which satisfies.*

(1) *$\nabla_{(E,h)}$ is compatible with h. I.e.*

$$d(h(\xi,\eta)) = h(\nabla_{(E,h)}\xi,\eta) + h(\xi,\nabla_{(E,h)}\eta) \tag{1.11}$$

(2) *For every holomorphic section ξ of E over any $U \subset X$ open, it holds that*

$$\nabla^{0,1}_{(E,h)}\xi = 0 \tag{1.12}$$

In this case $\nabla_{(E,h)}$ is the so-called canonical connection.

PROOF. [**We**, 78-79]. □

2. Sheaves

2.1. Some definitions.

DEFINITION 2.1. *A* presheaf *$\mathcal{F}$ on a topological space X is an assignment of an Abelian group $\mathcal{F}(U)$ to every non-empty open $U \subset X$, together with a collection of* restriction homomorphisms *$\{\tau^U_V : \mathcal{F}(U) \to \mathcal{F}(V)\}_{V\subset U}$ for U, V open in X. The restriction homomorphisms satisfy:*

(i) *For every U open in X, $\tau^U_U = \mathrm{id}_U$ the identity on U.*
(ii) *For $W \subset V \subset U$ open in X, $\tau^U_W = \tau^V_W \tau^W_V$.*

If $\mathcal{F}$ is a presheaf, an element of $\mathcal{F}(U)$ is called a section *of the presheaf $\mathcal{F}$ over U.*

A subpresheaf *$\mathcal{G}$ of $\mathcal{F}$ is a presheaf on X such that for all U open in X, $\mathcal{G}(U) \subset \mathcal{F}(U)$ and, if $\{\rho^U_V\}_{V\subset U}$ are the restriction functions for $\mathcal{G}$, then $\rho^U_V = \tau^U_V|_{\mathcal{G}(U)}$.*

DEFINITION 2.2. *Given two presheaves $\mathcal{F}$ and $\mathcal{G}$, a sheaf morphism $h : \mathcal{F} \to \mathcal{G}$ is a collection of maps*

$$h_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

defined for each open set $U \subset X$ such that the h_U commute naturally with the restriction homomorphisms τ^U_V on $\mathcal{F}$ and ρ^U_V on $\mathcal{G}$. That is, if $V \subset U$, with U, V open in X then

$$\rho^U_V h_U = h_V \tau^U_V.$$

DEFINITION 2.3. *A* sheaf *is a presheaf $\mathcal{F}$ such that for every collection $\{U_i\}_i$ of open sets of X with $U = \bigcup_i U_i$, the following axioms are satisfied:*

(1) *For $s, t \in \mathcal{F}(U)$ such that $\tau^U_{U_i}(s) = \tau^U_{U_i}(t)$ for all i, it holds that $s = t$.*
(2) *Given $s_i \in \mathcal{F}(U_i)$ such that*

$$\tau^{U_i}_{U_i\cap U_j} s_i = \tau^{U_j}_{U_i\cap U_j} s_j,$$

there exists an $s \in \mathcal{F}(U)$ which satisfies $\tau^U_{U_i} s = s_i$ for all i.

A subsheaf *$\mathcal{G}$ of a sheaf $\mathcal{F}$ is a subpresheaf of $\mathcal{F}$ which satisfies the sheaf axioms 1.2.3 (1), and 12.3 (2).*

REMARK 2.4. If $\mathcal{F}$ is a sheaf over X then $\mathcal{F}(\emptyset)$ is the group consisting of exactly one element.

EXAMPLE 2.5. Given a topological space X, we note that, for $K = \mathbb{R}, \mathbb{C}$, and $U \subset X$ open, the space $C(U, K)$ of continuous K-valued functions on U is a K-algebra. So, we can define the presheaf $\mathcal{C}_X$ by $\mathcal{C}_X(U) = C(U, K)$. For $V \subset U$ open subsets of X, the restriction homomorphisms τ_V^U are given by $\tau_V^U(f) = f|_V$, $f \in C(U, K) = \mathcal{C}_X(U)$. It can easily be checked that this is a sheaf of K-algebras.

EXAMPLE 2.6. For $\mathcal{S} = \mathcal{E}, \mathcal{O}$, if X is an $\mathcal{S}$-manifold, then we can define the sheaf $\mathcal{S}_X$ by $\mathcal{S}_X(U) := \mathcal{S}(U, K)$. Then $\mathcal{S}_X \subset \mathcal{C}_X$ and $\mathcal{S}_X$ is called the *structure sheaf* of the manifold X.

DEFINITION 2.7. *Let $\mathcal{R}$ be a sheaf of commutative rings over X. Say $\mathcal{F}$ is a sheaf such that, for every U open in X, we have given $\mathcal{F}(U)$ the structure of a module over $\mathcal{R}(U)$ in a manner compatible to the sheaf structure, i.e., for $\alpha \in \mathcal{R}(U)$ and $f \in \mathcal{F}(U)$,*

$$\tau_V^U(\alpha f) = \rho_V^U(\alpha)\tau_V^U(f)$$

where $V \subset U$ open in X and τ_V^U, resp. ρ_V^U are the corresponding $\mathcal{F}$, resp. $\mathcal{R}$ restrictions. Then we call $\mathcal{F}$ a sheaf of $\mathcal{R}$-modules.

Now, for $p \geq 1$, we define the presheaf $\mathcal{R}^p$ by

$$U \to \mathcal{R}^p(U) := \underbrace{\mathcal{R}(U) \oplus \cdots \oplus \mathcal{R}(U)}_{p \text{ times}}, \quad (\rho^p)_V^U := \underbrace{\rho_V^U \oplus \cdots \oplus \rho_V^U}_{p \text{ times}}.$$

A sheaf $\mathcal{G}$ over X is called a locally free sheaf of $\mathcal{R}$-modules of rank p *if $\mathcal{G}$ is sheaf of $\mathcal{R}$- modules and, for each $x \in X$, there is a neighbourhood $U \ni x$, such that, for all open $U' \subset U$, $\mathcal{G}(U') \cong \mathcal{R}^p(U')$ as $\mathcal{R}$-modules.*

THEOREM 2.8. *Given a $\mathcal{S}$-manifold X there is a natural equivalence between the category of $\mathcal{S}$-vector bundles on X of dimension p and the category of locally free sheaves of $\mathcal{S}$-modules on X of finite rank p. So, given a vector bundle E on X, we can define uniquely the locally free sheaf of rank p, $\mathcal{S}(E)_X$ on X where $\mathcal{S}(E)_X(U) := \mathcal{S}(U, E)$.*

PROOF. For proof that there is a natural one-to-one correspondence, see [**We**, 40-41]. It is then easy to see that this correspondence induces an equivalence of categories. □

DEFINITION 2.9. *Let $\mathcal{F}$ be a sheaf over X. For, $x \in X$, we define an equivalence relation on the disjoint union $\coprod_{U \ni x} \mathcal{F}(U)$ where U runs over all open neighbourhoods $U \subset X$ of x:*

If $U, V \subset X$ are open neighbourhoods of x, we say that two elements $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are equivalent if there exists and open neighbourhood W of x with $W \subset U \cap V$ and $s|_W = t|_W$.

The set of equivalence classes is called the stalk *of $\mathcal{F}$ at x and is denoted by $\mathcal{F}_x$. In other words, $\mathcal{F}_x$ is the direct limit of the groups $\mathcal{F}(U)$ $(x \in U)$ with respect to the restriction homomorphisms $\tau_V^U, x \in V \subset U$, i.e.*

$$\mathcal{F}_x = \lim_{U \ni x} \mathcal{F}(U) = \left(\coprod_{U \ni x} \mathcal{F}(U) \right) \Big/ \sim.$$

If $\mathcal{F}$ is a sheaf of Abelian groups or commutative rings then $\mathcal{F}_x$ will also inherit that structure.

An element of the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at $x \in X$ is called a germ.

Definition 2.10. *A sheaf $\mathcal{F}$ over a paracompact Haussdorff space is called* fine *if given any locally finite open cover $\mathcal{U} = \{U_i\}_i$ of X, there exists a partition of unity on $\mathcal{F}$ subordinate to $\mathcal{U}$. That is, there exists a family of sheaf morphisms $\{\phi_i : \mathcal{F} \to \mathcal{F}\}_i$ such that*

(i) $\operatorname{supp}(\phi_i) \subset U_i$ *for all i,*
(ii) $\sum_i \phi_i = \operatorname{id}_{\mathcal{F}}$.

Example 2.11. If E is a differentiable vector bundle over a differential manifold X and $\mathcal{E}(E)_X$ is the sheaf associated to E via theorem 1.2.8, then $\mathcal{E}(E)_X$ is fine.

Namely for any locally finite open cover $\mathcal{U} = \{U_i\}$ of X, there exists a partition of unity $\{\phi_i\}$ on X subordinate to $\mathcal{U}$ where each ϕ_i is a globally defined differentiable function and therefore multiplication by ϕ_i of elements of $\mathcal{E}(E)_X$ gives a sheaf homomorphism which induces a partition of unity on $\mathcal{E}(E)_X$.

If $K \subset X$ is a closed subspace of X and $\mathcal{F}$ is a sheaf over X, we define $\mathcal{F}(K)$ as the direct limit of $\mathcal{F}(U)$ over all open $U \subset X$ such that $K \subset U$. That is

$$\mathcal{F}(K) := \lim_{U \supset K} \mathcal{F}(U).$$

Definition 2.12. *A sheaf $\mathcal{F}$ over a space X is called* soft *if for any closed subset $K \subset X$, the natural restriction map*

$$\mathcal{F}(X) \to \mathcal{F}(K)$$

is surjective. That is, any section over K of a soft sheaf $\mathcal{F}$ can be extended to a global section of $\mathcal{F}$.

Proposition 2.13. *Fine sheaves are soft. In particular the sheaf $\mathcal{E}(E)$ associated to a vector bundle $E \to X$ by theorem 1.2.8 is soft.*

Example 2.14. Below are some commonly occurring examples of sheaves:

- *Constant sheaves*
  If $\mathcal{F}$ is a sheaf such that $\mathcal{F}(U) = G$ for some Abelian group G and for every non-zero connected open set $U \subset X$, then $\mathcal{F}$ is a *constant sheaf.* Examples are the sheaves $\mathcal{F} = \mathbb{Z}_X, \mathbb{R}_X, \mathbb{C}_X$ given by $\mathcal{F}(U) = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ respectively (so the restriction functions on $\mathcal{F}(U)$ are simply the identity on $\mathcal{F}(U)$). Constant sheaves on a manifold of dimension greater than zero are not soft and therefore also not fine. See [**We**, 53].
- *Sheaves of functions and forms*
  We have seen in example 1.2.11 that, if $E \to X$ is a vector bundle, the sheaf $\mathcal{E}(E)_X$ is fine. Similarly we can show that $\mathcal{C}(E)_X$ is fine for a paracompact differential manifold X and $\mathcal{E}^{p,q}(E)_X$ is fine for a paracompact complex manifold X.

  The sheaf $\mathcal{O}(E)_X$ of locally holomorphic sections of a complex bundle $E \to X$ is, in general, not soft and therefore also not fine. The same applies to the sheaf $\mathcal{O}^*(E)_X$ of nowhere vanishing locally holomorphic sections of E.

  In particular, if E is the trivial bundle $\mathbf{1} := ((\mathbb{C} \times X) \to X)$, the sheaves $\mathcal{C}_X(:= \mathcal{C}(\mathbf{1})_X)$, $\mathcal{E}_X$ and $\mathcal{E}_X^{p,q}$ are fine and, if X is a manifold of dimension at least 1, $\mathcal{O}_X, \mathcal{O}_X^*$ are neither soft nor fine.

2.2. Cohomology groups. Most of the proofs of the results in this section can be found in e.g. [**Fo**]. However, it is worthwhile to note that in some cases the results and definitions are given here in a more general form than in [**Fo**].

DEFINITION 2.15. *Let $\mathcal{F}$ be a sheaf on a topological space X and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. For $q = 0, 1, 2, \ldots,$ a q-* cochain *is an element of the q-th* cochain group *of $\mathcal{F}$, $C^q(\mathcal{U}, \mathcal{F})$, defined by*

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(i_0,\ldots,i_q)\in I^{q+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q})$$

(where I^{q+1} is the direct product of $q+1$ copies of I). The group operation on $C^q(\mathcal{U}, \mathcal{F})$ is componentwise addition.

DEFINITION 2.16. *For $q = 0, 1, \ldots,$ the coboundary operators*

$$\delta_q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$$

are defined by

$$\{\delta_q(f)\}_{i_0,\ldots,i_q,i_{q+1}} = \{g_{i_0,\ldots,i_q,i_{q+1}}\}_{i_0,\ldots,i_q,i_{q+1}}$$

where

$$g_{i_0,\ldots,i_q,i_{q+1}} = \sum_{k=0}^{q+1} (-1)^k f_{i_0,\ldots,\widehat{i_k},\ldots,i_q,i_{q+1}} \quad \text{on} \quad \bigcap_{k=0,\ldots,q+1} U_{i_k}.$$

(Here $f_{i_0,\ldots,\widehat{i_k},\ldots,i_q,i_{q+1}} := f_{i_0,\ldots,i_{k-1},i_{k+1},\ldots,i_q,i_{q+1}}$.)

So, in particular $\delta_0(f)_{i,j} = f_j - f_i$ on $U_i \cap U_j$ and $\delta_1(g)_{i,j,k} = g_{j,k} - g_{i,k} + g_{i,j}$ on $U_i \cap U_j \cap U_k$.

Where there is no possibility of confusion, δ_q will be referred to simply as δ.

It is easily checked that the coboundary operators are group homomorphisms.

DEFINITION 2.17. *Let*

$$Z^q(\mathcal{U}, \mathcal{F}) := \operatorname{Ker}(\delta_q)$$

and

$$B^q(\mathcal{U}, \mathcal{F}) := \operatorname{Im}(\delta_{q-1}).$$

*The elements of $Z^q(\mathcal{U}, \mathcal{F})$ are called q-*cocycles *and the elements of $B^q(\mathcal{U}, \mathcal{F})$ are called q-*coboundaries*.*

LEMMA 2.18. *For $q = 0, 1, \ldots,$ $B^q \subset Z^q$.*

PROOF. This follows immediately from the definitions. [**We**, 63]. □

DEFINITION 2.19. *For $q = 0, 1, \ldots$ the q-*th cohomology group *$H^q(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined by*

$$H^q(\mathcal{U}, \mathcal{F}) := {Z^q(\mathcal{U}, \mathcal{F})}/{B^q(\mathcal{U}, \mathcal{F})}.$$

DEFINITION 2.20. *Given two coverings $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_k\}_{k\in K}$ of X, $\mathcal{V}$ is called* finer *than $\mathcal{U}$, written $\mathcal{V} < \mathcal{U}$, if, for every $k \in K$, there exists an $i \in I$ such that $V_k \subset U_i$. In other words, there exists a* refining map *$\tau : K \to I$ such that $V_k \subset U_{\tau(k)}$ for all $k \in K$.*

Given a sheaf $\mathcal{F}$ on X and covers $\mathcal{V} < \mathcal{U}$ of X, the refining map τ enables us to construct a homomorphism $t_{\mathcal{V}}^{\mathcal{U}} : Z^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F})$ given by

$$t_{\mathcal{V}}^{\mathcal{U}} : \{f_{i_0,\dots,i_q}\}_{i_0,\dots,i_q} \mapsto f_{\tau(k_0),\dots,\tau(k_q)}\}_{i_0,\dots,k_q}.$$

We note that $t_{\mathcal{V}}^{\mathcal{U}}(B^q(\mathcal{U}, \mathcal{F})) \subset B^q(\mathcal{V}, \mathcal{F})$ for all q, so $t_{\mathcal{V}}^{\mathcal{U}}$ defines a homomorphism of cohomology groups

$$t_{\mathcal{V}}^{\mathcal{U}} : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F}).$$

Lemma 2.21. *The map $t_{\mathcal{V}}^{\mathcal{U}} : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F})$ is independent of the choice of refining map $\tau : K \to I$.*

Proof. [**Fo**, 98]. □

Lemma 2.22. *$t_{\mathcal{V}}^{\mathcal{U}} : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F})$ is injective.*

Proof. [**Fo**, 99]. □

Given three open coverings $\mathcal{W} < \mathcal{V} < \mathcal{U}$, the above implies that

$$t_{\mathcal{W}}^{\mathcal{V}} t_{\mathcal{V}}^{\mathcal{U}} = t_{\mathcal{W}}^{\mathcal{U}}.$$

Therefore, we can define an equivalence relation ($\sim$) on the disjoint union $\coprod H^q(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ runs over all open coverings of X, by $\xi \sim \eta$ for $\xi \in H^q(\mathcal{U}, \mathcal{F})$ and $\eta \in H^q(\mathcal{V}, \mathcal{F})$ if there is a covering $\mathcal{W} < \mathcal{U}, \mathcal{W} < \mathcal{V}$ such that $t_{\mathcal{W}}^{\mathcal{U}}\xi = t_{\mathcal{W}}^{\mathcal{V}}\eta$.

Definition 2.23. *The q-th* cohomology group *of X with coefficients in $\mathcal{F}$ is defined as the set of all the equivalence classes of $H^q(\mathcal{U}, \mathcal{F})$ running over all open coverings $\mathcal{U}$ of X. That is, $H^q(\mathcal{U}, \mathcal{F})$ is the direct limit of the cohomology groups $H^q(\mathcal{U}, \mathcal{F})$ over all open coverings $\mathcal{U}$ of X.*

$$H^q(\mathcal{U}, \mathcal{F}) := \lim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F}) = \left(\coprod H^q(\mathcal{U}, \mathcal{F})\right)\Big/ \sim .$$

Proposition 2.24. *Let $\mathcal{F}$ be a sheaf over X. For any covering $\mathcal{U} = \{U_i\}_i$ of open subsets of X,*

$$H^0(X, \mathcal{F}) \cong H^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X).$$

Proof. [**Fo**, 103]. □

Proposition 2.25. *If $\mathcal{F} \subset \mathcal{G}$ are sheaves then there is a well-defined natural homomorphism $\Theta : H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$,$i \geq 0$ induced by the inclusions $Z^i(\mathcal{U}, \mathcal{F}) \subset Z^i(\mathcal{U}, \mathcal{G})$ and $C^{i-1}(\mathcal{U}, \mathcal{F}) \subset C^{i-1}(\mathcal{U}, \mathcal{G})$ relative to an open cover $\mathcal{U} = \{U_i\}_i$ for X*

Proof. Since $\mathcal{F} \subset \mathcal{G}$, an element $\alpha \in Z^i(\mathcal{U}, \mathcal{F})$ is in $Z^i(\mathcal{U}, \mathcal{G})$ and can therefore be mapped onto the corresponding cohomology class in $H^i(X, \mathcal{G})$.

Now, let α, α' be representatives of the same class in $H^i(X, \mathcal{F})$. Then there is a cover $\mathcal{U}$ for X such that $\alpha - \alpha' = \delta_{i-1}(\beta)$ for all i, j, k and some $\beta \in C^{i-1}(\mathcal{U}, \mathcal{F})$. But, $C^{i-1}(\mathcal{U}, \mathcal{F}) \subset C^{i-1}(\mathcal{U}, \mathcal{G})$ so, α, α' are mapped onto the same element in $H^i(X, \mathcal{G})$.

That this is a homomorphism follows directly from the definition and the algebraic structure on $\mathcal{F}$ and $\mathcal{G}$.

□

REMARK 2.26. Occasionally we use the subscript $(\cdot)_X$ to distinguish a sheaf F_X over X with $F_X(U) = F(U)$ from $F = F_X(X)$, the global sections $F_X(X)$ of F_X. Examples are the constant sheaves, $\mathbb{Z}_X$, $\mathbb{R}_X$ and $\mathbb{C}_X$ and the sheaves of functions $\mathcal{C}_X$, $\mathcal{E}_X$ and $\mathcal{O}_X$. We will drop the subscript when referring to the associated cohomology groups $H^k(X,F) := H^k(X,F_X)$ (and similarly when referring to $C^K(\mathcal{U},F), Z^k(\mathcal{U},F)$ and $B^k(\mathcal{U},F)$) since there is no possibility of confusion. For example, we will write $H^k(X,\mathbb{R})$ rather than $H^k(X,\mathbb{R}_X)$.

DEFINITION 2.27. *A sequence*

$$\cdots \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \ldots$$

where $\alpha, \beta, \ldots$ are sheaf morphisms, is called exact *if, for every $x \in X$, the corresponding sequence of stalks and restriction maps*

$$\cdots \to \mathcal{F}_x \xrightarrow{\alpha|_{\mathcal{F}_x}} \mathcal{G}_x \xrightarrow{\beta|_{\mathcal{G}_x}} \ldots$$

is exact.

It is not necessarily the case that

$$\cdots \to \mathcal{F}(U) \xrightarrow{\alpha} \mathcal{G}(U) \xrightarrow{\beta} \ldots$$

is exact for every U open in X. However, the following does hold:

PROPOSITION 2.28. *If*

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is an exact sequence of sheaves then,

$$0 \to \mathcal{F}(U) \xrightarrow{\alpha} \mathcal{G}(U) \xrightarrow{\beta} \mathcal{H}(U)$$

is exact for every U open in X.

PROOF. [**Fo**, 121]. □

EXAMPLE 2.29. *The Dolbeault sequence*
Let X be a Riemann surface. As usual, $\mathcal{E}_X^{0,1}$ is the sheaf of local differentiable (0,1)-forms on X. If the Dolbeault operator $\overline{\partial}$ denotes the antiholomorphic component of the exterior derivative, then the Dolbeault sequence

$$0 \to \mathcal{O}_X \hookrightarrow \mathcal{E}_X \xrightarrow{\overline{\partial}} \mathcal{E}_X^{0,1} \to 0 \tag{2.1}$$

where $\hookrightarrow$ denotes inclusion, is a short exact sequence of sheaves. This follows from the *Dolbeault lemma* [**Fo**, 105], which says that every differentiable function g on X is locally of the form $g = \frac{\partial f}{\partial \bar{z}}$ for some differentiable function f on X.

THEOREM 2.30. *If*

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$$

is an exact sequence of sheaves over a paracompact Hausdorff space X, then, for $q = 1, 2, \ldots$, there exists a connecting homomorphism

$$\delta^* := \delta_q^* : H^{q-1}(X,\mathcal{H}) \to H^q(X,\mathcal{F})$$

so that

$$\cdots \to H^{q-1}(X,\mathcal{G}) \to H^{q-1}(X,\mathcal{H}) \xrightarrow{\delta^*} H^q(X,\mathcal{F}) \to H^q(X,G) \to \ldots \tag{2.2}$$

is an exact sequence.

Proof. [**We**, 56-58].

□

Theorem 2.31. *If $\mathcal{F}$ is a soft sheaf over X, then, for $q = 1, 2, \ldots$, the cohomology groups $H^q(X, \mathcal{F})$ vanish. In particular, if $E \to X$ is a $\mathcal{S}$-vector bundle with $\mathcal{S} = \mathcal{C}, \mathcal{E}$, $H^q(X, E) := H^q(X, \mathcal{S}(E))$ vanishes for $q \geq 1$.*

Proof. [**We**, 56-57], [**Hi**, 34]. □

Definition 2.32. *Let*

$$0 \to \mathcal{F} \xrightarrow{h} \mathcal{F}^0 \xrightarrow{h^0} \mathcal{F}^1 \xrightarrow{h^1} \mathcal{F}^2 \xrightarrow{h^2} \ldots \xrightarrow{h^{p-1}} \mathcal{F}^p \xrightarrow{h^p} \ldots \tag{2.3}$$

be an exact sequence of sheaves over a compact space X. If $H^q(X, \mathcal{F}^p) \cong 0$ for $p \geq 0$ and $q \geq 1$, then 1.(2.3) is called a resolution *of $\mathcal{F}$. In particular, by theorem 1.2.31 , this is the case if $\mathcal{F}^p$ is fine for all $p \geq 0$. In this case, 1.(2.3) is called a* fine resolution *of $\mathcal{F}$.*

Example 2.33. The sequence 1.(2.1) in example 1.2.29 is a fine resolution for $\mathcal{O}_X$.

Theorem 2.34. *Let*

$$0 \to \mathcal{F} \xrightarrow{h} \mathcal{F}^0 \xrightarrow{h^0} \mathcal{F}^1 \xrightarrow{h^1} \mathcal{F}^2 \xrightarrow{h^2} \ldots \xrightarrow{h^{p-1}} \mathcal{F}^p \xrightarrow{h^p} \ldots \tag{2.4}$$

be a resolution of a sheaf over a compact manifold X. This defines naturally a sequence

$$0 \to \mathcal{F}(X) \xrightarrow{h_*} \mathcal{F}^0(X) \xrightarrow{h^0_*} \mathcal{F}^1(X) \xrightarrow{h^1_*} \ldots \xrightarrow{h^{p-1}_*} \mathcal{F}^p(X) \xrightarrow{h^p_*} \ldots . \tag{2.5}$$

There are natural isomorphisms

$$H^q(X, \mathcal{F}) \cong {}^{\mathrm{Ker}\,(h^q_*)}\!\big/_{\mathrm{Im}\,(h^{q-1}_*)}, \quad q \geq 1$$

and

$$H^0(X, \mathcal{F}) \cong \mathrm{Ker}\,(h^0_*).$$

Proof. By proposition 1.2.28

$$0 \to \mathcal{F}(X) \xrightarrow{h_*} \mathcal{F}^0(X) \xrightarrow{h^0_*} \mathcal{F}^1(X)$$

is exact so $\mathcal{F}(X) = H^0(X, \mathcal{F}) = \mathrm{Ker}\,(h^0_*)$ as required.

Now let $\mathcal{K}^p$ denote the kernel of $h^p : \mathcal{F}^p \to \mathcal{F}^{p+1}$. Then, for all p,

$$0 \to \mathcal{K}^p \hookrightarrow \mathcal{F}^p \to \mathcal{K}^{p+1} \to 0 \tag{2.6}$$

is a short exact sequence of sheaves on X.

