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Simple Modules of Reductive Groups

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Simple Modules of Reductive Groups

Master's thesis, defended on June 22nd, 2009

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Contents

Introduction

Introduction

This master's thesis, or mémoire, is in fact a reading report on J.C.Jantzen's book "Representations of Algebraic Groups". The aim of this thesis is to offer a quick way to understand the simple modules for reductive groups, so little of this material is original.

The thesis contains three main parts: the correspondence between simple modules and dominant weights of reductive groups, Steinberg's Tensor Product Formula, and the Linkage theorem.

The classification of all simple modules is the main context of chapter 1. The main conclusion is given by corollary 1.2.7:

Proposition 0.0.1. *The $L(\lambda)$ with $\lambda \in X(T)_+$ are a system of representatives for the isomorphism classes of all simple G-modules.*

Here $X(T)_+$ is a subset of the characters of the reductive group given by G. And we will see that $L(\lambda)$'s which we construct later are all the simple modules of G.

Based on chapter 1, Steinberg's Tensor Formula (corollary 2.2.14), which we will prove in chapter 2, offers a way to treat simple modules $L(\lambda)$ as the tensor products of some simple modules of its Frobenius Kernels, namely:

Proposition 0.0.2. *Let $\lambda_0, \lambda_1, \dots, \lambda_m \in X_1(T)$ and set $\lambda = \sum_{i=0}^m p^i \lambda_i$. Then:*

$$L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_m)^{[m]}.$$

Here the upper index means some twist by Frobenius morphism.

On the other hand, the Linkage theorem (corollary 3.2.11) offers a necessary condition for the weights of a given simple module:

Corollary 0.0.3. *Let λ, $\mu \in X(T)_+$. If $\operatorname{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$, then $\lambda \in W_p \cdot \mu$.*

The proposition comes from some careful observation of the actions of reflections on the character space $X(T)$ and the study of higher cohomology groups of simple modules. We will give a detailed description of these in chapter 3. But this part may be not so satisfactory, as the comprehension of this theorem will be part of my future study.

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Chapter 0

Preliminaries

As the title hinters, there is no main context that dominates the whole chapter. The general aim of this chapter is to do some preparations for the later three chapters, and clarify some concepts and notations. The reader may just sketch the concepts and definitions, or even jump this chapter if they are already familiar with, and go back when they need a reference in the later three chapters.

0.1 Representations of Algebraic Groups

We will give a quick introduction on representation theory for general algebraic groups which is the basis of the whole thesis. And the induction functor Ind introduced here will play an important role in the latter part of this thesis.

0.1.1 Conventions

Let k be an arbitrary ring, and let A denote a k-algebra. A *k-functor* is a functor from the category of k-algebras to the category of sets. Obviously, any scheme X defines a k-functor by $X(A) = \mathrm{Hom}_{k-\mathrm{schemes}}(X, \mathrm{Spec}(A))$. Any k-module M also defines a k-functor M_a by $M_a(A) = M \otimes_k A$.

A *k-group functor* is a functor from the category of all k-algebras to the category of groups. For convenience, we define a *k-group scheme* to be a k-group functor that is represented by an affine scheme over k. An *algebraic k-group* is a k-group scheme which can also be represented by an algebraic affine scheme. It is well-known that each k-group scheme has an Hopf algebra structure of its coordinates ring $k[G]$. For example, we have the *additive group* $\mathbf{G}_\mathrm{a} = \mathrm{Spec}\, k[T]$ and *multiplicative group* $\mathbf{G}_\mathrm{m} = \mathrm{Spec}\, k[T, T^{-1}]$.

Now let G be a k-group scheme and M a k-module. A *representation of* G on M (or a *G-module structure on* M) is an operation of G on the k-functor M_a such that each $G(A)$ operates on $M_a(A) = M \otimes A$ through A-linear maps. Such a representation gives for each A a group homomorphism $G(A) \to \mathrm{End}_A(M \otimes A)^\times$. There is an obvious notion of a G-module homomorphism between two G-modules M and M'. The k-module of all such homomorphism is denoted by $\mathrm{Hom}_G(M, M')$.

For example, for G a k-group scheme, we have the *left regular representations* derived from the action of G on itself by left multiplications. We shall denote the corresponding homomorphisms $G \to \mathbf{GL}(k[G])$ by ρ_l. Similarly, we have the *right regular representations* and ρ_r. Furthermore, the conjugation action of G on itself gives rise to the *conjugation representation* of G on $k[G]$.

The representations of G on the k-module k, for example, correspond bijectively to the group homomorphisms form G to $\mathbf{GL}_1 = \mathbf{G}_{\mathrm{m}}$, i.e., to the elements of $X(G) = \mathrm{Hom}(G, \mathbf{G}_{\mathrm{m}})$. For each $\lambda \in X(G)$ we denote k considered as a G-module via λ by k_λ. In case $\lambda = 1$ we simply write k.

It is a fact that giving a G-module M is equivalent to giving a comodule M, which is given by a linear map $\Delta_M : M \to M \otimes k[G]$ (For example, see [M], proposition 3.2).

0.1.2 Twisting with Ring Endomorphism

A representation over k of a group can also be twisted by a ring endomorphism ϕ of k. If M is a k-module, then let $M^{(\phi)}$ be the k-module that coincides with M as an abelian group, but where $a \in k$ acts as $\phi(a)$ does on M. Now let $\psi : \mathbb{A}^1 \to \mathbb{A}^1$ be a morphism such that each $\psi(A)$ is a ring endomorphism on $\mathbb{A}^1(A) = A$ and such $\psi(k)$ is bijective. So let $\phi = \psi(k)^{-1}$. Then $\psi_* : f \to \psi \circ f$ is a ring endomorphism of $k[G] = \mathrm{Mor}(G, \mathbb{A}^1)$, but not, in general, k-linear. If we change the k-structure on $k[G]$ to that of $k[G]^{(\phi)}$, then ψ_* is a k-algebra homomorphism: $k[G]^{(\phi)} \to k[G]$. If M is a G-module, then the comodule map $\Delta_M : M \to M \otimes k[G]$ can also be regarded as a k-linear map:

$$M^{(\phi)} \to (M \otimes k[G])^{(\phi)} \cong M^{(\phi)} \otimes k[G]^{(\phi)}$$

If we compose with $id_M \otimes \psi_*$ we get a k-linear map $M^{(\phi)} \to M^{(\phi)} \otimes k[G]$. We can check that it gives a comodule of $k[G]$, and hence a G-module $M^{(\phi)}$.

0.1.3 Induction Functor

Let G be a k-group functor and H a subgroup functor of G. Every G-module M is an H-module in a natural way: restrict the action of $G(A)$ for each k-algebra to $H(A)$, In this way we get a functor:

$$\mathrm{Res}_H^G : \quad \{G-\mathrm{modules}\} \to \{H-\mathrm{modules}\}$$

which is obviously exact.

Now we define the right adjoint functor Ind_H^G as:

$$\begin{aligned} \mathrm{Ind}_H^G = &\{f \in \mathrm{Mor}(G, M_a) \mid f(gh) = h^{-1} f(g) \\ &\text{for all } \ g \in G(A),\, h \in H(A) \text{ and all } k\text{-algebras } A\} \end{aligned}$$

There is an equivalent way to define the induction functor. By regarding $M \otimes k[G]$ as a $G \times H$-module: G operates trivially on M and left regular representation on $k[G]$ and H acts normally on M and right regular representations on $k[G]$, and we can prove that $(M \otimes k[G])^H$ is a G-module and is exactly $\mathrm{Ind}_H^G M$. For details, see [J] I 3.3.

We can prove that the functor Ind is in fact a right adjoint functor of functor Res, and we denote it the formula *Frobenius Reciprocity*:

$$\operatorname{Hom}_G(N, \operatorname{Ind}_H^G M) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G N, M) \tag{0.1}$$

Let G' be a flat k-group scheme operating on G through automorphisms and let H be a flat subgroup scheme of G stable under G'. We can form the semi-direct products $H \rtimes G'$ and $G \rtimes G'$ naturally, and we can regard $H \rtimes G'$ as a subgroup scheme of $G \rtimes G'$. On the other hand, as G' normalizes H, it operates also on $\operatorname{Ind}_H^G M$. Therefore we get on $\operatorname{Ind}_H^G M$ a structure as a $G \rtimes G'$-module. We have the following isomorphism of $(G \ltimes G')$-modules:

$$\operatorname{Ind}_H^G M \cong \operatorname{Ind}_{H \rtimes G'}^{G \rtimes G'} M$$

for any $H \rtimes G'$-module M which acquires a H-module structure naturally.

0.1.4 Induction Functor, Geometric Interpretation

Now let G be a flat k-group scheme acting on X a flat k-scheme X such that X/G is a scheme. (Here we understand X/G to be the k-functor $A \mapsto X(A)/G(A)$). We have a canonical map $\pi : X \to X/G$. For each G-module M, we associate a sheaf $\mathscr{L}(M) = \mathscr{L}_{X/G}(M)$ on X/G:

$$\begin{aligned}\mathscr{L}(M)(U) = &\{f \in \operatorname{Mor}(\pi^{-1}U, M_a) \mid f(xg) = g^{-1}f(x) \\ &\text{for all } x \in (\pi^{-1}U)(A), g \in G(A) \text{ and all } A\}\end{aligned} \tag{0.2}$$

If $\pi^{-1}U$ is affine, then we have $\operatorname{Mor}(\pi^{-1}U, M_a) = M_a(k[\pi^{-1}U] = M \otimes k[\pi^{-1}U]$. This is a G-module via the given action on M and the operation on $k[\pi^{-1}U]$ derived from the action on $\pi^{-1}U \subset X$. So obviously we have:

$$\mathscr{L}(M)(U) = (M \otimes k[\pi^{-1}U])^G.$$

In fact, by an elementary argument, we have that $\mathscr{L}(M)$ is a sheaf of $\mathscr{O}_{X/G}$-modules. It is called the *associated sheaf* to M on X/G.

Following the notations of last subsection, it is easy to see that we have:

$$\operatorname{Ind}_H^G M = \mathscr{L}(M)(G/H) = H^0(G/H, \mathscr{L}(M)). \tag{0.3}$$

Note the last cohomology group is the cohomology of sheaves.

0.1.5 Simple Modules

In this subsection, we assume k is a field. As usual, a G-module is called *simple* if $M \neq 0$ and if M has no G-submodules other than 0 and M. It is called *semi-simple* if it is a direct sum of simple G-submodules. For any M the sum of all its simple submodules is called the *socle* of M and denoted by $\operatorname{soc}_G M$ (or equivalently $\operatorname{soc} M$ if it is clear which G is considered). It is the largest semi-simple submodule of M. For a given simple G-module E, the sum of all simple G-submodules of M isomorphic to E is called the *E-isotypic component* of $\operatorname{soc}_G M$ and denoted by $(\operatorname{soc}_G M)_E$.

The *socle series* or (ascending) *Loewy series* of M

$$0 \subset \operatorname{soc}_1 M = \operatorname{soc}_G M \subset \operatorname{soc}_2 M \subset \operatorname{soc}_3 M \subset \dots$$

is defined iteratively through $\operatorname{soc}(M/\operatorname{soc}_{i-1} M) = \operatorname{soc}_i M/\operatorname{soc}_{i-1} M$.

Any finite dimensional G-module M has a composition series (or *Jordan-Hölder series*). The number of factor isomorphic to a given simple G-module E is independent of the choice of the series. It is called the *multiplicity* of E as a composition factor of M and usually denoted by $[M : E]$ or $[M : E]_G$.

For any G-module M and any simple G-module E, the map $\phi \otimes e \mapsto \phi(e)$ is an isomorphism:

$$\operatorname{Hom}_G(E, M) \otimes_D E \cong (\operatorname{soc}_G M)_E \tag{0.4}$$

where $D = \operatorname{End}_G(E)$.

The *radical* $\operatorname{rad}_G M$ of a G-module M is the intersection of all maximal submodules. If $\dim M < \infty$, then $\operatorname{rad}_G M$ is the smallest submodule of M with $M/\operatorname{rad}_G M$ semi-simple.

0.1.6 Injective Modules

We define an *injective* G-module to be an injective object in the category of all G-modules. We give without proof the following propositions about injective modules, which gives a clear description of these objects:

Proposition 0.1.1.

1. *For each flat subgroup scheme H of G the functor Ind_H^G maps injective H-modules to injective G-modules.*
2. *Any G-module can be embedded into an injective G-module.*
3. *A G-module M is injective if and only if there is an injective k-module I such that M is isomorphic to a direct summand of $I \otimes k[G]$ with I regarded as a trivial G-module.*

Proof. See [J] I 3.9. □

Proposition 0.1.2.

1. *For each simple G-module E there is an injective G-module Q_E (unique up to isomorphism) with $E \cong \operatorname{soc} Q_E$.*
2. *An injective G-module is indecomposable (as a direct sum of injective submodules) if and only if it is isomorphic to Q_E for some simple G-module E.*

Proof. See [J] I 3.16. □

The module Q_E mentioned above is called the *injective hull* of E. More generally, we can find for each G-module M an injective G-module Q_M(unique up to isomorphism) with $\operatorname{soc} M \cong \operatorname{soc} Q_M$.

0.1.7 Cohomology

Now we assume that G is a flat group scheme over k and H a flat subgroup scheme of G. It is easy to see that the G-modules form an abelian category, and by proposition 0.1.1, the category contains enough injective modules. So we can apply general theory of cohomology theories. For example, the fixed point functor from $\{G\text{-modules}\}$ to $\{k\text{-modules}\}$ is left exact. We denote its derived functors by $M \to H^n(G, M)$, and call $H^n(G, M)$ the n^{th} *cohomology* group of M.

In this memoir, another very important functor is Ind, by subsection 0.1.4, we have:

Proposition 0.1.3. *Let G be a flat k-group scheme and H a subgroup scheme of G such that G/H is a scheme (e.g. H is closed in G).*

1. *There is for each H-module M and each $n \in \mathbb{N}$ a canonical isomorphism of k-modules:*
$$R^n \operatorname{Ind}_H^G M \cong H^n(G/H, \mathscr{L}_{G/H}(M)). \tag{0.5}$$
2. *If G/H is noetherian, then $R^n \operatorname{Ind}_H^G = 0$ for all $n > \dim(G/H)$.*
3. *Suppose that k is noetherian and that G/H is a projective scheme. For any H-module M that is finitely generated over k, each $R^n \operatorname{Ind}_H^G M$ is also finitely generated over k.*

Another important tool is to study the functor Ind's *Grothendieck's Spectral Sequence*:

Proposition 0.1.4. *Let $\mathscr{F} : \mathscr{C} \to \mathscr{C}'$ and $\mathscr{F}' : \mathscr{C}' \to \mathscr{C}''$ are additive functors where $\mathscr{C}$, $\mathscr{C}'$ and $\mathscr{C}''$ are abelian categories having enough injectives. If $\mathscr{F}'$ is left exact and if $\mathscr{F}$ maps injective objects in category $\mathscr{C}$ to acyclic for $\mathscr{F}'$, then there is a spectral sequence for each object M in $\mathscr{C}$ with differential d_r with bidegree $(r, 1-r)$, and*

$$E_2^{n,m} = (R^n \mathscr{F}') \circ (R^m \mathscr{F})M \Rightarrow R^{n+m}(\mathscr{F}' \circ \mathscr{F})M.$$

Note that if H' and H are flat subgroup scheme of G with $H \subset H'$, we have $\operatorname{Ind}_H^G = \operatorname{Ind}_{H'}^G \circ \operatorname{Ind}_H^{H'}$, then applying Grothendieck's spectral sequence, we have:

Proposition 0.1.5. *We have spectral sequence:*

$$E_2^{n,m} = (R^n \operatorname{Ind}_{H'}^G)(R^m \operatorname{Ind}_H^{H'})M \Rightarrow (R^{n+m} \operatorname{Ind}_H^G)M$$

Similarly, we give the following propositions without proof:

Proposition 0.1.6. *Let N be a G-module that is flat as a k-module. Then we have for each H-module M and each $n \in \mathbb{N}$ an isomorphism:*

$$R^n \operatorname{Ind}_H^G(M \otimes N) \cong (R^n \operatorname{Ind}_H^G M) \otimes N.$$

Proof. See [J] I 4.8. □

Proposition 0.1.7. *Let H be a flat subgroup scheme of G with $N \subset H$. Suppose that both G/N and H/N are affine. Then one has for each H/N-module M and each $n \in \mathbb{N}$ an isomorphism of G-modules:*

$$(R^n \operatorname{Ind}_H^G)M \cong (R^n \operatorname{Ind}_{H/N}^{G/N})M.$$

Proof. See [J] I 6.11. □

Proposition 0.1.8. *Let G' be a flat k-group scheme that operates on G. We can therefore form the semi-direct product $G \rtimes G'$. We assume that G' stablizes the subgroup scheme H of G, we have:*

$$\mathrm{Res}_G^{G\rtimes G'} \circ R^n \,\mathrm{Ind}_{H\rtimes G'}^{G\rtimes G'} \cong R^n \,\mathrm{Ind}_H^G \circ \mathrm{Res}_H^{H\rtimes G'} .$$

Proof. See [J] I 4.11. □

0.2 Algebras of Distributions

Here we will talk on the distributions of algebraic groups and some elementary properties. The algebra of distributions over an algebraic group is an important tool for our further study of algebraic groups and their representations.

0.2.1 Distributions on a Scheme

Let X be an affine scheme over k and $x \in X(k)$. Set $I_x = \{f \in k[X] \mid f(x) = 0\}$. A *distribution* on X with support in x of order $\leq n$ is a linear map $\mu : k[X] \to k$ with $\mu(I_x^{n+1}) = 0$. These distributions form a k-module that we denote by $\mathrm{Dist}_n(X, x)$. We have:

$$(k[X]/I_x^{n+1})^* = \mathrm{Hom}_k(k[X]/I_x^{n+1}, k) \cong \mathrm{Dist}_n(X, x) \subset k[X]^*.$$

Obviously $\mathrm{Dist}_0(X, x) \cong k^* = k$, and for any n:

$$\mathrm{Dist}_n(X, x) = k \oplus \mathrm{Dist}_n^+(X, x),$$

where

$$\mathrm{Dist}_n^+(X, x) = \{\mu \in \mathrm{Dist}_n(X, x) | \mu(1) = 0\} \cong (I_x/I_x^N)^*.$$

For a $\mu \in \mathrm{Dist}_n(X, x)$, we call $\mu(1)$ its *constant term* and elements in $\mathrm{Dist}_n^+(X, x)$ are called distributions without constant term. The k-module $\mathrm{Dist}_1^+(X, x) = (I_x/I_x^2)^*$ is called the *tangent space* to X at x and is denoted by T_xX.

The union of all $\mathrm{Dist}_n(X, x)$ in $k[X]^*$ is denoted by $\mathrm{Dist}(X, x)$ and its elements are called *distributions* on X with support in x:

$$\mathrm{Dist}(X, x) = \{\mu \in k[X]^* | \exists n \in \mathbb{N} : \mu(I_x^{n+1}) = 0\} = \bigcup_{n\geq 0} \mathrm{Dist}_n^+(X, x)$$

This is obviously a k-module. Similarly, $\mathrm{Dist}^+(X, x) = \bigcup_{n\geq 0} \mathrm{Dist}_n^+(X, x)$ is a k-module.

For each $f \in k[X]$ and $\mu \in k[X]^*$, we define $f\mu \in k[X]^*$ through $(f\mu)(f_1) = \mu(ff_1)$ for all $f_1 \in k[X]$. In this way $k[X]^*$ is a $k[X]$-module. As each I_x^{n+1} is an ideal in $k[X]$, obviously each $\mathrm{Dist}_n(X, x)$ and hence also $\mathrm{Dist}(X, x)$ is a $k[X]$-submodule of $k[X]^*$.

