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Moduli Spaces of p-divisible Groups and Period Morphisms

Master's thesis, defended on June 22, 2009

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Contents

Introduction

In their book [33] Rapoport and Zink fix an isocrystal (D, φ_D) over $\overline{\mathbb{F}_p}$ and consider the partial flag variety $\breve{\mathcal{F}}$ over $K_0 := W(\overline{\mathbb{F}_p})[1/p]$ parametrizing filtrations of D with fixed Hodge-Tate weights. They show that the weakly admissible locus $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ (the period space) is a rigid analytic subspace of $\breve{\mathcal{F}}$. They conjecture the existence of a rigid analytic subspace $(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}}$ of $\breve{\mathcal{F}}^{\mathrm{rig}}$, an étale morphism $(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}} \to (\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ of rigid analytic spaces which is bijective on rigid analytic points, and of an interesting local system of $\mathbb{Q}_p$-vector spaces on $(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}}$, see Conjecture 3.3.9. A.J. de Jong [29] pointed out that to study local systems it is best to work in the category of Berkovich spaces rather than rigid analytic spaces.

If the Hodge-Tate weights all are 0 and 1, Rapoport and Zink consider a moduli problem of p-divisible groups and show that it is representable by a formal scheme $\breve{\mathcal{M}}$. We give the proof in Chapter 2. They also construct a morphism called the *period morphism* of rigid analytic spaces from the generic fiber $\breve{\mathcal{M}}^{\mathrm{rig}}$ of $\breve{\mathcal{M}}$ to the period space. The period morphism is étale and surjective on rigid points. However, in order to determine the precise image of the period morphism, one should look at Berkovich spaces again.

The aim of this thesis is to understand Urs Hartl's construction [25] of an admissible locus $\breve{\mathcal{F}}_b^{a}$ in the case where the Hodge-Tate weights are 0 and 1. The first main theorem is the following

Theorem 0.0.1. *The set $\breve{\mathcal{F}}_b^{a}$ is an open $\breve{E}$-analytic subspace (in the sense of Berkovich, see Definition 2.3.19 (i)) of $\breve{\mathcal{F}}^{\mathrm{an}}$, where $\breve{\mathcal{F}}^{\mathrm{an}}$ is the Berkovich space associated to $\breve{\mathcal{F}}$.*

Moreover, Hartl [25] and Faltings [16] show that the period morphism factors through this admissible locus and is surjective on analytic points. This is our second main theorem.

Theorem 0.0.2. *The period morphism $\breve{\pi}^{\mathrm{an}} : \breve{\mathcal{M}}^{\mathrm{an}} \to \breve{\mathcal{F}}^{\mathrm{an}}$ factors through $\breve{\mathcal{F}}_b^{a}$ and surjective on analytic points of $\breve{\mathcal{F}}_b^{a}$.*

We will explain that in the case where the Hodge-Tate weights are 0 and 1 the rational Tate module of the universal p-divisible group on $\breve{\mathcal{M}}^{\mathrm{an}}$ gives conjecturally the answer to Rapoport-Zink's conjecture. We will try to explain the necessary background for these results in this thesis.

Organization of thesis

This thesis is organized as follows.

In Chapter 1, we define p-divisible groups and recall Grothendieck-Messing's deformation theory which are necessary in Rapoport-Zink's construction of p-adic period mappings. The main reference is Messing [32].

In Chapter 2, first we introduce the moduli spaces of p-divisible groups and prove its representability. Then we briefly recall the theory of rigid analytic geometry before defining period morphisms. The main reference of this chapter is Rapoport-Zink [33].

In Chapter 3, we introduce the weakly admissible locus of certain flag varieties and state precisely the conjecture of Rapoport-Zink. This is also from Rapoport-Zink [33].

In Chapter 4, the final chapter, we follow Hartl's construction of the admissible locus of a p-adic period space possessing period morphisms. This is rather technical. The main references are Hartl [25] and [26], Faltings [16].

Acknowledgements

I am grateful to my thesis supervisor Prof. Jean-François Dat (Université Pierre et Marie Curie) for introducing me to this interesting subject and for innumerable helpful discussions and suggestions, to Prof. Bas Edixhoven (Universiteit Leiden) and my tutor Robin de Jong (Universiteit Leiden) for their constant care, support and suggestion throughout my stay in Leiden.

I would like to thank the ALGANT committee for giving me such an wonderful opportunity to learn higher Mathematics in Europe.

I am also thankful to my friends: CHEN Miaofen, LU Chengyuan, WANG Shanwen... for their mathematical discussions and generous help.

Chapter 1

Grothendieck-Messing Deformation Theory

In this chapter, we explain the definitions and basic properties of p-divisible groups (or Barsotti-Tate groups in the terminology of [32]). We refer to [32] for details. These are necessary in the Rapoport-Zink's construction of period mappings for p-divisible groups.

1.1 p-divisible Groups

We fix a prime number p. Let S be a general base scheme, we identify the schemes X over S with the f.p.p.f. sheaves they represent. We say G is an *S-group* if G is a commutative f.p.p.f. sheaf of groups on the site $\mathrm{Sch}(S)$.

Definition 1.1.1. (Grothendieck) An S-group G is said to be a *p-divisible group on S* if it satisfies the following three properties:

(i) G is p-divisible, i.e. the morphism $p : G \to G$ is an epimorphism.

(ii) G is p-torsion, i.e. $G = \varinjlim_n G(n)$, where $G(n) = \ker(p^n : G \to G)$.

(iii) The S-groups $G(n)$ are representable by finite locally free S-group schemes.

Remark 1.1.2. (1) In fact, one can replace condition (iii) above by

$$(iii)' \text{ The group } G(1) \text{ is a finite locally free } S\text{-group scheme,}$$

as for every n, $G(n)$ is a multiple extension of groups isomorphic to $G(1)$.

(2) Since $G(1)$ is finite locally free over S, it follows from the elementary theory of finite group schemes over a field that the rank of $G(1)$ is of the form p^h, where $h = \mathrm{ht}(G)$ is a locally constant function on S with values in $\mathbb{N}$. Then for every n, the group $G(n)$ has rank p^{nh}. The integer h (whenever it is a constant) is called the *height* of the p-divisible group G.

We have an equivalent definition by Tate.

Definition 1.1.3. (Tate) A p-divisible group on S is an inductive system $(G_n)_{n\in\mathbb{N}}$ of finite locally free S-group schemes such that:

(i) $G_n = G_{n+1}(n)$

(ii) The rank of the fiber of $G(n)$ at s is $p^{nh(s)}$, where h is a locally constant function on S.

The equivalence of Definition 1.1.1 and 1.1.3 is given by:

Grothendieck's p-divisible group $G \rightsquigarrow$ Tate's p-divisible group $(G_n)_{n\in\mathbb{N}}$ where $G_n = G(n)$

Tate's p-divisible group $(G_n)_{n\in\mathbb{N}} \rightsquigarrow$ Grothendieck's p-divisible group $G = \varinjlim_{n\in\mathbb{N}} G_n$

The notion of morphism between two p-divisible groups is easily defined. In Grothendieck's definition any map $f : G \to H$ where G and H are p-divisible groups on S is a morphism of p-divisible groups if f is a morphism of f.p.p.f. sheaves of groups. In Tate's terminology, we require $f = (f_n)_{n\in\mathbb{N}}$ where $f_n : G_n \to H_n$ are morphisms of group schemes and compatible with the transition maps. Therefore all the p-divisible groups on a base scheme S form a category denoted by $pdiv(S)$.

Remark 1.1.4. The category $pdiv(S)$ is not abelian. Indeed, if we consider the multiplication by p from G to itself, it is easy to see that this morphism has both trivial kernel and cokernel in $pdiv(S)$. But it is not an isomorphism, hence $pdiv(S)$ cannot be an abelian category.

Definition 1.1.5. Let $G = (G(n))_{n\in\mathbb{N}}$ be a p-divisible group on S. Since the $G(n)$ are finite locally free S-group schemes, the dual group schemes $G(n)^* = \underline{\mathrm{Hom}}_{S-gr}(G(n), \mathbb{G}_{mS})$ are also finite and locally free. The epimorphism $p : G(n+1) \to G(n)$ gives a monomorphism $p^* : G(n)^* \hookrightarrow G(n+1)^*$. Then the inductive system $(G(n)^*)_{n\in\mathbb{N}}$ with respect to p^* gives a p-divisible group G^* over S (in the sense of Tate). We call G^* the *Cartier dual* of G.

Remark 1.1.6. The assignment $G \mapsto G^*$ gives a duality on the category of p-divisible groups on S.

Proposition 1.1.7. *([32]) The p-divisible groups are stable under base change and extensions. More precisely,*

(i) If $S' \to S$ is a morphism and G is in $pdiv(S)$, then $f^*(G)$ is in $pdiv(S')$.

(ii) If $0 \to G_1 \to G_2 \to G_3 \to 0$ is an exact sequence of S-groups and G_1 and G_3 are in $pdiv(S)$, then G_2 is in $pdiv(S)$ also and $\mathrm{ht}(G_2) = \mathrm{ht}(G_1) + \mathrm{ht}(G_3)$.

Example 1.1.8. (1) The constant formal group $(\mathbb{Q}_p/\mathbb{Z}_p)_S = \varinjlim_n (\frac{1}{p^n}\mathbb{Z}/\mathbb{Z})_S$ is an ind-étale p-divisible group over S.

(2)([32] Chapter I 3.4) Let A be an abelian scheme on S, i.e. a commutative group scheme $f : A \to S$ with f proper, smooth and having geometrically connected fibers. Then $\varinjlim A(n) = \varinjlim(\ker p^n)$ is a p-divisible group of rank $2d$ on S, where d is the relative dimension of A/S.

In the set of morphisms of p-divisible groups we have a particular subset, the isogenies of p-divisible groups.

Definition 1.1.9. Let G and G' be two p-divisible groups over S, a morphism $f : G \to G'$ is called an *isogeny* if it is an f.p.p.f. epimorphism with finite locally free kernel. Two p-divisible groups are called *isogeneous* if there exists such an f.

Proposition 1.1.10. *([19]) Suppose S is connected or quasi-compact. A morphism $f : G \to G'$ between two p-divisible groups over S is an isogeny if and only if there exists a morphism $g : G' \to G$ and an integer N such that $g \circ f = p^N \operatorname{Id}_G$ and $f \circ g = p^N \operatorname{Id}_{G'}$.*

We have a converse to the Definition 1.1.9.

Proposition 1.1.11. *([33] 2.7) Let G be a p-divisible group on S. Let H be a finite locally free S group scheme and $H \hookrightarrow G$ a monomorphism over S. Then the f.p.p.f. sheaf G/H is a p-divisible group.*

The multiplication by p on a p-divisible group is obviously an isogeny. It follows that for p-divisible groups, the group $\operatorname{Hom}_S(G, G')$ is a torsion free $\mathbb{Z}_p$-module. Let $\underline{\operatorname{Hom}}_S(G, G')$ be the Zariski sheaf of germs of morphisms.

Definition 1.1.12. A *quasi-isogeny* of p-divisible groups from G to G' is a global section ρ of the Zariski sheaf $\underline{\operatorname{Hom}}_S(G, G') \otimes_{\mathbb{Z}} \mathbb{Q}$ such that there exists locally an integer n for which $p^n\rho$ is an isogeny. We denote the group of quasi-isogenies by $\operatorname{Qisog}_S(G, G')$.

Quasi-isogenies of p-divisible groups have the following rigidity property.

Theorem 1.1.13. *([1] 2.2.3) Let S' be a closed subscheme of S with locally nilpotent defining sheaf of ideals J. Assume moreover that p is locally nilpotent on S. Then the canonical homomorphism*

$$\operatorname{Qisog}_S(X, Y) \longrightarrow \operatorname{Qisog}_{S'}(X_{S'}, Y_{S'})$$

is bijective.

In the sequel, we shall have to deal with p-divisible groups over formal schemes. Our formal scheme $\mathfrak{X}$ will be adic, locally noetherian (see Chapter II), hence there is a largest ideal of definition $\mathfrak{J}$, and $\mathfrak{X} = \varinjlim_n \mathfrak{X}_n$ where $\mathfrak{X}_n$ is locally written as $\operatorname{Spec}(O_{\mathfrak{X}}/\mathfrak{J}^{n+1})$. In particular $\mathfrak{X}_{\text{red}} = \mathfrak{X}_0$ is locally isomorphic to $\operatorname{Spec}(O_{\mathfrak{X}}/\mathfrak{J})$.

Definition 1.1.14. A *p-divisible group G over $\mathfrak{X}$* is an compatible system of p-divisible groups G_n over $\mathfrak{X}_n$, which means that we have $G_{n+1} \times_{\mathfrak{X}_{n+1}} \mathfrak{X}_n \cong G_n$ for every n.

Proposition 1.1.15. *[32] If $\mathfrak{X} = \operatorname{Spf} A$ is an affine formal scheme, the functor $G \mapsto (G \bmod \mathfrak{J}^n)_{n \in \mathbb{N}}$ induces an equivalence between the category of p-divisible groups over $\operatorname{Spec}(A)$ and the category of p-divisible groups over $\operatorname{Spf}(A)$.*

1.2 Relations with Formal Lie Groups

Definition 1.2.1. Let G be an S-group, for any $k \in \mathbb{N}$ we define a sub f.p.p.f. sheaf $\operatorname{Inf}^k(G)$ of G over S. For each S scheme T, the T sections of $\operatorname{Inf}^k(G)$ is the subset of elements $t \in$

$\Gamma(T, G) = G(T)$ satisfying that there is a covering $\{T_i \to T\}$ for the f.p.p.f. topology and for each T_i a closed subscheme T_i' defined by an ideal whose $k+1$-th power is 0 with the property that $t_{T_i'} \in \Gamma(T_i', G)$ factors through the unit section $e : S \hookrightarrow G$.

Remark 1.2.2. If G is an S-group scheme, $\mathrm{Inf}^k(G)$ is the k-th infinitesimal neighborhood of G along $e : S \hookrightarrow G$ in [13] IV 16.

Definition 1.2.3. Let G be an S-group, G is said to be *formally smooth* if for any affine scheme X and any closed subscheme $i : X_0 \hookrightarrow X$ defined by an ideal I with $I^2 = 0$, any morphism $f_0 : X_0 \to G$ lifts to a morphism (not necessarily unique) $f : X \to G$ such that $f_0 = f \circ i$.

Theorem 1.2.4. *([32] Chapter II 3.3.13) Assume p is locally nilpotent on S, then any p-divisible group on S is formally smooth.*

Definition 1.2.5. An S-group G is a *formal Lie group* if

(i) $G = \varinjlim_k \mathrm{Inf}^k(G)$, i.e. G is ind-infinitesimal,

(ii) G is formally smooth,

(iii) For any integer k, $\mathrm{Inf}^k(G)$ is representable.

One can prove that if G is a formal Lie group then, locally on S, G is of the form $\mathrm{Spf}(O_S[[X_1, \cdots, X_n]])$.

Definition 1.2.6. Let G be an S-group scheme with unit section $e : S \hookrightarrow G$. The O_S-module $\omega_G := e^*\Omega^1_{G/S}$ is called the *differential* of G.

Definition 1.2.7. Let G be a formal Lie group on S with unit section $e : S \hookrightarrow G$, then we define the *differential* of G as $\omega_G := e^*\Omega^1_{\mathrm{Inf}^k(G)/S}$ for sufficiently large k. We note that this definition is independent of the choice of $k >> 0$.

Remark 1.2.8. One can see that ω_G is a finite locally free O_S-module, we call its rank the *dimension* of G.

Theorem 1.2.9. *([32] Chapter II 3.3.18) Let p be locally nilpotent on S and G be a p-divisible group on S. Then $\overline{G} := \varinjlim_k \mathrm{Inf}^k(G)$ is a formal Lie group.*

Remark 1.2.10. In general, $\overline{G}$ is not a p-divisible group, as $\overline{G}(1)$ is not necessarily flat. For example, let E be an elliptic curve over $k[[t]]$, with k a finite field of characteristic p, such that the fibre over k is supersingular and the fibre over $k((t))$ is ordinary. Let G be the p-divisible group of E. Then the $(\overline{G(1)})_k$ has rank p^2, whereas $(\overline{G(1)})_{k((t))}$ has rank p, and hence $\overline{G}$ is not a p-divisible group.

Definition 1.2.11. We define the *differential* ω_G of a p-divisible group G on S (where p is locally nilpotent) as $\omega_{\overline{G}}$. We have $\omega_G = \omega_{G(n)}$ for $n >> 0$, since for any $n >> 0$ there exists an integer n' such that $\mathrm{Inf}^n(\overline{G}) = \mathrm{Inf}^{n'}(G)$. The rank of ω_G is called the *dimension* of G.

1.3 The Crystals Associated to p-divisible Groups

We first recall the classical theory of Dieudonné crystal associated to a p-divisible group G over a perfect field k of characteristic $p > 0$. For the details see [14].

Let $W(k)$ be the Witt ring of k, $K_0 = W(k)_{\mathbb{Q}}$ be the fraction field of $W(k)$. The Frobenius map $x \mapsto x^p$ in k extends to a Frobenius automorphism φ on $W(k)$ and K_0.

Definition 1.3.1. A *crystal over* k is a free $W(k)$-module M of finite rank, together with an injective φ-linear endomorphism F and $pM \subset FM$, i.e. $F : M \to M$ is injective, additive and $F(\lambda x) = \varphi(\lambda)F(x)$ for any $\lambda \in W(k)$, $x \in M$.

Definition 1.3.2. An *isocrystal over* k is a finite dimensional K_0-vector space N equipped with a bijective φ-linear automorphism F. Let $V = pF^{-1}$ be the *Verschiebung*.

Remark 1.3.3. (1) If M is a crystal over k, then $K_0 \otimes_{W(k)} M$ is an isocrystal over k.

(2) Let M be a lattice contained in an isocrystal N, then M is a crystal if and only if M is stable under F and V.

(3) It is easily seen that V is φ^{-1}-linear and $FV = VF = p\,\mathrm{Id}$.

(4) The crystals (resp. isocrystals) over k form a category. The morphisms between two objects are $W(k)$ (resp. K_0) linear maps which commute with the φ-linear endomorphisms F. This category is a $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) linear category, i.e. the Hom are $\mathbb{Z}_p$-modules (resp. $\mathbb{Q}_p$ vector spaces) and the composition is $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) bilinear.

Definition 1.3.4. All the schemes in this definition are assumed to be over $\mathbb{F}_p$.

(i) Let S be a scheme, the *absolute Frobenius* f_S of S is defined to be an endomorphism of S which is identical on base points and sends a section s of O_S to the section s^p.

(ii) Let S be a fixed base scheme and X be an S-scheme. We denote $X^{(p/S)}$ or simply $X^{(p)}$ the inverse image of X by the base change $f_S : S \to S$, i.e. we have the following commutative diagram.

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "f_S"] & S \end{tikzcd}$$

(iii) We define $F_{X/S} : X \to X^{(p)}$ the unique morphism making the following diagram commutative. This is called the *Frobenius* morphism of X over S.

$$\begin{tikzcd} X \arrow[dr, dotted, "F_{X/S}"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right] & & \\ & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "f_S"] & S \end{tikzcd}$$

If G is a flat commutative S-group scheme, one can define a canonical homomorphism functorial on G

$$V_{G/S} : G^{(p/S)} \longrightarrow G$$

called the *Verschiebung* morphism of G over S satisfying the following properties:

$$F_{G/S} \circ V_{G/S} = p\,\mathrm{Id}_{G^{(p)}} \quad \text{and} \quad V_{G/S} \circ G_{G/S} = p\,\mathrm{Id}_G$$

For our use, we assume that $S = \operatorname{Spec} k$ and G be a commutative group scheme over k. Then we denote $F_G = F_{G/k}$ and $V_G = V_{G/k}$. In this case $V_G : G^{(p)} \to G$ is the Cartier dual of $F_{G^*} : G^* \to (G^*)^{(p)} = (G^{(p)})^*$.

The classical Dieudonné theory associates to every p-divisible group G over k a Dieudonné crystal $D(G)$ and an isocrystal $E(G)$. The Dieudonné crystal $D(G) := \mathrm{Hom}(G, \mathrm{CW})$, where CW is the co-Witt vectors over k. The Frobenius and Verschiebung in Definition 1.3.2 are given by $F := E(F_G)$ and $V := E(V_G)$.

Theorem 1.3.5. *([14]) The functor $G \mapsto D(G)$ provides an anti-equivalence of categories between p-divisible groups over k and Dieudonné crystals. The rank of $D(G)$ is the height* $\mathrm{ht}(G)$ *of G.*

Remark 1.3.6. Assume G and H are two p-divisible groups over k of the same height and $f : G \to H$ be a homomorphism and $E(f) : E(H) \to E(G)$ is the K_0-linear map induced from the functoriality. One can show that f is an isogeny if and only if $E(f)$ is an isomorphism. These are also equivalent to the condition that $D(f)$ is an injection.

Example 1.3.7. (1) Let k be algebraically closed and $\lambda \in \mathbb{Q}$, $\lambda = r/s$ with $r, s \in \mathbb{Z}, (r, s) = 1$ and $s > 0$. We define an isocrystal $E_\lambda = K_0 < T > /(T^s - p^r, T\lambda = \varphi(\lambda)T, \lambda \in K_0)$, where $K_0 < T >$ is the non commutative polynomial ring, i.e. the elements in K_0 are not commutative with the indeterminate T. We can also write

$$E_\lambda = (K_0^s, \begin{bmatrix} 0 & \dots\dots & p^r \\ 1 & 0 & \dots\dots \\ \dots\dots & & \\ \dots\dots & 1 & 0 \end{bmatrix} \cdot \varphi)$$

Then $D_\lambda = \mathrm{End}(E_\lambda)$ is the unique division algebra with center $\mathbb{Q}_p$ and invariant λ. Moreover, we have $\dim E_\lambda = (1 - \lambda)\,\mathrm{ht}(E_\lambda)$.

(2) From [14] we have that $\lambda \in [0, 1] \cap \mathbb{Q}$ if and only if there is a p-divisible group G_λ such that $E(G_\lambda) \cong E_\lambda$.

From Remark 1.3.6 we see that after inverting p it is possible to work with vector spaces over a field and the classification of isocrystals over k of the form $E(G)$ is therefore equivalent to the classification of p-divisible groups up to isogeny.

Theorem 1.3.8. *(Manin)([14]) Let k be algebraically closed. The category of isocrystals over k is semi-simple. Its simple objects are the E_λ's, i.e. any isocrystal N over k is isomorphic to a direct sum $\sum(E_\lambda)^{m_\lambda}$. This is called the* slope decomposition *of N.*

The problem of generalizing Dieudonné theory to p-divisible groups over more general base S over which p is locally nilpotent has been tackled and advertised by Grothendieck ([24]). Grothendieck's proposal was to define $\mathbb{D}(G)$ as a $\mathcal{F}$-crystal on the crystalline site of S. As Grothendieck commented ([23]), there are two different ways to construct the generalized Dieudonné functor, the method of exponential and the method of $\natural$ extensions. The first gives a direct application to the theory of infinitesimal extension of p-divisible groups and the second clears easily the connection to the classical Dieudonné theory. In the case of p-divisible groups over a perfect field of characteristic $p > 0$, this gives a canonical isomorphism between them.

We give here the main results of Messing's covariant Dieudonné theory by using exponentials. The covariant theory and contravariant theory are connected via Cartier duality.

To define the crystaline site over a scheme S, we first introduce the concept of divided powers.

Definition 1.3.9. Let A be a ring and I an ideal of A. We say that I is equipped with *divided powers* if we are given a family of mappings $\gamma_n : I \to I$ for $n \geq 1$ which satisfy the following conditions:

(i) $\gamma_1(x) = x$, for all $x \in I$

(ii) $\gamma_n(x + y) = \gamma_n(x) + \sum_{i=1}^{n-1} \gamma_{n-i}(x)\gamma_i(y) + \gamma_n(y)$

(iii) $\gamma_n(xy) = x^n \gamma_n(y)$ for $x \in A$ and $y \in I$

(iv) $\gamma_m(\gamma_n(x)) = \frac{(mn)!}{(n!)^m m!}\gamma_{mn}(x)$

(v) $\gamma_m(x)\gamma_n(x) = \frac{(m+n)!}{m!n!}\gamma_{m+n}(x)$

Given such a system we define γ_0 via $\gamma_0(x) = 1$ for all $x \in I$ and refer to (I, γ) as an ideal with divided powers.

Remark 1.3.10. By the axiom (v), we have

$$\gamma_{m_1+m_2+\cdots+m_p}(x) \cdot \frac{(m_1 + m_2 + \cdots + m_p)!}{m_1!m_2!\cdots m_p!} = \prod_{i=1}^{p} \gamma_{m_i}(x)$$

In particular, we have $x^n = (\gamma_1(x))^n = n!\gamma_n(x)$. This formula is the main motivation to introduce the divided powers. If A is a $\mathbb{Q}$-algebra or a torsion free $\mathbb{Z}$-module, we have $\gamma_n(x) = x^n/n!$ for all $n \geq 0$. Hence every ideal has a unique structure of divided powers. We sometimes write the map γ_n by $x \mapsto x^{(n)}$.

Definition 1.3.11. Given (A, I, γ) an ideal with divided powers, we say that the divided powers are *nilpotent* if there is an integer N such that the ideal generated by elements of the form $\gamma_{i_1}(x_1)\cdots\gamma_{i_k}(x_k)$, $i_1 + \cdots + i_k \geq N$ is zero. This implies that $I^N = 0$ (taking $k = N$, $i_1 = \cdots = i_N = 1$).

Definition 1.3.12. Let (A, I, γ) be an ideal with nilpotent divided powers. We define two homomorphisms *exponential* and *logarithm* as

$$\exp : J \to 1 + J, \quad \exp(x) = \sum_{n \geq 0} x^{(n)},$$

$$\log : 1 + J \to J, \quad \log(1+x) = \sum_{n\geq 1} (-1)^{n-1}(n-1)! x^{(n)}$$

These two homomorphisms give an isomorphism $J^+ \cong (1+J)^*$.

Example 1.3.13. (1) Consider $W = W(k)$ the Witt ring with coefficients in a perfect field k of characteristic $p > 0$ and $I = pW$. Then by the classical method of Gauss, assume $n \geq 1$ is an integer and

$$n = a_0 + a_1 p + \cdots + a_l p^l$$

with $0 \leq a_j \leq p-1$, $j = 0, \ldots, l$ and let $s_n = \sum_{j=0}^{l} a_j$. Then the p-adic valuation of $n!$ is given by

$$\mathrm{ord}_p(n!) = \frac{n - s_n}{p-1} \leq n - 1$$

Then we define $\gamma_n(p) = p^n/n! \in pW$ giving the unique divided power structure on pW.