Then, for $p \geq 0, q \geq 2$,

$$\cdots \to H^{q-1}(X, \mathcal{F}^p) \to H^{q-1}(X, \mathcal{K}^{p+1}) \to H^q(X, \mathcal{K}^p) \to H^q(X, \mathcal{F}^p) \ldots \tag{2.7}$$

is exact by theorem 1.2.30, and since $H^q(X, \mathcal{F}^p) = 0$ for $q \geq 1, p \geq 0$, it follows that

$$H^{q-1}(X, \mathcal{K}^{p+1}) \cong H^q(X, \mathcal{K}^p). \tag{2.8}$$

Letting $p = q - 1$, we obtain

$$H^q(X, \mathcal{F}) = H^1(X, \mathcal{K}^{q-1}),\ q \geq 1.$$

by repeated application of equation (2.8). For $q = 1$ and letting $p = 0$, $\mathcal{F} = \mathrm{Ker}\, h^0$, so$H^1(X, \mathcal{F}) = H^1(X, \mathcal{K}^0)$.

Since

$$0 \to \mathcal{K}^{q-1} \hookrightarrow \mathcal{F}^{q-1} \to \mathcal{K}^q \to 0$$

is a short exact sequence, the sequence

$$\cdots \to H^0(X, \mathcal{F}^{q-1}) \to H^0(X, \mathcal{K}^q) \to H^1(X, \mathcal{K}^{q-1}) \to (H^1(X, \mathcal{F}^q) = 0)$$

is exact for $q \geq 1$ by theorem 1.2.30. Rewriting gives

$$\cdots \to \mathcal{F}^{q-1}(X) \xrightarrow{h^*_{q-1}} \mathcal{K}^q(X) \xrightarrow{h^*_q|_{\mathcal{K}^q}} \to H^q(X, \mathcal{F}) \to 0, \ q \geq 1.$$

In other words, for $q \geq 1$,

$$H^q(X, \mathcal{F}) \cong {}^{\operatorname{Ker}(h^q_*)}\big/_{\operatorname{Im}(h^{q-1}_*)}$$

as required.

□

CHAPTER 2

The Analytic Index of an Elliptic Complex

Most of the material in this chapter can be found in [**We**].

1. Elliptic differential operators

Let K denote the field $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 1.1. *A* linear differential operator P *acting on differentiable functions defined on an open set* $U \subset \mathbb{R}^n$ *is an operator of the form*

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{1.1}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i = 0, 1, \dots$ *is a multi-index* ($|\alpha| := \sum_i \alpha_i$) *and* $D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1 \dots n$. *For each* α, a_α *is a differentiable function on* U.

Similarly, a linear differential operator P *acting on differentiable functions defined on an open set* $U \subset \mathbb{C}^n$ *has the form*

$$P(z, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{1.2}$$

where $\alpha = (\alpha_1, \alpha_1', \dots, \alpha_n, \alpha_n')$, $\alpha_i, \alpha_i' = 0, 1, \dots$ *and* $D^\alpha = \partial_1^{\alpha_1} \overline{\partial}_1^{\alpha_1'} \dots \partial_n^{\alpha_n} \overline{\partial}_n^{\alpha_n'}$, $\partial_j = \frac{\partial}{\partial z_j}$, $\overline{\partial}_j = \frac{\partial}{\partial \overline{z}_j}$ $j = 1 \dots n$.

The order k *of the operator* P *in equation 2.*(1.2) *is the greatest integer such that there is an* α *with* $|\alpha| = k$ *and* $a_\alpha \not\equiv 0$.

Let X be a compact differentiable n-dimensional manifold without boundary and $E \to X$ and $F \to X$ vector bundles of rank p and q respectively. As usual, $\mathcal{E}(E)$ resp. $\mathcal{E}(F)$ denote the spaces of differentiable sections of E resp. F.

DEFINITION 1.2. *A* K*-linear operator* $P : \mathcal{E}(E) \to \mathcal{E}(F)$ *is a* differential operator of order k *if it has the following form: Let* $U \subset X$ *such that* $(e_j)_{j=1}^p$, *resp* $(f_i)_{i=1}^q$ *are frames for* E, *resp.* F *over* U, *and let* $\lambda = \left(\sum_{j=1}^p \lambda_j e_j\right) \in \mathcal{E}(U, E)$, *then* P *can locally be expressed in the form*

$$P(x, D)(\lambda) = P(x, D)\left(\sum_{j=1}^p \lambda_j e_j\right) = \sum_{i=1}^q \left(\sum_{\substack{j=1 \\ |\alpha| \leq k}}^p a_\alpha^{i,j}(x) D^\alpha(\lambda_j)(x)\right) f_i(x),$$

and there is at least one i *such that* $\sum_{\substack{j=1 \\ |\alpha|=k}}^p a_\alpha^{i,j} \not\equiv 0$ *on* U. *Here* $(a_\alpha^{i,j})_{i,j}$ *is a* $q \times p$ *matrix of smooth functions, so each component of the above is a differential operator in the sense of definition 2.1.1.*

The space of differential operators $P : \mathcal{E}(E) \to \mathcal{E}(F)$ of order less than or equal to k is denoted by $\mathrm{Diff}_k(E,F)$, and the space of all differential operators $\mathcal{E}(E) \to \mathcal{E}(F)$ is denoted by $\mathrm{Diff}(E,F) := \bigcup_k \mathrm{Diff}_k(E,F)$.

If T^*X is the real cotangent bundle of X we define $T'X$ as the collection of nonzero cotangent vectors on X:

$$T'X := \bigcup_{x\in X} (T^*_x X/\{0\}) \subset T^*X.$$

Let $\pi : T'X \to X$ be the canonical projection and π^*E, π^*F the pullbacks of E and F over $T'X$. Then π^*E (resp. π^*F) can be regarded as the subset of $E \times T'X$ (resp. $F \times T'X$) consisting of the elements (e,ξ) with $\xi \in T'X$ and $e \in E_{\pi\xi}$ (resp. $e \in F_{\pi\xi}$).

Definition 1.3. *For $k \in \mathbb{Z}$, the k-*symbol $\sigma_k(P) : \pi^*E \to \pi^*F$ *of a differential operator $P : E \to F$ is defined as follows:*

For $x \in X$, let $\xi \in T'X_x$, and $s \in \mathcal{E}(E)$ be a differentiable section of E with $s(x) = e$. If f is a differentiable function on an open neighbourhood of x with $df(x) = \xi$, then we define

$$\sigma_k(P)(s(x),\xi) := P\left(\frac{i^k}{k!}(f - f(x))^k s\right)(x) \in F_x. \tag{1.3}$$

Proposition 1.4. *The k-*symbol $\sigma_k(P) : \pi^*E \to \pi^*F$ *of P is a well-defined homomorphism which is homogeneous of degree k, i.e. for $\rho > 0$,*

$$\sigma_k(P)(s(x),\rho\xi) = \rho^k \sigma_k(P)(s(x),\xi).$$

Proof. The symbol $\sigma_k(P)$ is well-defined. That is, it is independent of the choices of $f \in \mathcal{E}(X)$ and $s \in \mathcal{E}(E) = \mathcal{E}(X,E)$:

Let f' be another differential function on X with $df'(x) = df(x) = \xi$ and s' another section of E with $s'(x) = s(x) \in E_x$. Locally, P has the form $P = \sum_{|\alpha|\le k} A_\alpha D^\alpha$ where $\{A_\alpha\}$ are $q \times p$ matrices of locally smooth functions. So, in applying P to $\left(\frac{i^k}{k!}(f' - f'(x))^k s'\right)$ and evaluating at x, derivatives of order $< k$ will vanish (by the chain rule) since a factor of $(f' - f'(x))\,(x)s(x) = 0$ will remain. For derivatives of order k we notice that the kth derivative of $(f' - f'(x))^k\, s$ depends only on $df'(x) = df(x) = \xi$ and $s'(x) = s(x)$. So, the k-symbol is independent of the choices of f and s.

That $\sigma_k(P)$ it is linear (in P) is immediate. If $t \in \mathcal{E}(E)$ denotes another section of E,

$$\sigma_k(P)((\lambda s + \mu t)(x),\rho\xi) = P\left(\frac{i^k}{k!}(\rho f - \rho f(x))^k(\lambda s + \mu t)\right)(x)$$
$$= \lambda\sigma_k(P)(s(x),\rho\xi) + \mu\sigma_k(P)(t(x),\rho\xi), \quad \lambda,\mu \in \mathbb{C}$$

by the linearity of P, so $\sigma_k(P)(s(x),\xi)$ is a homomorphism from E_x to F_x.

Finally, for $\rho > 0$

$$\sigma_k(P)(s(x),\rho\xi) = P\left(\frac{i^k}{k!}(\rho f - \rho f(x))^k s\right)(x)$$

$$= P\left(\frac{i^k}{k!}\rho^k(f - f(x))^k s\right)(x) = \rho^k\sigma_k(P)(s(x),\xi)$$

by the linearity of P. In other words $\sigma_k(P)$ is homogeneous of degree k.

□

Clearly $\sigma_k(P) \equiv 0$ if and only if the degree of P is less than k. In fact, for a differential operator P of degree k, we will only be interested in the k-symbol $\sigma_k(P)$ and so, henceforth, unless otherwise stated, we shall refer to this simply as the symbol of P and write $\sigma(P) := \sigma_k(P)$.

Definition 1.5. *The operator P is said to be* elliptic *if for all $x \in X$ and for all $\xi \in T'X_x$, $\sigma(P)(\xi)$ is an isomorphism from E_x to F_x.*

Of course, if an elliptic operator $P : E \to F$ exists then $\operatorname{rk} E = \operatorname{rk} F$.

Example 1.6. *The Laplacian.*
As an example, we will show that the *Laplace-Beltrami* operator for Riemannian manifolds is elliptic.

(X, g) is a n-dimensional Riemannian manifold with Levi-Civita connection ∇ (see example 1.1.33). $\mathcal{E}(TX)$ is the space of differentiable vector fields on X. For a smooth real valued function $f \in C^\infty(X, \mathbb{R})$, the operator

$$Hf : \mathcal{E}(TX) \times \mathcal{E}(TX) \to C^\infty(X, \mathbb{R})\ ,\ Hf(V_1, V_2) := L_{V_1}(L_{V_2}(f)) - L_{(\nabla_{V_1} V_2)}(f)$$

(where $L_{V_1} f$ denotes the Lie derivative of f with respect to V_1) is the called the *Hessian* of f.

The *Laplace-Beltrami* operator, or *Laplacian* Δ on X is the trace of this operator. That is

$$\Delta : C^\infty(X, \mathbb{R}) \to C^\infty(X, \mathbb{R}),\ \Delta(f)(x) := \sum_{j=1}^{n} Hf(v_j, v_j)$$

with $(v_1, \dots, v_n)$ a g- orthonormal basis of T_xX. Clearly the degree of Δ is 2. So, we wish to show that for any $\xi \in T'X$, $x = \pi(\xi) \in X$ and function $s \in C^\infty(X, \mathbb{R})$, $\sigma(\Delta)(s(x), \xi) : \mathbb{R} \to \mathbb{R}$ is an isomorphism.

Let $f \in C^\infty(X, \mathbb{R})$ such that $df(x) = \xi$ and $f(x) = 0$ and choose an orthonormal frame $\{v_1, \dots v_n\}$ of T_xX such that $\xi(v_j) = \|\xi\|\delta_{1,j},\ j = 1, \dots, n$. Locally, we can extend this to an orthonormal frame of vector fields $\{V_1, \cdots, V_n\}$, with $V_1(x) = v_1$, on some neighbourhood U of x.

Let $s \in C^\infty(X, \mathbb{R})$. Then,

$$\sigma(\Delta)(s(x), \xi) = \Delta\left(-\frac{1}{2}(f^2 s)\right)(x) = -\frac{1}{2}\sum_{j=1}^{n}\left(L_{V_j} L_{V_j}(f^2 s) - L_{\nabla_{V_j} V_j}(f^2 s)\right)(x).$$

Using the Leibniz property of the Lie derivative we have, for all $j = 1, \dots n$,

$$L_{V_j}(f^2 s) = f^2 L_{V_j}(s) + 2fsL_{V_j}(f)$$

so

$$\begin{aligned} L_{V_j} L_{V_j}(f^2 s) &= L_{V_j}\left(f^2 L_{V_j}(s) + 2fsL_{V_j}(f)\right) \\ &= 4fL_{V_j}(f)L_{V_j}(s) + f^2 L_{V_j} L_{V_j} s + 2s(L_{V_j}(f))^2 + 2fsL_{V_j} L_{V_j}(f). \end{aligned}$$

By construction, $L_{V_j}(f) = \|\xi\| \cdot \delta_{1,j}$ at x. So

$$\sum_{j=1}^{n} L_{V_j} L_{V_j}(f^2 s) = 4fL_{V_1} fL_{V_1} s + 2fsL_{V_1} L_{V_1}(f) + 2s(L_{V_1}(f))^2 + \sum_{j=1}^{n} f^2 L_{V_j} L_{V_j} s.$$

Using the fact that $f(x) = 0$, we get

$$\sum_{j=1}^{n} L_{V_j} L_{V_j}(f^2 s)(x) = 2s(L_{V_1}(f))^2 = 2s(x) \cdot \|\xi_x\|^2.$$

Again, by the Leibniz rule and the fact that $f(x) = 0$ we have

$$L_{\nabla_{V_j} V_j}(f^2 s)(x) = f^2(x) L_{\nabla_{V_j} V_j}(s)(x) + 2f(x)s(x) L_{\nabla_{V_j} V_j}(f)(x) = 0.$$

So,

$$\sigma(\Delta)(s(x),\xi) = \frac{-1}{2} \sum_{j=1}^{n} \Big(L_{V_j} L_{V_j}(f^2 s) - L_{\nabla_{V_j} V_j}(f^2 s) \Big)(x) = -\|\xi_x\|^2 s(x) = 0$$

if and only if $s(x) = 0$.

In other words, the endomorphism $\sigma(\Delta)(\cdot, \xi),\ \xi \in T'_x X$ is injective and hence an isomorphism and we have proved that $\Delta : C^\infty(X, \mathbb{R}) \to C^\infty(X, \mathbb{R})$ is elliptic.

2. Elliptic complexes

DEFINITION 2.1. *Given a finite number of differentiable vector bundles $(E_i)_{i=1}^l$ on X and differential operators $d_i : \mathcal{E}(E_i) \to \mathcal{E}(E_{i+1})$, $E = (E_i, d_i)_{i=1}^l$ is called a* complex *if $d_{i+1} \circ d_i \equiv 0$.*

A complex is said to be elliptic *if for any $\xi \in T'_x X$, the sequence*

$$\cdots \to E_{i,x} \xrightarrow{\sigma(d_i)(\xi)} E_{i+1,x} \to \cdots$$

is exact. (In particular, an elliptic operator is an elliptic complex of the form $0 \to \mathcal{E}(E) \xrightarrow{P} \mathcal{E}(F) \to 0$.)

EXAMPLE 2.2. *The de Rham complex on a complex manifold*

For a differentiable manifold X of real dimension m, the de Rham complex is given by

$$0 \to \mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{E}^m(X)$$

where d denotes exterior differentiation. We let T^*X denote the real cotangent bundle and $T^*X^{\mathbb{C}} = T^*X \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Using the notation above we write the de Rham complex as

$$0 \to \mathcal{E}(E_0) \xrightarrow{d} \mathcal{E}(E_1) \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{E}(E_m) \to 0$$

where $E_k := \wedge^k T^*X^{\mathbb{C}}$.

d is a differential operator of degree 1 so we calculate, for each $k = 1, \ldots, nm$, the associated 1-symbol homomorphisms

$$E_{0,x} \xrightarrow{\sigma(d)(s_0(x),\xi)} E_{1,x} \xrightarrow{\sigma(d)(s_1(x),\xi)} \cdots \xrightarrow{\sigma(d)(s_{m-1}(x),\xi)} E_{m,x}. \tag{2.1}$$

Given $\xi \in T'_x X \subset T'X$, we choose an $f \in \mathcal{E}(X)$ such that $df(x) = \xi$ and $f(x) = 0$. Let $s_i \in \mathcal{E}(E_i)$ such that $s_i(x) = e \in E_{i,x}$. Then

$$\sigma(d)(s(x), \xi) = id(fs_i)(x) = idf(x) \wedge e + if(x)ds_i(x) = i\xi \wedge e$$

by the Leibniz property for exterior differentiation.

So, the sequence 2.2.1 is exact. Namely $\xi \wedge e = 0$ if and only if e and ξ are linearly dependent, i.e. $e = \xi \wedge \alpha$ for some $\alpha \in E_{i-1,x}$.

EXAMPLE 2.3. *The Dolbeault complex on a complex manifold*
We generalise example 1.2.29 to higher dimensional complex manifolds and show that the complex thus obtained is elliptic.

For a complex n-dimensional differentiable manifold X the de Dolbeault complex $\overline{\partial}(X)$ is given by

$$\mathcal{E}^{p,0}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}(X) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathcal{E}^{p,n}(X) \to 0$$

where the *Dolbeault operator* $\overline{\partial} : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X)$ is the antiholomorphic component of the exterior derivative.

Let $\mathbf{T}$ denote the complex cotangent bundle and $\overline{\mathbf{T}}$ its conjugate. Then, we may write the Dolbeault complex as

$$\mathcal{E}(E_0) \xrightarrow{\overline{\partial}} \mathcal{E}(E_1) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathcal{E}(E_n) \to 0$$

where $E_k := \wedge^p \mathbf{T} \otimes \wedge^k \overline{\mathbf{T}}$.

$\overline{\partial}$ is a differential operator of degree 1 so we calculate, for each $k = 1, \dots, n$, the associated 1-symbol homomorphisms

$$E_{0,x} \xrightarrow{\sigma(\overline{\partial})(s_0(x),\xi)} E_{1,x} \xrightarrow{\sigma(\overline{\partial})(s_1(x),\xi)} \dots \xrightarrow{\sigma(\overline{\partial})(s_{n-1}(x),\xi)} E_{n,x}. \tag{2.2}$$

Given $\xi \in T'X_x$, we choose an $f \in C^\infty(X)$ such that $df(x) = \xi = \xi^{1,0} + \xi^{0,1}$ (with $\xi^{1,0} \in \mathbf{T}$ and $\xi^{0,1} \in \overline{\mathbf{T}}$) and $f(x) = 0$. Let $s_i \in \mathcal{E}(E_i)$ such that $s_i(x) = e \in E_{i,x}$. Then

$$\sigma(\overline{\partial})(s_i(x),\xi) = i\overline{\partial}(fs_i)(x) = i(\overline{\partial}f(x) \wedge e + f(x)\overline{\partial}s(x)) = i\xi^{0,1} \wedge e$$

by the Leibniz property.

So, the symbol sequence is exact. Namely $\xi^{0,1} \wedge e = 0$ if and only if e and ξ_x are linearly dependent, i.e. $e = \xi^{0,1} \wedge \alpha$ for some $\alpha \in E_{i-1,x}$.

Now let $E \to X$ be a holomorphic bundle over a paracompact manifold X. Choose a holomorphic frame $f = (f_i)_{i=1}^{\mathrm{rk}\,E}$ for E on a neighbourhood $U \subset X$ which is small enough that it is contained in a chart neighbourhood for X. Recall that $\mathcal{E}^{p,q}(E) = \mathcal{E}(E \otimes \wedge^p \mathbf{T} \otimes \wedge^q \overline{\mathbf{T}})$, so for an element $\alpha \in \mathcal{E}^{p,q}(E)$, $\alpha|_U = \sum_i f_i \otimes \beta_i$ where each β_i is a (p,q)-form on U.

Then define

$$\overline{\partial}_E(\alpha|_U) := \sum_i \overline{\partial}(f_i \otimes \beta_i). = \sum_i f_i \otimes \overline{\partial}(\beta_i)$$

Since f_i is holomorphic, $\overline{\partial}_E(\alpha|_U) = \sum_i f_i \otimes \overline{\partial}(\beta_i)$.

PROPOSITION 2.4. *The operator $\overline{\partial}_E$ of a complex vector bundle $E \to X$ is a well-defined global operator. That is, it is independent of the choice of frame f.*

PROOF. Let $f' = (f'_j)_j$ be another holomorphic frame for E on U. Then, there is a change of frame matrix $(t_{i,j})_{i,j}$ $i,j = 1, \dots, \mathrm{rk}\,E$ such that each of the maps $t_{i,j}$ is holomorphic and $f_i = \sum_j t_{i,j} f'_j$.

Now

$$\alpha|_U = \sum_j f'_j \otimes \left(\sum_i t_{i,j}\beta_i \right),$$

so, in terms of f',

$$\overline{\partial}(\alpha|_U) = \sum_j f'_j \otimes \overline{\partial} \left(\sum_i t_{i,j}\beta_i \right) = \sum_j f'_j \otimes \left(\sum_i t_{i,j}\overline{\partial}(\beta_i) \right)$$

$$\sum_{i,j} f'_j \otimes f_{i,j}\overline{\partial}(\beta_i) = \sum_i f_i \otimes \overline{\partial}(\beta_i).$$

In other words, $\overline{\partial}(\alpha) \in \mathcal{E}^{p,q+1}(E)$ is globally defined. So $\overline{\partial}_E : \mathcal{E}^{p,q}(E) \to \mathcal{E}^{p,q+1}(E)$ is a well-defined map.

□

DEFINITION 2.5. *Given a holomorphic vector bundle $E \to X$, the operator $\overline{\partial}_E$ is called the* Dolbeault operator of E.

EXAMPLE 2.6. *The Dolbeault complex for a holomorphic bundle*

We wish to show that the Dolbeault complex $\overline{\partial}(E)$ of E given by

$$\mathcal{E}^{p,0}(E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,1}(E) \xrightarrow{\overline{\partial}_E} \dots \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,q}(E) \to 0 \tag{2.3}$$

is elliptic. In fact the argument of example 2.2.3 carries over to this more general case.

$\overline{\partial}_E$ is a differential operator of degree 1 so we calculate, for each $k = 1, \dots, n$, the associated 1-symbol homomorphisms:

Let $s_i \in \mathcal{E}(\otimes \wedge^p \mathbf{T} \otimes \wedge^i \overline{\mathbf{T}})$ such that $s(x) = e \in E \otimes \wedge^p \mathbf{T} \wedge^i \overline{\mathbf{T}}$ and $\xi \in T'X_x$. We choose an $g \in C^\infty(X)$ such that $dg(x) = \xi = \xi^{1,0} + \xi^{0,1}$ (with $\xi^{1,0} \in \mathbf{T}$ and $\xi^{0,1} \in \overline{\mathbf{T}}$) and $g(x) = 0$.

Then

$$\sigma(\overline{\partial}_E)(s_i(x), \xi) = i\overline{\partial}_E(gs_i)(x) = i(\overline{\partial}g(x) \wedge e + g(x)\overline{\partial}_E s(x)) = i\xi^{0,1} \wedge e$$

by the Leibniz property.

As before $\xi^{0,1} \wedge e = 0$ if and only if e and ξ are linearly dependent, i.e. $e = \xi^{0,1} \wedge \alpha$ for some $\alpha \in E \otimes \wedge^p \mathbf{T} \otimes \wedge^{i-1}\overline{\mathbf{T}}$.

REMARK 2.7. If $E \to X$ is a holomorphic vector bundle, we consider the kernel of $\overline{\partial}_E : (\mathcal{E}(E) = \mathcal{E}^{0,0}(E)) \to \mathcal{E}^{0,1}(E)$:

As above, we choose a holomorphic frame $f = (f_i)_{i=1}^{\operatorname{rk} E}$ for E on a neighbourhood $U \subset X$ which is small enough that it is contained in a chart neighbourhood for X. Then, given $\alpha \in \mathcal{E}(E)$, $\alpha = \sum_{i=1}^{\operatorname{rk} E} \alpha_i f_i$ with $\alpha_i \in \mathcal{E}(U)$ on U. In this case α is holomorphic if and only if α_i is holomorphic for all i and, by the construction of $\overline{\partial}_E$ above, this is precisely when

$$\overline{\partial}_E(\alpha) = \overline{\partial}(\sum_i \alpha_i f_i) = \sum_i (\overline{\partial}\alpha_i) \otimes f_i = 0$$

(where the first equality follows from the definition of $\overline{\partial}_E$ in example 2.2.3).

DEFINITION 2.8. *The* cohomology groups $H^i(E)$ *of a complex* $E = (E_i, d_i)_i$ *are defined by*

$$H^i(E) = {}^{\operatorname{Ker} d_i}\!/_{\operatorname{Im} d_{i-1}}.$$

PROPOSITION 2.9. *The cohomology groups $H^i(E)$ of an elliptic complex $E = (E_i, d_i)_i$ over a compact manifold X are finite dimensional for all i.*

PROOF. This is an analytic proof which rests on some deep results in functional analysis. These are described in [**Ho**]. Given the findings of Hörmander, there is an elegant proof in [**AB**, 395-398]. [**We**, 119-153] also gives a good overview. □

DEFINITION 2.10. *The* analytic index *of an elliptic complex E of length l is defined to be* $ind_a(E) := \sum_{i=0}^{l}(-1)^i \dim H^i(E)$.

In particular, the analytic index of an elliptic operator $P : E \to F$ where $E \to X$ and $F \to X$ are vector bundles over X, is given by

$$ind_a(P) = \dim \mathrm{Ker}\,(P) - \dim \mathrm{Coker}\,(P).$$

CHAPTER 3

The Riemann-Roch Theorem

1. Divisors

1.1. Definitions.

DEFINITION 1.1. *A* divisor *on a Riemann surface X is an integer valued function $D : X \to \mathbb{Z}$ with discrete support. If $D(x) = n_x$, we write formally*
$D = \sum_{x \in X} n_x \cdot x$.
A divisor is called effective *if $n_x \geq 0$ for every x in X.*

We can define a partial ordering $\leq$ on the set of divisors on X by $D \leq D'$ if and only if $D(x) \leq D'(x)$ for all $x \in X$. ($D < D'$ if $D \leq D'$ and there is at least one $x \in X$ such that $D(x) < D'(x)$.)