Now let $\phi : X \to Y$ be a morphism of affine schemes and we have $\phi^* : k[Y] \to k[X]$. Then $(\phi^*)^{-1} I_x = I_{\phi(x)}$ for all $x \in X(k)$, hence $\phi^*(I_{\phi(x)}^{n+1}) \subset I_x^{n+1}$ and ϕ^* induces a linear map $k[Y]/I_{\phi(x)}^{n+1} \to k[X]/I_x^{n+1}$ which gives a linear map:

$$(d\phi)_x : \mathrm{Dist}(X, x) \to \mathrm{Dist}(Y, \phi(x))$$

with $(d\phi)_x : \mathrm{Dist}_n(X, x) \subset \mathrm{Dist}_n(Y, \phi(x))$ and $(d\phi)_x(\mathrm{Dist}_n^+(X, x)) \subset \mathrm{Dist}_n^+(Y, \phi(x))$ for all n.

We can also prove that if Y is a open subscheme of X containing x, then the open immersion $\phi : Y \to X$ induces an isomorphism: $(d\phi)_x : \mathrm{Dist}(Y, x) \to \mathrm{Dist}(X, x)$.

0.2.2 Infinitesimal Flatness

Let X be an affine scheme over k and let $x \in X(k)$. We call X *infinitesimal flat* at x if each $k[X]/I_x^{n+1}$ with $n \in \mathbb{N}$ is a finitely presented and flat k-module. In this case, we have the following properties:

1. If X is infinitesimally flat at x, then $X_{k'}$ is infinitesimally flat in x for each k-algebra k'. There are natural isomorphisms: $\mathrm{Dist}_n(X, x) \otimes k' \cong \mathrm{Dist}_n(X_{k'}, x)$ and $\mathrm{Dist}(X, x) \otimes k' \cong \mathrm{Dist}(X_{k'}, x)$.
2. If X and X' are infinitesimally flat in x resp. x', then $X \times X'$ is infinitesimally flat in (x, x'). There is an isomorphism $\mathrm{Dist}(X, x) \otimes \mathrm{Dist}(X', x') \cong \mathrm{Dist}(X \times X', (x, x'))$ mapping $\sum_{m=0}^{n} \mathrm{Dist}_m(X, x) \otimes \mathrm{Dist}_{n-m}(X', x')$ onto $\mathrm{Dist}_n(X \times X', (x, x'))$ for each $n \in \mathbb{N}$.
3. By applying (2), we consider the diagonal morphism $\delta_X : X \to X \times X$. Let us regard the tangent map $(d\delta_X)_x$ as a map $\Delta'_{X,x} : \mathrm{Dist}(X, x) \to \mathrm{Dist}(X, x) \otimes \mathrm{Dist}(X, x)$. it makes $\mathrm{Dist}(X, x)$ into a coalgebra. In fact, we have: if X is infinitesimally flat, then $\mathrm{Dist}(X, x)$ has a natural structure as a cocommutative coalgebra with a counit. Tangent maps are homomorphisms for these structures.

0.2.3 Distributions on a Group Scheme

Let G be a group scheme over k. In this case we set:

$$\mathrm{Dist}(G) = \mathrm{Dist}(G, 1).$$

We can make $\mathrm{Dist}(G)$ into an associative algebra over k. For any $\mu, \nu \in k[G]^*$ we can define product $\mu\nu$ as:

$$\mu\nu : k[G] \xrightarrow{\Delta} k[G] \otimes k[G] \xrightarrow{\mu \otimes \nu} k \otimes k \to k.$$

So we have an associative algebra structure with $\epsilon_G : \mu \to \mu(1)$ its neutral element.

By computation, we have: if $\mu \in \mathrm{Dist}_n(G)$ and $\nu \in \mathrm{Dist}_m(G)$, then: $[\mu, \nu] = \mu\nu - \nu\mu \in \mathrm{Dist}_{n+m-1}(G)$. So $\mathrm{Dist}(G)$ has a structure as a filtered associative algebra over k such that the associated graded algebra is commutative. We call Dist(G) the *algebra of distributions* on G. On the other hand, we can prove $[\mathrm{Dist}_n^+(G), \mathrm{Dist}_m^+(G)] \subset \mathrm{Dist}_{n+m-1}^+(G)$. This shows in particular that $\mathrm{Dist}_1^+(G)$ is a Lie algebra, which we denote by $\mathrm{Lie}\, G$, and call it *the Lie algebra of G*.

The conjugation action of G on itself yields a representation of G on $k[G]$ that stabilizes I_1, which is the ideal defining 1, hence also all I_1^{n+1}. We get thus G-structure on all $k[G]/I_1^{n+1}$, hence also on all $\mathrm{Dist}_n(G) = (k[G]/I_1^{n+1})^*$, provided that G is infinitesimally flat. The representation of G on $\mathrm{Lie}(G) = \mathrm{Dist}_1^+(G)$ constructed is called the *adjoint representation* of G. We use the notation Ad for this representation of G on $\mathrm{Dist}(G)$ and all $\mathrm{Dist}_n(G)$, $\mathrm{Dist}_n^+(G)$.

0.2.4 Distributions on $\mathbb{A}^n$

First let us consider as an example $X = \mathbb{A}^1 = \mathrm{Spec}_k\, k[T]$. Note at point $x = 0$, and $I_x = (T)$. The k-module $k[X]/I_x^{n+1}$ is free and have residue classes of $1 = T^0$, $T = T^1$,T^2,$\ldots$,T^n. Define $\gamma_r \in k[T]^*$ through $\gamma_r(T^n) = 0$ for $n \neq r$ and $\gamma_r(T^r) = 1$. Then obviously $\mathrm{Dist}(\mathbb{A}^1, 0)$ is a free k-module with basis $(\gamma_r)_{r\in\mathbb{N}}$ and each $\mathrm{Dist}_n(\mathbb{A}^1, 0)$ is a free k-module with basis $(\gamma_r)_{0\leq r\leq n}$.

This can be generalized to $\mathbb{A}^m = \mathrm{Spec}_k\, k[T_1, \ldots, T_m]$ for all m. For each multi-index $a = (a(1), a(2), \ldots, a(m)) \in \mathbb{N}^m$, set $T^a = T_1^{a(1)} \cdots T_m^{a(m)}$ and denote by γ_a the linear map with $\gamma_a(T^b) = 0$ if $a \neq b$ and $\gamma_a(T^a) = 1$. One easily checks that $\mathrm{Dist}(\mathbb{A}^n, 0)$ is free over k with all γ_a its basis, and that $\mathrm{Dist}_n(\mathbb{A}^m, 0)$ is free over k with all γ_a with $|a| = \sum_{i=1}^m a(i) \leq n$ as a basis.

0.2.5 Distributions on $\mathbf{G}_\mathrm{a}$ and $\mathbf{G}_\mathrm{m}$

Firstly let us look at the additive group $G = \mathbf{G}_\mathrm{a}$. As a scheme we may identify $\mathbf{G}_\mathrm{a} = \mathrm{Spec}\, k[T]$ with $\mathbb{A}^1$. Therefore we have already described $\mathrm{Dist}(\mathbf{G}_\mathrm{a})$ as a k-module in section 0.2.4. We have $\Delta(T) = 1 \otimes T + T \otimes 1$, hence $\Delta(T^n) = \sum_{i=0}^n T^i \otimes T^{n-i}$. This implies:

$$\gamma_n \gamma_m = \binom{n+m}{n} \gamma_{n+m}$$

Hence:

$$\gamma_1^n = n! \gamma_n.$$

So $\mathrm{Dist}(G_{a,\mathbb{C}})$ can be identified with the polynomial ring $\mathbb{C}[\gamma_1]$, and $\mathrm{Dist}(G_{a,\mathbb{Z}})$ with the $\mathbb{Z}$-lattice spanned by all $\frac{\gamma_1^n}{n!}$. In general $\mathrm{Dist}(\mathbf{G}_\mathrm{a}) = \mathrm{Dist}(G_{a,\mathbb{Z}}) \otimes_\mathbb{Z} k$.

Let us now consider the multiplicative group $\mathbf{G}_\mathrm{m} = \mathrm{Spec}_k\, k[T, T^{-1}]$. Then I_1 is generated by $T - 1$. The residue classes of 1, $(T - 1)$, $(T - 1)^2$,$\ldots$, $(T - 1)^n$ form a basis of $k[\mathbf{G}_\mathrm{m}]/I_1^{n+1}$. There is unique $\delta_n \in \mathrm{Dist}(\mathbf{G}_\mathrm{m})$ with $\delta_n((T - 1)^i) = 0$ for $0 \leq i < n$ and $\delta_n((T - 1)^n) = 1$. From this and binomial expansion of $T^n = ((T - 1) + 1)^n$ one gets $\delta_r(T^n) = \binom{n}{r}$ for all $n \in Z$ and $r \in \mathbb{N}$.

So all δ_r with $r \in \mathbb{N}$ form a basis of $\mathrm{Dist}(\mathbf{G}_\mathrm{m})$, and all δ_r with $r \leq n$ form a basis of $\mathrm{Dist}_n(\mathbf{G}_\mathrm{m})$. One get $\Delta(T - 1) = (T - 1) \otimes (T - 1) + (T - 1) \otimes 1 + 1 \otimes (T - 1)$ from $\Delta(T) = T \otimes T$, hence

$$\delta_r \delta_s = \sum_{i=0}^{min(r,s)} \frac{(r + s - 1)!}{(r - i)!(s - i)!i!}\, \delta_{r+s-i}$$

In particular, we have:

$$r!\delta_r = \delta_1(\delta_1 - 1)\dots(\delta_1 - r + 1)$$

So if k is a $\mathbb{Q}$-algebra, then $\delta_r = \binom{\delta_1}{r}$. Therefore $\mathrm{Dist}(G_{m,\mathbb{C}}) \cong \mathbb{C}[\delta_1]$, and $\mathrm{Dist}(G_{m,\mathbb{Z}})$ is a $\mathbb{Z}$-lattice in $\mathrm{Dist}(G_{m,\mathbb{C}})$ generated by all. In general $\mathrm{Dist}(\mathbf{G}_\mathrm{m}) = \mathrm{Dist}(G_{m,\mathbb{Z}}) \otimes k$.

0.2.6 G-modules and $\mathrm{Dist}(G)$-Modules

Let G be a group scheme over k. Then any G-module M carries a natural structure as a $\mathrm{Dist}(G)$-module. One sets for each $\mu \in \mathrm{Dist}(G)$ and $m \in M$:

$$\mu m = (id_M \otimes \mu) \circ \Delta_M(m),$$

i.e., the operation of μ on M is given by

$$M \overset{\Delta_M}{\to} M \otimes k[G] \overset{id_M \otimes \mu}{\to} M \otimes k \cong M.$$

It is trivial to verify that this gives M a $\mathrm{Dist}(G)$-module structure. And obviously we have:

$$\mathrm{Hom}_G(M, M') \subset \mathrm{Hom}_{\mathrm{Dist}(G)}(M, M').$$

Applying the description above we have:

Proposition 0.2.1. *1. Any G-submodule of a G-module M is also a $\mathrm{Dist}(G)$-submodule of M.*

2. If $m \in M^G$, then $\mu m = \mu(1)m$ for all $\mu \in \mathrm{Dist}(G)$.

3. If $m \in M_\lambda$, then $\mu m = \mu(\lambda)m$ for all $\mu \in \mathrm{Dist}(G)$ and $\lambda \in X(G) \subset k[G]$.

0.3 Finite Algebraic Groups

The main aim of this section is to prepare for the chapter 2. There we will treat some "Frobenius Kernels", which are in general a special kind of finite algebraic groups. So the propositions proved here will give a corresponding version of propositions in chapter 2.

0.3.1 Finite Algebraic Groups and Measures

A k-group scheme G is called a *finite* algebraic group if $\dim k[G] < \infty$. It is called *infinitesimal* if it is finite and its ideal $I_1 = \{f \in k[G] \mid f(1) = 0\}$ is nilpotent.

Recall $k[G]$ is a Hopf algebra. It has both an algebra structure and a coalgebra structure. As $\dim k[G] < \infty$, its dual space $k[G]^*$ hence acquire an algebra structure from the coalgebra structure of $k[G]$ and a coalgebra structure from the algebra structure of $k[G]$.

In fact we have:

Proposition 0.3.1. *The functor $R \mapsto R^*$, $\psi \mapsto \psi^*$ is a self-duality on the category of all finite dimensional Hopf algebra.*

As far as we know, there is an anti-equivalence of categories:

$$\{ \text{ group schemes over } k \ \} \to \{ \text{ commutative Hopf algebras over } k \ \}.$$

So we have

$$\{\text{finite algebraic } k\text{-groups}\} \to \{\text{finite dimensional cocommutative Hopf algebras over k}\}.$$

We denote this Hopf algebra $k[G]^*$ by $M(G)$ and call it the algebra of all *mesures* on G. We have an obvious embedding $G(k) = \mathrm{Hom}_{k-\mathrm{algebra}}(k[G], k) \hookrightarrow M(G)$: To each $g \in G(k)$, there is a *Dirac measure* $\delta_g : f \to f(g)$. We can check that the multiplication in $G(k)$ agrees with the multiplication in $M(G)$.

As G is finite, we obviously have $\mathrm{Dist}(G) \subset M(G)$ and G is infinitesimal if and only if $M(G) = \mathrm{Dist(G)}$.

Let R be a finite dimensional Hopf algebra. If M is an R-module, then M is an R^*-comodule in a natural way: Define the comodule map $M \to M \otimes R^* \cong \mathrm{Hom}(R, M)$ by mapping m to $a \to am$. If M is an R-module, then M is an R^*-module in a natural way: Define the action of any $\mu \in R^*$ as $(id_M \otimes \mu) \circ \Delta_M$ where $\Delta_M : M \to M \otimes R$ is the comodule map. For two such comodules M_1, M_2 a linear map $\psi : M_1 \to M_2$ is a homomorphism of R-comodules if and only if it is a homomorphism of R^*-modules. So we have equivalence of categories:

$$\{R\text{-comodules}\} \cong \{R^*\text{-modules}\}$$

In particular, we have:

$$\{G\text{-comodules}\} \cong \{M(G)\text{-modules}\}$$

It is clear that $\mathrm{Dist(G)} \subset M(G)$ give the same operation as we have given in 0.2.6 and the statement in 0.2.6 also works for $M(G)$.

The representation of G on $k[G]$ through ρ_l and ρ_r leads to two representations of G on $M(G)$, hence to two structures of $M(G)$-modules on $M(G)$. One can checks that any $\mu \in M(G)$ operates on $M(G)$ as left multiplication by μ when we deal with ρ_l, and as right multiplication with μ^{-1} when we deal with ρ_r.

In fact we have the following lemma describing $M(G)$ as a G-module:

Lemma 0.3.2. *If we regard $M(G)$ and $k[G]$ as G-module by the same action of G(e.g. left regular representation or right regular representation), then the G-modules $M(G)$ and $k[G]$ are isomorphic. In particular, we have* $\dim M(G)^G = 1$

Proof. See [J] 8.7. □

Now we call a projective object in the category of all G-modules simply a *projective* G-module. So we see they correspond under the equivalence of categories to the projective $M(G)$-modules. This shows that each G-module is a homomorphic image of a projective G-module. The representation theory of finite dimensional algebras shows that for each simple G-module E, there is a unique(up to isomorphism) projective G-module Q with $Q/\operatorname{rad}(Q) = E$. It is called the *projective cover* of E.

On the other hand, as we have $k[G] \cong M(G)$, and with proposition 0.1.1, we see that a finite dimensional G-module is projective if and only if it is injective. So there is a bijection $E \mapsto E'$ on the set of simple G-modules such that the injective hull Q_E is the projective cover of E', i.e.,

$$Q_E/\operatorname{rad}(Q_E) \cong E'.$$

This bijection will be described at the end of next subsection.

0.3.2 Invariant Measures

We call an element in $M(G)_l^G$ (resp. $M(G)_r^G$) a *left invariant measure* (resp. *right invariant measure*) on G. The description of left and right representations of $M(G)$ on itself in 0.3.1 implies:

$$M(G)_l^G = \{\mu_0 \in M(G) | \mu\mu_0 = \mu(1)\mu_0\} \text{ for all } \mu \in M(G)$$

and

$$M(G)_l^G = \{\mu_0 \in M(G) | \mu_0\mu = \mu(1)\mu_0\} \text{ for all } \mu \in M(G)$$

So $M(G)_l^G$ is stable under right multiplication by elements of $M(G)$, hence an $M(G)$- and G-submodule of $M(G)$ with respect to the right regular representation. As $\dim M(G)_l^G = 1$, this gives a character $\delta_G \in X(G) \subset k[G]$. So we have for $g \in G(A)$ and any A:

$$\rho_r(g)(\mu_0 \otimes 1) = \mu_0 \otimes \delta_G(g) \text{ for all } \mu_0 \in M(G)_l^G$$

This character δ_G is called the *modular function* of G. We call G *unimodular* if $\delta_G = 1$.

There is also a natural structure as a $k[G]$-module on $M(G)$: For any $f \in k[G]$ and $\mu \in M(G)$ we define $f\mu$ through

$$(f\mu)(f_1) = \mu(ff_1)$$

for all $f_1 \in k[G]$. We claim that for any $f \in k[G]$,$\mu \in M(G)$, and $g \in G(A)$:

$$\rho_l(g)(f\mu) = (\rho_l(g)f)(\rho_l(g)\mu)$$

Indeed we have:

$$\begin{aligned}\rho_l(g)(f\mu)(f') &= (f\mu) \circ \rho_l(g^{-1})(f') = \mu(ff' \circ \rho_l(g^{-1})) \\ &= \mu((\rho_l(g)f)(\rho_l(g)f')) = (\rho_l(g)f)(\rho_l\mu)(f')\end{aligned}$$

Now if M is a G-module, then we denote by M^l the $(G \times G)$-module that is equal to M as a vector space and where the first factor G operates as on M and the second factor operates trivially. Similarly M^r is defined. For $\lambda \in X(G)$ we shall usually write λ^l and λ^r instead of $(k_\lambda)^l$ and $(k_\lambda)^r$. We regard $k[G]$ and $M(G)$ as $(G \times G)$-modules with the first factor operating via ρ_l and the second on via ρ_r. And we have the following proposition describing $M(G)$ as a $k[G]$-module.

Proposition 0.3.3. *Let $\mu_0 \in M(G)_l^G$, $\mu_0 \neq 0$. Then $f \mapsto f\mu_0$ is an isomorphism of $k[G]$-modules and of $(G \times G)$-modules:*

$$k[G] \otimes (\delta_G)^r \cong M(G).$$

Proof. See [J] 8.12. □

Remark 0.1. It is obvious rom the definition and from left regular G-action described above shows that the map is a homomorphism of $k[G]$ and of $(G \times G)$-modules.

Furthermore, we have:

Proposition 0.3.4. *Let E be a simple G-module and Q a projective cover of E. Then:*

$$\operatorname{soc} Q \cong E \otimes \delta_G.$$

Proof. [J] 8.13. □

0.3.3 Coinduced Modules

Any closed subgroup H of finite algebraic group G is itself a finite algebraic k-group. We can identify $M(H)$ with the subalgebra $\{\mu \in M(G) \mid \mu(I(H)) = 0\}$ where $I(H)$ is the ideal corresponds to H.

Now we define a functor from $\{H\text{-modules }\}$ to $\{G\text{-modules}\}$ by

$$\operatorname{Coind}_H^G M = M(G) \otimes_{M(H)} M$$

for any H-module M. We call this functor the *coinduction* from H to G.