(2) We can replace W by any separated and complete noetherian adic ring A of characteristic zero with p contained in an ideal of definition. Then the ideal pA can be equipped with a canonical divided power structure.

Definition 1.3.14. Let (A, I, γ) and (A', I', γ') be two ideals with divided powers. A *divided power homomorphism* $\phi : (A, I, \gamma) \to (A', I', \gamma')$ is a homomorphism of rings $\phi : A \to A'$ such that $\phi(I) \subset I'$ and $\phi(x^{(n)}) = \phi(x)^{(n)}$ for any $x \in I$.

Definition 1.3.15. Let (A, I, γ) be an ideal with divided powers and let $\phi : A \to B$ be a ring homomorphism. We say that γ *extends to* B if there exists a divided powers structure γ' on IB such that the mapping $\phi : (A, I, \gamma) \to (B, IB, \gamma')$ is a divided power homomorphism.

We have two cases when the divided powers structure extends successfully.

Proposition 1.3.16. *([24] or [32] Chapter 3 (1.8)) Let (A, I, γ) be as above and $\phi : A \to B$ be a ring homomorphism, then*

(i) If I is principal, then γ extends to IB

(ii) If B is a flat A-algebra, γ extends to IB.

Remark 1.3.17. Our construction can be globalized as follows: we replace A by a scheme S, I by a quasi-coherent ideal sheaf $\mathfrak{I}$ of O_S, divided powers on $\mathfrak{I}$ are given by assigning to each open subset U a system of divided powers on $\Gamma(U, \mathfrak{I})$ commuting with the restriction maps.

Given a divided power morphism between $(S, \mathfrak{I}, \gamma)$ and $(S', \mathfrak{I}', \gamma')$ is the same as to give a morphism of schemes $f : S \to S'$ such that $f^{-1}(\mathfrak{I}')$ maps into $\mathfrak{I}$ under the map $f^{-1}(O_{S'}) \to O_S$ and the divided powers induced on the image of $f^{-1}(\mathfrak{I}')$ "coincide" with those defined by γ'.

Definition 1.3.18. For a scheme X, we define the *crystalline site* $\mathrm{Crys}(X)$ as a category whose objects are triples $T := (U \hookrightarrow T, \gamma)$ where:

(i) U is a Zariski open subscheme of X

(ii) $U \hookrightarrow T$ is a locally nilpotent immersion

(iii) $\gamma = (\gamma_n)$ are locally nilpotent divided powers on the defining ideal I of U in T.

The morphisms from $(U \hookrightarrow T, \gamma)$ to $(U' \hookrightarrow T', \gamma')$ are the commutative diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"'] & T \arrow[d, "\overline{f}"] \\ U' \arrow[r] & T' \end{tikzcd} \tag{1.1}$$

where $f : U \to U'$ is the inclusion and $\overline{f} : T \to T'$ is a divided power morphism, i.e. the morphism of sheaf of rings $\overline{f}^{-1}(O_{T'}) \to O_T$ is a divided power morphism.

A covering family of an object $(U \hookrightarrow T, \gamma)$ is a collection of morphisms $\{(U_i \hookrightarrow T_i, \gamma_i) \to (U \hookrightarrow T, \gamma)\}$ such that T_i is the open subscheme of T whose underlying set is U_i an open subset of U and $\bigcup U_i = U$.

Definition 1.3.19. A *sheaf* (of sets for example) on this site is a contravariant functor F : $\mathrm{Crys}(X)^{op} \to$ (Sets) such that for every covering family $\{T_i \to T\}$, the following sequence of sets is exact

$$0 \to F(T) \to \prod_i F(T_i) \rightrightarrows \prod_{i,j} F(T_i \times_T T_j)$$

Remark 1.3.20. Sheaves on this site admit the following description: to give a sheaf F is equivalent to giving an ordinary sheaf $F_{(U \hookrightarrow T, \gamma)}$ on T for each object $(U \hookrightarrow T, \gamma)$, and for every morphism $u : (U_1 \hookrightarrow T_1, \gamma_1) \to (U \hookrightarrow T, \gamma)$ in $\mathrm{Crys}(X)$, a map $\rho_u : u^{-1}(F_{(U \hookrightarrow T, \gamma)}) \to F_{(U_1 \hookrightarrow T_1, \gamma_1)}$ such that

(i) If $v : (U_2 \hookrightarrow T_2, \gamma_2) \to (U_1 \hookrightarrow T_1, \gamma_1)$ is another morphism, then we have a commutative diagram

$$\begin{tikzcd} v^{-1}(u^{-1}(F_{(U \hookrightarrow T, \gamma)})) \arrow[r, "v^{-1}(\rho_u)"] \arrow[dr, "\rho_{u \circ v}"'] & v^{-1}(F_{(U_1 \hookrightarrow T_1, \gamma_1)}) \arrow[d, "\rho_v"] \\ & F_{(U_2 \hookrightarrow T_2, \gamma_2)} \end{tikzcd}$$

(ii) If $u : (U_1 \hookrightarrow T_1, \gamma_1) \to (U \hookrightarrow T, \gamma)$ a morphism satisfying $\overline{u} : T_1 \to T$ is an open immersion, the map $\rho_u : u^{-1}(F_{(U \hookrightarrow T, \gamma)}) \to F_{(U_1 \hookrightarrow T_1, \gamma_1)}$ is an isomorphism.

In Grothendieck's term: "crystals grow and are rigid".

Remark 1.3.21. (1) The site $\mathrm{Crys}(X)$ is ringed in a natural way, namely the sheaf of rings $O_{X_{\mathrm{Crys}}}$ corresponds to the system $O_{(U \hookrightarrow T, \gamma)} = O_T$.

(2) A sheaf of modules M on the site $\mathrm{Crys}(X)$ is given by a family M_T of O_T-modules satisfying the similar properties as in Remark 1.3.20. Such an M is said to be *special* if for any diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"'] & T \arrow[d, "\overline{f}"] \\ U' \arrow[r] & T' \end{tikzcd}$$

we have $\overline{f}^*(M_{T'}) = M_T$. A module M is said to be *quasi-coherent* if M is special and all M_T are quasi-coherent O_T-modules.

Definition 1.3.22. Let $\mathcal{F}$ be a fibred category on (Sch) which is a stack with respect to the Zariski topology. An *$\mathcal{F}$-crystal* on X is a Cartesian section of the fibred category $\mathcal{F} \times_{(\mathrm{Sch})} \mathrm{Crys}(X)$, where $\mathrm{Crys}(X) \to (\mathrm{Sch})$ is given by $(U \hookrightarrow T, \gamma) \mapsto T$. A morphism of $\mathcal{F}$-crystals is a morphism of Cartesian sections. This means that for each object $(U \hookrightarrow T, \gamma)$ in $\mathrm{Crys}(X)$ we are given an object $Q_{(U \hookrightarrow T, \gamma)}$ in $\mathcal{F}_T$ and that for each morphism (1.1) in $\mathrm{Crys}(X)$ we are given an isomorphism

$$u_{\bar{f}} : Q_{(U \hookrightarrow T, \gamma)} \longrightarrow \bar{f}^* Q_{(U' \hookrightarrow T', \delta)}$$

These isomorphisms are to satisfy $\bar{f}^*(u_{\bar{g}}) \circ u_{\bar{f}} = u_{\bar{g} \circ \bar{f}}$ where $\bar{g}$ comes from a morphism in $\mathrm{Crys}(X)$

$$\begin{array}{ccc} U' & \longrightarrow & T' \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle \bar{f}}\big\downarrow \\ U'' & \longrightarrow & T'' \end{array}$$

In particular, an $\mathcal{F}$-crystal is a sheaf on $\mathrm{Crys}(X)$.

Remark 1.3.23. A special $O_{X_{\mathrm{Crys}}}$-module M is a crystal in modules. Here $\mathcal{F}_T = \mathrm{QCoh}(T)$ is the category of quasi-coherent O_T-modules.

Let S_0 be our base scheme with p locally nilpotent on it. In order to generalize the classical Dieudonné theory (in covariant form), we hope to define a functor

$$\mathbb{D} : pdiv(S_0) \longrightarrow (\text{Crystals in finite locally free } O_{S_{0\mathrm{Crys}}}\text{-modules})$$

By the method of exponentials, one can associate to certain p-divisible groups on S_0 a crystal in finite locally free $O_{S_{0\mathrm{Crys}}}$-modules. The word "certain" means that our p-divisible groups in question are locally liftable to infinitesimal neighborhoods. More precisely, we define $pdiv(S_0)'$ to be the full subcategory of $pdiv(S_0)$ consisting of those p-divisible groups G_0 with the property that there is an open cover of S_0 (depending on G_0) formed of affine open subsets $U_0 \subset S_0$ such that for any nilpotent immersion $U_0 \hookrightarrow U$ there is a p-divisible group G_U on U with $G_U|_{U_0} = G_0|_{U_0}$.

By the arguments of Grothendieck and Illusie, every p-divisible group over S_0 is locally liftable to infinitesimal neighborhoods, i.e. $pdiv(S_0)' = pdiv(S_0)$. To such a p-divisible group G Messing had defined:

(1) a crystal in (f.p.p.f.) groups: $\mathbb{E}(G)$

(2) a crystal in formal Lie groups: $\overline{\mathbb{E}(G)}$

(3) a crystal in finite locally free modules: $\mathbb{D}(G)$

The crystal $\mathbb{E}(G)$ is our basic crystal to construct and $\overline{\mathbb{E}(G)}$ is obtained from $\mathbb{E}(G)$ by "completing along the unit section", while $\mathbb{D}(G)$ will be obtained from $\mathbb{E}(G)$ by applying Lie functor (Definition 1.3.32). To construct $\mathbb{E}(G)$ we now arrive to introduce the universal extension of a p-divisible group by vector groups.

Definition 1.3.24. Let S be a scheme and M be a quasi-coherent O_S-module. One can associate to M a f.p.p.f. S-group $\widetilde{M}$ whose section over an S-scheme T is given by $\Gamma(T,\widetilde{M}) = \Gamma(T, O_T \otimes_{O_S} M)$. If moreover M is a locally free O_S-module of finite rank, then $\widetilde{M}$ is representable by the group scheme defined by the symmetric algebra $\mathrm{Sym}(M^\vee)$ which is locally isomorphic to a finite product of $\mathbb{G}_a$'s and $\widetilde{M}$ is called a *vector group* over S.

Proposition 1.3.25. *([32] Chapter IV 1.3) Suppose G is an S-group scheme with G^* representable (e.g. G is a finite locally free S-group). Then the functor (on quasi-coherent O_S-modules): $M \mapsto \mathrm{Hom}_{S-gr}(G,\widetilde{M})$ is represented by ω_{G^*}, i.e. there is a morphism $d : G \to \omega_{G^*}$ such that the natural map $\mathrm{Hom}_{O_S-\mathrm{mod}}(\omega_{G^*}, M) \to \mathrm{Hom}_{S-gr}(G,\widetilde{M})$ is a bijection for any quasi-coherent O_S-module M.*

From now on, we assume p^N is zero on S and G is a p-divisible group on S. Then for any quasi-coherent O_S-module M, $\mathrm{Hom}_{S-gr}(G,\widetilde{M}) = 0$. This is because $p^N : G \to G$ is an epimorphism and p^N times any homomorphism $f : G \to \widetilde{M}$ is zero and we have the following commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "p^N"'] & \widetilde{M} \arrow[d, "p^N"] \\ G \arrow[r] & \widetilde{M} \end{tikzcd}$$

Definition 1.3.26. An *extension* of G by a vector group V is an exact sequence of commutative S-groups:

$$0 \to V \to E \to G \to 0$$

Such an extension is said to be *universal* if for any vector group M the natural mapping $\mathrm{Hom}_{O_S-\mathrm{mod}}(V, M) \to \mathrm{Ext}^1_S(G, M)$ is a bijection.

By an automorphism of an extension $0 \to V \to E \to G \to 0$ we mean a morphism $\alpha : E \to E$ such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "\mathrm{Id}"'] & E \arrow[r] \arrow[d, "\alpha"'] & G \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\ 0 \arrow[r] & V \arrow[r] & E \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

Since $\mathrm{Hom}(G, V) = 0$, an extension of G by a vector group V admits no non trivial automorphism.

Theorem 1.3.27. *([32] Chapter IV 1.10) Assume that $p^N O_S = 0$ and G is a p-divisible group on S, then there is a universal extension $E(G)$ of G by a vector group $V(G)$.*

Remark 1.3.28. (1) Here $V(G)$ is actually $\omega_{G(N)^*} = \omega_{G^*}$.

(2) The universal extension commutes with an arbitrary base change $S' \to S$. Hence this can be generalized to base schemes S where p is locally nilpotent.

Corollary 1.3.29. *([32] Chapter IV 1.14) Assume that p is locally nilpotent on S and G is a p-divisible group on S. Then there exists a universal extension $0 \to V(G) \to E(G) \to G \to 0$ of G with $V(G) = \omega_{G^*}$.*

Proposition 1.3.30. *([32] Chapter IV 1.15) Let p be locally nilpotent on S and G, H two p-divisible groups on S with $u : G \to H$ a homomorphism. Then there is a unique homomorphism $E(u) : E(G) \to E(H)$ such that we obtain a morphism of extensions:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(G) & \longrightarrow & E(G) & \longrightarrow & G & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle V(u)} & & {\scriptstyle E(u)}\Big\downarrow & & {\scriptstyle u}\Big\downarrow & & \\ 0 & \longrightarrow & V(H) & \longrightarrow & E(H) & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

where $V(u)$ is the map induced on the invariant differentials by the Cartier dual of u.

Proposition 1.3.31. *([32] Chapter IV 1.19) Assume p is locally nilpotent on S and G be a p-divisible group on S, then $\overline{E(G)} := \varinjlim_k \mathrm{Inf}^k E(G)$ is a formal Lie group.*

Definition 1.3.32. We define $\mathrm{Lie}(E(G)) = \mathrm{Lie}(\overline{E(G)}) = (\omega_{\overline{E(G)}})^\vee$.

Proposition 1.3.33. *([32] Chapter IV 1.22) By taking the Lie functor of the universal extension $0 \to V(G) \to E(G) \to G \to 0$, we get an exact sequence $0 \to V(G) \to \mathrm{Lie}(E(G)) \to \mathrm{Lie}(G) \to 0$.*

We state the main theorem which allows the construction of $\mathbb{E}(G)$.

Theorem 1.3.34. *([32] Chapter IV 2.2) Let $S = \mathrm{Spec}(A)$, $p^N \cdot 1_S = 0$, $S_0 = \mathrm{Var}(I)$ where I is an ideal of A with nilpotent divided powers. Let G and H be two p-divisible groups on S and assume $u_0 : G_0 \to H_0$ is a homomorphism between their restrictions to S_0. By Proposition 1.3.30 u_0 defines a morphism $E(u_0) : E(G_0) \to E(H_0)$ of extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(G_0) & \longrightarrow & E(G_0) & \longrightarrow & G_0 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle V(u_0)} & & {\scriptstyle E(u_0)}\Big\downarrow & & {\scriptstyle u_0}\Big\downarrow & & \\ 0 & \longrightarrow & V(H_0) & \longrightarrow & E(H_0) & \longrightarrow & H_0 & \longrightarrow & 0 \end{array}$$

Then there is a unique morphism $v : E(G) \to E(H)$ (not necessarily respecting the structure of extensions) with the following properties:

(i) v is a lifting of $E(u_0)$

(ii) Given $w : V(G) \to V(H)$, a lifting of $V(u_0)$, denote by i the inclusion $V(H) \to E(H)$, such that $d = i \circ w - v|_{V(G)} : V(G) \to E(H)$ induces zero on S_0. Then d is an exponential in the sense of [32] Chapter 3.

We have several corollaries that are needed for the construction of the crystal $\mathbb{E}$.

Corollary 1.3.35. *([32] Chapter IV 2.4.1) Let K be a third p-divisible group on S and $u_0' : H_0 \to K_0$ a homomorphism. Then $E_S(u_0' \circ u_0) = E_S(u_0') \circ E_S(u_0)$.*

Corollary 1.3.36. *([32] Chapter IV 2.4.2) If $G = H$ and $u_0 = \mathrm{Id}_{G_0}$, then $E_S(u_0) = \mathrm{Id}_G$.*

The above two corollaries are proved by showing that the right hand sides of the equalities are actually satisfying the condition in Theorem 1.3.34. The following corollary follows from them.

Corollary 1.3.37. *([32] Chapter IV 2.4.3) Let G and H be p-divisible groups on S and $u_0 : G_0 \to H_0$ an isomorphism. Then $E_S(u_0)$ is an isomorphism.*

Let S_0 be an arbitrary scheme with p locally nilpotent on it and G_0 be in $pdiv(S_0)'$. Since the f.p.p.f. groups form a stack with respect to the Zariski topology, it suffices to give the value of the crystal $\mathbb{E}(G_0)$ on "sufficiently small" objects $(U_0 \hookrightarrow U)$ of the crystalline site of S_0. Take U_0 affine, we lift $G_0|_{U_0}$ to a p-divisible group G_U on U. From Corollaries 1.3.35 and 1.3.37, $E(G_U)$ is independent of the choice of lifting up to canonical isomorphism.

If $V_0 \hookrightarrow V$ is a second object of the crystalline site and there is morphism

$$\begin{array}{ccc} U_0 & \longrightarrow & U \\ {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow \\ V_0 & \longrightarrow & V \end{array}$$

then for a lifting G_U of $G_0|_{U_0}$ to U and a lifting G_V of $G_0|_{V_0}$ to V the same corollaries give a canonical isomorphism $\bar{f}^*(E(G_U)) \cong E(G_V)$.

Definition 1.3.38. We define the value of the crystal $\mathbb{E}(G_0)$ on a sufficiently small object $(U_0 \hookrightarrow U)$ is simply $E(G_U)$ for some choice of lifting of $G_0|_{U_0}$ to U. We see that $\mathbb{E}$ is functorial.

Remark 1.3.39. Given $T_0 \to S_0$ the diagram is commutative

$$\begin{array}{ccc} pdiv(S_0)' & \xrightarrow{\mathbb{E}} & (\text{Crystals in f.p.p.f. groups on } S_0) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ pdiv(T_0)' & \xrightarrow{\mathbb{E}} & (\text{Crystals in f.p.p.f. groups on } T_0) \end{array}$$

Definition 1.3.40. Define other two crystals

$\overline{\mathbb{E}(G_0)}_{(U_0 \hookrightarrow U)} := \overline{(\mathbb{E}(G_0)_{(U_0 \hookrightarrow U)})}$

$\mathbb{D}(G_0)_{(U_0 \hookrightarrow U)} := \mathrm{Lie}(\overline{\mathbb{E}(G_0)}_{(U_0 \hookrightarrow U)})$.

Since $\mathbb{E}$ is functorial, we see that $\overline{\mathbb{E}}$ and $\mathbb{D}$ are functorial.

Remark 1.3.41. We call $\mathbb{D}$ our *covariant Dieudonné functor*. It can be shown that $\mathbb{D}(G)$ is a finite locally free crystal on S of rank the height of G.

Remark 1.3.42. If k is a perfect field of characteristic $p > 0$ and G is a p-divisible group over k. Let $W(k)$ be the Witt ring with coefficients in k, then $W(k)/pW(k) = k$. For $p \geq 3$ we define $W_n := W(k)/p^nW(k)$. Then the surjective ring homomorphism $W_n \to W(k)/pW(k) = k$

gives a nilpotent immersion $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} W_n$ with nilpotent divided powers on the defining ideal $pW(k)/p^nW(k)$, see Example 1.3.13. The relation of the classical Dieudonné crystal and the Grothendieck-Messing crystal is given by

$$D(G) = \varprojlim_n \mathbb{D}(G^*)$$

If $p = 2$ we take $W_n := W(k)/4^nW(k)$.

1.4 Deformation Theory

Let p be locally nilpotent on S and $S_0 \hookrightarrow S$ be a nilpotent immersion defined by an ideal I which is endowed with locally nilpotent divided powers. For $G_0 \in pdiv(S_0)$, we denote $\mathbb{D}(G_0)_S$ the value of the Lie algebra crystal on $(S_0 \hookrightarrow S)$.

Definition 1.4.1. A filtration $\mathrm{Fil}^1 \subset \mathbb{D}(G_0)_S$ is said to be *admissible* if it is a locally direct factor vector subbundle of $\mathbb{D}(G_0)_S$ which reduces to $V(G_0) \subset \mathrm{Lie}(E(G_0))$ on S_0.

Definition 1.4.2. We define a category whose objects are pairs (G_0, Fil^1) with G_0 a p-divisible group on S_0 and Fil^1 an admissible filtration of $\mathbb{D}(G_0)_S$. The morphisms between two objects are the pairs (u_0, ξ) where $u_0 : G_0 \to H_0$ is a morphism of S_0-group and $\xi : \mathrm{Fil}^1(G_0) \to \mathrm{Fil}^1(H_0)$ which satisfying the following commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^1(G_0) \arrow[r, "i"] \arrow[d, "\xi"'] & \mathbb{D}(G_0)_S \arrow[d, "\mathbb{D}(u_0)_S"] \\
\mathrm{Fil}^1(H_0) \arrow[r, "i"] & \mathbb{D}(H_0)_S
\end{tikzcd}$$

and reduces to the commutative diagram

$$\begin{tikzcd}
V(G_0) \arrow[r, "i"] \arrow[d, "V(u_0)"'] & \mathrm{Lie}(E(G_0)) \arrow[d, "\mathrm{Lie}(E(u_0))"] \\
V(H_0) \arrow[r, "i"] & \mathrm{Lie}(E(H_0))
\end{tikzcd}$$

the morphism $\mathbb{D}(u_0)_S = \mathrm{Lie}(\mathbb{E}(u_0)_S)$ where $E(u_0)_S$ is the unique morphism in Theorem 1.3.34

Theorem 1.4.3. *(Grothendieck-Messing)([32]) The functor $G \mapsto (G_0 = G|_{S_0}, V(G) \hookrightarrow \mathbb{D}(G_0)_S)$ establishes an equivalence of categories between $pdiv(S)$ and the category of admissible pairs (G_0, Fil^1).*

By passing to the limit, one can consider the deformation theory on formal schemes which are complete with respect to the p-adic topology. For example, let A be a complete discrete valuation ring with residue field k perfect of characteristic $p > 0$ and K the fraction field of A, which is of characteristic zero. Then A is p-adic and denote $A_n = A/p^{n+1}A$. For any p-divisible group G_0 over $S_0 = \mathrm{Spec}(A_0)$, we define by passage to limit a finite locally free A-module M

such that $M \otimes_A A_n = \mathbb{D}(G_0)_{(\mathrm{Spec}(A_0) \hookrightarrow \mathrm{Spec}(A_n))}$. Here we equip $(\mathrm{Spec}(A_0) \hookrightarrow \mathrm{Spec}(A_n))$ with the canonical divided powers as in Example 1.3.7. Then to give a p-divisible group G over $\mathrm{Spf}\, A$ is the same as to give

(1) A p-divisible group $G_0 = G \otimes_A A_0$ over A_0.

(2) A system of admissible filtration V_n of $\mathbb{D}(G_0)_{(\mathrm{Spec}\, A_0 \hookrightarrow \mathrm{Spec}\, A_n)}$ for each $n \in \mathbb{N}$, which is compatible in the sense that $V_{n+1} \otimes_{A_{n+1}} A_n \cong V_n$.

Now we state a question of Grothendieck. Fix a p-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}_p}$ of height h and dimension d. Let $W := W(\overline{\mathbb{F}_p})$ be the ring of Witt vectors and let K_0 be its fraction field. Let O_K be a complete discrete valuation ring with residue field $\overline{\mathbb{F}_p}$ and fraction field K of characteristic 0. To every p-divisible group X over O_K with $\mathbb{X} \cong X \otimes_{O_K} \overline{\mathbb{F}_p}$ we associate an extension

$$0 \longrightarrow (\mathrm{Lie}\, X^*)_K^\vee \longrightarrow \mathbb{D}(\mathbb{X})_K \longrightarrow (\mathrm{Lie}\, X)_K \longrightarrow 0$$

We denote by $\mathcal{F} = \mathrm{Grass}_{h-d}(\mathbb{D}(\mathbb{X})_{K_0})$ the Grassmannian of $(h-d)$-dimensional subspaces of $\mathbb{D}(\mathbb{X})_{K_0}$. Grothendieck [23] raised the following question:

Describe the subset of $\mathcal{F}$ formed by the points $(\mathrm{Lie}\, X^*)_K^\vee$ where X is any deformation of $\mathbb{X}$ over any complete discrete valuation ring O_K with residue field $\overline{\mathbb{F}_p}$ and fraction field K of characteristic 0.

We will return to this question in Proposition 3.2.13.

Chapter 2

Moduli Spaces for p-divisible Groups and the Period Morphisms

In this chapter, we first give some generalities on formal schemes. Then we state the moduli problem of p-divisible groups considered in [33] and prove its representability. Before introducing the period morphism we recall the theory of rigid analytic geometry which is necessary in the sequel. Then by using Grothendieck-Messing's deformation theory we can describe the construction of period morphism as in [33], we will prove that the period morphism is étale.

2.1 Generalities on Formal Schemes

We assume in this section that all the rings we consider are commutative.

Definition 2.1.1. Let A be a topological ring and $\{I_\alpha\}$ a set of open ideals of A that form a fundamental system of neighborhoods of zero in A. We say that A is a *linear topological ring* if for any $a \in A$, $\{a + I_\alpha\}$ form a fundamental system of neighborhoods of a. An element $x \in A$ is called *topologically nilpotent* if x^n goes to zero as n tends to infinity.

Definition 2.1.2. An ideal I of A is called an *ideal of definition* of A if I is an open ideal and for any open neighborhood V of 0 there exists and integer n such that $I^n \subset V$. A linear topological ring A having an ideal of definition is called a *preadmissible ring*. A preadmissible ring is *admissible* if it is separated and complete.

Remark 2.1.3. A preadmissible noetherian ring admits a maximal ideal of definition. ([13]I Chapter 0 7.1.7).

Definition 2.1.4. A preadmissible ring A is said to be *preadic* if there is an ideal of definition I of A such that I^n form a fundamental system of neighborhoods of zero. Moreover if it is separated and complete, A is called I-adic. In this case A is the projective limit of the discrete rings $A_n = A/I^{n+1}, n \geq 0$. This topology is independent of the choice of the ideals of definition, since for any other ideal of definition J there exist positive integers p, q such that $J \supset I^p \supset J^q$.

