R-equivalence and Zero-Cycles on 3-Dimensional Tori
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Conventions

Unless otherwise explicitly stated to the contrary, the following notations and conventions will be in force.

1. The symbol $\subseteq$ denotes inclusion of sets, while $\subset$ denotes a ***strict*** inclusion, and similarly for $\supseteq$ and $\supset$.

2. All rings and algebras are assumed to be commutative with 1 and homomorphisms between them always send 1 to 1. For any ring A, A^* denotes its group of units. If $\mathfrak{p}$ is a prime ideal of a ring A, we write $\operatorname{ht}\mathfrak{p}$ for the ***height*** of $\mathfrak{p}$.

3. If $\mathcal{C}$ is a category, $\mathcal{C}^{op}$ denotes its opposite category. For objects $X, Y \in \mathcal{C}$, $\operatorname{Mor}_{\mathcal{C}}(X, Y)$ will denote the set of all morphisms from X to Y. When $\mathcal{C}$ is additive, we will often write Hom instead of Mor. Notations for some categories are the following:

$\mathfrak{Set}$: the category of sets;

$\mathfrak{Group}$: the category of groups;

$\mathfrak{Top}$: the category of topological spaces;

$\mathfrak{Alg}_{/k}$: the category of algebras over a field k;

$\mathfrak{Field}_{/k}$: the category of field extensions of a field k.

The symbols $\prod$ and $\coprod$ are standard notations respectively for direct product and coproduct in categories. For example, in $\mathfrak{Set}$ or $\mathfrak{Top}$, $\coprod$ means disjoint union. In additive categories, $\bigoplus$ will be used in place of $\coprod$.

4. A diagram of morphisms in a category of the following form

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

is called a ***fibre square*** if (X', g', f') is a fibre product of $X \xrightarrow{f} Y$ and $Y' \xrightarrow{g} Y$.

5. Except in the Appendix, by a ***scheme*** we will always mean an algebraic scheme over a field, that is, a scheme of finite type over a field. A morphism of schemes over a field k always means a k-morphism. If X and Y are k-schemes, $\operatorname{Mor}_k(X, Y)$ denotes the set of k-morphisms from X to Y.

6. A ***variety*** over a field k is a separated (algebraic) scheme over k. A ***curve*** is a variety of dimension 1. A ***surface*** is a variety of dimension 2. $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ denote respectively the n-dimensional affine space and projective space over k. The subscript k is often omitted when it is clear from the context.

7. ***Subschemes*** or ***subvarieties*** are always assumed to be closed. As a subscheme, an ***irreducible component*** of a scheme will be given the reduced subscheme structure.

8. If X is an integral k-scheme, the field of rational functions on X will be denoted by $k(X)$.

9. If X is a scheme and $x \in X$, the residue field of x will be denoted $\kappa(x)$. If V is an irreducible subscheme of X, we write $\mathscr{O}_{X,V}$ for the local ring of X at V, that

is, $\mathscr{O}_{X,V} = \mathscr{O}_{X,\xi}$, where ξ is the generic point of V. The maximal ideal of $\mathscr{O}_{X,V}$ will be written $\mathfrak{m}_{X,V}$.

10. If A is an algebra over a field k and X is a k-scheme, we write $X \times_k A$, or simply X_A, for the fibre product $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, A$. $\bar{k}$ usually denotes a separable algebraic closure of k, and we will write $\overline{X} := X \times_k \bar{k}$.

11. If X, Y are schemes over a field k, the fibre product $X \times_k Y$ will be often denoted simply by $X \times Y$ when the ground field is clear from the context.

12. A scheme X is called ***normal*** (resp. ***regular***) if all its local rings $\mathscr{O}_{X,x}$, $x \in X$ are integrally closed domains (resp. regular). A scheme X over a field k is called ***smooth*** if $X \times_k k^{\mathrm{ac}}$ is regular, where k^{ac} is an algebraic closure of k. A morphism $f : X \to Y$ is said to be ***smooth*** at $x \in X$, if f is flat at x and if the scheme-theoretic fibre X_y, where $y = f(x)$, is smooth over the residue field $\kappa(y)$. f is said to be smooth if it is smooth at every $x \in X$.

Contents

Introduction

Let X be an algebraic variety over a field k. For each integer $p \geq 0$, the group $Z_p(X)$ of *cycles* of dimension p on X is the free abelian group with basis the set of all integral subvarieties of dimension p. For any $(p+1)$-dimensional integral subvariety W of X, there is a well-defined group homomorphism $k(W)^* \longrightarrow Z_p(X)$, written as $f \mapsto [\mathrm{div}(f)]$. The *Chow group* $\mathrm{CH}_p(X)$ is by definition the quotient of $Z_p(X)$ divided by the subgroup generated by all the images of such homomorphisms. Two cycles α, $\beta \in Z_p(X)$ are called *rationally equivalent* if their images in $\mathrm{CH}_p(X)$ coincide. If X is purely of dimension d, we will write $\mathrm{CH}^{d-p}(X) = \mathrm{CH}_p(X)$ when the grading by codimension is more convenient. Note that the group $\mathrm{CH}_0(X)$ is nothing but the free abelian group based on closed points modulo the rational equivalence.

Suppose X has rational points. We say two points x, $y \in X(k)$ are *directly R-equivalent* if there is a k-rational map $f : \mathbb{P}^1 \dashrightarrow X$ such that $f(0) = x$ and $f(\infty) = y$. The *R-equivalence* on $X(k)$ is the equivalence relation generated by direct R-equivalence. The set of R-equivalence classes on $X(k)$ will be written as $X(k)/R$. For points in $X(k)$, R-equivalence is stronger than rational equivalence.

Assume further that the variety X is proper. Then there is a *degree* homomorphism: $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$ which maps the class $[x]$ of a closed point x to the degree of the field extension $\kappa(x)/k$. Denote by $A_0(X)$ the kernel of the degree homomorphism. When a point $x_0 \in X(k)$ is fixed, there exists a well-defined map $X(k)/R \longrightarrow A_0(X)\,;\ x \mapsto [x] - [x_0]$. Things become even more interesting if X is a smooth compactification of an algebraic torus T (namely, X is smooth projective and contains T as a dense open subset). In this case, the inclusion $T \to X$ induces a natural map $T(k)/R \to X(k)/R$. Let 1 denote the identity element of the group $T(k)$. We have a well-defined map

$$\varphi : T(k)/R \longrightarrow A_0(X)\,; \quad t \mapsto [t] - [1]\,.$$

The set $T(k)/R$ inherits naturally a group structure. So we are interested in the following questions: is φ a group homomorphism, and is it an isomorphism?

The main result to be discussed in this thesis is the following theorem, recently proved by Merkurjev.

Main Theorem. *Let T be an algebraic torus over a field k and let X be a smooth compactification of T. If $\dim T \leq 3$, the map*

$$\varphi : T(k)/R \longrightarrow A_0(X)\,; \quad t \mapsto [t] - [1]$$

is an isomorphism of groups.

In addition to the fundamental work by Colliot-Thélène and Sansuc [7] on the R-equivalence on tori, results from the K-theory of toric models, worked out by Merkurjev and Panin [24], have turned out to be probably the most important ingredients in the proof of the main theorem. A little bit of the theory of Chow motives is also needed in at least the following contexts: morphisms of Chow motives induce natural isomorphisms $\mathrm{CH}_0(X) \cong \mathrm{CH}_0(X')$ and $A_0(X) \cong A_0(X')$ for birationally equivalent smooth projective varieties X and X', this allows us to take X any toric model of the torus T so that K-theory of toric models may be applied to derive useful information, and the point which makes it possible to prove good results in this direction is that when X is a toric model the Chow motive of $\overline{X} = X \times_k \bar{k}$ splits, where $\bar{k}$ is a separable closure of k.

According to a theorem by Colliot-Thélène and Sansuc, the group $T(k)/R$ is trivial whenever T is rational over k, and in that case $A_0(X)$ also vanishes because it is birationally invariant for smooth projective varieties. It is known that tori of dimension at most 2 are all rational. So the nontrivial case for the main theorem is the 3-dimensional case.

Here we explain in a very rough idea how the K-theory, which seems to stand far away from behind the statement of main theorem, has found an important role to play in the proof. The starting point is the BGQ-spectral sequence

$$E_2^{pq} = H^p(X,\, \mathscr{K}_{-q}) \Longrightarrow K_{-p-q}(X)$$

which gives natural isomorphisms $\mathrm{CH}^p(X) \cong E_2^{p,\,-p}$, natural maps

$$K_1(X)^{(1)} \longrightarrow H^1(X,\, \mathscr{K}_2) \longrightarrow \mathrm{CH}^3(X)\,, \tag{1}$$

where $K_p(X)^{(i)}$ denotes the i-th term in the topological filtration of $K_p(X)$, as well as the edge homomorphism

$$g:\ \ \mathrm{CH}^d(X) \longrightarrow K_0(X)$$

where $d = \dim X$. The map g factors as

$$\mathrm{CH}^d(X) \xrightarrow{\ \eta\ } K_0(X)^{(d)} \longrightarrow K_0(X) \tag{2}$$

where the map η takes the class of a closed point x to the class of the sheaf $\mathscr{O}_x$ in $K_0(X)$. There is an isomorphism $K_0(X)^{(d)} \cong \mathbb{Z}$ and the composition

$$\mathrm{CH}_0(X) = \mathrm{CH}^d(X) \xrightarrow{\ \eta\ } K_0(X)^{(d)} \cong \mathbb{Z}$$

coincides with the degree map $\mathrm{CH}_0(X) \to \mathbb{Z}$. This is how the K-groups relate to $\mathrm{CH}_0(X)$ and $A_0(X)$.

We may also consider the BGQ-spectral sequence for $\overline{X}$. Various objects attached to $\overline{X}$ have natural action by the Galois group $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$. The $\mathfrak{g}$-modules such as $K_0(\overline{X})$, $K_1(\overline{X})^{(1)}$ and $H^1(\overline{X},\, \mathscr{K}_2)$ are already well studied in [24]. Interesting things may happen in the 3-dimensional case because then the maps in (1) and (2) can be joined together. The BGQ-spectral sequence for $\overline{X}$ yields an isomorphism of $\mathfrak{g}$-modules

$$K_0(\overline{X})^{(1/2)} = K_0(\overline{X})^{(1)}/K_0(\overline{X})^{(2)} \cong \mathrm{CH}^1(\overline{X})\,.$$

Colliot-Thélène and Sansuc proved that the torus T has a flasque resolution

$$1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1$$

with $\hat{S} = \mathrm{CH}^1(\overline{X})$, and there is a natural isomorphism of groups $T(k)/R \cong H^1(k,\, S)$. This finally provides opportunities for the K-groups to interact with the group $T(k)/R$. Details of the above discussion and the proof of main theorem occupy the major part of Chapter 4.

The organization of the thesis is as follows. The first two chapters introduce the notions of rational equivalence and Chow groups. Chapter 1 focuses on basic constructions and prove some most important and useful results. In Chapter 2, Chern classes are defined and used to give a nice description of rational equivalence on vector bundles and projective bundles. Chapter 3 is aimed at a quick introduction to Chow motives. Deep results will not be given proofs, but expositions on the basic concepts are expected to be clear enough. As mentioned before, Chapter 4 deals with things we are mainly concerned with. We begin with reviews of basics on algebraic tori and then introduce R-equivalence and flasque resolutions of tori. After that we will be concentrated on things that are related to the main theorem and fill in the details of the proof. Finally, as an application of main theorem, we obtain a theorem that gives a way to compute the Chow group $\mathrm{CH}_0(T)$ for lower dimensional tori, provided that the group $T(k)/R$ is known. We will carry out some calculations for concrete examples in the end of §4.8. As the attempt to give a comprehensive exposition of higher algebraic K-theory is not necessary for us and will get us totally lost, we will only give a brief survey on this subject in the Appendix.

The central part of the thesis is motivated by Merkurjev's recent paper [23]. For Chow groups and rational equivalence we follow Fulton's book [11], and for R-equivalence on tori we have referred to [7]. Quillen's lecture [27] is the basic reference for higher algebraic K-theory of schemes and Manin's paper [21] is a main source of our knowledge about motives.

Chapter 1

Rational Equivalence and Chow Groups

In this chapter we introduce basic constructions about the rational equivalence and Chow groups. Almost all material in this chapter has its origin in Chapter 1 of Fulton's book [11], except for the last section; there we state a classical result on zero-cycles and follow the proof given in [4].

1.1 Cycles and Rational Equivalence

1.1.1 The Order Function

Lemma 1.1.1. *Let A be a 1-dimensional Noetherian domain, $a \in A$ a nonzero element with $a \notin A^*$. Then $A/(a)$ is an Artinian ring. In particular,*

$$\ell_A\big(A/(a)\big) = \ell_{A/(a)}\big(A/(a)\big) < \infty\,,$$

where ℓ_A denotes the length of the A-module in parentheses.

Proof. We need to show for any prime ideal $\overline{\mathfrak{p}}$ of $A/(a)$, $\operatorname{ht}\overline{\mathfrak{p}} = 0$. Let $\mathfrak{p}$ be the prime ideal of A corresponding to $\overline{\mathfrak{p}}$. If $\operatorname{ht}\overline{\mathfrak{p}} > 0$, then there is a prime ideal $\mathfrak{q}$ of A such that $\mathfrak{p} \supset \mathfrak{q} \supseteq (a) \neq 0$. Since A is a domain, 0 is a prime ideal of A. Then the chain $\mathfrak{p} \supset \mathfrak{q} \supset 0$ contradicts the hypothesis $\dim A = 1$. □

Let X be an integral scheme over a field k and V an integral subscheme of X of codimension 1. The local ring $A = \mathscr{O}_{X,V}$ is a 1-dimensional Noetherian domain. The ***order*** of vanishing along V is the unique group homomorphism

$$\operatorname{ord}_V : k(X)^* \longrightarrow \mathbb{Z}$$

such that for all $a \in A$, $a \neq 0$,

$$\operatorname{ord}_V(a) = \ell_A(A/(a))\,.$$

Lemma 1.1.1 shows that $\operatorname{ord}_V(a)$ is finite indeed. That this determines a well-defined homomorphism is proved in [11, §§A.2–3]. For a fixed $r \in k(X)^*$, there are only finitely many codimension 1 integral subschemes V with $\operatorname{ord}_V(r) \neq 0$ ([11, B.4.3]).

If X is regular along V (for example this happens when X is normal), then $A = \mathscr{O}_{X,V}$ is a discrete valuation ring. Then any $r \in k(X)^*$ has the form $r = ut^m$ where $u \in A^*$, t is a generator of the maximal ideal of A, and $m \in \mathbb{Z}$. In this case, we have $\operatorname{ord}_V(r) = m$ ([11, Example A.3.2]).

Let $\widetilde{X} \to X$ be the normalization of X in its function field. Then for any $r \in k(X)^* = k(\widetilde{X})^*$, one has

$$\mathrm{ord}_V(r) = \sum_{\widetilde{V}} \mathrm{ord}_{\widetilde{V}}(r)[k(\widetilde{V}) : k(V)]$$

where the sum is over all integral subschemes $\widetilde{V}$ of $\widetilde{X}$ which map onto V ([11, Example A.3.1]). The order function on normal integral schemes thus determines the order function on arbitrary integral schemes.

For $r \in A = \mathscr{O}_{X,V}$, one has

$$\mathrm{ord}_V(r) \geq \min\{ n \in \mathbb{Z} \,|\, r \in \mathfrak{m}_{X,V}^n \}.$$

The inequality is an equality if X is regular along V, but is strict if $r \in \mathfrak{m}_{X,V}$ and X is singular along V ([11, Example 1.2.4]).

1.1.2 Rational Equivalence of Cycles

Let X be a scheme, d an integer ≥ 0. A ***d-cycle*** on X is a finite formal sum $\sum n_i[V_i]$ where the V_i are d-dimensional integral subschemes of X and $n_i \in \mathbb{Z}$. The group of d-cycles, denoted $Z_d(X)$, is the free abelian group with basis the d-dimensional integral subschemes of X.

For any $(d+1)$-dimensional integral subschemes W of X, and any $r \in k(W)^*$, define a d-cycle $[\mathrm{div}(r)]$ on X by

$$[\mathrm{div}(r)] := \sum_V \mathrm{ord}_V(r)[V],$$

the sum being over all codimension 1 integral subschemes V of W, here ord_V is the order function on $k(W)^*$ defined by the local ring $\mathscr{O}_{W,V}$.

A d-cycle α is called ***rationally equivalent to zero***, written $\alpha \overset{\mathrm{rat}}{\sim} 0$, if there are a finite number of $(d+1)$-dimensional integral subschemes W_i of X and $r_i \in k(W_i)^*$ such that $\alpha = \sum[\mathrm{div}(r_i)]$. Since $[\mathrm{div}(r^{-1})] = -[\mathrm{div}(r)]$, the cycles rationally equivalent to zero form a subgroup of $Z_d(X)$, which we denote by $\mathrm{Rat}_d(X)$. The ***Chow group*** $\mathrm{CH}_d(X)$ of dimension d of X is defined to be the quotient group

$$\mathrm{CH}_d(X) := Z_d(X)/\mathrm{Rat}_d(X).$$

If X is purely dimensional, we put

$$Z^d(X) := Z_{\dim X - d}(X), \quad \text{and} \quad \mathrm{CH}^d(X) := \mathrm{CH}_{\dim X - d}(X).$$

Define

$$Z_\bullet(X) := \bigoplus_{d\geq 0} Z_d(X), \quad \text{and} \quad \mathrm{CH}_\bullet(X) := \bigoplus_{d\geq 0} \mathrm{CH}_d(X),$$

and similarly if X is purely dimensional,

$$Z^\bullet(X) := \bigoplus_{d\geq 0} Z^d(X), \quad \text{and} \quad \mathrm{CH}^\bullet(X) := \bigoplus_{d\geq 0} \mathrm{CH}^d(X).$$

When we don't care the gradings, we also write $Z(X)$ (resp. $\mathrm{CH}(X)$) for $Z_\bullet(X)$ or $Z^\bullet(X)$ (resp. $\mathrm{CH}_\bullet(X)$ or $\mathrm{CH}^\bullet(X)$). An element of $Z(X)$ (resp. $\mathrm{CH}(X)$) is called a ***cycle*** (resp. ***cycle class***) on X. A more classical definition of $\mathrm{CH}(X)$ will be given in §1.2.3. A cycle is called ***positive*** if it is not zero and each of its coefficients is nonnegative. A cycle class is ***positive*** if it can be represented by a positive cycle.

A scheme X and its underlying reduced scheme X_{red} have the same integral subschemes. So $Z_d(X) = Z_d(X_{\text{red}})$ and $\mathrm{CH}_d(X) = \mathrm{CH}_d(X_{\text{red}})$ for all $d \geq 0$.

Let X_1 and X_2 be closed subschemes of X. Then for each d we have an exact sequence

$$0 \longrightarrow Z_d(X_1 \cap X_2) \xrightarrow{\varphi} Z_d(X_1) \oplus Z_d(X_2) \xrightarrow{\psi} Z_d(X_1 \cup X_2) \longrightarrow 0$$

where the map φ is defined by $\alpha \mapsto (\alpha\,, \alpha)$ and ψ is defined by $\psi(\alpha\,, \beta) = \alpha - \beta$. The restriction of ψ gives a homomorphism $\widetilde{\psi} : \mathrm{Rat}_d(X_1) \oplus \mathrm{Rat}_d(X_2) \longrightarrow \mathrm{Rat}_d(X_1 \cup X_2)$ which is again surjective. Moreover, as subgroups of $Z_d(X_1) \oplus Z_d(X_2)$,

$$\mathrm{Rat}_d(X_1 \cup X_2) \subseteq \mathrm{Ker}\,\widetilde{\psi} \subseteq \mathrm{Ker}\,\psi = Z_d(X_1 \cap X_2)\,.$$

Thus we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\,\widetilde{\psi} & \longrightarrow & \mathrm{Rat}_d(X_1) \oplus \mathrm{Rat}_d(X_2) & \xrightarrow{\widetilde{\psi}} & \mathrm{Rat}_d(X_1 \cup X_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_d(X_1 \cap X_2) & \longrightarrow & Z_d(X_1) \oplus Z_d(X_2) & \xrightarrow{\psi} & Z_d(X_1 \cup X_2) & \longrightarrow & 0
\end{array}$$

with exact rows. This yields by snake lemma an exact sequence

$$0 \longrightarrow Z_d(X_1 \cap X_2)/\mathrm{Ker}\,\widetilde{\psi} \longrightarrow \mathrm{CH}_d(X_1) \oplus \mathrm{CH}_d(X_2) \longrightarrow \mathrm{CH}_d(X_1 \cup X_2) \longrightarrow 0\,.$$

Since $\mathrm{Rat}_d(X_1 \cap X_2) \subseteq \mathrm{Ker}\,\widetilde{\varphi}$, there is a natural surjection

$$\mathrm{CH}_d(X_1 \cap X_2) \to Z_d(X_1 \cap X_2)/\mathrm{Ker}\,\widetilde{\psi}$$

whence an exact sequence

$$\mathrm{CH}_d(X_1 \cap X_2) \longrightarrow \mathrm{CH}_d(X_1) \oplus \mathrm{CH}_d(X_2) \longrightarrow \mathrm{CH}_d(X_1 \cup X_2) \longrightarrow 0\,.$$

If X is a disjoint union of subschemes $X_1\,, \ldots, X_m$, then for any $d \geq 0$, one has

$$Z_d(X) = \bigoplus_{i=1}^{m} Z_d(X_i)\,, \qquad \mathrm{Rat}_d(X) = \bigoplus_{i=1}^{m} \mathrm{Rat}_d(X_i)$$

and hence $\mathrm{CH}_d(X) = \oplus_{i=1}^{m} \mathrm{CH}_d(X_i)$.

Suppose the scheme X has dimension n. Then $Z_n(X) = \mathrm{CH}_n(X)$ is the free abelian group on the n-dimensional irreducible components of X. So if V is an n-dimensional irreducible component, then the coefficient at $[V]$ of any cycle on X only depends on the cycle class. This holds more generally for any irreducible component V of X, in other words, for any two cycles α and β on X, if $\alpha \overset{\text{rat}}{\sim} \beta$, then α and β have the same coefficient at $[V]$. Indeed, an irreducible component of X cannot be contained in any higher dimensional integral subscheme. So a cycle of the form $[\mathrm{div}(r)]$, with $r \in k(W)^*$ for some integral subscheme W, cannot include $[V]$. Thus, for any cycle class $\alpha \in \mathrm{CH}(X)$ and any irreducible component V of X, we can define the ***coefficient*** of V in α to be the coefficient of $[V]$ in any cycle representing α.

1.2 Proper Push-forward

1.2.1 Push-forward of Cycles

Let $f : X \to Y$ be a proper morphism of schemes. For any integral subscheme V of X, the reduced subscheme on the image $W = f(V)$ is then an integral subscheme of Y. There is an induced embedding of function fields $k(W) \to k(V)$, which is a finite extension if W has the same dimension as V. Set

$$\deg(V/W) = \begin{cases} [k(V) : k(W)] & \text{if } \dim V = \dim W \\ 0 & \text{if } \dim W < \dim V\,. \end{cases}$$

Define $f_*[V] = \deg(V/W)[W]$, and extend it by linearity to a ***push-forward*** homomorphism $f_* : Z_d(X) \to Z_d(Y)$. If $g : Y \to Z$ is another proper morphism, then $(g \circ f)_* = g_* \circ f_*$.

Theorem 1.2.1. *Let $f : X \to Y$ be a proper morphism and α a d-cycle on X. If $\alpha \overset{\text{rat}}{\sim} 0$ on X, then $f_*\alpha \overset{\text{rat}}{\sim} 0$ on Y. Therefore, there is an induced homomorphism $f_* : \mathrm{CH}_d(X) \to \mathrm{CH}_d(Y)$.*

Proof. We may assume $\alpha = [\mathrm{div}(r)]$, where r is a rational function on an integral subscheme W of X. Replacing X by W and Y by $f(W)$, we may assume X and Y are integral and f is surjective. Then the result follows from the next more explicit proposition. □

Proposition 1.2.2. *Let $f : X \to Y$ be a proper surjective morphism of integral schemes. Then for any $r \in k(X)^*$, one has*

$$f_*[\mathrm{div}(r)] = \begin{cases} 0 & \textit{if } \dim Y < \dim X \\ [\mathrm{div}(N_{k(X)/k(Y)}(r))] & \textit{if } \dim Y = \dim X \end{cases}$$

here when $\dim Y = \dim X$, $N_{k(X)/k(Y)}$ denotes the norm map corresponding to the field extension $k(X)/k(Y)$.