We have obviously:

Proposition 0.3.5. *The functor* Coind_H^G *is right exact.*

For any H-module the map $i_M : M \to \operatorname{Coind}_H^G M$ with $i_M(m) = 1 \otimes m$ is a homomorphism of H-modules. The universal property of the tensor product implies that for each G-module V we get an isomorphism:

$$\operatorname{Hom}_G(\operatorname{Coind}_H^G M, V) \cong \operatorname{Hom}_H(M, \operatorname{Res}_H^G V), \quad \phi \mapsto \phi \circ i_M.$$

So the functor Coind_H^G is left adjoint to Res_H^G.

Now we give the following proposition without proof about the relationship between induction functor and coinduction functor:

Proposition 0.3.6. *Let H be a closed subgroup of G. Then we have for each H-module M an isomorphism:*

$$\operatorname{Coind}_H^G \cong \operatorname{Ind}_H^G(M \otimes ((\delta_G)|_H \ \delta_H^{-1}))$$

Proof. [J] I 8.17. □

And we have the following dual proposition:

Proposition 0.3.7. *Let H be a closed subgroup of G and M a finite dimensional H-module, then:*

$$(\mathrm{Ind}_H^G M)^* = \mathrm{Ind}_H^G(M^* \otimes ((\delta_G)|_H\ \delta_H^{-1}))$$

Proof. [J] I 8.18. □

And we give the following proposition which is quite important for the study Frobenius kernels:

Proposition 0.3.8. *Let G' be a k-group scheme operating on G through group automorphisms. Then G' operates naturally on $k[G]$ and $M(G)$. The space $M(G)_l^G$ is a G'-submodule of $M(G)$ and the operation of G' on $M(G)_l^G$ is given by some $\chi \in X(G')$. If $\mu_0 \in M(G)_l^G$, $\mu \neq 0$, then the map $f \mapsto f\mu_0$ is an isomorphism $k[G] \otimes \chi \cong M(G)$ of G'-modules. If G is a closed normal subgroup of G' and if we take the action of G' by conjugation on G, then $\chi|_G = \delta_G$.*

Proof. We can form the semi-direct product $G \rtimes G'$ and make it operate on G such that G acts through left multiplication and G' as given. This yields representations of $G \rtimes G'$ on $k[G]$ and $M(G)$ that yield the operation considered in the proposition when restricted to G' and yield the left regular representation when restricted to G. Hence $M(G)_l^G$ are the fixed points of the normal subgroup G of $G \rtimes G'$, hence a G'-submodule.

It is now obvious that G' operates through some $\chi \in X(G')$ on $M(G)_l^G$ and that $f \mapsto f\mu_0$ is an isomorphism $k[G] \otimes \chi \cong M(G)$ of G'-modules. Suppose finally that G is a normal subgroup of G' and that we consider the conjugation action of G' on G. Then each $g \in G(A) \subset G'(A)$ acts through the composition of $\rho_l(g)$ and $\rho_r(g)$ on $M(G) \otimes A$, hence through $\rho_r(g)$ on $\mu_0 \otimes 1$. Therefore the definition shows $\chi(g) = \delta_G(g)$. □

Chapter 1

Simple Modules for Reductive Groups

Our main aim in this chapter is to prove corollary 1.2.7, which is the basis for our further study in the last two chapters. We will give a sketch introduction on reductive groups to clarify the notations and concepts at first. Then we go directly to construct $L(\lambda)$ which will be proved to be simple. We will prove that $L(\lambda)$ in fact determines all the simple modules of algebraic groups. At last, We will give the dual theory at the end of this chapter, which will be useful for our study in the last chapter, the Linkage Theorem.

1.1 Reductive Groups and Root Systems

We will give a quick tour of the main properties of reductive groups. It is not suitable for those people who are not familiar with them. For detailed study, the reader may refer to those famous books, like [B].

1.1.1 Reductive Groups

Here we assume k to be field. Now we assume $G_{\mathbb{Z}}$ to be a split and reductive algebraic $\mathbb{Z}$-group. Set $G_A = (G_{\mathbb{Z}})_A$ for any ring A and $G = G_k$.

Then G_K is for any algebraically closed field K a reduced K-group, and it is a connected and reductive K-group. The ring $\mathbb{Z}[G_Z]$ is free, so $k[G]$ is free and hence G is flat.

Let $T_{\mathbb{Z}}$ be a *split maximal torus* of $G_{\mathbb{Z}}$. Set $T_A = T_Z \times \mathrm{Spec}(A)$ for any ring A and $T = T_k$. Then T_Z is isomorphic to a direct product of, say, r copies of the multiplicative group over $\mathbb{Z}$. The integer r is uniquely determined and is called the *rank* of G.

For any algebraically closed field K, the group T_K is reduced and it is a maximal torus in G_K. The k-group T is isomorphic to ${\mathbf{G}_{\mathrm{m}}}^r$ and $X(T) = X(T_{\mathbb{Z}}) = \mathbb{Z}^r$. Any T-module M has a direct decomposition into weight spaces:

$$M = \bigoplus_{\lambda \in X(T)} M_\lambda.$$

Here the set of all λ with $M_\lambda \neq 0$ is called the weight space of M. and we can define its *formal character*:

$$\operatorname{ch} M = \sum_{\lambda \in X(T)} \operatorname{rk}(M_\lambda) e(\lambda). \tag{1.1}$$

Apply the argument above to the adjoint representation on $\operatorname{Lie}(G)$, then the decomposition has the form:

$$\operatorname{Lie} G = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in R} (\operatorname{Lie} G)_\alpha \tag{1.2}$$

Here R is the set of non-zero weights of $\operatorname{Lie} G$. So (1.2) amounts to

$$(\operatorname{Lie} G)_0 = \operatorname{Lie} T$$

The elements of R are called the *roots* of G with respect to T, and the set R is called *root system* of G with respect to T. For any $\alpha \in R$ the root subspace $(\operatorname{Lie}(G))_\alpha$ is a free k-module of rank 1. Let $\rho = \frac{1}{2} \sum_{\alpha \in R} \alpha$.

1.1.2 Root Systems

A subset R of an euclidean space E (with inner product $(\ ,\ )$) is called a *root system* in E if the following axioms are satisfied:

1. R is finite, spans E, and does not contain 0.
2. If $\alpha \in R$, the only scaler multiples of α in R are $\pm\alpha$.
3. If $\alpha \in R$, the *reflection* s_α defined by $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ leaves R invariant.
4. If α, $\beta \in R$, then $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

It is a fact that R, the set of the roots of G, contained in $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ forms a root system of G which coincide with the root system given by its semi-simple Lie algebra. In fact we can define:

$$Y(T) = \operatorname{Hom}(\mathbf{G}_{\mathrm{m}}, T)$$

which has a natural structure of an abelian group.Then for any $\lambda \in X(T)$ and any $\phi \in Y(T)$, we have $\lambda \circ \phi \in \operatorname{End}(\mathbf{G}_{\mathrm{m}}) \cong \mathbb{Z}$, which gives the pairing $\langle\ ,\ \rangle$ and induces the isomorphism $Y(T) = \operatorname{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$. It is also a fact that the root system contains a *base* S, which is defined by:

1. S is a basis of E.
2. Each root $\beta \in R$ can be written as $\beta = \sum k_\alpha \alpha$ with $\alpha \in S$ and $k_\alpha \in \mathbb{N}$.

The root in S is called *simple root*. If we choose a *positive system* $R^+ \subset R$, then it gives a relationship in R, and it also gives a unique base denoted by S.

Here we define the *Weyl group* of R to be $W = \langle s_\alpha \mid \alpha \in R \rangle$. Note any g contained in $N_G(T)(A)$, the normalizer of $T(A)$ acts through conjugation on $T(A)$, hence also linearly on $X(T(A))$. Here we assume A is integral, and hence $X(T(A)) = X(T)$. And we have $W \cong (N_G(T)/T)(A)) \cong N_G(T)(A)/T(A)$.

On the other hand, we see that the Weyl group is generated by the *simple reflections* with respect to the positive system R^+, i.e., by all s_α with $\alpha \in S$. So we can define the *length* $l(w)$ of any $w \in W$ to be the smallest m such that there exists $\beta_1, \beta_2, \ldots, \beta_m \in S$ with $w = s_{\beta_1} s_{\beta_2} \ldots s_{\beta_m}$. So $l(w) = 0$ if and only if $w = 1$ and $l(w) = 1$ if and only if $w = s_\alpha$ with $\alpha \in S$.

1.1.3 Regular Subgroups

For each $\alpha \in R$ there is a *root homomorphism*:

$$x_\alpha : \quad \mathbf{G}_\mathrm{a} \to G$$

with

$$t x_\alpha t^{-1} = x_a(\alpha(t)a)$$

for any k algebra A and all $t \in T(A)$, $a \in A$, such that the tangent map dx_α induces an isomorphism:

$$dx_\alpha : \quad \operatorname{Lie} G_\alpha \cong (\operatorname{Lie} G)_\alpha.$$

Such a root homomorphism is uniquely determined up to a unit in k.

The functor $A \mapsto x_\alpha(\mathbf{G}_\mathrm{a}(A))$ is a closed subgroup of G denoted by U_α. It is called the *root subgroup* of G corresponding to α. So x_α is an isomorphism $\mathbf{G}_\mathrm{a} \cong U_\alpha$ and we have:

$$\operatorname{Lie} U_\alpha = (\operatorname{Lie} G)_\alpha.$$

For any $\alpha \in R$ there is a another homomorphism:

$$\phi_\alpha : \mathbf{SL}_2 \to G$$

such that for a suitable normalizaiton of x_α and $x_{-\alpha}$:

$$\phi_\alpha \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = x_\alpha(a) \text{ and } \phi_\alpha \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = x_{-\alpha}(a).$$

For any A and $a \in A$. we have

$$n_\alpha(a) = x_\alpha(a) x_{-\alpha}(-a^{-1}) x_\alpha(a) = \phi_\alpha \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \in N_G(T)(A)$$

and

$$\alpha^\vee(a) = n_\alpha(a)n_\alpha(1)^{-1} = \phi_\alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in T(A)$$

for any $a \in A^\times$ and any A. Obviously $\alpha^\vee \in Y(T)$. It is the *coroot* or *dual root* corresponding to α.

A subset $R' \subset R$ is called closed if $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap R \subset R'$ for any $\alpha, \beta \in R'$. It is called *unipotent* (resp. *symmetric*) if $R' \cap (-R') = \varnothing$(resp. $R' = -R'$).

For any $R' \subset R$ unipotent and closed we denote by $U(R')$ the closed subgroup generated by all U_α with $\alpha \in R'$. In fact, we have an isomorphism of schemes(but not of group schemes):

$$\prod_{\alpha \in R'} U_\alpha \cong U(R')$$

And obviously:

$$\operatorname{Lie} U(R') = \bigoplus_{\alpha \in R'} (\operatorname{Lie} G)_\alpha$$

Each $U(R')$ is connected and unipotent. It is isomorphic to $\mathbb{A}^n$ with $n = |R'|$ as a scheme. It is normalized by T.

If $R' \subset R$ is symmetric and closed, then let $G(R')$ be the closed subgroup of G generated by T and by all U_α with $\alpha \in R'$. Then

$$\operatorname{Lie} G(R') = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in R'} (\operatorname{Lie} G)_\alpha \tag{1.3}$$

The k-group $G(R')$ is split, reductive, and connected. It contains T as a maximal torus. Its root system is exactly R'.

We can take in particular some $I \subset S$ and set $R_I = R \cap \mathbb{Z}I$. Then R_I is closed and symmetric. Set $L_I = G(R_I)$. Then L_I is split and reductive with Weyl group isomorphic to $W_I = \langle s_\alpha | \alpha \in I \rangle$.

1.1.4 Bruhat Decomposition

Both R^+ and $-R^+$ are unipotent and closed subsets of R. We set $U^+ = U(R^+)$ and $U = U(-R^+)$. Then $B^+ = TU^+ = T \ltimes U^+$ and $B = TU = T \ltimes U$ are *Borel subgroups* with $B \cap B^+ = T$. Note that B corresponds to the negative roots.

Let us choose for $w \in W$ a representative $\dot{w} \in N_G(T)(k)$, then we have $\dot{w}U_\alpha\dot{w}^{-1} = U_{w(\alpha)}$. As W permutes the positive systems simply transitively, there is a unique $w_0 \in W$ with $w_0(R^+) = -R^+$. Let $\dot{w_0} \in N_G(T)$ be a representative of w_0, then $\dot{w_0}U\dot{w_0}^{-1} = U^+$ and $\dot{w_0}B\dot{w_0}^{-1} = B^+$.

For any $I \subset S$ the subsets $R^+ - R_I$ and $(-R^+) - R_I$ of R are closed and unipotent, hence $U_I^+ = U(R^+ - R_I)$ and $U_I = U((-R^+) - R_I)$ are closed subgroups of G. In fact we have that L_I normalizes U_I^+ and U_I. One has $U_I^+ \cap L_I = 1 = U_I \cap L_I$, so we get semi-direct product inside G: $P_I = L_I U_I = L_I \ltimes U_I$ and $P_I^+ = L_I U_I^+ = L_I \ltimes U_I^+$.

The P_I (resp. P_I^+) with $I \subset S$ are called *the standard parabolic subgroups* containing B (resp. B^+), and L_I is called the standard *Levi factor* of P_I (and of P_I^+) containing T. Furthermore, U_I (resp. U_I^+) is the *unipotent radical* of P_I (resp. P_I^+).

For any k-algebra k' that is a field, $G(k')$ is decomposed as the disjoint union:

$$G(k') = \bigcup_{w \in W} B(k')\dot{w}B(k').$$

This is called the *Bruhat Decomposition* of $G(k')$. In fact, the Bruhat decomposition implies that the $\dot{w}U^+B$ with $w \in W$ form an open covering of G. In particular

$$\prod_{\alpha \in R^+} U_\alpha \times T \times \prod_{\alpha \in R^+} U_{-\alpha} \cong U^+B$$

is an open set which covers the unit of G.

And we also have:

Proposition 1.1.1. *Any regular function on $\bigcup_{\alpha \in S} \dot{s}_\alpha U^+(k)B(k) \cup U^+(k)B(k)$ can be extended to the whole $G(k)$.*

Proof. See [J] II 1.19 (8). □

1.1.5 The Algebra of Distributions

Set for any $\alpha \in R$

$$X_\alpha = (dx_\alpha)(1) \in (\operatorname{Lie} G_{\mathbb{Z}})_\alpha,$$

where we regard x_α as a homomorphism $G_{a,\mathbb{Z}} \to G_{\mathbb{Z}}$. Choose a basis $\phi_1, \cdots, \phi_r$ of $Y(T_{\mathbb{Z}}) = T(T)$ and set for each i

$$H_i = (d\phi)_i(1) \in \operatorname{Lie}(T_{\mathbb{Z}})$$

As we have seen, the origin of G is contained in the open subscheme U^+TU which is a product of copies of $\mathbf{G}_{\mathrm{m}}$ and $\mathbf{G}_{\mathrm{a}}$. So we have that $(H_i)_{1 \le i \le r}$ is a basis of $\operatorname{Lie}(T_{\mathbb{Z}})$ and $(H_i)_{1 \le i \le r}$, $(X_\alpha)_{\alpha \in R}$ is a basis of $\operatorname{Lie}(G_{\mathbb{Z}})$. By section 0.2, we have:

1. G is infinitesimal
2. The multiplication induces an isomorphism of k-modules

$$\otimes_{\alpha \in R^+} \operatorname{Dist}(U_\alpha) \otimes \operatorname{Dist}(T) \otimes_{\alpha \in R^+} \operatorname{Dist}(U_{-\alpha}) \cong \operatorname{Dist}(G) \tag{1.4}$$

1.1.6 G-Modules

Any T-module M has a weight space decomposition $M = \bigoplus_{\lambda \in X(T)} M_\lambda$. The operation of $\operatorname{Dist}(T)$ on M can be described as follows: Any $H_{i,m}$ acts on M_λ as multiplication by $\binom{\langle \lambda, \phi_i \rangle}{m}$ by the definition of $H_{i,m}$ and the action of $\operatorname{Dist}(\mathbf{G}_{\mathrm{m}})$.

If M is a G-module and if $\dot{w} \in N_G(T)(k)$ is a representative of some $w \in W$, then an elementary calculation shows:

$$\dot{w}M_\lambda = M_{w(\lambda)} \tag{1.5}$$

for all $\lambda \in X(T)$. So $\operatorname{rk} M_\lambda = \operatorname{rk} M_{w(\lambda)}$.

We can regard $\operatorname{Dist}(G)$ as a G-module under the adjoint action, hence also a T-module. Then we have for all $\alpha \in R$ and all $n \in N$:

$$X_{\alpha,n} \in \operatorname{Dist}(G)_{n,\alpha} \text{ and } \operatorname{Dist}(T) \subset \operatorname{Dist(G)}_0 \tag{1.6}$$

Indeed, we can identify $k[U_\alpha]$ with a polynomial ring in one variable Y_α, which is a weight vector of weight $-\alpha$ for the adjoint action of T as $kY_\alpha \cong (\operatorname{Lie} U_\alpha)^*$. Then Y_α^n has weight $-n\alpha$ and the "dual" vector $X_{\alpha,n}$ weight $n\alpha$.

If M is a TU_α-module for some $\alpha \in R$, then 1.6 implies for all λ and n:

$$X_{\alpha,n} M_\lambda \subset M_{n\alpha+\lambda}. \tag{1.7}$$

Let M be a B^+-module. Suppose $\lambda \in X(T)$ is maximal among all weights of M (with respect to some R^+). So $\lambda + n\alpha$ is not a weight of M for all $\alpha \in R^+$ and $n > 0$, hence $X_{\alpha,n} M_\lambda = 0$. Note

$$x_\alpha(a)(m \otimes 1) = \Delta_M(m \otimes 1)|_{T=a} = \sum_{n \geq 0} (m_n \otimes T^n)|_{T=a} = \sum_{n \geq 0} (X_{\alpha,n} m) \otimes a^n. \tag{1.8}$$

So finally we have:

Proposition 1.1.2. *If λ is maximal among the weights of M, then $M_\lambda \subset M^{U^+}$.*

1.1.7 The Case $\mathbf{GL}_n$

The general linear groups are the simplest examples of reductive groups. Fix $n \in \mathbb{N}$, $n \geq 2$ and take $G = \mathbf{GL}_n$. The conventions and notations introduced below will be used whenever we look at this example.

For all i, j $(1 \leq i, j \leq n)$ let E_{ij} be the $(n \times n)$-matrix over k with (i,j)-coefficient equal to 1 and all other coefficients equal to 0. The E_{ij} form a basis of $M_n(k)$. Let us denote the dual basis of $M_n(k)^*$ by X_{ij} $(1 \leq i, j \leq n)$. So the X_{ij} are the matrix coefficients on $M_n(k)$ and $k[G]$ is generated by the X_{ij} an by $\det(X_{ij})^{-1}$.

We choose $T \subset \mathbf{GL}_n$ as the subfunctor such that $T(A)$ consists of all diagonal matrices in $\mathbf{GL}_n(A)$ for all A, i.e., $T = V(\{X_{ij} | i \neq j\})$. Then T is isomorphic to a direct product of n-copies of $\mathbf{G}_\mathrm{m}$. Then $\epsilon_i = X_i|_T (1 \leq i \leq n)$ forms a basis of $X(T)$, and the $\epsilon_i'(1 \leq i \leq n)$ with $\epsilon_i'(a) = \sum_{j \neq i} E_{jj} + aE_{ii}$ form a basis of $Y(T)$. One has $\langle \epsilon_i, \epsilon_j' \rangle = \delta_{ij}$.