Definition 2.1.5. Let A be an I-adic ring and denote X the affine scheme $\mathrm{Spec}(A)$. Then one can associate a topological ringed space $(\mathfrak{X}, O_{\mathfrak{X}})$, where $\mathfrak{X}$ is the topological space $V(I)$ and $O_{\mathfrak{X}}$ is the sheaf of rings $\varprojlim O_X/I^{n+1}$. Here we consider each O_X/I^{n+1} as a sheaf of rings on $V(I)$, and make them into an inverse system in the natural way. Such a locally ringed space is called an *affine formal scheme.*

Definition 2.1.6. A *formal scheme* is a locally ringed space $(\mathfrak{X}, O_{\mathfrak{X}})$ which has an open covering $\{\mathfrak{U}_i\}$ such that for each i, the pair $(\mathfrak{U}_i, O_{\mathfrak{X}}|_{\mathfrak{U}_i})$ is isomorphic, as a locally ringed space, to an affine formal scheme.

Remark 2.1.7. (1) We can also define formal schemes as functor. For a preadmissible ring $(A, \{I_\alpha\})$, we define a contravariant functor $\mathrm{Spf}\, A$ on the category of schemes:

$$\mathrm{Spf}\, A(Z) := \varinjlim_\alpha \mathrm{Hom}(Z, \mathrm{Spec}(A/I_\alpha))$$

This is a local functor, i.e. a sheaf for the Zariski topology on the category of quasi-compact quasi-separated schemes. If the ring $(A, \{I^n\})$ is adic, $\mathrm{Spf}\, A$ is clearly our affine formal scheme. A formal scheme is then a local functor which has a covering by open subfunctors which are affine formal schemes. If a formal scheme is locally isomorphic to $\mathrm{Spf}\, A$, where A is a noetherian adic ring, we call it *locally noetherian.*

(2) If Λ is a p-adic ring, we may consider the category Nilp_Λ of schemes S over $\mathrm{Spec}\,\Lambda$ such that p is locally nilpotent on S. Then the category of formal schemes over $\mathrm{Spf}\,\Lambda$ is a full subcategory of the category of set valued sheaves on Nilp_Λ.

Definition 2.1.8. *A morphism* $f : \mathfrak{X} \to \mathfrak{Y}$ between two (locally noetherian) formal schemes is a morphism of corresponding locally ringed spaces, which is continuous on the sheaves of topological rings.

Definition 2.1.9. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of formal schemes is of *finite type, étale, lisse, etc* if for any scheme Z and any morphism $Z \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} Z$ is a scheme and $\mathfrak{X} \times_{\mathfrak{Y}} Z \to Z$ is of finite type, étale, smooth, etc in the usual sense.

Definition 2.1.10. Let $\mathfrak{X}$ be a locally noetherian formal scheme, then there is a unique reduced scheme $\mathfrak{X}_{\mathrm{red}}$ and a unique morphism of locally ringed spaces $\mathfrak{X}_{\mathrm{red}} \to \mathfrak{X}$, such that for any reduced scheme Z the natural map $\mathrm{Hom}(Z, \mathfrak{X}_{\mathrm{red}}) \to \mathrm{Hom}(Z, \mathfrak{X})$ is a bijection.

Definition 2.1.11. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally noetherian formal schemes, a morphism $\mathfrak{X} \to \mathfrak{Y}$ is called *formally of finite type* if $\mathfrak{X}_{\mathrm{red}} \to \mathfrak{Y}_{\mathrm{red}}$ is of finite type. We have the same definition of *locally formally of finite type.*

The following proposition gives a condition to ensure that the completion of a preadmissible ring $(A, \{I_\alpha\})$ is adic. It is a reformulation of [13]I Chapter 0 7.2.7.

Proposition 2.1.12. *Let A be a preadmissible ring with a fundamental system of neighborhoods given by a chain of ideals*

$$I_1 \supset I_2 \supset \cdots \supset I_r \supset \cdots$$

Let I be an ideal of definition of A such that I/I^2 is topologically of finite type, i.e. $I/I^2 + I_r$ is a A-module of finite type for all r. If for each $m \in \mathbb{N}$ the following chain of ideals stabilizes and equals to $\mathfrak{a}_m$

$$I_1 + I^m \supset I_2 + I^m \supset \cdots \supset I_r + I^m \supset \cdots ,$$

then the projective limit $\varprojlim A/\mathfrak{a}_m$ is an I-adic ring.

Proof. It is easy to see that $\mathfrak{a}_1 = I$, $\mathfrak{a}_m \supset \mathfrak{a}_{m+1}$, $\mathfrak{a}_{m+1} + \mathfrak{a}_1^m = \mathfrak{a}_m$. Moreover $\mathfrak{a}_1/\mathfrak{a}_2$ is an A-module of finite type. Then by [13]I Chapter 0 7.2.7, $\varprojlim A/\mathfrak{a}_m$ is an I-adic ring. □

2.2 Moduli Spaces of p-divisible Groups

This section is essentially Chapter 2 and 3 of [33]. We will state and prove the case we need in the sequel.

We fix a prime number p and let $W := W(\overline{\mathbb{F}_p})$ be the ring of Witt vectors with coefficients in $\overline{\mathbb{F}_p}$. We denote by K_0 the fraction field of W and φ the Frobenius automorphism on W and K_0. For any complete discrete valuation ring O of mixed characteristic $(0, p)$, we denote Nilp_O the category of locally noetherian schemes S over $\mathrm{Spec}\, O$ such that pO_S is locally nilpotent. We denote $\overline{S}$ the closed subscheme of S defined by the ideal pO_S.

Theorem 2.2.1. *Let $\mathbb{X}$ be a fixed p-divisible group over* $\mathrm{Spec}\,\overline{\mathbb{F}_p}$. *We define the contravariant functor $\mathcal{M}$ over* Nilp_W *which associates to S the set of isomorphism classes of the pairs (X, ρ) given by*

(i) A p-divisible group X over S,

(ii) A quasi-isogeny $\rho : \mathbb{X}_{\overline{S}} \to X_{\overline{S}}$.

Then the functor $\mathcal{M}$ is represented by a formal scheme locally formally of finite type over $\mathrm{Spf}\, W$. *Two points (X_1, ρ_1) and (X_2, ρ_2) are isomorphic if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \to X_1$.*

We prove the theorem in several steps. First we consider isocrystals over a perfect field k with characteristic $p > 0$.

Definition 2.2.2. An isocrystal (N, F) over k is called *decent*, if the vector space N is generated by elements n satisfying an equation $F^s n = p^r n$ for some integers $r, s > 0$.

Remark 2.2.3. An equation of the form $F^s n = p^r n$ implies that n lies in some slope component of N. Hence N is decent if and only if all slope components are decent. Any decent isocrystal N is obtained by base change from a decent isocrystal over some finite field. If k is algebraically closed, any isocrystal is decent. We call a p-divisible group over k is decent if the corresponding isocrystal is decent. Therefore in the case we are interested in, we can assume our fixed p-divisible group $\mathbb{X}$ is decent and defined over a finite field L.

We have a lemma to bound the points in $\mathcal{M}$.

Lemma 2.2.4. *Let N be a decent isocrystal over a finite field L. Then there is a natural number c and a finite extension L' of L such that for any perfect field P containing L' and for any crystal $M \subset N \otimes W(P)$, there is a crystal $M' \subset N \otimes W(L')_{\mathbb{Q}}$ such that $M \subset M' \otimes W(P)$ and has index smaller than c.*

Proof. The proof is reduced to the case where N is isoclinic, see [33] 2.18. $\square$

We give an alternative definition of the functor $\mathcal{M}$. Since $\mathbb{X}$ is defined over $L = W(L)/pW(L)$, by Grothendieck-Messing, one can choose a p-divisible group $\tilde{\mathbb{X}}$ over $\operatorname{Spf} W(L)$ such that $\tilde{\mathbb{X}}|_L \cong \mathbb{X}$. Then a point of $\mathcal{M}$ with values in $S \in \mathrm{Nilp}_W$ is given by the following data:

(1) A p-divisible group X on S

(2) A quasi-isogeny $\tilde{\rho} : \tilde{\mathbb{X}}_S \to X$ of p-divisible groups on S. Here we use Theorem 1.1.13.

Definition 2.2.5. Let $f : X \to Y$ be an isogeny of p-divisible groups on S. Then the kernel of f is of rank a power of p. If the rank is a constant and equal to p^h, we call $h := \mathrm{ht}(f)$ the *height of the isogeny.*

Lemma 2.2.6. *Define $\mathcal{M}^n$ the closed subfunctor of $\mathcal{M}$ with the set of S-valued point the isomorphism classes of pairs $(X, \tilde{\rho})$ where X is a p-divisible group on S, $\tilde{\rho} : \tilde{\mathbb{X}}_S \to X$ a quasi-isogeny such that $p^n\tilde{\rho}$ is an isogeny. Then $\mathcal{M} = \varinjlim \mathcal{M}^n$ and $\mathcal{M}^n$ is representable by the p-adic completion of a scheme locally of finite type over* $\operatorname{Spf} W(L)$.

Proof. We write $\mathcal{M}^n = \coprod \mathcal{M}^{n,m}$ as a union of closed and open subfunctors, where a point $(X, \tilde{\rho})$ in $\mathcal{M}^{n,m}(S)$ satisfies that $p^n\tilde{\rho}$ is an isogeny of height m. Then by Proposition 1.1.11, to give a point in $\mathcal{M}^{n,m}(S)$ is equivalent to giving a finite locally free S-group scheme $G \subset \tilde{\mathbb{X}}_S$ of rank p^m. By [34] every commutative finite flat S-group scheme G is killed by the rank of O_G as an O_S-module, we have $p^m G = 0$, which means that $G \subset \tilde{\mathbb{X}}(m)_S$. Since the conditions making G into a group schemes define a closed subscheme of the Grassmannian $\mathrm{Grass}_{p^{m\,\mathrm{ht}\,\mathbb{X}}-p^m}(\tilde{\mathbb{X}}(m))$, we see that the functor $\mathcal{M}^{n,m}$ is representable by the p-adic completion of a closed subscheme of the Grassmannian . Therefore we have the representability of $\mathcal{M}^n$ and $\mathcal{M} = \varinjlim \mathcal{M}^n$ follows obviously. $\square$

We still need another representability of $\mathcal{M}$ as a union of representable subfunctors. To do this we define for any field extension P of L a quasi-metric on the set $\mathcal{M}(P)$.

Definition 2.2.7. Let $\alpha : X \to Y$ be a quasi-isogeny of p-divisible groups over P. We define $q(\alpha) = \mathrm{ht}(p^n\alpha)$, where n is the smallest integer such that $p^n\alpha$ is an isogeny.

Remark 2.2.8. (1) By definition $q(\alpha) = q(p^n\alpha)$ for any integer n.

(2) Since the rank of a finite locally free group scheme is invariant under base change, we have $q(\alpha) = q(\alpha_{P'})$ for any field extension P' of P.

Lemma 2.2.9. *Let $\alpha : X \to Y$ be an isogeny of p-divisible groups on a scheme S. Then for any integer c the set of points $s \in S$ such that $q(\alpha_s) \leq c$ is closed.*

Proof. We prove that the set of points $s \in S$ such that $q(\alpha_s) > c$ is open. By Remark 2.2.8(1) we may assume that α_s is an isogeny, but $p^{-1}\alpha_s$ is not an isogeny. Then there is a neighborhood U of s such that $p^{-1}\alpha_t$ is not an isogeny for $t \in U$ and $\mathrm{ht}(\alpha_t)$ is a constant function on U. Let n_t be the smallest integer which makes $p^{n_t}\alpha_t$ into an isogeny. Then it is easily seen that $n_t \geq 0$ for any $t \in U$. Then we have

$$c < q(\alpha_s) = \mathrm{ht}(\alpha_s) = \mathrm{ht}(\alpha_t) \leq \mathrm{ht}(p^{n_t}\alpha_t) = q(\alpha_t)$$

□

Definition 2.2.10. Let $\alpha : X \to Y$ be a quasi-isogeny of p-divisible groups over P. We define the function $d(\alpha) = q(\alpha) + q(\alpha^{-1})$. For any two points of $\mathcal{M}(P)$ we define $d((X, \rho), (X', \rho')) = d(\rho'\rho^{-1})$.

Remark 2.2.11. If m_+ (resp. m_-) is the smallest integer such that $p^{m_+}\alpha$ (resp. $p^{m_-}\alpha^{-1}$) is an isogeny, then we have $d(\alpha) = (m_+ + m_-)\,\mathrm{ht}\,\mathbb{X}$. This is because the sequence of morphisms of p-divisible groups

$$p^{m_+ + m_-} : X \xrightarrow{p^{m_+}\alpha} Y \xrightarrow{p^{m_-}\alpha^{-1}} X$$

gives an exact sequence

$$0 \to \ker(p^{m_+}\alpha) \to \ker(p^{m_+ + m_-}) \to \ker(p^{m_-}\alpha^{-1}) \to 0$$

Corollary 2.2.12. *Lemma 2.2.9 remains valid with q replaced by d.*

Proof. We have

$$\{s \in S| \;\; d(\alpha_s) \leq c\} = \bigcup_{0 \leq n \leq c} (\{s \in S| \;\; q(\alpha_s^{-1}) \leq n\} \cap \{s \in S| \;\; q(\alpha_s) \leq c - n\})$$

□

Then Lemma 2.2.4 translates immediately into

Lemma 2.2.13. *There is a natural number c and a finite extension L' of L such that for any perfect field P containing L', and any point $X \in \mathcal{M}(P)$ there is a point $Y \in \mathcal{M}(L')$ such that $d(X, Y_P) \leq c$.*

Definition 2.2.14. Let $\rho : X \to Y$ be a quasi-isogeny and n be an integer such that $p^n\rho$ is an isogeny. We define the *height of ρ*

$$\mathrm{ht}\,\rho := \mathrm{ht}\,p^n\rho - \mathrm{ht}\,p^n$$

This is independent of the choice of n.

Because $d((X,\rho),(X,p\rho))=0$, the function d is not a metric on $\mathcal{M}(P)$. To get a metric, we should restrict ourselves to a subset of $\mathcal{M}(P)$. For $k\in\mathbb{Z}$ we consider the subfunctor $\mathcal{M}(k)\subset\mathcal{M}$ of quasi-isogenies of height k. We define

$$\tilde{\mathcal{M}}=\coprod_{h=0}^{\operatorname{ht}\mathbb{X}-1}\mathcal{M}(h)$$

Then the function d is a metric on $\tilde{\mathcal{M}}(P)$.

We define for any natural number c a subfunctor $\mathcal{M}_c$ of $\mathcal{M}$

$$\mathcal{M}_c(s)=\{(X,\rho)\in\mathcal{M}(S)|\ \ d(\rho_s)\leq c \text{ for any } s\in S\}$$

Lemma 2.2.15. *The functor $\mathcal{M}_c$ is representable by a formal scheme, which is locally formally of finite type over* $\operatorname{Spf}W(L)$.

Proof. Let $\mathcal{M}_c(h)$ be the open and closed subfunctor of $\mathcal{M}_c$ consists of points of quasi-isogenies of height h. Then $\mathcal{M}_c$ is a disjoint union of copies isomorphic to $\tilde{\mathcal{M}}_c=\coprod_{h=0}^{\operatorname{ht}\mathbb{X}-1}\mathcal{M}_c(h)$. Therefore it is enough to show that $\tilde{\mathcal{M}}_c$ is representable by a formal scheme formally of finite type.

We consider the functor $\tilde{\mathcal{M}}_c{}^n=\mathcal{M}^n\cap\tilde{\mathcal{M}}_c$. Let (X,ρ) be the universal p-divisible group on $\mathcal{M}^n$. Then the subfunctor $\tilde{\mathcal{M}}_c{}^n$ is representable by the completion of the scheme $\mathcal{M}^n$ along the closed set of points $s\in\mathcal{M}^n$ given by the condition that $d(\rho_s)\leq c$ and $0\leq\operatorname{ht}\rho_s\leq\operatorname{ht}\mathbb{X}-1$. Hence it is represented by a formal scheme formally of finite type over $\operatorname{Spf}W(L)$.

Let (X,ρ) be any point of $\tilde{\mathcal{M}}_c(P)$ where P is a field. Then $p^{-1}\rho$ is not an isogeny, otherwise we would have $\operatorname{ht}\rho=\operatorname{ht}p+\operatorname{ht}p^{-1}\rho\geq\operatorname{ht}\mathbb{X}$. Hence the smallest integer m_+ such that $p^{m_+}\rho$ is an isogeny must be non negative. Since $\operatorname{ht}\rho^{-1}=-\operatorname{ht}\rho$, we have $-\operatorname{ht}\mathbb{X}+1\leq\operatorname{ht}\rho^{-1}\leq 0$. Again we conclude that the smallest integer m_- such that $p^{m_-}\rho^{-1}$ is an isogeny is non negative. By Remark 2.2.11 we have $m_++m_-\leq c/\operatorname{ht}\mathbb{X}$. Hence m_+ is bounded by $c/\operatorname{ht}\mathbb{X}$.

Now we want to prove that for $n\geq c/\operatorname{ht}\mathbb{X}$, $(\tilde{\mathcal{M}}_c{}^n)_{\mathrm{red}}=(\tilde{\mathcal{M}}_c{}^{n+1})_{\mathrm{red}}$.

First we introduce a representability lemma and prove its consequence.

Lemma 2.2.16. *Let $\alpha:X\to Y$ be a quasi-isogeny of p-divisible groups on a scheme S. Then the functor $F(T)=\{\phi\in\operatorname{Hom}(T,S)|\ \ \phi^*\alpha$ is an isogeny$\}$ is representable by a closed subscheme of S.*

For the proof, see [33] 2.9.

Corollary 2.2.17. *If S is a reduced scheme, $\alpha:X\to Y$ be a quasi-isogeny of p-divisible groups on S, then α is an isogeny if and only if α_s is an isogeny for any $s\in S$.*

Proof. The hypothesis that α is an isogeny clearly implies that α_s is an isogeny for any $s\in S$.

If α_s is an isogeny for any $s\in S$, we denote $k(s)$ the residue field of s. Then

$$F(\operatorname{Spec}k(s))=\{\operatorname{Hom}(\operatorname{Spec}k(s),S)|\ \ \alpha_s \text{ is an isogeny}\}$$

Hence we see that F on the point level is actually S. Since S is reduced, we see that F is representable by S and the element $\mathrm{Id}\in F(S)$ gives that α is an isogeny. $\square$

In order to prove $(\tilde{\mathcal{M}_c}^n)_{\mathrm{red}} = (\tilde{\mathcal{M}_c}^{n+1})_{\mathrm{red}}$ for $n \geq c/\operatorname{ht}\mathbb{X}$, by 2.1.10 we only need to show that for any reduced scheme S, $(\tilde{\mathcal{M}_c}^n)(S) = (\tilde{\mathcal{M}_c}^{n+1})(S)$. By definition a point (X, ρ) in $\tilde{\mathcal{M}_c}^n(S)$ satisfies that $p^n\rho$ is an isogeny. After Corollary 2.2.17 this is equivalent to $p^n\rho_s$ is an isogeny for any $s \in S$. Since $n \geq c/\operatorname{ht}\mathbb{X}$, $p^{n+1}\rho_s$ is an isogeny if and only if $p^n\rho_s$ is an isogeny. Then by Corollary 2.2.17 again, we have $p^{n+1}\rho$ is an isogeny. Hence $(\tilde{\mathcal{M}_c}^n)(S) = (\tilde{\mathcal{M}_c}^{n+1})(S)$.

Now we prove that $\tilde{\mathcal{M}_c} = \varinjlim \tilde{\mathcal{M}_c}^n$ is a formal scheme. We fix an affine open subscheme $U \subset (\tilde{\mathcal{M}_c}^n)_{\mathrm{red}}$ for large n. For n big we get an affine open formal subscheme $\operatorname{Spf} R_n$ of $\tilde{\mathcal{M}_c}^n$ whose underlying set is U. Hence we have a projective system of surjective maps of adic rings $R_{n+1} \to R_n$. Let $R = \varprojlim R_n$ be the projective limit. We write $R_n = R/\mathfrak{a}_n$ and let J be the inverse image of an ideal of definition in some R_n. We have to prove that R is J-adic. Since R_n is J-adic, we may write $R = \varprojlim R/(\mathfrak{a}_n + J^m)$. We claim that for fixed m, the following descending sequence stabilizes

$$\mathfrak{a}_1 + J^m \supset \mathfrak{a}_2 + J^m \supset \cdots \supset \mathfrak{a}_n + J^m \supset \cdots$$

Let X_n be the universal p-divisible group on $\operatorname{Spf} R_n$. Then $X = \varinjlim X_n$ defines a p-divisible group on R/J^m. We get by the representability that there is a suitable N and a unique map $R_N \to R/J^m$ such that the pull-back of X_n gives X. For any $n \geq N$ the composite map

$$R_n \to R_N \to R/J^m \to R_n/J^m R_n$$

has to be the canonical one. This implies that the first arrow induces an isomorphism $R_n/J^m R_n \to R_N/J^m R_N$, then $\mathfrak{a}_n + J^m = \mathfrak{a}_N + J^m$. Therefore by Proposition 2.1.12 R is an J-adic ring. This completes the proof of Lemma 2.2.15. $\square$

Now we prove Theorem 2.2.1.

Proof. Let c and L' as in Lemma 2.2.13. It is enough to show that $\mathcal{M}$ is representable over L'. As in Lemma 2.2.15, it is enough to show that the subfunctor $\tilde{\mathcal{M}}$ is representable by a formal scheme locally of finite type. Obviously Lemma 2.2.13 remains valid for $\tilde{\mathcal{M}}$.

Let a be an integer and X be the universal p-divisible group over $\tilde{\mathcal{M}}_a$. For a point $(Y, y : \mathbb{X}_{L'} \to Y)$ of $\tilde{\mathcal{M}}(L')$ we denote by $\tilde{\mathcal{M}}_a(Y) \subset \tilde{\mathcal{M}}_a$ the closed subset of points $s \in \tilde{\mathcal{M}}_a$ such that $d(X_s, Y_s) \leq c$. By the triangular inequality, $\tilde{\mathcal{M}}_a(Y) = \emptyset$ if $d(\mathbb{X}_{L'}, Y) > a + c$.

Let U_a^f be the open formal subscheme of $\tilde{\mathcal{M}}_a$, whose underlying set is the complement of

$$\bigcup_{Y \in \tilde{\mathcal{M}}(L'), d(\mathbb{X}_{L'}, Y) \geq f} \tilde{\mathcal{M}}_a(Y)$$

Then U_a^f is locally formally of finite type over $\operatorname{Spf} W(L')$ as $\tilde{\mathcal{M}}_a$ is. We claim that $U_a^f = U_{a+1}^f$ if $a \geq f + c$.

First we show the equality for the underlying sets. Let $Z \in U_{a+1}^f(P)$ a point with values in some field P. We have to show that $Z \in U_a^f$, i.e. $d(\mathbb{X}_P, Z) \leq a$. By Lemma 2.2.13 there exists a point $Y \in \tilde{\mathcal{M}}(L')$ such that $d(Y_P, Z_P) \leq c$. By the definition of U_{a+1}^f it follows that $d(\mathbb{X}_P, Y_P) < f$. Hence $d(\mathbb{X}_P, Z) < f + c \leq a$.

The equality of formal schemes follows because $\tilde{\mathcal{M}}_a$ is the completion of $\tilde{\mathcal{M}}_{a+1}$ along the closed image of the inclusion. Indeed, this implies that U_a^f is the completion of U_{a+1}^f along the closed subset U_{a+1}^f. Hence the claim follows.

We set $U^f = U_a^f$ for any $a \geq f + c$. Clearly $U^f \to U^{f+1}$ is an open immersion of formal schemes of finite type. We have $\tilde{\mathcal{M}} = \bigcup_f U^f$, because any point s of $\tilde{\mathcal{M}}$ such that $d(\mathbb{X}_s, X_s) < f - c$ is contained in the open set U^f. Indeed, if s is in the complement of U^f, there is a $Y \in \tilde{\mathcal{M}}(L')$, such that $d(X_s, Y_s) \leq c$ and $d(\mathbb{X}_s, Y_s) \geq f$. Hence we get the contradiction $d(\mathbb{X}_s, X_s) \geq f - c$. Since we can write $\tilde{\mathcal{M}}$ as a union of increasing open sub formal schemes locally of finite type over $\operatorname{Spf} W(L')$, the theorem follows. $\square$

One can also consider the variant of the moduli functor for p-divisible groups with additional structures of type (E): endomorphisms or of type (PE): polarizations and endomorphisms.

Case (E): We consider a finite dimensional semi-simple $\mathbb{Q}_p$-algebra B and a maximal order O_B of B. Suppose $\mathbb{X}$ has an action of O_B and can be lifted to a p-divisible group $\widetilde{\mathbb{X}}$ over O_K with a compatible action of O_B where K is a finite extension of K_0. Let E be the field of definition of the isomorphism class of $\mathrm{Lie}(\widetilde{\mathbb{X}})$ as representation of B. E is a finite extension of $\mathbb{Q}_p$. Let $\breve{E} = EK_0$ and $O_{\breve{E}}$ the integer ring of $\breve{E}$.

We consider the functor $\breve{\mathcal{M}}$ on $\mathrm{Nilp}_{O_{\breve{E}}}$ which associates to $S \in \mathrm{Nilp}_{O_{\breve{E}}}$ the set of isomorphism classes of pairs (X, ρ) satisfying:

(1) A p-divisible group X over S with an action of O_B such that for any S-scheme S' and any element $a \in O_B \otimes O_{S'}$, we have

$$\det{}_{O_{S'}}(a, \mathrm{Lie}(X_{S'})) = \det{}_K(a, \mathrm{Lie}(\widetilde{\mathbb{X}})_K) \tag{2.1}$$

(2) An O_B-quasi-isogeny $\rho : \mathbb{X}_{\overline{S}} \to X_{\overline{S}}$

Case (PE): We suppose $p \neq 2$ and B with an involution $*$ stabilizes O_B. If X is a p-divisible group with an action of O_B, i.e. $i : O_B \to \mathrm{End}(X)$, we have on the dual $\widehat{X}$ of X an action of O_B given by $b \mapsto i(b^*)^\wedge$. We call an O_B symmetric quasi-isogeny $X \to \widehat{X}$ a $*$-polarization of X. We suppose that $\mathbb{X}$ has a $*$-polarization λ.

We consider the functor $\breve{\mathcal{M}}$ defined as above with an additional condition that there exists an O_B isomorphism $\lambda_X : X \to \widehat{X}$ and a constant $c_X \in \mathbb{Q}_p^\times$ such that $\hat{\rho} \circ \lambda_X \circ \rho = c_X \lambda$.

Theorem 2.2.18. *The functor $\breve{\mathcal{M}}$ is representable by a formal scheme locally formally of finite type over* $\operatorname{Spf}(O_{\breve{E}})$.