Proof. See [11, Prop. 1.4]. □

Let $Y_1, \dots, Y_n$ be closed subschemes of a scheme X. Given $\alpha_i \in \mathrm{CH}(Y_i)$, $i = 1, \dots, n$ and $\beta \in \mathrm{CH}(X)$, we will usually write "$\beta = \sum_{i=1}^n \alpha_i$ in $\mathrm{CH}(X)$" in place of the precise equation $\beta = \sum_{i=1}^n \varphi_{i*}(\alpha_i)$ where $\varphi_i : Y_i \to X$ is the natural inclusion.

Definition 1.2.3. Let X be a proper scheme over a field k. The ***degree*** of a 0-cycle $\alpha = \sum_P n_P[P]$ on X, denoted $\deg(\alpha)$ or $\int_X \alpha$, is defined by

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P[\kappa(P) : k]$$

where $\kappa(P)$ is the residue field of P.

In the above definition, if $p : X \to S := \mathrm{Spec}\ k$ is the structure morphism and if we identify $Z_0[S] = \mathbb{Z} \cdot [S]$ with $\mathbb{Z}$, then $\deg(\alpha) = p_*(\alpha)$ is in fact the push-forward of α by p. On S a cycle is rationally equivalent to 0 if and only if it is equal to 0. By Thm 1.2.1, there is an induced homomorphism

$$\deg = \int_X : \quad \mathrm{CH}_0(X) \longrightarrow \mathbb{Z} = \mathrm{CH}_0(S)\,.$$

The kernel of the degree map $\deg : \mathrm{CH}_0(X) \longrightarrow \mathbb{Z}$ will be denoted $A_0(X)$.

We can extend the degree homomorphism to the whole of $\mathrm{CH}_\bullet(X)$, by putting $\int_X \alpha = 0$ if $\alpha \in \mathrm{CH}_d(X)$, $d > 0$. If $f : X \to Y$ is a morphism between proper schemes, then for any $\alpha \in \mathrm{CH}_\bullet(X)$, we have $\int_X \alpha = \int_Y f_*(\alpha)$ since $p_X = p_Y \circ f$. We often write simply $\int$ in place of $\int_X$ when no confusion seems likely to result.

Examples 1.2.4. (1) Let X be an integral variety which is regular in codimension 1 (this means $\mathscr{O}_{X,V}$ is regular for every integral subscheme V of codimension 1). Let $n = \dim X$. Then $\mathrm{CH}_{n-1}(X)$ is isomorphic to the group of isomorphism classes of invertible sheaves on X. For affine and projective spaces, we have $\mathrm{CH}_{n-1}(\mathbb{A}^n) = 0$ and $\mathrm{CH}_{n-1}(\mathbb{P}^n) = \mathbb{Z}$ ([16, §II.6]).

(2) Let X be a connected smooth projective curve of genus g over an algebraically closed field k. Then $A_0(X)$ can be made into an abelian variety of dimension g. For example, when $g = 1$, X is a so-called elliptic curve and if O is a fixed rational point of X, we have an isomorphism $X(k) \to A_0(X)$; $P \mapsto [P] - [O]$.

1.2.2 Cycles of Subschemes

Let X be a scheme and let $X_1, \dots, X_t$ be the irreducible components of X. The local rings $\mathscr{O}_{X,X_i}$ are all Artinian rings. The number $m_i := \ell_{\mathscr{O}_{X,X_i}}(\mathscr{O}_{X,X_i})$ is called the ***geometric multiplicity*** of X_i in X. The (***fundamental***) ***cycle*** $[X]$ of X is defined to be the cycle

$$[X] := \sum_{i=1}^{t} m_i [X_i] \,.$$

This is regarded as an element in $Z(X)$, but by abuse of notation, we also write $[X]$ for its image in $\mathrm{CH}(X)$. If X is purely d-dimensional, i.e., $\dim X_i = d$ for all i, then $[X] \in Z_d(X)$. In this case, $Z_d(X) = \mathrm{CH}_d(X)$ is the free abelian group on the basis $[X_1], \dots, [X_t]$.

If X is a subscheme of a scheme Y, then $Z(X) \subseteq Z(Y)$, and we write $[X]$ also for the image of $[X]$ in $Z(Y)$ and for its image in $\mathrm{CH}(Y)$.

Example 1.2.5. Let V be an integral scheme of dimension $d+1$, and let $f : V \to \mathbb{P}^1$ be a dominant morphism. Let $0 = (1:0)$, $\infty = (0:1)$ be respectively the zero and infinite points of $\mathbb{P}^1$. Assume they are both in the image of f. The inverse image schemes $f^{-1}(0)$ and $f^{-1}(\infty)$ are purely d-dimensional subschemes of V, and $[f^{-1}(0)] - [f^{-1}(\infty)]$ is equal to the cycle $[\mathrm{div}(f)]$ defined in §1.1.1, where f also denotes the rational function in $k(V)$ determined by the morphism f.

Here and hereafter, when we write $f^{-1}(P)$ for a dominant morphism $f : V \to \mathbb{P}^1$ and a point $P \in \mathbb{P}^1$, we will always assume this fibre is nonempty.

1.2.3 Alternative Definition of Rational Equivalence

Let X be a scheme and let $p : X \times \mathbb{P}^1 \to X$ be the first projection. Let V be a $(d+1)$-dimensional integral subscheme of $X \times \mathbb{P}^1$ such that the second projection induces a dominant morphism f from V to $\mathbb{P}^1$. For any rational point P of $\mathbb{P}^1$, the scheme-theoretic fibre $f^{-1}(P)$ is a subscheme of $X \times \{P\}$, which p maps isomorphically onto a subscheme of X; we denote this subscheme by $V(P)$. Note in particular that

$$p_*[f^{-1}(P)] = [V(P)] \ \text{ in } \ Z_d(X)\,.$$

The morphism $f : V \to \mathbb{P}^1$ determines a rational function $f \in k(V)^*$. We have already seen that

$$[f^{-1}(0)] - [f^{-1}(\infty)] = [\mathrm{div}(f)]\,,$$

where $0 = (1:0)$ and $\infty = (0:1)$ are the zero and infinite points of $\mathbb{P}^1$. Therefore,

$$p_*[\mathrm{div}(f)] = [V(0)] - [V(\infty)]\,,$$

which is rationally equivalent to 0 on X by Thm. 1.2.1.

Proposition 1.2.6. *Let X be a scheme, $\alpha \in Z_d(X)$. The following conditions are equivalent:*

(i) $\alpha \overset{\mathrm{rat}}{\sim} 0$;

(ii) *there are $(d+1)$-dimensional integral subschemes $V_1, \ldots, V_t$ of $X \times \mathbb{P}^1$ such that the projections from V_i to $\mathbb{P}^1$ are dominant, with $\alpha = \sum_{i=1}^{t}([V_i(0)] - [V_i(\infty)])$ in $Z_d(X)$;*

(iii) *there are finitely many normal integral schemes V_i with rational functions f_i on V_i determined by some dominant morphisms $f_i : V_i \to \mathbb{P}^1$, and proper morphisms $p_i : V_i \to X$ such that $\alpha = \sum p_{i*}[\mathrm{div}(f_i)]$.*

Proof. (iii)⇒(i). This follows from Thm. 1.2.1.

(ii)⇒(iii). Let $\pi_i : \widetilde{V}_i \to V_i$ be the normalization of V_i. It gives by composite with the projection $V_i \to \mathbb{P}^1$ the morphism $f_i : V_i \to \mathbb{P}^1$; and by composite with the projection $V_i \to X$ the morphism $p_i : \widetilde{V}_i \to X$. Then we have $\alpha = \sum p_{i*}[\mathrm{div}(f_i)]$.

(i)⇒(ii) We may assume $\alpha = [\mathrm{div}(r)]$, where r is a rational function on a $(d+1)$-dimensional integral subscheme W of X. Then r defines a rational map $W \dashrightarrow \mathbb{P}^1$. Let V be the graph of this rational map. It is an integral subscheme of $X \times \mathbb{P}^1$ which the projection $p : X \times \mathbb{P}^1 \to X$ maps birationally and properly onto W. Let f be the induced morphism from V to $\mathbb{P}^1$. Then by Prop. 1.2.2, we get $[\mathrm{div}(r)] = p_*[\mathrm{div}(f)]$. The latter is equal to $[V(0)] - [V(\infty)]$ as was seen in the preceding argument. □

We say a cycle $Z = \sum n_i[V_i]$ on $X \times \mathbb{P}^1$ ***projects dominantly*** to $\mathbb{P}^1$ if each V_i which appears with nonzero coefficient in Z projects dominantly to $\mathbb{P}^1$. In this case, we set

$$Z(0) := \sum n_i[V_i(0)]\,, \quad Z(\infty) := \sum n_i[V_i(\infty)]\,.$$

Proposition 1.2.7. *Two d-cycles α, α' on a scheme X are rationally equivalent if and only if there is a positive $(d+1)$-cycle Z on $X \times \mathbb{P}^1$ projecting dominantly to $\mathbb{P}^1$, and a positive d-cycle β on X such that*

$$Z(0) = \alpha + \beta \quad \textit{and} \quad Z(\infty) = \alpha' + \beta\,.$$

Proof. The "if" part is obvious from Prop. 1.2.6. For the "only if" part, using Prop. 1.2.6, we can find some positive $(d+1)$-cycle Z' on $X \times \mathbb{P}^1$ projecting dominantly to $\mathbb{P}^1$ such that $\alpha - \alpha' = Z'(0) - Z'(\infty)$. Choose a positive d-cycle β so that $\gamma := \alpha - Z'(0) + \beta$ is positive. Write $\gamma = \sum n_i[V_i]$ and set $Z = Z' + \sum n_i[V_i \times \mathbb{P}^1]$. Then we have

$$Z(0) = Z'(0) + \gamma = \alpha + \beta \quad \text{and} \quad Z(\infty) = Z'(\infty) + \gamma = \alpha' + \beta\,.$$

This finishes the proof. □

1.3 Flat Pull-back

1.3.1 Pull-back of Cycles

We say a morphism $f : X \to Y$ has ***relative dimension*** n if for all integral subschemes V of Y, the inverse image scheme $f^{-1}(V) = X \times_Y V$ is purely of dimension $\dim V + n$.

Proposition 1.3.1. *Let $f : X \to Y$ be a flat morphism of algebraic schemes with Y irreducible and X purely of dimension $\dim Y + n$. Then f has relative dimension n, and all base extensions $X \times_Y Y' \to Y'$ have relative dimension n.*

Proof. See [16, Coro. III.9.6] or [14, IV.14.2]. □

In what follows, a ***flat*** morphism is always assumed to have some relative dimension.

The following are important examples of flat morphisms having relative dimension:

(1) An open immersion is flat of relative dimension $n = 0$.

(2) Let E be an affine bundle (cf. §1.4) of rank n, or a projective bundle (cf. §2.1) of rank $n+1$ over a scheme X. Then the natural projection $p : E \to X$ is flat of relative dimension n.

(3) If Z is a purely n-dimensional scheme, then for any scheme Y, the first projection $Y \times Z \to Y$ is flat of relative dimension n.

(4) Any dominant morphism from an $(n+1)$-dimensional integral scheme to a smooth 1-dimensional connected scheme is flat of relative dimension n.

(5) If $f : X \to Y$ and $g : Y \to Z$ are flat morphisms of relative dimensions m and n, then $g \circ f : X \to Z$ is flat of relative dimension $m+n$.

Now let $f : X \to Y$ be a flat morphism of relative dimension n. For any integral subscheme V of Y, set

$$f^*[V] = [f^{-1}(V)]\,.$$

Here $f^{-1}(V)$ is the inverse image scheme, a subscheme of X, of pure dimension $\dim V + n$, and $[f^{-1}(V)]$ is its cycle as defined in §1.2.2. This extends by linearity to ***pull-back*** homomorphisms

$$f^* : Z_d(Y) \longrightarrow Z_{d+n}(X)\,.$$

Lemma 1.3.2. *Let $f : X \to Y$ be a flat morphism of some relative dimension, then for any closed subscheme Z of Y,*

$$f^*[Z] = [f^{-1}(Z)]\,.$$

Proof. See [11, p.18, Lemma 1.7.1]. □

It follows from the above lemma that if $f : X \to Y$ and $g : Y \to Z$ are flat morphisms (having relative dimensions), then $(g \circ f)^* = f^* \circ g^*$. For if V is an integral subscheme of Z, then

$$(g \circ f)^*[V] = [f^{-1}(g^{-1}(V))] = f^*[g^{-1}(V)] = f^*g^*[V]\,.$$

Proposition 1.3.3. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a fibre square with f flat of relative dimension n and g proper. Then f' is flat of relative dimension n, g' is proper, and for all $\alpha \in Z(Y')$, one has

$$g'_* f'^* \alpha = f^* g_* \alpha \quad \text{in} \;\; Z(X)\,.$$

Proof. See [11, p.18, Prop. 1.7]. □

Theorem 1.3.4. *Let $f : X \to Y$ be a flat morphism of relative dimension n, and $\alpha \in Z_d(Y)$. If $\alpha \overset{\text{rat}}{\sim} 0$ on Y, then $f^*\alpha \overset{\text{rat}}{\sim} 0$ on X. There are therefore induced homomorphisms for all $d \geq 0$,*

$$f^* : \; \mathrm{CH}_d(Y) \longrightarrow \mathrm{CH}_{d+n}(X)\,.$$

Proof. By Prop. 1.2.6, we may assume $\alpha = [V(0)] - [V(\infty)]$, where V is a $(d+1)$-dimensional integral subscheme of $Y \times \mathbb{P}^1$ for which the projection $g : V \to \mathbb{P}^1$ is dominant, hence flat of relative dimension d. Let $W = (f \times \mathrm{Id})^{-1}(V)$, a closed subscheme of $X \times \mathbb{P}^1$, and let $h : W \to \mathbb{P}^1$ be the morphism induced by the projection to $\mathbb{P}^1$. Let $p : X \times \mathbb{P}^1 \to X$ and $q : Y \times \mathbb{P}^1 \to Y$ be the projections. Then we have

$$\alpha = [V(0)] - [V(\infty)] = q_*[\mathrm{div}(g)]$$

and by Prop. 1.3.3,

$$f^*\alpha = f^*q_*\big([g^{-1}(0)] - [g^{-1}(\infty)]\big) = p_*(f \times \mathrm{Id})^*\big([g^{-1}(0)] - [g^{-1}(\infty)]\big)\,.$$

The last term equals $p_*([h^{-1}(0)] - [h^{-1}(\infty)])$ by Lemma 1.3.2, in view of $h = g \circ (f \times \mathrm{Id})$. Note that $f \times \mathrm{Id}$ is flat, so it is also an open map, then we have h is dominant since g is. Let $W_1, \ldots, W_t$ be the irreducible components of W, h_i the restriction of h to W_i. Then every h_i is dominant. (It is a general fact that if $f : X \to Y$ is a flat morphism of algebraic schemes with Y irreducible, then every irreducible component of X dominates Y.) We then get $[h_i^{-1}(0)] - [h_i^{-1}(\infty)] = [\mathrm{div}(h_i)]$. Write $[W] = \sum m_i[W_i]$. Since p_* preserves rational equivalence, it suffices to verify that $[h^{-1}(P)] = \sum m_i[h_i^{-1}(P)]$ for $P = 0$ and $P = \infty$. This is a special case of the following general lemma. □

Lemma 1.3.5. *Let X be a purely n-dimensional scheme, with irreducible components $X_1, \ldots, X_r$, and geometric multiplicities $m_1, \ldots, m_r$. Let D be an effective Cartier divisor* (cf.§2.2.1) *on X, i.e., a closed subscheme of X whose ideal sheaf is locally generated by one non-zero-divisor. Let $D_i = D \cap X_i$ be the restriction of D to X_i, then*

$$[D] = \sum_{i=1}^{r} m_i[D_i] \quad \text{in} \quad Z_{n-1}(X)\,.$$

Proof. See [11, p.19, Lemma 1.7.2]. □

1.3.2 An Exact Sequence

Proposition 1.3.6. *Let Y be a closed subscheme of a scheme X, and let $U = X \backslash Y$. Let $i : Y \to X$, $j : U \to X$ be the natural inclusions. Then the sequence*

$$\mathrm{CH}_d(Y) \xrightarrow{i_*} \mathrm{CH}_d(X) \xrightarrow{j^*} \mathrm{CH}_d(U) \longrightarrow 0$$

is exact for all d.

Proof. Since any integral subscheme V of U extends to an integral subscheme $\overline{V}$ of X, the sequence

$$0 \longrightarrow Z_d(Y) \xrightarrow{i_*} Z_d(X) \xrightarrow{j_*} Z_d(U) \longrightarrow 0$$

is exact. If W is an integral subscheme of U of dimension $d+1$ and $r \in k(W)^*$, then r also gives a rational function $\overline{r}$ on $\overline{W}$ since $k(W) = k(\overline{W})$. We have $j^*[\mathrm{div}(\overline{r})] = [\mathrm{div}(r)]$ in $Z_d(U)$. Therefore, the restriction of j^* gives a surjective homomorphism $\widetilde{j^*} : \mathrm{Rat}_d(X) \to \mathrm{Rat}_d(U)$. Then we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\,\widetilde{j^*} & \longrightarrow & \mathrm{Rat}_d(X) & \xrightarrow{\widetilde{j^*}} & \mathrm{Rat}_d(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_d(Y) & \longrightarrow & Z_d(X) & \xrightarrow{j^*} & Z_d(U) & \longrightarrow & 0
\end{array}$$

which yields an exact sequence

$$0 \longrightarrow Z_d(Y)/\mathrm{Ker}\,\widetilde{j^*} \longrightarrow \mathrm{CH}_d(X) \longrightarrow \mathrm{CH}_d(U) \longrightarrow 0\,.$$

Clearly, $\mathrm{Rat}_d(Y) \subseteq \mathrm{Ker}\,\widetilde{j^*}$, so we have a natural surjection $\mathrm{CH}_d(Y) \to Z_d(Y)/\mathrm{Ker}\,\widetilde{j^*}$ whence the exact sequence

$$\mathrm{CH}_d(Y) \xrightarrow{i_*} \mathrm{CH}_d(X) \xrightarrow{j^*} \mathrm{CH}_d(U) \longrightarrow 0\,.$$

This completes the proof. □

1.4 Affine Bundles

A scheme E together with a morphism $p : E \to X$ is called an ***affine bundle*** of rank n over X if X has an open covering $\{ U_\lambda \}$ together with isomorphisms $p^{-1}(U_\lambda) \cong U_\lambda \times \mathbb{A}^n$ such that p restricted to $p^{-1}(U_\lambda)$ corresponds to the projection from $U_\lambda \times \mathbb{A}^n$ to U_λ.

Proposition 1.4.1. *Let $p : E \to X$ be an affine bundle of rank n over a scheme X. Then the flat pull-back $p^* : \mathrm{CH}_d(X) \to \mathrm{CH}_{d+n}(E)$ is surjective for all d.*

Proof. Choose a closed subscheme Y of X such that $U := X\backslash Y$ is an affine open set over which E is trivial (i.e. $p^{-1}(U) \cong U \times \mathbb{A}^n$). There is a commutative diagram

$$\begin{array}{ccccccc} \mathrm{CH}_d(Y) & \longrightarrow & \mathrm{CH}_d(X) & \longrightarrow & \mathrm{CH}_d(U) & \longrightarrow & 0 \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \\ \mathrm{CH}_d(p^{-1}(Y)) & \longrightarrow & \mathrm{CH}_d(E) & \longrightarrow & \mathrm{CH}_d(p^{-1}(U)) & \longrightarrow & 0 \end{array}$$

where the vertical maps are flat pull-backs and the rows are exact by Prop. 1.3.6. By a diagram chase, it suffices to prove the assertion for the restriction of E to U and to Y. By virtue of Noetherian induction, it suffices to prove it for $X = U$. Thus we may assume X is affine and $E = X \times \mathbb{A}^n$. The projection factors as

$$X \times \mathbb{A}^n \longrightarrow X \times \mathbb{A}^{n-1} \longrightarrow X\,,$$

so we may assume $n = 1$.

We want to show that $[V]$ is in $p^*\mathrm{CH}_d(X)$ for any $(d+1)$-dimensional integral subscheme V of E. We may replace X by the closure $\overline{p(V)}$, and E by $p^{-1}(\,\overline{p(V)}\,)$. So we may assume X is integral and p maps V dominantly to X. Let A be the coordinate ring of X, then the function field $K := k(X)$ is the field of fractions of A. Let $\mathfrak{q}$ be the prime ideal in $A[t]$ that defines V in $E = \mathrm{Spec}\,A[t]$. Note that $\dim X \le \dim V \le \dim E = \dim X + 1$. Hence $\dim X = \dim V = d+1$ or $\dim X = \dim V - 1 = d$. If $\dim X = d$, then $\dim V = \dim E$, so $V = E$ and $[V] = p^*[X]$. So we need only consider the case $\dim X = d+1$.

Since $V \to X$ is dominant, the ring homomorphism $A \to A[t]/\mathfrak{q}$ is injective. This means $S := A \setminus \{ 0 \}$ has no intersection with $\mathfrak{q}$. Thus $S^{-1}\mathfrak{q} = \mathfrak{q}K[t]$ is a prime ideal of $S^{-1}A[t] = K[t]$. Now $V \neq E$ implies $\mathfrak{q} \neq 0$. So $\mathfrak{q}K[t]$ has a nonzero generator r and we may assume $r \in A[t]$.

Now we claim that

$$[V] - [\mathrm{div}(r)] = \sum n_i [V_i]$$

for some $(d+1)$-dimensional integral subschemes V_i of E whose projections to X are not dominant.

Indeed, $[\mathrm{div}(r)]$ is a $\mathbb{Z}$-linear combination of $[V_i]$ for some $(d+1)$-dimensional integral subschemes V_i of E. These integral subschemes are defined by some $\mathfrak{p}_i \in$

Spec $A[t]$ with $\operatorname{ht}\mathfrak{p}_i = 1$. The coefficient of $[V_i]$ in $[\operatorname{div}(r)]$ is $\ell_{A[t]_{\mathfrak{p}_i}}(A[t]_{\mathfrak{p}_i}/(r))$. Since $rK[t] = \mathfrak{q}K[t]$, we get

$$r \in (rK[t]) \cap A[t] = (\mathfrak{q}K[t]) \cap A[t] = \mathfrak{q}$$

and thus $rA[t]_{\mathfrak{q}} \subseteq \mathfrak{q}A[t]_{\mathfrak{q}}$. For any $\frac{q}{f} \in \mathfrak{q}A[t]_{\mathfrak{q}}$, $q \in \mathfrak{q}$, $f \in A[t] \setminus \mathfrak{q}$, we can write $q = r \cdot \frac{g}{\alpha}$ with $g \in A[t]\,, \alpha \in S = A \setminus \{\,0\,\}$. Since $\mathfrak{q} \cap S = \emptyset$, we have $\alpha f \notin \mathfrak{q}$ showing $\frac{q}{f} = \frac{rg}{\alpha f} \in rA[t]_{\mathfrak{q}}$. So we see $rA[t]_{\mathfrak{q}} = \mathfrak{q}A[t]_{\mathfrak{q}}$ is the maximal ideal of $A[t]_{\mathfrak{q}}$. This implies

$$\ell_{A[t]_{\mathfrak{q}}}(A[t]_{\mathfrak{q}}/(r)) = 1\,.$$

In other words, the coefficient of $[V]$ in $[\operatorname{div}(r)]$ is equal to 1. Write $[V] - [\operatorname{div}(r)] = \sum n_i[V_i]$ with V_i integral subschemes of dimension $d+1$ and $n_i \neq 0$. We need prove $p : V_i \to X$ is not dominant. This is equivalent to saying that $\mathfrak{p}_i \cap A \neq 0$, where $\mathfrak{p}_i \in \operatorname{Spec} A[t]$ is the prime ideal defining V_i in E. The coefficient $n_i \neq 0$ means $r \in \mathfrak{p}_i$, hence

$$\mathfrak{q}K[t] = rK[t] \subseteq \mathfrak{p}_iK[t]\,.$$

If $\mathfrak{p}_i \cap A = 0$, we would get $(\mathfrak{p}_iK[t]) \cap A[t] = \mathfrak{p}_i$ which implies $\mathfrak{q} = (\mathfrak{q}K[t]) \cap A[t] \subseteq \mathfrak{p}_i$. But this is absurd for $\mathfrak{p}_i \neq \mathfrak{q}$ since $\operatorname{ht}\mathfrak{p}_i = \operatorname{ht}\mathfrak{q} = 1$. This proves our claim.