The set of divisors $\mathrm{Div}\,(X)$ on a Riemann surface X defines in a natural way an Abelian group with operation $+$ such that, for $x \in X$, $(D+D')(x) := D(x)+D'(x)$.

Let X now be a Riemann surface and let $\mathcal{M}(X)$ denote the field of meromorphic functions on X. If $\phi \in \mathcal{M}(X)$ is not identically zero, then, at every $x \in X$, if z is a local coordiinate for X centered at x, then ϕ has the unique form $\phi(z) = z^k g(z)$ with $k \in \mathbb{Z}$ and $g(z)$ holomorphic and non-zero at $z = 0$. k is the *order of ϕ at x*, $\mathrm{ord}\,_x(\phi)$ and $\mathrm{ord}\,_z(\phi) \neq 0$ only on a discrete set in X. So, ϕ defines a divisor $\mathrm{div}\,(\phi)$ by

$$\mathrm{div}\,(\phi) := \sum_{x \in X} \mathrm{ord}\,_x(\phi) \cdot x.$$

In particular, a non-zero holomorphic function on X defines an effective divisor.

PROPOSITION 1.2. *Let X be a compact Riemann surface. Given points $\{x_i\}_{i=1}^N \subset X$ and complex numbers $\{c_i\}_{i=1}^N$, there exists a function $\phi \in \mathcal{M}(X)$ such that $\phi(x_i) = c_i$, $i = 1, \ldots, N$.*

PROOF. [**Fo**, 116].

□

DEFINITION 1.3. *A divisor of the form* $\mathrm{div}\,(\phi)$ *for $\phi \in \mathcal{M}(X)^*$ (where $\mathcal{M}(X)^*$ is the group of non-zero elements of $\mathcal{M}(X)$) is called a* principal divisor.

DEFINITION 1.4. *We say that two divisors D and D' on X are* linearly equivalent *if there is a meromorphic function ϕ on X such that $D' = D + \mathrm{div}\,(\phi)$.*

DEFINITION 1.5. *A* meromorphic differential *on a Riemann surface X is a holomorphic 1-form ω on $X - S$ where $S \subset X$ is discrete, with the following local description: Since S is discrete, for all $s \in S$, there exists an open neighbourhood U_s of s with $U_s \cap S = \{s\}$ and U_s is contained in a chart neighbourhood of X with local coordinate z. Then*

$$\omega|_{U_s} = \phi dz$$

where ϕ is meromorphic on U_s.

We denote the set of meromorphic differentials on X by $\mathcal{M}^1(X)$.

REMARK 1.6. It follows from definition 3.1.5 that the meromorphic differentials on X correspond to the meromorphic sections (see definition 1.1.8) of the holomorphic cotangent bundle $\mathbf{T}$ of X.

LEMMA 1.7. *If ω is a non-zero meromorphic differential on a Riemann surface X, then every meromorphic differential η on X is of the form $\psi\omega$ with $\psi \in \mathcal{M}(X)$.*

PROOF. Let $\{(U_i, z_i)\}_i$ be a holomorphic atlas for X. On $U_i \cap U_j$, $g_{i,j} := \frac{dz_j}{dz_i}$ is a transition function for the cotangent bundle $\mathbf{T}$. So

$$\omega|_{U_j} = \phi_j dz_j = g_{j,i}\phi_i dz_i, \text{ and } \eta|_{U_j} = \theta_j dz_j = g_{j,i}\theta_i dz_i \text{ on } U_i \cap U_j.$$

But

$$\frac{\theta_j}{\phi_j}\frac{dz_j}{dz_j} = \frac{\theta_j}{\phi_j}$$

and

$$\frac{\theta_j}{\phi_j}\frac{dz_j}{dz_j} = \frac{g_{j,i}\theta_i}{g_{j,i}\phi_i}\frac{dz_i}{dz_i} = \frac{\theta_i}{\phi_i}$$

for all i, j. So, ψ, given locally by $\psi|_{U_i} = \frac{\theta_i}{\phi_i}$, is a globally defined meromorphic function such that $\eta = \psi\omega$.

□

REMARK 1.8. Proposition 3.1.2 asserts the existence of a non-constant meromorphic function ϕ on X. Therefore, the set of meromorphic differentials on a Riemann Surface X is non-empty. Namely, $\partial\phi$ given locally by $\partial\phi := \frac{\partial\phi}{\partial z}dz$ is a meromorphic differential on X.

DEFINITION 1.9. *Given a mermorphic differential ϕdz on an open $V \subset \mathbb{C}$, we define it's order at $x \in V$ as the order of ϕ at x.*

If X is a Riemann surface, $x \in X$ and z a local coordinate at x, the order $\operatorname{ord}_x(\omega)$ *of a meromorphic differential ω with $\omega = \phi dz$ in a neighbourhood of x is* $\operatorname{ord}_x(\omega) := \operatorname{ord}_x(\phi)$. *So, we can define*

$$\operatorname{div}(\omega) = \sum_{x \in X} \operatorname{ord}_x(\omega) \cdot x.$$

A divisor of the form $D = \operatorname{div}(\omega)$ with ω a meromorphic differential on X is a called a canonical divisor *on X.*

DEFINITION 1.10. *On a compact Riemann Surface X a divisor D has compact support and therefore the map* $\deg : \operatorname{Div}(X) \to \mathbb{Z}$ *defined by* $\deg(D) := \sum_{x \in X} n_x$ *is well defined and clearly a homomorphism. This is called the* degree homomorphism.

PROPOSITION 1.11. *Every principal divisor has degree 0 and so linearly equivalent divisors have the same degree.*

PROOF. [**Fo**, 80-81]. This is a consequence of the residue theorem on Riemann surfaces. □

An immediate consequence of lemma 3.1.7 and proposition 3.1.11 is

COROLLARY 1.12. *All canonical divisors on a Riemann surface X have the same degree.*

1.2. The line bundle L_D. Let D be a divisor on a compact Riemann Surface X. We associate to D a (holomorphic) line bundle L_D over X.

Since X is compact, $\operatorname{supp}(D) = \{p_j \in X | n_{p_j} \neq 0\}$ is finite. We choose a finite covering $\{U_k\}_k$ of X such that for each U_k, there is a meromorphic function g_k on U_k with $\operatorname{ord}_{g_k}(p) = n_p$ for every $p \in U_k$. For example, we could choose U_k so that at most one point $p_k \in \operatorname{supp}(D)$ is contained in U_k. (See e.g. [**Jo**, 228-229] for why it is possible to construct such functions $\{g_k\}_k$.)

Then, the zero and polar sets of $\{g_i\}$ are well defined since, by construction, g_i and g_j have the same zero and polar sets on $U_i \cap U_j$. In particular, $\frac{g_i}{g_j}$ is a nowhere zero holomorphic function on $U_i \cap U_j$ and so the covering $\{U_i\}_i$ together with the transition functions $\{g_{i,j}\}_{i,j} := \{\frac{g_i}{g_j} | U_i \cap U_j\}_{i,j}$ specify a one dimensional holomorphic vector bundle (a complex line bundle) L_D on X. That is, we can take each open set U_i as a trivialising neighbourhood for L_D so that $L_D|_{U_i} \equiv U_i \times \mathbb{C}$. Fibres over points $x \in U_i \cap U_j$ are then identified by the function $g_{i,j} = \frac{g_i}{g_j}$. So L_D is a holomorphic line bundle over X.

Conversely, we shall see that all holomorphic line bundles on a compact Riemann surface X can be associated to a divisor on X.

Definition 1.13. *If $s \in \mathcal{M}(L)$ is a non-zero meromorphic section of a holomorphic line bundle L represented by functions $s_i \in \mathcal{M}(U_i)$ relative to an open cover $\{U_i\}_i$ of X, then the order* $\operatorname{ord}_x(s)$ *of s at $x \in U_i$ is given by*

$$\operatorname{ord}_x(s) := \operatorname{ord}_x(s_i).$$

This is clearly independent of the choice of cover $\{U_i\}_i$.

Lemma 1.14. *Every holomorphic line bundle $L \to X$ on a compact Riemann surface X admits a global meromorphic section s such that $s \not\equiv 0$ on X. In particular, since s has only isolated zeroes and poles, we can associate a divisor (s) on X to s by $(s)(x) = \operatorname{ord}_x(s)$ for $x \in X$.*

Proof. [**Fo**, 225]. □

Lemma 1.15. *Let L be a holomorphic line bundle on a compact Riemann surface X. If s is a global meromorphic section of L and (s) the divisor associated to s, then (s) defines a holomorphic line bundle $L_{(s)}$ on X.*

In this case $L_{(s)}$ is isomorphic to L.

Proof. Let L have transition functions $\{g_{i,j}\}_{i,j}$ and $s = \{s_i\}_i$ be a global meromorphic section of L. Then, for all, i, j, $\frac{s_i}{s_j}$ is non-vanishing and holomorphic on $U_i \cap U_j$ so we can define $L_{(s)}$ by means of the meromorphic functions s_i on U_i. Since $s = \{s_i\}_i$ is a section, $s_i = g_{i,j} s_j$ so $L_{(s)}$ has transition functions $s_{i,j} := \frac{s_i}{s_j} = g_{i,j}$. □

Definition 1.16. *The* degree $\deg(L)$ *of a holomorphic line bundle L is defined as* $\deg(L) := \deg(s)$ *where s is a global meromorphic section of L. By the above lemma 3.1.15, this is well defined.*

The following theorem says that there is a one to one correspondence between the linear equivalence classes of divisors on X and the isomorphism classes of line bundles on X. In particular, the line bundle L_D associated to a divisor D is well-defined up to isomorphism.

THEOREM 1.17. *If $L = L_D$ and $L' = L_{D'}$, then L and L' are isomorphic if and only if D and D' are linearly equivalent.*

PROOF. Let $D' = D + \operatorname{div}(\phi)$, $\phi \in \mathcal{M}(X)$. Then, according to the above method, if L is associated to the data $\{U_i, g_i\}_i$, D' can be associated to $g_i\phi$ and so we obtain transition functions $g'_{i,j} = \frac{g_i\phi}{g_j\phi} = g_{i,j}$ for L'. By proposition 1.1.7, $L \cong L'$.

For the converse, let L have transition functions $\{g_{i,j}\}_{i,j} = \{\frac{g_i}{g_j}\}_{i,j}$ and L' has transition functions $\{g'_{i,j}\}_{i,j} = \{\frac{g'_i}{g'_j}\}_{i,j}$ with respect to a cover $\mathcal{U} = \{U_i\}_i$, and with $g_i, g'_i \in \mathcal{M}(U_i)$ for all i. The divisors corresponding to $\{g_i\}_i$ and $\{g'_i\}_i$ are denoted by D and D' respectively.

Say $f : L \to L'$ is an isomorphism of holomorphic line bundles. We wish to show that $D' = D + \operatorname{div}(\phi)$ for some $\phi \in \mathcal{M}(X)$. That is, that

$$g'_i = \phi g_i, \quad \text{for all } i.$$

For all i, let s_i, resp. s'_i denote the holomorphic section of $L|_{U_i}$ resp. $L'|_{U_i}$ which is mapped to the constant function $1 \in \mathbb{C}$ by a trivialising map. Then, in particular, s_i, resp. s'_i is nowhere vanishing on U_i, so, since f is linear on fibres

$$f(s_i) = f_i \cdot s'_i$$

where f_i is a nowhere vanishing holomorphic function on U_i.

So, on $U_i \cap U_j$ we have

$$f(s_j) = f_j \cdot s'_j = f_j \frac{g'_j}{g'_i} s'_i$$

and also

$$f(s_j) = f\left(\frac{g_j}{g_i} s_i\right) = \frac{g_j}{g_i} f(s_i) = \frac{g_j}{g_i} f_i s'_i.$$

So, since $s'_i \neq 0$ on $U_i \cap U_j$

$$\frac{g'_j}{g'_i} f_j = \frac{g_j}{g_i} f_i$$

on $U_i \cap U_j$ and therefore

$$\phi_i := \frac{g'_i}{g_i} f_i$$

defines a meromorphic function ϕ on X. Furthermore, on U_i, the divisor associated to ϕ is the divisor associated to $\frac{g'_i}{g_i}$ (since f_i is non-vanishing and holomorphic on U_i). I.e. $\operatorname{div}(\phi) = D' - D$. This is what we wished to prove.

□

EXAMPLE 1.18. By proposition 3.1.11, linearly equivalent divisors on a compact Riemann surface have the same degree. On $\mathbb{CP}^1$ the converse also holds. That is, if divisors D and D' on $\mathbb{CP}^1$ have the same degree, they are linearly equivalent:

Let D be a divisor on $\mathbb{CP}^1$ such that $\deg(D) = 0$. We cover $\mathbb{CP}^1$ with the open sets U_0 and U_1 as in example 1.1.5. On U_0, we have the local coordinate $[1 : z] \mapsto z$ and on U_1 we have the local coordinate $[w : 1] \mapsto w$. By translating if necessary, we may assume that D is supported on $(U_0 \cap U_1) \subset U_0$ (since the complement of $U_0 \cap U_1$ in $\mathbb{CP}^1$ is a discrete set of two points) and we may therefore write $D = \sum_{z \in \mathbb{C}^*} n_z \cdot z$.

If $\mathrm{supp}\,(D) = \{z_i\}_i$, then $\sum_i n_{z_i} = 0$. We wish to associate a meromorphic function ϕ on $\mathbb{CP}^1$ to the divisor D. On U_0,define

$$\phi_0(z) := \prod_i (z - z_i)^{n_{z_i}}.$$

Clearly $\mathrm{div}\,(\phi_0) = D|_{U_0}$. Now, on $U_0 \cap U_1$, $w = \frac{1}{z}$, $w_i := \frac{1}{z_i}$, and we have

$$\begin{aligned}\phi_0(w) &= \prod_i \left(\frac{1}{w} - z_i\right)^{n_{z_i}} \\ &= \left(\frac{1}{w}\right)^{\sum_i n_{z_i}} \prod_i (1 - wz_i)^{n_{z_i}} \\ &= \prod_i \left(1 - \frac{w}{w_i}\right)^{n_{z_i}} \\ &= \prod_i \left(1 - \frac{w}{w_i}\right)^{n_{z_i}}, \\ &= \prod_i \left(\frac{1}{w_i}\right)^{n_{z_i}} (w_i - w)^{n_{z_i}}.\end{aligned}$$

Since $\prod_i \left(\frac{1}{w_i}\right)^{n_{z_i}} (w_i - 0)^{n_{z_i}}$ is finite, we can extend ϕ_0 to all of U_1. Furthermore, ϕ is non-zero at $w = 0$ so, we have defined a global meromorphic function ϕ on $\mathbb{CP}^1$ with $\mathrm{div}\,(\phi) = D$. We have shown that if D is a divisor on $\mathbb{CP}^1$ such that $\deg D = 0$, then $D = \mathrm{div}\,(\phi)$ for some meromorphic function ϕ on $\mathbb{CP}^1$, and therefore that two divisors on $\mathbb{CP}^1$ have the same degree if and only if they are linearly equivalent.

This, together with theorem 3.1.17, implies that two line bundles on $\mathbb{CP}^1$ have the same degree if and only if they are isomorphic.

Let $\pi : \mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1$ be as in examples 1.1.5 and 1.1.17. We define a global meromorphic section s on $\mathcal{O}_{\mathbb{CP}^1}(-1)$ via the covering $\mathbb{CP}^1 = U_0 \cup U_1$. Define

$$s_0 : [1 : z] \mapsto (1, z) \ \text{ on } U_0, \quad \text{and } s_1 : [w : 1] \mapsto (\frac{1}{w}, \frac{1}{w}) \text{ on } U_1.$$

Then $s = \{s_0, s_1\}$ transforms according to $s_1 = g_{1,0}s_0$ on $U_0 \cap U_1$ and s is a nowhere vanishing global meromorphic section of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ with a single pole of order 1 in the point given by $[0 : 1]$.

Therefore we have $\deg(\mathcal{O}_{\mathbb{CP}^1}(-1)) = -1$ and, if D is a divisor on $\mathbb{CP}^1$ with $\deg(D) = -1$, there is an isomorphism $L_D \xrightarrow{\sim} \mathcal{O}_{\mathbb{CP}^1}(-1)$.

Proposition 1.19. *Let K be a canonical divisor on a compact Riemann Surface X. Then L_K is isomorphic to the holomorphic cotangent bundle $\mathbf{T}$ of X. We call $L_K \cong \mathbf{T}$ the* canonical line bundle *on X.*

Proof. A canonical divisor is the divisor of a meromorphic differential on X. The statement follows from definition 3.1.5 (see remark 3.1.6). □

Definition 1.20. *By theorem 1.2.8, we associate to L_D a sheaf $\mathcal{O}_D := \mathcal{O}(D)$ of local holomorphic sections of L_D.*

Proposition 1.21. *There is a natural isomorphism between $\mathcal{O}_D$ and the sheaf $\mathcal{F}$ where $\mathcal{F}(U) = \{\phi \in \mathcal{M}(U) : D|_U + \mathrm{div}\,(\phi) \geq 0\}$ and the restriction functions $\tau^U_V, V \subset U$ are simply the restrictions $\tau^U_V(\phi) = \phi|_V,\ \phi \in \mathcal{M}(U)$.*

PROOF. D is the divisor of a meromorphic section s_0 of $L = L_D$. $\phi \in \mathcal{F}(U)$ if and only if $\operatorname{div}(\phi \cdot s_0) = \operatorname{div}(\phi) + \operatorname{div}(s_0) \geq 0$. That is $\operatorname{div}(\phi \cdot s_0)$ is effective on U and therefore $\phi \cdot s_0 \in \mathcal{O}_D(U)$.

Conversely, given a local holomorphic section $s \in \mathcal{O}_D(U)$,

$$\operatorname{div}(s) - \operatorname{div}(s_0) \geq -D|_U$$

so the formal expression $\frac{s}{s_0}$ defines an element of $\mathcal{F}(U)$.

□

2. The Riemann-Roch Theorem and the *analytic index* of a divisor

DEFINITION 2.1. *The* genus g *of a compact Riemann Surface* X *is defined by*

$$g := \dim H^1(X, \mathcal{O}).$$

THEOREM 2.2. The Riemann-Roch theorem
If D *is a divisor on a* compact *Riemann surface* X *and* $\mathcal{O}_D$ *is the sheaf of local holomorphic sections of* L_D *(or, by proposition 3.1.21, the sheaf of local meromorphic functions* ϕ *on* U *open in* X *such that* $\operatorname{div}(\phi) + D|_U \geq 0$*), then* $H^0(X, \mathcal{O}_D)$ *and* $H^1(X, \mathcal{O}_D)$ *are finite dimensional vector spaces with dimensions* $h^0(X, \mathcal{O}_D)$ *and* $h^1(X, \mathcal{O}_D)$ *respectively and*

$$h^0(X, \mathcal{O}_D) - h^1(X, \mathcal{O}_D) = 1 - g + \deg D. \tag{2.1}$$

It is not the aim of the current thesis to prove this theorem (refer to e.g. [**Fo**, 129-130]). In this chapter we are interested in showing that the integer quantity given on the left hand side of the equation can be interpreted as the analytic index of a differential operator associated to the divisor D.

2.1. Serre duality. The Serre duality theorem provides a powerful tool for calculating with the Riemann-Roch theorem.

THEOREM 2.3. Serre duality theorem
There is an isomorphism

$$\Phi : H^0(X, \mathcal{O}_{K-D}) \to \left(H^1(X, \mathcal{O}_D)\right)^*.$$

PROOF. [**Fo**, 132-138]. □

COROLLARY 2.4. *If* K *is a canonical divisor on a compact Riemann surface* X*, then*

$$\deg K = 2g - 2.$$

PROOF. By Serre duality, theorem 3.2.3,

$$H^1(X, \mathcal{O}_K) = (H^0(X, \mathcal{O}_{K-K}))^* = (H^0(X, \mathcal{O}))^*.$$

$H^0(X, \mathcal{O})$ consists of the holomorphic functions on X but since X is compact, $H^0(X, \mathcal{O}) = \mathbb{C}$ so $h^1(X, \mathcal{O}_K) = h^0(X, \mathcal{O}) = 1$.

Furthermore, $H^0(X, \mathcal{O}_K) = (H^1(X, \mathcal{O}_{K-K}))^* = (H^1(X, \mathcal{O}))^*$ but $h^1(X, \mathcal{O}) =: g$. So the Riemann-Roch theorem gives

$$h^0(X, \mathcal{O}_K) - h^1(X, \mathcal{O}_K) = g - 1 = 1 - g + \deg K.$$

That is $\deg K = 2g - 2$.

□

EXAMPLE 2.5. The projective line $\mathbb{CP}^1$ has genus 0. Therefore a canonical divisor K on $\mathbb{CP}^1$ has degree -2. In particular, by proposition 3.1.19, the degree $\deg(\mathbf{T})$ of the holomorphic cotangent bundle $\mathbf{T}$ of $\mathbb{CP}^1$ is -2.

2.2. The analytic index of a divisor.

DEFINITION 2.6. *If D is a divisor on a compact Riemann surface X then the* Dolbeault operator $\overline{\partial}_D$ *of D is defined as $\overline{\partial}_D := \overline{\partial}_{L_D} : \mathcal{E}(L_D) \to \mathcal{E}^{0,1}(L_D)$ (see definition 2.2.5).*

REMARK 2.7. Where the bundle is clear from the context, we shall simply refer to the operators $\overline{\partial}_L, \overline{\partial}_D$ as $\overline{\partial}$.

We will use the notation

$$\mathcal{E}_D := \mathcal{E}(L_D) \text{ and } \mathcal{E}_D^{0,1} := \mathcal{E}^{0,1}(L_D). \tag{2.2}$$

In example 2.2.6, we have seen that the Dolbeault operator $\mathcal{E}^{p,q}(E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,q+1}(E)$ for a holomorphic vector bundle $E \to X$ is elliptic with symbol

$$\sigma(\overline{\partial}_E)(s_q(x), \xi) = i\overline{\partial}_E(gs_q)(x) = i(\overline{\partial}g(x) \wedge e + g(x)\overline{\partial}_E s_q(x)) = i\xi^{0,1} \wedge e$$

for $s_q \in \mathcal{E}^{p,q}(E)$ such that $s_q(x) = e \in E \otimes \wedge^p \mathbf{T} \wedge^q \overline{\mathbf{T}}$, $\xi \in T'X$ and $g \in C^\infty(X)$ such that $dg(x) = \xi = \xi^{1,0} + \xi^{0,1}$ (with $\xi^{1,0} \in \mathbf{T}$ and $\xi^{0,1} \in \overline{\mathbf{T}}$) and $g(x) = 0$.

In particular if D is a divisor on a compact Riemann surface X, the operator $\overline{\partial}_D$ is elliptic. Furthermore, by remark 2.2.7, $\operatorname{Ker}(\mathcal{E}_D \xrightarrow{\overline{\partial}_D} \mathcal{E}_D^{0,1}) = \mathcal{O}_D$.

LEMMA 2.8. *The sequence*

$$0 \to (\mathcal{O}_D)_X \hookrightarrow (\mathcal{E}_D)_X \xrightarrow{\overline{\partial}_D} (\mathcal{E}_D^{0,1})_X \xrightarrow{\overline{\partial}} 0 \tag{2.3}$$

(where the arrow $\hookrightarrow$ denotes inclusion) is an exact sequence of sheaves. Here we denote the sheaf morphism induced by $\overline{\partial}_D$ also by $\overline{\partial}_D$.

PROOF. Clearly the map $\mathcal{O}_D(U) \hookrightarrow \mathcal{E}_D(U)$ is injective for all $U \subset X$ open, and we have just seen that, by remark 2.2.7, $\operatorname{Ker}(\mathcal{E}_D \xrightarrow{\overline{\partial}_D} \mathcal{E}_D^{0,1}) = \mathcal{O}_D$. So, by the definition of the Dolbeault operator $\overline{\partial}_D$, $\operatorname{Ker}(\mathcal{E}_D(U) \xrightarrow{\overline{\partial}_D} \mathcal{E}_D^{0,1}(U)) = \mathcal{O}_D(U)$ for all $U \subset X$ open.

It therefore remains to show that $(\mathcal{E}_D)_X \xrightarrow{\overline{\partial}_D} (\mathcal{E}_D^{0,1})_X$ is surjective: Let $x \in X$ and $\alpha \in E_D^{0,1}(V)$ where $V \ni x$ is an open neighbourhood. On a sufficiently small neighbourhood $U \subset V$ of x, $\alpha = s \otimes \omega$ where $s \in \mathcal{E}_D(U)$ and $\omega \in \mathcal{E}^{0,1}(U)$. The Dolbeault lemma ([**Fo**, 105]) says that, in a small enough neighbourhood $U' \subset U \subset X$, $\omega = \overline{\partial} f$ where $f \in \mathcal{E}$. Therefore, by the definitions 3.2.6, and 2.2.5 of $\overline{\partial}_D = \overline{\partial}_{L_D}$, $s \otimes \omega = \overline{\partial}(f \cdot s) =: \overline{\partial}_D(\beta)$ for some $\beta \in (\mathcal{E}_D)(U)$.

□

Lemma 3.2.8 can be generalised in an obvious fashion to holomorphic vector bundles of rank r over compact complex manifolds of any dimension.