Example 2.2.19. (1) We consider the Lubin-Tate case. Let $\mathbb{X}$ be a p-divisible group of dimension 1 and height h over $\overline{\mathbb{F}_p}$. We assume moreover that the isocrystal of $\mathbb{X}$ is isolinic of slope $1/h$. We consider the functor $\mathcal{M}^{\mathrm{LT}}$ defined in Theorem 2.2.1. Then $\mathcal{M}^{LT} \cong \coprod_{n \in \mathbb{Z}} \operatorname{Spf} W(\overline{\mathbb{F}_p})[[T_1, \ldots, T_{h-1}]]$, c.f. [33] Proposition 3.79.

(2) We consider the Drinfeld's example which is of case (E). Let F be a finite extension of $\mathbb{Q}_p$ and B:=D be a division algebra of center F with invariant $1/d$. We denote $\widetilde{F}$ an unramified extension of F contained in D of degree d and τ the Frobenius automorphism of $\widetilde{F}$ over F. Let

π be a uniformizer of O_F, q the cardinality of the residue field of F and Π an element of O_D such that

$$O_D = O_{\widetilde{F}} < \Pi > /(\Pi^d = \pi, \Pi a = \tau(a)\Pi \text{ for any } a \in O_{\widetilde{F}})$$

Definition 2.2.20. Let S be an O_F-scheme with p nilpotent on it and X be a p-divisible group on S. We call X a *p-divisible O_F-module* over S if it has an action $\iota : O_F \to \mathrm{End}(X)$ such that the action of O_F on the tangent space $\mathrm{Lie}\, X$ induced by ι is given by the structure morphism $O_F \to O_S$.

Definition 2.2.21. The *F-height* of a p-divisible O_F-module X is the quotient of $\mathrm{ht}\, X$ by $[F : \mathbb{Q}_p]$. The F-height is actually the integer h such that $\mathrm{rk}(\ker \pi : X \to X) = q^h$.

Definition 2.2.22. Let X be a p-divisible O_F-module over S. We call X a *special formal O_D-module* if it has an action of O_D and

(i) X is a p-divisible O_F-module by the induced action of O_F

(ii) the action of $O_{\widetilde{F}}$ on $\mathrm{Lie}\, X$ makes it into an $O_S \otimes_{O_F} O_{\widetilde{F}}$-module locally free of rank one.

We assume $S = \mathrm{Spec}\, k$, where k is an algebraically closed field in the following propositions.

Proposition 2.2.23. *([7]) A special formal O_D-module if of F-height a multiple of d^2.*

Proposition 2.2.24. *([33] 3.60) Any two special formal O_D-modules of F-height d^2 are isogeneous. The group of O_D-quasi-isogenies of such a special formal O_D-module is isomorphic to $\mathbf{GL}_d(F)$.*

Now we define Drinfeld's functor. We choose an embedding $\varepsilon : F \hookrightarrow \overline{\mathbb{Q}_p}$ of F. Let $k = \overline{\mathbb{F}_p}$ be the residue field of $\overline{\mathbb{Q}_p}$, $K_0 = W(\overline{\mathbb{F}_p})_{\mathbb{Q}}$. Let $E = F$ and $\breve{E} = EK_0$. Fix $\mathbb{X}$ a special formal O_D-module over $\overline{\mathbb{F}_p}$. We consider a functor $\breve{\mathcal{M}}^{\mathrm{Dr}}$ on $\mathrm{Nilp}_{O_{\breve{E}}}$ which associates to every $S \in \mathrm{Nilp}_{O_{\breve{E}}}$ the set of isomorphism classes of pairs (X, ρ) satisfying:

(1) a special formal O_D-module X over S of F-height d^2

(2) an O_D-quasi-isogeny $\rho : \mathbb{X}_{\overline{S}} \to X_{\overline{S}}$.

Remark 2.2.25. The condition (1) can be expressed as a condition of the determinant of the action of O_D on $\mathrm{Lie}(X)$ as in 2.1, see [33] 3.58.

Proposition 2.2.26. *In the Drinfeld example $\breve{\mathcal{M}}^{Dr}$ is a p-adic formal scheme locally of finite type over* $\mathrm{Spf}\, O_{\breve{E}}$.

Geometrically, the generic fiber of $\mathcal{M}^{\mathrm{Dr}}$ can be used to p-adically uniformize the rigid analytic spaces corresponding to Shimura varieties associated to certain unitary groups, cf. [8].

Remark 2.2.27. The formal schemes $\breve{\mathcal{M}}$ are not necessarily p-adic, as $pO_{\breve{\mathcal{M}}}$ may not be an ideal of definition.

2.3 Rigid Analytic Geometry

Let O be a complete discrete valuation ring which is an extension of $\mathbb{Z}_p$ of finite type and F be its fraction field. Let k be the residue field and π a uniformizer.

Definition 2.3.1. The F-algebra

$$F\{y_1, \dots, y_n\} := \{f = \sum a_{i_1,\dots,i_n} y_1^{i_1} \cdots y_n^{i_n} | \ a_i \in F, i_j \geq 0 \text{ and } |a_{i_1,\dots,i_n}| \to 0 \text{ for } i_1 + \cdots + i_n \to \infty\}$$

is called the *Tate algebra* in n variables over F.

A Tate algebra is Noetherian and carries the Gauss norm $|f| := \sup\{|a_{i_1,\dots,i_n}|\}$ with respect to which it is complete. The Gauss norm extends the valuation on F.

Definition 2.3.2. An *affinoid F-algebra* is an F-algebra B which can be described as a quotient of a Tate algebra for some n.

Remark 2.3.3. For every presentation $B = F\{y_1, \dots, y_n\}/I$, the residue norm of the Gauss norm is a complete F-algebra norm on B. All these norms induce the same topology on B. More generally any F-algebra norm on B defining this topology is called an *F-Banach norm* on B.

Let B be an affinoid F-algebra. In rigid analytic geometry one equips the set $\operatorname{Spm} B$ of all maximal ideals of B with a Grothendieck topology and a structure sheaf. One calls $\operatorname{Spm} B$ an *affinoid rigid analytic space.* Every point $x \in \operatorname{Spm} B$ has *residue field* B/x which is a finite extension of F. Every F-algebra homomorphism $B \to C$ induces a map $\operatorname{Spm} C \to \operatorname{Spm} B$ and these are precisely the *morphisms* between affinoid rigid analytic spaces.

Definition 2.3.4. An *affinoid subdomain* of $\operatorname{Spm} B$ is an affinoid space $\operatorname{Spm} B'$ together with an F-algebra homomorphism $B \to B'$ which identifies the set $\operatorname{Spm} B'$ with a subset U of $\operatorname{Spm} B$ and which is universal for morphisms $\operatorname{Spm} A \to \operatorname{Spm} B$ of affinoid rigid analytic spaces whose image lies in U.

Any covering in the definition of the Grothendieck topology is called an *admissible covering* and any finite covering of $\operatorname{Spm} B$ by affinoid subdomains is admissible.

Definition 2.3.5. A *rigid analytic space* over F is a set X carrying a Grothendieck topology and a structure sheaf, such that X possesses an admissible covering (covering in the Grothendieck topology) by affinoid rigid analytic spaces.

Definition 2.3.6. An admissible covering of a rigid analytic space is said to be of *finite type* if every member of the covering meets only finitely many of the other members.

Definition 2.3.7. A rigid analytic space over F is called *quasi-paracompact* if it possesses an admissible affinoid covering of finite type.

Definition 2.3.8. A rigid analytic space is called *quasi-separated* if the intersection of any two affinoid subdomains is a finite union of affinoid subdomains.

Definition 2.3.9. A morphism of rigid spaces $f : Y \to X$ is *smooth (resp. étale)* if there exist admissible affinoid coverings $\{Y_i\}_i$ and $\{X_i\}_i$ of Y and X such that

(i) $f(Y_i) \subset X_i$

(ii) If $A_i = \Gamma(X_i, O_X)$, $B_i = \Gamma(Y_i, O_Y)$, there exists an isomorphism

$$B_i = A_i\{T_1, \ldots, T_n\}/(f_1, \ldots, f_r)$$

and locally on X_i a suitable $r \times r$ minor of $(\partial f_k/\partial T_l)_{1\le k\le r, 1\le l\le n}$ with determinant an invertible element of B_i (resp. and $r = n$).

Proposition 2.3.10. *([33] 5.10) The morphism of rigid spaces $f : X \to Y$ is étale if and only if the following condition is satisfied. Let Z be a rigid analytic space with only one point and $Z_0 \subset Z$ be a closed subspace. Then any commutative diagram of morphism below with solid arrows can be completed in a unique way by a dotted arrow into a commutative diagram*

$$\begin{tikzcd} Z_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Rigid analytic spaces can be viewed as generic fibers of formal schemes which are locally formally of finite type over $\operatorname{Spf} O$. This is Raynaud-Berthelot's functor. We first explain Raynaud's construction.

Definition 2.3.11. A *topologically finitely presented (tfp) O-algebra* is an O-algebra of the form

$$A = O\{X_1, \ldots, X_n\}/I$$

where I is a finitely generated ideal in the ring $O\{X_1, \ldots, X_n\}$ of restricted power series in n variables over O: power series $\sum a_J X^J \in O[[X_1, \ldots, X_n]]$ with $X^J = X_1^{j_1} \cdots X_n^{j_n}$ such that $a_J \to 0$ as $||J|| := j_1 + \cdots + j_n \to \infty$.

Such an A is a π-adic noetherian ring, see [13] I Chapter 0 7.5.5.

Definition 2.3.12. (i) A *tfp affine formal scheme* over O is a locally ringed space isomorphic to $\operatorname{Spf} A$, where A is a tfp O-algebra.

(ii) A *locally of finite type (resp. finite type) tfp O-formal scheme* is a locally ringed space with an open covering (resp. finite covering) by tfp affine O-formal schemes.

To every tfp O-formal scheme $\mathfrak{X}$ locally of finite type one can associate an F-rigid analytic space $\mathfrak{X}^{\mathrm{rig}}$ called Raynaud's *generic fiber.* To describe the construction we start with the affine case.

If $\mathfrak{X} = \operatorname{Spf} A$ is a tfp affine formal scheme over O, $A \otimes F$ is clearly an affinoid F-algebra. Then we define $\mathfrak{X}^{\mathrm{rig}} = \operatorname{Spm}(A \otimes F)$. We observe that the points of $\mathfrak{X}^{\mathrm{rig}}$ is in bijection with the quotients of A which are integral domains, finite and flat over O. Moreover we have a *specialization* map $sp : \mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}$. On the set level, sp is constructed as follows: For any

$x \in \mathrm{Spm}(A \otimes F)$ defines a homomorphism $A \otimes F \to k(x)$ where $k(x)$ is the residue field of x and a finite extension of F. This homomorphism maps A to the valuation ring $O(x)$ of $k(x)$ and we have a homomorphism $A/(\pi) \to \tilde{k}(x)$ where $\tilde{k}(x)$ is the residue field of $k(x)$. This gives a point $sp(x) \in \mathrm{Spec}\, A/(\pi) \subset \mathfrak{X}$.

Then for general locally of finite type tfp O-formal scheme $\mathfrak{X}$, we define the set $\mathfrak{X}^{\mathrm{rig}}$ to be the set of closed formal subschemes $\mathfrak{Z}$ which are irreducible reduced finite flat over O. The support of such a formal subscheme $\mathfrak{Z}$ is a closed point of $\mathfrak{X}$, called the *specialization* of the point $x \in \mathfrak{X}^{\mathrm{rig}}$ corresponding to $\mathfrak{Z}$. This gives the specialization map $sp : \mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}$. For any affine open $\mathfrak{U} = \mathrm{Spf}(A) \subset \mathfrak{X}$, $sp^{-1}(\mathfrak{U})$ can be identified with $\mathrm{Spm}(A \otimes F)$.

Proposition 2.3.13. *([33]) Let $\mathfrak{X}$ be a tfp formal scheme locally of finite type*

(i) There exists a unique rigid analytic structure on $\mathfrak{X}^{\mathrm{rig}}$ over F with the following properties:

(a) The inverse image under $sp : \mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}$ of an open subscheme (resp. of an open covering) of $\mathfrak{X}$ is an admissible open subset (resp. an admissible covering) of $\mathfrak{X}^{\mathrm{rig}}$.

(b) For any affine open subscheme $\mathfrak{U} = \mathrm{Spf}\, A \subset \mathfrak{X}$ the structure on $\mathfrak{U}^{\mathrm{rig}} = sp^{-1}(\mathfrak{U})$ induced from $\mathfrak{X}^{\mathrm{rig}}$ coincides with the one on $\mathrm{Spm}(A \otimes F)$.

(ii) The map sp defines a morphism of ringed sites $\mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}$ with $sp_*(O_{\mathfrak{X}^{\mathrm{rig}}}) = O_{\mathfrak{X}} \otimes F$. This morphism has the following universal property, let $\mathfrak{Y}$ be any rigid analytic space and let $u : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of ringed sites. Then u factors in a unique way through sp.

(iii) The functor $\mathfrak{X} \to \mathfrak{X}^{\mathrm{rig}}$ has the following properties,

(a) If $\mathfrak{X}$ is of finite type, then $\mathfrak{X}^{\mathrm{rig}}$ is quasi-compact.

(b) It commutes with products and transforms open (resp. closed) immersions to open (resp. closed) immersions.

Remark 2.3.14. For any coherent $O_{\mathfrak{X}}$-module $\mathcal{F}$, we denote $\mathcal{F}^{\mathrm{rig}}$ the coherent $O_{\mathfrak{X}^{\mathrm{rig}}}$-module $sp^*(\mathcal{F})$.

Now we consider the case of a locally noetherian formal scheme $\mathfrak{X}$ locally formally of finite type over $\mathrm{Spf}\, O$. As before we start with the affine case.

Let $\mathfrak{X} = \mathrm{Spf}\, A$ and $f_1, \ldots, f_r$ be a system of generators of a defining ideal. For each n, take

$$B_n = A\{T_1, \ldots, T_r\}/(f_1^n - \pi T_1, \ldots, f_r^n - \pi T_r) \tag{2.2}$$

where $A\{T_1, \ldots, T_r\}$ is the π-adic completion of $A[T_1, \ldots, T_r]$. The hypothesis implies that B_n is topologically finitely presented over O, hence $B_n \otimes F$ is an affinoid F-algebra. For $n' \geq n$ we have a canonical homomorphism $B_{n'} \to B_n$ by sending T_i' to $f_i^{n'-n} T_i$. The corresponding morphism $\mathrm{Spm}(B_n \otimes F) \to \mathrm{Spm}(B_{n'} \otimes F)$ identifies $\mathrm{Spm}(B_n \otimes F)$ with the special domain defined by $|f_i(x)| \leq |\pi|^{1/n}$. The rigid space $\mathfrak{X}^{\mathrm{rig}}$ is then defined as the union of $\mathrm{Spm}(B_n \otimes F)$, with the $\mathrm{Spm}(B_n \otimes F)$ as an admissible open coverings. One shows easily that this definition is independent of the choice of the defining ideal and of the set of generators. This definition coincides with the usual one in the case where $\mathfrak{X}$ is a tfp O-formal scheme.

We can also define the specialization map in this case. For $n' \geq n$ we have natural homomorphism $A \to B_{n'} \to B_n$ which gives the commutative diagram

$$\begin{tikzcd} \mathrm{Spm}(B_n \otimes F) \arrow[r, "sp"] \arrow[d] & \mathrm{Spf}(B_n) \arrow[d] \\ \mathrm{Spm}(B_{n'} \otimes F) \arrow[r, "sp"] & \mathrm{Spf}(B_{n'}) \end{tikzcd}$$

Since we have $\mathrm{Spf}(A) = \varinjlim \mathrm{Spf}(B_n)$, by passing to the limit, we get the a morphism of ringed sites $sp : \mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}$.

As before, we can generalize the construction to general formal schemes and the Proposition 2.3.13 carries over.

Example 2.3.15. (1) Let $\mathfrak{X} = \mathrm{Spf}\, O\{T_1, \ldots, T_n\}$ then $\mathfrak{X}^{\mathrm{rig}}$ is the closed unit ball.

(2) Let $\mathfrak{X} = \mathrm{Spf}\, O[[T_1, \ldots, T_n]]$ with ideal of definition $(\pi, T_1, \ldots, T_n)$ then $\mathfrak{X}^{\mathrm{rig}}$ is the open unit ball regarded as the increasing union of closed balls of radius $|\pi|^{1/n}$.

There is another variant of rigid analytic geometry which remedies the problem that rigid analytic spaces are not topological spaces in the classical sense and provides more underlying points. The theory was developed by Berkovich.

Let B be an affinoid F-algebra with F-Banach norm $|\cdot|$. B is called *strictly F-affinoid algebra* by Berkovich.

Definition 2.3.16. An *analytic point* x of B is a semi-norm $|\cdot|_x : B \to \mathbb{R}_{\geq 0}$ which satisfies

(i) $|f + g|_x \leq \max\{|f|_x, |g|_x\}$ for all $f, g \in B$

(ii) $|fg|_x = |f|_x |g|_x$ for all $f, g \in B$

(iii) $|\lambda|_x = |\lambda|$ for all $\lambda \in F$

(iv) $|\cdot|_x : B \to \mathbb{R}_{\geq 0}$ is continuous with respect to the norm $|\cdot|$ on B.

The set of all analytic points of B is denoted by $\mathcal{M}(B)$. On $\mathcal{M}(B)$ one associates the coarsest topology such that for every $f \in B$ the map $\mathcal{M}(B) \to \mathbb{R}_{\geq 0}$ given by $x \mapsto |f|_x$ is continuous. Then $\mathcal{M}(B)$ is a compact Hausdorff space, such a space is called a *strictly F-affinoid space.*

Every morphism $\varphi : B \to C$ of affinoid F-algebras is automatically continuous and hence induces a continuous morphism $\mathcal{M}(\varphi) : \mathcal{M}(C) \to \mathcal{M}(B)$ by mapping the semi-norm $C \to \mathbb{R}_{\geq 0}$ to the composition $B \to C \to \mathbb{R}_{\geq 0}$. By definition the $\mathcal{M}(\varphi)$ are the morphisms in the category of strictly F-affinoid spaces. In particular, for an affinoid subdomain $\mathrm{Spm}\, B' \subset \mathrm{Spm}\, B$ this morphism identifies $\mathcal{M}(B')$ with a closed subset of $\mathcal{M}(B)$.

For every analytic point $x \in \mathcal{M}(B)$ we let $\ker|\cdot|_x := \{b \in B| \quad |b|_x = 0\}$. It is a prime ideal in B.

Definition 2.3.17. We define the *(complete) residue field $k(x)$ of x* as the completion with respect to $|\cdot|_x$ of the fraction field of $B/\ker|\cdot|_x$.

Hence there is a natural continuous homomorphism $B \to k(x)$ of F-algebra. Conversely, let K be a complete extension of F, by which we mean a field extension of F equipped with an absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$ which restricts on F to the norm of F and K is complete with respect to $|\cdot|$. Any continuous F-algebra homomorphism $B \to K$ defines on B a semi-norm which is an analytic point.

Remark 2.3.18. Every maximal ideal of B, i.e. every rigid analytic point of $\operatorname{Spm} B$, clearly defines an analytic point with the residue field a finite extension of F. For general analytic point $x \in \mathcal{M}(B)$, $k(x)$ may be quite large. For example, one can see [28] A 2.2 (d) that there exists a point in $\mathcal{M}(F\{y\})$ with residue field $\widehat{\overline{F}}$ the completion of an algebraic closure of F.

Definition 2.3.19. (i) A *strictly F-analytic space* is a topological space which admit an atlas (covering by compact subsets) homeomorphic to strictly affinoid F-analytic charts.

(ii) A *good* strictly F-analytic space is a strictly F-analytic space such that every point has a strictly F-analytic neighborhood.

Definition 2.3.20. We call good strictly F-analytic spaces as *Berkovich spaces.* A covering $\{U_i\}_i$ of X by strictly F-affinoid subspaces $U_i \subset X$ is an *affinoid covering* if the open interiors of the U_i in X still cover X.

One can associate a Berkovich space Y^{an} to schemes Y which are locally of finite type over F. Moreover Y^{an} is Hausdorff if and only if the scheme is separated. To every strictly F-analytic space X which is Hausdorff one can associate a quasi-separated rigid analytic space

$$X^{\mathrm{rig}} := \{x \in X| \ \ k(x) \text{ is a finite extension of } F\}$$

Definition 2.3.21. A topological Hausdorff space is called *paracompact* if every open covering $\{U_i\}_i$ has a locally finite refinement $\{V_j\}_j$, where *locally finite* means that every point has a neighborhood which meets only finitely many of the V_j.

The relation between Berkovich spaces, rigid analytic spaces, and formal schemes is explained in the following theorem

Theorem 2.3.22. *The following three categories are equivalent:*

(i) the category of paracompact strictly F-analytic spaces,

(ii) the category of quasi-separated quasi-paracompact rigid analytic spaces over F, and

(iii) the category of quasi-paracompact admissible formal O-schemes, localized by admissible formal blowing-ups.

Remark 2.3.23. Parallel to Berthelot's construction, we can associate to $\mathfrak{X}$ a formal scheme locally formally of finite type over $\operatorname{Spf} O$ an F-analytic space $\mathfrak{X}^{\mathrm{an}}$ as a union of the Berkovich spectra $\mathcal{M}(B_n \otimes F)$, where B_n is as in (2.2).

Regarding paracompactness the following lemma will be useful.

Lemma 2.3.24. *[28] Let X be a Berkovich space over F. Assume that X admits a countable affinoid covering. Then X possesses a countable fundamental system of neighborhoods consisting of affinoid Berkovich subspaces. In particular if X is Hausdorff every open subset of X is a paracompact Berkovich space.*

2.4 Period Morphisms

First we prove a main theorem that enables the construction of the period morphisms.

Let (F, O, k, π) be a complete discrete valuation ring of unequal characteristic with perfect residue field of characteristic $p > 0$. Let $\mathcal{M}$ be a formal scheme locally formally of finite type over $\operatorname{Spf} O$. Let X be a p-divisible group over $\mathcal{M}$ and M_X the Lie algebra of the universal extension of X. Let $\mathbb{X}$ be a (fixed) p-divisible group over k and we are given a quasi-isogeny

$$\rho : \mathbb{X}_{\mathcal{M}_0} \longrightarrow X_{\mathcal{M}_0}$$

where $\mathcal{M}_0$ denotes the k-scheme defined by an ideal of definition of $\mathcal{M}$ containing the uniformizer π. By the rigidity of quasi-isogenies (Theorem 1.1.13), ρ is independent of the choice of such an ideal of definition.

Theorem 2.4.1. *The quasi-isogeny ρ induces a canonical and functorial isomorphism of locally free $O_{\mathcal{M}^{\mathrm{rig}}}$-modules of finite rank, compatible with base change*

$$\tilde{\rho} : E(\mathbb{X}) \otimes_{W(k)_{\mathbb{Q}}} O_{\mathcal{M}^{\mathrm{rig}}} \cong (M_X)^{\mathrm{rig}}$$

Here $E(\mathbb{X})$ denotes the isocrystal associated to the p-divisible group $\mathbb{X}$, cf Section 1.3.

Proof. We first assume $\mathcal{M}$ is a tfp π-adic formal scheme. Let $\mathcal{M}_0$ be defined by the image of π and $\mathcal{M}_0'$ be defined by the image of p. Then $\mathcal{M}_0 \subset \mathcal{M}_0'$ is a nilpotent immersion. By the rigidity of quasi-isogenies the quasi-isogeny ρ extends uniquely to a quasi-isogeny of p-divisible groups over $\mathcal{M}_0'$

$$\rho' : \mathbb{X}_{\mathcal{M}_0'} \longrightarrow X_{\mathcal{M}_0'}$$

Then by Proposition 1.1.10 there exist $n \geq 0$,$m \geq 0$ and an isogeny $f : X_{\mathcal{M}_0'} \to \mathbb{X}_{\mathcal{M}_0'}$ such that $\rho_n' = p^n \rho'$ is an isogeny and $\rho_n' \circ f = f \circ \rho_n' = p^m \operatorname{Id}$.

Since the closed immersion $\mathcal{M}_0' \subset \mathcal{M}$ has a canonical divided power structure. We may apply the theory of Grothendieck-Messing. Therefore $\mathbb{D}(\rho_n')$ and $\mathbb{D}(f)$ induces an isogeny of locally free $O_{\mathcal{M}}$-modules of finite rank below and $\mathbb{D}(\rho_n') \circ \mathbb{D}(f) = \mathbb{D}(f) \circ \mathbb{D}(\rho_n') = p^m \operatorname{Id}$

$$D(\mathbb{X}) \otimes O_{\mathcal{M}} \longrightarrow M_X$$

Applying sp^* on each side we get an isomorphism of $O_{\mathcal{M}^{\mathrm{rig}}}$-modules

$$\mathbb{D}(\rho_n')^{\mathrm{rig}} : E(\mathbb{X}) \otimes O_{\mathcal{M}^{\mathrm{rig}}} \cong (M_X)^{\mathrm{rig}}$$

this is because the consideration of rigid spaces means tensoring F with the algebra and hence we can invert p. Then we define our isomorphism $\tilde{\rho} = \frac{1}{p^n} \mathbb{D}(\rho_n')^{\mathrm{rig}}$.

For $p = 2$, we consider $\mathcal{M}_0'$ is the subscheme defined by 4.

For the general case, i.e. $\mathcal{M}$ is locally noetherian locally formally of finite type over $\operatorname{Spf} O$. We may assume $\mathcal{M} = \operatorname{Spf} A$ is affine. Then $\mathcal{M}^{\mathrm{rig}}$ is the union of the open subspace $\operatorname{Spm}(B_n \otimes F)$

and $\operatorname{Spf} B_n$ comes with a morphism to $\mathcal{M}$. Since $\operatorname{Spf} B_n$ is a tfp π-adic formal scheme, we take X_n the pull-back of X to $\operatorname{Spf} B_n$. Then we have canonical isomorphisms

$$\tilde{\rho_n} : E(\mathbb{X}) \otimes O_{\operatorname{Spm}(B_n \otimes F)} \longrightarrow (M_{X_n})^{\mathrm{rig}}$$

By passing to the limit we can define the isomorphism

$$\tilde{\rho} : E(\mathbb{X}) \otimes O_{\mathcal{M}^{\mathrm{rig}}} \longrightarrow (M_X)^{\mathrm{rig}}$$

□

Remark 2.4.2. When considering Berkovich's F-analytic space, we have $E(\mathbb{X}) \otimes_{W(k)_{\mathbb{Q}}} O_{\mathcal{M}^{\mathrm{an}}} \cong (M_X)^{\mathrm{an}}$.