The root system has the form:

$$R = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n, i \neq j\}.$$

For Lie algebra, we have $(\operatorname{Lie} G)_{\epsilon_i - \epsilon_j} = kE_{ij}$ (for $i \neq j$) and $(\operatorname{Lie} G)_0 = \sum_{i=1}^n kE_{ii}$. We take (for $i \neq j$):

$$x_{\epsilon_i - \epsilon_j}(a) = 1 + aE_{ij}$$

and

$$\phi_{\epsilon_i-\epsilon_j}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{h\neq i,j} E_{hh} + aE_{ii} + bE_{ij} + cE_{ji} + dE_{jj}.$$

One has $(\epsilon_i - \epsilon_j)^\vee = \epsilon'_i - \epsilon'_j$ for all i, j. The Weyl group permutes $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ and can be identified with the symmetric group S_n: Map any $w \in W$ to the permutation σ with $w(\epsilon_i) = \epsilon_{\sigma(i)}$ for all i. Then $s_{\epsilon_i-\epsilon_j}$ is mapped to the transposition (i, j). The composition of this isomorphism with the canonical map $N_G(T) \to W$ admits a section: Map any σ to the permutation matrix $\sum_{i=1}^n E_{\sigma(i),i}$. The element w_0 corresponds to the permutation σ_0 with $\sigma_0(i) = n+1-i$ for all i. We choose as system of positive roots:

$$R^+ = \{\epsilon_i - \epsilon_j \mid i \leq i < j \leq n\}. \tag{1.9}$$

Then

$$S = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n\}. \tag{1.10}$$

The centre of G is isomorphic to $\mathbf{G}_\mathrm{m}$:

$$\mathbf{G}_\mathrm{m} \cong Z(G),\ a \mapsto a\sum_{i=1}^n E_{ii}$$

The Borel subgroup B (resp. B^+) is the functor associating to each A the group of lower (resp. upper) triangular matrices in $\mathbf{GL}_n(A)$. Furthermore, $U(A)$ (resp. $U^+(A)$) consists of all matrices in $B(A)$ (resp. $B^+(A)$) such that all diagonal entries are equal to 1.

Let us identify $\mathbf{GL}_n$ and $\mathbf{GL}(k^n)$ via the canonical basis $e_1, e_2, \ldots, e_n$ of k^n. Set $V_i = \langle e_n, e_{n-1}, \ldots, e_{n+1-i}\rangle$ for $1 \leq i \leq n$. Then B is the stabilizer of the flag $(V_1 \subset V_2 \subset \cdots \subset V_{n-1})$, i.e., $B = \cap_{i=1}^{n-1}\mathrm{Stab}_G(V_i)$. The stabilizer of any partial flag $(V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_r})$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r$ is the parabolic subgroup P_I with $I = \{\alpha_i \mid 1 \leq i < n,\ i \neq n - i_h \text{ for } 1 \leq h \leq r\}$. One has in particular

$$\mathrm{Stab}_G(V_i) = P_{S-\alpha_{n-i}}.$$

1.2 Simple G-Modules

The contents of this section may be divided into three parts, the first three subsections give the general theory for simple modules, which will play an important role both in chapter 2 and chapter 3. The dual theory have corresponding theories for Frobenius kernels in chapter 2, but it will be only used in chapter 3 for the study of higher cohomology groups. And at last we will give an example for simple modules.

1.2.1 Simple Modules for Borel Groups

Before talking about simple modules for reductive groups, we look into simple modules for Borel groups, or more generally, TU-modules at a special case at first.

Suppose $G'' = H \ltimes G'$ with H a diagonalizable and G' a unipotent group scheme. We set for each $\lambda \in X(H)$:

$$Y_\lambda = \operatorname{Ind}_H^{G''} k_\lambda.$$

We have $k[G''] \cong \operatorname{Ind}_1^{G''} k \cong \operatorname{Ind}_H^{G''} \operatorname{Ind}_1^H k \cong \operatorname{Ind}_H^{G''} k[H]$ by transitivity of induction. As H is diagonalizable, we have $k[H] = \oplus_{\lambda \in X(H)} k_\lambda$, hence:

$$k[G''] = \oplus_{\lambda \in X(H)} Y_\lambda.$$

As Y_λ is isomorphic to $k[G']$ as a G'-module (by ρ_l for $k[G']$). Note the only simple module of the unipotent group is k, hence $\operatorname{soc}'_G k[G'] = k[G']^{G'} = k$, i.e.

Proposition 1.2.1. *Y_λ is indecomposable and injective G-module.*

Each $\lambda \in X(H)$ can be extended to an element of $X(G'')$ having G' as kernel. We also denote this extension by λ and the corresponding G''-module, by k_λ. For each G''-module M the subspace $M^{G'}$ is a H-submodule. Because H is diagonalizable, it is a direct sum of one dimensional H (note also a G''-)-submodules of the form k_λ with $\lambda \in X(H)$. This shows in particular that $M^{G'}$ is a semi-simple G''-module. As G' is unipotent, $M^{G'} \neq 0$ for any simple G''-module. There for k_λ with $\lambda \in X(H)$ are all simple G''-modules and we have:

$$\operatorname{soc}_{G''} M = M^{G'}.$$

For any G''-module M. And we have

$$Y_\lambda = \operatorname{Ind}_H^{G''} k_\lambda = \operatorname{Ind}_H^{H \ltimes G'} k_\lambda = \operatorname{Ind}_1^{G'} k_\lambda = k[G'] \otimes k_\lambda.$$

.

1.2.2 The Simple Module $L(\lambda)$

As in the first section, we assume that k is a field and G is a reductive group coming from reductive group over $\mathbb{Z}$. Note U and U^+ are unipotent, so we have for any G-module $V \neq 0$:

$$V^U \neq 0 \text{ and } V^{U^+} \neq 0. \tag{1.11}$$

As T normalizes U and U^+, these two subspaces are T-submodules of V, hence direct sums of their weight spaces. For any $\lambda \in X(T)$ the λ-weight space of V^U is the sum of all simple B-submodules in V isomorphic to k_λ (similarly for U^+ and B^+). We shall write λ instead of k_λ whenever no confusion is possible. So we can also express 1.11 as follows:

There are $\lambda, \lambda' \in X(T)$ with $\operatorname{Hom}_B(\lambda, V) \neq 0 \neq \operatorname{Hom}_{B^+}(\lambda', V)$.

If $\dim V \leq \infty$, then there are $\lambda, \lambda' \in X(T)$ with:

$$\operatorname{Hom}_B(V, \lambda) \neq 0 \neq \operatorname{Hom}_{B^+}(V, \lambda').$$

Using Frobenius reciprocity (0.1), this implies

$$\operatorname{Hom}_G(V, \operatorname{Ind}_B^G \lambda) \neq 0 \neq \operatorname{Hom}_G(V, \operatorname{Ind}_{B^+}^G \lambda') \tag{1.12}$$

Now we use the notation:

$$H^i(M) = (R^i \operatorname{Ind}_B^G)(M)$$

for any B-module M and any $i \in \mathbb{N}$. And we shall write $H^i(\lambda) = H^i(k_\lambda)$.

Proposition 1.2.2. *Let $\lambda \in X(T)$ with $H^0(\lambda) \neq 0$.*

1. *We have $\dim H^0(\lambda)^{U^+} = 1$ and $H^0(\lambda)^{U^+} = H^0(\lambda)_\lambda$*
2. *Each weight μ of $H^0(\lambda)$ satisfies $w_0\lambda \leq \mu \leq \lambda$.*

Proof. Recall that

$$H^0(\lambda) = \{f \in k[G] \mid f(gb) = \lambda(b)^{-1} f(g) \text{ for all } g \in G(A),\ b \in B(A) \text{ and all } A\}.$$

The operation of G is given by left translation. So any $f \in H^0(\lambda)^{U^+}$ satisfies

$$f(u_1 t u_2) = \lambda(t)^{-1} f(1)$$

for $u_1 \in U^+(A)$, $t \in T(A)$, $u_2 \in U(A)$ and for all A. Thus $f(1)$ determines the restriction of f to U^+B, hence also f, as U^+B is dense in G. This implies $\dim H^0(\lambda)^{U^+} \leq 1$. The equality follows from 1.11 and our assumption $H^0(\lambda) \neq 0$.

Furthermore, the evaluation map $\epsilon : H^0(\lambda) \to \lambda$, $f \mapsto f(1)$ is a homomorphism of B-modules and is injective on $H^0(\lambda)^{U^+}$. This implies

$$H^0(\lambda) \subset H^0(\lambda)_\lambda.$$

Assume μ to be a maximal weight of $H^0(\lambda)$, then $H^0(\lambda)_\mu \subset H^0(\lambda)^{U^+}$ by proposition 1.1.2. So finally we have $H^0(\lambda)_\lambda = H^0(\lambda)^{U^+}$ and $\mu \leq \lambda$ for each weight μ of $H^0(\lambda)$.

If μ is a weight of $H^0(\lambda)$, then so is $w_0\mu$ by 1.5, and hence $w_0\mu \leq \lambda$ and $w_0\lambda \leq \mu$. □

Corollary 1.2.3. *If $H^0(\lambda) \neq 0$, then $\operatorname{soc}_G H^0(\lambda)$ is simple.*

Proof. If L_1, L_2 are two different simple submodules of $H^0(\lambda)$, then $L_1 \oplus L_2 \subset H^0(\lambda)$, hence $L_1^{U^+} \oplus L_2^{U^+} \subset H^0(\lambda)^{U^+}$ and $\dim H^0(\lambda)^{U^+} \geq 2$ and leads to a contradiction. Therefore $\operatorname{soc}_G H^0(\lambda)$ has to be simple. □

Now we set

$$L(\lambda) = \operatorname{soc}_G H^0(\lambda)$$

for any $\lambda \in X(T)$ with $H^0(\lambda) \neq 0$.

Proposition 1.2.4.

1. *Any simple G-module is isomorphic to exactly one $L(\lambda)$ with $\lambda \in X(T)$ and $H^0(\lambda) \neq 0$.*
2. *Let $\lambda \in X(T)$ with $H^0(\lambda) \neq 0$. Then $L(\lambda)^{U^+} = L(\lambda)_\lambda$ and $\dim L(\lambda)^{U^+} = 1$. Any weight μ of $L(\lambda)$ satisfies $w_0\lambda \leq \mu \leq \lambda$. The multiplicity of $L(\lambda)$ as a composition factor of $H^0(\lambda)$ is equal to one.*

Proof. For 2, As $L(\lambda)^{U^+} \neq 0$ the formulas $L(\lambda)^{U^+} = L(\lambda)_\lambda$ and $\dim L(\lambda)^{U^+} = 1$ follow immediately form proposition 1.2.2. The same is true for $w_0\lambda \leq \mu \leq \lambda$. Finally, the multiplicity of $L(\lambda)$ in $H^0(\lambda)$ is at least one by construction, but cannot be strictly larger as $\dim L(\lambda)_\lambda = 1 = \dim H^0(\lambda)_\lambda$ and as $V \to V_\lambda$ is an exact functor.

For 1, the existence follows from 1.12, the uniqueness from the formula $L(\lambda)^{U^+} = L(\lambda)_\lambda$ in 2.

□

***Remark* 1.1.**

1. This proposition shows that λ is the largest weight of $L(\lambda)$ with respect to $\leq$. It is custom to call it the *highest weight* of $L(\lambda)$ and to call $L(\lambda)$ the simple G-module with highest weight λ.

2. Using $\dot{w}_0 U \dot{w}_0^{-1} = U^+$ we see:

$$\dim L(\lambda)^U = 1 \text{ and } L(\lambda)^U = L(\lambda)_{w_0\lambda}$$

Corollary 1.2.5. *Let $\lambda \in X(T)$ with $H^0(\lambda) \neq 0$. The module dual to $L(\lambda)$ is $L(-w_0\lambda)$.*

Proof. For any finite dimensional T-module V the μ-weight space of V^* is naturally identified with $(V_{-\mu})^*$. So the weights of V^* are exactly the $-\mu$ with μ a weight of V.

In the case of $L(\lambda)$ this implies that the weights μ of $L(\lambda)^*$ satisfy $w_0\lambda \leq -\mu\lambda$, hence $-\lambda \leq \mu - w_0\lambda$, and that $-w_0\lambda$ will occur. As $L(\lambda)^*$ is simple, it has to be isomorphic to $L(-w_0\lambda)$. □

1.2.3 Determination of Simple G-Modules

Set

$$\begin{aligned} X(T)_+ =& \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in S\} \\ =& \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\} \end{aligned} \tag{1.13}$$

The elements of $X(T)_+$ are called the *dominant weights* of T(with respect to R^+).

Proposition 1.2.6. *Let $\lambda \in X(T)$. The following are equivalent:*

1. *λ is dominant*
2. *$H^0(\lambda) \neq 0$*
3. *There is a G-module V with $(V^{U^+})_\lambda \neq 0$*

Proof. (3) $\Rightarrow$ (2) Using a composition series of a suitable finite dimensional submodule of V we reduce to the case V is a simple. Now (2) follows from 1.2.4.

(2) $\Rightarrow$ (1) Suppose $H^0(\lambda) \neq 0$. So $s_\alpha\lambda$ with $\alpha \in R^+$ is a weight of $H^0(\lambda)$, hence $s_\alpha\lambda \leq \lambda$. Now $s_\alpha\lambda = \lambda - \langle \lambda, \alpha^\vee \rangle\alpha$, so $\langle \lambda, \alpha^\vee \rangle\alpha \geq 0$, hence $\langle \lambda, \alpha^\vee \rangle \geq 0$.

(1) ⇒ (2) We may assume k is algebraically closed. We can regard $G(k)$ as a variety over k and have to find a regular function $f : G(k) \to k$, $f \neq 0$ such that $f(gtu) = \lambda(t)^{-1}f(g)$ for all $g \in G(k)$, $t \in T(k)$, $u \in U(k)$

Consider the function f_λ on the open subvariety $U^+(k)T(k)U(k) \subset G(k)$ given by $f_\lambda(u_1tu_2) = \lambda(t)^{-1}$ for all $u_1 \in U^+(k)$, $t \in T(k)$, $u_2 \in U(k)$. Obviously $f_\lambda \neq 0$. As the restriction map $k[G] \to k[U^+TU]$ is injective, We just have to show $f_\lambda \in k[G]$. Then automatically $f_\lambda \in H^0(\lambda)$ and $H^0(\lambda) \neq 0$.

We want to show that f_λ can be extended to a regular function on each $\dot{s}_\alpha U^+(k)B(k) \cup U^+(k)B(k)$. Then it can be extended also to $Y = U^+(k)B(k) \cup \bigcup_{\alpha\in S} \dot{s}_\alpha U^+(k)B(k)$, as for all $\alpha, \beta \in S$ the extensions coincide on $\dot{s}_\alpha U^+(k)B(k) \cap \dot{s}_\beta U^+(k)B(k) \cup U^+(k)B(k)$ which is dense in $G(k)$, hence also on $\dot{s}_\alpha U^+(k)B(k) \cap \dot{s}_\beta U^+(k)B(k)$.If f_λ extends to Y, then it can extends to the whole $G(k)$ by proposition 1.1.1.

Let us consider now a fixed simple root $\alpha \in S$. Set $U_1^+ = \langle U_\beta | \beta \rangle 0, \beta \neq \alpha\rangle$. Then $U^+ = U_1^+ U_\alpha = U_1^+ \times U_\alpha$ and $\dot{s}_\alpha$ normalizes U_1^+. So

$$\dot{s}_\alpha U^+(k)B(k) = U_1^+(k)\dot{s}_\alpha U_\alpha(k)B(k).$$

The group $U_\alpha(k)$ consists of all $x_\alpha(a)$ with $a \in k$. The map $(u_1, a, t, u) \mapsto u_1\dot{s}_\alpha x_\alpha(a)tu$ is an isomorphism of varieties $U_1^+(k) \times k \times T(k) \times U(k) \to \dot{s}_\alpha U^+(k)B(k)$. By computation we have:

$$\dot{s}_\alpha x_\alpha(a) = x_\alpha(-a^{-1})\alpha^\vee(-a^{-1})x_{-\alpha}(a^{-1})$$

for all $a \neq 0$. Then

$$u_1\dot{s}_\alpha x_\alpha(a)tu = u_1 x_\alpha(-a^{-1})\alpha^\vee(-a^{-1})t x_{-\alpha}(\alpha(t)a^{-1})u \in U^+(k)B(k)$$

and

$$f_\lambda(u_1\dot{s}_\alpha x_\alpha(a)tu) = \lambda(t)^{-1}\lambda(\alpha^\vee(-a^{-1}))^{-1} = \lambda(t)^{-1}(-a)^{\langle\lambda,\alpha^\vee\rangle}.$$

As $\langle\lambda, \alpha^\vee\rangle \geq 0$ this function on $U_1^+(k) \times (k - \{0\}) \times T(k) \times U(k)$ can be uniquely extended to a regular function on $U_1^+(k) \times k \times T(k) \times B(k)$ hence f_λ can be extended to $\dot{s}_\alpha U^+(k)B(k)$ as regular function.

(2) ⇒ (3) This is obvious. □

Corollary 1.2.7. *The $L(\lambda)$ with $\lambda \in X(T)_+$ are a system of representatives for the isomorphism classes of all simple G-modules.*

Proposition 1.2.8. *Let $\lambda \in X(T)_+$. Then*

$$\mathrm{End}_G H^0(\lambda) = k = \mathrm{End}_G(L(\lambda))$$

Proof. Using Frobenius reciprocity, we have:

$$\begin{aligned} \mathrm{End}_G H^0(\lambda) &\cong \mathrm{Hom}_B(H^0(\lambda), \lambda) \subset \mathrm{Hom}_T(H^0(\lambda), \lambda) \\ &\cong \mathrm{Hom}(H^0(\lambda)_\lambda, \lambda) \cong k \end{aligned} \tag{1.14}$$

On the other hand, the identity map is a non-zero element in $\mathrm{End}_G H^0(\lambda)$, so $\mathrm{End}_G H^0(\lambda) = k$.