THEOREM 2.9. *If D is a divisor on a compact Riemann surface X and $\overline{\partial}_D : \mathcal{E}_D \to \mathcal{E}_D^{0,1}$ is the Dolbeault operator on L_D, then*

$$\operatorname{Ker}(\mathcal{E}_D \xrightarrow{\overline{\partial}} \mathcal{E}_D^{0,1}) \cong \mathcal{O}_D = H^0(X, \mathcal{O}_D)$$

and

$$\operatorname{Coker}(\mathcal{E}_D \xrightarrow{\overline{\partial}} \mathcal{E}_D^{0,1}) \cong H^1(X, \mathcal{O}_D).$$

PROOF. The first statement has been proved in lemma 3.2.8 above. Again by lemma 3.2.8,

$$0 \to (\mathcal{O}_D)_X \hookrightarrow (\mathcal{E}_D)_X \xrightarrow{\overline{\partial}} (\mathcal{E}_D^{0,1})_X \xrightarrow{\overline{\partial}} 0$$

is a short exact sequence of sheaves. So, by theorem 1.2.30

$$0 \to \mathcal{O}_D \hookrightarrow \mathcal{E}_D \xrightarrow{\overline{\partial}} \mathcal{E}_D^{0,1} \xrightarrow{\delta^*} H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{E}_D)$$

is exact

Furthermore, by theorem 1.2.31, $H^1(X, \mathcal{E}_D) = H^1(X, \mathcal{E}_D^{0,1}) = 0$. So, it follows directly from theorem 1.2.34 that

$$H^1(X, \mathcal{O}_D) \cong \operatorname{Coker}(\mathcal{E}_D \xrightarrow{\overline{\partial}} \mathcal{E}_D^{0,1}).$$

□

DEFINITION 2.10. *The* analytic index of a divisor D *on a compact Riemann surface* X *is given as*

$$ind_a(D) := ind_a(\overline{\partial}(L_D)),$$

the analytic index of the elliptic operator $\overline{\partial}_D : \mathcal{E}_D \to \mathcal{E}_D^{0,1}$.

COROLLARY 2.11. *By theorem 3.2.9,*

$$ind_a(D) = h^0(X, \mathcal{O}_D) - h^1(X, \mathcal{O}_D),$$

the left hand side of equation 3.(2.1).

3. The Euler characteristic and Hirzebruch-Riemann-Roch

DEFINITION 3.1. *Let* $E \to X$ *be a holomorphic vector bundle of rank* r *over a compact complex manifold* X *of dimension* n. *The* Euler characteristic $\chi(E)$ *of* E *is given by*

$$\chi(E) := \sum_{i \geq 0} (-1)^i h^i(X, E),$$

where $h^i(X, E) := \dim H^i(X, E)$ *and* $H^i(X, E) := H^i(X, \mathcal{O}(E))$ *as in theorem 1.2.8.*

LEMMA 3.2. $H^i(X, E) = 0$ *for* $i \geq n$.

PROOF. Let $E \to X$ be a holomorphic vector bundle with X is a compact complex manifold of complex dimension n. It follows from remark 2.2.7 that the sequence of sheaves

$$0 \to \mathcal{O}(E)_X \hookrightarrow \mathcal{E}(E)_X \xrightarrow{\overline{\partial}_E} \mathcal{E}^{0,1}(E)_X \xrightarrow{\overline{\partial}_E} \dots,$$

is a resolution for $\mathcal{O}(E)_X$. In fact, since the sheaves $\mathcal{E}^{0,k}(E)_X$ are fine for all k (by example 1.2.14), the above sequence is a fine resolution for $\mathcal{O}(E)_X$. So the sequence of global sections

$$0 \to \mathcal{O}(E) \hookrightarrow \mathcal{E}(E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{0,1}(E) \xrightarrow{\overline{\partial}_E} \dots$$

is a complex. Therefore, by theorem 1.2.34

$$H^i(X, E) := H^i(X, \mathcal{O}(E)) \cong \operatorname{Ker}\left(\overline{\partial}_E(E \otimes \wedge^i \overline{\mathbf{T}})\right) \Big/ \operatorname{Im}\left(\overline{\partial}_E(E \otimes \wedge^{i-1} \overline{\mathbf{T}})\right) = 0$$

for $i > n$ since $\wedge^i \overline{\mathbf{T}} = 0$ in that case.

□

COROLLARY 3.3. *The analytic index of a divisor D on a Riemann surface X is equal to the Euler characteristic $\chi(L_D)$ of the bundle L_D over X.*

PROOF. This is a restatement of corollary 3.2.11. □

REMARK 3.4. As a special case of this, we have already seen in 3.2.4 that the analytic index of a canonical divisor K is equal to

$$1 - g + \deg K = 1 - g + (2g - 2) = g - 1,$$

the Euler number of X.

In fact as a direct corollary of theorem 1.2.34, we have the following general result:

THEOREM 3.5. *Let E be a holomorphic vector bundle over a compact complex manifold X . Then*

$$\chi(E) = ind_a(\overline{\partial}(E)).$$

CHAPTER 4

The Topological Index of a Divisor

We now turn to the right hand side of the Riemann-Roch equation:

$$T(D) := 1 - g + \deg D. \tag{0.1}$$

The aim of this chapter will be to show that, if L is a holomorphic line bundle on a compact Riemann surface X, and D is the divisor of a meromorphic section of L, $T(D)$ is dependent only on the topological, and not the analytic, structure of L.

1. De Rham Cohomology

Let X be a real n-dimensional differentiable manifold. As before (page 32), for $p \geq 0$, $\mathcal{E}^p = \mathcal{E}^p(K)$ denotes the space of differential p-forms on X with coefficients in the field $K = \mathbb{R}, \mathbb{C}$. In particular, $\mathcal{E}^0$ is the space of differentiable K-valued functions on X.

Recall (equation 1.(1.1)) that if $K = \mathbb{C}$ and X is complex, $\mathcal{E}^p := \bigoplus_{q+r=p} \mathcal{E}^{q,r}$ where $\mathcal{E}^{q,r} := \mathcal{E}(\wedge^q \mathbf{T} \otimes \wedge^r \overline{\mathbf{T}})$.

The exterior derivative $d_p : \mathcal{E}^p \to \mathcal{E}^{p+1}$ is a K-homomorphism.

For $p \geq 1$, we define

$$Z^p := \mathrm{Ker}\,(\mathcal{E}^p \xrightarrow{d_p} \mathcal{E}^{p+1})$$

and

$$B^p := \mathrm{Im}\,(\mathcal{E}^{p-1} \xrightarrow{d_{p-1}} \mathcal{E}^p).$$

Then Z^p is the space of closed p-forms on X, and B^p the space of exact p-forms on X.

Definition 1.1. *For $K = \mathbb{R}, \mathbb{C}$, the p-th* de Rham cohomology group $H^p_{deRh}(X) := H^p_{deRh}(X, K)$ *of X is defined by*

$$H^p_{deRh}(X)\,(= H^p_{deRh}(X, K)) := {Z^p}/{B^p}.$$

Theorem 1.2. de Rham's theorem.

For X, a real paracompact n-dimensional differentiable manifold with $K_X = \mathbb{R}_X, \mathbb{C}_X$ the constant real or complex sheaf on X, there is a natural isomorphism

$$H^p(X, K) \xrightarrow{\sim} H^p_{deRh}(X, K), \quad p \geq 0.$$

Proof. We consider the resolution

$$0 \to K_X \hookrightarrow \mathcal{E}^0_X \xrightarrow{d_0} \mathcal{E}^1_X \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \mathcal{E}^n_X \xrightarrow{d_n} 0, \tag{1.1}$$

where $\hookrightarrow$ denotes inclusion. The result follows immediately from theorem 1.2.34. □

Henceforth we shall often identify $H^p(X, K)$ and $H^p_{deRh}(X, K)$.

REMARK 1.3. For a Riemann surface X, we describe the map $\Phi : H^2(X,\mathbb{R}) \to H^2_{deRh}(X)(= H^2_{deRh}(X,\mathbb{R}))$ explicitly:

Let $\mathcal{U} = (U_i)_i$ be a covering of X by open sets, and choose an $\alpha = \{\alpha_{i,j,k}\}_{i,j,k}$ in $Z^2(\mathcal{U},\mathbb{R}) \subset Z^2(\mathcal{U},\mathcal{E})$.

$\mathcal{E}_X$ is a fine sheaf (example 1.2.11) so, by theorem 1.2.31 we have

$$Z^2(\mathcal{U},\mathcal{E}) = B^2(\mathcal{U},\mathcal{E}) := \delta_1(C^1(\mathcal{U},\mathcal{E})).$$

In other words, there is a $\beta := \{\beta_{i,j}\}_{i,j} \in C^1(\mathcal{U},\mathcal{E})$ such that $\alpha = \delta_1(\beta)$.

Now, exterior differentiation $d : \mathcal{E} \to \mathcal{E}^1$ is well defined on $\mathcal{E}$ and, $\delta d = d\delta$. Therefore, since α is locally constant we have

$$0 = d\alpha = d\delta_1(\beta) = \delta_1(d\beta)$$

so $d\beta \in Z^1(\mathcal{U},\mathcal{E}^1)$.

Since $\mathcal{E}^1_X$ is also fine, $d\beta = \delta_0(\mu)$ for some $\mu \in C^0(\mathcal{U},\mathcal{E}^1)$. But then $\delta_1 d\mu = d^2\beta = 0$ so $\Phi(\alpha) := d\mu \in Z^0(\mathcal{U},\mathcal{E}^2) = \mathcal{E}^2(X)$ is a global 2-form on X. Since $d^2 = 0$, $d\mu$ is closed and therefore represents an element of $H^2_{deRh}(X)$.

To show that this is independent of the choices made, it is sufficient to show that, given a representative $\alpha = \{\alpha_{i,j,k}\}_{i,j,k}$ of $0 \in H^2(\mathcal{U},\mathbb{R})$, the two-form $d\mu$ constructed in the above manner is exact. Namely, if α is a representative of $0 \in H^2(\mathcal{U},\mathbb{R}), \alpha = \delta_1\beta$ for some $\beta \in C^1(\mathcal{U},\mathbb{R}) \subset C^1(\mathcal{U},\mathcal{E})$. So, $d\beta = (d\beta_{i,j})_{i,j} = 0$. Therefore, if $\mu \in C^0(\mathcal{U},\mathcal{E})$ is such that $\delta_0\mu = d\beta = 0$, $\mu_i = \mu_j$ for all i, j.

In other words, μ is a global 1-form and $d\mu$ is exact.

PROPOSITION 1.4. *If X is a compact, connected Riemann surface, then a differential 2-form $\alpha \in \mathcal{E}^2(X)$ is exact if and only if $\int_X \alpha = 0$. In particular $H^2_{deRh}(X,K) \cong K$.*

PROOF. If $\alpha \in \mathcal{E}^2(X)$ is exact, $\int_X \alpha = 0$ by Stokes' theorem since X is without boundary.

For the converse see [**Lo**, 35-36]. □

We will need the following lemma in the final chapter.

LEMMA 1.5. *If X, X' are homotopy equivalent topological spaces, then*

$$H^k_{deRh}(X,\mathbb{R}) \cong H^k_{deRh}(X',\mathbb{R})$$

for all k.

PROOF. [**BT**, 36] □

2. The genus of a Riemann surface

In the preceding (definition 3.2.1) we have defined the *genus g* of a compact Riemann surface X as the dimension $h^1(X,\mathcal{O}_X)$ of the first cohomology group $H^1(X,\mathcal{O})$ of the sheaf of local holomorphic functions (the sheaf with locally convergent power series as germs) on X. In what follows, we consider the equivalence of alternative definitions in order to arrive at a more obviously topological characterisation.

Let $\mathcal{O}^1 := \mathcal{O}^1(X)$ denote the space of holomorphic 1-forms on X and $\mathcal{O}^1_X$ the sheaf of locally holomorphic 1-forms on X. Then, if K is a canonical divisor on X, $\mathcal{O}^1 := \mathcal{O}(\mathbf{T}) = \mathcal{O}_K$ by proposition 3.1.19.

PROPOSITION 2.1.
$$h^1(X, \mathcal{O}) = h^0(X, \mathcal{O}^1).$$

PROOF. Since $\mathcal{O} = \mathcal{O}_0$ where 0 is the divisor with empty support, and
$$H^0(X, \mathcal{O}^1) = \mathcal{O}^1 = \mathcal{O}_K,$$
this follows from Serre Duality, proposition 3.2.3. □

DEFINITION 2.2. *Let X be a Riemann surface. A differentiable function $f : X \to \mathbb{C}$ is said to be* harmonic *if*
$$\partial\overline{\partial}(f) = 0.$$

EXAMPLE 2.3. Let f be a non-vanishing holomorphic function on $U \subset X$. Then $\log |f|^2 = \log f\overline{f}$ is harmonic. Namely,
$$\frac{\partial^2}{\partial z \partial \overline{z}} \log |f|^2 = \frac{\partial^2}{\partial z \partial \overline{z}}(\log f + \log \overline{f}) = \frac{\partial}{\partial z}(\log \overline{f}) = 0.$$

DEFINITION 2.4. *A complex differential 1-form on a Riemann surface X is a* harmonic 1-form *if it is locally of the form df with f a harmonic function on $U \subset X$.*

PROPOSITION 2.5. *A differential 1-form ω on a Riemann surface X is harmonic if and only if it is of the form*
$$\omega = \eta_1 + \overline{\eta_2}$$
with η_1, η_2 holomorphic differentials (locally of the form ϕdz, ϕ holomorphic). Therefore, the space $\mathcal{H}^1(X)$ of harmonic differentials on X is equal to the direct sum $\mathcal{O}^1(X) \oplus \overline{\mathcal{O}^1(X)}$.

PROOF. [**Fo**, 154]. □

In particular, it follows from the above proposition 4.2.5 that
$$\dim \mathcal{H}^1(X) = 2 \dim \mathcal{O}^1(X) = 2g.$$

LEMMA 2.6. *The space $\mathcal{H}^1(X)$ is isomorphic to the first de Rham cohomology group $H^1_{deRh}(X, \mathbb{C})$ of X.*

PROOF. Since $\mathcal{H}^1(X)$ consists of closed differentials, there is an obvious natural map $\mathcal{H}^1(X) \to H^1_{deRh}(X, \mathbb{C})$.

The proof that this is an isomorphism rests on Hodge theory which we will not discuss here. See [**Lo**, 39]. □

THEOREM 2.7. *$H_1(X)$ denotes the abelianised fundamental group of X, that is the group obtained by taking the quotient of the fundamental group with its commutator subgroup. It holds that*
$$H^1_{deRh}(X, \mathbb{C}) \cong \mathrm{Hom}\,(H_1(X), \mathbb{C}).$$
In particular, $H^1_{deRh}(X, \mathbb{C})$ has dimension $2g$.

PROOF. This a theorem of de Rham.

The map $\int : H^1_{deRh}(X, \mathbb{C}) \to \mathrm{Hom}\,(H_1(X), \mathbb{C})$ is given by
$$\int([\eta])([\alpha]) := \int_\alpha \eta$$

where η is a representative of $[\eta] \in H^1_{deRh}(X, \mathbb{C})$ and α a representative of $[\alpha] \in H_1(X)$. A proof that this is an isomorphism can be found in [**Lo**, 29-30].

□

Therefore the genus g of a Riemann surface X is a topological invariant of X, independent of its holomorphic structure. We have

THEOREM 2.8.

$$g := \dim H^1(X, \mathcal{O}) = \dim H^0(X, \mathcal{O}^1) = \frac{1}{2} \dim \mathrm{Hom}\,(H_1(X), \mathbb{C}).$$

From the description of g as half the number of generators of the Abelianised fundamental group, we are able to come to the intuitive idea of genus as the number of 'handles' or 'holes' in a Riemann surface.

3. The degree of a divisor

We wish to show that, if L is a holomorphic line bundle on a compact Riemann surface X, and D is the divisor of a meromorphic section of L, the degree $\deg(D)$ of D is only dependent on the topological structure of L.

Let $\mathrm{Pic}\,(X)$ denote the space of isomorphism classes of holomorphic line bundles[1] on X. $\mathbb{C}^* = \mathbf{GL}(1, \mathbb{C})$ denotes the non-zero complex numbers.

LEMMA 3.1. $\mathrm{Pic}\,(X)$ *is a group with operation* $\otimes$.

PROOF. $\mathrm{Pic}\,(X)$ has identity $\mathrm{id}_{\,\mathrm{Pic}\,(X)} = [X \times \mathbb{C} \to X]$.

If L, L' are holomorphic line bundles over X then $[L] \otimes [L'] := [L \otimes L']$ is well-defined and an element of $\mathrm{Pic}\,(X)$. Namely, if L is defined by $\{g_{i,j}\}_{i,j}$ and L' by $\{g'_{i,j}\}_{i,j}$ with respect to $\mathcal{U} = \{U_i\}_i$, $L \otimes L'$ is defined by $\{t_{i,j}\}_{i,j}$,

$$t_{i,j} := g_{i,j} g'_{i,j} : U_i \cap U_j \to \mathbb{C}^*$$

and so is also a holomorphic line bundle on X.

In the proof of theorem 3.1.17, we have seen that isomorphic line bundles can be represented by the same transition functions. Therefore the class of $[L \otimes L']$ is independent of the choices of representatives L and L' for $[L]$ and $[L']$.

Finally, by proposition 1.1.20, $[L]^{-1} \in \mathrm{Pic}\,(X)$ is given by $[L^*]$ where L^* is the holomorphic line bundle dual to L.

□

Let $\mathcal{O}^*_X$ denote the sheaf of non-vanishing locally holomorphic functions on X.

PROPOSITION 3.2. *For a compact, connected Riemann surface* X, *there is a natural isomorphism*

$$H^1(X, \mathcal{O}^*) \cong \mathrm{Pic}\,(X).$$

PROOF. A representative of an isomorphism class $\tilde{L}$ of line bundles on X is defined by a covering $\mathcal{U} = (U_i)_{i \in I}$ of X and non-zero holomorphic functions $g_{i,j}$ on $U_i \cap U_j$ such that $g_{i,k} = g_{i,j} g_{j,k}$ for all $i, j, k \in I$. So, L corresponds to an element $(g_{i,j})_{i,j}$ of $Z^1(\mathcal{U}, \mathcal{O}^*)$. Conversely, an element $(g_{i,j})_{i,j}$ of $Z^1(\mathcal{U}, \mathcal{O}^*)$ defines the transition functions relative to $\mathcal{U}$ of a line bundle L on X.

---

[1] The notation $\mathrm{Pic}\,(X)$ stands for the *Picard group* of the space X.

Now, $L' \in \tilde{L}$ if and only if there exist non-zero holomorphic functions f_i on each U_i such that, if L' is defined by $(g'_{i,j})_{i,j}$, $g'_{i,j} = \frac{f_i}{f_j} g_{i,j}$. But this is precisely when $(g_{i,j})_{i,j}$ and $(g'_{i,j})_{i,j}$ belong to the same class in $H^1(\mathcal{U}, \mathcal{O}^*)$.

Therefore, there is a well defined natural one to one correspondence between $\mathrm{Pic}\,(X)$ and $H^1(X, \mathcal{O}^*)$.

Since the trivial bundle on X can be defined by the transition functions $g_{i,j}(x) = \mathrm{id} = 1$ for all $x \in U_i \cap U_j$, and the correspondence is defined in terms of the multiplicative property of the transition functions, it is straightforward to show that this is an isomorphism.

□

In what follows we will therefore often use isomorphism classes of holomorphic line bundles and elements of $H^1(X, \mathcal{O}^*)$ interchangeably. We may also use the same notation when referring to representatives of the classes of $\mathrm{Pic}\,(X) \cong H^1(X, \mathcal{O}^*)$, though, of course, only where we have shown the results to be independent of the particular choice of representative.

REMARK 3.3. In particular $\deg(L_D) = \deg(D)$ by lemma 3.1.15.

LEMMA 3.4. *The degree map* $\deg : \mathrm{Pic}\,(X) \to \mathbb{Z}$ *is a surjective group homomorphism. In other words*

(1) $\deg(L \otimes M) = \deg(L) + \deg(M)$
(2) $\deg(L^*) = -\deg(L)$

PROOF. Given two line bundles L, and L' over X defined via the same covering $\mathcal{U} = \{U_i\}$ by the transition functions $\{g_{i,j}\} = \{g_i/g_j\}$ and $\{g'_{i,j}\} = \{g'_i/g'_j\}$ respectively, we calculate the degree of the tensor product $L \otimes L'$. A section of $L \otimes L'$ has locally the form $\xi_i \xi'_i$ and on $U_i \cap U_j$ we have $\xi_j \xi'_j = (g_{j,i}\xi_i)(g'_{j,i}\xi'_i) = g_{j,i}g'_{j,i}\xi_i\xi'_i$. But $g_{j,i}g'_{j,i} = \frac{g_j g'_j}{g_i g'_i}$ and so $\deg(L \otimes L') = \deg(L) + \deg(L')$.

We have already seen that L^* can be described by the transition functions $\{g_{i,j}^{-1} = \frac{g_j}{g_i}\}_{i,j}$. So $\deg(L^*) = -\deg(L)$ as required

The map is surjective since for all $n \in \mathbb{Z}$ we can define a divisor D_n with $\deg(D_n) = \deg(\{D_n\}) = n$, for example we take the point divisor $D_n = n \cdot x$ for some $x \in X$.

□

EXAMPLE 3.5. In example 3.1.18 we have seen that two holomorphic line bundles on $\mathbb{CP}^1$ are isomorphic if and only if they have the same degree. Therefore, by the above lemma 4.3.4 and examples 1.1.5 and 3.1.18, if $\mathbf{T}$ is the holomorphic cotangent bundle on $\mathbb{CP}^1$,

$$\mathbf{T} \cong \mathcal{O}_{\mathbb{CP}^1}(-1) \otimes \mathcal{O}_{\mathbb{CP}^1}(-1).$$

In particular, if $\mathbb{CP}^1 = U_0 \cup U_1$ is the covering of $\mathbb{CP}^1$ given in example 1.1.5, then relative to this covering, $\mathbf{T}$ has transition function $g_{0,1}[1 : z] = (\frac{1}{z})^2 = \frac{1}{z^2}$.

Clearly, if $\mathcal{T}$ is the holomorphic tangent bundle of $\mathbb{CP}^1$, $\deg(\mathcal{T}) = -\deg(\mathbf{T}) = 2$ and

$$\mathcal{T} = \mathbf{T}^* = \mathcal{O}_{\mathbb{CP}^1}(-1)^* \otimes \mathcal{O}_{\mathbb{CP}^1}(-1)^*.$$

Moreover, by 3.1.18, two holomorphic line bundles on $\mathbb{CP}^1$ are isomorphic if and only if they have the same degree. So, if we define

$$\mathcal{O}_{\mathbb{CP}^1}(k) := \begin{cases} \otimes^k \mathcal{O}_{\mathbb{CP}^1}(-1)^* =: \otimes^k \mathcal{O}_{\mathbb{CP}^1}(1), & k > 0 \\ \otimes^k \mathcal{O}_{\mathbb{CP}^1}(-1), & k \leq 0 \end{cases}$$

(where $\otimes^0 \mathcal{O}_{\mathbb{CP}^1}(-1) = \mathbb{C} \times X \to X$ is the trivial line bundle), then, for every holomorphic line bundle L on $\mathbb{CP}^1$,

$$\deg(L) = k \;\Rightarrow\; L \cong \mathcal{O}_{\mathbb{CP}^1}(k).$$[2]

By proposition 4.3.2, a holomorphic line bundle L over X can be viewed as an element of the cohomology group $H^1(X, O^*)$ represented by the cocycle $\{g_{i,j}\}_{i,j}$ (with respect to a covering $\mathcal{U} = \{U_i\}_i$ of X).

Let (L, h) be a holomorphic line bundle with Hermitian metric h over a compact Riemann Surface X and let $\lambda = \{\lambda_i\}_i$ be a collection of positive functions λ_i on U_i obtained as in remark 1.1.15 relative to a cover $\mathcal{U} = \{U_i\}_i$ of X.

Proposition 3.6. *There is a global closed 2-form $\tilde{c}_1(L)$ on X defined by*

$$\tilde{c}_1(L)|_{U_i} = \frac{i}{2\pi} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda_i dz \wedge d\overline{z}.$$

Proof. On $U_i \cap U_j$

$$\frac{i}{2\pi} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda_j dz \wedge d\overline{z} = \frac{i}{2\pi} \frac{\partial^2}{\partial z \partial \overline{z}} \log(g_{i,j} \overline{g_{i,j}} \lambda_i) dz \wedge d\overline{z}$$

$$= \frac{i}{2\pi} \frac{\partial^2}{\partial z \partial \overline{z}} (\log |g_{i,j}|^2 + \log \lambda_i) dz \wedge d\overline{z} = \frac{i}{2\pi} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda_i dz \wedge d\overline{z}.$$

So we can write $\tilde{c}_1(L) = \frac{i}{2\pi} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda dz \wedge d\overline{z}$ is a globally defined 2-form on X. By definition $\tilde{c}_1(L)$ is closed.

□

Since $\tilde{c}_1(L)$ is closed it represents an element of the de Rham cohomology group $H^2_{deRh}(X)$.

Proposition 3.7. *The class of $\tilde{c}_1(L)$ in $H^2_{deRh}(X)$ is independent of the choice of metric h on L.*

Proof. Let ${}_1h$ and ${}_2h$ be Hermitian metrics on L which induce collections of positive functions ${}_1\lambda, {}_2\lambda$ respectively. Then $\sigma = (\sigma_i)_i = \left(\frac{{}_1\lambda_i}{{}_2\lambda_i}\right)_i$ is positive. Furthermore, σ is globally defined since

$$\sigma_j = \frac{{}_1\lambda_j}{{}_2\lambda_j} = \frac{{}_1\lambda_i g_{i,j} \overline{g_{i,j}}}{{}_2\lambda_i g_{i,j} \overline{g_{i,j}}} = \frac{{}_1\lambda_i}{{}_2\lambda_i} = \sigma_i$$

on $U_i \cap U_j$ for all i, j. We have

$$\frac{i}{2\pi} \left(\frac{\partial^2}{\partial z \partial \overline{z}} \log {}_1\lambda_i dz \wedge d\overline{z} - \frac{\partial^2}{\partial z \partial \overline{z}} \log {}_2\lambda_i dz \wedge d\overline{z} \right)$$

---

[2] In fact, if $E \to \mathbb{CP}^1$ is holomorphic, then it can be proved that E is a direct sum of holomorphic line bundles on $\mathbb{CP}^1$. Therefore, if $\mathrm{rk}\,(E) = r$, there exist unique $k_1 \leq \cdots \leq k_r$ such that

$$E \cong \mathcal{O}_{\mathbb{CP}^1}(k_1) \oplus \cdots \oplus O_{\mathbb{CP}^1}(k_r).$$

This is the *Grothendieck splitting theorem* ([**Gr**]).

$$= \frac{i}{2\pi}\frac{\partial^2}{\partial z \partial \overline{z}} \log \sigma dz \wedge d\overline{z} = d\left(\frac{i}{2\pi}\frac{\partial}{\partial \overline{z}} \log \sigma d\overline{z}\right)$$

which is exact.