Now consider the locally noetherian formal scheme $\breve{\mathcal{M}}$ as in Section 2.2. Then $\breve{\mathcal{M}}$ is locally formally of finite type over $\operatorname{Spf} O_{\breve{E}}$. We take $(X^{\mathrm{univ}}, \rho^{\mathrm{univ}})$ as our universal p-divisible group on $\breve{\mathcal{M}}$. Then we have

$$E(\mathbb{X}) \otimes O_{\breve{\mathcal{M}}^{\mathrm{rig}}} \cong (M_{X^{\mathrm{univ}}})^{\mathrm{rig}}$$

The kernel of the epimorphism

$$E(\mathbb{X}) \otimes O_{\breve{\mathcal{M}}^{\mathrm{rig}}} \cong (M_{X^{\mathrm{univ}}})^{\mathrm{rig}} \twoheadrightarrow (\operatorname{Lie}(X^{\mathrm{univ}}))^{\mathrm{rig}}$$

defines an $\breve{\mathcal{M}}^{\mathrm{rig}}$-valued point of the Grassmannian $\operatorname{Grass}_{h-d}(E(\mathbb{X}))$, if our fixed p-divisible group $\mathbb{X}$ is of height h and dimension d.

Definition 2.4.3. By the universal property of the construction of rigid analytic spaces associated to locally of finite type $\breve{E}$-schemes, the $\breve{\mathcal{M}}^{\mathrm{rig}}$-valued point defined above gives a morphism of rigid analytic spaces $\breve{\pi} : \breve{\mathcal{M}}^{\mathrm{rig}} \to \operatorname{Grass}_{h-d}(E(\mathbb{X}))^{\mathrm{rig}}$. We call $\breve{\pi}$ the *period morphism.*

Remark 2.4.4. The period morphism in Berkovich's sense is denoted by $\breve{\pi}^{\mathrm{an}} : \breve{\mathcal{M}}^{\mathrm{an}} \to \operatorname{Grass}_{h-d}(E(\mathbb{X}))^{\mathrm{an}}$.

Theorem 2.4.5. *The period morphism $\breve{\pi}$ is étale. In particular $\breve{\mathcal{M}}^{\mathrm{rig}}$ is a smooth rigid analytic space.*

Proof. We are going to use the infinitesimal criterion 2.3.10. Let Z be as in the statement of this criterion. Then Z is of the form $Z = \operatorname{Spm}(R \otimes \breve{E})$, where $R \otimes \breve{E}$ is the affinoid algebra associated to a finite flat $O_{\breve{E}}$-algebra R. Denote by $\mathfrak{n} \subset R$ the nilradical, $R/\mathfrak{n}$ is a complete discrete valuation ring. The difficult of lifting the morphism is to choose formal models with generic fibers our critical rigid spaces. Consider the set of R-algebras R' which are finite flat $O_{\breve{E}}$-algebras with $R/\mathfrak{n} \cong R'/\mathfrak{n}R'$ and with $R \otimes_{O_{\breve{E}}} \breve{E} \cong R' \otimes_{O_{\breve{E}}} \breve{E}$. Then these form in an obvious way an inductive set $\mathcal{S}$ under the inclusion relation.

We fix a free $R/\mathfrak{n}$-module M_0 of finite rank. There is an obvious functor (associated rigid module) from the category of inductive system of locally free R'-modules M' of finite rank isomorphic to M_0 after tensoring $R/\mathfrak{n}$ to the category of locally free O_Z-modules M of finite rank isomorphic to M_0^{rig} after tensoring with O_{Z_0}. This functor is exact and an equivalence of categories, compatible with base change.

We now prove the existence of the dotted arrow in the diagram of Proposition 2.3.10. Replacing R by a larger $R' \in \mathcal{S}$, we may assume that the morphism $Z_0 \to \breve{\mathcal{M}}^{\mathrm{rig}}$ is induced from a morphism of formal schemes

$$\operatorname{Spf} R/\mathfrak{a} \longrightarrow \breve{\mathcal{M}},$$

where $\mathfrak{a} \subset \mathfrak{n}$ is a nilpotent ideal. We may assume $\mathfrak{a}^2 = 0$. By pull-back of the universal p-divisible group along the above morphism, we obtain an object (X_0, ρ_0) of the moduli problem over $\operatorname{Spf} R/\mathfrak{a}$. The morphism $Z \to \mathrm{Grass}_{h-d}(E(\mathbb{X}))^{\mathrm{rig}}$ making the solid diagram commutative defines a locally free factor module

$$E(\mathbb{X}) \otimes O_Z \longrightarrow L'$$

We equip $\mathfrak{a}$ with trivial divided powers. Let M be the value of the crystal associated to X_0 on $\operatorname{Spf} R$ and M_0 be the universal extension of X_0. Then we have a natural identification

$$E(\mathbb{X}) \otimes O_Z = M^{\mathrm{rig}}$$

Then at least after replacing R by a larger $R' \in \mathcal{S}$, the surjective morphism $M^{\mathrm{rig}} \to L'$ is induced from a unique surjective homomorphism of locally free modules

$$M \longrightarrow L$$

which reduces to $M_0 \to \mathrm{Lie}(X_0)$ after tensoring $R/\mathfrak{a}$. By Grothendieck-Messing (Theorem 1.4.3), there is a unique p-divisible group X over $\operatorname{Spf} R$ such that the above homomorphism $M \to L$ is actually $M_X \to \mathrm{Lie}(X)$ and restricting to X_0. The quasi-isogeny ρ_0 lifts automatically by rigidity. It is obvious that (X, ρ) is an object of $\breve{\mathcal{M}}(\operatorname{Spf} R)$. The induced rigid analytic morphism $Z \to \breve{\mathcal{M}}^{\mathrm{rig}}$ renders the diagram commutative. The uniqueness assertion is also clearly seen from above. $\square$

Remark 2.4.6. There are different notions of étaleness for morphisms between rigid spaces and morphisms between analytic spaces. For example the inclusion of the unit ball $\mathbb{D}(0,1) \hookrightarrow \mathbb{A}^1$ is not étale in Berkovich's theory but is in the classical theory of Tate (since it is an open immersion). But for our period morphisms both notions coincide.

Chapter 3

The Conjecture of Rapoport-Zink

In this chapter, we first introduce some basic constructions in Fontaine's theory of p-adic Galois representations. Then we define the p-adic period spaces $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ through which our period morphisms factors. After that we state the conjecture of Rapoport-Zink which conjectures the existence of an étale morphism $(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}} \to (\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ of rigid analytic spaces with interesting local system on $(\breve{\mathcal{F}}_b^{a})^{\mathrm{rig}}$.

3.1 Fontaine's Rings

We recall some of the rings used in p-adic Hodge theory. Let O_K be a complete valuation ring of rank one which is an extension of $\mathbb{Z}_p$ and K be its fraction field. Let v_p be the valuation on O_K which we assume to be normalized so that $v_p(p) = 1$. Furthermore we assume that for some perfect field k of characteristic p the Witt vectors $W(k)$ are contained in O_K, with fraction field K_0. Then K_0 admits a Frobenius automorphism φ. Let $\mathbb{C}$ be the completion of an algebraic closure $\overline{K}$ of K and let $O_{\mathbb{C}}$ be the valuation ring of $\mathbb{C}$. Let μ_m denote the subset of $\overline{K}$ defined by $\mu_m = \{x \in \overline{K}, x^m = 1\}$. We will choose once and for all a compatible sequence of primitive p^n-th roots of unity, $\varepsilon^{(0)} = 1$, and $\varepsilon^{(n)} \in \mu_{p^n} \subset \overline{K}$, such that $\varepsilon^{(1)} \neq 1$ and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$. This means we choose an "orientation" in p-adic Hodge theory. Let $G_K := \mathrm{Gal}(\overline{K}/K)$.

Definition 3.1.1. We define the ring

$$\widetilde{\mathbf{E}}^+ := \widetilde{\mathbf{E}}^+(\mathbb{C}) := \{x = (x^{(n)})_{n\in\mathbb{N}} | \ \ x^{(n)} \in O_{\mathbb{C}}, (x^{(n+1)})^p = x^{(n)}\}$$

with multiplication $xy := (x^{(n)}y^{(n)})_{n\in\mathbb{N}}$ and addition $x+y := (\lim_{m\to\infty}(x^{(m+n)}+y^{(m+n)})^{p^m})_{n\in\mathbb{N}}$

Remark 3.1.2. If we define the valuation on $\widetilde{\mathbf{E}}^+$ by $v_{\mathbf{E}}(x) := v_p(x^{(0)})$, then $\widetilde{\mathbf{E}}^+$ becomes a complete valuation ring of rank one. One can show that $\widetilde{\mathbf{E}}^+$ is a perfect ring of characteristic p and with algebraically closed fraction field, called $\widetilde{\mathbf{E}} := \widetilde{\mathbf{E}}(\mathbb{C})$. It is obvious that $\varepsilon := (\varepsilon^{(n)})_{n\in\mathbb{N}} \in \widetilde{\mathbf{E}}^+$ and gives the cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ by the relation $g(\varepsilon) = \varepsilon^{\chi(g)}$ for any $g \in G_K$.

Definition 3.1.3. We define the following rings

$$\begin{aligned}
&\widetilde{\mathbf{A}}^+ := \widetilde{\mathbf{A}}^+(\mathbb{C}) := W(\widetilde{\mathbf{E}}^+(\mathbb{C}))\\
&\widetilde{\mathbf{A}} := \widetilde{\mathbf{A}}(\mathbb{C}) := W(\widetilde{\mathbf{E}}(\mathbb{C}))\\
&\widetilde{\mathbf{B}}^+ := \widetilde{\mathbf{B}}^+(\mathbb{C}) := \widetilde{\mathbf{A}}^+(\mathbb{C})[1/p]\\
&\widetilde{\mathbf{B}} := \widetilde{\mathbf{B}}(\mathbb{C}) := \widetilde{\mathbf{A}}(\mathbb{C})[1/p] \text{ the fraction field of } \widetilde{\mathbf{A}}(\mathbb{C})
\end{aligned}$$

Since $\widetilde{\mathbf{E}}^+$ and $\widetilde{\mathbf{E}}$ are of characteristic p, we have the absolute Frobenius automorphism $x \mapsto x^p$ on them. Furthermore, we have an Galois action of G_K on $\widetilde{\mathbf{E}}^+$ and $\widetilde{\mathbf{E}}$ given by the natural action on each coordinate. By functoriality of Witt rings, the Frobenius automorphism and Galois action extends to $\widetilde{\mathbf{A}}^+$, $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$. We also denote the extended Frobenius map by φ. It is easily seen that φ commutes with the action of G_K.

For $x \in \widetilde{\mathbf{E}}(\mathbb{C})$ we let $[x] \in \widetilde{\mathbf{A}}(\mathbb{C})$ the Teichmuller lift of x. Then every element $x \in \widetilde{\mathbf{B}}^+(\mathbb{C})$ can be written uniquely as $\sum_{k>>-\infty} p^k[x_k]$, with $x_k \in \widetilde{\mathbf{E}}^+$. We define a homomorphism of rings

$$\begin{aligned}
\theta : \widetilde{\mathbf{A}}^+ &\longrightarrow O_{\mathbb{C}}\\
\sum_{k\geq 0} p^k[x_k] &\longmapsto \sum_{k\geq 0} p^k x_k^{(0)}
\end{aligned}$$

Proposition 3.1.4. *([22]) The homomorphism θ is surjective. The kernel of θ, $\ker\theta$ is a principal ideal. An element $x \in \widetilde{\mathbf{A}}^+$ is a generator of $\ker\theta$ if and only if $v_{\mathbf{E}}(x^{(0)}) = 1$. For example the element $\omega = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}$ is a generator of $\ker\theta$. Moreover, $\bigcap(\ker\theta)^n = 0$.*

Remark 3.1.5. The surjective homomorphism θ extends naturally to a surjective homomorphism $\theta : \widetilde{\mathbf{B}}^+(\mathbb{C}) \to \mathbb{C}$.

Definition 3.1.6. The ring $\mathbf{B}_{\mathrm{dR}}^+$ is defined to be

$$\mathbf{B}_{\mathrm{dR}}^+ := \varprojlim_{n\in\mathbb{N}} \widetilde{\mathbf{B}}^+/(\ker\theta)^n$$

Note that $\theta([\varepsilon]-1) = 0$, the series $\sum_{n\geq 1}(-1)^{n-1}\frac{([\varepsilon]-1)^n}{n}$ converges to an element $t \in \mathbf{B}_{\mathrm{dR}}^+$. One should think of t as $t = \log([\varepsilon])$. It can be shown that $\theta(t) = 0$ and t is a generator of $\ker\theta$. t is a period for the cyclotomic character as $g(t) = \chi(g)t$ for any $g \in G_K$. We define

$$\mathbf{B}_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}}^+[1/t]$$

Since G_K stabilize $\ker\theta$, we have an action of G_K on $\mathbf{B}_{\mathrm{dR}}$. We can show that $\mathbf{B}_{\mathrm{dR}}$ is a field with filtration $\mathrm{Fil}^i\,\mathbf{B}_{\mathrm{dR}} = t^i\mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{B}_{\mathrm{dR}}^{G_K} = K$. Note that it is impossible to extend the Frobenius map φ on $\mathbf{B}_{\mathrm{dR}}$. In order to extend φ, we need to introduce the ring $\mathbf{B}_{\mathrm{cris}}$.

Definition 3.1.7. (i) We define the ring $A_{\mathrm{cris}} := A_{\mathrm{cris}}(\mathbb{C})$ to be the p-adic completion of the divided power envelope of $\widetilde{\mathbf{A}}^+(\mathbb{C})$ with respect to $\ker\theta$, i.e. the p-adic completion of $\widetilde{\mathbf{A}}^+(\mathbb{C})[\frac{\omega^n}{n!}, n \in \mathbb{N}]$.

(ii) The ring $\mathbf{B}_{\mathrm{cris}}^+ := \mathbf{B}_{\mathrm{cris}}^+(\mathbb{C})$ is defined to be $A_{\mathrm{cris}}(\mathbb{C})[1/p]$.

(iii) We define $\mathbf{B}_{\mathrm{cris}} := \mathbf{B}_{\mathrm{cris}}(\mathbb{C}) = \mathbf{B}_{\mathrm{cris}}^+(\mathbb{C})[1/t]$.

If we write $[\varepsilon]-1=b\cdot\omega$ for some $b\in\widetilde{\mathbf{A}}^+$, $\frac{([\varepsilon]-1)^n}{n}=(n-1)!b^n\frac{\omega^n}{n!}$ and $(n-1)!\to 0$ p-adically. From this we see that t is in A_{cris}. We can view A_{cris}, $\mathbf{B}^+_{\text{cris}}$ and $\mathbf{B}_{\text{cris}}$ as subrings of $\mathbf{B}_{\text{dR}}$. Since they are stable under G_K, we have the G_K action on them. Moreover, φ extends to A_{cris}, $\mathbf{B}^+_{\text{cris}}$ and $\mathbf{B}_{\text{cris}}$, see [22]. The Frobenius map φ commutes with the action of G_K and $\mathbf{B}^{G_K}_{\text{cris}}\supset K_0$.

We now introduce the concept of filtered isocrystals.

Definition 3.1.8. (i) A *filtered isocrystal* over K is an isocrystal (D,φ_D) over k as in Definition 1.3.2 with an exhaustive separated decreasing filtration of $D_K:=D\otimes_{K_0}K$ by K-subspaces, i.e. there exist $r,s\in\mathbb{Z}$ with $s<r$ such that $\operatorname{Fil}^r D_K=D_K$ and $\operatorname{Fil}^s D_K=(0)$. We denote this filtered isocrystal $\underline{D}:=(D,\varphi_D,\operatorname{Fil}^\bullet D_K)$.

(ii) The integers h for which $\operatorname{Fil}^{-h}D_K\neq\operatorname{Fil}^{-h+1}D_K$ are called the *Hodge-Tate weights* of $\underline{D}$.

(iii) Let $t_N(\underline{D})=v_p(\det\varphi_D)$ (the *Newton slope* of $\underline{D}$) be the p-adic valuation of $\det\varphi_D$ (with respect to any basis of D) and let $t_H(\underline{D})=\sum_{i\in\mathbb{Z}}i\cdot\dim_K\operatorname{gr}^i_{\operatorname{Fil}^\bullet}(D_K)$ the *Hodge slope* of $\underline{D}$. The filtered isocrystal $\underline{D}$ is called *weakly admissible* if

$$t_H(\underline{D})=t_N(\underline{D})\quad\text{and}\quad t_H(\underline{D'})\le t_N(\underline{D'})$$

for any subobject $\underline{D'}=(D',\varphi_D|_{D'},\operatorname{Fil}^\bullet D'_K)$ of $\underline{D}$, where D' is any φ_D stable K_0-subspace of D equipped with the induced filtration $\operatorname{Fil}^\bullet D'_K$ on $D'_K:=D'\otimes K$. The category of weakly admissible filtered isocrystals over K is denoted by $\operatorname{MF}(K)$.

Example 3.1.9. Let $\mathbb{X}$ be a p-divisible group over $\overline{\mathbb{F}_p}$ of height h and dimension d. Set $D:=\mathbb{D}(\mathbb{X})_{K_0}$ and $\varphi_D:=\mathbb{D}(\operatorname{Frob}_{\mathbb{X}})_{K_0}$. Let X be any deformation of $\mathbb{X}$ over any complete discrete valuation ring O_K with residue field $\overline{\mathbb{F}_p}$ and fraction field K of characteristic 0. Let $L_K:=(\operatorname{Lie}X^*)^\vee_K$ and view it as a K-subspace of $D_K:=D\otimes_{K_0}K$. We define the filtered isocrystal $\underline{D}:=(D,\varphi_D,\operatorname{Fil}^\bullet D_K)$ of Hodge-Tate weights 0 and 1 by $\operatorname{Fil}^{-1}D_K:=D_K$, $\operatorname{Fil}^0 D_K:=L_K$ and $\operatorname{Fil}^1 D_K=(0)$. Then the Newton slope $t_N(\underline{D})=v_p(\det\varphi_D)$ and the Hodge slope $t_H(\underline{D})=\dim_K L_K-\dim_K D_K=-d$. Then $\underline{D}$ is weakly admissible if

$$t_N(\underline{D})=-d\quad\text{and}\quad t_N(D',\varphi_D|_{D'},L_K\cap D'_K)\ge t_H(D',\varphi_D|_{D'},L_K\cap D'_K)$$

for all φ_D-stable K_0-subspaces $D'\subset D$.

Let $\operatorname{Rep}_{\mathbb{Q}_p}(G_K)$ be the category of p-adic representations of G_K, i.e. finite dimensional $\mathbb{Q}_p$-representations of G_K. For any $V\in\operatorname{Rep}_{\mathbb{Q}_p}(G_K)$, we can define a K-vector space $\mathbf{D}_{\text{dR}}(V):=(\mathbf{B}_{\text{dR}}\otimes_{\mathbb{Q}_p}V)^{G_K}$ with a filtration $\operatorname{Fil}^i\mathbf{D}_{\text{dR}}(V):=(\operatorname{Fil}^i\mathbf{B}_{\text{dR}}\otimes_{\mathbb{Q}_p}V)^{G_K}$. To the same $\mathbb{Q}_p$-vector space V, we can similarly associate a K_0-vector space $\mathbf{D}_{\text{cris}}(V):=(\mathbf{B}_{\text{cris}}\otimes_{\mathbb{Q}_p}V)^{G_K}$.

Definition 3.1.10. (i)A p-adic representation V of G_K is called *de Rham* if

$$V\otimes_{K_0}\mathbf{B}_{\text{dR}}\cong\mathbf{B}^d_{\text{dR}}$$

as a $\mathbf{B}_{\text{dR}}[G_K]$-module which respects filtration.

(ii) A p-adic representation V of G_K is called *crystalline* if

$$V \otimes_{K_0} \mathbf{B}_{\text{cris}} \cong \mathbf{B}_{\text{cris}}^d$$

as a $\mathbf{B}_{\text{cris}}[G_K]$-module which respects filtration and Frobenius.

We denote $\mathrm{Rep}_{\mathbb{Q}_p}^{\text{cris}}(V)$ the category of crystalline representations of G_K.

Remark 3.1.11. If K is a finite extension of K_0, the above definitions are equivalent to the usual definitions of being de Rham and crystalline. That is to say a p-adic representation is de Rham if $\dim_K \mathbf{D}_{\text{dR}}(V) = \dim_{\mathbb{Q}_p} V$ and is crystalline if $\dim_{K_0} \mathbf{D}_{\text{cris}}(V) = \dim_{\mathbb{Q}_p} V$.

Remark 3.1.12. A crystalline representation is always de Rham and in this case $\mathbf{D}_{\text{dR}}(V) = \mathbf{D}_{\text{cris}}(V) \otimes_{K_0} K$.

Let V be any p-adic representation of G_K. We can define a filtered isocrystal over K

$$(\mathbf{D}_{\text{cris}}(V), \varphi_{\mathbf{D}_{\text{cris}}(V)}, \mathrm{Fil}^{\bullet} \mathbf{D}_{\text{cris}}(V)_K)$$

Indeed, the filtration is given by $\mathrm{Fil}^i \mathbf{D}_{\text{cris}}(V)_K = \mathbf{D}_{\text{cris}}(V)_K \cap \mathrm{Fil}^i \mathbf{D}_{\text{dR}}(V)$. For the Frobenius map, first one can define an endomorphism φ on $\mathbf{B}_{\text{cris}} \otimes_{\mathbb{Q}_p} V$ which sends $b \otimes v$ to $\varphi(b) \otimes v$. This endomorphism commutes with the Galois action and hence stabilize $\mathbf{D}_{\text{cris}}(V)$. The Frobenius map $\varphi_{\mathbf{D}_{\text{cris}}(V)}$ comes from the restriction of φ on $\mathbf{D}_{\text{cris}}(V)$. Such a filtered isocrystal is always weakly admissible.

Definition 3.1.13. A filtered isocrystal over K is called *admissible* if it comes from a crystalline representation of G_K. We denote by $\mathrm{MF}^{ad}(K)$ the category of admissible filtered isocrystals over K.

Hence we have an exact $\otimes$-functor

$$\mathbf{D}_{\text{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\text{cris}} G_K \longrightarrow \mathrm{MF}^{ad}(K)$$

Fontaine also constructed the quasi-inverse of $\mathbf{D}_{\text{cris}}$

$$\mathcal{F} : \mathrm{MF}^{ad}(K) \longrightarrow \mathrm{Rep}_{\mathbb{Q}_p}^{\text{cris}}(G_K)$$

by setting $\mathcal{F}(D) := (\mathrm{Fil}^0(D \otimes_{K_0} \mathbf{B}_{\text{cris}}))^{\varphi=1}$, which inherits the action of G_K from $\mathbf{B}_{\text{cris}}$.

Example 3.1.14. Let X be a p-divisible group over O_K and define the Tate module $T_p(X) := \varprojlim X(n)(\overline{K}) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, X_{\overline{K}})$ of X. The vector space $V_p(X) := T_p(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ gives a p-adic representation of G_K. Fontaine proved that $\mathbf{D}_{\text{cris}}(V_p(X)) = (\mathbb{D}(X_k)_{K_0}, \varphi, \mathrm{Fil}^{\bullet} \mathbb{D}(X_k)_K)$, where X_k is the special fiber of X and the filtration is given by $\mathrm{Fil}^{-1} \mathbb{D}(X_k)_K = \mathbb{D}(X_k)_K$, $\mathrm{Fil}^0 \mathbb{D}(X_k)_K = (\mathrm{Lie}\, X^*)_K^{\vee}$ and $\mathrm{Fil}^1 \mathbb{D}(X_k)_K = (0)$. In particular $\mathbf{D}_{\text{cris}}(V_p(X))$ is a weakly admissible filtered isocrystal over K and $(\mathrm{Fil}^0(\mathbf{D}_{\text{cris}}(V_p(X)) \otimes \mathbf{B}_{\text{cris}}))^{\varphi=1} = V_p(X)$.

Remark 3.1.15. The category of weakly admissible filtered isocrystals over K is an abelian category which is closed under extensions and under passage to the dual object. Faltings proved that it is closed under tensor product. The subcategory consisting of admissible filtered isocrystals is clear closed under extensions, dual and tensor product.

If K is a finite extension of K_0, we have the following theorem of Colmez-Fontaine.

Theorem 3.1.16. *([10]) If K is a finite extension of K_0, any weakly admissible filtered isocrystal is admissible.*

Now we define more Fontaine's rings which will be used in our construction in Chapter 4.

Definition 3.1.17. (i) For $x = \sum_{k=0}^{\infty} p^k[x_k] \in \widetilde{\mathbf{A}}(\mathbb{C})$, $x_i \in \widetilde{\mathbf{E}}$, $k \in \mathbb{N}$, define

$$w_k(x) := \min_{i \leq k}\{v_{\mathbf{E}}(x_i)\}.$$

(ii) For a real number $r > 0$, define

$$v^{(0,r]}(x) := \inf\{w_k(x) + \frac{k}{r}, k \in \mathbb{N}\} = \inf\{v_{\mathbf{E}}(x_k) + \frac{k}{r}, k \in \mathbb{N}\} \in \mathbb{R} \cup \{\pm\infty\}$$

(iii) Define

$$\widetilde{\mathbf{A}}^{(0,r]} := \widetilde{\mathbf{A}}^{(0,r]}(\mathbb{C}) := \{x \in \widetilde{\mathbf{A}}(\mathbb{C}) | \lim_{k\to\infty} w_k(x) + \frac{k}{r} = +\infty\}$$

One can easily prove the following

Proposition 3.1.18. $\widetilde{\mathbf{A}}^{(0,r]}$ *is a ring and* $v^{(0,r]}$ *satisfies the following properties:*
(i) $v^{(0,r]}(x) = +\infty \Leftrightarrow x = 0$
(ii) $v^{(0,r]}(xy) \geq v^{(0,r]}(x) + v^{(0,r]}(y)$
(iii) $v^{(0,r]}(x+y) \geq \min\{v^{(0,r]}(x), v^{(0,r]}(y)\}$
(iv) $v^{(0,r]}(\varphi(x)) = pv^{(0,pr]}(x)$.