The subscheme $W_i := \overline{p(V_i)}$ is defined by the ideal $P_i := \mathfrak{p}_i \cap A \in \operatorname{Spec} A$, and $p^{-1}(W_i)$ is defined by the ideal $P_iA[t]$. Since $A[t]/P_iA[t] \cong (A/P_i)[t]$ is an integral domain, $P_iA[t]$ is a prime ideal of $A[t]$. Now $0 \neq P_iA[t] \subseteq \mathfrak{p}_i$. It follows from the fact $\operatorname{ht}\mathfrak{p}_i = 1$ that $P_iA[t] = \mathfrak{p}_i$. Hence $V_i = p^{-1}(W_i)$ so that

$$[V] = [\operatorname{div}(r)] + \sum n_i p^*[W_i]$$

as desired. The proposition is thus proved. □

Remark 1.4.2. In the above proof, we can even prove $p(V_i)$ is closed itself. Indeed, if P' is a prime ideal of A containing $P_i = \mathfrak{p}_i \cap A$, then $P'A[t]$ is a prime ideal of $A[t]$ such that $P'A[t] \cap A = P'$. We have seen that $P_iA[t] = \mathfrak{p}_i$. So $P'A[t] \supseteq \mathfrak{p}_i$. This means $P'A[t]$ is an element of V_i which projects to $P' \in X$. So we get $\overline{p(V_i)} = p(V_i)$.

Remark 1.4.3. We will see that if E is a vector bundle over X, p^* is in fact an isomorphism (cf. Thm. 2.4.5).

Corollary 1.4.4. *We have* $\mathrm{CH}_n(\mathbb{A}^n) = \mathbb{Z}$ *and* $\mathrm{CH}_d(\mathbb{A}^n) = 0$ *for* $0 \le d < n$.

Proof. The first assertion is clear. For the second, we may use Prop. 1.4.1 to reduce to the case $d = 0$. So we need only to show $\mathrm{CH}_0(\mathbb{A}^n) = 0$ for $n \ge 1$. For $n = 1$, we know that $\mathrm{CH}_0(\mathbb{A}^1) = 0$. Assume $n \ge 2$. Given any closed point $P \in \mathbb{A}^n$, we can find in $\mathbb{A}^n$ a line $L \cong \mathbb{A}^1$ passing through P. Using $\mathrm{CH}_0(L) = \mathrm{CH}_0(\mathbb{A}^1) = 0$, we can find a function $f \in k(L)$ such that $[\operatorname{div}(f)] = [P]$. This means every point on $\mathbb{A}^n$ is rationally equivalent to 0, whence $\mathrm{CH}_0(\mathbb{A}^n) = 0$. □

Example 1.4.5. Let X be a scheme with a "cellular decomposition", i.e., X admits a filtration $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \emptyset$ by closed subschemes with each $X_i \setminus X_{i-1}$ a finite disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. Then $\mathrm{CH}_\bullet(X)$ is finitely generated by $\{\,[V_{ij}]\,\}$, where V_{ij} is the closure of U_{ij} in X.

This can be seen from the exact sequences

$$\mathrm{CH}_\bullet(X_{i-1}) \longrightarrow \mathrm{CH}_\bullet(X_i) \longrightarrow \mathrm{CH}_\bullet\Big(\coprod U_{ij}\Big) = \bigoplus \mathbb{Z}\cdot[U_{ij}] \longrightarrow 0\,, \quad \text{for} \quad i = 1\,,\dots,n$$

and by induction on i.

Example 1.4.6. Let L^d be a d-dimensional linear subspace of $\mathbb{P}^n$, $d = 0, 1 \dots, n$. Then $\mathrm{CH}_d(\mathbb{P}^n) = \mathbb{Z} \cdot [L^d] = \mathbb{Z}$.

Indeed, applying Prop. 1.3.6 with $X = \mathbb{P}^n$, $Y = L^{n-1}$, $U = \mathbb{A}^n$, we have exact sequences

$$\mathrm{CH}_d(Y) \longrightarrow \mathrm{CH}_d(X) \longrightarrow \mathrm{CH}_d(\mathbb{A}^n) \longrightarrow 0\,, \quad \text{for} \quad d = n\,, n-1\,, \dots, 0\,.$$

By induction on d and using Coro. 1.4.4, we see that $\mathrm{CH}_d(\mathbb{P}^n)$ is generated by $[L^d]$.

For $d = n$, we have $\mathrm{CH}_n(\mathbb{P}^n) = \mathbb{Z} \cdot [\mathbb{P}^n]$. For $d = n-1$, we already know $\mathrm{CH}_{n-1}(\mathbb{P}^n) = \mathbb{Z} \cdot [L^{n-1}]$ ([16, Prop. II.6.4]). Now assume $d < n-1$. Suppose there is an $m \geq 0$ such that $m[L^d] = \sum n_i [\mathrm{div}(r_i)]$, $r_i \in k(V_i)^*$ for some $(d+1)$-dimensional integral subschemes V_i. Let Z be the union of all the V_i, we can find a $(n-d-2)$-dimensional linear subspace H which is disjoint with Z. Let $f : Z \to \mathbb{P}^{d+1}$ be the projection from H. Apply Thm. 1.2.1 to f and use $\mathrm{CH}_d(\mathbb{P}^{d+1}) = \mathbb{Z}[L^d]$), then we see $m = 0$. This shows $\mathrm{CH}_d(\mathbb{P}^n) = \mathbb{Z}[L^d]$ in general.

Example 1.4.7. Let H be a hypersurface of degree m in $\mathbb{P}^n$. Then $[H] = m[L]$ with L a hyperplane. So it follows from Prop. 1.3.6 that $\mathrm{CH}_{n-1}(\mathbb{P}^n \setminus H) = \mathbb{Z}/m\mathbb{Z}$.

1.5 A Useful Result on Zero-Cycles

Included here is a classical result on zero-cycles, which has been used in several articles and will also be useful later in this thesis. The proof of this result, however, seems rarely given explicitly in the literature except for [4]. Our proof below follows the one given in the last paragraph of [4, §3], with only a few minor modifications.

Proposition 1.5.1. *Let V be an integral regular k-variety and let U be a nonempty open subset of V. Suppose one of the following two conditions is verified:*

(i) *the field k is perfect*;

or

(ii) *V is quasi-projective.*

Then every zero-cycle on V is rationally equivalent to a zero-cycle with support in U.

Proof. Let $Z = V \setminus U$. It suffices to prove the result for a closed point $x \in Z$. Let $d = \dim V$. In the regular local ring $\mathscr{O}_{V,\,x}$, there exists an element $g \neq 0$ which defines locally a closed subset containing Z. We can find a chain of regular parameters $f_1, \dots, f_{d-1}$, i.e., a subset of a system of generators of the maximal ideal $\mathfrak{m}_x$ of $\mathscr{O}_{V,\,x}$, such that the image of g in the regular local ring $\mathscr{O}_{V,\,x}/(f_1, \dots, f_{d-1})$ is nonzero. Writing out the equations locally defining the point x and taking the scheme-theoretic closure in V, we obtain a regular integral curve C which is closed in V, such that C is regular at x and Z does not contain C. Let $D \to C$ be the normalization of C. There is a point y on D on a neighborhood of y the natural morphism $D \to C$ is an isomorphism sending y to x. Let $\pi : D \to V$ be the composition $D \to C \to V$ and let $Z_1 = \pi^{-1}(Z)$. This is a proper closed subset of D. So Z_1 consists of finitely many closed points.

In case (i), D is smooth and is an open subset of a smooth complete integral curve $\overline{D}$. Then D is quasi-projective since $\overline{D}$ is projective. In case (ii), the curve C is quasi-projective and hence D also. So in both cases, D is a quasi-projective integral curve. Thus, there is an affine open subset of D containing Z_1. Let A be the semi-local ring at the points in Z_1 of such an affine open set. It is a semi-local Dedekind domain, hence is a PID. It then follows easily that there is a function $f \in A$ that has a simple zero at $y \in Z_1$ and takes a nonzero value at any other point in Z_1 different from y. Hence y is rationally equivalent to a zero-cycle on D that has support in $D \setminus Z_1$. The natural map $\pi_* : Z_0(D) \to Z_0(V)$ induced by the proper morphism $\pi : D \to V$ respects rational equivalence. Hence, $x = \pi_*(y)$ is rationally equivalent to a zero-cycle whose support is contained in U. $\square$

Chapter 2

Chern Classes of Vector Bundles

The most typical examples for which the rational equivalence may be nicely described are the cases of vector bundles and projective bundles. This description is only one of the numerous applications of Chern classes.

Material in this chapter is mostly extracted from Chapters 2 and 3 of [11].

2.1 Vector Bundles and Projective Bundles

A ***vector bundle*** E of rank r on a scheme X is a scheme E equipped with a morphism $\pi : E \to X$ together with an open covering $\{ U_i \}$ of X and isomorphisms $\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{A}^r$ over U_i, satisfying the following property: on each nonempty intersection $U_i \cap U_j$ the morphism

$$\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{A}^r \longrightarrow (U_i \cap U_j) \times \mathbb{A}^r$$

is given by $(x, y) \mapsto (x, g_{ij}(x)y)$, where g_{ij} is a morphism from $U_i \cap U_j$ to the general linear group (cf. §4.1.2) $\mathbf{GL}_r$ (over the same base as X). We call the g_{ij} ***transition functions*** of the vector bundle E. These transition functions satisfy $g_{ii} = \mathrm{Id}$, $g_{ji} = g_{ij}^{-1}$ and $g_{ij} g_{jl} = g_{ij}$ for all indices i, j, l. An ***isomorphism*** $f : (E, \pi, \{ U_i \}, \{ \varphi_i \}) \to (E', \pi', \{ U'_j \}, \{ \varphi'_j \})$ of vector bundles of the same rank is an isomorphism $f : E \to E'$ of schemes such that $\pi = \pi' \circ f$, and such that $(E, \pi, \{ U_i, U'_j \}, \{ \varphi_i, \varphi'_j \circ f \})$ is also a vector bundle. An open covering together with a collection of transition functions determines a vector bundle, unique up to unique isomorphism.

Let E be a vector bundle on X. A ***section*** of E is a morphism $s : X \to E$ such that $\pi \circ s = \mathrm{Id}$. If E is determined by transition functions g_{ij}, a section of E is determined by a collection of morphisms $s_i : U_i \to \mathbb{A}^r$ such that $s_i = g_{ij} s_j$ on $U_i \cap U_j$ for all i, j. The ***zero scheme*** of a section s of E, denoted $Z(s)$, is defined as follows. On each U_i, let $s_i : U_i \to \mathbb{A}^r$ be given by $s_i = (s_{i1}, \ldots, s_{ir})$ with each $s_{im} \in \Gamma(U_i, \mathscr{O}_X)$; then $Z(s)$ is defined in U_i by the ideal generated by $s_{i1}, \ldots, s_{ir}$. For $x \in X$, we say the section s ***vanishes*** at x if $x \in Z(s)$.

Let $\Gamma(E/X)$ be the set of sections of E over X. For each open set U of X, the restriction of E to U is the vector bundle $\pi^{-1}(U) \to U$ over U. The assignment $U \mapsto \Gamma(\pi^{-1}(U)/U)$ defines a sheaf $\mathscr{E}$ of sets on X, called the ***sheaf of sections*** of E. It turns out that $\mathscr{E}$ has a structure of $\mathscr{O}_X$-module and is locally free of rank r. Conversely, a locally free sheaf $\mathscr{E}$ (of finite constant rank) comes from a vector bundle E, which is unique up to isomorphism. This may be seen by using transition functions. For any affine open set U of X with coordinate ring A, $\pi^{-1}(U)$ is an affine open set of E, whose coordinate ring is the symmetric algebra $\mathrm{Sym}_A \Gamma(U, \mathscr{E}^\vee)$, where $\mathscr{E}^\vee := \mathscr{H}om(\mathscr{E}, \mathscr{O}_X)$ is the dual sheaf of $\mathscr{E}$.

Some basic operations are defined for vector bundles compatibly with the corresponding notion for sheaves. For example, if E and F are two vector bundles on X with sheaves of sections $\mathscr{E}, \mathscr{F}$ respectively, then the ***direct sum*** $E \oplus F$, the ***tensor product*** $E \otimes F$ and the ***dual bundle*** $E^\vee$ are respectively the vector bundles with sheaves of sections $\mathscr{E} \oplus \mathscr{F}$, $\mathscr{E} \otimes \mathscr{F}$ and $\mathscr{E}^\vee$. The ***trivial vector bundle*** of rank r is just $X \times \mathbb{A}^r$, whose sheaf of sections is $\mathscr{O}_X^{\oplus r}$. A ***line bundle*** a vector bundle of rank 1. The trivial line bundle on X is often denoted simply by $\mathbf{1}$.

Let E and F be vector bundles over X, with sheaves of sections $\mathscr{E}$ and $\mathscr{F}$ respectively. A morphism $E \to F$ as schemes over X corresponds to a morphism of $\mathscr{O}_X$-modules $\mathscr{E} \to \mathscr{F}$. A sequence of morphisms of schemes over X

$$\cdots \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow \cdots$$

with each E_i a vector bundle on X with sheaf of sections $\mathscr{E}_i$, is called ***exact*** if the corresponding sequence of sheaves

$$\cdots \longrightarrow \mathscr{E}_{i+1} \longrightarrow \mathscr{E}_i \longrightarrow \cdots$$

is exact.

Let $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. Then there is a scheme Y over X, unique up to a unique isomorphism over X, such that $\pi_* \mathscr{O}_Y = \mathscr{B}$, where $\pi : Y \to X$ is the structural morphism. This scheme is often denoted by $\mathbf{Spec}\, \mathscr{B}$, motivated by the following property: for each affine open subset U of X, there is an isomorphism $\varphi_U : \pi^{-1}(U) \xrightarrow{\sim} \mathrm{Spec}\, \Gamma(U, \mathscr{B})$ over U such that if $V \subseteq U$ is another affine open set, the following diagram with natural morphisms

$$\begin{array}{ccc} \pi^{-1}(V) & \xrightarrow{\varphi_V} & \mathrm{Spec}\, \Gamma(V, \mathscr{B}) \\ \downarrow & & \downarrow \\ \pi^{-1}(U) & \xrightarrow{\varphi_U} & \mathrm{Spec}\, \Gamma(U, \mathscr{B}) \end{array}$$

commutes (cf. [14, II.1.3.1]). The structural morphism $\pi : \mathbf{Spec}\, \mathscr{B} \to X$ is clearly an affine morphism.

Let $\mathscr{S} = \bigoplus_{n \geq 0} \mathscr{S}_n$ be a quasi-coherent graded $\mathscr{O}_X$-algebra, such that $\mathscr{S}_0 = \mathscr{O}_X$ and $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{O}_X$-algebra. On the one hand, we have the notion of the spectrum of $\mathscr{S}$ over X, i.e. the scheme $C := \mathbf{Spec}\, \mathscr{S}$, which is called the ***cone*** of the graded $\mathscr{O}_X$-algebra $\mathscr{S}$. On the other hand, there is a scheme $\mathbf{Proj}\, \mathscr{S}$ over X, unique up to a unique isomorphism over X, called the ***projective cone*** of $\mathscr{S}$, with the following property: for each affine open subset U of X, there is an isomorphism $\eta_U : p^{-1}(U) \xrightarrow{\sim} \mathrm{Proj}\, \Gamma(U, \mathscr{S})$ over U such that if $V \subseteq U$ is another affine open subset, the following diagram with natural morphisms

$$\begin{array}{ccc} p^{-1}(V) & \xrightarrow{\eta_V} & \mathrm{Proj}\, \Gamma(V, \mathscr{S}) \\ \downarrow & & \downarrow \\ p^{-1}(U) & \xrightarrow{\eta_U} & \mathrm{Proj}\, \Gamma(U, \mathscr{S}) \end{array}$$

commutes (cf. [14, II.3.1.2]). This $\mathbf{Proj}\, \mathscr{S}$ is also called the ***projective cone*** of $C = \mathbf{Spec}\, \mathscr{S}$, and denoted $\mathbb{P}(C) := \mathbf{Proj}\, \mathscr{S}$. The structural morphism $p : \mathbb{P}(C) \to X$ is proper. On $\mathbb{P}(C)$ there is a ***canonical line bundle*** often denoted $\mathscr{O}_C(1)$. If $X = \mathrm{Spec}\, A$ is affine, then $\mathscr{S}$ is determined by a graded A-algebra S which is generated by S_1. If S_1 is generated by a finite number of elements, then $S \cong A[x_0, \ldots, x_n]/I$ for some homogeneous ideal I of a polynomial ring over A. In this case, $\mathbb{P}(C)$ is the closed subscheme of $X \times \mathbb{P}^n$ defined by I, and $\mathscr{O}_C(1)$ is the pull-back of the standard line bundle on $\mathbb{P}^n$ by the natural projection.

If $\mathscr{S} \to \mathscr{S}'$ is a surjective graded homomorphism of graded $\mathscr{O}_X$-algebras with the above properties, and if $C = \mathbf{Spec}\,\mathscr{S}$, $C' = \mathbf{Spec}\,\mathscr{S}'$, then there are closed immersion $C' \to C$ and $\mathbb{P}(C') \to \mathbb{P}(C)$ such that $\mathscr{O}_C(1)$ restricts to $\mathscr{O}_{C'}(1)$. In particular, the natural map $\mathscr{S} \to \mathscr{O}_X$, which vanishes on $\mathscr{S}_n$ for all $n > 0$ and is the identity of $\mathscr{O}_X$ on $\mathscr{S}_0$, determines a closed immersion $X \to C$ which is also a section of the structural morphism $\pi : C \to X$, called the ***zero section*** of C.

A vector bundle E on X is the cone associated to the symmetric algebra $\mathbf{Sym}\,\mathscr{E}^\vee$, where $\mathscr{E}$ is the sheaf of sections of E. The ***projective bundle*** of $\mathscr{E}$ is defined to be the projective cone $\mathbb{P}(E) = \mathbf{Proj}(\mathbf{Sym}\,\mathscr{E}^\vee)$ of E. The direct sum $E \oplus \mathbf{1}$ of E with a trivial line bundle has sheaf of sections $\mathscr{E} \oplus \mathscr{O}_X$. We have

$$\mathbf{Sym}(\mathscr{E}^\vee \oplus \mathscr{O}_X) = \mathbf{Sym}(\mathscr{E}^\vee) \otimes_{\mathscr{O}_X} \mathbf{Sym}\mathscr{O}_X = \mathbf{Sym}(\mathscr{E}^\vee) \otimes_{\mathscr{O}_X} \mathscr{O}_X[T]$$

as graded $\mathscr{O}_X$-algebras, where T is an indeterminate, and the degree n component of $(\mathbf{Sym}\mathscr{E}^\vee) \otimes_{\mathscr{O}_X} \mathscr{O}_X[T]$ is

$$\big(\mathbf{Sym}(\mathscr{E}^\vee) \otimes_{\mathscr{O}_X} \mathscr{O}_X[T]\big)_n = \bigoplus_{i=0}^{n} \mathbf{Sym}^i(\mathscr{E}^\vee) \otimes T^{n-i}\,.$$

The natural projection $\mathscr{E}^\vee \oplus \mathscr{O}_X \to \mathscr{E}^\vee$ induces a surjection $\mathbf{Sym}(\mathscr{E}^\vee \oplus \mathscr{O}_X) \to \mathbf{Sym}(\mathscr{E}^\vee)$ and hence determines a closed immersion $i : \mathbb{P}(E) \to \mathbb{P}(E \oplus \mathbf{1})$. With this embedding, $\mathbb{P}(E)$ is called the ***hyperplane at infinity*** in $\mathbb{P}(E \oplus 1)$. Furthermore, there is a canonical open immersion $j : E \to \mathbb{P}(E \oplus \mathbf{1})$ which induces an isomorphism from E to the complement of $\mathbb{P}(E)$ in $\mathbb{P}(E \oplus 1)$. We say $\mathbb{P}(E \oplus \mathbf{1})$ is the ***projective completion*** of E.

2.2 Divisors and Pseudo-divisors

2.2.1 Cartier Divisors and Weil Divisors

Let X be a scheme. For each open set U of X, let $S(U)$ denote the set of elements of $\Gamma(U,\, \mathscr{O}_X)$ which are not zero divisors in each local ring $\mathscr{O}_{X,\,x}$ for $x \in U$. Then the rings $S(U)^{-1}\Gamma(U,\, \mathscr{O}_X)$ form a presheaf, whose associated sheaf of rings $\mathscr{K}_X$ we call the ***sheaf of total quotient rings*** of $\mathscr{O}_X$. We denote by $\mathscr{K}_X^*$ the sheaf (of multiplicative groups) of invertible elements in $\mathscr{K}_X$, and $\mathscr{O}_X^*$ the sheaf of invertible elements in $\mathscr{O}_X$.

A ***Cartier divisor*** D on X is a global section of the sheaf $\mathscr{K}_X^*/\mathscr{O}_X^*$, it is determined by a collection of affine open sets U_i, which cover X, and elements $f_i \in \Gamma(U_i,\, \mathscr{K}_X^*)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j,\, \mathscr{O}_X^*)$ for all i, j. These f_i are called ***local equations*** for D. We will write the group of Cartier divisors on X as

$$\mathrm{Div}(X) := \Gamma(X,\, \mathscr{K}_X^*/\mathscr{O}_X^*)$$

and write its group law additively.

A Cartier divisor is said to be ***principal*** if it is in the image of the natural map

$$\Gamma(X,\, \mathscr{K}_X^*) \longrightarrow \Gamma(X,\, \mathscr{K}_X^*/\mathscr{O}_X^*) = \mathrm{Div}(X)\,.$$

Two Cartier divisors D_1 and D_2 are said to be ***linearly equivalent*** if $D_1 - D_2$ is principal. The ***support*** of a Cartier divisor D, denoted $\mathrm{Supp}\,(D)$ or $|D|$[†], is the union of all integral subschemes Z of X such that a local equation for D is not in

[†]This shorthand for $\mathrm{Supp}\,(D)$ should not be confused with a notation for complete linear systems, which does not occur in this thesis.

$\mathscr{O}^*_{X,\,Z}$. This is a closed subset of X, which will be given the reduced subscheme structure when necessary.

There is a line bundle on X associated to a given Cartier divisor D, denoted $\mathscr{O}_X(D)$. The sheaf of sections of the line bundle $\mathscr{O}_X(D)$ may be defined to be the $\mathscr{O}_X$-submodule of $\mathscr{K}_X$ which is generated on each U_i by f_i^{-1}. Equivalently, transition functions for $\mathscr{O}_X(D)$ with respect to the covering $\{ U_i \}$ are $g_{ij} = f_i/f_j$. Let U denote the complement of $\mathrm{Supp}\,(D)$ in X. Then the local equations f_i are regular functions on $U \cap U_i$, which clearly satisfy $f_i = g_{ij} f_j$ on $U \cap U_i \cap U_j$. So they determine a ***canonical section*** s_D of the line bundle $\mathscr{O}_X(D)$ over U. This canonical section is nowhere vanishing on U.

A Cartier divisor D is called ***effective*** if all the local equations f_i are in $\Gamma(U_i,\, \mathscr{O}_X)$. In this case, the collection of functions f_i defines a ***canonical section*** of $\mathscr{O}_X(D)$ over X, which we also denote by s_D. This s_D vanishes only on the support of D.