□

DEFINITION 3.8. *The 2-form* $\tilde{c}_1(L)$ *is called the* first Chern form *of* L.

We denote the class of $\tilde{c}_1(L)$ *in* $H^2_{deRh}(X)$*, also by* $\tilde{c}_1(L)$*. This is the* first Chern class *of the line bundle* L.

LEMMA 3.9. *The map* $\tilde{c}_1 : \left(\mathrm{Pic}\,(X) \cong H^1(X, \mathcal{O}^*)\right) \to H^2_{deRh}(X)$ *is a homomorphism of groups.*

PROOF. Let $\mathbf{1}$ denote the trivial line bundle on X. Then we can choose a metric h on $\mathbf{1}$ such that $\lambda \equiv 1$ (since $\mathbf{1}$ is trivial we can choose $\lambda = \lambda_i$ for all i relative to any cover $\{U_i\}$ of X, λ is as in remark 1.1.15). Since $\log 1 = 0$, $\tilde{c}_1(\mathbf{1}) = 0 \in \mathcal{E}^2(X)$ so $\tilde{c}_1(\mathbf{1}) = 0$ in the additive group $H^2_{deRh}(X)$.

If L is a vector bundle with metric h and local frames f_i and L' is a vector bundle with metric h' and local frames f'_i, we obtain positive functions (as in remark 1.1.15)

$$\{\lambda_i := h_i(f_i, f_i) : U_i \to \mathbb{R}^+\}_i =: \lambda\} \quad \text{and} \quad \{\lambda'_i := h'_i(f'_i, f'_i) : U_i \to \mathbb{R}^+\}_i =: \lambda'.$$

Furthermore hh' defines a metric on $L \otimes L'$ and $f_i f'_i$ are local frames for $L \otimes L'$ so hh' is given by $\lambda\lambda' = \{\lambda_i \lambda'_i : U_i \to \mathbb{R}^+\}_i$. Then

$$\begin{aligned} \tilde{c}_1(L \otimes L') &= \frac{i}{2\pi} \log(\lambda\lambda') dz \wedge d\overline{z} = \frac{i}{2\pi} \log \lambda dz \wedge d\overline{z} + \frac{i}{2\pi} \log \lambda' dz \wedge d\overline{z} \\ &= \tilde{c}_1(L) + \tilde{c}_1(L') \in H^2_{deRh}(X) \end{aligned}$$

as required.

□

EXAMPLE 3.10. Let $\mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1$ be the bundle defined in example 1.1.5. Then, if z is a local coordinate, we have

$$\tilde{c}_1(\mathcal{O}_{\mathbb{CP}^1}(-1)) = \frac{i}{2\pi}\frac{\partial^2}{\partial z \partial \overline{z}} \log(1 + |z|^2) dz \wedge d\overline{z} = \frac{i}{2\pi}\frac{1}{(1 + |z|^2)^2} dz \wedge d\overline{z}.$$

We consider the short exact sequence of sheaves

$$0 \to \mathbb{Z}_X \hookrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0.$$

where $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ is the map defined by $\exp(f) := e^{2if}$ for $f \in \mathcal{O}(U)$ and $U \subset X$ open.

The connecting homomorphism theorem 1.2.30, says that the sequence

$$\cdots \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow{\delta^*} H^2(X, \mathbb{Z}) \to \ldots \tag{3.1}$$

is exact. In proposition 1.2.25 we have seen that the obvious map $\Theta : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ induced by the inclusions $Z^2(\mathcal{U}, \mathbb{Z}) \subset Z^2(\mathcal{U}, \mathbb{R})$ and $C^1(\mathcal{U}, \mathbb{Z}) \subset C^1(\mathcal{U}, \mathbb{R})$ is a well-defined homomorphism.

THEOREM 3.11. *Identifying* $H^2(X, \mathbb{R})$ *and* $H^2_{deRh}(X)$ *via theorem 4.2.7, it holds that*

$$\tilde{c}_1 = \Theta \circ \delta^* (: H^1(X, \mathcal{O}^*) \to H^2_{deRh}(X))$$

where $\delta^* : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ *is the connecting homomorphism.*

PROOF. By the footnote in [**We**, 104], there is an open covering $\mathcal{U} = \{U_i\}_i$ of X such that each intersection $U_i \cap U_j$ is simply connected. Let $\{g_{i,j}\}_{i,j}$ denote the corresponding transition functions.

We denote a map $Z^1(\mathcal{U}, \mathcal{O}^*) \to Z^2(\mathcal{U}, \mathbb{Z})$ which induces the connecting homomorphism δ^* also by δ^* and, given $g = \{g_{i,j}\}_{i,j} \in Z^1(\mathcal{U}, \mathcal{O}^*)$, we wish to construct a $\delta^*(g) \in Z^2(\mathcal{U}, \mathbb{Z})$.

For all $x \in X$, $\exp_x : (\mathcal{O}_X)_x \to (\mathcal{O}_X^*)_x$ is a surjective homomorphism so, since connected components of $U_i \cap U_j$ are simply connected for all i, j, there is a $f = (f_{i,j})_{i,j} \in C^1(\mathcal{U}, \mathcal{O})$ such that $f_{i,j} = \frac{i}{2\pi} \log g_{i,j} = \exp^{-1}(g_{i,j})$.

Then $\delta_1 f \in C^2(\mathcal{U}, \mathcal{O})$ and, in particular, $\delta_1 f \in Z^1(\mathcal{U}, \mathcal{O})$ (since $\delta^2 = \delta_2\delta_1 = 0$). Using the fact that $g_{i,k} = g_{i,j}g_{j,k}$ for all i, j, k we have

$$(\delta_1 f)_{i,j,k} = \frac{i}{2\pi}(\log g_{j,k} - \log g_{i,k} + \log g_{i,j}) \in \mathbb{Z} \subset \mathcal{O}$$

on each connected component of $U_i \cap U_j \cap U_k$ for all i, j, k.

So, $\delta_1 f \in Z^2(\mathcal{U}, \mathbb{Z})$ and is a representative for an element $\delta_1 f \in H^2(X, \mathbb{Z})$ as well as for the $\Theta\delta_1 f \in H^2(X, \mathbb{R})$.

We construct the corresponding element of $H^2_{deRh}(X)$: Using the method of remark 4.1.3, we let $\alpha := \delta_1 f$ and $\beta = f$. By the inclusion $\mathcal{O} \hookrightarrow \mathcal{E}$, $f = \{f_{i,j}\}_{i,j}$ is in $C^1(\mathcal{U}, \mathcal{E})$.

$L \to X$ is a holomorphic line bundle defined by $g = \{g_{i,j}\}_{i,j}$ and h is a Hermitian metric on L. $\lambda = \{\lambda_i\}_i$ is a collection of positive functions obtained from h as in remark 1.1.15. We choose $\mu = \mu_i \in C^0(\mathcal{U}, \mathcal{E}^1)$, $\mu_i = \frac{i}{2\pi}\frac{\partial}{\partial z} \log \lambda_i dz$. Then

$$(\delta_0\mu)_{i,j} = \mu_i - \mu_j = \frac{i}{2\pi}\frac{\partial}{\partial z} \log\left(\frac{\lambda_i}{\lambda_j}\right) dz$$

$$= \frac{i}{2\pi}\frac{\partial}{\partial z} \log\left(g_{i,j}\overline{g_{i,j}}\right) dz = \frac{i}{2\pi}\frac{\partial}{\partial z} \log(g_{i,j}) dz = df_{i,j}.$$

Therefore

$$\delta d\mu = d\delta\mu = d^2 f = 0$$

so

$$d\mu = \frac{i}{2\pi}\frac{\partial^2}{\partial z\partial\overline{z}} \log\lambda dz \wedge d\overline{z} \in Z^0(\mathcal{U}, \mathcal{E}^2) = \mathcal{E}^2$$

is a global 2-form and equal to $\tilde{c}_1(L)$.

□

REMARK 3.12. Sometimes it will also be convenient to refer to the element of $H^2(X, \mathbb{R})$ represented by $\delta_1(f) := \{(\delta_1 f)_{i,j,k}\}_{i,j,k}$ where

$$(\delta_1 f)_{i,j,k} = \frac{i}{2\pi}(\log g_{j,k} - \log g_{i,k} + \log g_{i,j}) \in \mathbb{Z}$$

also as the first Chern form $\tilde{c}_1(L)$ of L.

THEOREM 3.13. $\int_X \tilde{c}_1(L) = \deg(L)$.

PROOF. By lemmas 4.3.4 and 4.3.9, it suffices to prove this for $L = \{(p)\}$ a line bundle associated to a simple point divisor.

For such an L there exists a global holomorphic section s of L which is non-zero on $X - \{p\}$ and which vanishes to first degree at $p \in X$.

Since the result is independent of the choice of metric on L by proposition 4.3.7, we can choose a covering $\mathcal{U} = \{U_i\}_{i=0}^N$ for X, and a metric h for L as follows:

Let $\mathcal{U}$ be an atlas for X such that, relative to a local coordinate, $U_0 = B_1$ is a disc centered at $\{p\}$ with radius 1 relative to the metric on X (with scaling if necessary), $B_{\frac{1}{2}} \subset B_1$ is the concentric disc with radius $\frac{1}{2}$, and $B_{\frac{1}{2}} \cap U_i = \emptyset$ for all $i \neq 0$. We also choose $s = z$ on U_0, and $s \equiv 1$ on U_i, $i \neq 0$. Via a partition of unity we can construct a Hermitian metric h on L and choose a frame f for $U_0 = B_1$ such that $\lambda_0|_{B_{\frac{1}{2}}} := h(f)|_{B_{\frac{1}{2}}} \equiv 1$.

$\{g_{i,j}\}_{i,j}$ is the set of corresponding transition functions for L.

We let $B_r := B(p, r)$ be a disc about p with radius r in a coordinate neighbourhood of X.

There is a positive function $|s|^2$ on X given by

$$|s|^2 = \lambda_i s_i \overline{s_i}, \quad \text{on } U_i.$$

Then $|s|^2$ is globally defined since

$$\lambda_j s_j \overline{s_j} = \lambda_i g_{i,j} \overline{g_{i,j}} g_{j,i} s_i \overline{g_{j,i} s_i} = \lambda_i s_i \overline{s_i}.$$

s is holomorphic and non-zero on $X \backslash B_r$. So, on $U_i \backslash B_r$, $\log |s|^2 = \log \lambda_i + \log |s_i|^2$, and therefore

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda = \frac{\partial^2}{\partial z \partial \overline{z}} \log |s|^2. \tag{3.2}$$

Although $\tilde{c}_1(L, \lambda) = \frac{i}{2\pi} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda \, dz \wedge d\overline{z}$ is well-defined on X, $\frac{\partial}{\partial z} \log \lambda dz$ is not, in general, globally defined (as can be easily seen by writing out how it transforms under the transition functions $g_{i,j}$). However, since $|s|^2$ is globally defined

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log |s|^2 dz d\overline{z} = d(\frac{\partial}{\partial z} \log |s|^2 dz).$$

so by equation 4.(3.2),

$$\int_{X \backslash B_r} \tilde{c}_1(L) = \frac{i}{2\pi} \int_{X \backslash B_r} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda \; dz \; d\overline{z} = \frac{i}{2\pi} \int_{X \backslash B_r} d\left(\frac{\partial}{\partial z} \log |s|^2 \; dz \right).$$

We may apply Stoke's theorem to obtain

$$\int_{X \backslash B_r} \tilde{c}_1(L) = \frac{i}{2\pi} \int_{X \backslash B_r} d\left(\frac{\partial}{\partial z} \log |s|^2 \; dz \right) = -\frac{i}{2\pi} \int_{\partial B_r} \frac{\partial}{\partial z} \log |s|^2 \; dz.$$

For $r < \frac{1}{2}$, $|s|^2 = |z|^2$ so

$$-\frac{i}{2\pi} \int_{\partial B_r} \frac{\partial}{\partial z} \log |s|^2 \; dz = -\frac{i}{2\pi} \int_{\partial B_r} \frac{\partial}{\partial z} \log |z|^2 \; dz = -\frac{i}{2\pi} \int_{\partial B_r} \frac{1}{z} \; dz.$$

Since, by the residue theorem $\oint \frac{1}{z} \, dz = 2\pi i$ when the closed curve of integration contains $z = 0$,

$$\int_X \tilde{c}_1(L) = \lim_{r \to 0} \frac{i}{2\pi} \int_{X \backslash B_r} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda \; dz \; d\overline{z} = \lim_{r \to 0} -\frac{i}{2\pi} \int_{\partial B_r} \frac{1}{z} \; dz = 1$$

as required.

□

EXAMPLE 3.14. In example 4.3.10, we saw that

$$\tilde{c}_1(\mathcal{O}_{\mathbb{CP}^1}(-1))(z) = \frac{1}{(1+|z|^2)^2} dz \wedge d\overline{z}.$$

Since $\lim_{z\to\infty} \frac{1}{(1+|z|^2)^2} = 0$, theorem 4.3.13 gives

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{(1+|z|^2)^2} dz \wedge d\overline{z} = 1.$$

Let $\mathcal{C}_X$ denote the sheaf of locally continuous functions on X and $\mathcal{C}_X^*$ denote the sheaf of locally continuous functions on X which are nowhere vanishing on their domain. A holomorphic line bundle $L \in H^1(X, \mathcal{O}^*)$ (see proposition 4.3.2) is, in particular, a continuous line bundle. That is, $L \in H^1(X, \mathcal{C}^*)$.

Since the sequence

$$0 \to \mathbb{Z}_X \to \mathcal{C}_X \xrightarrow{\exp} \mathcal{C}_X^* \to 0$$

is exact, so too is the sequence

$$\cdots \to H^1(X,\mathcal{C}) \to H^1(X,\mathcal{C}^*) \xrightarrow{\delta_*} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{C}) \to \ldots. \tag{3.3}$$

LEMMA 3.15. $H^1(X,\mathcal{C}) = H^2(X,\mathcal{C}) = \{0\}$.

PROOF. $\mathcal{C}_X$ is a fine sheaf since there exists a continuous partition of unity on X. The result follows from theorem 1.2.31. □

THEOREM 3.16. *The degree* $\deg D$ *of a divisor* D *on a compact Riemann surface* X *is a topological invariant of the line bundle* L_D *and not dependent on the holomorphic structure of* L_D.

PROOF. The commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_X & \xrightarrow{\iota} & \mathcal{O}_X & \xrightarrow{\exp_{\mathcal{O}}} & \mathcal{O}_X^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \mathbb{Z}_X & \xrightarrow[\iota]{} & \mathcal{C}_X & \xrightarrow[\exp_{\mathcal{C}}]{} & \mathcal{C}^* & \longrightarrow & 0.
\end{array} \tag{3.4}$$

induces a commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H^1(X,\mathcal{O}) & \xrightarrow{\exp_{\mathcal{O}}} & H^1(X,\mathcal{O}^*) & \xrightarrow{\delta^*_{\mathcal{O}}} & H^2(X,\mathbb{Z}) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
 & & (0 = H^1(X,\mathcal{C})) & \xrightarrow[\exp_{\mathcal{C}}]{} & H^1(X,\mathcal{C}^*) & \xrightarrow[\delta^*_{\mathcal{C}}]{} & H^2(X,\mathbb{Z}) & \longrightarrow & 0.
\end{array} \tag{3.5}$$

([**We**, 56-57]) where ι denotes inclusion in each case, and

$$\alpha := \big(\Theta : H^1(X,\mathcal{O}) \to H^1(X,\mathcal{C})(=0)\big)$$

and

$$\beta := \big(\Theta : H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{C}^*)\big)$$

are natural homomorphism induced by the inclusions (proposition 1.2.25). In particular $\alpha \equiv 0$ is the zero-map.

Since the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ in diagram 4.(3.5) is the identity, so, in particular a bijection, by theorem 4.3.11, the following diagram commutes

$$\begin{array}{ccccccc} & & & & H^2(X, \mathbb{R}) & \xrightarrow{\int} & \mathbb{R} \\ & & & \nearrow^{\tilde{c}_1} & \uparrow{\scriptstyle\Theta} & & \\ \dots & \longrightarrow & H^1(X, \mathcal{O}^*) & \xrightarrow{\delta^*_{\mathcal{O}}} & H^2(X, \mathbb{Z}) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\sim} & & \\ \dots & \longrightarrow & H^1(X, \mathcal{C}^*) & \xrightarrow[\delta^*_{\mathcal{C}}]{} & H^2(X, \mathbb{Z}) & \longrightarrow & \dots . \end{array} \tag{3.6}$$

That is, the map obtained by first ignoring the holomorphic structure of a line bundle $L \to X$ and then applying the connecting homomorphism $\delta^*_{\mathcal{C}}$ and $\int\ \Theta$ is the same as $\int\ \tilde{c}_1$. By theorem 4.3.13 $(\int\ \tilde{c}_1 \equiv \deg) : \mathrm{Pic}\,(X) \to \mathbb{Z}$. In other words, the degree of a holomorphic and therefore continuous line bundle is a topological property independent of its holomorphic structure.

□

It has therefore been shown that the right-hand side of the Riemann-Roch equation 3.(2.1) can be characterised by the topology of the line bundle L_D associated to the divisor D on the Riemann surface X.

CHAPTER 5

Some aspects of algebraic topology and the T-characteristic

1. Chern classes

We generalise the Chern classes defined in chapter 4 to characteristic classes of complex differentiable vector bundles of rank r on smooth complex manifolds of dimension n.

1.1. Curvature. Connections on a vector bundle $E \to X$ have been defined in definition 1.1.30.

DEFINITION 1.1. *A connection ∇ on a vector bundle $E \to X$ defines in a natural fashion an element $K_\nabla \in \mathcal{E}^2(X, Hom(E,E))$ called the* curvature tensor *of ∇.*

If f is a frame at x, and A is the matrix for ∇ with respect to f at x, then K_∇ at x is given by the $r \times r$ matrix of 2-forms

$$\Theta_\nabla(f) = dA + A \wedge A.$$

LEMMA 1.2. *If g is a change of frame then we have*

$$\Theta_\nabla(gf) = g^{-1}\Theta_\nabla(f)g$$

and so $K_\nabla : \mathcal{E}(E) \to \mathcal{E}^2(E)$ is globally defined,

PROOF. [**We**, 72-73]. □

REMARK 1.3. Given vector fields V_1, V_2 defined on an open set of X and a differentiable section $\xi \in \mathcal{E}(E)$, we have

$$K(V_1,V_2)(\xi) = \nabla(V_1)(\nabla(V_2)(\xi)) - \nabla(V_2)(\nabla(V_1)(\xi)) - \nabla([V_1,V_2])(\xi).$$

We have already defined (definition 1.1.34) the canonical connection $\nabla_{(E,h)}$ with respect to a Hermitian metric h on a holomorphic bundle E.

DEFINITION 1.4. *The* canonical curvature K_h *on a holomorphic vector bundle E with hermitian metric h is the curvature form $K_h := K_{\nabla_{(E,h)}}$.*

For simplicity of notation, we will usually refer to the canonical connection $\nabla_{(E,h)}$ simply as ∇ and the canonical curvature as K.

Now let $f = (f_i)_{i=1}^r$ be a holomorphic frame for E on $U \subset X$ open and ∇ the canonical connection with respect to h. A is the corresponding connection matrix on U. Since f is a frame, $h(f)$ is invertible. Then, on U,

LEMMA 1.5. (1) *$A(f) = h(f)^{-1}\partial h(f)$,*
(2) *$\Theta(f) = \overline{\partial} A$.*

PROOF. As in (1.3), page 13, we set $h_{\alpha,\beta} := h(f_\beta, f_\alpha), \alpha, \beta = 1 \dots r$ and similarly $dh_{\alpha,\beta} := dh(f_\beta, f_\alpha)$, and use the notation $h := h(f) = (h_{\alpha,\beta})_{\alpha,\beta}$,
$dh := dh(f) = (dh_{\alpha,\beta})_{\alpha,\beta} = (\partial h_{\alpha,\beta})_{\alpha,\beta} + (\overline{\partial} h_{\alpha,\beta})_{\alpha,\beta} =: \partial h + \overline{\partial} h$. Then h, resp. dh (and ∂h, $\overline{\partial} h$) are $r \times r$ matrices of functions, resp. differentials on U.

If ∇ is the canonical connection with respect to h it holds by equation 1.(1.11) that

$$\begin{aligned} dh_{\alpha,\beta} &= h(\nabla f_\beta, f_\alpha) + h(f_\beta, \nabla f_\alpha) \\ &= h(\sum_{\delta=1}^{r} A_{\delta,\beta} f_\delta, f_\alpha) + \overline{h(\sum_{\delta=1}^{r} A_{\delta,\alpha} f_\delta, f_\beta)} \\ &= hA + \overline{A}^t h. \end{aligned}$$

Now by equation 1.(1.12), we have $\nabla'' \xi = 0$ for $\xi \in \mathcal{E}(E)$ holomorphic. Therefore, in particular $A = A(f)$ is of type (1,0) so we have $\partial h = hA$ and $\overline{\partial} h = \overline{A}^t h$.

In other words, if f is a holomorphic frame, $A = \partial h(f) \cdot h(f)^{-1}$ so (1) is proved.

Since

$$0 = \partial(1) = \partial \left(h \cdot h^{-1}\right) = \partial h \cdot h^{-1} + h \partial h^{-1},$$

it holds that

$$\partial h^{-1} = -h^{-1} \cdot \partial h \cdot h^{-1}. \tag{1.1}$$

Therefore, by part (1) and 5.(1.1)

$$\begin{aligned} \partial A &= \partial(h^{-1} \partial h) \\ &= \partial h^{-1} \wedge \partial h \\ &= -h^{-1} \cdot \partial h \cdot h^{-1} \wedge \partial h \\ &= -h^{-1} \partial h \wedge h^{-1} \partial h \\ &= -A \wedge A. \end{aligned}$$

It follows that

$$\Theta(f) : dA + A \wedge A = \partial A + \overline{\partial} A + A \wedge A = -A \wedge A + \overline{\partial} A + A \wedge A = \overline{\partial} A$$

as required.

□

1.2. Invariant Polynomials. Let $\mathcal{M}_r$ denote the set of $r \times r$ complex matrices.

DEFINITION 1.6. *A multi-linear form* $\tilde{\phi} : \mathcal{M}_r \times \cdots \times \mathcal{M}_r \to \mathbb{C}$ *is called* invariant *if*

$$\tilde{\phi}(gA_1 g^{-1}, \dots, gA_k g^{-1}) = \tilde{\phi}(A_1, \dots, A_k)$$

for all $g \in \mathbf{GL}(r, \mathbb{C})$ *and all* $A_i \in \mathcal{M}_r$.

Now, let X be a complex manifold and $\omega_i \in \mathcal{E}^p = \mathcal{E}^p(X)$ for $i = 1, \dots k$. We can extend the action of $\tilde{\phi}$ to $\mathcal{M}_r \otimes \mathcal{E}^p$ by settting

$$\tilde{\phi}(A_1 \otimes \omega_i, \dots, A_k \otimes \omega_k) := (\omega_1 \wedge \cdots \wedge \omega_k) \tilde{\phi}(A_1, \dots, A_k) \in \mathcal{E}^{pk}. \tag{1.2}$$

It is simple to check that $\tilde{\phi}$ so defined is well defined and multi-linear on $\mathcal{M}_r \otimes \mathcal{E}^p$.

Let $\pi : E \to X$ be a complex vector bundle with $\operatorname{rk} E = r$. Given a choice of frame f at $x \in X$, the restriction of a homomorphism $E \to E$ to a single fibre E_x

can be written as an element of $\mathcal{M}_r$ and it is possible to further extend the action of $\tilde{\phi}$ to $\mathcal{E}^p(\text{ Hom}(E,E)) := \mathcal{E}((\text{ Hom}(E,E) \otimes \wedge^p T^*X)$ to obtain a map

$$\tilde{\phi}_X : \mathcal{E}^p(\text{ Hom}(E,E)) \times \cdots \times \mathcal{E}^p(\text{ Hom}(E,E)) \to \mathcal{E}^{pk}(X). \tag{1.3}$$

To this end, let $U \subset X$ be an open subset over which E is trivial. If $\xi_i \in \mathcal{E}^p(U, \text{ Hom}(E,E))$, for $i = 1, \ldots k$ and f is a frame for E over U , $\xi_i(f)$ has the form $A_i\omega_i$, with $A_i \in \mathcal{M}_r$ and $\omega_i \in \mathcal{E}^p(U)$, so it is natural to define

$$\tilde{\phi}_U(\xi_1, \ldots, \xi_k) := \tilde{\phi}_U(\xi_1(f), \ldots, \xi_k(f)). \tag{1.4}$$

Lemma 1.7. *This definition is independent of the choice of frame so $\tilde{\phi}_U$ can be extended to all of X to obtain a map $\tilde{\phi}_X$ as in 5.(1.5).*

Proof. Given a choice of frame f for E on $U \subset X$ open, another frame on U has the form $g \circ f$ where g has values in $\mathbf{GL}(r, \mathbb{C})$. If $\alpha \in \text{Hom}\,(E,E)$ then $\alpha(g \circ f) = g^{-1}\alpha(f)g$. Therefore, by definition, if $\tilde{\phi}$ is an invariant multi-linear form and $\xi_i \in \mathcal{E}^p(U, \text{ Hom}(E,E))$ as above,

$$\begin{aligned} \tilde{\phi}_U(\xi_1(g \circ f), \ldots, \xi_k(f \circ f)) &= \tilde{\phi}_U(g^{-1}\xi_1(f)g, \ldots, g^{-1}\xi_k(f)g) \\ &= \tilde{\phi}_U(\xi_1(g \circ f), \ldots, \xi_k(f \circ f)). \end{aligned}$$

In other words, $\tilde{\phi}_U$ is independent of the choice of frame.