Remark 3.1.19. If we define $\widetilde{\mathbf{B}}^{(0,r]} := \widetilde{\mathbf{B}}^{(0,r]}(\mathbb{C}) := \widetilde{\mathbf{A}}^{(0,r]}(\mathbb{C})[1/p]$, then $v^{(0,r]}$ extends to a valuation on $\widetilde{\mathbf{B}}^{(0,r]}(\mathbb{C})$ by defining

$$v^{(0,r]}(x) := \min\{w_k(x) + \frac{k}{r}, k \in \mathbb{Z}\} = \min\{v_{\mathbf{E}}(x_k) + \frac{k}{r}, k \in \mathbb{Z}\}$$

for any $x = \sum_{k>>-\infty}^{+\infty} p^k[x_k] \in \widetilde{\mathbf{B}}^{(0,r]}(\mathbb{C})$.

Definition 3.1.20. Since for $r \geq s$, $\widetilde{\mathbf{B}}^{(0,r]} \subset \widetilde{\mathbf{B}}^{(0,s]}$, we define

$$\widetilde{\mathbf{B}}^{\dagger} := \widetilde{\mathbf{B}}^{\dagger}(\mathbb{C}) := \bigcup_{r>0} \widetilde{\mathbf{B}}^{(0,r]}(\mathbb{C})$$

the field of overconvergent elements.

For $0 < s \leq r$, and $x \in \widetilde{\mathbf{B}}^{(0,r]}(\mathbb{C})$, set

$$v^{[s,r]}(x) := \min\{v^{(0,s]}(x), v^{(0,r]}(x)\}$$

Let $\widetilde{\mathbf{B}}^{]0,r]}$ be the completion of $\widetilde{\mathbf{B}}^{(0,r]}$ by the Fréchet topology induced by the family of semi-valuations $v^{[s,r]}$ for all $0 < s \leq r$. In concrete terms this means that a sequence of elements $x_n \in \widetilde{\mathbf{B}}^{(0,r]}(\mathbb{C})$ converges in $\widetilde{\mathbf{B}}^{]0,r]}(\mathbb{C})$ if and only if $\lim_{n\to\infty} v^{[s,r]}(x_{n+1} - x_n) = +\infty$ for all $0 < s \leq r$. Also if $0 < s \leq r$ we let $\widetilde{\mathbf{B}}^{[s,r]}(\mathbb{C})$ be the completion of $\widetilde{\mathbf{B}}^{(0,r]}(\mathbb{C})$ with respect to $v^{[s,r]}$. Hence we view $\widetilde{\mathbf{B}}^{]0,r]}(\mathbb{C})$ as a subring of $\widetilde{\mathbf{B}}^{I}(\mathbb{C})$ for any closed subinterval $I \subset (0,r]$.

If $I =]0,r]$ or $I = [s,r]$ the functions $f_i : \widetilde{\mathbf{B}}^{(0,r]}(\mathbb{C}) \to \widetilde{\mathbf{E}}(\mathbb{C})$ defined by $x = \sum_{i>>-\infty}^{\infty} p^i [f_i(x)]$ extend by continuity to $\widetilde{\mathbf{B}}^{I}(\mathbb{C})$ and for any $x \in \widetilde{\mathbf{B}}^{I}(\mathbb{C})$ the sum $\sum_{i=-\infty}^{\infty} p^i [f_i(x)]$ converges to x in $\widetilde{\mathbf{B}}^{I}(\mathbb{C})$; see [31]. Let

$$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} := \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(\mathbb{C}) := \bigcup_{r>0} \widetilde{\mathbf{B}}^{]0,r]}(\mathbb{C})$$

By Proposition 3.1.18, the homomorphism φ gives rise to bicontinuous isomorphisms $\varphi : \widetilde{\mathbf{B}}^{]0,pr]}(\mathbb{C}) \cong \widetilde{\mathbf{B}}^{]0,r]}(\mathbb{C})$ and $\varphi : \widetilde{\mathbf{B}}^{[ps,pr]}(\mathbb{C}) \cong \widetilde{\mathbf{B}}^{[s,r]}(\mathbb{C})$ defining an automorphism of $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(\mathbb{C})$. The restriction of θ to $\widetilde{\mathbf{B}}^{(0,1]}(\mathbb{C})$ defines a homomorphism $\theta : \widetilde{\mathbf{B}}^{(0,1]}(\mathbb{C}) \to \mathbb{C}$ which extends by continuity to $\widetilde{\mathbf{B}}^{]0,1]}(\mathbb{C})$, we have $t = \log([\varepsilon]) \in \widetilde{\mathbf{B}}^{]0,1]}(\mathbb{C})$.

Finally, we define

$$\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} := \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}(\mathbb{C}) = \bigcap_{n\in\mathbb{N}} \varphi^n \mathbf{B}^{+}_{\mathrm{cris}}(\mathbb{C})$$

For any $r > 0$, $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}(\mathbb{C}) \subset \widetilde{\mathbf{B}}^{]0,r]}(\mathbb{C})$. More precisely, $\widetilde{\mathbf{B}}^{]0,r]}(\mathbb{C})$ equals the p-adic completion of $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}(\mathbb{C})[\frac{p}{[\varepsilon-1]}]$ and is hence a flat $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}(\mathbb{C})$-algebra.

3.2 p-adic Period Spaces

In this section, all the isocrystals we consider are assumed to be over $\overline{\mathbb{F}_p}$. Let $K_0 := W(\overline{\mathbb{F}_p})[1/p]$ the fraction field of the ring of Witt vectors over $\overline{\mathbb{F}_p}$. Let φ be the Frobenius lift on K_0. We denote by $\mathrm{Isoc}(K_0)$ the category of isocrystals over $\overline{\mathbb{F}_p}$.

Let G be a reductive linear algebraic group over $\mathbb{Q}_p$ and let $\mathrm{Rep}_{\mathbb{Q}_p}(G)$ the category of finite dimensional $\mathbb{Q}_p$-rational representations of G.

Definition 3.2.1. An exact faithful $\otimes$-functor $\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathrm{Isoc}(K_0)$ is called an *isocrystal with G-structure* over K_0.

Let $b \in G(K_0)$, then the functor

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \longrightarrow \mathrm{Isoc}(K_0)$$

associated to b, defined by $V \mapsto (V \otimes_{\mathbb{Q}_p} K_0, b(\mathrm{Id} \otimes \varphi))$, is an isocrystal with G-structure over K_0. Two elements b and b' in $G(K_0)$ are conjugate if and only if there is an element $g \in G(K_0)$ such that $gb\varphi(g)^{-1} = b'$. In this case g defines an isomorphism between the isocrystals with G-structure associated to b and b'. If G is connected, any isocrystal with G-structure over K_0 is associated to an element $b \in G(K_0)$ as above.

Let $\mathbb{D} = \varprojlim \mathbb{G}_m$ be the pro-algebraic group over $\mathbb{Q}_p$ whose character group is $\mathbb{Q}$. For any element $b \in G(K_0)$, Kottwitz defined a morphism of algebraic groups over K_0

$$\nu : \mathbb{D} \longrightarrow G_{K_0}$$

which is characterized by the property that for any object V in $\mathrm{Rep}_{\mathbb{Q}_p}(G)$, the $\mathbb{Q}$-grading on the vector space $V \otimes K_0$ is the slope decomposition of $(V \otimes_{\mathbb{Q}_p} K_0, b(\mathrm{Id} \otimes \varphi))$ an isocrystal over $\overline{\mathbb{F}_p}$, cf Theorem 1.3.8. The group $\mathbb{Q}^\times$ acts on $\mathbb{D}$, since it acts on the character group $\mathbb{Q}$. For $s \in \mathbb{Q}^\times$ we denote $s\nu$ for the composite $\mathbb{D} \xrightarrow{s} \mathbb{D} \xrightarrow{\nu} G$. Let $\mathbb{D} \to \mathbb{G}_m$ be the projection to the multiplicative group induced by the inclusion of the character group $\mathbb{Z} \subset \mathbb{Q}$. Then for a suitable positive integer s the morphism $s\nu$ factors through this projection $s\nu : \mathbb{G}_m \to G$. Hence $s\nu$ is regarded as a one parameter subgroup of G over K_0.

Definition 3.2.2. A φ-conjugacy class $\bar{b}$ of $G(K_0)$ is *decent* if there is an element $b \in \bar{b}$ such that

$$(b\varphi)^s = s\nu(p)\varphi^s$$

for some positive integer s.

Proposition 3.2.3. *[33] Assume that $\bar{b}$ is decent and that b and s are from Definition 3.2.2. Then $b \in G(\mathbb{Q}_{p^s})$ and ν is defined over $\mathbb{Q}_{p^s}$. If G is connected, any φ-conjugacy class is decent.*

We now fix a conjugacy class of a one parameter subgroup $\mu : \mathbb{G}_m \to G$ over $\overline{K_0}$. Here two one parameter subgroups μ, μ' are conjugate if and only if there exists some $g \in G(\overline{K_0})$ such that $g\mu g^{-1} = \mu'$. Then there is a finite extension E of $\mathbb{Q}_p$ in $\overline{K_0}$ such that the conjugacy class $\{\mu\}$ of μ is defined.

For a one parameter subgroup $\mu : \mathbb{G}_m \to G$ over K we can define a filtration on $V_K := V \otimes_{\mathbb{Q}_p} K$, where V is any $\mathbb{Q}_p$-representation of G. We let $V_{K,\mu,j}$ be the subspace of V_K of weight j with respect to μ, i.e.

$$V_{K,\mu,j} := \{v \in V_K|\ \ \mu(z) \cdot v = z^j \cdot v \text{ for all } z \in \mathbb{G}_m(K)\}$$

Then we define the filtration $\mathrm{Fil}^i_\mu V_K := \bigoplus_{j \geq i} V_{K,\mu,j}$.

Definition 3.2.4. Two one parameter subgroups of G over $\overline{K_0}$ are called *equivalent* if they define the same weight filtration for any object in $\mathrm{Rep}_{\mathbb{Q}_p}(G)$. Note that two equivalent one parameter subgroups belong to the same conjugacy class.

Consider the functor

$$R \longmapsto \{ \text{ the equivalent classes in the conjugacy class } \{\mu\} \text{ defined over } R\}$$

on the category of E-algebras. If one defines an algebraic subgroup of G over E by

$$P(\mu)(\overline{K_0}) = \{g \in G(\overline{K_0})|\ \ g\mu g^{-1} \text{ is equivalent to } \mu\}$$

then $P(\mu)$ is parabolic and the functor above is representable by the projective variety $G_E/P(\mu)$. We denote this homogeneous space over E by $\mathcal{F}$. If V is a faithful representation in $\mathrm{Rep}_{\mathbb{Q}_p}(G)$

and if we denote by $\mathrm{Flag}(V)$ the partial flag variety over $\mathbb{Q}_p$ which represents the functor which associate to any $\mathbb{Q}_p$-algebra R the filtration $\mathrm{Fil}^\bullet$ of $V \otimes R$ as R-modules such that Fil^i is a direct summand and $\mathrm{rk}_R \mathrm{Fil}^i = \dim_K \mathrm{Fil}^i_\mu V_K$. Then there is a natural E-closed immersion

$$\mathcal{F} \longrightarrow \mathrm{Flag}(V) \otimes_{\mathbb{Q}_p} E$$

Combine the discussion above, to any pair (μ, b) where μ is a one parameter subgroup over K and $b \in G(K_0)$, we have an exact $\otimes$-functor

$$\begin{aligned} \mathcal{I} : \mathrm{Rep}_{\mathbb{Q}_p}(G) &\longrightarrow \mathrm{MF}(K) \\ V &\longmapsto (V \otimes K_0, b(\mathrm{Id} \otimes \varphi), \mathrm{Fil}^\bullet_\mu V_K) \end{aligned}$$

Definition 3.2.5. We call a pair (μ, b) weakly admissible if the filtered isocrystal $\mathcal{I}(V)$ over K is weakly admissible for any object V in $\mathrm{Rep}_{\mathbb{Q}_p}(G)$.

Remark 3.2.6. To see the weakly admissibility for (μ, b), it is enough to check the weak admissibility of $\mathcal{I}(V)$ for a faithful representation V of G. Indeed, any representation of G appears as a direct summand of $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ and $\mathcal{I}(V)^{\otimes m} \otimes (\mathcal{I}(V)^\vee)^{\otimes n}$ is weakly admissible by Faltings. Here $V^\vee$ (resp. $\mathcal{I}(V)^\vee$) is the dual of V (resp. $\mathcal{I}(V)$).

In what follows, we fix an element b in $G(K_0)$ and a conjugacy class of one parameter subgroup $\{\mu\}$ of field of definition E. Let $\breve{E} = EK_0 = \widehat{E^{\mathrm{ur}}}$ be the completion of the maximal unramified extension of E. We denote $\breve{\mathcal{F}} = \mathcal{F} \otimes_E \breve{E}$. We consider one parameter subgroups μ defined over finite extensions K of $\breve{E}$.

Definition 3.2.7. A point ξ in $\breve{\mathcal{F}}(K)$ is called *weakly admissible* if the pair (ξ, b) is weakly admissible. This condition is independent of the choice of the representative in the equivalence class of ξ. We denote by $\breve{\mathcal{F}}^{wa}_b(K)$ the subset of weakly admissible points associated with $(G, b, \{\mu\})$.

Definition 3.2.8. For any point $\xi \in \breve{\mathcal{F}}^{wa}_b(K)$ where K is a finite extension of K_0, we have a fiber functor $\mathrm{Rep}_{\mathbb{Q}_p}(G) \to (\mathbb{Q}_p - \text{vector spaces})$ given by the composition

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathrm{MF}^{ad}(K) \xrightarrow{\mathcal{F}} \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \subset (\mathbb{Q}_p - \text{vector spaces})$$

Here we use Theorem 3.1.16 and $\mathcal{F}$ is the quasi-inverse functor of $\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}^{ad}(K)$.

Definition 3.2.9. We define the p-adic period space associated with $(G, b, \{\mu\})$ as

$$(\breve{\mathcal{F}}^{wa}_b)^{\mathrm{rig}} := \{\xi \in \breve{\mathcal{F}}^{\mathrm{rig}} | \ \ (\xi, b) \text{ is weakly admissible}\}$$

Proposition 3.2.10. *([33] 1.34) The set $(\breve{\mathcal{F}}^{wa}_b)^{\mathrm{rig}}$ of weakly admissible points with respect to b in $\breve{\mathcal{F}}(\mathbb{C}_p)$ is an admissible open subset of $\breve{\mathcal{F}}$ as a rigid analytic space.*

One may specify the algebraic groups G over $\mathbb{Q}_p$ used in this section as in [33]1.38 which are related to classifying p-divisible groups with additional structures as in Chapter II. Here for our use, we only assume that $G = \mathbf{GL}(V)$ for a finite dimensional $\mathbb{Q}_p$-vector space V. Let $b \in G(K_0)$ and $(D, \varphi_D) = (V_{K_0}, b(\mathrm{Id} \otimes \varphi))$. Assume that there exists a p-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}_p}$ of dimension d whose covariant Dieudonné isocrystal is (D, φ_D). Then we consider the period morphism, we have

Theorem 3.2.11. *The period morphism factors through $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ and surjective on rigid points.*

Proof. We first consider the period morphism associated to the moduli problem without additional structures. Let X^{univ} be the universal p-divisible group over $\breve{\mathcal{M}}$. For any point $x \in \breve{\mathcal{M}}^{\mathrm{rig}}$, $k(x)$ is a finite extension of $\breve{E}$. Then the image $\breve{\pi}(x) \in \breve{\mathcal{F}}^{\mathrm{rig}}$ is the point associated to the pull back of $(\mathrm{Lie}\, X^{\mathrm{univ}\,*})^{\vee\,\mathrm{rig}} \hookrightarrow E(\mathbb{X}) \otimes O_{\breve{\mathcal{M}}^{\mathrm{rig}}} \cong (M_{X^{\mathrm{univ}}})^{\mathrm{rig}} \to (\mathrm{Lie}(X^{\mathrm{univ}}))^{\mathrm{rig}}$ via $\mathrm{Spf}\, O_{k(x)} \to \breve{\mathcal{M}}$. This is actually the canonical filtration associated to the p-divisible group which is the pull back of the universal p-divisible group. Then from Example 3.1.14 $\breve{\pi}(x) \in (\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$.

It remains to show that every K-valued point $y := (D, \varphi_D, L_K)$ of $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ for $K/\breve{E}$ finite lies in the image of $\breve{\pi}$. By the theorem of Colmez-Fontaine (Theorem 3.1.16) the filtered isocrystal y is admissible, i.e. arises from a crystalline p-adic Galois representation V. By Breuil [9] Theorem 1.4 there is a p-divisible group X over O_K and an isomorphism $T_p(X_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V$ if $p > 2$. Kisin extended Breuil's theorem to $p = 2$ and reproved the Colmez-Fontaine Theorem. Fontaine's functor $\mathbf{D}_{\mathrm{cris}}$ transforms the isomorphism $T_p(X_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V$ into an isomorphism

$$\bar{\rho}_* : (\mathbb{D}(X)_{\breve{E}}, \mathbb{D}(\mathrm{Frob}_X)_{\breve{E}}, (\mathrm{Lie}\, X^*)_K^\vee) \longrightarrow (D, \varphi_D, L_K)$$

of filtered isocrystals. This defines a quasi-isogeny $\bar{\rho} : \mathbb{X} \to X_{\overline{\mathbb{F}_p}}$ which by rigidity lifts to a unique quasi-isogeny $\rho : \mathbb{X}_{O_K/(p)} \to X_{O_K/(p)}$. So y is given by $(X, \rho : \mathbb{X}_{O_K/(p)} \to X_{O_K/(p)})$, hence lies in the image of the period morphism.

For the p-divisible group $\mathbb{X}$ with additional structures the proof is similar, see [33]. $\square$

However, the rigid analytic structure is not decided by the set of rigid points. For example, we consider the rigid space $X_1 \coprod X_2$ over K where $X_1 = \{x \in \overline{K} |\ \ |x| < 1\}$ and $X_2 = \{x \in \overline{K} |\ \ |x| = 1\}$. Let f be the natural morphism from $X_1 \coprod X_2$ to the unit ball $X = \{x \in \overline{K} |\ \ |x| \leq 1\}$. Then f is bijective on points but never an isomorphism, since X is connected. This indicates that we need to work with Berkovich's $\breve{E}$-analytic spaces instead of rigid analytic spaces, since in Berkovich spaces the analytic structure is completely determined by the underlying points.

In the spirit that it is better to work with Berkovich spaces, we have the following

Proposition 3.2.12. *([25] 1.3) There exists an open $\breve{E}$-analytic subspace $\breve{\mathcal{F}}_b^{wa}$ of $\breve{\mathcal{F}}^{\mathrm{an}}$ whose associated rigid analytic space is the period space $(\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$.*

To end this section, we reture to Grothendieck's question at the end of Chapter 1.

Proposition 3.2.13. *The subset of $\mathcal{F}$ formed by the points $(\mathrm{Lie}\, X^*)_K^\vee$ where X is any deformation of $\mathbb{X}$ over any complete discrete valuation ring O_K with residue field $\mathbb{F}_p$ and fraction field K of characteristic 0 is contained in the subset $(\mathcal{F}^{wa})^{\mathrm{rig}}$.*

Proof. Let $L_K = (\mathrm{Lie}\, X^*)^\vee_K$ be a point in Grothendieck set, given by a p-divisible group X over O_K with $\mathbb{X} \cong X_{\overline{\mathbb{F}_p}}$. Over $O_K/(p)$ this isomorphism lifts by rigidity to a quasi-isogeny $\rho : \mathbb{X}_{O_K/(p)} \to X_{O_K/(p)}$ and (X, ρ) gives a point of $\mathcal{M}(O_K)$. By construction $\breve{\pi}^{\mathrm{rig}}(X, \rho)_K = L_K$. So the point L_K belongs to the image of $\breve{\pi}^{\mathrm{rig}}$ which in turn lies in $(\mathcal{F}^{wa})^{\mathrm{rig}}$.

□

3.3 The Conjecture of Rapoport-Zink

Let (F, O, k, π) be a complete discrete valuation ring. We first recall the definition of the étale site X_{et} of a rigid analytic space X over F from [30].

The underlying category of the site X_{et} is the category of all étale morphisms $f : Y \to X$ of rigid analytic spaces over F. A morphism from f to f' is a morphism $g : Y \to Y'$ such that $f' \circ g = f$. The morphism g is automatically étale.

Definition 3.3.1. A family of étale morphisms $\{g_i : Z_i \to Y\}_{i \in I}$ is a *covering for the étale topology* if for every (some) choice of admissible affinoid covering $Z_i = \bigcup_j Z_{i,j}$ one has $Y = \bigcup_{i,j} g_i(Z_{i,j})$, and this is an admissible covering in the Grothendieck topology of Y.

Remark 3.3.2. The above definition is local on Y in the following sense: if $Y = \bigcup Y_l$ is an admissible affinoid covering, then $\{g_i : Z_i \to Y\}$ is a covering for the étale topology if and only if for all l the same is true for the covering $\{g_i : g_i^{-1}(Y_l) \to Y_l\}$. This implies that if $\{Z_i \to Y\}$ and $\{W_{i,j} \to Z_i\}$ for all i are coverings for the étale topology, then $\{W_{i,j} \to Y\}$ is a covering for the étale topology.

Clearly any admissible covering of Y is a covering for the étale topology.

Definition 3.3.3. The category X_{et} equipped with the family of coverings for the étale topology is thus a site, called the *étale site of* X. The sheaves on this site are called *étale sheaves on* X.

Definition 3.3.4. Let X be a quasi-separated quasi-paracompact rigid analytic space over F whose associated strictly F-analytic space X^{an} is good. A *geometric point* $\bar{x}$ of X is a morphism $\bar{x} : \mathrm{Spm}\, K \to X \otimes_F K$ for an algebraically closed complete extension K of F.

Remark 3.3.5. The geometric point $\bar{x}$ can be viewed as a morphism $B \to K$ for a suitable affinoid subdomain $\mathrm{Spm}\, B \subset X$. Then the absolute value on K defines an analytic point $x \in \mathcal{M}(B)$ which we call the *underlying analytic point* of $\bar{x}$. We may even choose B such that $\mathcal{M}(B)$ is an affinoid neighborhood of x in X^{an}.

Definition 3.3.6. Let X be a rigid analytic space and $\bar{x}$ be a geometric point of X. A pair $(f, \bar{y})$, where $f : U \to X$ is an étale morphism of rigid analytic spaces and $\bar{y}$ is a geometric point of U, is called an *étale neighborhood of* $\bar{x}$ if the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Spm}\, K \arrow[r, "\bar{y}"] \arrow[dr, "\bar{x}"'] & U \otimes_F K \arrow[d, "f"] \\ & X \otimes_F K \end{tikzcd}$$

Definition 3.3.7. If $\mathcal{F}$ is a sheaf on X_{et} and $\bar{x}$ is a geometric point of X the *stalk* $\mathcal{F}_{\bar{x}}$ of $\mathcal{F}$ at $\bar{x}$ is the inductive limit

$$\mathcal{F}_{\bar{x}} := \varinjlim_U \mathcal{F}(U)$$

over all étale neighborhoods U of $\bar{x}$.

Corresponding to the definitions above one can also define the étale site of a Berkovich space; see [5] Section 4.1.

Now we define the notion of local systems of $\mathbb{Q}_p$-vector spaces on X.

Definition 3.3.8. Let X be a rigid analytic space. We define a *local system of* $\mathbb{Z}_p$*-lattices* on X as a projective system $\mathcal{F} = (\mathcal{F}_n, i_n)$ of sheaves $\mathcal{F}_n$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules on X_{et} such that $\mathcal{F}_n$ is a locally constant free $\mathbb{Z}/p^n\mathbb{Z}$-module of finite rank and i_n induces an isomorphism of sheaves of $\mathbb{Z}/p^{n-1}\mathbb{Z}$-modules

$$i_n \otimes \mathrm{Id} : \mathcal{F}_n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^{n-1}\mathbb{Z} \longrightarrow \mathcal{F}_{n-1}$$

(Of course locally constant means locally for the étale topology.) The category $\mathbb{Z}_p - \underline{\mathrm{Loc}}_X$ of local systems of $\mathbb{Z}_p$-lattices with the obvious morphisms is an additive $\mathbb{Z}_p$-linear tensor category. If $\bar{x}$ is a geometric point of X we define the *stalk* $\mathcal{F}_{\bar{x}}$ of $\mathcal{F}$ at $\bar{x}$ as

$$\mathcal{F}_{\bar{x}} := \varprojlim (\mathcal{F}_{n,\bar{x}}, i_n).$$

It is a finite free $\mathbb{Z}_p$-module. Starting from $\mathbb{Z}_p$-lattices one defines local systems of $\mathbb{Q}_p$-vector spaces as in [29]. In concrete terms a *local system of* $\mathbb{Q}_p$*-vector spaces* on X is given by the following data

$$\mathcal{V} = (\{U_i \to X\}, \mathcal{F}_i, \varphi_{ij})$$

where

- $\{U_i \to X\}$ is a covering for the étale topology on X,
- $\mathcal{F}_i$ is a local system of $\mathbb{Z}_p$-lattices over U_i for each i,
- for each pair i, j, φ_{ij} is an invertible section over $U_i \times_X U_j$ of the sheaf

$$\mathrm{Hom}_{\mathbb{Z}_p - \underline{\mathrm{Loc}}_X}(\mathcal{F}_i|_{U_i \times_X U_j}, \mathcal{F}_j|_{U_i \times_X U_j}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

These data are subject to the cocycle condition $pr_{ij}^*(\varphi_{ij}) \circ pr_{jk}^*(\varphi_{jk}) = pr_{ik}^*(\varphi_{ik})$ on the triple product $U_i \times_X U_j \times_X U_k$.

A refinement of the covering gives by definition an isomorphic object. Therefore morphisms $\mathcal{V} \to \mathcal{V}'$ need only be defined for systems given over the same covering $\{U_i \to X\}$. In this case after possibly refining the covering, such a morphism is defined by a collection of sections φ_i of the sheaf $\mathrm{Hom}_{\mathbb{Z}_p - \underline{\mathrm{Loc}}_X}(\mathcal{F}_i, \mathcal{F}'_i) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ over U_i satisfying $\varphi'_{ij} \circ pr_j^*(\varphi_i) = pr_j^*(\varphi_j) \circ \varphi_{ij}$ over $U_i \times_X U_j$.

If $\bar{x}$ is a geometric point of X we define the *stalk* $\mathcal{V}_{\bar{x}}$ of $\mathcal{V}$ at $\bar{x}$ as follows. Choose a lift $\bar{y}$ of $\bar{x}$ in some U_i and put

$$\mathcal{V}_{\bar{x}} := \mathcal{F}_{i,\bar{y}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

One easily verifies that $\mathcal{V}_{\bar{x}}$ is a well defined finite dimensional $\mathbb{Q}_p$-vector space.