Now let X be an integral scheme. The sheaf of total quotient rings $\mathscr{K}_X$ is then the constant sheaf associated to the function field $K = k(X)$ of X. For any $f \in k(X)^*$, we write $\mathrm{div}(f)$ for the corresponding principal Cartier divisor. Let $n = \dim X$. A ***Weil divisor*** on X is an element of $Z_{n-1}(X)$. If D is a Cartier divisor on X and V is an integral subscheme of codimension 1, we write

$$\mathrm{ord}_V(D) := \mathrm{ord}_V(f_\alpha)$$

where f_α is a local equation for D on any affine open set U_α with $U_\alpha \cap V \neq \emptyset$, and ord_V is the order function on $k(X)$ defined by V. This number is well-defined since f_α is determined up to units. The ***associated Weil divisor*** $[D]$ of D is defined by

$$[D] := \sum_V \mathrm{ord}_V(D)[V]\,,$$

the sum being over all codimension 1 integral subschemes V of X; as usual, there are only finitely many V with $\mathrm{ord}_V(D) \neq 0$. By the additivity of the order functions, we see that $D \mapsto [D]$ defines a group homomorphism

$$\mathrm{Div}(X) \longrightarrow Z_{n-1}(X)\,.$$

Note that the Weil divisor associated to a principal divisor $\mathrm{div}(f)$ coincides with the cycle $[\mathrm{div}(f)]$ defined earlier. From the definition of rational equivalence, it follows that Weil divisors associated to linearly equivalent Cartier divisors are rationally equivalent. So if $\mathrm{Pic}(X)$ denotes the group of linear equivalence classes of Cartier divisors, there is an induced homomorphism

$$\mathrm{Pic}(X) \longrightarrow \mathrm{CH}_{n-1}(X)\,.$$

Remark 2.2.1. It is also standard to denote by $\mathrm{Pic}(X)$ the group of isomorphism classes of invertible sheaves on X. Since X is integral in our case, this group is isomorphic to the group of Cartier divisors modulo linear equivalence ([16, Chapt.2, §6], see also Lemma 2.2.3).

2.2.2 Pseudo-divisors

For our use, Cartier divisors have a drawback that the pull-back f^*D of a Cartier divisor D on X by a morphism $f : X' \to X$ is defined only under certain assumptions, for example in the case f is flat ([14, IV.21.4]). The notion of pseudo-divisor is a simple generalization of Cartier divisor, which will not have this defect, but will still carry enough information to determine intersections for cycle classes.

Definition 2.2.2. A ***pseudo-divisor*** on a scheme X is a triple (L, Z, s), consisting of a line bundle L on X, a closed subset Z of X, and a nowhere vanishing section of L over $X \setminus Z$. We call L the ***line bundle***, Z the ***support***, and s the ***section*** of the pseudo-divisor. Data (L', Z', s') define the same pseudo-divisor if $Z = Z'$ and there is an isomorphism σ of L with L' such that the restriction of σ to $X \setminus Z$ takes s to s'.

Note that a pseudo-divisor with support X is simply an isomorphism class of line bundles on X.

Any Cartier divisor D on a scheme X determines a pseudo-divisor $(\mathscr{O}_X(D), |D|, s_D)$, where $\mathscr{O}_X(D)$ is the line bundle of D, $|D|$ is the support of D, and s_D is the canonical section of $\mathscr{O}_X(D)$. We say that a pseudo-divisor (L, Z, s) is ***represented*** by a Cartier divisor D if $|D| \subseteq Z$, and if there is an isomorphism from $\mathscr{O}_X(D)$ to L, which off Z, takes s_D to s. Note that we allow Z to be larger than $|D|$; for example if $Z = X$, all linearly equivalent Cartier divisors represent the same pseudo-divisor.

A general pseudo-divisor will often be denoted by a single letter D, and we write $\mathscr{O}_X(D)$ for its line bundle, $|D|$ for its support, and s_D for its section. This agrees with the notation for Cartier divisors, except that a Cartier divisor may have smaller support than a pseudo-divisor it represents.

If $D = (L, Z, s)$ and $D' = (Z', L', s')$ are pseudo-divisors on a scheme X, their ***sum*** $D + D'$ is defined to be the pseudo-divisor

$$D + D' := (L \otimes L', Z \cup Z', s \otimes s')$$

and the pseudo-divisor $-D$ is defined to be

$$-D := (L^{-1}, Z, s^{-1}).$$

For a fixed closed subset Z of X, the pseudo-divisors with support Z form a group, denoted $\mathrm{Div}_Z(X)$.

If $f : X' \to X$ is a morphism of schemes, then the ***pull-back*** f^*D of a pseudo-divisor $D = (L, Z, s)$ on X is the pseudo-divisor $(f^*L, f^{-1}(Z), f^*s)$ on X'. This pull-back induces a group homomorphism $f^* : \mathrm{Div}_Z(X) \to \mathrm{Div}_{Z'}(X')$ with $Z' = f^{-1}(Z')$. Moreover, if $g : X \to Y$ is another morphism, one has $(g \circ f)^* = f^* \circ g^*$.

Lemma 2.2.3. *Let X be an integral scheme. Then every pseudo-divisor (L, Z, s) on X is represented by some Cartier divisor D. Moreover,*
(i) *if $Z \neq X$, D is uniquely determined;*
(ii) *if $Z = X$, D is determined up to linear equivalence.*

Proof. Let $g_{\alpha\beta}$ be transition functions for the line bundle L, with respect to some affine open covering $\{ U_\alpha \}$ of X. Fix one index α_0 and set $f_\alpha = g_{\alpha\alpha_0}$. Then $f_\alpha / f_\beta = g_{\alpha\beta}$, so the data (U_α, f_α) define a Cartier divisor D with $\mathscr{O}_X(D) \cong L$. In case $Z = X$, this gives the existence of a Cartier divisor representing the given pseudo-divisor.

If $Z \neq X$, $U := X \setminus Z$ is a nonempty open set of X. The section s is given by a collection of regular functions s_α on $U \cap U_\alpha$ such that $s_\alpha = g_{\alpha\beta} s_\beta$ for all α, β. Since $s_\alpha / f_\alpha = s_\beta / f_\beta$ for all α, β, there is a rational function $r \in k(X)^*$ with $r = s_\alpha / f_\alpha$ for all α. Set $D' = D + \mathrm{div}(r)$. The local equations for D' are given by $f'_\alpha = f_\alpha \cdot r = s_\alpha$. Now we have $\mathscr{O}_X(D') \cong L$ and the canonical section $s_{D'}$ of D' is given by the functions f'_α so that it corresponds to s. This proves the existence of the desired Cartier divisor in case $Z \neq X$.

For the uniqueness, if D and D', with local equations f_α and f'_α, both represent (L, Z, s), then $f'_\alpha / f'_\beta = g_{\alpha\beta} = f_\alpha / f_\beta$ for all α, β. So there is a rational function $f \in k(X)^* = k(U_\alpha)^*$ such that $f'_\alpha = f_\alpha f$ for all α. If $U = \emptyset$, this means exactly D' is linearly equivalent to D. If $U \neq \emptyset$, f'_α and f_α must agree on $U \cap U_\alpha$ for all α since $s_{D'} = s = s_D$. Hence $f = 1$ on U, i.e. $f = 1$ in $k(X)$, which shows $D = D'$. ☐

Definition 2.2.4. Let D be a pseudo-divisor on an n-dimensional integral scheme X, and let $|D|$ be its support. Define the ***Weil divisor class*** $[D] \in \mathrm{CH}_{n-1}(|D|)$ of D as follows. Take a Cartier divisor which represents D, and let $[D]$ be the class in $\mathrm{CH}_{n-1}(|D|)$ of the associated Weil divisor. In case $|D| = X$, the Caritier divisor is determined up to linear equivalence, but the class of its associated Weil divisor in $\mathrm{CH}_{n-1}(|D|) = \mathrm{CH}_{n-1}(X)$ is always well-defined. In case $|D| \neq X$, this Cartier divisor is unique and its associated Weil divisor $[D]$ is in fact a cycle on $|D|$. Now $\dim |D| < n = \dim X$, hence $Z_{n-1}(|D|) = \mathrm{CH}_{n-1}(|D|)$. So the Weil divisor class $[D] \in \mathrm{CH}_{n-1}(|D|)$ is again well-defined.

The mapping $D \mapsto [D]$ clearly defines a group homomorphism $\mathrm{Div}_Z(X) \to \mathrm{CH}_{n-1}(Z)$.

2.3 Intersection with Divisors

2.3.1 Intersection Classes

Definition 2.3.1. Let D be a pseudo-divisor on a scheme X, and let V be d-dimensional integral subscheme of X. We define the ***intersection class*** $D \cdot [V]$ in $\mathrm{CH}_{d-1}(|D| \cap V)$ as follows. Let $j : V \to X$ be the natural inclusion. The restriction j^*D is a pseudo-divisor on V, with support $|D| \cap V$, so we can define $D \cdot [V]$ to be the Weil divisor class of j^*D:

$$D \cdot [V] := [j^*D] \in \mathrm{CH}_{d-1}(|D| \cap V)\,.$$

When D is a Cartier divisor, this may be rephrased as follows: if $V \not\subseteq |D|$, D restricts to a Cartier divisor j^*D on V, and $D \cdot [V]$ is its associated Weil divisor regarded as an element in $\mathrm{CH}_{d-1}(|D| \cap V) = Z_{d-1}(|D| \cap V)$; if $V \subseteq |D|$, $D \cdot [V]$ is the class in $\mathrm{CH}_{d-1}(V) = \mathrm{CH}_{d-1}(|D| \cap V)$ of the associated Weil divisor $[C]$ of any Cartier divisor C on V whose line bundle $\mathscr{O}_V(C)$ is isomorphic to $j^*\mathscr{O}_X(D)$.

In line with our earlier convention, we will write $D \cdot [V]$ also for the image of the above class in $\mathrm{CH}_{d-1}(Y)$, for any subscheme Y of X which contains $|D| \cap V$.

Let $\alpha = \sum n_V[V]$ be a d-cycle on X. The ***support*** of α, written $|\alpha|$, is the union of the integral subschemes V appearing with nonzero coefficients in α. For a pseudo-divisor D on X, each $D \cdot [V]$ is a class in $\mathrm{CH}_{d-1}(|D| \cap |\alpha|)$. Thus we can define the ***intersection class*** $D \cdot \alpha$ in $\mathrm{CH}_{d-1}(|D| \cap |\alpha|)$ by setting

$$D \cdot \alpha := \sum n_V (D \cdot [V])\,.$$

As above, we also regard $D \cdot \alpha$ as an element of $\mathrm{CH}_{d-1}(Y)$ for any subscheme Y containing $|D| \cap |\alpha|$.

Proposition 2.3.2. *Let D and D' be pseudo-divisors on a scheme X.*

(i) *For any two d-cycles α, α' on X,*

$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha'$$

in $\mathrm{CH}_{d-1}(|D| \cap (|\alpha| \cup |\alpha'|))$.

(ii) *For any d-cycle α on X,*

$$(D + D') \cdot \alpha = D \cdot \alpha + D \cdot \alpha'$$

in $\mathrm{CH}_{d-1}((|D| \cup |D'|) \cap |\alpha|)$.

(iii) (*Projection formula*) *If $f : X' \to X$ is a proper morphism and α' is a d-cycle on X', then*

$$f_*(f^*D \cdot \alpha') = D \cdot f_*(\alpha')$$

in $\mathrm{CH}_{d-1}(|D| \cap f(|\alpha'|))$, *where by abuse of notation, the induced morphism from* $f^{-1}(|D|) \cap |\alpha'|$ *to* $|D| \cap f(|\alpha'|)$ *is also denoted by* f.

(iv) *If* $f : X' \to X$ *is a flat morphism of relative dimension* n *and* α *is a* d*-cycle on* X*, then*

$$f^*D \cdot f^*\alpha = f^*(D \cdot \alpha)$$

in $\mathrm{CH}_{d+n-1}(f^{-1}(|D| \cap |\alpha|))$, *where by abuse of notation, the induced morphism from* $f^{-1}(|D| \cap |\alpha|)$ *to* $|D| \cap |\alpha|$ *is also denoted by* f.

(v) *If the line bundle* $\mathscr{O}_X(D)$ *is trivial and* α *is a* d*-cycle on* X*, then*

$$D \cdot \alpha = 0$$

in $\mathrm{CH}_{d-1}(|\alpha|)$.

Proof. (i) follows directly from the definition. In proving the other assertions, we may therefore assume $\alpha = [V]$, with V an integral subscheme. Then (ii) follows from the fact that restricting to V and forming associated Weil divisor classes preserve sums.

For (iii), by functoriality of pull-back and pull-forward, we may assume $\alpha' = [V]$, $V = X'$ and $f(V) = X$. The pseudo-divisor D is then represented by a Cartier divisor, which we also denote by D. What we need prove becomes the identity of cycles on X:

$$f_*[f^*D] = \deg(X'/X)[D]\,.$$

This identity is local on X, so we may assume $D = \mathrm{div}(r)$ for some $r \in k(X)^*$. Then Prop. 1.2.2 shows that

$$f_*[f^*\mathrm{div}(r)] = f_*[\mathrm{div}(f^*r)] = [\mathrm{div}(N_{k(X')/k(X)}(f^*r))] = [\mathrm{div}(r^d)] = d[\mathrm{div}(r)]$$

where $d = \deg(X'/X)$, f^* also denotes the field embedding $f^* : k(X) \to k(X')$ and $N_{k(X')/k(X)}$ denotes the norm related to this field extension. This gives the desired result.

For (iv), we may also assume $V = X$ so that D is represented by a Cartier divisor. The identity to prove is now

$$[f^*D] = f^*[D]$$

as cycles on X'. Again as a local assertion on X, we may assume D is principal and hence the difference of two effective divisors. Since both sides are additive, it suffices to prove the identity for D effective. By the definition of associated Weil divisor, $[D]$ coincides in this case with cycle of the closed subscheme associated to the divisor D. Thus the result is seen from Lemma 1.3.2.

Finally, for (v), we may assume $V = X$ so that D is represented by a Cartier divisor on X. The assertion is then that $[D] = 0$ in $\mathrm{CH}_{d-1}(X)$ when D is principal, which we have already seen earlier. □

Theorem 2.3.3. *Let* D *and* D' *be Cartier divisors on an* n*-dimensional integral scheme* X*. Then*

$$D \cdot [D'] = D' \cdot [D]$$

in $\mathrm{CH}_{n-2}(|D| \cap |D'|)$.

Proof. See [11, p.35, Thm. 2.4]. □

Corollary 2.3.4. *Let* D *be a pseudo-divisor on a scheme* X*, and* α *a* d*-cycle on* X *which is rationally equivalent to* 0*. Then*

$$D \cdot \alpha = 0$$

in $\mathrm{CH}_{d-1}(|D|)$.

Proof. Suppose $\alpha = [\mathrm{div}(r)]$, $r \in k(V)^*$, with V an integral subscheme of X. We may replace X by V and then D by a representing Caritier divisor. Then

$$D \cdot [\mathrm{div}(r)] = \mathrm{div}(r) \cdot [D] = 0$$

in $\mathrm{CH}_{d-1}(|D|)$, by Thm. 2.3.3 and Prop. 2.3.2 (v). □

By Coro. 2.3.4, if D is a pseudo-divisor on a scheme X and Y is a subscheme of X, the group homomorphism

$$Z_d(Y) \longrightarrow \mathrm{CH}_{d-1}(|D| \cap Y)\,; \quad \alpha \mapsto D \cdot \alpha$$

induces a homomorphism

$$\mathrm{CH}_d(Y) \longrightarrow \mathrm{CH}_{d-1}(|D| \cap Y)$$

which is also denoted $\alpha \mapsto D \cdot \alpha$.

Corollary 2.3.5. *Let D and D' be pseudo-divisors on a scheme X. Then for any d-cycle α on X,*

$$D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$$

in $\mathrm{CH}_{d-2}(|D| \cap |D'| \cap |\alpha|)$.

Proof. Taking $\alpha = [V]$ and restricting D and D' to V, one is reduced to Thm. 2.3.3. □

Definition 2.3.6. Let $D_1, \dots, D_r$ be pseudo-divisors on a scheme X. For any $\alpha \in Z_d(X)$, define the ***intersection class*** $D_1 \cdots D_r \cdot \alpha$ in $\mathrm{CH}_{d-r}(\cap_{i=1}^r |D_i| \cap |\alpha|)$ by induction:

$$D_1 \cdots D_r \cdot \alpha := D_1 \cdot (D_2 \cdots D_r \cdot \alpha)\,.$$

By Coro. 2.3.5, this is independent of the ordering of the D_i, and by Prop. 2.3.2, it is linear in each variable D_i and α. More generally, for any homogeneous polynomial $P(T_1, \dots, T_r)$ of degree m with coefficients in $\mathbb{Z}$ and any subscheme Z of X containing $(\cup_{i=1}^r |D_i|) \cap |\alpha|$, the class

$$P(D_1, \dots, D_r) \cdot \alpha \in \mathrm{CH}_{d-m}(Z)$$

can be defined in an obvious way.

If $r = d$ and $Y := \cap_{i=1}^r |D_i| \cap |\alpha|$ is a proper scheme, we define the ***intersection number*** $(D_1 \cdots D_r \cdot \alpha)_X$ by

$$(D_1 \cdots D_r \cdot \alpha)_X = \int_Y D_1 \cdots D_r \cdot \alpha\,.$$

Similarly, if $Z := (\cup_{i=1}^r |D_i|) \cap |\alpha|$ is a proper scheme and P is a homogeneous polynomial of degree $m = d$, we define

$$(P(D_1, \dots, D_r) \cdot \alpha)_X := \int_Z P(D_1, \dots, D_r) \cdot \alpha\,.$$

2.3.2 Chern Class of a Line Bundle

Definition 2.3.7. Let L be a line bundle on a scheme X. For any d-dimensional integral subscheme V of X, we define an element $c_1(L) \cap [V]$ in $\mathrm{CH}_{d-1}(X)$ as follows. The restriction $L|_V$ of L to V is isomorphic to $\mathscr{O}_V(C)$ for some Cartier divisor C on V, determined up to linear equivalence. The associated Weil divisor $[C]$ determines an element in $\mathrm{CH}_{d-1}(X)$, which we take as our $c_1(L) \cap [V]$. Namely,

$$c_1(L) \cap [V] := [C] \ \text{ in } \ \mathrm{CH}_{d-1}(X)\,.$$

This is extended by linearity to define a homomorphism $\alpha \mapsto c_1(L) \cap \alpha$ from $Z_d(X)$ to $\mathrm{CH}_{d-1}(X)$.

Note that if $L = \mathscr{O}_X(D)$ for a pseudo-divisor D on X, it follows from the definition of the intersection class that

$$c_1(\mathscr{O}_X(D)) \cap \alpha = D \cdot \alpha \text{ in } \mathrm{CH}_{d-1}(X)\,.$$

Proposition 2.3.8. *Let L, L' be line bundles on a scheme X, and $\alpha \in Z_d(X)$.*

(i) *If $\alpha \overset{\text{rat}}{\sim} 0$, then $c_1(L) \cap \alpha = 0$. There is therefore an induced homomorphism*

$$c_1(L) \cap - : \mathrm{CH}_d(X) \longrightarrow \mathrm{CH}_{d-1}(X)\,.$$

(ii) (*Commutativity*) $c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$ *in* $\mathrm{CH}_{d-2}(X)$.

(iii) (*Projection formula*) *If $f : X' \to X$ is a proper morphism and $\alpha' \in Z_d(X')$, then*

$$f_*(c_1(f^*L) \cap \alpha') = c_1(L) \cap f_*(\alpha')$$

in $\mathrm{CH}_{d-1}(X)$.

(iv) *If $f : X' \to X$ is a flat morphism of relative dimension n, then*

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$$

in $\mathrm{CH}_{d+n-1}(X')$.

(v) *One has*

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$$

and

$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha$$

in $\mathrm{CH}_{d-1}(X)$.

Proof. Since a line bundle on X determines a pseudo-divisor on X with support X, the assertions follow from the corresponding facts for intersection classes with pseudo-divisors. ☐

If $L_1, \dots, L_r$ are line bundles on X, $\alpha \in \mathrm{CH}_d(X)$, and $P(T_1, \dots, T_r)$ is a homogeneous polynomial of degree m with integer coefficients , then

$$P(c_1(L_1), \dots, c_1(L_r)) \cap \alpha$$

can be defined in $\mathrm{CH}_{d-m}(X)$. In particular, for a line bundle L on X and $\alpha \in \mathrm{CH}_d(X)$, the element $c_1(L)^r \cap \alpha$ in $\mathrm{CH}_{d-r}(X)$ is defined inductively by $c_1(L)^r \cap \alpha = c_1(L) \cap (c_1(L)^{r-1} \cap \alpha)$.

2.3.3 Gysin Map for Divisors

Definition 2.3.9. Let D be an effective Cartier divisor on a scheme X. Its associated closed subscheme will be also denoted D. Let $i : D \to X$ be the inclusion. The ***Gysin homomorphisms*** $i^* : Z_d(X) \to \mathrm{CH}_{d-1}(D)$ are defined by

$$i^*(\alpha) := D \cdot \alpha\,.$$

Proposition 2.3.10. *Let D be an effective Cartier divisor on a scheme X and let i be the Gysin homomorphism.*

(i) *If $\alpha \in Z_d(X)$ is rationally equivalent to 0, then $i^*(\alpha) = 0$. Therefore, there are induced homomorphisms*

$$i^* : \mathrm{CH}_d(X) \longrightarrow \mathrm{CH}_{d-1}(D)\,.$$

(ii) *For $\alpha \in Z_d(X)$, one has*

$$i_* i^*(\alpha) = c_1(\mathscr{O}_X(D)) \cap \alpha \;\textit{ in }\; \mathrm{CH}_{d-1}(X)\,.$$

(iii) *If α is a d-cycle on D, then*

$$i^* i_*(\alpha) = c_1(i^*\mathscr{O}_X(D)) \cap \alpha \quad \textit{in} \;\; \mathrm{CH}_{d-1}(D)\,.$$

(iv) *If X is purely n-dimensional, then*

$$i^*[X] = [D] \;\; \textit{in} \;\; \mathrm{CH}_{n-1}(D).$$

(v) *If L is a line bundle on X and $\alpha \in Z_d(X)$, then*

$$i^*(c_1(L) \cap \alpha) = c_1(i^*L) \cap i^*(\alpha)$$

in $\mathrm{CH}_{d-2}(D)$.

Proof. (i) and (v) are special cases of Coros. 2.3.4 and 2.3.5 respectively. (ii) and (iii) follow from the definitions; in both cases, both sides are represented by the intersection class $D \cdot \alpha$. (iv) says that $[D] = D \cdot [X]$, which is a restatement of Lemma 1.3.5. □

Example 2.3.11. Let L be a line bundle on a scheme X, $p : L \to X$ the projection, and $i : X \to L$ the embedding of X into L by the zero section. Then $i^*p^*\alpha = \alpha$ for all $\alpha \in \mathrm{CH}_d(X)$. Indeed, we may assume $\alpha = [V]$ and $V = X$. Then $p^{-1}(V)$ is purely dimensional, hence by Prop. 2.3.10 (iv),

$$i^*p^*[V] = i^*[p^{-1}(V)] = [X] = [V]\,.$$

Combining this with Prop. 1.4.1, one concludes that the flat pull-back

$$p^* : \mathrm{CH}_d(X) \longrightarrow \mathrm{CH}_{d-1}(L)$$

is an isomorphism. One will see a generalization in Thm. 2.4.5.

2.4 Chern Classes

2.4.1 Segre Classes of Vector Bundles

Definition 2.4.1. Let E be a vector bundle of rank $r = e + 1$ on a scheme X. Let $P = \mathbb{P}(E)$ be the projective bundle of E, $p = p_E : \mathbb{P}(E) \to X$ the projection, and $\mathscr{O}_E(1)$ the canonical line bundle on P. For any integer i, define homomorphisms $\alpha \mapsto s_i(E) \cap \alpha$ from $\mathrm{CH}_d(X)$ to $\mathrm{CH}_{d-i}(X)$ by the formula

$$s_i(E) \cap \alpha := p_*(c_1(\mathscr{O}_E(1))^{e+i} \cap p^*\alpha)\,.$$

Note that the morphism p is proper, as well as flat of relative dimension e. So the flat pull-back $p^* : \mathrm{CH}_d(X) \to \mathrm{CH}_{d+e}(P)$ and the proper push-forward $p_* : \mathrm{CH}_{d-i}(P) \to \mathrm{CH}_{d-i}(X)$ make sense.