Similarly,

$$\begin{aligned}\operatorname{End}_G L(\lambda) &\subset \operatorname{Hom}_G(L(\lambda), H^0(\lambda)) \cong \operatorname{Hom}_B(L(\lambda), \lambda) \\ &\subset \operatorname{Hom}_T(L(\lambda), \lambda) \cong \operatorname{Hom}(L(\lambda)_\lambda, \lambda) = k.\end{aligned} \tag{1.15}$$

As $\operatorname{End}_G L(\lambda) \neq 0$, we have $\operatorname{End}_G L(\lambda) = k$. □

So we see that the center $Z(G)$ of G acts on $H^0(\lambda)$ through scalars. As $Z(G) \subset T$ and as $H^0(\lambda) \neq 0$, this scalar has to be the restriction of λ to $Z(G)$. More generally, we have:

Proposition 1.2.9. *For each $\lambda \in X(T)$ the group $Z(G)$ acts on each $H^0(\lambda)$through the restriction of λ to $Z(G)$.*

1.2.4 Dual Theory for Simple Modules

First, we introduce an important automorphism for reductive groups G:

Proposition 1.2.10. *There is an anti-automorphism τ of G with $\tau^2 = id_G$ and $\tau|_T = id_T$ and $\tau(U_\alpha) = U_{-\alpha}$ for all $\alpha \in R$.*

So for a G-module M, we define the G-module ${}^\tau M$ through the following ways: Take ${}^\tau M = M^*$ as a vector space, but define the action of G on $\phi \in M^*$ via $g\phi = \phi \circ \tau(g)$. So it is easy to see $\operatorname{ch} M = \operatorname{ch} {}^\tau M$ and ${}^\tau({}^\tau M) = M$ as $\tau^2 = 1$. If M is simple, then is ${}^\tau M$. As they share the same maximal weight, we have:

Proposition 1.2.11. ${}^\tau L(\lambda) = L(\lambda)$ *for all $\lambda \in X(T)_+$.*

Note an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of finite dimensional G-modules gives the exact sequence: $0 \to {}^\tau M_2 \to {}^\tau M \to {}^\tau M_1 \to 0$

By the knowledge of cohomology theory, we have:

$$\operatorname{Ext}^1_G(M_2, M_1) = \operatorname{Ext}^1_G({}^\tau M_1, {}^\tau M_2)$$

Applying this to simple modules, we have:

Proposition 1.2.12. $\operatorname{Ext}^1_G(L(\lambda), L(\mu)) = \operatorname{Ext}^1_G(L(\mu), L(\lambda))$

By proposition 0.1.3, we have $\dim H^0(\lambda) > 0$. So we can define for each $\lambda \in X(T)_+$ a G-module by:

$$V(\lambda) = H^0(-w_0\lambda)^* \tag{1.16}$$

The automorphism σ of G defined by $\sigma(g) = \tau(\dot{w}_0 g^{-1} \dot{w}_0^{-1})$ stabilizes B and induces $-w_0$ on $X(T)$. So ${}^\sigma H^0(\lambda) \cong H^0(-w_0\lambda)$. Twisting a module with the conjugation of $(\dot{w}_0)$ produces an isomorphic module. Hence we also get:

$$V(\lambda) \cong {}^\tau H^0(\lambda)$$

Lemma 1.2.13. *Let $\lambda \in X(T)_+$.*

1. *There are for each G-module M functorial isomorphisms:*

$$\mathrm{Hom}_G(V(\lambda), M) \cong \mathrm{Hom}_{B^+}(\lambda, M) = (M^{U^+})_\lambda$$

2. *The G-module $V(\lambda)$ is generated by a B^+-stable line of weight λ. Any G-module generated by a B^+-line of weight λ is a homomorphic image of $V(\lambda)$.*

Proof. For 1., note $H^0(-w_0\lambda) \subset k[G] \otimes (-w_0\lambda) = \mathrm{Mor}(G, (-w_0\lambda)_a)$, and we consider map ϵ which sends $f \in \mathrm{Mor}(G, (-w_0\lambda)_a)$ to $f(1) \in -w_0\lambda$. We get the canonical isomorphisms:

$$\begin{aligned} \mathrm{Hom}_G(V(\lambda), M) &\cong \mathrm{Hom}_G(M^*, H^0(-w_0\lambda)) \cong \mathrm{Hom}_B(M^*, -w_0\lambda) \\ &\cong \mathrm{Hom}_B(w_0\lambda, M) \cong \mathrm{Hom}_{B^+}(\lambda, M) \end{aligned} \tag{1.17}$$

mapping any ϕ at first to ϕ^*, then to $\epsilon \circ \phi^*$, to $\phi \circ \epsilon^*$, and finally to $\dot{w}_0 \circ \phi \circ \epsilon^* = \phi \circ \dot{w}_0 \circ \epsilon^*$. So we get an isomorphism $\gamma : \mathrm{Hom}_G(V(\lambda), M) \cong \mathrm{Hom}_{B^+}(\lambda, M)$ which is given by: $\gamma(\lambda)(\phi) = \phi \circ \gamma(\mathrm{id}_{V^*(\lambda)})$.

For 2., set $v = \gamma(\mathrm{id}_{V(\lambda)})(1)$. Obviously v is a B^+-eigenvector of weight λ. If $\phi : V(\lambda) \to M'$ is a homomorphism of G-modules with $\phi(v) = 0$, then $\gamma(\phi) = \phi \circ \gamma(\mathrm{id}_{V(\lambda)}) = 0$, hence $\phi = 0$. This shows that $V(\lambda)$ is generated by v. For any G-module M and any B^+-eigenvector $m \in M$ of weight λ we have a B^+-homomorphism $\lambda \to M$ with $1 \mapsto m$, hence a G-homomorphism $V(\lambda) \to M$ with $v \mapsto m$. If m generates M, then M is a homomorphic image of $V(\lambda)$. □

By duality, we obviously have:

$$V(\lambda)/\operatorname{rad}_G V(\lambda) \cong L(\lambda). \tag{1.18}$$

Proposition 1.2.14. *Let $\lambda, \mu \in X(T)_+$ with $\mu \not\geq \lambda$. Then:*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = \mathrm{Hom}_G(\operatorname{rad}_G V(\lambda), L(\mu)).$$

Proof. We get from the short exact sequence

$$0 \to \operatorname{rad}_G V(\lambda) \to V(\lambda) \to L(\lambda) \to 0$$

a long exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_G(L(\lambda), L(\mu)) \to \mathrm{Hom}_G(V(\lambda), L(\mu)) \to \mathrm{Hom}_G(\operatorname{rad}_G V(\lambda), L(\mu)) \\ &\to \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_G(V(\lambda), L(\mu)) \to \dots \end{aligned} \tag{1.19}$$

Any homomorphism from $V(\lambda)$ to the simple G-module $L(\mu)$ has to factor through $V(\lambda)/\operatorname{rad}_G V(\lambda)$. So we see that the first map in 1.19 is an isomorphism $\mathrm{Hom}_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(V(\lambda), L(\mu))$. So the proposition will follow from 1.19 as soon as we show that $\mathrm{Ext}^1_G(V(\lambda), L(\mu)) = 0$.

Consider an exact sequence of G-modules

$$0 \to L(\mu) \to M \to V(\lambda) \to 0. \tag{1.20}$$

Choose some $v \in M_\lambda$ that is mapped to a B^+-eigenvector generating $V(\lambda)$. By our assumption λ is a maximal weight of M, therefore v is a B^+-eigenvector of weight λ. So the G-submodule M' of M generated by v is a homomorphic image of $V(\lambda)$ by lemma 1.2.13. On the other hand, it is mapped onto $V(\lambda)$. So M' has to be isomorphically onto $V(\lambda)$ in 1.20 and has to be a complement to the kernel $L(\mu)$. So the exact sequence splits, hence $\mathrm{Ext}^1_G(V(\lambda), L(\mu)) = 0$ □

By duality, we have:

Corollary 1.2.15. $\operatorname{Ext}^1_G(L(\mu), L(\lambda)) \cong \operatorname{Hom}_G(L(\mu), H^0(\lambda)/\operatorname{soc}_G H^0(\lambda))$.

1.2.5 Examples of Simple Modules

Now we describe the simple modules for a special case. Let $G = \mathbf{GL}_n$ with $n \geq 2$, let us use the notations introduced in subsection 1.1.7 and set $V = k^n$. Let $P = P_{\alpha_1,\alpha_2,\ldots,\alpha_{n-2}}$ be the stabilizer in G of the line $ke_n \subset k^n = V$. There is a character $\omega \in X(P)$ such that

$$g(e_n \otimes 1) = e_n \otimes \omega(g)^{-1}$$

for all $g \in P(A)$ and any A.

We have for any $r \in \mathbb{N}$:

$$H^0(r\omega|_T) = \operatorname{Ind}_B^G(r\omega) \cong \operatorname{Ind}_P^G \operatorname{Ind}_B^P(r\omega) \cong \operatorname{Ind}_P^G(r\omega \otimes \operatorname{Ind}_B^P k) \cong \operatorname{Ind}_P^G(r\omega),$$

using the transitivity of induction, and $\operatorname{Ind}_B^p k = k[P/B] = k$.

Let $\bar{k}$ be an algebraically closed extension of k. As all groups concerned are reduced we have:

$$\operatorname{Ind}_P^G(r\omega) = \{f \in k[G] \mid f(gg_1) = \omega(g_1)^{-r} f(g) \text{ for all } g \in G(\bar{k}), g_1 \in P(\bar{k})\}.$$

Set H equal to the kernel of ω in P. We see that every $f \in \operatorname{Ind}_P^G(r\omega)$ is invariant under right translation by elements of $H(\bar{k})$. The map $g \mapsto ge_n$ induces an isomorphism of varieties $G(\bar{k})/H(\bar{k}) \cong G(\bar{k})e_n = (V \otimes \bar{k}) - \{0\}$ as the tangent map is obviously surjective. So $\bar{k}[G(\bar{k})/H(\bar{k})] = \bar{k}[(V \otimes \bar{k}) - 0]$. But $V \otimes \bar{k}$ is a smooth, hence normal variety of dimension at least two. Any regular function on $(V \otimes \bar{k}) - 0$ extends to $V \otimes \bar{k}$. So $\bar{k}[G(\bar{k})/H(\bar{k})] = k[G/H] \otimes \bar{k}$ and there is a natural map $S(V^*) \cong k[G/H]$, here $S(V^*)$ denotes the symmetric algebra of V^*. As it becomes an isomorphism after tensoring with $\bar{k}$, it was one already before. Hence

$$S(V^*) \cong k[G/H].$$

To any $f \in S(V^*)$, there corresponds $g \mapsto f(ge_n)$. If $g_1 \in P(\bar{k})$, then $g_1 e_n = \omega(g_1)^{-1} e_n$, hence $f(gg_1e_n) = f(\omega(g_1)^{-1} ge_n)$. So we have:

$$\operatorname{Ind}_P^G(r\omega) = \{f \in S(V^*) | f(av) = a^r f(v) \text{ for all } a \in \bar{k} \text{ and } v \in V \otimes \bar{k}\}.$$

This space is obviously equal to $S^r(V^*)$, the homogeneous part of degree r in $S(V^*)$:

$$\operatorname{Ind}_P^G(r\omega) \cong H^0(r\omega) \cong S^r(V^*).$$

If we denote the basis of V^* dual to $e_1, \ldots, e_n$ by $X_1, \ldots, X_n$, then we can identify $S(V^*)$ with the polynomial ring in $X_1, \ldots, X_n$ and $S^r(V^*)$ with the subspace of homogeneous polynomials of degree r. The weight of each e_i is ϵ_i, so each X_i has weight $-\epsilon_i$ and any monomial $X_1^{a(1)} \ldots X_n^{a(n)}$ has weight $-\sum_{i=1}^n a(i)\epsilon_i$. Therefore all different monomials in $S^r(V^*)$ have a different weight, each weight space in $S^r(V^*)$ has dimension one and is spanned by one monomial $X_1^{a(1)} \ldots X_n^{a(n)}$

with $\sum_{i=1}^n a(i) = r$. We have $\omega = \epsilon_n$, So X_n^r is the unique monomial of weight $r\omega$ that has to span $H^0(r\omega)_{r\omega} = H^0(r\omega)^{U^+}$. So,

$$L(r\omega) = \mathrm{Dist}(U)X_n^r$$

The representation of G on $S(V^*)$ and on any $S^r(V^*)$ can be constructed via base change from the representation of $G_{\mathbb{Z}} = \mathbf{GL}_{n,\mathbb{Z}}$ on $S(V_{\mathbb{Z}}^*)$ resp. $S^r(V_{\mathbb{Z}}^*)$, where $V_{\mathbb{Z}}$ is the natural representation of $G_{\mathbb{Z}}$. These $G_{\mathbb{Z}}$-modules are lattices in the corresponding representations of $G_{\mathbb{Q}}$ on $S(V_{\mathbb{Q}}^*)$ resp. $S^r(V_{\mathbb{Q}}^*)$. So we can compute the action of any $X_{\alpha,m}$ ($\alpha \in R$, $m \in \mathbb{N}$) on $S^r(V^*)$ from the action of corresponding element in $\mathrm{Dist}(G_{\mathbb{Q}})$.

We have $\mathrm{Lie}\, G = M_n(k)$ and $X_\alpha = E_{ij}$ if $\alpha = \epsilon_i - \epsilon_j$. Then $X_{\alpha,n} = (E_{ij})^n/n!$ in $\mathrm{Dist}(G_{\mathbb{Q}})$.

From $E_{ij}e_l = \delta_{jl}e_i$ we get $E_{ij}X_l = -\delta_{il}X_j$, hence

$$E_{ij,m}X_1^{a(1)}\dots X_n^{a(n)} = (-1)^m \binom{a(i)}{m} (\prod_{l\neq i,j} X_l^{a(l)}) X_i^{a(i)-m} X_j^{a(j)+m}.$$

In particular, for the case $n=2$, this implies:

$$L(r\omega) = \sum_{m\geq 0} kE_{21,m}X_2^r = \sum_{m=0}^r k\binom{r}{m} X_1^m X_2^{r-m}$$

So if $\mathrm{char}(k) = p \neq 0$, then $L(r\omega)$ is spanned by the $X_1^m X_2^{r-m}$ with $p \not| \binom{r}{m}$. If $\mathrm{char}(k) = 0$, then $L(r\omega) = S^r(V^*)$.

Chapter 2

Frobenius Kernels

In this chapter, we give a proof for Steinberg's Tensor Formula, i.e., corollary 2.2.14. We first give the general theory for Frobenius kernels which is quite elementary in the first section. Then in the second section we will give a detailed study on the structure of the Frobenius kernels based on the knowledge of reductive groups and the modules of Frobenius kernels based on the knowledge of finite algebraic groups. And finally we will prove corollary 2.2.14.

2.1 General Theory of Frobenius Kernels

2.1.1 Frobenius Morphism

Now we assume k to be a perfect field of characteristic p. For each k-algebra A and each $m \in \mathbb{Z}$ we define $A^{(m)}$ as a k-algebra that coincide with A as a ring but where each $b \in k$ operates as $b^{p^{-m}}$ does on A. Trivially $A^{(0)} = A$. So we have:

$$\mathrm{Hom}_{k-alg}(A^{(-m)}, A') \cong \mathrm{Hom}_{k-alg}(A, A'^{(m)}) \tag{2.1}$$

for all $m \in \mathbb{Z}$. For each k-algebra, each $m \in \mathbb{Z}$ and $r \in \mathbb{N}$ the map:

$$\gamma_r : A^{(m)} \to A^{(m-r)}, a \mapsto a^{p^r}$$

is a homomorphism of k-algebras.

We now define for any k-functor X and any $r \in \mathbb{N}$ a new k-functor $X^{(r)}$ by

$$X^{(r)}(A) = X(A^{(-r)})$$

for all k-algebras A.

Furthermore, we define a morphism $F_X^r : X \to X^{(r)}$ through

$$F_X^r(A) = X(\gamma_r) : X(A) \to X(A^{(-r)}) = X^{(r)}(A)$$

and by (2.1), we have

$$(\mathrm{Spec}_k R)^{(r)} \cong \mathrm{Spec}_k(R^{(r)})$$

Furthermore, F_X^r has as comorphism $R^{(r)} \to R$, $f \to f^{p^r}$.

If $k = \mathbb{F}_p$, then obviously $X^{(r)} = X$ for all r and any k-functor X. If X is affine and if F_X is an endomorphism of X with $F_X^*(f) = f^p$ for all $f \in k[X]$, then obviously $F_X^r = (F_X)^r$. More generally, if X has an $\mathbb{F}_p$-structure, then we can identify $X^{(r)}$ with X.

2.1.2 Frobenius Kernels

Let G be a k-group functor. Then obviously each $G^{(r)}$ is also a k-group functor and F_G^r is a homomorphism of k-group functors. Its kernel $G_r = \ker(F_G)^r$ is a normal subgroup functor of G that we call the r^{th} *Frobenius kernel* of G. So we have an ascending chain:

$$G_1 \subset G_2 \subset G_3 \subset \dots$$

of normal subgroup functors of G.

For example the r^{th} Frobenius kernel of $\mathbf{G}_{\mathrm{a}}$ is $G_{a,r} = \mathrm{Spec}\, k[T]/T^r$ and the r^{th} Frobenius kernel of $\mathbf{G}_{\mathrm{m}}$ is $G_{m,r} = \mu_{p^r} = \mathrm{Spec}\, k[T]/(T^r - 1)$.

Let I_1 be the ideal defining 1. Obviously G_r is the closed subscheme of G defined by $\sum_{f \in I_1} k[G] f^{p^r}$. Therefore $k[G_r]$ is finite dimensional and the ideal of 1 in $k[G_r]$ is nilpotent. Hence

Proposition 2.1.1. *Each G_r is an infinitesimal k-group.*

Choose $f_1, \dots, f_m \in I_1$ such that $f_i + I_1^2$ forms a basis of I_1/I_1^2. Then $m = \dim \mathrm{Lie}\, G$ and f_i generate I_1 as an ideal. One has obviously $\dim k[G_r] \leq p^{rm}$ for all r, and equality holds if 1 is a simple point of G. So we have

Proposition 2.1.2. *If G is reduced, then $\dim k[G_r] = p^{r \dim G}$ for all $r \in \mathbb{N}$.*

And for the Frobenius map, we have the following proposition:

Proposition 2.1.3. *If G is a reduced algebraic k-group, then each F_G^r induces isomorphisms $G/G_r \cong G^r$ and $G'_r/G_r \cong (G^{(r)})_{r'-r}$ for all $r' \geq r$.*

Proof. See [J] I 9.5. □

Proposition 2.1.4. *Let G be a reduced algebraic k-group and $r \in \mathbb{N}$. Then G operates on $\mathrm{Dist}(G_r)_l^{G_r}$ through character:*

$$g \mapsto \det(\mathrm{Ad}(g))^{p^r - 1},$$

where Ad *denotes the adjoint representation of G on* $\mathrm{Lie}\, G$.

Proof. Recall from 0.3.8 that the conjugation action of G on G_r leads to representations of G on $k[G_r]$ and $M(G_r) = \mathrm{Dist}(G_r)$ is a one dimensional submodule on which G has to operate through some character $\chi \in X(G)$.

Set $q = p^r$ and choose $f_1, \ldots, f_m \in I_1$ such that the $f_i + I_1^2$ form a basis of I_1/I_1^2. Let $\bar{f}_i$ be the image of f_i in $k[G_r]$. As G is reduced, hence 1 a simple point, the monomials $\bar{f_1}^{a_1} \bar{f_2}^{a_2} \ldots \bar{f_m}^{a_m}$ with $0 \le a(i) \le q$ for all i form a basis of $k[G_r]$.

We can identify $k[G_r]$ with the factor ring $k[T_1, \ldots, T_m]/(T_1^q, \ldots, T_m^q)$ of the polynomial ring $k[T_1, \ldots, T_m]$. It is therefore a graded ring in a natural way. Any endomorphism ϕ of the vector space $\sum_{i=1}^m k\bar{f}_i$ induces an endomorphism of the graded algebra $k[G_r]$. As $F = \prod_{i=1}^m \bar{f_i}^{q-1}$ is the only basis element of degree $m(q-1)$, it has to be mapped under ϕ into a multiple $c(\phi)F$ of itself. Obviously $\phi \mapsto c(\phi)$ has to be multiplicative. This implies $c(\phi) = \det(\phi)^{q-1}$ for all ϕ, as this is obviously true for ϕ in upper or lower triangular form, hence for all ϕ by multiplicativity. This extends easily to any k-algebra A and any endomorphism of $\sum_{i=1}^m k\bar{f}_i \otimes A$ as $c(\phi)$ is obviously a polynomial in the matrix coefficients of ϕ.

This can be applied in particular to the operation of any $g \in G(A)$ for any k-algebra A on $k[G_r] \otimes A$ derived from the conjugation action on G_r. Then the action of g on

$$\sum_{i=1}^m k\bar{f}_i \otimes A \cong (I_1/I_1^2) \otimes A \cong \operatorname{Lie} G^* \otimes A$$

is dual to the adjoint action on $\operatorname{Lie} G \otimes A$, hence has determinant equal to $\det(\operatorname{Ad}(g))^{-1}$. So this implies:

$$gF = \det(\operatorname{Ad}(g))^{-(q-1)} F.$$

Consider now $\mu_0 \in \operatorname{Dist}(G_r)_l^{G_r}$, $\mu_0 \neq 0$. If $\mu_0(F) = 0$, then $\mu_0(k[G_r]f) = 0$ by the definition of the $k[G_r]$-module structure on $\operatorname{Dist}(G_r)$ as in 0.3.2. So $\operatorname{Dist}(G_r)(F) = 0$ by proposition 0.3.3. This is a contradiction, so we must have $\mu_0(F) \neq 0$. Then

$$\chi(g)\mu_0(F) = (g\mu_0)(F) = \mu_0(g^{-1}F) = \det(\operatorname{Ad}(g))^{q-1}\mu_0(F)$$

implies $\chi(g) = \det(\operatorname{Ad}(g))^{q-1}$. □

Note that the $\operatorname{Dist}(G_r)$ forms an ascending chain of subalgebras of G, and one has:

$$\operatorname{Dist(G)} = \bigcup_{r>0} \operatorname{Dist}(G_r).$$

For some group scheme G, e.g., G is irreducible, the set of Dist(G)-submodules of G-modules M is exactly the set of G-submodules, so we give the following claims without proofs.