The local systems of $\mathbb{Q}_p$-vector spaces form a category $\mathbb{Q}_p - \underline{\mathrm{Loc}}_X$. It is an abelian $\mathbb{Q}_p$-linear tensor category. Rapoport and Zink make the

Conjecture 3.3.9. ([33] 1.37) There exists an étale morphism $(\breve{\mathcal{F}}^a_b)^{\mathrm{rig}} \to (\breve{\mathcal{F}}^{wa}_b)^{\mathrm{rig}}$ of rigid analytic spaces over $\breve{E}$ which is bijective on (rigid analytic) points and there exists a tensor functor from $\mathrm{Rep}_{\mathbb{Q}_p}(G)$ to the category $\mathbb{Q}_p - \underline{\mathrm{Loc}}_{(\breve{\mathcal{F}}^a_b)^{\mathrm{rig}}}$ of local systems of $\mathbb{Q}_p$-vector spaces on $(\breve{\mathcal{F}}^a_b)^{\mathrm{rig}}$ with the following property:

For any point $\mu \in (\breve{\mathcal{F}}^{wa}_b)^{\mathrm{rig}}(K)$ with $K/\breve{E}$ finite, the fiber functor which associates with a representation in $\mathrm{Rep}_{\mathbb{Q}_p} G$ the fiber at the corresponding point of μ in $(\breve{\mathcal{F}}^a_b)^{\mathrm{rig}}$ of the local system is isomorphic to the fiber functor defined in Definition 3.2.8.

Correspondingly one can define the category $\mathbb{Q}_p - \underline{\mathrm{Loc}}_{X^{\mathrm{an}}}$ of local systems of $\mathbb{Q}_p$-vector spaces on the Berkovich space X^{an} associated to X as in [29] Section 4. A.J. de Jong pointed out that it is best done working with Berkovich spaces rather than rigid analytic spaces. We have the following propositions indicating the relations.

Proposition 3.3.10. *([29] 5.1) There is a natural tensor equivalence of categories* $\mathbb{Q}_p - \underline{\mathrm{Loc}}_{X^{\mathrm{an}}} \to \mathbb{Q}_p - \underline{\mathrm{Loc}}_X$.

Proposition 3.3.11. *([29] 4.4) A local system of $\mathbb{Q}_p$-vector spaces $\mathcal{V}$ on X can always be given as $\mathcal{V} = (\{U_i \to X\}, \mathcal{F}_i, \varphi_{ij})$ where the U_i are affinoid subdomains of X and the associated Berkovich spaces U_i^{an} form an affinoid covering of X^{an}.*

Chapter 4

Hartl's Construction

In this chapter, we give Hartl's construction of proposed substitute for Rapoport-Zink's conjecture. We give the proof that the period morphism factors through this subspace and surjective on points. The construction is inspired by solving the same problem in equal characteristic case [28].

4.1 From Filtered isocrystals to φ-Modules

We recall some definitions and facts for φ^a-modules over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$.

Definition 4.1.1. Let a be a positive integer. A φ^a-module over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is a finite free $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-module $\mathbf{M}$ with a φ^a-semilinear map $\varphi_{\mathbf{M}} : \mathbf{M} \to \mathbf{M}$. The rank of $\mathbf{M}$ as a $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-module is denoted by $\operatorname{rk}\mathbf{M}$. A *morphism of φ^a-modules* is a morphism of the underlying $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-modules which commutes with the $\varphi_{\mathbf{M}}$'s. We denote the set of morphisms between two φ^a-modules $\mathbf{M}$ and $\mathbf{M}'$ by $\mathrm{Hom}_{\varphi^a}(\mathbf{M}, \mathbf{M}')$.

Definition 4.1.2. For any positive integer b, we define a *restriction of Frobenius functor* $[b]_*$ from φ^a-modules over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ to φ^{ab}-modules over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ sending $(\mathbf{M}, \varphi_{\mathbf{M}})$ to $(\mathbf{M}, \varphi^b_{\mathbf{M}})$.

Example 4.1.3. (1) Let $c, d \in \mathbb{Z}$ with $d > 0$ and $(c, d) = 1$. Define the φ^a-module $\mathbf{M}(c, d)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ as $\mathbf{M}(c, d) = \bigoplus_{i=1}^d \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \mathbf{e}_i$ equipped with

$$\varphi_{\mathbf{M}}(\mathbf{e}_1) = \mathbf{e}_2, \quad \ldots, \varphi_{\mathbf{M}}(\mathbf{e}_{d-1}) = e_d, \quad \varphi_{\mathbf{M}}(\mathbf{e}_d) = p^c \mathbf{e}_1. \tag{4.1}$$

(2) We have a rank one $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-module $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \cdot t$. Since $\varphi t = pt$, $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \cdot t$ is isomorphic to $\mathbf{M}(1, 1)$.

Lemma 4.1.4. *([31] 4.1.2 and 3.2.4) The φ^a-modules $\mathbf{M}(c, d)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ satisfy*

(i) $\mathbf{M}(c, d)^\vee \cong \mathbf{M}(-c, d)$

(ii) $[d]_* \mathbf{M}(c, d) \cong \mathbf{M}(c, 1)^{\oplus d}$.

Definition 4.1.5. For a φ^a-module $\mathbf{M}$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, we define the set of *$\varphi^a$-invariants* as

$$\mathbf{H}^0_{\varphi^a}(\mathbf{M}) := \{x \in \mathbf{M} | \ \ \varphi_{\mathbf{M}}(x) = x\}$$

Remark 4.1.6. $\mathbf{H}^0_{\varphi^a}(\mathbf{M})$ is a vector space over $\mathbf{H}^0_{\varphi^a}(\mathbf{M}(0,1)) = W(\mathbb{F}_{p^a})[1/p]$.

Proposition 4.1.7. *([31] 4.1.3 and 4.1.4) The φ^a-modules $\mathbf{M}(c,d)$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ satisfy*

(i) $\mathrm{Hom}_{\varphi^a}(\mathbf{M}(c,d), \mathbf{M}(c',d')) \neq (0)$ *if and only if* $\frac{c}{d} \geq \frac{c'}{d'}$.

(ii) $\mathbf{H}^0_{\varphi^a}(\mathbf{M}(c,d)) \neq (0)$ *if and only if* $\frac{c}{d} \leq 0$.

Proposition 4.1.8. *([25] 3.6) If $c > 0$, then the φ^a-module $\mathbf{M}(-c,1)$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ has φ^a-invariants*

$$\mathbf{H}^0_{\varphi^a}(\mathbf{M}(-c,1)) = \{\sum_{\nu \in \mathbb{Z}} p^{c\nu} \sum_{j=0}^{c-1} p^j \varphi^{-a\nu}([x_j])| \ \ x_0, \ldots, x_{c-1} \in \widetilde{\mathbf{E}}, v_{\mathbf{E}}(x_j) > 0\}$$

For φ^a-modules over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, Kedlaya proved the following structure theorem.

Theorem 4.1.9. *([31] 4.5.7) Any φ^a-module $\mathbf{M}$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is isomorphic to a direct sum of φ^a-modules $\mathbf{M}(c_i,d_i)$ for uniquely determined pairs (c_i,d_i) up to permutation. It satisfies $\wedge^{\mathrm{rk}\,\mathbf{M}}\mathbf{M} \cong \mathbf{M}(c,1)$ where $c = \sum_i c_i$ and $\mathrm{rk}\,\mathbf{M} = \sum_i d_i$.*

Definition 4.1.10. We define $\det \mathbf{M} := \wedge^{\mathrm{rk}\,\mathbf{M}}\mathbf{M} \cong \mathbf{M}(c,1)$ and we call $\deg \mathbf{M} := c$ the *degree* of $\mathbf{M}$ and $\mathrm{wt}\,\mathbf{M} := \frac{\deg \mathbf{M}}{\mathrm{rk}\,\mathbf{M}}$ the *weight* of $\mathbf{M}$.

Proposition 4.1.11. *([31] 3.4.9) Every φ^a-submodule $\mathbf{M}' \subset \mathbf{M}(c,d)^{\oplus n}$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ satisfies* $\mathrm{wt}\,\mathbf{M}' \geq \mathrm{wt}\,\mathbf{M}(c,d)^{\oplus n} = \frac{c}{d}$.

Let $\mathbf{M}^{]0,r]}$ be a finite free $\widetilde{\mathbf{B}}^{]0,r]}$-module and for $0 < s \leq r$ let $\mathbf{M}^{]0,s]} := \mathbf{M}^{]0,r]} \otimes_{\widetilde{\mathbf{B}}^{]0,r]},\iota} \widetilde{\mathbf{B}}^{]0,s]}$ be obtained by base change via the natural inclusion $\iota : \widetilde{\mathbf{B}}^{]0,r]} \hookrightarrow \widetilde{\mathbf{B}}^{]0,s]}$. Furthermore we assume an isomorphism

$$\varphi_{\mathbf{M}}^{]0,rp^{-a}]} : \mathbf{M}^{]0,r]} \otimes_{\widetilde{\mathbf{B}}^{]0,r]},\varphi^a} \widetilde{\mathbf{B}}^{]0,rp^{-a}]} \longrightarrow \mathbf{M}^{]0,rp^{-a}]}$$

By tensoring $\iota : \widetilde{\mathbf{B}}^{]0,rp^{-a}]} \to \widetilde{\mathbf{B}}^{]0,rp^{-na}]}$ for any $n \in \mathbb{N}$, this gives an isomorphism of $\widetilde{\mathbf{B}}^{]0,rp^{-na}]}$-modules

$$\mathbf{M}^{]0,rp^{-(n-1)a}]} \otimes_{\widetilde{\mathbf{B}}^{]0,rp^{-(n-1)a}]},\varphi^a} \widetilde{\mathbf{B}}^{]0,rp^{-na}]} \longrightarrow \mathbf{M}^{]0,rp^{-na}]}$$

Since $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \bigcup_{s>0} \widetilde{\mathbf{B}}^{]0,s]}$, $\varphi_{\mathbf{M}}^{]0,rp^{-a}]}$ extends to an isomorphism of $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$-modules

$$\mathbf{M} \otimes_{\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}},\varphi^a} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \longrightarrow \mathbf{M}$$

where $\mathbf{M} := \mathbf{M}^{]0,r]} \otimes_{\widetilde{\mathbf{B}}^{]0,r]},\iota} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$. Therefore this gives a φ^a-semilinear map $\varphi_{\mathbf{M}} : \mathbf{M} \to \mathbf{M}$.

Definition 4.1.12. A φ^a-module $(\mathbf{M}, \varphi_{\mathbf{M}})$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is said to be *represented* by the pair $(\mathbf{M}^{]0,r]}, \varphi_{\mathbf{M}}^{]0,rp^{-a}]})$ if $(\mathbf{M}, \varphi_{\mathbf{M}})$ is isomorphic to the φ^a-module $\mathbf{M}^{]0,r]} \otimes_{\widetilde{\mathbf{B}}^{]0,r]},\iota} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ constructed from $(\mathbf{M}^{]0,r]}, \varphi_{\mathbf{M}}^{]0,rp^{-a}]})$ as above.

Proposition 4.1.13. *([25] 3.10) If $\mathbf{M}$ is represented by $(\mathbf{M}^{]0,r]}, \varphi_{\mathbf{M}}^{]0,rp^{-a}]})$, then*

$$\mathbf{H}^0_{\varphi^a}(\mathbf{M}) = \{x \in \mathbf{M}^{]0,r]} \mid \ \varphi_{\mathbf{M}}^{]0,rp^{-a}]}(x \otimes_{\varphi^a} 1) = x \otimes 1\}$$

Now assume that K is a (not necessarily finite) extension of K_0 and C be the completion of a fixed algebraic closure of K. To any filtered isocrystal $\underline{D}$ over K of Hodge-Tate weights 0 and 1, we will associate a φ^a-module $\mathbf{M}(\underline{D})$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$. The construction is based on the following

Proposition 4.1.14. *([25] 4.1) Let $\mathbf{N}$ be a φ-module over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ represented by a finite free $\widetilde{\mathbf{B}}^{]0,1]}$-module $\mathbf{N}^{]0,1]}$ and an isomorphism $\varphi_{\mathbf{N}}^{]0,p^{-1}]} : \mathbf{N}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\varphi} \widetilde{\mathbf{B}}^{]0,p^{-1}]} \cong \mathbf{N}^{]0,p^{-1}]}$. Let $W_{\mathbf{M}}$ be a C-subspace of $W_{\mathbf{N}} := \mathbf{N}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C$. Then there exists a uniquely determined φ-submodule $\mathbf{M} \subset \mathbf{N}$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ with $t\mathbf{N} \subset \mathbf{M}$ which is represented by a $\widetilde{\mathbf{B}}^{]0,1]}$-submodule $t\mathbf{N}^{]0,1]} \subset \mathbf{M}^{]0,1]} \subset \mathbf{N}^{]0,1]}$ such that $\mathbf{M}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C = W_{\mathbf{M}}$.*

Now let (D, φ_D) be an isocrystal over $\overline{\mathbb{F}_p}$ and let $\mathrm{Fil}^0 D_K$ be a K-subspace of $D_K = D \otimes_{K_0} K$. Let $\mathrm{Fil}^{-1} D_K = D_K$ and $\mathrm{Fil}^1 D_K = (0)$. We denote $\underline{D} = (D, \varphi_D, \mathrm{Fil}^{\bullet} D_K)$ the filtered isocrystal over K and set $\mathbf{D}^{]0,1]} := D \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,1]}$, $\varphi_{\mathbf{D}}^{]0,1]} := \varphi_D \otimes \mathrm{Id}$, $\mathbf{D} := D \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ and $\varphi_{\mathbf{D}} := \varphi_D \otimes \varphi$. Then the $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$-module $(\mathbf{D}, \varphi_{\mathbf{D}})$ is represented by the $\widetilde{\mathbf{B}}^{]0,1]}$-module $(\mathbf{D}^{]0,1]}, \varphi_{\mathbf{D}}^{]0,1]})$.

Definition 4.1.15. By Proposition 4.1.14, we define $\mathbf{M}(\underline{D})$ be the φ-submodule of $\mathbf{D}$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ represented by the $\widetilde{\mathbf{B}}^{]0,1]}$-module $\mathbf{M}^{]0,1]}$ with $\mathbf{M}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C = (\mathrm{Fil}^0 D_K) \otimes_K C$ inside $D_C := \mathbf{D}^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C$.

Lemma 4.1.16. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be φ-module over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ represented by $\widetilde{\mathbf{B}}^{]0,1]}$-module $\mathbf{M}_i^{]0,1]}$. Assume that $\mathbf{M}_1^{]0,1]} \supset \mathbf{M}_2^{]0,1]} \supset t\mathbf{M}_1^{]0,1]}$. Then*

$$\deg \mathbf{M}_2 - \deg \mathbf{M}_1 = \dim_C(\mathbf{M}_1^{]0,1]} / \mathbf{M}_2^{]0,1]}) \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C$$

Proof. Clearly the equality holds for $\mathbf{M}_2 = t\mathbf{M}_1 \cong \mathbf{M}_1 \otimes \mathbf{M}(1,1)$, since $\deg t\mathbf{M}_1 - \deg \mathbf{M}_1 = \mathrm{rk}\, \mathbf{M}_1$. We claim that is suffices to prove the inequality

$$\deg \mathbf{M}_2 - \deg \mathbf{M}_1 \geq \dim_C V \tag{4.2}$$

where we abbreviate $V := (\mathbf{M}_1^{]0,1]} / \mathbf{M}_2^{]0,1]}) \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C$. Indeed we apply the inequality to the two inclusions

$$\mathbf{M}_1^{]0,1]} \supset \mathbf{M}_2^{]0,1]} \supset t\mathbf{M}_1^{]0,1]} \ \text{ and } \ \mathbf{M}_2^{]0,1]} \supset t\mathbf{M}_1^{]0,1]} \supset t\mathbf{M}_2^{]0,1]}$$

and the lemma follows from the exact sequence of $\widetilde{\mathbf{B}}^{]0,1]}$-modules

$$0 \longrightarrow \mathbf{M}_2^{]0,1]}/t\mathbf{M}_1^{]0,1]} \longrightarrow \mathbf{M}_1^{]0,1]}/t\mathbf{M}_1^{]0,1]} \longrightarrow \mathbf{M}_1^{]0,1]}/\mathbf{M}_2^{]0,1]} \longrightarrow 0$$

To prove the inequality (4.2) we argue by induction on $\dim_C V$. Let $\dim_C V = 1$. Since $\det \mathbf{M}_2 \hookrightarrow \det \mathbf{M}_1$ is an inclusion, we have from Proposition 4.1.7 that $\deg \mathbf{M}_2 \geq \deg \mathbf{M}_1$. If we had $\deg \mathbf{M}_2 = \deg \mathbf{M}_1$ then $\mathbf{M}_2 = \mathbf{M}_1$ by [31] 3.4.2. Hence $\deg \mathbf{M}_2 - \deg \mathbf{M}_1 \geq 1 = \dim_C V$. Let now $\dim_C V > 1$ and choose a C-subspace V' of dimension 1 of V. By Proposition 4.1.14 there is a unique $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$-submodule $\mathbf{M}_2 \subset \mathbf{M}_3 \subset \mathbf{M}_1$ corresponding to V'. By induction $\deg \mathbf{M}_3 - \deg \mathbf{M}_1 \geq \dim_C(V/V')$ and $\deg \mathbf{M}_2 - \deg \mathbf{M}_3 \geq \dim_C V'$ and then the inequality follows. □

Theorem 4.1.17. $\deg \mathbf{M}(\underline{D}) = t_N(\underline{D}) - t_H(\underline{D})$

Proof. By the Dieudonné-Manin's classification [14] there exists an isocrystal $(D', \varphi_{D'})$ over $\overline{\mathbb{F}_p}$ of rank one with $\varphi_{D'} = p^{t_N(\underline{D})} \cdot \varphi$ which is isomorphic to $\det(D, \varphi_D)$. Then by construction $\deg \mathbf{D} = t_N(\underline{D})$. From Lemma 4.1.16 we have

$$\begin{aligned}\deg \mathbf{M}(\underline{D}) - \deg \mathbf{D} &= \dim_C(\mathbf{D}^{]0,1]}/\mathbf{M}(\underline{D})^{]0,1]}) \otimes_{\widetilde{\mathbf{B}}^{]0,1]},\theta} C \\ &= \dim_C(D_K/\operatorname{Fil}^0 D_K) \otimes_K C = -t_H(\underline{D}) \quad \text{(Example 3.1.9)}\end{aligned}$$

□

Remark 4.1.18. Since we do not know how to construct $\mathbf{M}(\underline{D})$ for filtered isocrystal $\underline{D}$ with Hodge-Tate weights other than 0 and 1. We cannot make $\underline{D} \mapsto \mathbf{M}(\underline{D})$ into a tensor functor.

If K/K_0 is a finite extension, one can check that $\mathbf{M}(\underline{D})$ equals the φ-module over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ constructed by Berger [3]. One of Berger's main theorem is the following criterion.

Theorem 4.1.19. *([3]) Let K be a finite extension of K_0. Then $\underline{D}$ is admissible if and only if*

$$\mathbf{M}(\underline{D}) \cong \mathbf{M}(0,1)^{\oplus \dim D}$$

We make explicit what happens otherwise.

Proposition 4.1.20. *Assume that $t_N(\underline{D}) = t_H(\underline{D})$. Then $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0,1)^{\oplus \dim D}$ if and only if for some (any) integer $e \geq (\dim D) - 1$ there exists a non zero $x \in \mathbf{H}^0_{\varphi^e}([e]_*\mathbf{D} \otimes \mathbf{M}(1,1))$ with $\theta(\varphi^m_{\mathbf{D}}(x)) \in (\operatorname{Fil}^0 D_K) \otimes_K C$ for all $m = 0, \ldots, e-1$.*

Proof. Let $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0,1)^{\oplus \dim D}$. By Theorems 4.1.19 and 4.1.9 there is a φ-module $\mathbf{M}(c,d)$ over $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}(C)$ with $c < 0$ and $d < \dim D$ which is a summand of $\mathbf{M}(\underline{D})$. Then by Proposition 4.1.7 for any $e \geq (\dim D) - 1$ there exists a non zero morphism of φ-modules $f : \mathbf{M}(-1,e) \to \mathbf{M}(c,d) \subset \mathbf{M}(\underline{D}) \subset \mathbf{D}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_e$ be a basis of $\mathbf{M}(-1,e)$ satisfying the relation (4.1) in Example 4.1.3 and let x be the image of $\mathbf{e}_1$ in $\mathbf{D}$ under f. Then $\varphi^e_{\mathbf{D}}(x) = p^{-1}x$, that is $x \in \mathbf{H}^0_{\varphi^e}([e]_*\mathbf{D} \otimes \mathbf{M}(1,1))$. Moreover $f(\mathbf{e}_{m+1}) = \varphi^m_{\mathbf{D}}(x)$ in $\mathbf{D}$ for $0 \leq m < e$. Now the fact that the morphism f factors through $\mathbf{M}(\underline{D})$ amounts by Proposition 4.1.14 to $\theta(\varphi^m_{\mathbf{D}}(x)) \in (\operatorname{Fil}^0 D_K) \otimes_K C$.

Conversely assume that for some integer $e \geq (\dim D) - 1$ there exists a non zero element x in $\mathbf{H}^0_{\varphi^e}([e]_*\mathbf{D} \otimes \mathbf{M}(1,1))$ with $\theta(\varphi^m_{\mathbf{D}}(x)) \in (\mathrm{Fil}^0 D_K) \otimes_K C$ for all $m = 0, \ldots, e-1$. Define the non trivial morphism of φ-modules $f : \mathbf{M}(-1, e) \to \mathbf{D}$ by $f(\mathbf{e}_{m+1}) := \varphi^m_{\mathbf{D}}(x)$ for $0 \leq m < e$. Since $\theta(\varphi^m_{\mathbf{D}}(x)) \in (\mathrm{Fil}^0 D_K) \otimes_K C$, the morphism f factors through $\mathbf{M}(\underline{D})$ by Proposition 4.1.14. By Proposition 4.1.7 we have $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0,1)^{\oplus \dim D}$. $\square$

4.2 Construction of $\breve{\mathcal{F}}^a_b$

Let G be a reductive group and $\{\mu\}$ be a conjugacy class of one parameter subgroups of G. We make the following assumption on the pair $(G, \{\mu\})$ and assume that this assumption is satisfied throughout this section:

There exists a faithful $\mathbb{Q}_p$-rational representation V of G such that all the weights of $\{\mu\}$ on V are 0 or -1.

Let $G' = \mathbf{GL}(V)$ and $b \in G(K_0)$. Then G is a closed subgroup of G' and b can be viewed as an element of $G'(K_0)$. We have a closed embedding $\mathcal{F} \hookrightarrow \mathcal{F}' := \mathrm{Flag}(V)$ of flag varieties. Here $\mathrm{Flag}(V)$ is actually a Grassmannian. We denote by $\breve{\mathcal{F}}^{\mathrm{an}}$ the $\breve{E}$-analytic space associated with $\mathcal{F} \otimes_E \breve{E}$.

Let $\mu \in \breve{\mathcal{F}}^{\mathrm{an}}$ be an analytic point. Let $K = k(\mu)$ be the (complete) residue field of μ and let C be the completion of an algebraic closure of K. Let $\underline{D_\mu} := (V_{K_0}, b \cdot \varphi, \mathrm{Fil}^\bullet_\mu V_K)$. In particular, $\mathrm{Fil}^{-1} V_K = V_K$ and $\mathrm{Fil}^0 V_K = (0)$. We let $\mathbf{M}_\mu := \mathbf{M}(\underline{D_\mu})$ be the φ-module over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C)$ in Definition 4.1.15.

Definition 4.2.1. We define

$$\breve{\mathcal{F}}^a_b := \{\mu \in \breve{\mathcal{F}}^{\mathrm{an}} \text{ analytic points} \mid \mathbf{M}_\mu \cong \mathbf{M}(0,1)^{\oplus d}\}$$

where $d = \dim_{\mathbb{Q}_p} V$.

If $b' = gb\varphi(g^{-1})$ for some $g \in G(K_0)$ the map $\mu \mapsto g^{-1}\mu g$ maps $\breve{\mathcal{F}}^a_b$ isomorphically onto $\breve{\mathcal{F}}^a_{b'}$. The Newton slope and Hodge slope are determined by the conjugacy class $\{\mu\}$. If $t_N(\underline{D_\mu}) \neq t_H(\underline{D_\mu})$ the sets $\breve{\mathcal{F}}^a_b$ and $(\breve{\mathcal{F}}^{wa}_b)^{\mathrm{rig}}$ are empty. So from now on we assume $t_N(\underline{D_\mu}) = t_H(\underline{D_\mu})$ for all $\mu \in \breve{\mathcal{F}}^{\mathrm{an}}$. The main theorem is the following

Theorem 4.2.2. *(Hartl) The set $\breve{\mathcal{F}}^a_b$ is an open $\breve{E}$-analytic subspace of $\breve{\mathcal{F}}^{\mathrm{an}}$. If b is decent with the integer s, then $\breve{\mathcal{F}}^a_b$ has a natural structure of open E_s-analytic subspace of $(\mathcal{F} \otimes_E E_s)^{\mathrm{an}}$ from which it arises by base change to $\breve{E}$.*

Proof. Step 1: Let V be a faithful representation of G satisfying the assumption. We may reduce to prove the case where $G' = \mathbf{GL}(V)$. Indeed, G is a closed subgroup of G' and this identifies a closed embedding $\mathcal{F} \hookrightarrow \mathcal{F}' := \mathrm{Flag}(V) \otimes_{\mathbb{Q}_p} E$. Here $\mathrm{Flag}(V)$ is a Grassmannian isomorphic to G'/S', where $S' = \mathrm{Stab}_{G'}(V_0)$ is the stabilizer of an appropriate subspace V_0 of V. By definition $\breve{\mathcal{F}}^a_b = \breve{\mathcal{F}}^{\mathrm{an}} \cap \breve{\mathcal{F}}'^a_b$. So it suffice to prove the theorem for G' instead of G. Since G' is connected we may assume by Proposition 3.2.3 that b is decent, say with integer s. We let $\mathcal{F}'_s := (\mathcal{F}' \otimes_E E_s)^{\mathrm{an}}$ and define the subset $\mathcal{F}'^{a}_b \subset \mathcal{F}'^{\mathrm{an}}_s$ by the same condition as in Definition 4.2.1. We only need to show that it is open.