Proposition 2.4.2. *Let E, F be vector bundles on a scheme X, of rank $e+1$, $l+1$ respectively, and let $\alpha \in \mathrm{CH}_\bullet(X)$.*

(i) *One has*

$$s_i(E) \cap \alpha = 0 \;\; \textit{for all} \;\; i < 0 \quad \textit{and} \;\; s_0(E) \cap \alpha = \alpha\,.$$

(ii) *For all i, j,*

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha)\,.$$

(iii) *If $f : X' \to X$ is a proper morphism, $\alpha' \in \mathrm{CH}_\bullet(X')$, then for all i,*

$$f_*(s_i(f^*E) \cap \alpha') = s_i(E) \cap f_*(\alpha')\,.$$

(iv) *If $f : X' \to X$ is a flat morphism of some relative dimension, then for all i,*

$$s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha)\,.$$

(v) *If E is a line bundle, then*

$$s_1(E) \cap \alpha = -c_1(E) \cap \alpha\,.$$

Proof. We first prove (iii) and (iv). Given a morphism $f : X' \to X$ and a vector bundle E on X, there is a fibre square

$$\begin{array}{ccc} \mathbb{P}(f^*E) & \xrightarrow{f'} & \mathbb{P}(E) \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

such that $f'^*\mathscr{O}_E(1) = \mathscr{O}_{f^*E}(1)$.

If f is proper, we have

$$\begin{aligned} f_*(s_i(f^*E) \cap \alpha') &= f_*p'_*\big(c_1(\mathscr{O}_{f^*E}(1))^{e+i} \cap p'^*\alpha'\big) \\ &= p_*f'_*\big(c_1(f'^*\mathscr{O}_E(1)^{e+i}) \cap p'^*\alpha'\big) \\ &= p_*(c_1(\mathscr{O}_E(1))^{e+i} \cap f'_*p'^*\alpha') \quad \text{(Prop. 2.3.8 (iii))} \\ &= p_*\big(c_1(\mathscr{O}_E(1))^{e+i} \cap p^*f_*\alpha'\big) \quad \text{(Prop. 1.3.3)} \\ &= s_i(E) \cap f_*\alpha'\,. \end{aligned}$$

This proves (iii).

If f is flat of some relative dimension, we have

$$\begin{aligned} s_i(f^*E) \cap f^*\alpha &= p'_*\big(c_1(\mathscr{O}_{f^*E}(1))^{e+i} \cap p'^*f^*\alpha\big) \\ &= p'_*\big(c_1(f'^*\mathscr{O}_E(1)^{e+i}) \cap f'^*p^*\alpha\big) \\ &= p'_*f'^*\big(c_1(\mathscr{O}_E(1))^{e+i} \cap p^*\alpha\big) \quad \text{(Prop. 2.3.8 (iv))} \\ &= f^*p_*\big(c_1(\mathscr{O}_E(1))^{e+i} \cap p^*\alpha\big) \quad \text{(Prop. 1.3.3)} \\ &= f^*(s_i(E) \cap \alpha)\,. \end{aligned}$$

This proves (iv).

To prove (i), we may assume $\alpha = [V]$, with V a d-dimensional integral subscheme of X. By (iii), we may assume $X = V$. Then $\mathrm{CH}_{d-i}(X) = 0$ for $i < 0$, which proves the first assertion. Moreover, since $\mathrm{CH}_d(X) = \mathbb{Z} \cdot [X]$, $s_0(E) \cap \alpha = m[X]$ for some $m \in \mathbb{Z}$. To show $m = 1$, by (iv) we may restrict to an open set of X, so we may assume E is trivial. Then $\mathbb{P}(E) = X \times \mathbb{P}^e$, and $\mathscr{O}_E(1)$ is isomorphism to the line bundle of the Cartier divisor $X \times \mathbb{P}^{e-1}$ of $\mathbb{P}(E)$. Hence

$$c_1(\mathscr{O}_E(1)) \cap [X \times \mathbb{P}^e] = [X \times \mathbb{P}^{e-1}]$$

by definition of the Chern class of a line bundle. By induction it follows that

$$m[X] = s_0(E) \cap \alpha = p_*\big(c_1(\mathscr{O}_E(1))^e \cap [\mathbb{P}(E)]\big) = p_*([X \times \mathbb{P}^0]) = [X]\,,$$

hence, $m = 1$. The assertion (i) is thus proved.

For (ii), consider the fibre square

$$\begin{array}{ccc} Q & \xrightarrow{p'} & \mathbb{P}(F) \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{P}(E) & \xrightarrow{p} & X \end{array}$$

with natural morphisms. Then we have

$$\begin{aligned}
&s_i(E)\cap(s_j(F)\cap\alpha)\\
=&p_*\left(c_1(\mathscr{O}_E(1))^{e+i}\cap p^*q_*\big((c_1(\mathscr{O}_F(1)))^{l+j}\cap q^*\alpha\big)\right)\\
=&p_*\left(c_1(\mathscr{O}_E(1))^{e+i}\cap q'_*p'^*\big(c_1(\mathscr{O}_F(1))^{l+j}\cap q^*\alpha\big)\right)\qquad\text{(Prop. 1.3.3)}\\
=&p_*q'_*\left(c_1(q'_*\mathscr{O}_E(1))^{e+i}\cap p'^*\big(c_1(\mathscr{O}_F(1))^{l+j}\cap q^*\alpha\big)\right)\quad\text{(Prop. 2.3.8 (iii))}\\
=&p_*q'_*\left(c_1(q'_*\mathscr{O}_E(1))^{e+i}\cap\big(c_1(p'^*\mathscr{O}_F(1))^{l+j}\cap p'^*q^*\alpha\big)\right)\quad\text{(Prop. 2.3.8 (iv))}
\end{aligned}$$

and similarly,

$$s_j(F)\cap(s_i(E)\cap\alpha)=q_*p'_*\left(c_1(p'^*\mathscr{O}_F(1))^{l+j}\cap\big(c_1(q'^*\mathscr{O}_E(1))^{e+1}\cap q'^*p^*\alpha\big)\right)\,.$$

Then the result follows from Prop. 2.3.8 (ii) and the facts that $q_*p'_*=p_*q'_*$ and $p'^*q^*=q'^*p^*$.

Finally, to prove (v), we may assume E is trivial by (iv). Note that $\mathbb{P}(E)=X$ and $\mathscr{O}_E(1)=E^\vee$ in this case. So we have

$$s_1(E)\cap\alpha=c_1(\mathscr{O}_E(1))\cap\alpha=c_1(E^\vee)\cap\alpha=-c_1(E)\cap\alpha$$

by Prop. 2.3.8 (v). □

Corollary 2.4.3. *With notation as above, the flat pull-back*

$$p^*:\mathrm{CH}_d(X)\longrightarrow\mathrm{CH}_{d+e}(\mathbb{P}(E))$$

is a split injection.

Proof. By Prop. 2.4.2 (i), the homomorphism $\beta\mapsto p_*(c_1(\mathscr{O}_E(1))^e\cap\beta)$ is a left inverse of p^*. □

2.4.2 Chern Classes of Vector Bundles

Let E be a vector bundle on a scheme X. Consider the formal power series

$$s_t(E):=\sum_{i=0}^{\infty}s_i(E)t^i=1+s_1(E)t+s_2(E)t^2+\cdots\,.$$

The $s_i(E)$ can be regarded as endomorphisms of the group $\mathrm{CH}_\bullet(X)$, so multiplication can be defined as composition of endomorphisms. Thanks to Prop. 2.4.2 (ii), the $s_i(E)$ generate a commutative $\mathbb{Z}$-algebra $R:=\mathbb{Z}[s_i(E)]$ and $s_t(E)$ can be regarded as an element in the ring $R[[t]]$ of formal power series over R. There is an inverse power series of $s_t(E)$ in $R[[t]]$, which we denote by

$$c_t(E):=\sum_{i=0}^{\infty}c_t(E)t^i=1+c_1(E)t+c_2(E)t^2+\cdots\,.$$

Explicitly, one finds that

$$\begin{aligned}
c_0(E)&=1\,,\ \ c_1(E)=-s_1(E)\,,\\
c_2(E)&=-c_1(E)s_1(E)-s_2(E)\,,\\
c_n(E)&=-\sum_{i=0}^{n}s_i(E)c_{n-i}(E)\\
&=-s_1(E)c_{n-1}(E)-s_2(E)c_{n-2}(E)-\cdots-s_n(E)\,.
\end{aligned}$$

We will see that $c_t(E)$ is in fact a polynomial (Thm. 2.4.4 (i)). We call it the ***Chern polynomial*** of the vector bundle E. For $\alpha \in \mathrm{CH}_d(X)$, we write $c_i(E) \cap \alpha$ for the element in $\mathrm{CH}_{d-i}(X)$ obtained by applying the endomorphism $c_i(E)$ to α.

In view of Prop. 2.4.2 (v), for a line bundle E, the first Chern class defined here agrees with the definition given earlier. So if X is an integral scheme and $E \cong \mathscr{O}_X(D)$ for some Cartier divisor D on X, then $c_1(E) \cap [X] = [D]$.

Theorem 2.4.4. *Let E and F be vector bundles on a scheme X, and let α be any cycle on X.*

(i) (*Vanishing*) *For all $i > \mathrm{rank}(E)$, $c_i(E) = 0$.*

(ii) (*Commutativity*) *For all i, j, one has*

$$c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha)\,.$$

(iii) (*Projection formula*) *If $f : X' \to X$ is a proper morphism, then for all cycles α' on X', one has*

$$f_*(c_i(f^*E) \cap \alpha') = c_i(E) \cap f_*(\alpha')\,.$$

(iv) *If $f : X' \to X$ is a flat morphism of some relative dimension, then*

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)\,.$$

(v) (*Whitney sum*) *For any exact sequence*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of vector bundles on X, one has

$$c_t(E) = c_t(E')c_t(E'')\,,$$

in other words,

$$c_l(E) = \sum_{i+j=l} c_i(E')c_j(E'')\,, \quad \forall\, l \geq 0\,.$$

Proof. Properties (ii), (iii) and (iv) follow easily from corresponding facts for Segre classes. For (i) and (v), see [11, p.50, Thm. 3.2]. ☐

2.4.3 Application: Rational Equivalence on Bundles

Theorem 2.4.5. *Let E be a vector bundle of rank $r = e + 1$ on a scheme X, with projection $\pi : E \to X$. Let $\mathbb{P}(E)$ be the associated projective bundle, $p : \mathbb{P}(E) \to X$ the projection, and $\mathscr{O}_E(1)$ the canonical line bundle on $\mathbb{P}(E)$.*

(i) *The flat pull-back*

$$\pi^* : \mathrm{CH}_{d-r}(X) \longrightarrow \mathrm{CH}_d(E)$$

is an isomorphism for every d.

(ii) *Each element β in $\mathrm{CH}_d(\mathbb{P}(E))$ can be expressed uniquely in the form*

$$\beta = \sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^*\alpha_i$$

with $\alpha_i \in \mathrm{CH}_{d-e+i}(X)$. Thus there are canonical isomorphisms

$$\bigoplus_{i=0}^{e} \mathrm{CH}_{d-e+i}(X) \cong \mathrm{CH}_d(\mathbb{P}(E))\,.$$

Proof. The surjectivity of π^* was proved in Prop. 1.4.1. Let θ_E be the homomorphism

$$\bigoplus_{i=0}^{e} \mathrm{CH}_\bullet(X) \longrightarrow \mathrm{CH}_\bullet(\mathbb{P}(E))\,; \quad \theta_E(\oplus\alpha_i) := \sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^*\alpha_i\,.$$

To show the surjectivity of θ_E, the same Noetherian induction argument used in the proof of Prop. 1.4.1 reduces it to the case when E is trivial. By induction on the rank, it suffices to prove that θ_F is surjective when θ_E is known to be surjective, where $F = E \oplus \mathbf{1}$ is the direct sum of E and a trivial line bundle.

Let $P = \mathbb{P}(E)\,, Q = \mathbb{P}(F) = \mathbb{P}(E \oplus \mathbf{1})$, $q : Q \to X$ the projection. We have a commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\ \iota\ } & Q & \xleftarrow{\ j\ } & E \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle q} & \swarrow{\scriptstyle \pi} & \\ & & X & & \end{array}$$

identifying Q as the projective completion of E, P as the hyperplane at infinity. By Prop. 1.3.6, the row in the following commutative diagram is exact

$$\begin{array}{ccccccc} \mathrm{CH}_d(P) & \xrightarrow{\ \iota_*\ } & \mathrm{CH}_d(Q) & \xrightarrow{\ j^*\ } & \mathrm{CH}_d(E) & \longrightarrow & 0 \\ & & {\scriptstyle q^*}\uparrow & \nearrow{\scriptstyle \pi^*} & & & \\ & & \mathrm{CH}_{d-r}(X) & & & & \end{array}$$

As for the link between p^* and q^*, we have the following formula:

$$c_1(\mathscr{O}_F(1)) \cap q^*\alpha = \iota_* p^*\alpha\,, \ \ \forall\, \alpha \in \mathrm{CH}_\bullet(X)\,. \tag{2.1}$$

Indeed, it suffices to check this for $\alpha = [V]$, with V an integral subscheme of X. Since $\mathscr{O}_F(1)$ has a section vanishing precisely on P, it is isomorphic to the line bundle of the Cartier divisor $P = \mathbb{P}(E)$ on $Q = \mathbb{P}(F)$. It follows from the definition of the first Chern class of line bundles that

$$c_1(\mathscr{O}_F(1)) \cap q^*[V] = \mathbb{P}(E) \cdot [q^{-1}(V)] = [p^{-1}(V)] = \iota_* p^*[V]$$

as required.

Now let $\beta \in \mathrm{CH}_\bullet(Q)$ and write $j^*\beta = \pi^*\alpha$ for some $\alpha \in \mathrm{CH}_\bullet(X)$. Then $\beta - q^*\alpha$ lies in the kernel of j^*. Since $\mathrm{Ker}\,(j^*) = \mathrm{Im}\,(\iota_*)$, and by induction hypothesis we have the surjectivity of θ_E, we obtain

$$\beta - q^*\alpha = \iota_* \left(\sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^*\alpha_i \right)$$

for some $\alpha_i \in \mathrm{CH}_\bullet(X)$. Since $\iota^*\mathscr{O}_F(1) = \mathscr{O}_E(1)$, the projection formula rewrites the right side as

$$\begin{aligned} & \sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^i \cap \iota_* p^*\alpha_i \\ = & \sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^i \cap (c_1(\mathscr{O}_F(1)) \cap q^*\alpha_i) \quad \text{(by (2.1))} \\ = & \sum_{i=1}^{e+1} c_1(\mathscr{O}_F(1))^i \cap q^*\widetilde{\alpha}_i \end{aligned}$$

where $\widetilde{\alpha}_i = \alpha_{i-1}$. Hence, putting $\widetilde{\alpha}_0 = \alpha$, we get

$$\beta = q^*\alpha + \sum_{i=1}^{e+1} c_1(\mathscr{O}_F(1))^i \cap q^*\widetilde{\alpha}_i = \sum_{i=0}^{e+1} c_1(\mathscr{O}_F(1))^i \cap q^*\widetilde{\alpha}_i$$

showing that θ_F is surjective.

Let us now prove the uniqueness of the expression in (ii). Suppose there is a nontrivial relation

$$\beta = \sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^*\alpha_i = 0\,.$$

Let l be the largest integer with $\alpha_l \neq 0$ in $\mathrm{CH}_\bullet(X)$. Then we find

$$0 = p_*(c_1(\mathscr{O}_E(1))^{e-l} \cap \beta) = \alpha_l$$

using Prop. 2.4.2, whence a contradiction.

Finally, to see that π^* is injective, let $F = E \oplus \mathbf{1}$, $Q = \mathbb{P}(F)$ with other notation as before. If $\pi^*\alpha = 0$ for some $\alpha \neq 0$ in $\mathrm{CH}_\bullet(X)$, then $j^*q^*\alpha = 0$, so

$$\begin{aligned} q^*\alpha &= \iota_*\left(\sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^*\alpha_i\right) \\ &= \sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^i \cap \iota_* p^*\alpha_i \quad \text{(Projection formula)} \\ &= \sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^i \cap \big(c_1(\mathscr{O}_F(1)) \cap q^*\alpha_i\big) \\ &= \sum_{i=1}^{e+1} c_1(\mathscr{O}_F(1))^i \cap q^*\widetilde{\alpha}_i\,. \end{aligned}$$

But this contradicts the uniqueness part of (ii) for the bundle $F = E \oplus \mathbf{1}$. □

Chapter 3

Introduction to Chow Motives

3.1 Category of Correspondences

Let k be a field. Let $\mathfrak{V}_k$ denote the category of smooth projective varieties over k.

Definition 3.1.1. Let X and Y be objects in $\mathfrak{V}_k$. A (***Chow***) ***correspondence*** between X and Y is an element of the group

$$\bigoplus_i \mathrm{CH}_{\dim X_i}(X_i \times Y)\,,$$

where $X = \coprod_i X_i$ is the decomposition of X into irreducible components. We will write $\mathbf{Cor}(X, Y)$ for the group of correspondences between the varieties X and Y.

Let Z be a third object in $\mathfrak{V}_k$. Let $\alpha \in \mathbf{Cor}(X, Y)$ and $\beta \in \mathbf{Cor}(Y, Z)$ be correspondences. The ***composition*** $\beta \circ \alpha$ is the correspondence defined as

$$\beta \circ \alpha := \pi_{13*}(\pi_{12}^*(\alpha) \cdot \pi_{23}^*(\beta)) \in \mathbf{Cor}(X, Z)\,.$$

Here and in what follows notation of the type $\pi_{13} : X \times Y \times Z \to X \times Z$ means the projection onto the product of the first and third factors. The product $\pi_{12}^*(\alpha)\cdot\pi_{23}^*(\beta)$ is the ***intersection product*** in the graded ring $\mathrm{CH}^\bullet(X \times Y \times Z)$ (cf. [11, Chapt. 8]).

For $X \in \mathfrak{V}_k$, let $\delta_X : X \to X \times X$ be the diagonal morphism and let $\Delta_X \subseteq X \times X$ be its image. It defines a correspondence $[\Delta_X] \in \mathbf{Cor}(X, X)$.

Lemma 3.1.2. *With notation as above,*

(i) *for any correspondences* $\alpha \in \mathbf{Cor}(X, Y)$*,* $\beta \in \mathbf{Cor}(Y, X)$*, one has*

$$\alpha \circ [\Delta_X] = \alpha\,, \quad [\Delta_X] \circ \beta = \beta\,.$$

(ii) *composition of correspondences is associative.*

Proof. See [21, p.446, Lemma]. □

This lemma permits us to make the following definition.

Definition 3.1.3. The ***category of Chow correspondences*** over k, denoted $\mathfrak{CV}_k$, is defined by the following data.

(i) The objects of $\mathfrak{CV}_k$ are the objects of $\mathfrak{V}_k$. A variety X as an object of $\mathfrak{CV}_k$ will be denoted by the symbol $\underline{X}$.

(ii) For any two objects $\underline{X}$, $\underline{Y}$ in $\mathfrak{CV}_k$,

$$\mathrm{Hom}(\underline{X}\,,\,\underline{Y}) := \mathbf{Cor}(X, Y)\,.$$

(iii) Composition of morphisms in $\mathfrak{CV}_k$ is composition of correspondences.

There is a natural functor: $\mathfrak{V}_k \longrightarrow \mathfrak{CV}_k$ sending an object $X \in \mathfrak{V}_k$ to $\underline{X} \in \mathfrak{CV}_k$ and a morphism $f : X \to Y$ to $[\Gamma_f] \in \mathbf{Cor}(X, Y)$, where Γ_f is the ***graph*** of the morphism f, i.e., the image of the composite morphism

$$X \xrightarrow{\delta_X} X \times X \xrightarrow{\mathrm{Id} \times f} X \times Y \,.$$

The category $\mathfrak{CV}_k$ is an additive category in which the direct sum of two objects $\underline{X}$ and $\underline{Y}$ is

$$\underline{X} \oplus \underline{Y} = \underline{X \amalg Y} \,.$$

Furthermore, $\mathfrak{CV}_k$ possesses a ***tensor product***: for objects $\underline{X}\,,\,\underline{Y}$, one defines

$$\underline{X} \otimes \underline{Y} := \underline{X \times Y}$$

and for morphisms $\alpha \in \mathrm{Hom}(\underline{X}\,,\,\underline{Y}\,)\,, \beta \in \mathrm{Hom}(\underline{X'}\,,\,\underline{Y'}\,)$,

$$\alpha \otimes \beta := \pi^*(\alpha) \cdot \varpi^*(\beta) \in \mathrm{Hom}(\underline{X} \otimes \underline{X'}\,,\,\underline{Y} \otimes \underline{Y'}\,)\,,$$

where $\pi : X \times X' \times Y \times Y' \to X \times Y$ and $\varpi : X \times X' \times Y \times Y' \to X' \times Y'$ are the canonical projections.

3.2 Category of Chow Motives

Recall that a ***projector*** p in a category $\mathcal{C}$ is an element of $\mathrm{Hom}_{\mathcal{C}}(X,\, X)$ for some object X of $\mathcal{C}$ such that $p^2 = p$.

Definition 3.2.1. An additive category $\mathcal{C}$ is called ***pseudo-abelian*** if for every object $X \in \mathcal{C}$ and every projector $p \in \mathrm{Hom}_{\mathcal{C}}(X,\, X)$, the kernel $\mathrm{Ker}\,(p)$ exists and the natural morphism

$$\mathrm{Ker}\,(p) \oplus \mathrm{Ker}\,(\mathrm{Id} - p) \longrightarrow X$$

is an isomorphism.

Proposition 3.2.2. *Let $\mathcal{C}$ be any additive category. There exists a pseudo-abelian category $\mathcal{C}_{ps}$ together with a fully faithful additive functor $\eta : \mathcal{C} \longrightarrow \mathcal{C}_{ps}$ satisfying the following property:*

if $F : \mathcal{C} \to \mathcal{D}$ is an additive functor into a pseudo-abelian category $\mathcal{D}$, then there is an additive functor $F_{ps} : \mathcal{C}_{ps} \to \mathcal{D}$, unique up to isomorphism, such that $F_{ps} \circ \eta \cong F$.

The category $\mathcal{C}_{ps}$ is unique up to equivalence, and is called the **pseudo-abelian envelop** *of $\mathcal{C}$.*

Proof. We construct the category $\mathcal{C}_{ps}$ as follows. Take as objects pairs $(X,\, p)$, where $X \in \mathcal{C}$ and $p \in \mathrm{Hom}_{\mathcal{C}}(X,\, X)$ is a projector. For two objects $(X,\, p)$ and $(Y,\, q)$, define

$$\mathrm{Hom}_{\mathcal{C}_{ps}}\big((X,\, p)\,,\,(Y,\, q)\big) := q\,\mathrm{Hom}_{\mathcal{C}}(X,\, Y)p \subseteq \mathrm{Hom}_{\mathcal{C}}(X,\, Y)\,.$$

The composition of morphisms in $\mathcal{C}_{ps}$ is induced by composition in $\mathcal{C}$. The identity in $\mathrm{Hom}_{\mathcal{C}_{ps}}((X,\, p)\,,\,(X,\, p))$ is the morphism $p = p \circ \mathrm{Id} \circ p$.

Let's prove that $\mathcal{C}_{ps}$ is pseudo-abelian.