□

Definition 1.8. *We call a map $\phi : \mathcal{M}_r \to \mathbb{C}$ an* invariant polynomial *of degree k if for every $g \in GL(r, \mathbb{C})$ and for every $A \in \mathcal{M}_r$, $\phi(gAg^{-1}) = \phi(A)$ and $\phi(A)$ is a homogeneous polynomial of degree k in the entries of A.*

Example 1.9. The determinant map $\det : \mathcal{M}_r \to \mathbb{C}$ is an invariant polynomial of degree r. Furthermore

$$\det(I + A) = \sum_{k=0}^{r} \Phi_k(A)$$

where each Φ_k is an invariant polynomial of degree k in the entries of A.

Remark 1.10. We note that an invariant k-linear form $\tilde{\phi}$ naturally defines an invariant polynomial ϕ by setting

$$\phi(A) := \tilde{\phi}(A, \ldots, A),\ A \in \mathcal{M}_r.$$

The converse is also true (see [**We**, 85]): Every invariant polynomial ϕ of degree k acting on $\mathcal{M}_r$ defines a linear map

$$\tilde{\phi} : \underbrace{\mathcal{M}_r \times \cdots \times \mathcal{M}_r}_{k \text{ times}} \to \mathbb{C}$$

such that $\tilde{\phi}$ is invariant, and $\tilde{\phi}(A, \ldots, A) = \phi(A)$ for all A. In particular there is a 1-1 correspondence between the invariant k-linear forms on $\mathcal{M}_r$ and the invariant homogeneous polynomials of degree k acting on $\mathcal{M}_r$. So we can use the same symbol ϕ for both the invariant k-linear form $\tilde{\phi}$ and its restriction to the diagonal ϕ.

Remark 5.1.10 implies that, given an invariant polynomial ϕ of degree k, we can extend the action of ϕ to $\mathcal{E}^p(\operatorname{Hom}(E,E))$ by constructing the map (5.1.3) corresponding to $\tilde{\phi}$ and then evaluating this on the diagonal. That is, for $\xi \in \mathcal{E}^p(\operatorname{Hom}(E,E))$,

$$\phi_X(\xi) : \mathcal{E}^p(\operatorname{Hom}(E,E)) \to \mathcal{E}^{pk}(X),\ \phi_X(\xi) = \tilde{\phi}_X(\xi,\dots,\xi). \tag{1.5}$$

Given a connection $\nabla : \mathcal{E}(E) \to E^1(E)$ on E, its curvature K is an element of $\mathcal{E}^2(\operatorname{Hom}(E,E))$ given locally by an $r \times r$ matrix Θ of two forms. So, if $\phi : \mathcal{M}_r \to \mathbb{C}$ is an invariant polynomial of degree k, $\phi_X(K) \in \mathcal{E}^{2k}(X)$ is well-defined by lemma 5.1.7.

Proposition 1.11. *If ϕ is an invariant homogeneous polynomial of degree k acting on $\mathcal{M}_r$, then the $2k$-form $\phi(K) \in \mathcal{E}^{2k}$ is closed.*

Proof. [**We**, 86-87] or [**MS**, 296-298]. ☐

Therefore $\phi(K)$ defines an element of the de Rham cohomology group $H^{2k}_{deRh}(X)$.

Proposition 1.12. *If ϕ is an invariant homogeneous polynomial of degree k acting on $\mathcal{M}_r$, and ∇, ∇' are connections on a complex vector bundle $E \to X$, the forms $\phi(K_\nabla)$ and $\phi(K_{\nabla'})$ represent the same element in the de Rham cohomology group $H^{2k}_{deRh}(X)$.*

Proof. [**We**, 86-87] or [**MS**, 298]. ☐

1.3. Chern classes. As before let $E \xrightarrow{\pi} X$ be a complex diferentiable vector bundle and let $\nabla : \mathcal{E}(E) \to \mathcal{E}^1(E)$ be a connection on E with curvature

$$K_\nabla : \mathcal{E}(E) \to \mathcal{E}^2(E).$$

As in example 5.1.9 above, for $k = 0, \dots, r$, we define the invariant polynomials $\Phi_k(A)$ given by $\det(I + A) = \sum_{k=0}^r \Phi_k(A)$.

Definition 1.13. *The* k-th Chern form $c_k(E,\nabla)$ of E relative to the connection ∇ *is the closed differential $2k$-form given by*

$$c_k(E,\nabla) := (\Phi_k)_X\left(\frac{i}{2\pi} K_\nabla\right)$$

where $(\Phi_k)_X(\frac{i}{2\pi}K_\nabla) := (\Phi_k)_X(\frac{i}{2\pi}\Theta_\nabla)$ is well-defined by lemma 5.1.7.

The total Chern form *of E relative to ∇ is*

$$c(E,\nabla) = \bigoplus_{k=0}^{r} c_k(E,\nabla) \in \bigoplus_{k=0}^{r} H^{2k}_{deRh}(X)(X).$$

By proposition 5.1.11, we can define the k-th Chern class $c_k(E)$ *of E relative to ∇ as the cohomology class of $c_k(E,\nabla)$ in $H^{2k}_{deRh}(X,\mathbb{C})$. By proposition 5.1.12, this is independent of the connection ∇. In particular, if E is holomorphic, we can henceforth assume that the Chern classes are calculated relative to the canonical connection.*

The total Chern class $c(E)$ of E *is then given by*

$$c(E) := \bigoplus_{k=0}^{r} c_k(E) \in H^*_{deRh}(X).$$

In what follows, we will use the same notation to refer to the (total or k-th) Chern class in $H^*_{deRh}(X)$ and a representative of this class.

The Chern classes $c_i(X)$ of a complex manifold X are defined to be the Chern classes $c_i(\mathcal{T})$ of the holomorphic tangent bundle of X.

REMARK 1.14. Note that the definition of the Chern classes implies that, if $E \to X$ is a complex bundle with $\operatorname{rk} E = r$, then $c_i(E) = 0$ for all $i > r$.

PROPOSITION 1.15. *The Chern classes have the following properties:*

(1) *Let E be a complex vector bundle over a differentiable manifold X. For all i, $c_i(E)$ is only dependent on the isomorphism class of E.*
(2) *If Y is also a differentiable manifold and $\varphi : Y \to X$ is a differentiable map then*
$$c(\varphi^* E) = \varphi^* c(E)$$
where $\varphi^* c(E)$ is the pullback of the cohomology class $c(E) \in H^*_{deRh}(X, \mathbb{C})$.
(3) *Let E, F be complex differentiable bundles over a differentiable manifold X. Then*
$$c(E \oplus F) = c(E) \cdot c(F)$$
where the product is defined in terms of the wedge product in the de Rham cohomologies. That is
$$c_k(E \oplus F) = \bigoplus_{i+j=k} c_i(E) \wedge c_j(F).$$
(4) *For all i,*

$$c_i(E^*) = (-1)^i c_i(E). \tag{1.6}$$

PROOF. [**We**, 92]. □

THEOREM 1.16. *For a holomorphic line bundle L over a Riemann surface X, the first Chern class $\tilde{c}_1(L)$ defined in definition 4.3.8 corresponds to $c_1(L)$ according to definition 5.1.13 above.*

PROOF. Let $\{U_i\}_i$ be a trivialising cover for L and f^i be a holomorphic frame for L over U_i. Furthermore, let h be a Hermitian metric on L, and $h(f) = \lambda := \{\lambda_i\}_i$ be as in remark 1.1.15. By, lemma 5.1.5 the matrix A for the connection is given on U_i by
$$A(f) = \frac{1}{\lambda_i}\frac{\partial \lambda_i}{\partial z} dz = \frac{\partial}{\partial z} \log \lambda_i dz$$
and so the curvature K is represented on U_i by the matrix
$$\Theta_i = \overline{\partial} A(f^i) = \frac{\partial}{\partial \overline{z}} \left(\frac{\partial}{\partial z} \log \lambda_i \right) dz \wedge d\overline{z}.$$

Since L is one dimensional, $\det(I + \alpha) = 1 + \alpha$ for $\alpha \in \mathcal{M}_1 = \mathbb{C}$. So by definition 5.1.13 the Chern form $c_1(L)$ is given by
$$c_1(L) = \frac{i}{2\pi} \frac{\partial}{\partial \overline{z} \partial z} \log \lambda_i dz \wedge d\overline{z} = \tilde{c}_1(L) \in H^2_{deRh}(X).$$

□

2. Multiplicative sequences and the Todd polynomials

[**Hi**, 9-16] is the authorative reference for this section.

2.1. Definitions and basic properties. Let R be a commutative ring with identity $\mathrm{id}_R =: 1$, and let $p_0 = 1 \in R$, and $\{p_i\}_{i=1}^{\infty}$ be variables.

The ring of polynomials in the variables p_i with coefficients in R is denoted by $\mathcal{R} = R[p_1, p_2, \dots]$ which can be graded as follows:

The *weight* of the product $p_{j_1} p_{j_2} \dots p_{j_r}$ is given by $\sum_{i=1}^{r} j_i$. We let $\mathcal{R}_0 := R$ and $\mathcal{R}_k$ be the group of polynomials consisting only of terms with weight k. That is, $\mathcal{R}_k$ consists of linear combinations of products of weight k. So, $\mathcal{R}_k$ is the R-module spanned by all products $p_{j_1} p_{j_2} \dots p_{j_r}$ of weight k.

Clearly $\mathcal{R} = \sum_{k=0}^{\infty} \mathcal{R}_k$.

Definition 2.1. *Let $(K_j)_{j=0}^{\infty}$ be a sequence of polynomials in p_i such that $K_0 = 1$ and $K_i \in \mathcal{R}_i$ (so K_i is a polynomial in the variables $p_1, \dots, p_i$). $(K_j)_{j=0}^{\infty}$ is called a* multiplicative (or m-) sequence *if every identity of the form*

$$1 + p_1 z + p_2 z^2 + \cdots = (1 + q_1 z + q_2 z^2 + \dots)(1 + r_1 z + r_2 z^2 + \dots) \tag{2.1}$$

implies that

$$\sum_{j=0}^{\infty} K_j(p_1, p_2, \dots) z^j = \sum_{i=0}^{\infty} K_i(q_1, q_2, \dots, q_i) z^i \sum_{k=0}^{\infty} K_k(r_1, r_2, \dots, r_k) z^k. \tag{2.2}$$

Definition 2.2. *We write*

$$K\left(\sum_{i=0}^{\infty} p_i z^i\right) := 1 + \sum_{i=1}^{\infty} K_i(p_1, \dots, p_i) z^i.$$

The characteristic power series *associated to the m-sequence $(K_j)_j$ is given by*

$$K(1+z) = \sum_{i=0}^{\infty} b_i z^i, (b_0 = 1, b_i = K_i(1, 0, \dots, 0) \in R, i \geq 1).$$

Proposition 2.3. *Every formal power series $Q(z) = \sum_{i=0}^{\infty} b_i z^i$ is the characteristic power series of a unique m-sequence $(K_j)_j$.*

Proof. [**Hi**, 10]. □

Example 2.4. The sequence $(p_j)_{j=0}^{\infty}$ is an m-sequence since in this case equations (2.1) and (2.2) are equivalent. It follows immediately from the definition (5.2.2) that $(p_j)_{j=0}^{\infty}$ has characteristic power series $1 + z$.

2.2. Todd polynomials. In what follows we will need the m-sequence of *Todd polynomials* $(T_j)_j$

Definition 2.5. *The* Todd polynomials *$(T_j)_j$ are the elements of the m-sequence associated to the characteristic power series*

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2} x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}$$

where B_k is the k^{th} Bernoulli number. [1]

[1] The Bernoulli numbers $(B_k)_k$ are the coefficients of $(-1)^{k-1} \frac{z^{2k}}{2k!}$ in the power series expansion of

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1}{2!} z^2 - \dots .$$

The first few Todd polynomials are given by [2]

$$
\begin{aligned}
T_1(p_1) &= \frac{1}{2}p_1,\\
T_2(p_1,p_2) &= \frac{1}{12}(p_2+p_1^2),\\
T_3(p_1,p_2,p_3) &= \frac{1}{24}p_2p_1,\\
T_4(p_1,p_2,p_3,p_4) &= \frac{1}{720}(-p_4+p_3p_1+3p_2^2+4p_2p_1^2-p_1^4).
\end{aligned}
\tag{2.3}
$$

In what follows, we will only need the first two Todd polynomials T_1 and T_2. However, the polynomials $T_3, T_4, \ldots,$ are also required for generalising the results of this paper to higher dimensions.

Remark 2.6. Let $E \to X$ be a complex differentiable bundle with Chern classes $\{c_k \in H^{2k}_{deRh}(X)\}_{k=1}^{\mathrm{rk}\,E=r}$. If we take $\{p_k = c_k\}_k$ and define the product $c_i \cdot c_j := c_i \wedge c_j$, then

$$
\begin{aligned}
T_1(c_1) &= \frac{1}{2}c_1,\\
T_2(c_1,c_2) &= \frac{1}{12}(c_2 \oplus c_1 \wedge c_1), \text{ etc.}
\end{aligned}
\tag{2.4}
$$

These are elements of $H^*_{deRh}(X) = \bigoplus_{k=0}^{\dim_{\mathbb{R}}(X)} H^k_{deRh}(X)$.

3. The Todd class and the Chern Character

Let X be a locally compact complex manifold and $E \to X$ be a continuous complex bundle with Chern classes $c_i \in H^{2i}_{deRh}(X)$.

Definition 3.1. *The* (total) Todd class *of E is defined by*

$$
\mathrm{td}\,(E) = \sum_{j=0}^{\infty} T_j(c_1, \ldots c_j) \in H^*_{deRh}(X)
$$

where $(T_j(c_1, \ldots, c_j))_j$ is the m-sequence of Todd polynomials (definition 5.2.5). The Todd class, $\mathrm{td}\,(X)$*, of a compact complex manifold X, is defined as*

$$
\mathrm{td}\,(X) := \mathrm{td}\,(\mathcal{T}),
$$

the Todd class of its holomorphic tangent bundle $\mathcal{T} = \mathcal{T}(X)$.

Lemma 3.2. *If E and F are differentiable complex bundles over X, then*

$$
\mathrm{td}\,(E \oplus F) = \mathrm{td}\,(E)\mathrm{td}\,(F). \tag{3.1}
$$

Proof. This follows from the defining property of m-sequences 5.(2.2) and proposition 5.1.15, 3. □

Remark 3.3. Since $\mathrm{td}\,(E)$ is a finite series starting with 1 (the constant function 1 on X), the inverse $(\mathrm{td}\,(E))^{-1}$ exists.

[2] See [**Hi**, 14].

EXAMPLE 3.4. Let L be a continuous complex line bundle over X. Then $c_i(E) = 0$ for $i > 1$ (by remark 5.1.14), so, by the definition 5.2.2 of the characteristic formal series for an m-sequence, we have that $\mathrm{td}\,(E)$ is equal to the characteristic formal series associated to $(T_j(c_1, \dots c_j))_j$. So, formally

$$\mathrm{td}\,(E) = Q(d) := \frac{d}{1 - e^{-d}},$$

where $d := c_1(E)$.

EXAMPLE 3.5. The Todd class $T(L)$ of a complex line bundle $L \to X$ over a Riemann surface X where $c_1 = c_1(L)$ is the first Chern class of L, is given by

$$\mathrm{td}\,(L) = 1 + T_1(c_1) = \frac{d}{1 - e^{-d}} = 1 \oplus \frac{1}{2} c_1 \in H^*_{deRh}(X).$$

The formal polynomial $P(x)$ given by $P(x) = \sum_{j=0}^{q} c_j x^j$, has a unique formal factorisation

$$\sum_{j=0}^{q} c_j x^j = \prod_{i=0}^{q} (1 + \gamma_i x). \tag{3.2}$$

The c_i's are symmetric polynomials in the γ_i's.

REMARK 3.6. Let $\sum_{j=0}^{q} c_j x^j = \prod_{i=0}^{q} (1 + \gamma_i x)$ be a formal factorisation. Then,

$$\mathrm{td}\,(E) = \prod_{i=1}^{q} \frac{\gamma_i}{1 - e^{-\gamma_i}} \in H^*_{deRh}(X). \tag{3.3}$$

See [**Hi**, 91].

DEFINITION 3.7. *If, E is a continuous complex vector bundle of rank q over X, with Chern classes $c_i, i = 1, \dots q$, and such that $\sum_{j=0}^{q} c_j x^j$ has the factorisation 5.(3.2), we define the* (total) Chern character $\mathrm{ch}\,(E)$ *of E by*

$$\mathrm{ch}\,(E) := \sum_{i=1}^{q} e^{\gamma_i} \in H^*_{deRh}(X).$$

If L_D is the line bundle associated with a divisor D on a compact Riemann surface X, then we denote $\mathrm{ch}\,(L_D)$ *by* $\mathrm{ch}\,(D)$.

LEMMA 3.8. *Let E and F be continuous rank q complex vector bundles over X. Then*

$$\mathrm{ch}\,(E \oplus F) = \mathrm{ch}\,(E) + \mathrm{ch}\,(F)$$

and

$$\mathrm{ch}\,(E \otimes F) = \mathrm{ch}\,(E)\mathrm{ch}\,(F).$$

PROOF. [**Hi**, 91,(64)]. □

PROPOSITION 3.9. *X is a compact complex manifold and E is a continuous complex bundle of rank q over X. Then*

$$\sum_{k=0}^{q} (-1)^k \mathrm{ch}\,(\wedge^k E^*) = (\mathrm{td}\,(E))^{-1} c_q(E)$$

where $c_q(E)$ is the q-th Chern class of E.

PROOF. [**Hi**, 92]. □

4. The T-characteristic

DEFINITION 4.1. *If X has complex dimension n and $\alpha \in H^*_{deRh}(X)$, with $\alpha = \alpha_< + \alpha_n$ where α_n is an n-form and $\alpha_<$ is a linear combination of k-forms with $k < n$, where defined, the evaluation of the form α over the* fundamental class *of X is defined by*

$$\{\alpha\}[X] := \int_X \alpha_n.$$

DEFINITION 4.2. *The T-characteristic $T(E)$ of a complex vector bundle $E \to X$ is given by*

$$T(E) := \{\mathrm{ch}\,(E)\mathrm{td}\,(X)\}[X]. \tag{4.1}$$

THEOREM 4.3. The Hirzebruch-Riemann-Roch theorem
For a holomorphic vector bundle E over a compact complex manifold X, the Euler characteristic $\chi(E)$ of E is equal to the T-characteristic $T(E)$ of E.

PROOF. The proof of this theorem provides the subject matter of most of [**Hi**]. □

If X is a Riemann surface and L is a holomorphic line bundle over X, then definition 5.3.7 gives

$$\mathrm{ch}\,(L) = 1 + c_1(L).$$

THEOREM 4.4. *The topological index of a divisor D on a compact Riemann surface X of genus g is equal to the T-characteristic of L_D. That is*

$$1 - g + \deg D = \{\mathrm{ch}\,(D)\mathrm{td}\,(X)\}[X].$$

PROOF. Using the equations above, we have

$$\mathrm{td}\,(X) = 1 + T_1(c_1(X)) = 1 + \frac{1}{2}c_1(X)$$

since the complex tangent bundle $\mathcal{T}$ has complex rank 1. Now, by proposition 5.1.15, (1.6), $c_1(X) := c_1(\mathcal{T}) = -c_1(\mathbf{T})$, and since, if K is a canonical divisor, $L_K = \mathbf{T}$ (as in proposition 3.1.19). So, $\int_X c_1(X) = -\int_X c_1(\mathbf{T}) = -\deg K$. We have

$$\mathrm{ch}\,(D)\mathrm{td}\,(X) = (1 + c_1(D))\left(1 - \frac{1}{2}c_1(\mathbf{T})\right) = 1 + c_1(D) - \frac{1}{2}c_1(\mathbf{T})$$

and so (by an abuse of notation)

$$T(L) = \int_X \left(c_1(D) + \frac{c_1(X)}{2}\right) = \deg D - \frac{1}{2}\deg K.$$

We have already seen in corollary 3.2.4 that $\deg K = 2g - 2$ and so, it follows immediately that $T(L) = \deg D - g + 1$ as required. □

CHAPTER 6

The Topological Index of the Dolbeault operator

THEOREM 0.5. The Atiyah-Singer Index formula

X is a compact manifold of real dimension m and E, F are differentiable complex vector bundles over X and $P : \mathcal{E}(E) \to \mathcal{E}(F)$ is an elliptic operator. Let η be an almost complex structure (definition 1.1.27) for the total space of the cotangent bundle T^*X. The orientation of T^*X is given by local coordinates $(x_1, \xi_1, \ldots, x_m, \xi_m)$, where $(x_1, \ldots, x_m)$ are chart coordinates for some $x \in X$ and $\xi_i = dx_i$ for all i. Then the analytic index $\chi(P)$ of P is equal to its topological index $ind_t(P)$ where

$$ind_t(P) := \{\mathrm{ch}\,(P) \cdot \mathrm{td}\,(\eta)\}[T^*X].$$

$\mathrm{ch}\,(P)$ is defined as the Chern character of the *difference bundle* associated to the operator P. This will be constructed later in this chapter.[1]

1. Elements of topological K-theory

DEFINITION 1.1. *Let X be a compact topological space. It can be shown that the isomorphism classes of continuous complex vector bundles over X form an Abelian semi-group with semi-group operation $\oplus$.*

The induced Abelian group is called the topological K-group *$K(X)$ of X.*[2] *If E is a vector bundle over X the element of $K(X)$ associated to E is denoted by $[E]$.*

We will sometimes refer to an element of $K(X)$ as a virtual bundle *over X.*

EXAMPLE 1.2. For a point space $\{x\}$ we have $K(\{x\}) \cong \mathbb{Z}$ since two complex vector spaces are isomorphic if and only if they have the same dimension.

REMARK 1.3. The elements of $K(X)$ are the classes of formal differences of the form $E - F$ where E, F are complex vector bundles on X.

$E - F$ is equivalent to $E' - F'$ if and only if there exists another complex bundle G on X such that

$$E \oplus F' \oplus G \cong E' \oplus F \oplus G.$$

The bundle G is necessary to ensure transitivity of the relation since the cancellation rule as in $\mathbb{Z}$ doesn't, in general, apply to vector bundles. That is, it is not, in general the case that $a + c = b + c \Rightarrow a = b$ where a, b, c are isomorphism classes of complex bundles over a compact manifold X.

It is straightforward to show that $K(X)$ is, in fact, a ring under the operations of tensor product $\otimes$ and sum $\oplus$.

---

[1]For an elliptic complex E, we can associate a differential operator P_E to E and then the Atiyah-Singer index formula says that $\mathrm{ind}_a(E) = \mathrm{ind}_t(P_E)$. This will be discussed briefly in the appendix, page 81.

[2]K-theory was introduced by Grothendieck in 1957. The definitions here are part of the *topological K-theory* refined by Hirzebruch and Atiyah in 1959, rather than the more general *algebraic K-theory* still associated with Grothendieck and further developed from the 1960's.

REMARK 1.4. Let X, X' be compact spaces and $f : X \to X'$ a continuous map. Then f induces in a natural fashion a ring homomorphism $f^! : K(X') \to K(X)$ as follows:

Let $\{U_i\}_i$ be an open covering for X' and $E' \to X'$ a vector bundle over X' given by transition functions $\{g_{i,j}\}_{i,j}$ with respect to $\{U_i\}_i$. Since f is continuous, $\{f^{-1}U_i\}_i$ is a cover for X and we can define a vector bundle $E \to X$ by transition functions $\{g_{i,j}\}_{i,j}$ with $g_{i,j} := g'_{i,j} \circ f|_{U_i \cap U_j}$. Then $f^! : K(X') \to K(X)$ is the group homomorphism induced by the semi-group map $f^!([E']) := [E]$.

PROPOSITION 1.5. *Let X, X' be homotopy equivalent compact spaces with homotopy equivalence $f : X \to X'$. Then, $f^! : K(X') \to K(X)$ is an isomorphism.*

PROOF. [**At**, 16-18]. □

DEFINITION 1.6. *If we choose $x_0 \in X$, then the inclusion $\iota : x_0 \hookrightarrow X$ induces a homomorphism $\iota^! : K(X) \to K(\{x_0\}) \cong \mathbb{Z}$. When X is connected, this is independent of the choice of basepoint x_0. We define*

$$\tilde{K}(X) := \operatorname{Ker} \iota^!.$$

Now, let $Y \subset X$ be a closed non-empty subspace. The space obtained by contracting Y to a basepoint $y_0 \in Y$ is denoted by X/Y, and $\iota : \{y_0\} \to X/Y$ is the inclusion map. We define the relative K-group *$K(X, Y)$ by*

$$K(X, Y) := \tilde{K}\left(X/Y \right).$$

In particular, $K(X, Y)$ is an ideal of $K\left(X/Y \right)$.

If $Y = \emptyset$, we define $X/\emptyset$ as the disjoint sum $X + \{p\}$ of X and a point $\{p\}$. Then

$$K(X, \emptyset) = \tilde{K}(X + \{p\}) = \operatorname{Ker}(\iota^! : K(X + \{p\}) \to K(\{p\})) = K(X).$$

DEFINITION 1.7. *For a locally compact space W we define $K(W) := \tilde{K}(W^+)$ where $W^+ := W + \{p\}$ is the one-point compactification of W and*

$$\tilde{K}(W^+) := \operatorname{Ker}(\iota^! : K(W^+) \to K(\{p\})).$$

Using this definition, if X is a compact or locally compact space and $f : W \to X$ is a proper map (that is $f^{-1}(K) \subset W$ is compact for all $K \subset X$ compact), we can define $f^! : K(X) \to K(W)$ as in remark 6.1.4.