Choose an integer $e \geq (\dim V) - 1$ which is also a multiple of s. Then by Proposition 4.1.20, the set $\mathcal{F}'^{a}_b \subset \mathcal{F}'^{\mathrm{an}}_s$ equals the set of analytic points $\mu \in \mathcal{F}'^{\mathrm{an}}_s$ such that there exists an algebraically closed complete extension C of $k(\mu)$ and a non zero element $x \in \mathbf{H}^0_{\varphi^e}([e]_*\mathbf{D}_C \otimes \mathbf{M}(1,1))$ with $\theta(\varphi^m_{\mathbf{D}}(x)) \in (\mathrm{Fil}^0_\mu V_{k(\mu)}) \otimes_{k(\mu)} C$ for all $m = 0, \ldots, e-1$. Here $[e]_*\mathbf{D}_C$ is the φ^e-module $(D, \varphi^e_D) \otimes_{K_0} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C)$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C)$.

Step 2: We identify $V \otimes_{\mathbb{Q}_p} \mathbb{A}^1_{E_s}$ with affine d-space $\mathbb{A}^d_{E_s}$ over E_s. For $\eta \in E_s$ consider the $\breve{E}$-analytic polydisc with radii $(|\eta|, \cdots, |\eta|)$

$$\mathbb{D}(\eta)^{de} = \mathcal{M}(E_s\{\frac{h_{im}}{\eta}| \quad i = 1, \ldots, d; \quad m = 0, \ldots, e-1\}) \subset (\mathbb{A}^d_{E_s})^e$$

We will construct a constant $\eta \in E_s$ and a compact subset Z of $\mathbb{D}(\eta)^{de}$ in step 3 with the following property: If C is an algebraically closed complete extension of E_s and $x \in \mathbf{H}^0_{\varphi^e}([e]_*\mathbf{D}_C \otimes \mathbf{M}(1,1))$ with $x \neq 0$, then for some integer N

$$(h_m)^{e-1}_{m=0} := ((h_{1m}, \ldots, h_{dm})^T)^{e-1}_{m=0} := (p^N \theta(\varphi^m_{\mathbf{D}}(x)))^{e-1}_{m=0}$$

is a C valued point of Z and Z consists precisely of those points.

Now let G'^{an}_s be the E_s-analytic space associated with the group scheme $G' \otimes_{\mathbb{Q}_p} E_s$ and consider the morphism of E_s-analytic spaces

$$\begin{aligned} \beta : G'^{\mathrm{an}}_s \times_{E_s} \mathbb{D}(\eta)^{de} &\longrightarrow (\mathbb{A}^d_{E_s})^e \cong (V \otimes_{\mathbb{Q}_p} \mathbb{A}^1_{E_s})^e \\ (g, (h_m)^{e-1}_{m=0}) &\longmapsto (g^{-1}h_m)^{e-1}_{m=0} \end{aligned}$$

Let Y be the closed subset of $G'^{\mathrm{an}}_s \times_{E_s} \mathbb{D}(\eta)^{de}$ defined by the condition that $(h_m)^{e-1}_{m=0}$ belongs to Z and that $\beta(Y) \subset (V_0 \otimes_{\mathbb{Q}_p} \mathbb{A}^1_{E_s})^e$. Furthermore consider the projection map

$$pr_1 : G'^{\mathrm{an}}_s \times_{E_s} \mathbb{D}(\eta)^{de} \longrightarrow G'^{\mathrm{an}}_s$$

onto the first factor and the canonical map $\gamma : G'^{\mathrm{an}}_s \to \mathcal{F}'^{\mathrm{an}}_s$ coming from the isomorphism $\mathcal{F}'^{\mathrm{an}}_s \cong G'^{\mathrm{an}}_s / \mathrm{Stab}_{G'_s}(V_0)$. Then $\mu \in \mathcal{F}'^{\mathrm{an}}_s$ does not belong to $\mathcal{F}'^{a}_b$ if and only if $\mu \in \gamma \circ pr_1(Y)$. Since $\mathbb{D}(\eta)^{de}$ is quasi-compact the projection pr_1 is a proper map of topological Hausdorff spaces, $pr_1(Y)$ is closed. Note that $\mathcal{F}'^{\mathrm{an}}_s$ carries the quotient topology under γ since γ is a smooth morphism of schemes, hence open by [5] Proposition 3.5.8 and Corollary 3.7.4. Since by construction $pr_1(Y) = \gamma^{-1}(\gamma \circ pr_1(Y))$ we conclude that $\mathcal{F}'^{a}_b = \mathcal{F}'^{\mathrm{an}}_s - \gamma \circ pr_1(Y)$ is open in $\mathcal{F}'^{\mathrm{an}}_s$ as desired.

Step 3: It remains to construct the compact set Z. Since b is decent, the φ-module $[e]_*\mathbf{D} \otimes \mathbf{M}(1,1)$ is isomorphic to $\bigoplus_{i=1}^d \mathbf{M}(-c_i, 1)$ for suitable integers c_i. We assume that the identification of $V \otimes_{\mathbb{Q}_p} \mathbb{A}^1_{E_s}$ with $\mathbb{A}^d_{E_s}$ in Step 2 was chosen compatible with this direct sum decomposition. Let $c_1, \dots, c_k > 0 = c_{k+1} = \dots = c_l > c_{l+1}, \dots, c_d$. Then by Proposition 4.1.8

$$\begin{aligned}\mathbf{H}^0_{\varphi^e}([e]_*\mathbf{D} \otimes \mathbf{M}(1,1)) \cong & \bigoplus_{i=1}^{k} \{ \sum_{\nu} p^{c_i \nu} \sum_{j=0}^{c_i - 1} p^j \varphi^{-e\nu}([u_{ij}]) | \ \ u_{ij} \in \widetilde{\mathbf{E}}, v_{\mathbf{E}}(u_{ij}) > 0 \} \\ & \oplus \bigoplus_{i=k+1}^{l} W(\mathbb{F}_{p^e})[1/p] \\ & \oplus \bigoplus_{i=l+1}^{d} (0)\end{aligned}$$

For $1 \le i \le k, 0 \le j \le c_i - 1$ consider the compact sets

$$\begin{aligned}U_{ij}^{(0)} &:= \mathcal{M}(E_s\{u_{ij}^{(0)}/p\}) = \{|u_{ij}^{(0)}| \le |p|\} \ \text{ and} \\ U_{ij}^{(n)} &:= \mathcal{M}(E_s\{u_{ij}^{(n)}\}) = \{|u_{ij}^{(n)}| \le 1\}. \ \text{ for } n \ge 1\end{aligned}$$

Then the sets

$$\begin{aligned}U_{ij} &:= \{(u_{ij}^{(n)})_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} U_{ij}^{(n)} | \ \ (u_{ij}^{n+1})^p = u_{ij}^{(n)} \text{ for all } n \ge 0\} \ \text{ and} \\ U &:= \prod_{i=1}^{k} \prod_{j=0}^{c_i - 1} U_{ij} \times \prod_{i=k+1}^{l} W(\mathbb{F}_{p^e}) \times \prod_{i=l+1}^{d} \{0\}\end{aligned}$$

are compact by Tychonoff's theorem. For an arbitrary algebraically closed extension C of E_s consider a C-valued point u of U given by

$$(((u_{ij}^{(n)})_{n \in \mathbb{N}})_{i=1,\dots,k; j=0,\dots,c_i-1}, (a_i)_{i=k+1,\dots,l}, (0)_{i=l+1,\dots,d})$$

with $u_{ij}^{(n)} \in C$ and $a_i \in W(\mathbb{F}_{p^e})$. We assign to u the C-valued point y of $\mathbb{A}^{de}_{E_s}$ with $(h_m)_{m=0}^{e-1} = (\theta(\varphi^m_{\mathbf{D}}(x)))_{m=0}^{e-1}$ where x is the element of $\mathbf{H}^0_{\varphi^e}([e]_*\mathbf{D} \otimes \mathbf{M}(1,1))$ associated with the $u_{ij}^{(n)}$ and a_i. This defines a map

$$\begin{aligned}\alpha : U &\longrightarrow \mathbb{A}^{de}_{E_s} \\ u &\longmapsto y\end{aligned}$$

of topological Hausdorff spaces. One can prove that α is a continuous map, for this we refer to [25].

Now multiplying $(h_{im})_{i,m}$ with p amounts to replacing $u_{ij}^{(n)}$ by $u_{i,j-1}^{(n)}$ for $j = 1, \dots, c_i - 1$, and $u_{i,0}^{(n)}$ by $(u_{i,c_i-1}^{(n)})^{p^e}$, and a_i by pa_i. Thus we may take

$$Z := \alpha(U - \{u \in U| \ \ |u_{ij}^{(0)}| < |p|^{p^e}, a_i \in pW(\mathbb{F}_{p^e}) \text{ for all } i \text{ and } j\}).$$

Then Z is the continuous image of a compact set and satisfies the property required in Step 2. This proves the theorem. $\square$

Proposition 4.2.3. *Let $\breve{\mathcal{F}}_b^{wa}$ be as in Theorem 3.2.12. Then the set $\breve{\mathcal{F}}_b^a$ is an open $\breve{E}$-analytic subspace of $\breve{\mathcal{F}}_b^{wa}$.*

Proof. After Theorem 4.2.2 we only need to show that $\breve{\mathcal{F}}_b^a$ is contained in $\breve{\mathcal{F}}_b^{wa}$. Let $\mu \in \breve{\mathcal{F}}_b^a$ be an analytic point and set $K = k(\mu)$.

Let $D' \subset D$ be a φ_D-stable K_0-subspace and let $\mathrm{Fil}^i_\mu D'_K := D'_K \cap \mathrm{Fil}^i_\mu D_K$. We have to show that $t_H(\underline{D'}) \leq t_N(\underline{D'})$ for any subobject $\underline{D'} := (D', \varphi_D|_{D'}, \mathrm{Fil}^\bullet_\mu D'_K) \subset \underline{D_\mu}$ and with equality if $\underline{D'} = \underline{D_\mu}$. Consider the φ-submodule $\mathbf{M}' := \mathbf{M}(\underline{D'}) \subset \mathbf{M}(\underline{D_\mu})$. Then by Theorem 4.1.17 $t_N(\underline{D'}) - t_H(\underline{D'}) = \deg \mathbf{M}' = \mathrm{rk}\, \mathbf{M}' \cdot \mathrm{wt}\, \mathbf{M}'$. If $\underline{D'} = \underline{D_\mu}$, since $\mu \in \breve{\mathcal{F}}_b^a$, we have $\mathbf{M}(\underline{D_\mu}) \cong \mathbf{M}(0,1)^{\oplus \dim D}$ and thus $t_H(\underline{D_\mu}) = t_N(\underline{D_\mu})$. If $\underline{D'} \subset \underline{D_\mu}$, we have $\mathrm{wt}\, \mathbf{M}' \geq \mathrm{wt}\, \mathbf{M}(\underline{D_\mu})$ by Proposition 4.1.11 finishing the proof. $\square$

Corollary 4.2.4. *The open immersion $\breve{\mathcal{F}}_b^a \subset \breve{\mathcal{F}}_b^{wa}$ induces an étale morphism of rigid analytic spaces $(\breve{\mathcal{F}}_b^a)^{\mathrm{rig}} \to (\breve{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ which is bijective on rigid analytic points. It is an isomorphism if and only if $\breve{\mathcal{F}}_b^a = \breve{\mathcal{F}}_b^{wa}$.*

Proof. The functor $(\cdot)^{\mathrm{rig}}$ takes étale morphisms to étale morphisms. The rigid analytic points are Berkovich analytic points with residue field finite over $\breve{E}$. Then by Definition 4.2.1, Theorem 4.1.19 and 3.1.16 that the morphism is bijective on rigid analytic points. The rest is a consequence of Theorem 2.3.22 since $\breve{\mathcal{F}}_b^a$ and $\breve{\mathcal{F}}_b^{wa}$ are paracompact by Lemma 2.3.24. $\square$

4.3 Relations with Period Morphisms

Let $G = \mathbf{GL}(V)$ for a finite dimensional $\mathbb{Q}_p$-vector space V. Let $\breve{\mathcal{F}}$ be the Grassmanian over K_0 of d-dimensional subspaces of V_{K_0}. Let $b \in G(K_0)$ and $(D, \varphi_D) = (V_{K_0}, b \cdot \varphi)$. Assume that there exists a p-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}_p}$ of dimension d whose covariant Dieudonné isocrystal is (D, φ_D). We consider the moduli problem of deformations of $\mathbb{X}$ as in Theorem 2.2.1 and the period morphism $\breve{\pi}^{\mathrm{an}} : \breve{\mathcal{M}}^{\mathrm{an}} \to \breve{\mathcal{F}}^{\mathrm{an}}$ in Remark 2.4.4.

Theorem 4.3.1. *(Hartl, Faltings) The period morphism factors through $\breve{\mathcal{F}}_b^a$ and surjective on analytic points of $\breve{\mathcal{F}}_b^a$.*

Proof. Let $x \in \mathcal{M}^{\mathrm{an}}$ be an analytic point and let $\mu = \breve{\pi}^{\mathrm{an}}(x) \in \breve{\mathcal{F}}^{\mathrm{an}}$. Let $K = k(x)$ and let C be the completion of an algebraic closure of K. Let X_x be the fiber of the universal p-divisible group X at x and consider the Tate module $T_p(X_x)$ of X_x. An element $\lambda \in T_p(X_x)$ corresponds to a morphism of p-divisible groups $\lambda : \mathbb{Q}_p/\mathbb{Z}_p \to X_{O_C}$ over O_C. By functoriality of the universal vector extension this yields the following diagram of C-vector spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_C & \xrightarrow{\mathrm{Id}} & \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_C & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & (\mathrm{Fil}^\bullet_\mu D_K) \otimes_K C & \longrightarrow & D \otimes_{K_0} C & \longrightarrow & (\mathrm{Lie}\, X_x)_C & \longrightarrow & 0 \end{array}$$

Note that $\mathrm{Lie}(\mathbb{Q}_p/\mathbb{Z}_p) = (0)$, since $\mathbb{Q}_p/\mathbb{Z}_p$ is ind-étale. By the crystalline natural of the covariant Dieudonné module, we evaluate $\mathbb{D}(\lambda) : \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{D}(X_{O_C})$ on the pd-thickening $\mathbf{B}^+_{\mathrm{cris}}(C)$ of O_C (here we use the crystalline theory of Berthelot-Messing) and get

$$\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbf{B}^+_{\mathrm{cris}}(C)} \longrightarrow \mathbb{D}(X_{O_C})_{\mathbf{B}^+_{\mathrm{cris}}(C)} = \mathbb{D}(X_x)_{\mathbf{B}^+_{\mathrm{cris}}(C)}$$

We have $\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbf{B}^+_{\mathrm{cris}}(C)} = \mathbf{B}^+_{\mathrm{cris}}(C)$ since the universal vector extension of $\mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbf{B}^+_{\mathrm{cris}}(C)$ is obtained from the sequence $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$ by pushout via $\mathbb{Z}_p \to \mathbf{B}^+_{\mathrm{cris}}(C)$. Then we have a morphism

$$\begin{aligned} T_pX_x \otimes_{\mathbb{Z}_p} \mathbf{B}^+_{\mathrm{cris}}(C) &\longrightarrow \mathbb{D}(X_x)_{\mathbf{B}^+_{\mathrm{cris}}(C)} \cong D \otimes_{K_0} \mathbf{B}^+_{\mathrm{cris}}(C) \\ \lambda \otimes a &\longmapsto \mathbb{D}(\lambda)(a) \end{aligned}$$

Here the isomorphism on the right arises from the quasi-isogeny ρ_x, since p is invertible in $\mathbf{B}^+_{\mathrm{cris}}(C)$ (Theorem 2.4.1). By Faltings [17] Theorem 7, the morphism on the left is injective. Since the elements of T_pX_x are φ-invariant inside $T_pX_x \otimes_{\mathbb{Z}_p} \mathbf{B}^+_{\mathrm{cris}}(C)$ and $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C)$ equals $\bigcap_{n\in\mathbb{N}} \varphi^n \mathbf{B}^+_{\mathrm{cris}}(C)$, we get a monomorphism

$$T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C) \longrightarrow D \otimes_{K_0} \widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C).$$

It gives rise to a monomorphism

$$\begin{aligned} T_pX_x \longrightarrow T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^{]0,1]}(C) &\longrightarrow D \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,1]}(C) \\ \lambda \longmapsto \lambda \otimes 1 &\longmapsto \mathbb{D}(\lambda)(1) \end{aligned}$$

since $\widetilde{\mathbf{B}}^{]0,1]}(C)$ is a flat $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C)$-algebra. Consider the morphism $\theta : D \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,1]} \to D \otimes_{K_0} C$. From Diagram 4.3 we see that $\theta(T_p(X_x)) \subset (\mathrm{Fil}^0_\mu D_K) \otimes_K C$ and we have $T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C) \hookrightarrow \mathbf{M}_\mu := \mathbf{M}(D, \varphi_D, \mathrm{Fil}^0_\mu D_K)$ by Proposition 4.1.14. This forces $\mathbf{M}_\mu \cong \mathbf{M}(0,1)^{\dim V}$. Otherwise we have $\mathbf{M}_\mu \cong \bigoplus_j \mathbf{M}(c_j, d_j)$ with $c_1 > 0$ (note that $\deg \mathbf{M} = \sum_j c_j = 0$). Since the elements of T_pX_x are φ-invariant and $\mathbf{H}^0_\varphi(\mathbf{M}(c_1, d_1)) = (0)$ by Proposition 4.1.7, the projection

$$T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C) \longrightarrow \mathbf{M}(c_1, d_1)$$

is zero. Thus $T_pX_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C) \hookrightarrow \bigoplus_{j>1} \mathbf{M}(c_j, d_j)$, but this is impossible since we have $\mathrm{rk}_{\mathbb{Z}_p} T_pX_x = \dim V > \mathrm{rk} \bigoplus_{j>1} \mathbf{M}(c_j, d_j)$. This proves that the image $\mu = \breve{\pi}^{\mathrm{an}}(x)$ of x is in $\breve{\mathcal{F}}^a_b$.

The proof of surjectivity of $\breve{\pi}^{\mathrm{an}}$ is parallel to Theorem 3.2.11 but the difficulty is to find the p-divisible group over O_K where $K/\breve{E}$ may be infinite. The proof is essentially due to Faltings. Let μ be any point in $\breve{\mathcal{F}}^a_b$. The morphism $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C))^{\oplus \dim V} \cong \mathbf{M}_\mu \hookrightarrow D \otimes_{K_0} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is represented, with respect to a K_0-basis of D, by a matrix $M \in \mathrm{Mat}_{h\times h}(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C))$ with $tM^{-1} \in \mathrm{Mat}_{h\times h}(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}(C))$. Then by Proposition I.4.1 of [4], we have in fact $M, tM^{-1} \in \mathrm{Mat}_{h\times h}(\widetilde{\mathbf{B}}^+_{\mathrm{rig}}(C)) \subset \mathrm{Mat}_{h\times h}(\mathbf{B}^+_{\mathrm{cris}})$. So M defines an isomorphism $\mathbf{B}_{\mathrm{cris}}(C)^{\oplus \dim V} \cong D \otimes_{K_0}$

$\mathbf{B}_{\text{cris}}(C)$ compatible with Frobenius, which maps $(\mathbf{B}^{+}_{\text{cris}}(C))^{\oplus \dim V}$ onto the preimage of $\operatorname{Fil}^0_\mu D_K \otimes_K C$ under the map $\operatorname{Id} \otimes \theta : D \otimes_{K_0} \mathbf{B}^{+}_{\text{cris}}(C) \to D \otimes_{K_0} C$. This means that $(D, \varphi_D, \operatorname{Fil}^0_\mu D_K)$ is admissible in the sense of Definition 3.1.10(ii). By [16] Theorem 9 and 14, there is a p-divisible group X over O_K and a quasi-isogeny $\rho : \mathbb{X}_{O_K/(p)} \to X_{O_K/(p)}$ such that

$$(V_{K_0}, \varphi, (\operatorname{Lie} X^*)^\vee_K \hookrightarrow V \otimes_{\mathbb{Q}_p} K) \cong (D, \varphi_D, \operatorname{Fil}^0_\mu D_K)$$

Therefore μ lies in the image of $\breve{\pi}^{\text{an}}$.

$\square$

For a p-adic period space $(\breve{\mathcal{F}}^{wa}_b)^{\text{rig}}$ possessing period morphism, we have the following consideration on Rapoport-Zink's conjecture 3.3.9.

We choose and fix a faithful p-adic representation V of G of Hodge-Tate weights 0 and 1. Consider the moduli problem associated to $\mathbb{X}$, where $\mathbb{X}$ is a p-divisible group over $\overline{\mathbb{F}_p}$ such that $(D(\mathbb{X}), D(\operatorname{Frob}_{\mathbb{X}})) = (V \otimes K_0, b(\operatorname{Id} \otimes \varphi))$. Let X be the universal p-divisible group over $\breve{\mathcal{M}}^{\text{an}}$ and the Tate module $T_p X$ gives a local system of $\mathbb{Q}_p$-vector spaces on $\breve{\mathcal{F}}^a_b$. We extend $V \mapsto T_p(X)$ to a functor $\operatorname{Rep}_{\mathbb{Q}_p} G \to \mathbb{Q}_p - \underline{\operatorname{Loc}}_{\breve{\mathcal{F}}^a_b}$ by the same reason as in 3.2.6.

Conjecture 4.3.2. ([25]) The set $\breve{\mathcal{F}}^a_b$ is the unique largest open $\breve{E}$-analytic subspace of $\breve{\mathcal{F}}^{wa}_b$ on which the tensor functor from $\operatorname{Rep}_{\mathbb{Q}_p} G$ to $\mathbb{Q}_p - \underline{\operatorname{Loc}}_{\breve{\mathcal{F}}^a_b}$ with property in Conjecture 3.3.9.

Remark 4.3.3. This was shown to be true by A.J. de Jong in the Lubin-Tate situation where $\breve{\mathcal{F}}^a_b = \breve{\mathcal{F}}^{wa}_b = \breve{\mathcal{F}}^{\text{an}}$.

Remark 4.3.4. In general $\breve{\mathcal{F}}^a_b$ is strictly open subspace of $\breve{\mathcal{F}}^{wa}_b$, for example one can see [25].

Bibliography

[1] Y. André, *Period mappings and differential equations. From $\mathbb{C}$ to $\mathbb{C}_p$*, available on arXiv:math/0203194v1.

[2] L. Berger, *An introduction to the theory of p-adic representations*, available at http://www.umpa.ens-lyon.fr/∼lberger/.

[3] L. Berger, *Équations différentielles p-adiques et (φ, N)-modules filtrés*, Preprint on arXiv:math.NT/0406601.

[4] L. Berger, *Limites de représentations cristallines*, Compos. Math. 140 (2004), no. 6, 1473-1498.

[5] V. Berkovich, *Etale cohomology for non-archimedean analytic spaces*, Publ. Math. IHES. 78 (1993), 5-161.

[6] P. Berthelot, *Cohomologie rigide et cohomologie rigide à support propre. Première partie*, Preprint IRMAR 96-03, available at http://perso.univ-rennes1.fr/pierre.berthelot/.

[7] J.-F. Boutot, *Uniformisation p-adique des variétés de Shimura*, Séminaire Bourbaki, 1996-1997, exp. n^o 831, pp. 307-322.

[8] J.-F. Boutot, H. Carayol, *Uniformisation p-adique des courbes de Shimura: les théorèm de Čerednik et Drinfeld*, Courbes Modulaires et Courbes de Shimura, Astérisque, no. 196-197, Société Mathématique de France, 1991.

[9] C. Breuil, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. 152 (2000), 489-549.

[10] P. Colmez, J.-M. Fontiane, *Construction des représentations p-adiques semi-stables*, Invent. Math. 140 (2000), no. 1, 1-43.

[11] B. Conrad, *Several approaches to non-archimedean geometry*, available at http://math.stanford.edu/∼conrad/.

[12] J.-F. Dat, *Groupes p-divisibles d'après Grothendieck-Messing*, Exposé IHES.

[13] J. Dieudonné, A. Grothendieck, *Éléments de Géométrie Algébrique*, Publ. Math. IHES, nos. 4, 8, 11, 17, 20, 24, 28, 32.

[14] M. Demazure, *Lectures on p-divisible groups*, LNM 302.

[15] V.G. Drinfeld, *Coverings of p-adic symmetric domains*, Funct. Anal. Appl. 10 (1976), 107-115.

[16] G. Faltings, *Coverings of P-adic Period Domains*, Preprint 2007.

[17] G. Faltings, *Integral crystalline cohomology over very ramified valuation rings*, J. Amer. Math. Soc. 12 (1999), no. 1, 117-144.

[18] Ł. Fargues, *L'isomorphisme entres les tours de Lubin-Tate et de Drinfeld et applications cohomologiques*, L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld, Birkhauser, Progress in Math, vol. 262.

[19] L. Fargues, *An introduction to Lubin-Tate spaces and p-divisible groups*, Course notes, avaiable at http://www.math.u-psud.fr/∼fargues

[20] J.-M. Fontaine, *Le corps des périodes p-adiques*, Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque 223 (1994), 59-111.

[21] J.-M. Fontaine, *Représentations p-adiques semi-stables*, Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque 223 (1994), 113-184.

[22] J.-M. Fontaine, Y. Ouyang, *Theory of p-adic Galois Representations*, available at http://staff.ustc.edu.cn/∼yiouyang/

[23] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 431-436, Gauthier-Villars, Paris, 1971.

[24] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonneé*, Séminaire de Mathématiques Supérieures 45, Presses de l'Iniversité de Montréal, Montréal 1974.

[25] U. Hartl, *On a Conjecture of Rapoport and Zink*, Preprint on arXiv:math.NT/0605254.

[26] U. Hartl, *On Period Spaces for p-divisible Groups*, Preprint on arXiv:0709.3444.

[27] U. Hartl, *A Dictionary between Fontaine-Theory and its Analogue in Equal Characteristic*, Journal of Number Theory 129, n. 7 (2009), 1734-1757.

[28] U. Hartl, *Period Spaces for Hodge Structures in Equal Characteristic*, Preprint on arXiv:math.NT/0511686.

[29] A.J. de Jong, *Étale Fundamental Groups of non-Archimedean analytic spaces*, Comp. Math. 97 (1995), 89-118.

[30] A.J. de Jong, M. van der Put, *Étale cohomology of rigid spaces*, Documenta Mathematica (electronic journal), 1 (1996), pp. 1–56.

[31] K. Kedlaya, *slope filtrations revisited*, Doc. Math. 10 (2005), 447-525.

[32] W. Messing, *The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes*, LNM 264, Springer-Verlag, Berlin-New York, 1972.

[33] M. Rapoport, Th. Zink, *Period Spaces for p-divisible Groups*, Ann. Math. Stud. 141, Princeton University Press, Princeton 1996.

[34] J. Tate, *Finite flat group schemes*, Modular Forms and Fermat's Last Theorem, G. Cornell, J. Silverman and G. Stevens, eds., Springer, New York, 1997, pp. 121-154.