Proposition 2.1.5. *If G is irreducible:*

1. *If M is a G-module, then $M^G = \cap_{r>0} M^{G_r}$.*
2. *If M, M' are G-modules, then $\operatorname{Hom}_G(M, M') = \cap_{r>0} \operatorname{Hom}_{G_r}(M, M')$.*
3. *Let M be a G-module and N a subspace of M. Then N is a G-submodule if any only if it is a G_r-submodule for all $r \in \mathbb{N}$.*
4. *If M is a G-module with $\dim M < \infty$, then there is an $n \in \mathbb{N}$ with $M^G = M^{G_r}$ for all $r > n$.*
5. *If M, M' are G-modules with $\dim(M \otimes M') < \infty$, then there is an $n \in \mathbb{N}$ with $\operatorname{Hom}_G(M, M') = \operatorname{Hom}_{G_r}(M, M')$ for all $r > n$.*

2.1.3 Frobenius Twists of Representations

For any vector space M over k and any $r \in \mathbb{N}$ we denote by $M^{(r)}$ the vector space that is equal to M as an abelian group and where any $a \in k$ operates as $a^{p^{-r}}$ does on M. If M is a G-module, then we have a natural structure as a G-module on each $M^{(r)}$ with $r \geq 0$ by section 0.1.2.

Suppose now that M has a fixed $\mathbb{F}_p$-structure, so we have an $\mathbb{F}_p$-subspace $M' \subset M$ with $M' \otimes_{\mathbb{F}_p} k = M$. We get then a Frobenius endomorphism F_M on M and on each $M \otimes A \cong M' \otimes_{\mathbb{F}_p} A$ through $F_m(m' \otimes a) = m' \otimes a^p$. Each F_M^r is an isomorphism of A-modules $M \otimes A \to M^{(r)} \otimes A$. Suppose that G is defined over $\mathbb{F}_p$ and denote the corresponding Frobenius endomorphism by $F_G : G \to G$. If the representation of G on M is defined over F_p (i.e., if $F_G(g)F_M(m) = F_M(gm)$ for all $m \in M$, $g \in G(A)$), then we can define a new representation of G on M by composing the given $G \to \mathbf{GL}(M)$ with $F_G^r : G \to G$. Then $F_M^{(r)} : M \to M^{(r)}$ is an isomorphism of G-modules if we take the new structure on M just defined and on $M^{(r)}$ as defined.

2.2 Frobenius Kernels for Reductive Groups

In this section we assume p to be a prime number and assume that k is a perfect field of characteristic p, let $q = p^r$. And G a reductive group scheme arise from $G_{\mathbb{Z}}$.

2.2.1 Structure of G_r

As G arises from $G_{\mathbb{Z}}$ through base change we also get G from $G_{\mathbb{F}_p}$ through base change. Therefore any $G^{(r)}$ is isomorphic to G, and there is a Frobenius endomorphism $F = F_G : G \to G$ such that we get any F_G^r as $G \xrightarrow{F^r} G \cong G^{(r)}$ using suitable isomorphism. We get in particular for all r:

$$G_r = \ker(F^r).$$

We also have $T = (T_{\mathbb{F}_p})_k$ and $U_\alpha = (U_{\alpha,\mathbb{F}_p})_k$ for all α. So F stabilizes T and all U_α. We get any x_α from $x_{\alpha,\mathbb{Z}} : G_{a,\mathbb{Z}} \cong U_{\alpha,\mathbb{Z}}$ and we have $X(T) = X(T_{\mathbb{Z}})$. Therefore the isomorphisms $\mathbf{G}_{\mathrm{a}} \cong U_\alpha$ and $T \cong (\mathbf{G}_{\mathrm{m}})^r$(where $r = \operatorname{rk} T$) are compatible with the usual Frobenius endomorphisms on $\mathbf{G}_{\mathrm{a}}$ and $\mathbf{G}_{\mathrm{m}}$. This implies (for all A):

$$F(t) = t^p \text{ for all } t \in T(A) \tag{2.2}$$

and

$$F(x_\alpha(a)) = x_\alpha(a^p) \text{ for all } \alpha \in R, a \in A. \tag{2.3}$$

All the groups introduced before (e.g.B,B^+,P_I,P_I^+,$U(R')$,L_I) are F-stable and F restricts to a Frobenius endomorphism on them.

Lemma 2.2.1. *The multiplication induces isomorphisms of schemes(for any r)*

$$\prod_{\alpha \in R^+} U_{\alpha,r} \times T_r \times \prod_{\alpha \in R^+} U_{-\alpha,r} \cong U_r^+ \times T_r \times U_r \cong G_r.$$

Proof. As 1 is contained in the open subscheme U^+TU of G its inverse image $G_r = F^{-r}(1)$ is contained in $F^{-r}(U^+TU) = U^+TU$. The multiplication induces isomorphism of schemes:

$$\prod_{\alpha\in R^+} U_\alpha \times T \times \prod_{\alpha\in R^+} U_{-\alpha} \cong U^+ \times T \times U \cong U^+TU$$

where 1 corresponds to the element having all components equal to 1. Now F stabilizes all factors and induces the Frobenius endomorphism on each of them. This implies the lemma. □

Lemma 2.2.2. *The elements*

$$\prod_{\alpha\in R^+} X_{\alpha,n(\alpha)} \prod_i H_{i,m(i)} \prod_{\alpha\in R^+} X_{-\alpha,n'(\alpha)}, \qquad 0 \le n(\alpha), m(i), n'(\alpha) \le p^r$$

form a basis of $\mathrm{Dist}(G_r)$

Proof. By lemma above we have an isomorphism of vector spaces:

$$\bigotimes_{\alpha\in R^+} \mathrm{Dist}(U_{\alpha,r}) \otimes \mathrm{Dist}(T_r) \otimes \bigotimes_{\alpha\in R^+} \mathrm{Dist}(U_{-\alpha,r}) \cong \mathrm{Dist}(G_r).$$

Now we use the notations in 0.2.5. Note $\mathrm{Dist}(G_{a,r})$ is the subalgebra of Dist(G) spanned by all $\mu \in$ Dist(G) such that $\mu(T^{q+i}) = 0$ for all $i \ge 0$. So $\mathrm{Dist}(G_{a,r}) = \sum_{n=0}^{q-1} k\gamma_n$. For μ_q, we see that $\mathrm{Dist}(\mu_q)$ consists of all $\nu \in \mathrm{Dist}(\mathbf{G}_{\mathrm{m}})$ with $\nu(T^i(T^q - 1)) = 0$ for all $i \in \mathbb{Z}$. By calculation we have $\delta_n(T^i(T^q - 1)) = 0$ for all $i \in Z$ if $0 \le n < q$. As $\dim \mathrm{Dist}(\mu_q) = q$, we have $\mathrm{Dist}(\mu_q) = \sum_{n=0}^{q-1} k\delta_n$. And the description above completes the proof if we take $X_{\alpha,n(\alpha)}$ and $X_{-\alpha,n'(\alpha)}$ to be the basis of $\mathrm{Dist}(G_{a,r})$ and $H_{m,m(i)}$ be the basis of $\mathrm{Dist}(G_{m,r})$ mentioned above. □

Recall that G (resp.B, B^+) operates on G_r (resp. B_r,B_r^+) through conjugation leading to a representation on $\mathrm{Dist}(G_r)$ (resp. $\mathrm{Dist}(B_r)$, $\mathrm{Dist}(B_r^+)$), and that then $\mathrm{Dist}(G_r)_l^{G_r}$ (resp. $\mathrm{Dist}(B_r)_l^{B_r}$, $\mathrm{Dist}(B_r^+)_l^{B_r^+}$) is a one dimensional submodule by propositions 0.3.8 and 2.1.4.

Proposition 2.2.3. *1. The action of G on $\mathrm{Dist}(G_r)_l^{G_r}$ is trivial.*

2. The action of $B = TU$ on $\mathrm{Dist}(B_r)_l^{B_r}$ is trivial on U and is given by $-2(p^r - 1)\rho$ on T.

3. The action of $B^+ = TU^+$ on $\mathrm{Dist}(B_r^+)_l^{B_r^+}$ is trivial on U and is given by $-2(p^r - 1)\rho$ on T.

Proof. 1. The adjoint representation of G on $\mathrm{Lie}\, G$ factors through $G/Z(G)$, which is semisimple and admits no character. Hence $\det \circ \mathrm{Ad} = 1$ and the claim follows from proposition 2.1.5

2. We have $\mathrm{Lie}\, B = \mathrm{Lie}\, T \bigoplus \oplus_{\alpha\in R^+} (\mathrm{Lie}\, G)_{-\alpha}$. Hence :

$$\det(\mathrm{Ad}(t)) = (\mathrm{rk}(T))0 + \sum_{\alpha\in R^+} (-\alpha))(t) = -2\rho(t)$$

for any $t \in T(A)$, so T acts by proposition 2.1.4 on $\mathrm{Dist}(B_r)_l^{B_r}$ via $(p^r - 1)(-2\rho)$. On the other hand, the unipotent group U admits no character, hence U has to operate trivially.

3. Proof is just like above.

□

Corollary 2.2.4. *Let M be a B_r-module and M' a B_r^+-module. Then we have*

1. $\operatorname{Coind}_{B_r}^{G_r} M \cong \operatorname{Ind}_{B_r}^{G_r}(M \otimes 2(p^r-1)\rho)$,
2. $\operatorname{Coind}_{B_r^+}^{G_r} M' \cong \operatorname{Ind}_{B_r^+}^{G_r}(M' \otimes (-2(p^r-1)\rho))$.

In case $\dim M < \infty$ *resp.* $\dim M' < \infty$*, we have*

1. $(\operatorname{Ind}_{B_r}^{G_r} M)^* \cong \operatorname{Ind}_{B_r}^{G_r}(M^* \otimes 2(p^r-1)\rho)$,
2. $(\operatorname{Ind}_{B_r^+}^{G_r} M')^* \cong \operatorname{Ind}_{B_r^+}^{G_r}(M'^* \otimes (-2(p^r-1)\rho))$.

Proof. This follows from proposition 2.2.3 and subsection 0.3.3. □

2.2.2 Induced and Coinduced Modules

Any $\lambda \in X(T)$ defines by restriction a character of T_r, which we usually also denote by λ. We get from the restriction an exact sequence:

$$0 \to p^r X(T) \to X(T) \to X(T_r) \to 0$$

where the first map is the inclusion.

By extending its restriction to T_r trivially to U_r or U_r^+, any $\lambda \in X(T)$ defines a one dimensional module (usually denoted by λ) for B_r and B_r^+. We can induce and coinduce these modules to G_r and give the following notations:

$$Z_r(\lambda) = \operatorname{Coind}_{B_r^+}^{G_r} \lambda,$$

$$Z'_r(\lambda) = \operatorname{Ind}_{B_r}^{G_r} \lambda.$$

So corollary 2.2.4 gives:

$$Z_r(\lambda) \cong \operatorname{Ind}_{B_r^+}^{G_r}(\lambda - 2(p^r-1)\rho), \tag{2.4}$$

$$Z'_r(\lambda) \cong \operatorname{Coind}_{B_r}^{G_r}(\lambda - 2(p^r-1)\rho), \tag{2.5}$$

$$Z_r(\lambda)^* \cong Z_r(2(p^r-1)\rho - \lambda), \tag{2.6}$$

$$Z'_r(\lambda)^* \cong Z'_r(2(p^r-1)\rho - \lambda). \tag{2.7}$$

And we have $\dim Z_r(\lambda) = \dim Z'_r(\lambda) = p^r|R^+|$ for all λ.

As $\lambda + p^r v$ and λ have the same restriction to T_r, we get:

$$Z_r(\lambda + p^r v) = Z_r(\lambda), \tag{2.8}$$

$$Z'_r(\lambda + p^r v) = Z'_r(\lambda). \tag{2.9}$$

To study $Z_r(\lambda)$ and $Z'_r(\lambda)$, now we give a more detailed study for induced and coinduced modules for Frobenius kernels.

Note that the isomorphism of schemes $U_r^+ \times B_r \cong G_r$ given by the multiplication is compatible with the action of U_r^+ by left multiplication on U_r^+ and G_r with the action of B_r by right multiplication on B_r and G_r. It is also compatible with the action of T_r by conjugation on U_r^+ and by left multiplication on B_r and G_r. Therefore, the isomorphisms $k[G_r] \cong k[U_r^+] \otimes k[B_r]$ and $\mathrm{Dist}(U_r^+) \otimes \mathrm{Dist}(B_r) \cong \mathrm{Dist}(G_r)$ of vector spaces are compatible with representations of U_r^+ (resp. B_r, T_r) induced by these actions.

We have for any B_r-module M:

$$\mathrm{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes (k[B_r] \otimes M)^{B_r},$$

hence:

$$\mathrm{Ind}_{B_r}^{G_r} M \cong k[U_r^+] \otimes M$$

This isomorphism is compatible with the representations of U_r^+ (acting via ρ_l on $k[U_r^+]$) and of T_r (acting as given on M and via the conjugation action on $k[U_r^+]$). Similarly,

$$\mathrm{Coind}_{B_r}^{G_r} M = \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} M \cong \mathrm{Dist}(U_r^+) \otimes \mathrm{Dist}(B_r) \otimes_{\mathrm{Dist}(B_r)} M,$$

hence:

$$\mathrm{Coind}_{B_r}^{G_r} M \cong \mathrm{Dist}(U_r^+) \otimes M.$$

Again this isomorphism of vector spaces is compatible with the representations of U_r^+ and T_r.

By interchanging the roles of U_r and U_r^+ we also get for each B_r^+-module M' isomorphisms:

$$\mathrm{Ind}_{B_r^+}^{G_r}(M') \cong k[U_r] \otimes M'$$

and

$$\mathrm{Coind}_{B_r^+}^{G_r}(M') \cong \mathrm{Dist}(U_r) \otimes M'$$

of U_r-modules and T_r-modules.

We have $\dim k[U_r] = p^{r \dim U} = p^{r|R^+|}$ as U is reduced. This is also the dimension of $k[U_r^+]$, $\mathrm{Dist}(U_r)$, and $\mathrm{Dist}(U_r^+)$. So all the induced or coinduced modules considered above have dimension equal to $\dim(M)p^{r|R^+|}$.

Recall our discussion at the end of subsection 0.3.1, we give a new proposition for the bijection mentioned in our special case:

Proposition 2.2.5. *Let $\lambda \in X(T)$.*

1. *Considered as a B_r-module, $Z_r(\lambda)$ is the projective cover of λ and the injective hull of $\lambda - 2(p^r - 1)\rho$.*
2. *Considered as a B_r^+-module, $Z'_r(\lambda)$ is injective hull of λ and the projective cover of $\lambda - 2(p^r - 1)\rho$.*

Proof. The injective hull of any simple module for groups like $B_r^+ = T_r \times U_r^+$ and $B_r = T_r \ltimes U_r$ has been determined in subsection 1.2.1. The injective hull, say, of λ for B_r^+ is $k[U_r^+] \otimes \lambda$, with U_r^+ operating only on the first factor (via ρ_l) and with T_r operating through the conjugation action on $k[U_r^+]$ tensored with the restriction of λ. On the other hand, $Z'_r(\lambda) = \operatorname{Ind}_{B_r}^{G_r} \lambda$ has exactly this form by description above. Thus we have first part of (2), and second part of (1) can be proved similarly. And for the projective cover we have proposition 0.3.4. □

2.2.3 Simple G_r-Modules

It follows from proposition 2.2.5 that all $\mathbb{Z}_r(\lambda)^{U_r}$ and $\mathbb{Z}_r(\lambda')^{U_r^+}$ have dimension one, as these spaces of fixed points are contained in the B_r-resp. the B_r^+-socle of the module. On the other hand, we have $M^{U_r} \neq 0 \neq M^{U_r^+}$ for any G_r-module $M \neq 0$, as U_r, U_r^+ are unipotent.

Arguing as in corollary 1.2.3 we have;

Proposition 2.2.6. *Any $Z_r(\lambda)$ and any $Z'_r(\lambda)$ has a simple socle when considered as a G_r-module.*

And by dualizing we get:

Proposition 2.2.7. *Any $Z_r(\lambda)/\operatorname{rad}_{G_r} Z_r(\lambda)$ and any $Z'_r(\lambda)/\operatorname{rad}_{G_r} Z'_r(\lambda)$ is a simple G_r-module.*

Generally we have:

Proposition 2.2.8. *For any simple G_r-module L there are λ_1, λ_2, λ_3, $\lambda_4 \in X(T)$ with:*

$$\begin{aligned} L &\cong \operatorname{soc}_{G_r} Z'_r(\lambda_1) \cong \operatorname{soc}_{G_r} Z_r(\lambda_2) \cong Z'_r(\lambda_3)/\operatorname{rad}_{G_r} Z'_r(\lambda_3) \\ &\cong Z_r(\lambda_4)/\operatorname{rad}_{G_r} Z_r(\lambda_4). \end{aligned} \tag{2.10}$$

Set for all $\lambda \in X(T)$

$$L_r(\lambda) = \operatorname{soc}_{G_r} Z'_r(\lambda).$$

.

Proposition 2.2.9. *We have for all $\lambda \in X(T)$:*

$$L_r(\lambda)^{U_r^+} = \lambda, \tag{2.11}$$

$$Z_r(\lambda)/\operatorname{rad}_{G_r} Z_r(\lambda) = L_r(\lambda), \tag{2.12}$$

$$\operatorname{End}_{G_r} L_r(\lambda) = k. \tag{2.13}$$

If Λ is a set of representatives in $X(T)$ for $X(T)/p^r X(T)$, then each simple G_r-module is isomorphic to exactly one $L_r(\lambda)$ with $\lambda \in \Lambda$.

Proof. The first formula follows immediately from the definition of $L_r(\lambda)$ and proposition 2.2.8. So we have:

$$\operatorname{Hom}_{B_r^+}(\lambda, L_r(\lambda)) \neq 0.$$

Recall that the functor Coind is right adjoint to functor Res, so we have:

$$0 \neq \operatorname{Hom}_{G_r}(\operatorname{Coind}_{B_r^+}^{G_r} \lambda, L_r(\lambda)) = \operatorname{Hom}_{G_r}(Z_r(\lambda), L_r(\lambda)).$$

As $L_r(\lambda)$ is simple, any non-zero homomorphism $Z_r(\lambda) \to L_r(\lambda)$ has to be surjective and to factor through $Z_r/\operatorname{rad}_{G_r} Z_r(\lambda)$ which we have shown to be a simple module. So we have the second formula.

Note that any $\phi \in \operatorname{End}_{G_r} L_r(\lambda)$ stabilizes the line $L_r(\lambda)^{U_r^+}$, hence has an eigenvalue in k. Now schur's lemma implies the last formula. □

Remark 2.1. Note that the last formula implies that $L_r(\lambda)$ is absolutely irreducible, i.e., that for each (perfect) field $k' \subset k$ the $(G_r)_{k'} = (G_{k'})_r$-module $L_r(\lambda) \otimes k'$ is the unique simple $(G_{k'})_r$-module such that $(T_{k'})_r$ operates via λ on the $(U_{k'}^+)_r$-fixed points. Furthermore, we have

$$\operatorname{soc}_{(G_k)_r}(M \otimes k') = (\operatorname{soc}_{G_r} M) \otimes k'$$

for any G_r-module M.