2. The difference bundle associated to an elliptic operator

Let $Y \subset X$ be a closed subset of a compact manifold X and let E_0, E_1 be continuous vector bundles over X such that there is an isomorphism $\alpha : E_0|_Y \to E_1|_Y$. We construct the *difference bundle* $d(E_0, E_1, \alpha)$. This will be an element of the relative K-group $K(X, Y)$.

Let

$$Z := (X \times 0) \cup (X \times 1) \cup (Y \times I) \subset X \times I$$

where $I = [0, 1]$ is the closed unit interval. $p : X \times I \to X$ is the projection. We define a vector bundle E over Z as follows:

Z is covered by the open sets

$$Z_0 = (X \times 0) \cup (Y \times [0, 1)) \text{ and } Z_1 = (X \times 1) \cup (Y \times (0, 1]).$$

For $i = 0, 1$, the restriction of p to Z_i is $p_i := p|_{Z_i}$.

Then $p_i^* E_i$ is a vector bundle over the open set Z_i and $p^*\alpha$ is an isomorphism on the open set $Z_0 \cap Z_1$. Identifying $p^* E_i$ via $p^*\alpha : p^* E_0|_{Y \times (0,1)} \xrightarrow{\simeq} p^* E_1|_{Y \times (0,1)}$ on $Z_0 \cap Z_1$, we glue these together to obtained the desired bundle $E \to Z$.

The projection $\tau : Z \to {}^{Z}\!/_{(X \times 0)}$ induces a homomorphism

$$\tau^! : K\left({}^{Z}\!/_{(X \times 0)} \right) \to K(Z)$$

and the inclusion $\iota : X = X \times 0 \to Z$ induces a homomorphism

$$\iota^! : K(Z) \to K(X \times 0)) = K(X).$$

Lemma 2.1. *The sequence*

$$0 \to K\left({}^{Z}\!/_{X \times 0} \right) \xrightarrow{\tau^!} K(Z) \xrightarrow{\iota^!} K(X) \to 0$$

is exact.

Proof. Given a vector bundle E on X we form a bundle $p^*(E)$ on $Z \subset X \times I$, by taking one copy of E over $X \times 0$ and another over $X \times 1$ and gluing them together with the identity map on $Y \times (0, 1)$. So $\iota^!$ is surjective.

Given a trivial virtual bundle E over Z, we observe that its restriction $E|_U$ to any open $U \subset Z$ is also trivial. In particular $E|_{Z-(X \times 0)}$ is trivial. So

$$(\tau^!)^{-1}(0_{K(Z)}) = 0_{K\left({}^{Z}\!/_{X \times 0} \right)}.$$

In other words, $\tau^!$ is injective.

Finally, let V be a vector bundle over Z. Then $\iota^!([V]) = 0$ if and only if V is isomorphic to a bundle which is trivial over $X \times 0 \subset Z$. That is $[V] \in \mathrm{Im}\ (\tau^!)$. □

Lemma 2.2. *The exact sequence*

$$0 \to K\left({}^{Z}\!/_{(X \times 0)} \right) \xrightarrow{\tau^!} K(Z) \xrightarrow{\iota^!} K(X) \to 0 \tag{2.1}$$

splits. So, there is a homomorphism $g^! : K(Z) \to K\left({}^{Z}\!/_{(X \times 0)} \right)$ *such that* $g^! \circ \tau^!$ *is the identity on* $K\left({}^{Z}\!/_{(X \times 0)} \right)$.

Proof. Let $f : Z \to X = (X \times 0)$ be given by $f(x, t) = (x, 0)$ for all $(x, t) \in Z$. Then f is a deformation retraction and $f^! : K(X) \to K(Z)$ is a group homomorphism such that $\iota^! \circ f^!$ is the identity on $K(X)$. Now, given $a \in K(Z)$,

$$a - (f^! \circ \iota^!)(a) \in \mathrm{Ker}\,(\iota^!) = \mathrm{Im}\,(\tau^!)$$

since $\iota^!(a - (f^! \circ \iota^!)(a)) = \iota^!(a) - \iota^!(a) = 0$. Therefore, since $\tau^!$ is injective, there is a unique $b \in K\left({}^{Z}\!/_{(X \times 0)} \right)$, such that $\tau^!(b) = a - (f^! \circ \iota^!)(a)$. We define $g^!(a) := b$. Then

$$\tau^!(g^1 \circ f^! \circ \iota^!(a)) = (f^! \circ \iota^!)(a) - (f^! \circ \iota^! \circ f^! \circ \iota^!(a)) = (f^! \circ \iota^!(a)) - (f^! \circ \iota^!)(a) = 0$$

and therefore, since $\tau^!$ is injective

$$(g^! \circ f^! \circ \iota^!)(a) = 0.$$

So, for $b \in K\left({}^{Z}\!/_{(X \times 0)} \right)$,

$$g^! \circ \tau^!(b) = b$$

and $g^! : K(Z) \to K\left({}^{Z}\!/_{(X \times 0)} \right)$ is the required homomorphism.

□

REMARK 2.3. Since $E|_{X \times 0} = p_0^* E_0$, the virtual bundle $E - p^* E_0$ is trivial over $X = X \times 0$, and therefore $E - p^* E_0 \in \operatorname{Ker} \iota^!$.

In particular $g^!(E - p^* E_0) \in K\left({}^{Z}\!/_{X \times 0} \right)$ is trivial near the base-point x_0. I.e. $g^!(E - p^* E_0) \in \operatorname{Ker}\left(K\left({}^{Z}\!/_{X \times 0} \right) \to K(\{x_0\}) \right)$.

LEMMA 2.4. *$K(Z, X \times 0)$ is isomorphic to $K(X, Y)$.*

PROOF. [**At**, 69] □

DEFINITION 2.5. *By the above remark 6.2.3, $g^!(E - p^* E_0) \in K(Z, (X \times 0))$. The* difference bundle *$d(E_0, E_1, \alpha)$ is defined as the image of the virtual bundle $g^!(E - p^* E_0) \in K(Z, X \times 0) \subset K\left({}^{Z}\!/_{(X \times 0)} \right)$ in $K(X, Y) \subset K\left({}^{X}\!/_{Y} \right)$.*

PROPOSITION 2.6. *Let X, X' be compact spaces and $Y \subset X, Y' \subset X'$ closed subspaces. Let E, F and E', F' be vector bundles over X and X' respectively, and $\alpha : E|_Y \to F|_Y$, $\alpha' : E'|'_Y \to F'|'_Y$ be isomorphisms.*

1. *If $f : (X, Y) \to (X', Y')$ is a map, then*
$$d(f^* E', f^* F', f^* \alpha') = f^! d(E', F', \alpha').$$
2. *$d(E, F, \alpha)$ is only dependent on the homotopy class of α.*
3. *If Y is the empty set then $d(E, F, \alpha) = E - F$.*
4. *Let $j : (X, \emptyset) \to (X, Y)$ be inclusion. j induces naturally a homomorphism $j^! : K(X, Y) \to (K(X, \emptyset) = K(X))$. If E and F are vector bundles over X and $\alpha : E|_Y \to F|_Y$ is an isomorphism, then*
$$j^! d(E, F, \alpha) = E - F.$$
5. *$d(E, F, \alpha) = 0$ if and only if there is a bundle $G \to X$ such that $\alpha \oplus \; id|_{G|_Y}$ extends to an isomorphism $E \oplus G \to F \oplus G$ over the whole of X.*
6. *$d(E \oplus E', F \oplus F', \alpha \oplus \alpha') = d(E, F, \alpha) + d(E', F', \alpha')$.*
7. *$d(E, F, \alpha) + d(E, F, \alpha^{-1}) = 0$.*
8. *If G is a bundle over X and $\beta : F|_Y \to G|_Y$ is an isomorphism, then*
$$d(E, G, \beta \circ \alpha) = d(E, F, \alpha) + d(F, G, \beta).$$

PROOF. [**AH**, 33-34]. □

DEFINITION 2.7. *If W is a rank k vector bundle over a compact manifold X of dimension n and W is equipped with a metric, then we define the* disk bundle *$B(W) \subset W$ as the closed $n + k$ dimensional submanifold with boundary consisting of the vectors $\xi \in W$ such that $|\xi| \leq 1$. Similarly, the* sphere bundle *$S(W)$ of W is the embedded $n + k - 1$ dimensional compact sub-manifold of $B(W)$ consisting of elements ξ of W with $|\xi| = 1$.*

We define $B(X) := B(T^* X)$ (where $T^* X$ is the real cotangent bundle of X) and similarly $S(X) := S(T^* X)$.

Let $\pi : B(X) \to X$ be the canonical projection. By definition, if E and F are complex bundles over X and $P : \mathcal{E}(E) \to \mathcal{E}(F)$ is an elliptic operator, the symbol, $\sigma_P : \pi^*E \to \pi^*F$ is an isomorphism on $S(X) := S(T^*X)$. Here, we let σ_P denote the restriction of the sumbol $\sigma(P)$ to the sphere bundle $S(X)$. Therefore, we can construct the difference bundle $d(P) := d(\pi^*E, \pi^*F, \sigma_P)$.

In particular we have already seen in example 2.2.6 that the Dolbeault operator $\overline{\partial}_L : \mathcal{E}(L) \to \mathcal{E}(L \otimes \overline{\mathbf{T}})$, where X is a Riemann surface and L is a holomorphic line bundle on X, is an elliptic operator. So we can construct the associated difference bundle $d(\pi^*(, \pi^*(L \otimes \overline{\mathbf{T}}), \sigma_{\overline{\partial}_L})$.

We will return to this shortly.

3. The Thom Isomorphism

3.1. The Thom isomorphism in topological K-theory. Let X be a compact space and $p : W \to X$ a vector bundle.

PROPOSITION 3.1. *There is an isomorphism*

$$\varphi^! : K(X) \to K(W)$$

called the Thom isomorphism.

PROOF. [**AS1**, 494]. □

(Note that, since W is only locally compact, $K(W) := K(W + \{p\}, \{p\})$ as in definition 6.1.7 above.)

REMARK 3.2. For a vector bundle $E \to X$ over X, the pullback bundle $p^!E$ over W is well defined. Therefore, $K(W)$ can be regarded as a module over $K(X)$.

COROLLARY 3.3. *For X compact, there is a Thom isomorphism*

$$\varphi^! : K(X) \to K(B(X), S(X))$$

PROOF. There is a natural isomorphism $(T^*(X))^+ \cong {}^{B(X)}/_{S(X)}$. The result then follows from the definitions 6.1.6 and 6.1.7 together with the above proposition. □

In particular, since $K(T^*X)$ is a module over $K(X)$, $K(B(X), S(X))$ is a module over $K(X)$.

3.2. The Thom isomorphism for cohomology. Let X be a compact space. By theorem 4.1.2 we can view the Chern character of a vector bundle $E \to X$ as an element of $H^*(X, \mathbb{R}) := \bigoplus_k H^k(X, \mathbb{R})$.

LEMMA 3.4. *The Chern character is a semi-group homorphism on the semi-group of isomorphism classes of complex vector bundles over X. It extends naturally to a ring homomorphism*

$$\text{ch} : K(X) \to H^*(X, \mathbb{R}).$$

PROOF. [**Hi**, 177] □

LEMMA 3.5. *Let $f : X \to X'$ be a continuous map between compact spaces. If $\mathcal{U} = \{U_i\}_i$ is an open cover for X' then $\{f^{-1}U_i\}_i$ is an open cover for X and we can define a homomorphism*

$$f^* : H^*(X', \mathbb{R}) \to H^*(X, \mathbb{R})$$

induced by

$$f^*(\{\alpha_{i_0,\dots,i_q}\}_{i_0,\dots,i_q}) := \{\alpha_{i_0,\dots,i_q} f\}_{i_0,\dots,i_q}$$

where $\alpha_{i_0,\dots,i_q} \circ f$ is defined on $f^{-1}(U_{i_0} \cap \cdots \cap U_{i_q})$.

The diagram

$$\begin{array}{ccc} K(X') & \xrightarrow{f!} & K(X) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(X', \mathbb{R}) & \xrightarrow[f^*]{} & H^*(X, \mathbb{R}) \end{array} \tag{3.1}$$

commutes

PROOF. [**Hi**, 177]. □

In particular, if X is a compact space and $Y \xrightarrow{\iota} X$ is a closed subspace, with inclusion map ι, and $\{y_0\} \xrightarrow{\iota'} X/Y$ the inclusion of the basepoint $\{y_0\}$ in X/Y, then

$$\begin{array}{ccccc} K(X,Y) & \longrightarrow & K\left(X/Y\right) & \xrightarrow{(\iota')!} & K(\{y_0\}) \\ & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ & & H^*\left(X/Y, \mathbb{R}\right) & \xrightarrow[(\iota')^*]{} & H^*(\{y_0\}, \mathbb{R}). \end{array} \tag{3.2}$$

commutes. And since $K(X,Y)$ is contained in $K\left(X/Y\right)$, the restriction of the Chern character is a well-defined homomorphism

$$\mathrm{ch}\,|_{K(X,Y)} : K(X,Y) \to H^*\left(X/Y, \mathbb{R}\right)$$

such that $\mathrm{Im}\,(\mathrm{ch}\,|_{K(X,Y)}) \subset \mathrm{Ker}\,((\iota')^*)$.

DEFINITION 3.6. *Let X be a compact space and $Y \xrightarrow{\iota} X$ a closed subspace, with inclusion map ι. For $k = 0, 1, \dots,$ the* relative cohomology groups *$H^k(X, Y, \mathbb{R})$ are defined by means of the long exact sequence*

$$\cdots \to H^k(X,Y,\mathbb{R}) \to H^k(X,\mathbb{R}) \xrightarrow{\iota^*} H^k(Y,\mathbb{R}) \to H^{k+1}(X,Y,\mathbb{R}) \to \dots.$$

PROPOSITION 3.7. The Poincaré duality theorem

Let X be an oriented manifold of real dimension n. Then there is an isomorphism

$$H^k_{deRh}(X) \cong (H^{n-k}_c(X))^*$$

for each $k \le n$.

PROOF. [**BT**, 44-46]. □

Now if X is a real manifold of dimension n and $\iota : Y \hookrightarrow X$ a closed orientable submanifold of dimension k (ι is the inclusion map), Poincaré duality says that there exists a unique cohomology class $[\eta_Y] \in H^{n-k}_{deRh}(X)$ called the *closed Poincaré dual of* Y in X, such that, if η_Y is a representative for $[\eta_Y]$

$$\int_Y \iota^* \omega = \int_X \omega \wedge \eta_Y$$

for every closed k-form ω with compact support on X.

If Y is compact in X, then

$$\int_Y \iota^* \omega$$

is well-defined for any differential k-form on X, not just those with compact support, and so n defines a functional $\int_Y$ on $H^k_{deRh}(X)$. I.e Y defines an element of $(H^k_{deRh}(X))^*$ and so by, Poincaré duality, there exists a corresponding $[\eta'_Y] \in H^{n-k}_c(X)$ called the *compact Poincaré dual of* Y.

If the differential $(n-k)$-form with compact support η'_Y is a representative of $[\eta'_Y]$, then we have the unique characterisation

$$\int_Y \iota^* \omega = \int_X \omega \wedge \eta_Y$$

for every closed k-form ω on X.

Now, if η'_Y is a representative for $[\eta'_Y]$ then clearly it is also a representative for $[\eta_Y]$. So, when $Y \subset X$ is a compact submanifold, it is possible to ensure that the closed Poincaré dual $[\eta_Y]$ has compact support.

For simplicity of notation, we will henceforth (except for extra emphasis) make no distinction between a closed form η and its class $[\eta]$ in the de Rham cohomology $H^*_{deRh}(X)$, and denote them both simply by η.

LEMMA 3.8. *Let $\iota : Y \hookrightarrow X$ be a k-dimensional compact orientable submanifold of an n-dimensional orientable manifold X, and η'_Y its compact Poincaré dual in X. Then the support of η'_Y may be shrunk into any open neighbourhood $U \supset Y$ of Y in X.*

PROOF. Let $\eta'_{Y,U}$ be the compact Poincaré dual of Y in U. This has compact support in U so we can extend it by 0 to a form $\eta'_Y \in H^{n-k}_c(X)$. Now, for $\omega \in H^k(X)$,

$$\int_Y \iota^* \omega = \int_U \omega \wedge \eta'_{Y,U} = \int_X \omega \wedge \eta'_Y$$

so η'_Y is the compact Poincaré dual of Y in X. □

Let $\pi : W \to X$ be a real vector bundle of rank r over a compact manifold X of dimension n. We view X as a compact submanifold of W, $\iota : X \hookrightarrow W$, by embedding it as the zero-section in W.

By Poincaré duality, the map

$$H^n_{deRh}(X) \ni [\omega] \mapsto [\pi^* \omega \wedge \eta_X] \in H^{r+n}_c(W)$$

is well-defined. In fact, the following proposition tells us that this map is an isomorphism, and that $H^k_{deRh}(X) \cong H^{k+r}_c(W)$ for all $k \geq 0$.

PROPOSITION 3.9. *There exists a Thom isomorphism*

$$\phi_* : H^k_{deRh}(X) \to H^{k+r}_c(W)$$

given by

$$\phi_*([\omega]) = [\pi^*\omega \wedge \eta_X]$$

where ω *is representative for* $[\omega] \in H^k_{deRh}(X)$ *and* η_X *is a representative for* $[\eta_X]$ *the Poincaré dual of* X.[3]

PROOF. [**BT**, 63-64]. □

Henceforth we will denote both η_X and $[\eta_X]$ by Φ and call Φ the *Thom class* of W. It should be clear from the context whether we mean the class or its representative.

By lemma 6.3.8 we can shrink the support of Φ to any open neighbourhood of $X \subset W$. In particular, we can ensure that the support of Φ is contained in $B(W)$.

PROPOSITION 3.10. *Let* $W \to X$ *be a real vector bundle of rank* r. *Then*

$$H^{k+r}_c(W) \cong H^{k+r}(B(W), S(W), \mathbb{R})$$

for all $k \geq 0$.

PROOF. [**LM**, 239]. □

By proposition 6.3.10, we may regard the Thom class $\Phi \in H^r_c(W)$ of W as an element of $H^r(B(W), S(W), \mathbb{R})$. We denote this also by Φ. In particular, we can also write the Thom isomorphism

$$\phi_* : H^k_{deRh}(X) \to H^{k+r}_c(W)$$

as an isomorphism

$$\phi_* : H^k(X, \mathbb{R}) \to H^{k+r}(B(W), S(W), \mathbb{R}).$$

We will use these two forms of the Thom isomorphism interchangeably in what follows.

DEFINITION 3.11. *As usual,* $\iota : X \hookrightarrow W$ *is the embedding of* X *as the zero section in* W. *The* Euler class $e(W)$ *of* W *is the pullback* $\iota^*\Phi$ *of* Φ *to* X.

PROPOSITION 3.12. *For a complex vector bundle* E *of rank* q *over a compact manifold* X*, the Euler class* $e(W)$ *of the underlying real manifold* W *is equal to the top Chern class* $c_q(E)$ *of* E.

PROOF. [**AS3**, 550]. □

Now let E be a complex vector bundle of rank q over X and $\pi : W \to X$ the underlying real vector bundle. Further let $j : (B(W), \emptyset) \to (B(W), S(W))$ be the natural embedding. The defining exact sequences of $H^*(B(W), S(W), \mathbb{R})$ and $H^*(B(W), \emptyset, \mathbb{R}) = H^*(B(W), \mathbb{R})$ induce a homomorphism

$$j^* : H^*(B(W), S(W), \mathbb{R}) \to H^*(B(W), \emptyset, \mathbb{R}) = H^*(B(W), \mathbb{R}).$$

LEMMA 3.13. $H^*(B(W), S(W), \mathbb{R})$ *is a module over* $H^*(B(W), \mathbb{R}) = H^*(W, R)$ *and* $j^* : H^*(B(W), S(W), \mathbb{R}) \to H^*(B(W), \mathbb{R})$ *is a module homomorphism.*

---

[3] If X is not compact we can use cohomology with compact vertical support instead and the following arguments can be carried over to apply to this case.

PROOF. [**Hi**, 180]. □

REMARK 3.14. Let X is a compact manifold and E, F complex vector bundles over X with $P : \mathcal{E}(E) \to \mathcal{E}(F)$ an elliptic operator. Furthermore, let $d(P) \in K(B(X), S(X))$ be the difference bundle associated to P. Then ([**Hi**, 187])

$$\mathrm{ch}\,(d(P)) \in H^*(B(X), S(X), \mathbb{R}).$$

In particular, if $\eta \in H^*(T^*X)$ then, by the above lemma 6.3.13, $\eta \cdot \mathrm{ch}\,(d(P)) \in H^*(B(X), S(X), \mathbb{R})$ and so, by proposition 6.3.10, we may evaluate the form of $\eta \cdot \mathrm{ch}\,(d(P))$ over the fundamental class of T^*X. That is,

$$\{\eta \cdot \mathrm{ch}\,(d(P))\}[T^*X]$$

is well-defined.

LEMMA 3.15. $\pi^* : H^k_{deRh}(X) \to H^k_{deRh}(W) \cong H^*(B(W), \mathbb{R})$ *is an isomorphism.*

PROOF. If X is embedded as the zero-section of W, then, $F : W \times [0,1] \to W$ given by $((x,\xi),t) \mapsto (x,(1-t)\xi),\ x \in X, \xi \in W_x, t \in [0,1]$, is a deformation retraction (as is $F|_{B(W)}$). So W, $B(W)$ and X are homotopy equivalent and the conclusion follows by application of lemma 4.1.5 and then theorem 4.1.2.

□

We assume the support of Φ is contained in the interior of $B(W)$ (that this is permitted follows from lemma 6.3.8). Denote the restriction of $\pi : W \to X$ to $B(W)$ also by π.

COROLLARY 3.16. *Using the notation above,*

$$j^*\Phi = \pi^* e(W) = \pi^* c_q(E).$$

This gives

$$j^*(\phi_*(\omega)) = \pi^*(\omega \cdot c_q(E))$$

for $\omega \in H^*_{deRh}(X)$.

PROOF. The following diagram naturally commutes:

$$\begin{array}{ccc} H^*(B(W), S(W), \mathbb{R}) & \xrightarrow{j^*} & H^*(B(W), \mathbb{R}) \\ \downarrow \iota^* & \nearrow \pi^* & \\ H^*(X, \mathbb{R}). & & \end{array} \tag{3.3}$$

In particular, we have already seen in proposition 6.3.10 that we can regard Φ as an element of $H^*(B(W), S(W), \mathbb{R})$. So

$$j^*\Phi = (\pi^* \circ \iota^*)\Phi = \pi^* e(W) = \pi^* c_q(E). \tag{3.4}$$

Therefore for $\omega \in H^*(X, \mathbb{R}) = H^*_{deRh}(X)$,

$$j^*(\phi_*(\omega)) = j^*(\pi^*\omega \cdot \Phi) = \pi^*\omega \cdot j^*\Phi = \pi^*(\omega \cdot \iota^*\Phi) = \pi^*(\omega \cdot e(W)) = \pi^*(\omega \cdot c_q(E))$$

where the second equality follows from lemma 6.3.13 and the third from equation 6.(3.4) above. □

We will need the following lemma in the next section.

COROLLARY 3.17. *Let E and F be complex vector bundles over a compact complex manifold X and let W be a even-dimensional real oriented vector bundle over X, with disc and sphere bundles $B(W)$ and $S(W)$ respectively and projection $\pi : B(W) \to X$. Furthermore, let $\alpha : \pi^*E|_{S(W)} \to \pi^*F|_{S(W)}$ be an isomorphism. Then,*

$$e(W) \cdot \phi_*^{-1} (\text{ch } d(\pi^*E, \pi^*F, \alpha)) = \text{ch}\,(E) - \text{ch}\,(F).$$

PROOF. By proposition 6.2.6, 4 above,

$$j^*(\text{ch}\,(d(\pi^*E, \pi^*F, \alpha)) = \text{ch}\,(j^!(d(\pi^*E, \pi^*F, \alpha))) = \text{ch}\,(\pi^*E) - \text{ch}\,(\pi^*F).$$

Using corollary 6.3.16,

$$\begin{aligned}\pi^*(\text{ch}\,(E) - \text{ch}\,(F)) = \text{ch}\,(\pi^*E) - \text{ch}\,(\pi^*F) &= j^*\phi_*\phi_*^{-1}\text{ch}\,(d(\pi^*E, \pi^*F, \alpha)) \\ &= \pi^*(\phi_*^{-1}\text{ch}\,(d(\pi^*E, \pi^*F, \alpha)) \cdot e(W)).\end{aligned}$$

Since π^* is an isomorphism,

$$e(W) \cdot \phi_*^{-1}\text{ch}\,(d(\pi^*E, \pi^*F, \alpha)) = \text{ch}\,(E) - \text{ch}\,(F)$$

as required. □

4. The Todd genus is a special case of the topological index

In the following we are interested in the case $W = T^*X$ the real cotangent bundle over X.

Let X be an n-dimensional complex compact manifold, so an $(m = 2n)$-dimensional real compact manifold. We choose a Hermitian metric h and frame e on the complex tangent bundle $\mathcal{T}$ such that $\mathcal{T}$, and therefore $\mathbf{T}(= \mathcal{T}^*)$, are described by unitary transition functions. Furthermore, the real tangent and cotangent bundles TX and T^*X are described by orthogonal transition functions under the real bundle metric g induced by h (as in remark 1.1.18) and relative to the frame of the underlying real bundle induced by e.