Let $q : (X,\, p) \to (X,\, p)$ be a projector. Since q is of the form $q = pfp$ with $f \in \mathrm{Hom}_{\mathcal{C}}(X,\, X)$, we have $qp = pq = q = q^2$. It follows that $p - q : X \to X$ is a projector in $\mathcal{C}$. The kernel of q in $\mathcal{C}_{ps}$ is the object $(X,\, p - q)$ together with the natural morphism

$$r_1 := p \circ \mathrm{Id}_X \circ (p - q) = p - q : \quad (X,\, p - q) \longrightarrow (X,\, p)\,.$$

It is readily seen that $qr_1 = 0$. Further, if $u : (Y, s) \to (X, p)$ is a morphism such that $qu = 0$, we have $(p - q)u = pu = u \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$. Thus, the morphism

$$w := (p - q) \circ u \circ s = u : \quad (Y, s) \longrightarrow (X, p - q)$$

satisfies $u = rw$. If $w' : (Y, s) \longrightarrow (X, p - q)$ is another morphism such that $u = r_1 w'$, then we have

$$w = u = r_1 w' = (p - q)w' = w'$$

since w' is of the form $w' = (p - q)vs$ for some $v \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$. This proves

$$\mathrm{Ker}\,(q) = \big((X, p - q)\,,\ r_1\big)$$

in $\mathcal{C}_{ps}$. Likewise, we find that (X, q) together with the morphism

$$r_2 := p \circ \mathrm{Id}_X \circ q : \quad (X, q) \longrightarrow (X, p)$$

is the kernel of the morphism

$$\mathrm{Id}_{(X,\, q)} - q \ : \quad (X, p) \longrightarrow (X, p)\,.$$

Now let's prove that the natural morphism

$$\varphi = (r_1\,,\, r_2) : \quad (X, p - q) \oplus (X, q) \longrightarrow (X, p)$$

is an isomorphism. We need find the inverse ψ of φ. Note that we have natural morphisms

$$r_1' := (p - 1) \circ \mathrm{Id}_X \circ p = p - q : \quad (X, p) \longrightarrow (X, p - q)$$

and

$$r_2' := q \circ \mathrm{Id}_X \circ p = q : \quad (X, p) \longrightarrow (X, q)$$

which induce a morphism

$$\phi : \quad (X, p) \longrightarrow (X, p - q) \prod (X, q)\,,$$

which satisfies $\pi_1 \phi = r_1'$ and $\pi_2 \phi = r_2'$, where $\pi_1\,, \pi_2$ are projections from the product $(X, p - q) \prod (X, q)$ onto its factors.

Let $\iota_1 : (X, p - q) \longrightarrow (X, p - q) \oplus (X, q)$ and $\iota_2 : (X, q) \longrightarrow (X, p - q) \oplus (X, q)$ be the natural morphisms. Then there is a canonical isomorphism

$$\iota_1 \pi_1 + \iota_2 \pi_2 : \quad (X, p - q) \prod (X, q) \longrightarrow (X, p - q) \oplus (X, q)\,.$$

The composition ϕ yields a morphism

$$\psi : \quad (X, p) \longrightarrow (X, p - q) \oplus (X, q)\,.$$

Now

$$\begin{aligned} \psi \circ \varphi \circ \iota_1 = \psi \circ r_1 &= (\iota_1 \pi_1 + \iota_2 \pi_2) \circ \phi \circ r_1 \\ &= \iota_1 \circ (\pi_1 \phi) \circ r_1 + \iota_2 \circ (\pi_2 \phi) \circ r_2 \\ &= \iota_1 \circ (r_1' \circ r_1) + \iota_2 \circ (r_2' \circ r_1)\,. \end{aligned}$$

But $r_1' \circ r_1 = (p - q) \circ (p - q) = \mathrm{Id}_{(X,\, p-q)}$ and $r_2' \circ r_1 = q(p - q) = 0$. So we get

$$\psi \circ \varphi \circ \iota_1 = \iota_1 \circ (r_1' \circ r_1) + \iota_2 \circ (r_2' \circ r_1) = \iota_1 \circ \mathrm{Id}_{(X,\, p-q)} = \iota_1\,.$$

Similarly, we have $\psi \circ \varphi \circ \iota_2 = \iota_2$, hence $\psi \circ \varphi = \mathrm{Id}$. On the other hand,

$$\begin{aligned}\varphi \circ \psi &= \varphi \circ (\iota_1 \pi + \iota_2 \pi_2) \circ \phi = (\varphi \iota_1) \circ (\pi_1 \phi) + (\varphi \iota_2) \circ (\pi_2 \phi) \\ &= r_1 \circ r_1' + r_2 \circ r_2' = p - q + q = p = \mathrm{Id}_{(X,\, p)}\ .\end{aligned}$$

So φ is indeed an isomorphism.

Finally, the assignment $X \mapsto (X\,, \mathrm{Id}_X)$ defines obviously an additive fully faithful functor $\eta : \mathcal{C} \to \mathcal{C}_{ps}$ and the universal property can be easily checked for the pair $(\mathcal{C}_{ps},\, \eta)$. The proof is thus completed. □

Definition 3.2.3. The category of ***effective Chow motives*** over k, denoted $\mathfrak{CM}_k^+$, is defined to be the pseudo-abelian envelop of the category $\mathfrak{CV}_k$.

The composite functor $\mathfrak{V}_k \to \mathfrak{CV}_k \to \mathfrak{CM}_k^+$ will be denoted by h:

$$h(X) = (\underline{X},\, \mathrm{Id}_{\underline{X}}) \in \mathfrak{CM}_k^+\ , \quad \text{for} \quad X \in \mathfrak{V}_k\,.$$

The category $\mathfrak{CM}_k^+$ inherits from $\mathfrak{CV}_k$ a tensor product which is defined as

$$(\underline{X}\,,\, p) \otimes (\underline{Y}\,,\, q) := (\underline{X} \otimes \underline{Y}\,,\, p \otimes q)\,.$$

Clearly, one has

$$h(X \amalg Y) = h(X) \oplus h(Y)\,, \quad h(X \times Y) = h(X) \otimes h(Y)$$

for all $X,\, Y \in \mathfrak{V}_k$. We denote by $\mathbb{1} := h(\mathrm{Spec}\ k)$ the neutral object for the tensor product.

Example 3.2.4. Let $X = \mathbb{P}^1$ be the projective line over k. Let $p = [\mathbb{P}^1 \times \{\,\mathrm{pt}\,\}]$ and $q = [\{\,\mathrm{pt}\,\} \times \mathbb{P}^1]$ be the elements in $\mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$ corresponding to the 1-cycles $\mathbb{P}^1 \times \{\,\mathrm{pt}\,\}$ and $\{\,\mathrm{pt}\,\} \times \mathbb{P}^1$, where "pt" designates a rational point of $\mathbb{P}^1$. Then $p\,, q$ define two morphisms $\underline{\mathbb{P}}^1 \to \underline{\mathbb{P}}^1$ in $\mathfrak{CV}_k$. We claim that they are projectors in $\mathfrak{CV}_k$ and $p + q = \mathrm{Id}_{\underline{\mathbb{P}}^1}$. In fact,

$$p^2 = \pi_{13*}\big([\mathbb{P}^1 \times \{\,\mathrm{pt}\,\} \times \mathbb{P}^1] \cdot [\mathbb{P}^1 \times \mathbb{P}^1 \times \{\,\mathrm{pt}\,\}]\big) = [\mathbb{P}^1 \times \{\,\mathrm{pt}\,\}] = p\,.$$

Similarly, one proves $q^2 = q$. Note that

$$\mathrm{Id}_{\underline{\mathbb{P}}^1} = [\Delta_{\mathbb{P}^1}] \in \mathrm{Hom}(\underline{\mathbb{P}}^1\,,\, \underline{\mathbb{P}}^1\,)\,.$$

We have

$$\mathrm{Id}_{\underline{\mathbb{P}}^1} - (p+q) = [\Delta_{\mathbb{P}^1}] - ([\mathbb{P}^1 \times \{\,\mathrm{pt}\,\}] + [\{\,\mathrm{pt}\,\} \times \mathbb{P}^1]) = [\mathrm{div}(\phi)]$$

where ϕ is the rational function on $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$([x_1 : x_2]\,,\, [y_1 : y_2]) \mapsto \frac{x_1 y_2 - x_2 y_1}{x_1 y_1}\,.$$

Hence, $\mathrm{Id}_{\underline{\mathbb{P}}^1} = p + q$ in $\mathrm{Hom}(\underline{\mathbb{P}}^1\,,\, \underline{\mathbb{P}}^1)$.

The motive $(\underline{\mathbb{P}}^1\,,\, q)$ is called the ***Tate motive*** and will be often denoted by $\mathbb{L}$.

Proposition 3.2.5. *With notation as in Example* 3.2.4*, we have canonical isomorphisms*

$$(\underline{\mathbb{P}}^1\,,\, p) \cong \mathbb{1}\,; \quad h(\mathbb{P}^1) \cong \mathbb{1} \oplus \mathbb{L}\,.$$

Proof. $(\underline{\mathbb{P}}^1\,,\,p)$ is the kernel of the projector

$$q : (\underline{\mathbb{P}}^1\,,\,\mathrm{Id}_{\underline{\mathbb{P}}^1}) \longrightarrow (\underline{\mathbb{P}}^1\,,\,\mathrm{Id}_{\underline{\mathbb{P}}^1})\,.$$

Hence, in the pseudo-abelian category $\mathfrak{CM}_k^+$, one has

$$h(\mathbb{P}^1) = (\underline{\mathbb{P}}^1\,,\,\mathrm{Id}_{\underline{\mathbb{P}}^1}) \cong (\underline{\mathbb{P}}^1\,,\,p) \oplus (\underline{\mathbb{P}}^1\,,\,q)\,.$$

It remains to show $\mathbb{1} = h(\mathrm{Spec}\ k) \cong (\underline{\mathbb{P}}^1\,,\,p)$. Indeed, the correspondence $[\{\,\mathrm{pt}\,\} \times \{\,\mathrm{pt}\,\}] \in \mathbf{Cor}(\mathrm{Spec}\ k,\, \mathbb{P}^1)$ gives a morphism

$$\rho := p \circ [\{\,\mathrm{pt}\,\} \times \{\,\mathrm{pt}\,\}] \circ [\{\,\mathrm{pt}\,\} \times \{\,\mathrm{pt}\,\}] = [\{\,\mathrm{pt}\,\} \times \{\,\mathrm{pt}\,\}]$$

from $\mathbb{1} = (\underline{\mathrm{Spec}\ k}\,,\,[\{\,\mathrm{pt}\,\} \times \{\,\mathrm{pt}\,\}])$ to $(\underline{\mathbb{P}}^1\,,\,p)$. On the other hand, the correspondence $[\mathbb{P}^1 \times \{\,\mathrm{pt}\,\}] \in \mathbf{Cor}(\mathbb{P}^1,\,\mathrm{Spec}\ k)$ provides a morphism

$$\tau := [\{\,\mathrm{pt}\,\} \times \{\,\mathrm{pt}\,\}] \circ [\mathbb{P}^1 \times \{\,\mathrm{pt}\,\}] \circ p = [\mathbb{P}^1 \times \{\,\mathrm{pt}\,\}]$$

from $(\underline{\mathbb{P}}^1\,,\,p)$ to $\mathbb{1}$. Now

$$\tau \circ \rho = [\{\,\mathrm{pt}\,\} \times \{\,\mathrm{pt}\,\}] = \mathrm{Id} :\ \ \mathbb{1} \longrightarrow \mathbb{1}$$

and

$$\rho \circ \tau = p = p \circ \mathrm{Id}_{\underline{\mathbb{P}}^1} \circ p = \mathrm{Id}_{(\underline{\mathbb{P}}^1,\,p)}\ :\quad (\underline{\mathbb{P}}^1\,,\,p) \longrightarrow (\underline{\mathbb{P}}^1\,,\,p)\,.$$

This proves the proposition. □

Clearly, if $X,\,Y \in \mathfrak{V}_k$ with X irreducible, then

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(h(X)\,,\,h(Y)) = \mathrm{CH}_{\dim X}(X \times Y)\,.$$

In particular,

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{1}\,,\,h(Y)) = \mathrm{Hom}_{\mathfrak{CM}_k^+}(h(\mathrm{Spec}\ k)\,,\,h(Y)) = \mathrm{CH}_0(Y)\,,$$

and

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(h(\mathbb{P}^1)\,,\,h(Y)) = \mathrm{CH}_1(\mathbb{P}^1 \times Y)\,.$$

Prop. 3.2.5 also implies that

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(h(\mathbb{P}^1))\,,\,h(Y)) = \mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{1}\,,\,h(Y)) \oplus \mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{L}\,,\,h(Y))$$

and we know from Thm. 2.4.5 that

$$\mathrm{CH}_1(\mathbb{P}^1 \times Y) = \mathrm{CH}_0(Y) \oplus \mathrm{CH}_1(Y)\,.$$

From this, we are reasonably convinced that

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{L}\,,\,h(Y)) = \mathrm{CH}_1(Y)\,.$$

For higher dimensional projective spaces, one has (cf. [21, §6, Formula (8)]):

$$h(\mathbb{P}^m) = \mathbb{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{\otimes m}\,,\quad \forall\ m \geq 1\,.$$

So it is not a surprising fact that

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{L}^{\otimes m}\,,\,h(Y)) = \mathrm{CH}_m(Y)\,,\quad \forall\, m \geq 1\,.$$

In like manner, we obtain

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(h(X)\,,\,\mathbb{1}) = \mathrm{CH}_{\dim X}(X) = \mathrm{CH}^0(X)\,,$$

and

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(h(X)\,,\ \mathbb{L}^{\otimes m}) = \mathrm{CH}^m(X)\,,\quad \forall\, m \geq 1\,.$$

Since

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{L}^{\otimes m}\,,\ \mathbb{1}) = \mathrm{CH}_m(\mathrm{Spec}\ k) = 0\,,\quad \forall\, m \geq 1\,,$$

we have for $m \geq 1$,

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{L}^{\otimes m}\,,\ \mathbb{L}) &= \mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{L}^{\otimes m}\,,\ h(\mathbb{P}^1)) \\ &= \mathrm{CH}_m(\mathbb{P}^1) = \begin{cases}\mathbb{Z} & \text{if } m = 1 \\ 0 & \text{if } m > 1\end{cases}\end{aligned}$$

As a matter of fact, we have in general

$$\mathrm{Hom}_{\mathfrak{CM}_k^+}(\mathbb{L}^{\otimes m}\,,\ \mathbb{L}^{\otimes n}) = \begin{cases}\mathbb{Z} & \text{if } m = n \\ 0 & \text{if } m \neq n\end{cases}\,,\quad \forall\, m,\, n \geq 0\,.$$

Definition 3.2.6. The category of ***Chow motives*** over k, denoted $\mathfrak{CM}_k$, is constructed as follows:

(i) the objects of $\mathfrak{CM}_k$ are couples $(M,\, m)$ where M is an object of $\mathfrak{CM}_k^+$ and $m \in \mathbb{Z}$;

(ii) the set of morphisms between two objects $(M,\, m)$ and $(N,\, n)$ is defined as

$$\mathrm{Hom}_{\mathfrak{CM}_k}((M,\, m)\,,\ (N,\, n)) := \varinjlim_{i \geq \max(-m,\, -n)} \mathrm{Hom}_{\mathfrak{CM}_k^+}(M \otimes \mathbb{L}^{\otimes(i+m)},\ N \otimes \mathbb{L}^{\otimes(i+n)})\,,$$

where the transition maps for the inductive system on the right are naturally obtained by applying the functor $-\otimes \mathbb{L}$.

We will often write $M(m)$ for the object $(M,\, m)$.

Morphisms between motives that come from varieties have explicit descriptions. For $X,\, Y \in \mathfrak{V}_k$, one has

$$\mathrm{Hom}_{\mathfrak{CM}_k}(h(X)(r)\,,\ h(Y)(s)) = \bigoplus_{i=1}^{n} \mathrm{CH}_{\dim X_i + r - s}(X_i \times Y)\,,\quad \forall\, r\,, s \in \mathbb{Z}$$

where $X_1\,, \ldots, X_n$ are all the irreducible components of X.

The category $\mathfrak{CM}_k$ is again additive. A motive $M \in \mathfrak{CM}_k$ is called ***split*** if it is isomorphic to a motive of the form $\oplus_{i=r}^{r}(\mathbb{1}\,,\, d_i)\,,\, d_i \in \mathbb{Z}$.

The next two results indicates somewhat links between Chow motives and Chow groups, that will be later useful for us. (See Appendix A for basic information about K-theory.)

Lemma 3.2.7 ([23, Lemma 1.4])**.** *Let $X \in \mathfrak{V}_k$ and suppose the Chow motive $h(X)$ is split. Then the product map*

$$\mathrm{CH}^p(X) \otimes K_q(k) \longrightarrow H^p(X,\, \mathscr{K}_{p+q})$$

is an isomorphism for any p, q.

Here for $X \in \mathfrak{V}_k$, the structure morphism $X \to \mathrm{Spec}\ k$ induces natural maps $K_q(k) \to K_q(X)$. We then have natural maps (cf. Remark A.3.4)

$$K_p(X) \otimes K_q(k) \longrightarrow K_p(X) \otimes K_q(X) \longrightarrow K_{p+q}(X)\,,$$

which induce product maps on the K-cohomology groups

$$H^i(X,\, \mathscr{K}_p) \otimes K_q(k) \longrightarrow H^i(X,\, \mathscr{K}_{p+q})\,.$$

The product map in Lemma 3.2.7 is obtained by identifying $\mathrm{CH}^p(X)$ with $H^p(X,\, \mathscr{K}_p)$.

Proposition 3.2.8 ([23, Prop. 1.5]). *Let X be an irreducible variety in $\mathfrak{V}_k$. Then the Chow motive $h(X)$ is split if and only if the following two conditions are satisfied:*

(i) *the Chow group* $\mathrm{CH}(X)$ *is free of finite rank over* $\mathbb{Z}$ *and the natural map* $\mathrm{CH}(X) \to \mathrm{CH}(X_L)$ *is an isomorphism for every field extension L/k;*

(ii) *the intersection pairing*

$$\mathrm{CH}^p(X) \otimes \mathrm{CH}_p(X) \longrightarrow \mathbb{Z}\,; \quad \alpha \otimes \beta \mapsto \deg(\alpha \cdot \beta)$$

is a perfect duality for every p.

Chapter 4

R-Equivalence and Zero-cycles on Tori

4.1 Some Reviews

The study of algebraic groups over not necessarily algebraically closed fields usually involves theories of group schemes or Galois descent. According to what is most convenient, we may employ both tools to a certain extent to make the exposition as clear and elementary as possible. The present section is thus aimed at a brief sketch of some basics in these spirits that may be helpful for our later discussions. For systematic exposition and detailed proofs, one may refer to monographs such as [14], [9] and [10].

4.1.1 Field of Definition and Galois Descent

In arithmetic geometry, one often hopes that a variety, probably also with various objects attached to it, defined over a larger field can be defined over a smaller field, and wants at the same time if possible that nice properties are preserved by this kind of descent. In this subsection, we give statements of useful facts in this respect. Basic references are [14, IV.4.8–4.9] and the very nice exposition of Galois descent in [18, §2].

Let k be a field, X, Y (algebraic) k-schemes, $\mathscr{F}$, $\mathscr{G}$ quasi-coherent $\mathscr{O}_X$-modules, and K/k a field extension. Let $\mathfrak{S}(X)$ be the set of all subschemes of X, and let $\Phi(\mathscr{F})$ be the set of all quasi-coherent submodules of $\mathscr{F}$.

For any subextension k'/k of K/k, we have canonical maps:

$$\begin{aligned} \Phi(\mathscr{F}\otimes_k k') &\longrightarrow \Phi(\mathscr{F}\otimes_k K)\,; \qquad \mathscr{H}\mapsto \pi^*\mathscr{H}\,,\\ \mathrm{Hom}(\mathscr{F}\otimes_k k',\,\mathscr{G}\otimes_k k') &\longrightarrow \mathrm{Hom}(\mathscr{F}\otimes_k K,\,\mathscr{G}\otimes_k K)\,; \qquad u\mapsto \pi^*(u)\,,\\ \mathfrak{S}(X_{k'}) &\longrightarrow \mathfrak{S}(X_K)\,; \qquad Z\mapsto \pi^{-1}(Z)=Z\times_k K\,,\\ \mathrm{Mor}_{k'}(X_{k'},\,Y_{k'}) &\longrightarrow \mathrm{Mor}_K(X_K,\,Y_K)\,; \qquad f\mapsto f\times \mathrm{Id}\,, \end{aligned} \tag{4.1}$$

where $\pi : X_K \to X_{k'}$ denotes the natural projection, $\mathscr{F}\otimes_k k'$ denotes the pull-back of $\mathscr{F}$ by the natural morphism $X_{k'}\to X$ and similarly for $\mathscr{F}\otimes_k K$ and so on. All the above maps are injective.

We say an object in $\Phi(\mathscr{F}\otimes_k K)$, resp. $\mathrm{Hom}(\mathscr{F}\otimes_k K,\,\mathscr{G}\otimes_k K)$, resp. $\mathfrak{S}(X_K)$, resp. $\mathrm{Mor}_K(X_K,\,Y_K)$, is defined over k', or k' is a ***field of definition*** of the corresponding object if it is in the image of the corresponding map in (4.1). If $\{X_\lambda\}$ is an open covering of X, such an object is defined over k' if and only if its restriction to each X_λ is defined over k'. Moreover, such an object always has a smallest field of definition in K/k. If $\widetilde{Z}$ is a closed subscheme of X_K, the smallest field of definition of $\widetilde{Z}$ is finitely generated over k. The same statement holds for any $\widetilde{f} \in \mathrm{Mor}_K(X_K, Y_K)$. If $\mathscr{F}$ and $\mathscr{G}$ are coherent, the smallest field of definition of an object in $\Phi(\mathscr{F} \otimes_k K)$, or in $\mathrm{Hom}(\mathscr{F} \otimes_k K, \mathscr{G} \otimes_k K)$, is finitely generated over k.

If a closed subscheme $\widetilde{Z}$ of X_K is defined over a subfield k', the open subset $\widetilde{U} := X_K \setminus \widetilde{Z}$ is also defined over k'. When K/k is an algebraic extension, the smallest field of definition of any closed subscheme $\widetilde{Z}$ of X_K is a finite separable extension of k. So any quasi-projective K-scheme $\widetilde{X}$ can be defined over a subfield k' which is finitely generated over k, and if moreover K/k is an algebraic extension, $\widetilde{X}$ can be defined over a finite separable extension of k.

Let L/k be a Galois extension and $\mathfrak{g} = \mathrm{Gal}(L/k)$ the Galois group. For any $\sigma \in \mathfrak{g}$, denote by $S(\sigma)$ the k-isomorphism Spec $L \to$ Spec L given by the isomorphism $\sigma^{-1} : L \to L$. Let $x_\sigma = \mathrm{Id} \times S(\sigma) : X_L \to X_L$ denote the natural morphism determined by the fibre square

$$\begin{array}{ccc} X_L & \xrightarrow{x_\sigma} & X_L \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, L & \xrightarrow{S(\sigma)} & \mathrm{Spec}\, L \end{array}$$

This x_σ is a k-automorphism on X_L. So $\mathfrak{g}$ operates as k-automorphisms of X_L and many objects attached to X_L thus admit a natural $\mathfrak{g}$-action. For example, the Chow groups $\mathrm{CH}^p(X_L)$ have a natural $\mathfrak{g}$-action, and since a closed subscheme can be defined over a finite subextension of L/k, this action is continuous for the discrete topology on $\mathrm{CH}^p(X_L)$ and the natural profinite topology on $\mathfrak{g}$.

If Y is another k-scheme, the set $\mathrm{Mor}_L(X_L, Y_L)$ has a $\mathfrak{g}$-action given by

$$\sigma \cdot f := y_\sigma \circ f \circ x_\sigma^{-1}\,; \quad \forall\, f \in \mathrm{Mor}_L(X_L, Y_L)\,,\ \sigma \in \mathfrak{g}\,.$$

Since any $f \in \mathrm{Mor}_L(X_L, Y_L)$ can be defined over a finite subextension of L/k, this $\mathfrak{g}$-action is also continuous. Furthermore, $f \in \mathrm{Mor}_L(X_L, Y_L)$ is defined over k if and only if $\sigma \cdot f = f$ for all $\sigma \in \mathfrak{g}$. In particular, for any k-algebra A, one has $X(A) = X(A \otimes_k L)^{\mathfrak{g}}$.

Here is a good place to introduce the following definitions.

Definition 4.1.1. Let Γ be a profinite group. By a ***Γ-set*** we mean a set S equipped with a left Γ-action that is continuous for the discrete topology on S and the profinite topology on Γ. A ***Γ-group*** is a discrete group equipped with a continuous left action of Γ which is compatible with the group structure of A, i.e. $\gamma(ab) = \gamma(a)\gamma(b)$ for all $\gamma \in \Gamma\,, a\,, b \in A$. A ***$\Gamma$-module*** is a commutative Γ-group.

So in the case discussed above, the Chow groups $\mathrm{CH}^p(X_L)$ are $\mathfrak{g}$-modules and $\mathrm{Mor}_L(X_L, Y_L)$ and $X(A \otimes_k L)$, $A \in \mathfrak{Alg}_{/k}$ are Γ-sets.