For all $g \in G(k)$ and any G_r-module M, as described in 0.1.2, we can construct a new G_r-module ${}^gM$.

Proposition 2.2.10. *For all $g \in G(k)$ and for all simple G_r-modules L we have ${}^gL = L$.*

Proof. Obviously the "twist" $M \to {}^gM$ commutes with field extension, so we may assume k is algebraic closed. Let $L = L_r(\lambda)$ and choose $v \neq 0$. Any $g \in B^+(k)$ normalizes U_r^+, hence v is also in ${}^gL^{U_r^+}$. If $t \in T_r(A)$ for some A and $g \in B^+(k)$, then $g^{-1}tg = t(t^{-1}g^{-1}tg) \in B_r^+(A)$ and $t^{-1}g^{-1}tg \in U^+(A)$, hence $t^{-1}g^{-1}tg \in U_r^+(A)$. The operation of t on ${}^gL \otimes A$ maps $v \otimes 1$ to $(g^{-1}tg)(v \otimes 1)$ in $L \otimes A$, hence to $t(v \otimes 1) = \lambda(ta)(v \otimes 1)$. So $({}^gL)^{U_r^+} = \lambda$ and ${}^gL = L$. As this is true for any simple G_r-module and any $g \in B^+(k)$, and as $G(k)$ is th union of its Borel subgroups, we get the proposition. □

Now we will show the relationship between the $L_r(\lambda)$ and $L(\lambda)$. First we introduce a lemma without proof.

Lemma 2.2.11. *Let $\lambda \in X(T)_+$ and choose $v \in L(\lambda)_\lambda$, $v \neq 0$. Then* $\operatorname{Dist}(G_r)v$ *is a simple G_r-module isomorphic to $L_r(\lambda)$.*

Set $X_r(T) = \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in S\}$, so we have:

$$X_1(T) \subset X_2(T) \subset \cdots \subset X_r(T) \subset \ldots X(T)_+.$$

Proposition 2.2.12. *For each $\lambda \in X_r(T)$, the simple G-module $L(\lambda)$ is also simple as a G_r-module and is isomorphic to $L_r(\lambda)$ for G_r.*

Proof. Choose v as in lemma 2.2.11 and set $L = \operatorname{Dist}(G_r)v$. We have to show that $L = L(\lambda)$. we can obviously assume that k is algebraically closed. For each $g \in G(k)$ the subspace $gL \cong {}^gL$ is another simple G_r-submodule of $L(\lambda)$, so either $gL = L$ or $gL \cap L = 0$.

Consider now a simple root α and a representative of s_α in $N_G(T)(k)$, e.g., let us take $n_\alpha(1)$. Then $n_\alpha(1)v \neq 0$. It is easy to show:

$$\bigoplus_{n \geq 0} L(\lambda)_{\lambda - n\alpha} = \operatorname{Dist}(U_{-\alpha})v = \sum_{n \geq 0} kX_{-\alpha,n}v,$$

hence $L(\lambda)_{\lambda-n\alpha} = kX_{-\alpha,n}v$ for all $n \in \mathbb{N}$, and

$$L(\lambda)_{\lambda-n\alpha} \subset \mathrm{Dist}(G_r)v$$

for all $n < p^r$. Now $s_\alpha\lambda = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha$ and $0 \leq \langle\lambda, \alpha^\vee\rangle < p^r$ by assumption. This yields $n_\alpha(a)v \in \mathrm{Dist}(G_r)v = L$ and, therefore, $n_\alpha(1)L = L$. As W is generated by the s_α with $\alpha \in S$ and $w \in W$ has a representative $\dot{w} \in N_G(T)(k)$ with $\dot{w}L = L$.

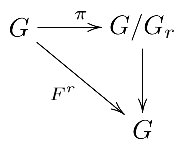
Using the Bruhat decomposition $G(k) = B^+(k)\dot{w}B^+(k)$ we get $gL = L$ for all $g \in G(k)$. So L is a non-zero $G(k)$-submodule of the simple G-module $L(\lambda)$. As G is reduced, this implies $L = L(\lambda)$, hence the simplicity of $L(\lambda)$ as a G_r-module. □

2.2.4 Steinberg's Tensor Product Theorem

If M is a G-module, then we can define another G-module structure on M by composing the given representation $G \to \mathbf{GL}(M)$ with $F^r : G \to G$. We denote this G-module by $M^{[r]}$.

On the other hand, look at a G-module V such that G_r acts trivially on V. Then the representation $G \to \mathbf{GL}(V)$ factors through G/G_r.

We know $G/G_r \cong G$, more precisely, we have a commutative diagram.



Therefore V has the form $M^{[r]}$ for some G-module M, usually denoted by $V^{[-r]}$.

For example, we can apply this G-module V to V^{G_r} and, more generally, for any G-modules V, V' with $\dim V' < \infty$ to $\mathrm{Hom}_{G_r}(V', V)$.

Suppose there is a system $X'_r(T)$ of representatives for $X(T)/p^rX(T)$ with $X'_r(T) \subset X_r(T)$. Then we have for any G-module M an isomorphism of G-module M an isomorphism of G-modules by 0.4 and 1.2.8:

$$\mathrm{soc}_{G_r} M \cong \bigoplus_{\lambda \in X'_r(T)} L(\lambda) \otimes \mathrm{Hom}_{G_r}(L(\lambda), M). \tag{2.14}$$

We can apply the remark above to all $\mathrm{Hom}_{G_r}(L(\lambda), M)$.

Proposition 2.2.13. *We have for all $\lambda \in X_r(T)$ and $\mu \in X(T)_+$:*

$$L(\lambda + p^r\mu) \cong L(\lambda) \otimes L(\mu)^{[r]}.$$

Proof. We may assume that there is $X'_r(T)$ as above and that $\lambda \in X'_r(T)$, hence that we can apply formula 2.14 to $M = L(\lambda+p^r\mu)$. As this module is simple, there is only one nonzero summand, and the corresponding $\mathrm{Hom}(\dots)^{[-r]}$ has to be simple. Therefore there are $\lambda' \in X'_r(T)$ and $\mu' \in X(T)_+$ with $L(\lambda + p^r\mu) \cong L(\lambda') \otimes L(\mu')^{[r]}$. Comparing the highest weights yields $\lambda + p^r\mu = \lambda' + p^r\mu'$. Now $\lambda - \lambda' \in p^rX(T)$ and λ, $\lambda' \in X'_r(T)$ imply $\lambda = \lambda'$ and then $\mu = \mu'$ and the proposition. □

By induction, we have

Corollary 2.2.14. (Steinberg's Tensor Product Theorem)

Let $\lambda_0, \lambda_1, \ldots, \lambda_m \in X_1(T)$ *and set* $\lambda = \sum_{i=0}^{m} p^i \lambda_i$. *Then:*

$$L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_m)^{[m]}.$$

Chapter 3

Linkage Theorem

In this chapter, our main aim is the last corollary: corollary 3.2.11. Here we only give the proof for this important Linkage Theorem. The proof based on two parts, which constitute the first and the second section: the theory of alcoves and the theory of higher cohomology for induciton functor. And in the end, we give a proof of linkage theorem by induction.

3.1 Theory for Alcoves

The theory of alcoves is quite independent in this thesis. It studies reflection groups in euclidean spaces. Here this theory will apply to the action of affine Weyl groups on character space of reductive groups. The main result of this section is to give the fundamental domain for the affine Weyl group W_p by proposition 3.1.2 and describe the new relation by proposition 3.1.3. The latter proposition will be used to prove the Linkage Theorem.

3.1.1 W_p and Alcoves

Let us denote by $s_{\beta,n}$ for all $\beta \in R$ and $n \in \mathbb{N}$ the affine reflection on $X(T)$ or $X(T) \otimes \mathbb{R}$ with:

$$s_{\beta,n}(\lambda) = \lambda - (\langle \lambda, \beta^\vee \rangle - n)\beta$$

for all λ. Set W_p equal to the group generated by all $s_{\beta,np}$ with $\beta \in R$ and $n \in \mathbb{Z}$. We call W_p the *affine Weyl group* (with respect to prime p). One easily shows that W_p is the semi-direct product of W and the group $p\mathbb{Z}R$ acting by translations on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ using $s_\beta s_{\beta,np}(\lambda) = \lambda - np\beta$ for all λ:

$$W_p \cong W \ltimes p\mathbb{Z}R$$

We shall consider the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ of W_p on $X(T)$ and $X(T) \otimes \mathbb{R}$. So we regard $s_{\beta,np}$ as a reflection with respect to the hyperplane:

$$\{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \beta^\vee \rangle = np\}.$$

The reflection group W_p acting on $X(T) \otimes \mathbb{R}$ defines a system of facets. A facet(for W_p) is a non-empty set of the form

$$\begin{aligned} F = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p \text{ for all } \alpha \in R_0^+(F), \\ (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p \text{ for all } \alpha \in R_1^+(F)\} \end{aligned} \tag{3.1}$$

for suitable integers $n_\alpha \in \mathbb{Z}$ and for a disjoint decomposition $R^+ = R_0^+(F) \cup R_1^+(F)$. Then the closure $\bar{F}$ of F is equal to:

$$\begin{aligned} F = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p \text{ for all } \alpha \in R_0^+(F), \\ (n_\alpha - 1)p \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq n_\alpha p \text{ for all } \alpha \in R_1^+(F)\} \end{aligned} \tag{3.2}$$

We call

$$\begin{aligned} \hat{F} = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p \text{ for all } \alpha \in R_0^+(F), \\ (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle \leq n_\alpha p \text{ for all } \alpha \in R_1^+(F)\} \end{aligned} \tag{3.3}$$

the upper closure of F. Obviously $\hat{F} \subset \bar{F}$ and both $\hat{F}$ and $\bar{F}$ are unions of facets.

Any facet F is an open subset in an affine subspace of $X(T) \otimes \mathbb{R}$, more precisely, in $\{\lambda \mid \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p$ for all $\alpha \in R_0^+(F)\}$ using the notations from above. The codimension of this subspace is equal to $\dim(\sum_{\alpha \in R_0^+(F)} \mathbb{R}\alpha 0$.

A facet F is called an *alcove* if $R_0^+(F) = \varnothing$. Or, equivalently, if F is an open subset of $X(T) \otimes \mathbb{R}$ of the union of all reflection hyperplanes, i.e., of

$$X(T) \otimes_{\mathbb{Z}} \mathbb{R} - \bigcup_{\alpha \in R^+} \bigcup_{n \in \mathbb{Z}} \{\lambda | \langle \lambda + \rho, \alpha^\vee \rangle = np\}$$

The union of the closures of the alcoves is $X(T) \otimes \mathbb{R}$. Any $\lambda \in X(T) \otimes \mathbb{R}$ and any facet belongs to the upper closure of exactly one alcove.

We have the following propositions for the action of W_p on alcoves.

Proposition 3.1.1. *If F is an alcove for W_p, then its closure $\bar{F}$ is a fundamental domain for W_p operating on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The group W_p permutes the alcoves simply transitively.*

Proposition 3.1.2. *If F is an alcove for W_p, then $\bar{F} \cap X(T)$ is a fundamental domain for W_p operating on $X(T)$.*

There is one alcove (*standard alcove*) which is always chosen as the fundamental domain: Set

$$C = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+\}.$$

As $\{\alpha^\vee | \alpha \in S\}$ is linearly independent and as $\beta^\vee \in \sum_{\alpha \in \mathbb{N}\alpha^\vee}$ for all $\beta \in R^+$, it is elementary to show that $C \neq 0$, hence C is an alcove.

3.1.2 Relations for Alcoves

Let us introduce an order relation $\uparrow$ on $X(T)$. We want $\lambda \uparrow \mu$ to hold if and only if there are $\mu_1, \ldots, \mu_r \in X(T)$ and reflections $s_1, \ldots, s_{r+1} \in W_p$ with

$$\lambda \leq s_1 \cdot \lambda = \mu_1 \leq s_2 \cdot \mu_1 = \mu_2 \leq \cdots \leq s_r \mu_{r-1} = \mu_r \leq s_{r+1} \mu_r = \mu,$$

or if $\mu = \lambda$. We have, obviously,

$$\lambda \uparrow \mu \Rightarrow \lambda \leq \mu \text{ and } \lambda \in W_p \cdot \mu.$$

We can also define an order relation $\uparrow$ on the set of alcoves for W_p. Let C_1, C_2 be two alcoves. For any $\lambda_1 \in X(T) \cap C_1$ there is a unique $\lambda_2 \in C_2 \cap W_p \lambda$, as C_2 is a fundamental domain for W_p. Then we want the following to hold:

$$\lambda_1 \uparrow \lambda_2 \Leftrightarrow C_1 \uparrow C_2.$$

It is elementary to show that the left hand side does not depend on the choice of λ_1 in $C_1 \cap X(T)$. As we are not certain that $C_1 \cap X(T) \neq \varnothing$, we go to another definition.

If $\alpha \in R^+$ and $n \in \mathbb{Z}$, then either $\langle \lambda + \rho, \alpha^\vee \rangle < np$ for all $\lambda \in C_1$, or $\langle \lambda + \rho, \alpha^\vee \rangle > np$ for all $\lambda \in C_1$. In the first case set $C_1 \uparrow s_{\alpha,np} \cdot C_1$, and in the second one set $s_{\alpha,np} C_1 \uparrow C_1$. This is a definition of $s \cdot C_1 \uparrow C_1$ for any reflection s in W_p and any alcove C_1. Now $C_1 \uparrow C_2$ if and only if there are reflections $s_1, \ldots, s_{r+1}$ in W_p with

$$C_1 \uparrow s_1 \cdot C_1 \uparrow (s_2 s_1) \cdot C_1 \uparrow \ldots \uparrow (s_{r+1} \ldots s_1) \cdot C_1 = C_2$$

or if $C_1 = C_2$. It is then obvious that we get an order relation.

Now we give the following proposition which is very important in proving the linkage principle. To prove this proposition only concerns elementary calculation of inner products and reflections, but it is really long and tedious. So I omit the proof, and people may refer to [J] II chapter 6 for the complete proof.

Proposition 3.1.3. *Let $\lambda \in X(T)$ with $\langle \lambda + \rho, \beta^\vee \rangle \geq 0$ for all $\beta \in S$. Let $\alpha \in R^+$ and $n \in \mathbb{N}$ with $0 < np < \langle \lambda + \rho, \alpha^\vee \rangle$. Let $w \in W$ with $\langle w s_{\alpha,np}(\lambda + \rho), \beta^\vee \rangle \geq 0$ for all $\beta \in S$. Then $w s_{\alpha,np} \cdot \lambda \uparrow \lambda$ and $w s_{\alpha,np} \cdot \lambda < \lambda$.*

3.2 Linkage Theorem

3.2.1 Higher Cohomology Groups

We have known much about $H^0(\lambda)$ in chapter 2. Now we will study its higher cohomologies for their structures and relations.

Let $I \subset S$ and let $P = P_I$, following the notations of subsection 1.1.3. As G/P_I is projective, proposition 0.1.3 implies:

Proposition 3.2.1. *If M is a finite dimensional P-module, then each $R^i \operatorname{Ind}_P^G M$ is finite dimensional G-module.*

Note that the dimension $n(P) = \dim G/P$ is equal to the number of roots α with $U_\alpha \not\subset P_I = P$, hence:

$$n(P) = |R^+ - R_I|.$$

So we have:

Proposition 3.2.2. $R^i \operatorname{Ind}_P^G = 0$ *for all* $i > n(P)$.

On the other hand, it is not hard to prove that:

Proposition 3.2.3. *The tangent sheaf on G/P is $\mathscr{L}(\operatorname{Lie} G/\operatorname{Lie} P)$*

So the canonical sheaf is:

$$\begin{aligned} \omega_{G/P} &= \bigwedge^{n(P)} \mathscr{L}(\operatorname{Lie} G/\operatorname{Lie} P)^\vee \\ &\cong \mathscr{L}(\bigwedge^{n(P)} (\operatorname{Lie} G/\operatorname{Lie} P)^*). \end{aligned} \tag{3.4}$$

The weights of T on $\operatorname{Lie} G/\operatorname{Lie} P$ are the $\alpha \in R^+ - R_I$. Therefore $\bigwedge^{n(P)}(\operatorname{Lie} G/\operatorname{Lie} P)^*$ is the one dimensional P-module corresponding to the weight $-2\rho_P$ where

$$\rho_P = \frac{1}{2} \sum_{\alpha \in R^+ - R_I} \alpha \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.5}$$

Hence:

$$\omega_{G/P} = \mathscr{L}(-2\rho_P),$$

In particular

$$\omega_{G/B} = \mathscr{L}(-2\rho),$$

so we have the following *Serre duality*:

Proposition 3.2.4. *For any finite dimensional P-module M, the G-module $R^i \operatorname{Ind}_P^G M$ is dual to $R^{n(P)-i} \operatorname{Ind}_P^G (M^* \otimes (-2\rho_P))$. In particular we have, for all $\lambda \in X(T)$,*

$$H^i(\lambda) \cong H^{n-i}(-(\lambda + 2\rho))^*,$$

where $n = |R^+|$.

As $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ and $\langle \rho, \beta^\vee \rangle = 1$ for all $\beta \in S$. So we see $s_\beta \rho - \rho \in \mathbb{Z}R$ for all $\beta \in S$, hence $w\rho - \rho \in \mathbb{Z}R \subset X(T)$ for all $w \in W = \langle s_\beta \mid \beta \in S \rangle$. This shows that if we define:

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

for all $w \in W$ and $\lambda \in X(T) \otimes \mathbb{R}$, then $\lambda \in X(T)$ implies $w \cdot \lambda \in X(T)$ for all $w \in W$. Recall that we have:

$$s_\alpha \cdot \lambda = s_\alpha \lambda - \alpha = \lambda - (\langle \lambda, \alpha^\vee \rangle + 1)\alpha.$$

Set $P(\alpha) = P_{\{\alpha\}}$.

Proposition 3.2.5. *Let $\alpha \in S$ and $\lambda \in X(T)$.*

1. *The unipotent radical of $P(\alpha)$ acts trivially on $R^i \operatorname{Ind}_B^{P(\alpha)} \lambda$ for all i.*
2. *If $\langle \lambda, \alpha^\vee \rangle = -1$, then $R^i \operatorname{Ind}_B^{P\alpha} \lambda = 0$ for all i*
3. *If $\langle \lambda, \alpha^\vee \rangle = r \geq 0$, then $R^i \operatorname{Ind}_B^{P(\alpha)} \lambda = 0$ for all $i \neq 0$ and $\operatorname{Ind}_B^{P(\alpha)} \lambda$ has a basis $v_0, v_1, \ldots, v_r$ such that for all $i (0 \leq i \leq r)$ and A:*

$$tv_i = (\lambda - i\alpha)(t)v_i \qquad \text{for all } t \in T(A), \tag{3.6}$$

$$x_\alpha(a)v_i = \sum_{j=0}^{i} \binom{i}{j} a^{i-j} v_j \qquad \text{for all } a \in A, \tag{3.7}$$

$$x_{-\alpha}(a)v_i = \sum_{j=i} r \binom{r-i}{r-j} a^{j-i} v_j \qquad \text{for all } a \in A. \tag{3.8}$$

4. *If $\langle \lambda, \alpha^\vee \rangle \leq -2$, then $R^i \operatorname{Ind}_B^{P(\alpha)} \lambda = 0$ for all $i \neq 1$ and $R^1 \operatorname{Ind}_B^{P(\alpha)} \lambda$ has a basis $v'_0, v'_1, \ldots, v'_r$ where $r = -\langle \lambda, \alpha^\vee \rangle - 2$ such that for all $i (0 \leq i \leq r)$ and A:*

$$tv'_i = (s_\alpha \cdot \lambda - i\alpha)(t)v'_i \qquad \text{for all } t \in T(A), \tag{3.9}$$

$$x_\alpha(a)v'_i = \sum_{j=0}^{i} \binom{r-j}{r-i} a^{i-j} v'_j \qquad \text{for all } a \in A, \tag{3.10}$$

$$x_{-\alpha}(a)v'_i = \sum_{j=i}^{r} \binom{j}{i} a^{j-i} v'_j \qquad \text{for all } a \in A. \tag{3.11}$$

Proof. As the unipotent radical of $P(\alpha)$ is contained in U and as U operates trivially on $k_\lambda = \lambda$, the (1) comes from proposition 0.1.7. Furthermore, we may form the quotients by this unipotent radical. So we may assume that $G = P(\alpha)$ has semi-simple rank 1. Then G is a factor group of some group of the form $\mathbf{SL}_2 \times T'_1$ with T'_1 a torus. This can be done in such a way that T'_1 is the image of $\{$diagonal matrices in $\mathbf{SL}_2\} \times T'_1$ and such that x_α and $x_{-\alpha}$ come form the "standard" root homomorphism in $\mathbf{SL}_2$. So by proposition 0.1.7 again, we may assume $G = \mathbf{SL}_2 \times T'_1$.