If TX is the real tangent bundle of X, and T^*X the total space of the real cotangent bundle, with projection $\pi : T^*X \to X$, then T^*X is a $2m$-dimensional manifold with tangent bundle $\pi^*(TX) \oplus \pi^*(T^*X)$. The real metric induced by h gives an isomorphism $TX \cong T^*X$.

Recall the maps 1.(1.5) $\upsilon : \mathbf{O}(m) \to \mathbf{U}(m)$ (the complexification of $\mathbf{O}(m)$, obtained by simply expressing a matrix with real coefficients as one with complex coefficients) and 1.(1.4) $\psi : \mathbf{U}(m) \to \mathbf{O}(2m)$. By proposition 1.1.24, if W is a real bundle described by orthogonal transition functions,

$$(\psi \circ \upsilon)(W) \cong W \oplus W.$$

Therefore,

$$\pi^*TX \oplus \pi^*T^*X \cong \pi^*TX \oplus \pi^*TX = \psi(\pi^*\upsilon(TX)).$$

In particular, the $\mathbf{GL}(m, \mathbb{C})$) bundle, $\eta = \pi^*\upsilon(TX)$ is an almost complex structure for T^*X.[4]

[4] Note that we could equivalently have chosen to define the almost complex structure for T^*X in terms of T^*X rather than TX. The choice of convention here is for simplicity in the last stage of the paper.

Now, by proposition 1.1.28, equation (1.6), the complexfication $\upsilon(TX) = TX^{\mathbb{C}}(\cong TX \oplus TX)$ of TX is isomorphic to $\mathcal{T} \oplus \overline{\mathcal{T}}$, so $\eta \cong \pi^*(\mathcal{T} \oplus \overline{\mathcal{T}})$.

REMARK 4.1. Note that for a given complex manifold M of real dimension m and complex dimension n the orientation of $TM^{\mathbb{C}}$ differs by a factor $(-1)^{m(m-1)/2} = (-1)^{n(2n-1)} = (-1)^n$ from $\mathcal{T}(M) \oplus \overline{\mathcal{T}}(M)$. Namely, if the orienation of $TM^{\mathbb{C}}$ is given by the coordinates $z_1, \xi_1, \ldots, z_m, \xi_m$ (with $(z_1, \ldots, z_m)$ chart coordinates for some $z \in X$ and $(\xi_1, \cdots, \xi_m) \in T_z^*X$), the orientation of $\mathcal{T}(M) \oplus \overline{\mathcal{T}}(M)$ is given by the coordinates $\xi_1, \ldots \xi_m, z_1, \ldots, z_m$.

LEMMA 4.2. *Given an elliptic operator $P : E \to F$ with E, F complex vector bundles over the $n = \frac{m}{2}$-dimensional complex manifold X, it holds*

$$\begin{aligned} ind_t(P) &:= {}_{2m}\{\mathrm{ch}\,(P) \cdot \mathrm{td}\,(\eta)\}[T^*X] \\ &= (-1)^n \{\phi_*^{-1}(\mathrm{ch}\,(P)) \cdot \mathrm{td}\,(\mathcal{T}) \cdot \mathrm{td}\,(\overline{\mathcal{T}})\}[X] \end{aligned} \tag{4.1}$$

PROOF. By Poincaré duality, proposition 6.3.7 and the definition of the Thom isomorphism in proposition 6.3.9

$$_{2m}\{\mathrm{ch}\,(P) \cdot \mathrm{td}\,(\eta)\}[T^*X] = (-1)^n \{\phi_*^{-1}(\mathrm{ch}\,(P) \cdot \mathrm{td}\,(\eta))\}[X].$$

Now $\mathrm{ch}\,(P) \in H_c^*(T^*X)$ so $\phi_*^{-1}(\mathrm{ch}\,(P))$ is well defined and, by definition

$$\mathrm{ch}\,(P) = \phi_*(\phi_*^{-1}(\mathrm{ch}\,(P))) = \pi^*(\phi_*^{-1}(\mathrm{ch}\,(P))) \cdot \Phi.$$

Therefore

$$\begin{aligned} \mathrm{ch}\,(P)\mathrm{td}\,(\eta) &= \pi^*(\phi_*^{-1}(\mathrm{ch}\,(P))) \cdot \Phi \cdot \mathrm{td}\,\eta = \pi^*(\phi_*^{-1}(\mathrm{ch}\,(P)) \cdot (\pi^*)^{-1}(\mathrm{td}\,(\eta))) \cdot \Phi \\ &= \phi_*(\pi^*(\phi_*^{-1}(\mathrm{ch}\,(P))) \cdot \Phi \cdot (\pi^*)^{-1}(\mathrm{td}\,(\eta))) \end{aligned}$$

since π^* is an isomorphism (lemma 6.3.15) and therefore $(\pi^*)^{-1}(\mathrm{td}\,(\eta)) = \mathrm{td}\,(\upsilon(T^*X))$ is well-defined.

Finally, by lemma 5.3.1, $\mathrm{td}\,(\upsilon(T^*X)) = \mathrm{td}\,(\mathcal{T} \oplus \overline{\mathcal{T}}) = \mathrm{td}\,(\mathcal{T}) \cdot \mathrm{td}\,(\overline{\mathcal{T}})$. Putting this together we get

$$\begin{aligned} \mathrm{ind}_t(P) &:= {}_{2m}\{\mathrm{ch}\,(P) \cdot \mathrm{td}\,(\eta)\}[T^*X] \\ &= (-1)^n \{\phi_*^{-1}(\mathrm{ch}\,(P) \cdot \mathrm{td}\,(\eta))\}[X] \\ &= (-1)^n \{\phi_*^{-1}(\mathrm{ch}\,(P)) \cdot \mathrm{td}\,(\upsilon(T^*X))\}[X] \\ &= (-1)^n \{\phi_*^{-1}(\mathrm{ch}\,(P)) \cdot \mathrm{td}\,(\mathcal{T}) \cdot \mathrm{td}\,(\overline{\mathcal{T}})\}[X]. \end{aligned} \tag{4.2}$$

□

Now let X be a compact Riemann surface with holomorphic cotangent bundle $\mathbf{T}$, and $L \to X$ a holomorphic line bundle. Then $\mathcal{E}(L \otimes \overline{\mathbf{T}}) = \mathcal{E}^{0,1}(L)$. In definition 2.2.5, we have defined the operator

$$\overline{\partial}_L : \mathcal{E}(L) \to \mathcal{E}(L \otimes \overline{\mathbf{T}})$$

and shown that it is an elliptic differential operator of order 1. As before we choose a Hermitian metric and a frame on $\mathcal{T}$ so that $\mathcal{T}$ is described by unitary transition functions, and we let T^*X be the real cotangent bundle on X, with $T^*X^{\mathbb{C}}$ its complexification.

By corollary 1.1.29, $T^*X \cong \overline{\mathbf{T}}$ so we may identify the (real) disc bundle $B(X) = B(T^*X)$ with $B(\overline{\mathbf{T}})$. We calculated the symbol $\sigma(\overline{\partial}_L)$ in example 2.2.6. Since $B(T^*X) \cong B(\overline{\mathbf{T}})$, we can express the symbol as

$$\sigma(\overline{\partial})(s(x), \overline{\partial}f(x)) = (is(x)\overline{\partial}f(x), \overline{\partial}f(x))$$

for $s \in \mathcal{E}(L)$ and f a non-constant differentiable function on X such that $f(x) = 0$ and $\overline{\partial} f$ is non-zero on a neighbourhood U of $x \in X$. In particular, $\sigma(\overline{\partial})|_{S(\overline{\mathbf{T}}}$ is an isomorphism.

So $\sigma(\overline{\partial}_L) = (i\tilde{\beta}, \text{ id })$ in relation to $B(\overline{\mathbf{T}})$ where $\tilde{\beta} : L \times \overline{\mathbf{T}} \to L \otimes \overline{\mathbf{T}}$ is the natural map $(s(x), \overline{\partial} f(x)) \mapsto s(x) \cdot \overline{\partial} f(x)$.

Now, since for a vector bundle $E \to X$, $E \cong iE$ with the orientation unchanged, we can henceforth ignore the factor i in the symbol isomorphism.

Lemma 4.3.

$$d(\pi^* L, (\pi^* L) \otimes (\pi^* \overline{\mathbf{T}}), \beta) = \pi^* L \otimes d(\pi^* \mathbb{C}, \pi^* \overline{\mathbf{T}}, \beta). \tag{4.3}$$

Proof. We simply follow the general construction of the difference bundle given above. The map

$$\beta : \pi^* \mathbb{C}|_{S(\overline{\mathbf{T}})} \to \pi^* \overline{\mathbf{T}}|_{S(\overline{\mathbf{T}})}$$

is given by $\beta(z, \alpha, x) := (z \cdot \alpha, \alpha, x)$ for $z \in \mathbb{C}$, $\alpha \in S(\overline{\mathbf{T}})|_x$ over $x \in X$. In particular, since $\alpha \neq 0$, this is an isomorphism.

Let $Z = (B(\overline{\mathbf{T}}) \times 0) \cup (B(\overline{\mathbf{T}}) \times 1) \cup (S(\overline{\mathbf{T}}) \times I)$ and $p : (B(\overline{\mathbf{T}}) \times I) \to B(\overline{\mathbf{T}})$ be the projection onto $B(\overline{\mathbf{T}})$. Z is covered by the open sets

$$Z_0 = (B(\overline{\mathbf{T}}) \times 0) \cup (S(\overline{\mathbf{T}}) \times [0, 1))$$

and

$$Z_1 = (B(\overline{\mathbf{T}}) \times 1) \cup (S(\overline{\mathbf{T}}) \times (0, 1]),$$

with $p_i := p|_{Z_i},\ i = 0, 1$ the restriction map.

Then $p_0^*(\pi^* L) = p_0^*(\pi^*(L \otimes \mathbb{C}))$ is a vector bundle over the open set Z_0, and $p_1 \pi^*(L \otimes \overline{\mathbf{T}})$is a vector bundle over Z_1. $p^* \tilde{\beta} = \text{id}_{\pi^* L} \otimes p^* \beta$ is an isomorphism on the open set $Z_0 \cap Z_1$.

We can therefore glue the bundles $p_0^*(\pi^*(L \otimes \mathbb{C}))$ over Z_0 and $p_1 \pi^*(L \otimes \overline{\mathbf{T}})$ over Z_1 together with the isomorphism $\text{id}_{\pi^* L} \otimes p^* \beta$ on $Z_0 \cap Z_1$ to obtain a vector bundle $E(L, L \otimes \overline{\mathbf{T}}, \tilde{\beta})$ over Z.

Since $L = L \otimes \mathbb{C}$ and the maps π^*, p_i^* and p^* are homomorphisms, we have $E(L, L \otimes \overline{\mathbf{T}}, \tilde{\beta}) = p^* \pi^* L \otimes E(\mathbb{C}, \overline{\mathbf{T}}, \beta)$ over Z and $E(L, L \otimes \overline{\mathbf{T}}, \tilde{\beta}) - p^* \pi^* L)$ is trivial over Z_0. Applying the splitting map $g^! : K(Z) \to K\left(Z/_{(\overline{\mathbf{T}} \times 0)} \right)$ (see lemma 6.2.1 above) to this bundle, we obtain

$$\begin{aligned} g^! \left(E(L, L \otimes \overline{\mathbf{T}}, \tilde{\beta}) - p^* \pi^* L \right) &= g^! \left(p^* \pi^* L \otimes E(\mathbb{C}, \overline{\mathbf{T}}, \beta) - p^* \pi^* (L \otimes \mathbb{C}) \right) \\ &= g^! p^* \pi^* L \otimes g^! \left(E(\mathbb{C}, \overline{\mathbf{T}} T, \beta) - p^* \pi^* \mathbb{C} \right) \\ &= \pi^* L \otimes d(\pi^* \mathbb{C}, \pi^* \overline{\mathbf{T}}, \beta). \end{aligned}$$

as required.

□

So, if $\phi_* : H^i_{deRh}(X) \to H^{i+2}_{deRh}(B(\overline{\mathbf{T}})/S(\overline{\mathbf{T}}))$ is the Thom isomorphism

$$\phi_*^{-1} \text{ch}\, (\overline{\partial}_L) = \phi_*^{-1} \text{ch } \ d(\pi^* L, \pi^*(L \otimes \overline{\mathbf{T}}), \beta) = \phi_*^{-1} \text{ch}\, (L \otimes \ d(\pi^* \mathbb{C}, \pi^* \overline{\mathbf{T}}, \beta)).$$

Lemma 4.4.

$$\begin{aligned} \phi_*^{-1} (\text{ch}\, (d(\pi^* L, \pi^*(L \otimes \overline{\mathbf{T}}), \beta))) &= \phi_*^{-1} \left(\text{ch}\, (\pi^* L) \cdot \text{ch}\, (d(\pi^* \mathbb{C}, \pi^* \overline{\mathbf{T}}, \beta)) \right) \\ &= \text{ch}\, (L) \left(\phi_*^{-1} \text{ch}\, (d(\pi^* \mathbb{C}, \pi^* \overline{\mathbf{T}}, \beta)) \right). \end{aligned} \tag{4.4}$$

PROOF. This follows from the definition of the Thom isomorphism and the fact that ch is a homomorphism. □

LEMMA 4.5.

$$\phi_*^{-1}(\mathrm{ch}\,(d(\pi^*\mathbb{C}, \pi^*\overline{\mathbf{T}}, \beta))) = -(\mathrm{td}\,(\overline{\mathcal{T}}))^{-1}.$$

PROOF. [**Hi**, 181-182]

□

Combining all the above we arrive at

THEOREM 4.6. *If X is a compact Riemann surface and L a holomorphic line bundle over X, then*

$$ind_t(\overline{\partial}_L) = T(L)$$

where $T(L)$ is the Todd genus of L.

PROOF. Let $\mathbf{T}$ be the complex cotangent bundle of X and $\eta \cong \mathcal{T} \oplus \overline{\mathcal{T}}$ be an almost complex structure for T^*X, the real cotangent bundle of X. By the above

$$\mathrm{ind}_t(\overline{\partial}_L) := {}_2\{\mathrm{ch}\,(\overline{\partial}_L) \cdot \mathrm{td}\,\eta\}[T^*X]$$

$$= (-1)\{\phi_*^{-1}\mathrm{ch}\,(\overline{\partial}_L) \cdot \mathrm{td}\,\eta))\}[X] \tag{4.5}$$

$$= (-1)(-1)\{\mathrm{ch}\,(L)(\mathrm{td}\,(\overline{\mathcal{T}}))^{-1}) \cdot \mathrm{td}\,(\mathcal{T})\mathrm{td}\,(\overline{\mathcal{T}})\}[X] \tag{4.6}$$

$$= \{\mathrm{ch}\,(L) \cdot \mathrm{td}\,(\mathcal{T})\}[X] = T(L). \tag{4.7}$$

Here 6.(4.5) follows from lemma 6.4.2 and 6.(4.6) follows from lemmas 6.4.4 and 6.4.5

□

We have therefore proved that, in the case that E is a holomorphic line bundle over a Rieman surface X, the T-characteristic $T(E)$ of E is equal to the topological index $\mathrm{ind}_t(E)$ of E. In doing so we have shown that the classical Riemann Roch theorem 3.2.2 is a special case of the Atiyah-Singer index formula 6.0.5.

Appendix: Elliptic complexes and the topological index

In the introduction (page 5) it was mentioned that the Atiyah-Singer index formula can be applied to elliptic complexes (definition 2.2.1) defined on compact complex manifolds. If E is an elliptic complex on a compact complex manifold X then, analogue to the operator case, the Atiyah-Singer index formula says that

$$\mathrm{ind}_a(E) = \mathrm{ind}_t(E).$$

The analytic index $\mathrm{ind}_a(E)$ of an elliptic complex E has been defined in definition 2.2.10. In this appendix we shall give a definition of the topological index $\mathrm{ind}_t(E)$ of an elliptic complex E over a compact complex manifold X which corresponds to the definition already given in chapter 6, page 67 for the operator case.

For the sake of completion, we shall briefly mention the Dolbeault complex $\overline{\partial}(E)$ associated to a holomorphic vector bundle E of rank r over a compact complex manifold X of dimension n, and its corresponding operator. Of course, since the Hirzebruch-Riemann-Roch theorem is a special case of the Atiyah-Singer index formula and we have seen that the analytic index of the Dolbeault complex $\overline{\partial}(E)$ of E is equal to the Euler characteristic $\chi(E)$ of E (theorem 3.3.5), it then also holds that $T(E) = \mathrm{ind}_t(\overline{\partial}(E))$.

PROPOSITION 7.7. *Let $(E, h), (F, h')$ be Hermitian bundles over a compact complex manifold X and $P \in \mathrm{Diff}_k(E, F)$ a differential operator. Then P has a unique formal adjoint $P^* \in \mathrm{Diff}_k(F, E)$ with respect to the metrics h, h'. That is, we can define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{E}(E)$ and an inner product $\langle \cdot, \cdot \rangle'$ on $\mathcal{E}(F)$ by*

$$\langle \xi, \eta \rangle = \int_X h(\xi(x), \eta(x)) \mathrm{dvol}, \;\; \xi, \eta \in \mathcal{E}(E)$$

and

$$\langle \xi', \eta' \rangle' = \int_X h'(\xi'(x), \eta'(x)) \mathrm{dvol}, \;\; \xi', \eta' \in \mathcal{E}(F)$$

where dvol *is a volume form on X induced by the underlying Riemannian structure. Then, there exists an unique operator $P^* : \mathcal{E}(F) \to \mathcal{E}(E)$ such that*

$$\langle P(\xi), \eta' \rangle' = \langle \xi, P^*(\eta') \rangle, \quad \text{for all } \xi \in \mathcal{E}(E), \eta' \in \mathcal{E}(F).$$

Furthermore, it holds that, if $\sigma_k(P)^*$ is the adjoint of the linear map $\sigma_k(P)(\cdot, v_x) : E_x \to F_x$,

$$\sigma_k(P^*) = \sigma_k(P)^*.$$

In particular, P^* is elliptic if and only if P is elliptic.

PROOF. [**We**, 117-118]. The last statement follows immediately from the definitions and the first two statements. □

LEMMA 7.8. *If $(E, h), (F, h')$ are Hermitian bundles over a complex manifold X and $P \in \mathrm{Diff}_k(E, F)$ has adjoint $P^* \in \mathrm{Diff}_k(F, E)$, then*

$$\mathrm{Ker}\,(P^*) = \mathrm{Coker}\,(P)$$

and

$$\mathrm{Ker}\,(P) = \mathrm{Coker}\,(P^*).$$

PROOF. Given $\eta \in \mathcal{E}(F)$,

$$\begin{array}{rcll} \eta \in \mathrm{Ker}\,(P^*) & \Leftrightarrow & P^*(\eta) = 0 & \\ & \Leftrightarrow & \langle(\xi, P^*(\eta)\rangle = 0, & \text{for all } \xi \in \mathcal{E}(E) \\ & \Leftrightarrow & \langle(P(\xi), (\eta)\rangle' = 0, & \text{for all } \xi \in \mathcal{E}(E) \\ & \Leftrightarrow & \eta \in \mathrm{Coker}\,(P). & \end{array}$$

The proof of the second statement follows exactly the same method. □

Given an elliptic complex $E = (E_i, d_i)_{i=0}^l$ (of length $l + 1$) over a compact complex manifold X:

$$0 \to \mathcal{E}(E_0) \xrightarrow{d_0} \mathcal{E}(E_1) \xrightarrow{d_1} \ldots \xrightarrow{d_{l-1}} \mathcal{E}(E_l) \xrightarrow{d_l} \to 0,$$

we can define a unique operator $P_E : \mathcal{E}(F) \to (F')$ by

$$\begin{array}{rcl} F & := & \bigoplus_{\substack{k=0 \\ 2k \leq l}} E_{2k} \\ F' & := & \bigoplus_{\substack{k=0 \\ 2k \leq l}} E_{2k+1} \end{array}$$

and

$$P_E := d_0 \oplus \left(\bigoplus_{\substack{k=1 \\ 2k \leq l}} \left(d_{2k} + d_{2k-1}^*\right) \right) : \mathcal{E}(F) \to \mathcal{E}(F').$$

That is

$$P_E(\xi_0, \xi_2, \ldots) = (d_0(\xi_0) + d_1^*(\xi_2), d_2(\xi_2) + d_3^*(\xi_4), \ldots).$$

Since P_E is a direct sum of elliptic differential operators, it is itself an elliptic differential operator and therefore the analytic index $\mathrm{ind}_a(P_E)$ and the topological index $\mathrm{ind}_t(P_E)$ of P_E are defined.

REMARK 7.9. If we view an elliptic operator P as a complex E_P of length 1, then trivially $P_{E_P} = P$.

EXAMPLE 7.10. If L is a holomorphic line bundle over a compact Riemann surface X, then L has Dolbeault sequence $\overline{\partial}(L)$ given by

$$0 \to \mathcal{E}(L) \xrightarrow{\overline{\partial}_L} \mathcal{E}(L \otimes \overline{\mathbf{T}}) \to 0$$

and so $P_{\overline{\partial}(L)} = \overline{\partial}_L$.

PROPOSITION 7.11. *The analytic index of the operator P_E is equal to the analytic index of the complex E. In particular, the analytic index $\mathrm{ind}_a(P_E)$ is independent of the choice of metrics h_i on E_i.*

Proof. This follows from the definitions 2.2.10 and lemma A.7.8 above: Say, ξ_k and ξ_{k+2} are such that $d_k(\xi_k) \equiv \pm d^*_{k+1}(\xi_{k+2})$. Then, for all x in X,

$$\begin{aligned} |d_k(\xi_k)(x)|^2 &= h_{k+1}(d_k(\xi_k)(x), d_k(\xi_k)(x)) \\ &= \pm h_k(d_k(\xi_k)(x), d^*_{k+1}(\xi_{k+2})(x)) \\ &= \pm h_{k+1}(d_{k+1} d_k(\xi_k)(x), \xi_{k+2})(x)) \\ &= 0. \end{aligned}$$

So $\mathrm{Im}\,(d^*_{k+1}) \cap \mathrm{Im}\,(d_k) \equiv \{0\}$ and the dimensions of the kernel and cokernel of P_E are obatained by summing the dimensions of the respective kernels and cokernels of the constituent maps $d_0, d^*_1, d_2, d^*_3, \ldots$. Therefore,

$$\begin{aligned} \mathrm{ind}_a(P_E) &:= \dim \mathrm{Ker}\,(P_E) - \dim \mathrm{Coker}\,(P_E) \\ &= \left(\dim \mathrm{Ker}\,(d_0) + \sum_{\substack{k=1 \\ 2k \le l}} \big(\dim \mathrm{Ker}\,(d_{2k}) + \dim \mathrm{Ker}\,(d^*_{2k-1}) \big) \right) \\ &\quad - \left(\dim \mathrm{Coker}\,(d_0) + \sum_{\substack{k=1 \\ 2k \le l}} \big(\dim \mathrm{Coker}\,(d_{2k} + \dim \mathrm{Coker}\,(d^*_{2k-1}) \big) \right) \\ &= \left(\dim \mathrm{Ker}\,(d_0) + \sum_{\substack{k=1 \\ 2k \le l}} (\dim \mathrm{Ker}\,(d_{2k}) + \dim \mathrm{Coker}\,(d_{2k-1})) \right) \\ &\quad - \left(\dim \mathrm{Coker}\,(d_0) + \sum_{\substack{k=1 \\ 2k \le l}} (\dim \mathrm{Coker}\,(d_{2k}) + \dim \mathrm{Ker}\,(d_{2k-1})) \right) \\ &= (\dim \mathrm{Ker}\,(d_0) - \dim \mathrm{Coker}\,(d_0)) + \sum_{\substack{k=1 \\ 2k \le l}} (\dim \mathrm{Ker}\,(d_{2k}) - \dim \mathrm{Coker}\,(d_{2k})) \\ &\quad - (\dim \mathrm{Ker}\,(d_{2k-1}) - \dim \mathrm{Coker}\,(d_{2k-1})) \\ &= \sum_{i=0}^{l} (\dim \mathrm{Ker}\,(d_i) - \dim \mathrm{Coker}\,(d_i)) \\ &=: \mathrm{ind}_a(E) \end{aligned}$$

as required.

□

Definition 7.12. *The* topological index $\mathrm{ind}_t(E)$ of an elliptic complex E *on a compact complex manifold* X *is defined as*

$$ind_t(E) := ind_t(P_E).$$

Proposition 7.13. *The topological index $ind_t(E)$ is independent of the choice of metrics h_i on E_i, $i = 0, \ldots, l$.*

Proof. For more details on the construction see [**AS1**, 489-508], [**AS3**, 552-559].

□

If (E, h) is a holomorphic Hermitian bundle over a complact complex manifold X with $\operatorname{rk} E = r$, as before, the Dolbeault complex, $\overline{\partial}(E)$, of E is given by

$$0 \to \mathcal{E}^{0,0}(E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{0,1}(E) \xrightarrow{\overline{\partial}_E} \dots \xrightarrow{\overline{\partial}_E} \mathcal{E}^{0,q}(E) \to 0$$

PROPOSITION 7.14. *The topological index of $\overline{\partial}(E)$ is equal to T-charateristic $T(E)$.*

PROOF. [**Hi**, 189-190]. □

In particular, the Hirzebruch-Riemann-Roch theorem (theorem 5.4.3) is a special case of the Atiyah-Singer index formula (theorem 6.0.5).

REMARK 7.15. Here we have defined the topological index of an elliptic complex by reducing the complex to an operator between vector bundles. In chapter 6, the topological K-group of a compact manifold X was defined (definition 6.1.1) as the Abelian group induced by the semi-group of isomorphism classes of vector bundles. Equivalently, $K(X)$ can be defined as the group of certain equivalence classes of complexes over X. Using this definition, the topological index of an elliptic complex is obtained directly without first reducing to the operator case. This was also the approach taken by Atiyah and Singer in their proof of the Atiyah-Singer index formula using topological K-theory. See [**AS1, AS3**]

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