Let $\mathscr{M}$ be a quasi-coherent (resp. coherent, resp. locally free of rank r) sheaf on X_L. Suppose $\mathscr{M}$ is defined over a finite subextension of L/k so that we may assume L/k itself is finite. The so-called "Galois descent" for sheaves answers the question when $\mathscr{M}$ can be defined by a quasi-coherent (resp. coherent, resp. locally free of rank r) sheaf over k. A sufficient and necessary condition is that there exists a system $(\iota_\sigma)_{\sigma\in\mathfrak{g}}$ of isomorphisms $\iota_\sigma : x_\sigma^*\mathscr{M} \to \mathscr{M}$ satisfying $\iota_\tau \circ x_\tau^*(\iota_\sigma) = \iota_{\sigma\tau}$ for all $\sigma,\, \tau \in \mathfrak{g}$.

Now suppose given a quasi-projective scheme $\widetilde{X}$ over L. As it can be defined over a finite subextension of L/k, we may assume L/k is already finite. The Galois descent for schemes asserts that $\widetilde{X}$ can be defined over k if and only if there is a system $(T_\sigma)_{\sigma\in\mathfrak{g}}$ of k-isomorphisms $\widetilde{X}\to\widetilde{X}$ such that for every $\sigma\in\mathfrak{g}$ the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{T_\sigma} & \widetilde{X} \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{Spec}\, L & \xrightarrow{S(\sigma)} & \mathrm{Spec}\, L \end{array}$$

Basically, as the question is local, descent for sheaves or schemes reduces to descent of purely algebraic structures. To be precise, suppose L/k is finite and $\widetilde{A}$ is a vector space (resp. an algebra) over L. Then there is a vector space (resp. an algebra) A over k such that $\widetilde{A}\cong A\otimes_k L$ if and only if $\widetilde{A}$ is equipped with a semi-linear $\mathfrak{g}$-action, i.e., $\widetilde{A}$ admits a $\mathfrak{g}$-action such that for each $\sigma\in\mathfrak{g}$, the map $\widetilde{A}\to\widetilde{A}$ given by $a\mapsto\sigma(a)$ satisfies

$$\sigma(\alpha a)=\sigma(\alpha)\sigma(a)\,,\qquad \forall\,\alpha\in L,\, a\in\widetilde{A}\,.$$

4.1.2 Algebraic Groups

Let k be a field. For our use, an ***algebraic k-group*** (or simply a ***k-group***) is an algebraic k-group scheme, where by a ***k-group scheme*** we mean a k-scheme X for which the functor of points $X(-):=\mathrm{Mor}_k(-,\,X):\mathfrak{Alg}_{/k}\to\mathfrak{Set}$ is given a factorization through the category $\mathfrak{Group}$ of groups.

Let $G\,,H$ be algebraic k-groups. A ***homomorphism of algebraic groups*** from H to G is a morphism $\varphi:H\to G$ of k-schemes such that for each $A\in\mathfrak{Alg}_{/k}$, the natural map $\varphi_A:H(A)\to G(A)$ is actually a group homomorphism. In contrast to the set $\mathrm{Mor}_k(H,\,G)$ of morphisms of schemes, the set of homomorphisms of algebraic groups from H to G will be denoted by $\mathrm{Hom}_k(H,\,G)$.

Let G be an algebraic k-group. An ***algebraic subgroup*** of G is a homomorphism $\varphi:H\to G$ such that $\varphi_A:H(A)\to G(A)$ is an injective group homomorphism for every $A\in\mathfrak{Alg}_{/k}$. If φ is moreover a closed immersion, we say H is a closed subgroup of G.

In what follows, algebraic subgroups are tacitly assumed to be closed subgroups unless otherwise stated.

Any group scheme over a field is separated, so algebraic k-groups are k-varieties. Group-theoretic and algebraic-geometric notions may be spoken of for algebraic groups. For example, a k-group G is called ***commutative*** if for every $A\in\mathfrak{Alg}_{/k}$, $G(A)$ is a commutative abstract group, an algebraic subgroup H is called ***normal*** in G if for every $A\in\mathfrak{Alg}_{/k}$, $H(A)$ is normal in $G(A)$ as abstract subgroups, and G is called ***smooth*** if it is smooth as a k-variety.

Examples 4.1.2. Let k be a field.

(1) Spec k is an algebraic k-group which takes any $A\in\mathfrak{Alg}_{/k}$ to the trivial group. We will usually denote this k-group by 1. Note that for any k-group G, there are two natural homomorphisms $G\to 1$ and $1\to G$; the first one is given by the structural morphism $G\to\mathrm{Spec}\,k$ and the second one is given by the identity element of the group $G(k)$.

(2) Spec $k[t]$ is an algebraic k-group whose functor of points is given by $A\in\mathfrak{Alg}_{/k}\mapsto(A,\,+)\in\mathfrak{Group}$, where $(A\,,+)$ means the underly additive group of A. The standard notation for this k-group is $\mathbb{G}_{a,\,k}$, or simply $\mathbb{G}_a$ when the ground field is clear from the context, and it is called the ***additive group*** over k.

(3) Spec $k[t,\,t^{-1}]$ is an algebraic k-group with functor of points given by $A\mapsto A^*$. This k-group is called the ***multiplicative group*** over k, and the standard notation is $\mathbb{G}_{m\,,k}$, or simply $\mathbb{G}_m$.

(4) The n-th ***general linear group*** over k is the algebraic k-group $\mathbf{GL}_n$ determined by the k-scheme Spec $(k[t, x_{ij}]_{1\le i,\, j\le n}/(t\det(x_{ij})-1))$. It represents the functor

$$A \mapsto \mathrm{GL}_n(A) = \{\, g\in M_n(A)\,|\, \det(g)\in A^*\,\}\,.$$

Note that $\mathbf{GL}_1 = \mathbb{G}_m$.

(5) Assume here $\mathrm{char}\, k = 0$. For any $n\ge 1$, Spec $(k[t]/(t^n-1))$ is a k-group, usually denoted μ_n, such that

$$\mu_n(A) = \{\, a\in A\,|\, a^n = 1\,\}\,,\quad \forall\, A\in \mathfrak{Alg}_{/k}\,.$$

We call μ_n the ***group of n-th roots of unity***. Since k is assumed to have characteristic 0, μ_n is smooth.

(6) Let G be a finite abstract group. Define $\mathcal{G} := \coprod_{g\in G} S_g$ where $S_g = \mathrm{Spec}\, k$ for all $g\in G$. Then for each $A\in\mathfrak{Alg}_{/k}$, $\mathcal{G}(A)$ is the set of locally constant functions from Spec A to G. So $\mathcal{G}$ becomes an algebraic k-group and $\mathcal{G}(K) = G$ for any field extension K/k. We call this $\mathcal{G}$ the ***constant group*** associated to G. The constant groups associated to the groups $\mathbb{Z}/n\mathbb{Z}$ will be simply written $\mathbb{Z}/n\mathbb{Z}$.

If G and G' are algebraic k-groups, then $G\times_k G'$ also has a natural structure of algebraic k-groups, and for any $A\in\mathfrak{Alg}_{/k}$,

$$(G\times_k G')(A) = G(A)\times G'(A)\,.$$

More generally, given homomorphisms of algebraic groups $\varphi : G\to H$ and $\psi : G'\to H$, the fibre product $G\times_H G'$ is an algebraic group and for $A\in\mathfrak{Alg}_{/k}$,

$$(G\times_H G')(A) = G(A)\times_{H(A)} G'(A) = \{\,(a,\, b)\in G(A)\times G'(A)\,|\,\varphi(a) = \psi(b)\,\}\,.$$

For a homomorphism of k-groups $\varphi : G\to H$, the kernel of φ exists as an algebraic group. In fact, this kernel is represented by the fibre product $G' = G\times_H 1$ of $\varphi : G\to H$ and $e : 1\to H$, where e is the natural homomorphism defined by the identity element of $H(k)$. The natural projection $\alpha : G'\to G$ identifies for each $A\in\mathfrak{Alg}_{/k}$ the group $G'(A)$ with the kernel of the group homomorphism $\varphi_A : G(A)\to H(A)$. This kernel is a closed subgroup of G, and is smooth if φ is smooth.

Definition 4.1.3. A sequence of homomorphisms of algebraic k-groups

$$1\longrightarrow G'\overset{\alpha}{\longrightarrow} G\overset{\beta}{\longrightarrow} G''\longrightarrow 1$$

is said to be ***exact*** if α is a closed immersion identifying G' with the kernel of β and β is a faithfully flat morphism. (This implies that for an algebraic closure k^{ac} of k, the corresponding sequence

$$1\longrightarrow G'(k^{\mathrm{ac}})\longrightarrow G(k^{\mathrm{ac}})\longrightarrow G''(k^{\mathrm{ac}})\longrightarrow 1$$

is an exact sequence of abstract groups. When k has characteristic 0, this condition is also sufficient for the exactness of the above sequence of algebraic groups.) The group G'' is called the ***quotient*** of G by G'.

Although much more subtle, the existence of quotients by normal subgroups in the category of k-groups can be proved. In other words, if $\alpha : N\to G$ is a closed normal subgroup of G, there is a homomorphism $\beta : G\to H$ such that the sequence

$$1\longrightarrow N\overset{\alpha}{\longrightarrow} G\overset{\beta}{\longrightarrow} H\longrightarrow 1$$

is exact.

A k-group G is called a ***linear algebraic group*** over k if it is a smooth closed subgroup of some $\mathbf{GL}_n$. This is equivalent to saying that G is a smooth affine k-group. All the k-groups given in Examples 4.1.2 are linear. The additive group $\mathbb{G}_a$ can be embedded into $\mathbf{GL}_2$ via the matrix representation:

$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

A ***algebraic k-torus*** (or simply a ***k-torus***) is a k-group T such that $\overline{T} \cong \mathbb{G}^n_{m,\bar{k}}$ as $\bar{k}$-groups. In particular, a torus is a linear algebraic group.

4.1.3 Torsors and Cohomology

The objective of this section is to review basic notions and facts on torsors. For a friendly introduction to this subject, we refer to [15].

Let k be a field, $\bar{k}$ a fixed separable closure of k and $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ the Galois group. Let X be a k-variety and G a smooth algebraic k-group. A ***torsor*** under G over X (or a G-torsor over X) is a k-variety Y, equipped with a surjective flat k-morphism $f : Y \to X$ and a right action (defined over k) of G on it: $(y, g) \mapsto y \cdot g$ such that the morphism

$$Y \times_k G \longrightarrow Y \times_X Y\,, \quad (y,\, g) \mapsto (y,\, y \cdot g)$$

is an isomorphism of k-varieties. This means that the action of G on Y preserves the fibres of f, and the action on each geometric fibre is faithful and transitive: for any $y_1,\, y_2 \in Y(\bar{k})$ with $f(y_1) = f(y_2)$, there exists a unique $g \in G(\bar{k})$ such that $y_1 \cdot g = y_2$. A torsor over Spec k will be called simply a torsor over k.

Two G-torsors $f : Y \to X$ and $f' : Y' \to X$ are said to be ***isomorphic*** if there is an isomorphism of k-varieties $\varphi : Y \to Y'$ such that $f' \circ \varphi = f$ and $\varphi(yg) = \varphi(y)g$ for all $y \in Y(\bar{k})\,, g \in G(\bar{k})$.

Examples 4.1.4. (1) Let X be a k-variety and G a smooth k-group. $Y = X \times_k G$ together with the natural projection $Y \to X$ is a G-torsor over X. The G-action is given by $(x,\, g) \cdot g' := (x,\, gg')$. A torsor isomorphic to $X \times_k G$ is called ***trivial***. A trivial G-torsor over k is isomorphic to G.

(2) Assume $\mathrm{char}\, k \neq 2$. Let $X = \mathbb{A}^1 \setminus \{0\}$, and let $Y \subseteq \mathbb{A}^2$ be the subvariety defined by $y^2 = x\,, x \neq 0$. The projection $(x,\, y) \mapsto x$ makes Y into a torsor over X under the finite constant group $G = \mathbb{Z}/2\mathbb{Z}$. The action of the nontrivial element of G is given by $(x,\, y) \mapsto (x,\, -y)$.

(3) Let $X = \mathrm{Spec}\; k$. A torsor Y over k is simply a k-variety, equipped with a faithful and transitive action of $G(\bar{k})$ on $Y(\bar{k})$ which is compatible with the left action of $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$, namely, $\gamma(yg) = \gamma(y)\gamma(g)$ for all $y \in Y(\bar{k})$, $g \in G(\bar{k})$ and $\gamma \in \mathfrak{g}$. (The last condition means that the action of G is defined over k, as may be seen by using Galois descent.)

If G is finite (meaning that $G(\bar{k})$ is finite), then Y is a finite k-variety since there is a faithful and transitive action of $G(\bar{k})$ on $Y(\bar{k})$. So $Y \cong \mathrm{Spec}\; A$ for some finite-dimensional k-algebra A. The set $Y(\bar{k})$ is just the set $\mathrm{Hom}_{\mathfrak{Alg}_{/k}}(A,\, \bar{k})$. Thus $Y = \mathrm{Spec}\; A$ is equipped with a G-torsor structure as soon as $\mathrm{Hom}_{\mathfrak{Alg}_{/k}}(A,\, k)$ is given a faithful and transitive action of $G(\bar{k})$ which is compatible with the left action of $\mathfrak{g}$.

For a specific example, consider a finite Galois extension L/k. Let G be the finite constant k-group associated to $\mathrm{Gal}(L/k)$, i.e., $G(\bar{k}) = \mathrm{Gal}(L/k)$ and the $\mathfrak{g}$-action on $G(\bar{k})$ is trivial. Then $Y = \mathrm{Spec}\; L$ is a G-torsor over k. Indeed, $Y(\bar{k}) = \mathrm{Hom}_{\mathfrak{Alg}_{/k}}(L,\, \bar{k}) = \mathrm{Gal}(L/k)$ and we can define the action of $G(\bar{k})$ on $Y(\bar{k}) = \mathrm{Hom}_{\mathfrak{Alg}_{/k}}(L,\, \bar{k})$ by the product in the group $\mathrm{Gal}(L/k)$. For any $\gamma \in \mathfrak{g}$ and

$y \in Y(\bar{k}) = \mathrm{Hom}_{\mathfrak{Alg}_{/k}}(L, \bar{k})$, $\gamma(y) = \widetilde{\gamma}y$, where $\widetilde{\gamma}$ is the image of γ in $\mathrm{Gal}(L/k)$. Thus

$$\gamma(y \cdot g) = \widetilde{\gamma}yg = \gamma(y)g = \gamma(y)\gamma(g)\,, \quad \forall\, y \in Y(\bar{k}),\ g \in G(\bar{k}),\ \gamma \in \mathfrak{g}$$

where the last equality holds because $\mathfrak{g}$ acts trivially on $G(\bar{k})$.

(4) Take $k = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt[3]{2})$. Fix a primitive cubic root of unity ξ in $\overline{\mathbb{Q}}$. Then $\mathrm{Hom}_{\mathfrak{Alg}_{/\mathbb{Q}}}(L, \overline{\mathbb{Q}}) = \{\, \sigma_0, \sigma_1\,, \sigma_2 \,\}$ where σ_i sends $\sqrt[3]{2}$ to $\xi^i \sqrt[3]{2}$. The cyclic group C_3 of order 3 acts faithfully and transitively on $\mathrm{Hom}_{\mathfrak{Alg}_{/\mathbb{Q}}}(L, \overline{\mathbb{Q}})$. But to obtain a Galois-compatible action, C_3 must be equipped with the nontrivial action of $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$. So Spec L is a torsor over $\mathbb{Q}$ under the finite nonconstant group μ_3, but it is not a torsor under the constant group $\mathbb{Z}/3\mathbb{Z}$.

A G-torsor Y over k is trivial if and only if $Y(k) \neq \emptyset$. Indeed, if $Y(k) \neq \emptyset$, we can pick a point $y_0 \in Y(k)$. Then G is isomorphic to Y via the morphism $g \mapsto y_0 g$. So any torsor over an algebraically closed field is trivial.

More generally, a G-torsor Y over a variety X is trivial if and only if the structural morphism $f : Y \to X$ has a section over k. Indeed, if s is a section of f, then the trivial torsor $X \times_k G$ is isomorphic to Y via the morphism $(x, g) \mapsto s(x)g$.

We want to relate torsors to cohomology sets. For this purpose, let's first review some basics on non-abelian group cohomology. For more details, see for example [29].

Let Γ be a profinite group and let A be a Γ-group. We will often denote by ${}^{\gamma}a$ the element $\gamma(a)$ for $\gamma \in \Gamma\,, a \in A$. Define the set

$$H^0(\Gamma, A) := A^{\Gamma} = \{\, a \in A \,|\, {}^{\gamma}a = a\,, \forall\, \gamma \in \Gamma \,\}\,.$$

A ***cocycle*** is a continuous map $c : \Gamma \to A$, $\gamma \mapsto c_\gamma$ such that $c_{\gamma_1\gamma_2} = c_{\gamma_1}{}^{\gamma_1}c_{\gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$. Let $Z^1(\Gamma, A)$ denote the set of cocyles. For a given element $a \in A$, the map $\gamma \mapsto c_\gamma := a^{-1}\,{}^{\gamma}a$ defines a cocycle. Such a cocycle is said to be ***trivial***. We define an equivalence relation on $Z^1(\Gamma, A)$ by setting

$$c \sim c' \iff \text{there exists } a \in A \text{ such that } c'_\gamma = a^{-1} c_\gamma {}^{\gamma}a\,, \quad \forall\, \gamma \in \Gamma\,.$$

The ***first cohomology set*** $H^1(\Gamma, A)$ is by definition the quotient of $Z^1(\Gamma\,, A)$ divided by this equivalence relation. The class of trivial cocycles forms a distinguished element of $H^1(\Gamma, A)$. When A is abelian, namely a Γ-module, the sets $H^i(\Gamma, A)\,, i = 0, 1$ have naturally an abelian group structure.

Now we take $\Gamma = \mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ and $A = G(\bar{k})$, where as before G is a smooth k-group and $\bar{k}$ is a fixed separable closure of k. We write $H^i(k, G)$ (resp. $Z^1(k, G)$) in place of $H^i(\mathfrak{g}, G(\bar{k}))$ for $i = 0, 1$ (resp. $Z^1(\mathfrak{g}, G(\bar{k}))$) .

The cohomology set $H^1(k, G)$ has the following functorial behaviours:

(1) A field extension $k_1 \subseteq k_2$ induces a continuous map $\mathrm{Gal}(\,\overline{k}_2/k_2) \to \mathrm{Gal}(\,\overline{k}_1/k_2)$. Since $G(\,\overline{k}_1) \subseteq G(\,\overline{k}_2)$, we obtain a map $Z^1(k_1, G) \longrightarrow Z^1(k_2, G)$, which is compatible with the equivalence relation defining H^1. Hence the field extension $k_1 \subseteq k_2$ induces a map of pointed sets $H^1(k_1, G) \to H^1(k_2, G)$.

(2) If $G \to G'$ is a homomorphism of k-groups, then an element of $Z^1(k, G)$ can be pushed to $Z^1(k, G')$. So there is an induced map $H^1(k, G) \to H^1(k, G')$.

For torsors there are similar functorial behaviours.

(1) Let $k \subseteq k'$ be a field extension and let Y be a G-torsor over k. Then $Y' := Y \times_k k'$ is a torsor under $G' := G \times_k k'$ over k'.

(2) Let $G \to G'$ be a homomorphism of k-groups. Although more subtle, there is a natural way to associate to each G-torsor Y over k a G'-torsor Y' over k. When $G' = G/H$ is the quotient of G by a normal subgroup H, then Y' is just the quotient of Y by the action of H.

Now we are ready to relate torsors over k to cohomology sets. Let $\mathcal{T}_k^G$ be the set of isomorphism classes of G-torsors over k. The trivial torsors define a distinguished element of $\mathcal{T}_k^G$. Given a G-torsor Y over k, and fix an element $y_0 \in Y(\bar{k})$. For any $\gamma \in \mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$, there exists a unique $c_\gamma \in G(\bar{k})$ such that $\gamma(y_0) = y_0 c_\gamma$. It's easy to check that c is a cocycle and that if one replaces y_0 by $y_0' = y_0 g$ with $g \in G(\bar{k})$, then c_γ is replaced by $g^{-1} c_\gamma {}^\gamma g$. Thus, the class $[c]$ of c in $H^1(k, G)$ does not depend on y_0. If $\varphi : Y \to Y'$ is an isomorphism of torsors, then the equality $\gamma(y_0) = y_0 c_\gamma$ implies $\gamma(\varphi(y_0)) = \varphi(y_0) c_\gamma$, so the class $[c] \in H^1(k, G)$ does not change when Y is replaced by any other isomorphic torsor. We obtain in this way a well-defined map $\lambda : \mathcal{T}_k^G \longrightarrow H^1(k, G)$.

Theorem 4.1.5. *With notation as above, the map $\lambda : \mathcal{T}_k^G \to H^1(k, G)$ is an isomorphism of pointed sets that is functorial in k and G.*

For torsors over a general variety X, there is an analogous result in which the étale cohomology set $H^1_{\acute{e}t}(X, G)$ has to be introduced as a generalization of $H^1(k, G)$ and some additional assumptions (which are satisfied if G is a linear algebraic group) are needed.

Theorem 4.1.6. *Let G be a linear k-group and let $\mathcal{T}_X^G$ be the set of isomorphism classes of G-torsors over a k-variety X. Then there is an isomorphism of pointed sets $\mathcal{T}_X^G \cong H^1_{\acute{e}t}(X, G)$ that is functorial in X and G.*

For a proof of Thms. 4.1.5 and 4.1.6, see for example [25, p.123, Prop. 4.6].

Standard facts on cohomology sets include the following.

Theorem 4.1.7. *Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of smooth k-groups. Then there is an exact sequence of pointed sets:*

$$1 \to G'(k) \to G(k) \to G''(k) \to H^1(k, G') \to H^1(k, G) \to H^1(k, G'') .$$

If G', G, G'' are commutative, the above sequence is an exact sequence of abelian groups. Similar results hold for the étale cohomology.

Proof. See [25, p.122, Prop. 4.5]. □

Theorem 4.1.8 (***Hilbert's Theorem*** **90**)**.** *For any field k, one has*

$$H^1(k, \mathbb{G}_m) = 0 .$$

More generally, $H^1_{\acute{e}t}(A, \mathbb{G}_m) = 0$ for any local ring A.

Proof. See [25, p.124, Lemma 4.10]. □

4.1.4 The Weil Restriction

Let F/k be a field extension. The usual base change procedure enables us to extend a k-scheme to an F-scheme. The so-called "***restriction of scalars***", introduced by Weil in [35], is a kind of operation in the opposite direction. For simplicity, we restricted here to the case where F/k is a finite separable extension.

Let X be an (algebraic) F-scheme. It gives a functor $X(-) : \mathfrak{Alg}_{/F} \longrightarrow \mathfrak{Set}$. By composing it with the tensor product functor $- \otimes_k F : \mathfrak{Alg}_{/k} \longrightarrow \mathfrak{Alg}_{/F}$, we obtain a functor

$$R_{F/k}X \;:\; \mathfrak{Alg}_{/k} \longrightarrow \mathfrak{Set}\,; \qquad A \mapsto X(A \otimes_k F) .$$

It often happens that this functor is representable by a k-scheme, namely, there is a k-scheme X' such that $R_{F/k}X \cong X'(-)$ as functors from $\mathfrak{Alg}_{/k}$ to $\mathfrak{Set}$. In that case, the k-scheme which represents this functor is also denoted by $R_{F/k}X$ and is called the ***Weil restriction***, or the ***restriction of scalars***, of X from F to k. It turns out that the scheme $R_{F/k}X$ always exists for any quasi-projective F-scheme X. Here let us work in a bit more details only in the affine case.