Now proposition 0.1.8 implies that $R^i \operatorname{Ind}_B^G$ is isomorphic to $R^i \operatorname{Ind}_{B \cap \mathbf{SL}_2}^{\mathbf{SL}_2}(\lambda|_{B \cap \mathbf{SL}_2})$ as an $\mathbf{SL}_2$-module. On the other hand, T'_1 acts on each $R^i \operatorname{Ind}_B^G \lambda$ through the restriction of λ to T'_1 by proposition 1.2.9. This is compatible with our results as α vanishes on T'_1 and as $s_\alpha \cdot \lambda - \lambda \in \mathbb{Z}\alpha$. We may therefore assume $G = \mathbf{SL}_2$.

Now $\operatorname{Ind}_B^G \lambda$ is described in subsection 1.2.5. The character denoted by ω there maps any $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ to a, hence $\langle \omega, \alpha^\vee \rangle = 1$ and $X(T) = \mathbb{Z}\omega$. So any $\lambda \in X(T)$ is equal to $\langle \lambda, \alpha^\vee \rangle \omega$. For $\langle \lambda, \alpha^\vee \rangle = r \geq 0$ we get as $\operatorname{Ind}_B^G \lambda$ the r^{th} symmetric power of the dual of the natural representation. Taking the basis consisting of all monomials and changing some signs, we get the formula (3.6), (3.7), (3.8) in 3. If $\langle \lambda, \alpha^\vee \rangle < 0$, then $H^0(\lambda) = \operatorname{Ind}_B^G \lambda = 0$.

Because of proposition 3.2.2 we have $R^i \operatorname{Ind}_B^G = 0$ for $i \neq 0, 1$. So we have to look only at $R^1 \operatorname{Ind}_B^G$ and can now use Serre duality:

$$H^1(\lambda) \cong H^0(-(\lambda + 2\rho))^*.$$

If $\langle -(\lambda + 2\rho), \alpha^\vee \rangle < 0$, i.e., if $\langle \lambda, \alpha^\vee \rangle > -2$, this is zero. For $\langle \lambda, \alpha^\vee, \rangle \leq -2$, we get (3.9), (3.10), (3.11) using the dual basis up to sign changes. □

3.2.2 The Proof for Linkage Theorem

We will prove the Linkage Theorem in this part.

Lemma 3.2.6. *Let $\alpha \in S$ and $\lambda \in X(T)$ with $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$. Then there are finite dimensional B-modules $N_0^\alpha(\lambda)$, $N_1^\alpha(\lambda)$, $N_2^\alpha(\lambda)$ with :*

$$\operatorname{ch} N_1^\alpha(\lambda) = \operatorname{ch} N_2^\alpha(\lambda) = \sum_n e(s_\alpha \cdot \lambda + np\alpha), \tag{3.12}$$

where the sum is over all $n \in \mathbb{N}$ with $0 < np < \langle \lambda + \rho, \alpha^\vee \rangle$ such that there are long exact sequence of G-modules:

$$\ldots \to H^i(s_\alpha \cdot \lambda) \to H^{i-1}(\lambda) \to H^i(N_0^\alpha(\lambda)) \to H^{i+1}(s_\alpha \cdot \lambda) \to \ldots \tag{3.13}$$

and

$$\ldots \to H^i(N_1^\alpha(\lambda)) \to H^i(N_0^\alpha(\lambda)) \to H^{i-1}(N_2^\alpha(\lambda)) \to H^{i+1}(N_1^\alpha(\lambda)) \to \ldots \tag{3.14}$$

Proof. Let us assume at first $\rho \in X(T)$. We shall write $H_\alpha^0(M) = \operatorname{Ind}_B^{P(\alpha)} M$ for any B-module M.Set $r = \langle \lambda + \rho, \alpha^\vee \rangle$. We have described $H_\alpha^0(\lambda + \rho)$ quite explicitly in proposition 3.2.5: There is a basis $(v_i)_{p \leq i \leq r}$ such that T acts on v_i through $\lambda + \rho - i\alpha$, we know how U_α, $U_{-\alpha}$ operate and that each $U_{-\beta}$ with $\beta \in R^+$, $\beta \neq \alpha$ acts trivially.

It is then clear that $H_\alpha^0(\lambda + \rho)^- = \sum_{i=1}^r kv_i$ and that $kv_r \cong s_\alpha(\lambda + \rho)$ are B-submodules of $H_\alpha^0(\lambda + \rho)$. Set $H_\alpha^0(\lambda + \rho)^m = H_\alpha^0(\lambda + \rho)^- / kv_r$. This yields exact sequences of B-modules that we tensor with $-\rho$, and thus get:

$$0 \to H_\alpha^0(\lambda + \rho)^- \otimes (-\rho) \to H_\alpha^0(\lambda + \rho) \otimes (-\rho) \to \lambda \to 0 \tag{3.15}$$

and

$$0 \to s_\alpha \cdot \lambda \to H_\alpha^0(\lambda + \rho)^- \otimes (-\rho) \to H_\alpha^0(\lambda + \rho)^m \otimes (-\rho) \to 0 \tag{3.16}$$

Suppose for the moment that $r \geq 2$. We can then also form $H_\alpha^0(\lambda + \rho - \alpha)$. Let us denote the corresponding basis by $\tilde{v}_i$ $(0 \leq i \leq r - 2)$. The formulas in proposition 3.2.5 imply that the map $v_i \to (r - i)\tilde{v}_{i-1}$ induces a homomorphism of B-modules $H_\alpha^0(\lambda + \rho)^m \to H_\alpha^0(\lambda + \rho - \alpha)$, hence also $H_\alpha^0(\lambda + \rho)^m \otimes (-\rho) \to H_\alpha^0(\lambda + \rho - \alpha) \otimes (-\rho)$. Let us denote the kernel, cokernel, and image of this homomorphism by $N_1^\alpha(\lambda)$, $N_2^\alpha(\lambda)$, $N_3^\alpha(\lambda)$. We have, therefore, short exact sequences of B-modules:

$$0 \to N_1^\alpha(\lambda) \to H_\alpha^0(\lambda + \rho)^m \otimes (-\rho) \to N_3^\alpha(\lambda) \to 0 \tag{3.17}$$

and

$$0 \to N_3^\alpha(\lambda) \to H_\alpha^0(\lambda + \rho - \alpha) \otimes (-\rho) \to N_2^\alpha(\lambda) \to 0. \tag{3.18}$$

Furthermore, the explicit description of the map shows that $\operatorname{ch} N_1^\alpha(\lambda) = \operatorname{ch} N_2^\alpha(\lambda)$.

We have $\langle -\rho, \alpha^\vee \rangle = -1$, hence $R^\bullet \operatorname{Ind}_B^{P(\alpha)}(-\rho) = 0$ by proposition 3.2.5, hence by using proposition 0.1.6

$$R^\bullet \operatorname{Ind}_B^{P(\alpha)}(M \otimes (-\rho)) \cong M \otimes R^\bullet \operatorname{Ind}_B^{P(\alpha)}(-\rho) = 0$$

For any $P(\alpha)$-module M, hence by proposition 0.1.5, we have finally $R^\bullet \operatorname{Ind}_B^G(M \otimes (-\rho)) = 0$. We can apply this to $M = H_\alpha^0(\lambda + \rho)$ and to $M = H_\alpha^0(\lambda + \rho - \alpha)$. Therefore formula (3.15) and (3.18) above give isomorphisms for each i:

$$H^i(\lambda) \cong H^{i+1}(H_\alpha^0(\lambda + \rho)^- \otimes (-\rho)) \tag{3.19}$$

$$H^i(N_2^\alpha(\lambda)) \cong H^{i+1}(N_3^\alpha(\lambda)). \tag{3.20}$$

We can apply Ind_B^G to formula (3.16), (3.17), and get long exact sequence. They contain the right hand sides of 3.19 resp. 3.20 and we use 3.19 and 3.20 to replace these terms by left hand sides. Then we get 3.13 and 3.14 with $N_0^\alpha(\lambda) = H_\alpha^0(\lambda \otimes \rho)^m \otimes (-\rho)$.

For $r \leq 1$ we set $N_i^\alpha(\lambda) = 0$ for $i = 0, 1, 2$. This is certainly compatible with 3.12 and 3.14. If $r = 0$, then $\langle \lambda, \alpha^\vee \rangle = -1$ and $s_\alpha \cdot \lambda = \lambda$, hence $H^\bullet(\lambda) = H^\bullet(s_\alpha \lambda) = 0$ by proposition 3.2.5, and 3.13 holds. If $r = 1$, then $H_\alpha^0(\lambda + \rho)^m = 0$, hence 3.13 follows from 3.16 and 3.19 as above.

This proof the lemma in case $\rho \in X(T)$. In general there is a central extension $G' \to G$ with a split maximal torus $T' \to T$ such that $\rho \in X(T') \subset X(T)$. We can then carry out the constructions as above for T'. Let $B' \subset G'$ be the inverse image of B. The B'-modules $N_0^i(\lambda)$ have all their weights in $X(T)$, hence the kernel Z of $B' \to B$ acts trivially and the $N_0^i(\lambda)$ are B-modules in a natural way. Using $R^i \operatorname{Ind}_B^G N \cong R^i \operatorname{Ind}_{B'}^{G'} N$ for any B-module N, we see that we have 3.13 and 3.14 for G. □

Proposition 3.2.7. *(*The Strong Linkage Principle*) Let $\lambda \in X(T)$ with $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$ for all $\alpha \in R^+$ and $\mu \in X(T)_+$. If $L(\mu)$ is a composition factor of some $H^i(w \cdot \lambda)$ with $w \in W$ and $i \in \mathbb{N}$, then $\mu \uparrow \lambda$.*

We shall prove this result via induction on λ for $\leq$ in the following context, always assuming λ, μ as in the proposition. The first step is the point where the induction hypothesis enters. We use the notations $N_i^\alpha(\lambda')$ as in lemma 3.2.6

Lemma 3.2.8. *Let $\alpha \in S$ and $w \in W$ with $\langle w(\lambda + \rho), \alpha^\vee \rangle \geq 0$. If $L(\mu)$ is a composition factor of some $H^i(N_0^\alpha(w \cdot \lambda))$, then $\mu \uparrow \lambda$.*

Proof. Because of lemma 3.2.6, it is enough to prove the corresponding result for $N_1^\alpha(w \cdot \lambda)$ and $N_2^\alpha(w \cdot \lambda)$, hence for each composition factor of these B-modules. Also due to lemma 3.2.6, we have to look at all $H^i(\lambda_1)$ with λ_1 of the form $\lambda_1 = s_\alpha w \cdot \lambda + np\alpha$ for some $n \in \mathbb{N}$ with $0 < np < \langle w(\lambda + \rho), \alpha^\vee \rangle = \langle \lambda + \rho, w^{-1}(\alpha)^\vee \rangle$. By our assumption on λ we must have $\beta = w^{-1}(\alpha) \in R^+$, and we can write $\lambda_1 = s_\alpha w \cdot (\lambda - np\beta)$. There are $\lambda_2 \in X(T)$ and $w' \in W$ with $\lambda_1 = w' \cdot \lambda_2$ and $\langle \lambda_2 + \rho, \gamma^\vee \rangle \geq 0$ for all $\gamma \in R^+$. As $\lambda_2 \in W \cdot \lambda - np\beta$ we get from proposition 3.1.3 that $\lambda_2 \uparrow \lambda$ and $\lambda_2 < \lambda$. We can, therefore, apply the induction hypothesis to λ_2 and composition factor $L(\mu)$ of some $H^i(w' \cdot \lambda_2)$. We get $\mu \uparrow \lambda_2$, hence $\mu \uparrow \lambda$ and $\mu \leq \lambda_2 < \lambda$. □

Proposition 3.2.9. *Let $i \in \mathbb{N}$ and $w \in W$ with $l(w) \neq i$. If $L(\mu)$ is a composition factor of $H^i(w \cdot \lambda)$, then $\mu \uparrow \lambda$ and $\mu < \lambda$.*

Proof. Let us suppose at first $i < l(w)$ and use induction on i. If $i = 0$, then $H^0(w \cdot \lambda) \neq 0$, hence $w \cdot \lambda \in X(T)_+$. Therefore $w \cdot \lambda = \lambda$, as $\{\lambda' \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} | \langle \lambda' + \rho, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}$ is a fundamental domain for the "dot " action of W on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Futhermore $\lambda \in X(T)_+$, hence λ has a trivial stabilizer in W(under dot action), hence $w \cdot \lambda = \lambda$ implies $w = 1$ and $l(w) = 0 = i$. This is a contradiction and settles the case $i = 0$.

Suppose $i > 0$. If $\langle w(\lambda + \rho), \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$, then $H^i(w \cdot \lambda) = 0$ by . Therefore we can find some $\alpha \in S$ with $\langle w(\lambda+\rho), \alpha^\vee \rangle < 0$ we now apply lemma 3.2.6 to $s_\alpha w \dot{\lambda}$ instead of λ. And we get that $L(\mu)$ is either a composition factor of $H^{i-1}(N_0^\alpha(s_a l w \cdot \lambda))$, or of $H^{i-1}(s_\alpha w \cdot \lambda)$. In the first case, we apply lemma 3.2.8, in the second one we use induction over i as $l(s_\alpha w) = l(w) - 1 > i - 1$.

This settles the case $i < l(w)$. The case $i > l(w)$ follows either using Serre duality, or by descending induction. □

Proposition 3.2.10. *Suppose $\lambda \in X(T)_+$. Then $L(\lambda)$ is a composition factor with multiplicity one of each $H^{l(w)}(w \cdot \lambda)$ with $w \in W$. Any composition factor $L(\mu)$ of $H^{l(w)}(w \cdot \lambda)$ satisfies $\mu \uparrow \lambda$.*

Proof. Suppose that either $\mu \uparrow \lambda$ does not hold or that $\mu = \lambda$.

For any $w \in W$ and any $\alpha \in S$ with $l(s_\alpha w) = l(w) + 1$ we have by lemma 3.2.6 an exact sequence:

$$H^{l(w)}(N_0^\alpha(w \cdot \lambda)) \to H^{l(w)+1}(s_\alpha w \cdot \lambda) \to H^{l(w)}(w \cdot \lambda) \to H^{l(w)+1}(N_0^\alpha(w \cdot \lambda)). \tag{3.21}$$

lemma 3.2.8 implies that $L(\mu)$ is not a composition factor of any $H^i(N_0^\alpha(w \cdot \lambda))$. It is therefore not a composition factor of the kernel or of the cokernel of the homomorphism $H^{l(w)+1}(s_\alpha w \cdot \lambda) \to H^{l(w)}(w \cdot \lambda)$.

We can choose a sequence $w'_0, w'_1, \ldots, w'_n \in W$ where $n = |R^+|$, such that $l(w'_i) = i$ for all i (hence $w'_0 = 1$, $w'_n = w_0$), such that there are simple roots of $l(w)$ with $w'_i = s_{\alpha_i} w'_{i-1}$ for all $1 \leq i \leq n$, and such that $w'_{l(w)} = w$.

By definition of $l(w)$ there are $\alpha_i \in S$ with $w = s_{\alpha_{l(w)}} \cdots s_{\alpha_2} s_{\alpha_1}$. Set $w'_i = s_{\alpha_i} \ldots s_{\alpha_1}$ for $i \leq l(w)$. Then $l(w'_i) = i$, because $l(w'_i) < i$ implies $l(w) = l(s_{\alpha_{l(w)}} \ldots s_{\alpha_{i+1}} w'_i) < l(w)$. We have $l(w w_0^{-1}) = l(w_0) - l(w^{-1}) = n - l(w)$. Therefore there are $\alpha_i \in S$ with $w_0 w^{-1} = s_{\alpha_n} \ldots s_{\alpha_{l(w)+1}}$. Then take $w'_i = s_{\alpha_i} \ldots s_{\alpha_{l(w)+1}} w$ for all $i > l(w)$. We have $l(w'_i) = i$ as $l(w'_i) < i$ implies $l(w_0) < n$.

This sequence $w'_0, w'_1, \ldots, w'_n$ leads, by applying the argument as above to each (w_i, α_{i+1}) instead of (w, α), to a sequence of homomorphisms of G-modules:

$$H^n(w'_n \cdot \lambda) \to H^{n-1}(w'_{n-1} \cdot \lambda) \to \ldots \to H^1(w'_1 \cdot \lambda) \to H^0(w'_0 \cdot \lambda). \tag{3.22}$$

From the exact sequence we see that $L(\mu)$ does not occur in the kernel or cokernel of any of the maps $H^{i+1}(w'_{i+1} \cdot \lambda) \to H^i(w'_i \cdot \lambda)$. Therefore it occurs in each $H^i(w'_i \cdot \lambda)$ with the same multiplicity as in the image M of the composed map:

$$H^n(w_0 \cdot \lambda) = H^n(w'_n \cdot \lambda) \to H^0(w'_0 \cdot \lambda) = H^0(\lambda). \tag{3.23}$$

As this applies in particular to $\mu = \lambda$ and as $L(\lambda)$ is composition factor of $H^0(\lambda)$, it is also one of M, hence $m \neq 0$. We have by Serre duality:

$$H^n(w_0 \cdot \lambda) = H^n(w_0\lambda - 2\rho) \cong H^0(-w_0\lambda)^* \cong V(\lambda). \tag{3.24}$$

Hence for homomorphic image M:

$$M/\operatorname{rad}_G M \cong L(\lambda). \tag{3.25}$$

On the other hand, as $M \subset H^0(\lambda)$, we have

$$\operatorname{soc}_G M \cong L(\lambda). \tag{3.26}$$

As $L(\lambda)$ occurs with multiplicity one in $H^0(\lambda)$, hence in M, this implies $M = L(\lambda)$.

So the only $L(\mu)$ as above, which is a composition factor of some $H^{l(w)}(w \cdot \lambda)$ is $L(\lambda)$ and it occurs exactly once. □

Corollary 3.2.11. (The Linkage Principle) *Let* λ, $\mu \in X(T)_+$. *If* $\operatorname{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$, *then* $\lambda \in W_p \cdot \mu$.

Proof. Because of proposition 1.2.12, we may assume $\mu \not> \lambda$. Therefore by corollary 1.2.15, we have $[H^0(\lambda) : L(\mu)] \neq 0$. Hence $\mu \in W_p \cdot \lambda$ by strong linkage principle. □

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