Suppose

$$X = \text{Spec } B\,, \quad \text{with} \quad B = F[x_1, \ldots, x_n]/(f_1, \ldots, f_m)\,, \quad f_i \in F[x_1, \ldots, x_n]\,.$$

For any $A \in \mathfrak{Alg}_{/k}$,

$$(R_{F/k}X)(A) = \{\, (\alpha_1, \ldots, \alpha_n) \in (A \otimes_k F)^n \mid f_i(\alpha_1, \ldots, \alpha_n) = 0\,,\ \forall\, i\,\}\,.$$

Fix a basis $v_1, \ldots, v_r$ for F as a k-module, and write

$$\alpha_l = \sum_{j=1}^{r} \beta_{lj} v_j\,, \quad \beta_{lj} \in A\,.$$

Then $f_i(\alpha_1, \ldots, \alpha_n)$ can be developed into the following form:

$$f_i(\alpha_1, \ldots, \alpha_n) = \sum_{j=1}^{r} \phi_{ij}(\beta_{lj}) v_j$$

where ϕ_{ij} are polynomials with coefficients in k in the variables β_{lj}. Hence, the functor $R_{F/k}X$ may be represented by the k-scheme Spec $(k[\beta_{lj}]/(\phi_{ij}))$.

In what follows, when we write down $R_{F/k}X$ we will always assume that it can be represented by a scheme.

Some more properties of the Weil restriction are listed below.

Let X be an F-scheme and F/k a finite separable field extension.

(1) There is an isomorphism

$$\text{Mor}_k(-\,,\, R_{F/k}X) \cong \text{Mor}_F((-) \times_k F\,,\, X)$$

of functors from k-schemes to sets.

(2) For any quasi-projective F-varieties X_1, X_2,

$$R_{F/k}(X_1 \times_F X_2) = (R_{F/k}X_1) \times_k (R_{F/k}X_2)\,.$$

(3) For a morphism $u : X \to Y$ of F-schemes, there is a natural k-morphism $R_{F/k}(u) : R_{F/k}X \longrightarrow R_{F/k}Y$.

(4) If X is affine, so is $R_{F/k}X$; if X is smooth, so is $R_{F/k}X$. $R_{F/k}\mathbb{A}^m_F = \mathbb{A}^{mn}_k$ with $n = [F : k]$. $\dim R_{F/k}X = [F : k] \dim X$.

(5) If U is an open subset of X, then $R_{F/k}U$ is an open subset of $R_{F/k}X$.

(6) For any field extension L/k, writing $L \otimes_k F = \prod_{i=1}^{s} M_i$ as a product of fields, there is a canonical isomorphism of L-schemes:

$$(R_{F/k}X) \times_k L \cong \prod_{i=1}^{s} R_{M_i/L}(X \times_F M_i)\,.$$

(7) If G is an F-group, then $R_{F/k}G$ is a k-group; G is connected if and only if $R_{F/k}G$ is connected; if G is commutative, then so is $R_{F/k}G$.

(8) If T is a F-torus, then $R_{F/k}T$ is k-tours and $(R_{F/k}T) \times_k \bar{k} \cong T_1 \times_k \cdots \times_k T_n$ with $n = [F : k]$.

(9) If $1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$ is an exact sequence of F-groups, then the corresponding sequence of k-groups

$$1 \longrightarrow R_{F/k}G' \longrightarrow R_{F/k}G \longrightarrow R_{F/k}G'' \longrightarrow 1$$

is also exact.

(10) (Shapiro's Lemma.) For any F-group G, one has $H^1(k,\, R_{F/k}G) = H^1(F,\, G)$.

4.1.5 Tori and Lattices

As before, let k be a field and $\bar{k}$ a fixed separable closure of k and $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ the Galois group. For a k-torus T, define its group of ***characters*** to be $\hat{T} := \mathrm{Hom}_{\bar{k}}(\overline{T}\,,\,\mathbb{G}_{m,\,\bar{k}})$. The Galois group $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ acts continuously on $\hat{T}$ and thus makes it into a $\mathfrak{g}$-module. As an abelian group, we have

$$\hat{T} = \mathrm{Hom}_{\bar{k}}(\overline{T},\,\mathbb{G}_{m\,,\bar{k}}) = \mathrm{Hom}_{\bar{k}}(\mathbb{G}^n_{m,\,\bar{k}},\,\mathbb{G}_{m\,,\bar{k}}) \cong \mathbb{Z}^n\,.$$

So $\hat{T}$ is free of finite rank as a $\mathbb{Z}$-module. We call such a $\mathfrak{g}$-module a $\mathfrak{g}$-***lattice***. The precise definition is as follows.

Definition 4.1.9. Let Γ be a profinite group. By a Γ-***lattice*** we mean a Γ-module M that is free of finite rank as a $\mathbb{Z}$-module. The ***dual*** of a Γ-lattice M is the lattice $M^0 := \mathrm{Hom}_{\mathbb{Z}}(M\,,\,\mathbb{Z})$ with the Γ-action given by

$$(\gamma\cdot f)(m) := f(\gamma^{-1}m)\,,\quad \forall\,\gamma\in\Gamma,\ f\in M^0\,,\ \ m\in M\,.$$

A remarkable fact is that the functor $T\mapsto \hat{T}$ from the category of k-tori to the category of $\mathfrak{g}$-lattices is an anti-equivalence. So up to isomorphism there is a unique k-torus T°, called the ***dual torus*** of T, whose character group is the dual of $\hat{T}$.

Now let F/k be a finite separable extension contained in $\bar{k}/k$. Consider the k-tours $T = R_{F/k}\mathbb{G}_m$. The $\mathfrak{g}$-module $\hat{T}$ is isomorphic to $\mathbb{Z}[\mathfrak{g}/\mathfrak{h}]$, where $\mathfrak{h} = \mathrm{Gal}(\bar{k}/F)$. There is an surjective homomorphism of $\mathfrak{g}$-modules, called the ***augmentation map***, given by

$$\varepsilon = \varepsilon_{\mathfrak{g}/\mathfrak{h}}\ :\ \mathbb{Z}[\mathfrak{g}/\mathfrak{h}] \longrightarrow \mathbb{Z}\,;\qquad \varepsilon\left(\sum n_i e_i\right) = \sum n_i\,.$$

where (e_i) is a fixed set of representatives for $\mathfrak{g}/\mathfrak{h}$ and $\mathbb{Z}$ is regarded as a trivial $\mathfrak{g}$-lattice. Let $I_{\mathfrak{g}/\mathfrak{h}} = \mathrm{Ker}\,(\varepsilon_{\mathfrak{g}/\mathfrak{h}})$. One has the exact sequence of $\mathfrak{g}$-lattices

$$0\longrightarrow I_{\mathfrak{g}/\mathfrak{h}} \longrightarrow \mathbb{Z}[\mathfrak{g}/\mathfrak{h}] \stackrel{\varepsilon}{\longrightarrow} \mathbb{Z}\longrightarrow 0\,. \tag{4.2}$$

Passing to dual lattices, we get an exact sequence

$$0\longrightarrow \mathbb{Z} \stackrel{\varepsilon^0}{\longrightarrow} \mathbb{Z}[\mathfrak{g}/\mathfrak{h}] \longrightarrow J_{\mathfrak{g}/\mathfrak{h}} \longrightarrow 0$$

where $J_{\mathfrak{g}/\mathfrak{h}} = I^0_{\mathfrak{g}/\mathfrak{h}}$ and $\varepsilon^0(1) = e_1+\cdots+e_n$ with $n = [F:k]$. These two exact sequences induce exact sequences of k-tori

$$1\longrightarrow \mathbb{G}_{m,\,k}\longrightarrow T = R_{F/k}\mathbb{G}_m \longrightarrow T_1\longrightarrow 1$$

and

$$1\longrightarrow T_2\longrightarrow T = R_{F/k}\mathbb{G}_m \stackrel{N}{\longrightarrow} \mathbb{G}_{m,\,k}\longrightarrow 1\,.$$

The torus $T = R_{F/k}\mathbb{G}_m$ is an open subset of $R_{F/k}\mathbb{A}^1 = \mathbb{A}^n_k$ and thus T_1 can be embedded as an open subset of the projective space $\mathbb{P}^{n-1}_k$. Hence T_1 is rational over k. From this point of view, the torus T_2 is much more interesting. For each k-algebra A, the natural map

$$N\ :\ R_{F/k}\mathbb{G}_m(A) = (A\otimes_k F)^* \longrightarrow \mathbb{G}_m(A) = A^*$$

is the usual norm map, and $T_2(A)$ is equal to the kernel of this map. Fix a k-basis $v_1,\dots,v_n$ for F/k, there is a universal polynomial $N(x_1,\dots,x_n)\in k[x_1,\dots,x_n]$, which is homogeneous of degree n, such that

$$N\left(\sum_{i=1}^n a_i\otimes v_i\right) = N(a_1,\dots,a_n)\,.$$

As a variety, T_2 may be represented as a hypersurface in $\mathbb{A}_k^n$ defined by the norm equation:

$$N(x_1, \dots, x_n) = 1$$

where $x_1, \dots, x_n$ denote the coordinates of an element of F with respect to some k-basis of F. We use the standard notation $R^1_{F/k}\mathbb{G}_m$ for the torus T_2, and often call it the ***norm one torus*** for F/k. It is Chevalley who first discovered that the torus $R^1_{F/k}\mathbb{G}_m$ may be not k-rational for some F/k.

4.2 R-Equivalence on Tori

Let k be a field and let X be an algebraic k-scheme. Two rational points $x, y \in X(k)$ are said to be ***directly R-equivalent***, written $x \overset{R}{\underset{d}{\sim}} y$, if there is a k-rational map $f : \mathbb{P}^1 \dashrightarrow X$ and two points $a, b \in \mathbb{P}^1(k)$ such that $f(a) = x$ and $f(b) = y$. We may always choose the points a, b to be $0, \infty$.

Let $\mathcal{O}_k$ denote the ring of all rational functions $h(t) = f(t)/g(t)$ over k with $g(0)g(1) \neq 0$. This is the ring of rational functions on $\mathbb{A}^1_k$ that are regular at two given rational points 0 and 1. The evaluation maps $\mathcal{O}_k \to k\,,\ h(t) \mapsto h(i)$ for $i = 0, 1$ give two homomorphisms of k-algebras. The direct R-equivalence between $x, y \in X(k)$ amounts the same as to saying that there is a point $\alpha(t) \in X(\mathcal{O}_k)$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

The ***R-equivalence*** on $X(k)$ is the equivalence relation generated by direct R-equivalence. Namely, two points $x, y \in X(k)$ are R-equivalent, written $x \overset{R}{\sim} y$, if and only if there are points $x_0, x_1, \dots, x_n \in X(k)$ such that

$$x = x_0 \overset{R}{\underset{d}{\sim}} x_1 \overset{R}{\underset{d}{\sim}} x_2 \overset{R}{\underset{d}{\sim}} \cdots \overset{R}{\underset{d}{\sim}} x_n = y\,.$$

The set of R-equivalence classes will be denoted by $X(k)/R$. When it is a singleton, we write it as $X(k)/R = 0$. For a field extension L/k, we will simply write $X(L)/R$ for the set $X_L(L)/R$ of R-equivalence classes of the L-scheme X_L. The following facts immediately follow from the definition.

Proposition 4.2.1. *Let X, Y be two k-schemes and let K/k be a finite separable extension.*

(i) *All rational points on an affine space $\mathbb{A}^n$ are directly R-equivalent to one fixed point $x_0 \in \mathbb{A}^n(k)$. The same is true for any open subset of $\mathbb{A}^n$.*

(ii) *$X(k)/R = 0$ for $X = \mathbb{A}^n$ or $\mathbb{P}^n$.*

(iii) *Any k-morphism $f : X \to Y$ induces a map of sets $f_R : X(k)/R \to Y(k)/R$.*

(iv) *The natural map $(X \times_k Y)(k)/R \longrightarrow X(k)/R \times Y(k)/R$ is a bijection.*

(v) *If X' is a K-scheme and $X = R_{K/k}X'$ is the Weil's restriction of scalars to k, one has a bijection $X(k)/R \xrightarrow{\sim} X'(K)/R$.*

Proof. (i) If X is an open subset of $\mathbb{A}^n$ and $x_0, x_1 \in X(k)$, we can find a line $\ell \cong \mathbb{A}^1$ in $\mathbb{A}^n$ connecting x_0, x_1. Viewing $X \cap \ell$ as an open subset of $\mathbb{A}^1$, there is a k-rational map $f : \mathbb{P}^1 \dashrightarrow X$ which restricts to the identity map on $X \cap \ell$.

(ii) This follows immediately from (i).

(iii) Clearly, two directly R-equivalent points in $X(k)$ are mapped to directly R-equivalent points in $Y(k)$. Hence the natural map $X(k) \to Y(k)$ respects R-equivalence.

(iv) Identify $(X \times_k Y)(k)$ with $X(k) \times Y(k)$. Let $z_0 = (x_0, y_0)$ and $z_1 = (x_1, y_1)$ be two points in $(X \times_k Y)(k)$. One sees easily from the definition that $z_0 \overset{R}{\underset{d}{\sim}} z_1$ if and only if $x_0 \overset{R}{\underset{d}{\sim}} x_1$ and $y_0 \overset{R}{\underset{d}{\sim}} y_1$.

(v) We have an isomorphism of functors $\mathrm{Mor}_k(- , X) \cong \mathrm{Mor}_K((-) \times_k K, X')$. So there is a bijection $\varphi : X'(K) \xrightarrow{\sim} X(k)$. Let $x', y' \in X'(K)$ and let $x = \varphi(x'), y = \varphi(y') \in X(k)$. Suppose $x \underset{d}{\overset{R}{\sim}} y$. There is an open subset of V of $\mathbb{P}^1_k$ and a k-morphism $f : V \to X$ such that $f(0) = x$ and $f(\infty) = y$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_K(V_K, X') & \longrightarrow & \mathrm{Mor}_k(V, X) \\ \downarrow & & \downarrow \\ \mathrm{Mor}_K(K, X') & \xrightarrow{\varphi} & \mathrm{Mor}_k(k, X) \end{array}$$

where the vertical maps can be the evaluation at 0 or ∞. If follows that $x' \underset{d}{\overset{R}{\sim}} y'$. Hence the natural map $X'(K) \xrightarrow{\sim} X(k) \to X(k)/R$ induces a bijection $X'(K)/R \xrightarrow{\sim} X(k)/R$. □

Proposition 4.2.2. *Let X be any (algebraic) k-scheme. Let $x, y \in X(k)$ be rational points of X. We can also regard them as elements in $Z_0(X)$. Then*

$$x \overset{R}{\sim} y \ \text{ in } \ X(k) \implies x \overset{\mathrm{rat}}{\sim} y \ \text{ in } \ Z_0(X)\,.$$

Proof. We may assume x and y are directly R-equivalent. There is a k-rational map $f : \mathbb{P}^1 \dashrightarrow X$ such that $f(0) = x$ and $f(\infty) = y$. Let U be the domain of definition of f. Let $V = \Gamma_f \subseteq \mathbb{P}^1 \times X$ be its graph. The first projection induces a birational morphism $p : V \to U$. We have

$$x - y = [V(0)] - [V(\infty)]$$

where, as in Prop. 1.2.6, $V(Q)$ is the image of $p^{-1}(Q)$ under the second projection $q : \mathbb{P}^1 \times X \to X$ for $Q = 0$ or ∞. So Prop. 1.2.6 implies that $x \overset{\mathrm{rat}}{\sim} y$ in $Z_0(X)$. □

According to the above proposition, there is a well-defined map $X(k)/R \to \mathrm{CH}_0(X)$. Let $A_0(X)$ denote the subgroup of $\mathrm{CH}_0(X)$ consisting of classes of degree 0 and fix a point $x_0 \in X(k)$. There is a well-defind map $X(k)/R \to A_0(X)$ sending a point x to $[x] - [x_0]$.

As to the birational invariance of the set $X(k)/R$, one has the following result due to Colliot-Thélène and Sansuc.

Proposition 4.2.3 ([7, p.195, Prop. 10])**.** *Let X, Y be smooth, projective, irreducible k-varieties. Any k-rational map $f : X \dashrightarrow Y$ defines a map $f_R : X(k)/R \to Y(k)/R$ which coincides with the natural one everywhere f is defined. If $f : X \to Y$ is a birational map, then $f_R : X(k)/R \to Y(k)/R$ is a bijection. So $X(k)/R$ is a birational invariant in the category of smooth projective irreducible k-varieties, and if such a variety X is k-rational, one has $X(k)/R = 0$.*

Definition 4.2.4. Let X be a smooth irreducible k-variety. A ***smooth compactification*** of X, if it exists, is an open immersion $i : X \to X'$ from X into a smooth projective irreducible variety X'.

By Hironaka's theorem (cf. [17]), a smooth compactification exists when $\mathrm{char}\, k = 0$.

Corollary 4.2.5. *Let X be a smooth irreducible k-variety. Supppose X has a smooth compactification $i : X \to X'$. Then the property that the map $i_R : X(k)/R \to X'(k)/R$ is bijective (resp. injective, resp. surjective) does not depend on the choice of smooth compactifications.*

Proof. If $i : X \to X'$ and $j : X \to X''$ are two smooth compactifications of X, there is a birational map $f : X' \dashrightarrow X''$ such that $f \circ i = j$ as birational maps. The result then follows from Prop. 4.2.3. □

Let G be an algebraic k-group. There is a well-defined map

$$G(k)/R \times G(k)/R = (G \times G)(k)/R \longrightarrow G(k)/R$$

induced by the group multiplication. So the set $G(k)/R$ can be equipped naturally with a group structure. Let $R(k, G)$ be the R-equivalence class of the identity element $1 \in G(k)$. This is a normal subgroup of $G(k)$ and $G(k)/R$ is isomorphic to $G(k)/R(k, G)$.

Proposition 4.2.6. *Let G be an algebraic k-group. Any two R-equivalent points in $G(k)$ are directly R-equivalent.*

Proof. Let $g, g' \in G(k)$ and $g \overset{R}{\sim} g'$. There is a chain of elements $g_0, g_1, \dots, g_n$ of $G(k)$ such that $g_0 = g$, $g_n = g'$ and $g_i \underset{d}{\overset{R}{\sim}} g_{i+1}$ for all $i = 0, 1, \dots, n-1$. By induction, we may assume $n = 2$ and $g_0 = 1$. There exist points $f_0(t), f_1(t) \in G(\mathcal{O}_k)$ such that $f_0(0) = 1$, $f_0(1) = g_1$, $f_1(0) = g_1$, $f_1(1) = g_2$. Let $h(t) = f_1(t)f_0(1-t)^{-1}$. Then $h(0) = 1 = g_0$ and $h(1) = g_2$. This proves the proposition. □

The set $G(k)/R$ for algebraic groups is birationally invariant in a somewhat weaker sense. This is the content of the following proposition.

Proposition 4.2.7 ([7, p.197, Prop. 11]). *Let k be an infinite field. Let G and G' be connected linear k-groups that are k-***unirational**. *If G and G' are k-birationally equivalent, then there is a bijection of sets $G(k)/R \xrightarrow{\sim} G'(k)/R$.*

Here, an integral k-variety X is said to be k-***unirational*** if there is a dominant k-rational map $\mathbb{P}^n \dashrightarrow X$. Using Chevalley's theorem on fibre dimensions, one can choose such a rational map with $n = \dim X$ (cf. [22, p.51, Def. 12.8]). A connected linear k-group G is k-unirational if G is reductive (meaning that $G(\bar{k}) \neq 1$ and the maximal connected normal unipotent subgroup of $G(\bar{k})$ is trivial) or if k is perfect (cf. [33, §4.1]). In particular, any k-torus is unirational.

Remark 4.2.8. After making translations, we may suppose the bijection $G(k)/R \xrightarrow{\sim} G'(k)/R$ respects the identity elements of groups. But it's not evident a priori that the map is a group isomorphism.

Proposition 4.2.9 ([7, p.203, Prop. 13]). *Let T be a k-torus.*

(i) *The rational equivalence on $T(k)$ coincides with the R-equivalence.*

(ii) *If $T \to X$ is a smooth compactification of T, then the natural map $T(k)/R \to X(k)/R$ is bijective. So if T is rational, $T(k)/R = 0$.*

4.3 Flasque Resolution of Tori

Let Γ be a profinite group. Recall that a Γ-lattice is a Γ-module M that is free of finite rank as a $\mathbb{Z}$-module. The Γ-action on the dual lattice $M^0 = \mathrm{Hom}(M, \mathbb{Z})$ is given by

$$(\gamma \cdot f)(m) := f(\gamma^{-1}m)\,, \quad \forall\, \gamma \in \Gamma,\ f \in M^0\,,\ \ m \in M\,.$$

Definition 4.3.1. A Γ-lattice M is said to be

(1) a ***permutation lattice*** if it admits a $\mathbb{Z}$-basis $X = \{ e_1, \dots, e_n \}$ that is stable under Γ-action (this implies that for any $\gamma \in \Gamma$, the map $X \to X\,;\, x \mapsto \gamma x$ is bijective. Indeed, if $\gamma e_1 = \gamma e_2$, then we get a contradiction: $e_1 - e_2 = \gamma^{-1}\gamma(e_1 - e_2) = 0$);

(2) ***invertible*** if there is another Γ-lattice N such that $M \oplus N$ is a permutation Γ-lattice;

(3) ***coflasque*** if $H^1(\Gamma', M) = 0$ for all open subgroup Γ' of Γ;

(4) ***flasque*** if its dual lattice M^0 is coflasque.

Proposition 4.3.2.

(i) *A permutation lattice is isomorphic to its dual.*

(ii) *An invertibel lattice is both flasque and coflasque.*

Proof. (i) Let M be a permutation lattice with a $\mathbb{Z}$-basis $X = \{ e_1, \dots, e_n \}$ stable under Γ. M^0 has a $\mathbb{Z}$-basis $Y = \{ f_1, \dots, f_n \}$ dual to X, that is, $f_i(e_j) = \delta_{ij}$. Since the Γ-action is given by $(\gamma \cdot f)(m) = f(\gamma^{-1} m)$, Y is stable under Γ. In fact, if $\gamma e_i = e_j$ then $\gamma f_i = f_j$. The map $M \longrightarrow M^0$ sending e_i to f_j is therefore an isomorphism of Γ-lattices.

(ii) It is enough to prove the result for permutation lattices. In view of (i), we need only prove a permutation lattice M is coflasque. We want to show that for any open subgroup $\Gamma' = \Gamma$ of Γ, $H^1(\Gamma', M) = 0$. Without loss of generality, we may assume $\Gamma' = \Gamma$ since M is also a permutation Γ'-lattice. Let $X = \{ e_1, \dots, e_n \}$ be a Γ-stable $\mathbb{Z}$-basis for M, and let $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_r$ be the decomposition of X into disjoint Γ-orbits. Then we have $M = M_1 \oplus M_2 \oplus \cdots \oplus M_r$, where M_i is the permutation Γ-lattice with $\mathbb{Z}$-basis X_i. It suffices to prove $H^1(\Gamma, M_i) = 0$ for each i. So we may assume Γ acts transitively on X. Now let $\Gamma' := \{ \gamma \in \Gamma \mid \gamma e_1 = e_1 \}$ be the stablizer of e_1. This is an open subgroup of Γ. We have $M = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \cong \mathbb{Z}[\Gamma/\Gamma']$ since Γ-acts transitively on $X = \{ e_1, \dots, e_n \}$. It follows that $M \cong \mathbb{Z}[\Gamma/\Gamma']$ is isomorphic to the induced module

$$\mathrm{Ind}_{\Gamma}^{\Gamma'}(\mathbb{Z}) := \{ \text{continuous map } f : \Gamma \to \mathbb{Z} \mid f(\gamma' s) = f(s), \ \forall \gamma' \in \Gamma', \ s \in \Gamma \}$$

where $\mathbb{Z}$ is given the trivial Γ'-action. By Shapiro's lemma, we obtain

$$H^1(\Gamma, M) \cong H^1(\Gamma', \mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Z}).$$

The last group vanishes because Γ' is profinite and

$$\mathrm{Hom}_{\mathrm{cont}}(\Gamma', \mathbb{Z}) = \varprojlim \mathrm{Hom}(\Gamma'/\Gamma'_\alpha, \mathbb{Z}) = 0,$$

where the limit is taken over all the open subgroups Γ'_α of Γ'. This completes the proof. □

Let M be a Γ-lattice. A ***flasque resolution*** of M is a short exact sequence of Γ-lattices

$$0 \longrightarrow M \longrightarrow P \longrightarrow F \longrightarrow 0$$

in which P is a permuation lattice and F is a flasque lattice; a ***coflasque resolution*** of M is a short exact sequence of Γ-lattices

$$0 \longrightarrow Q \longrightarrow R \longrightarrow M \longrightarrow 0$$

in which R is a permuation lattice and Q is a coflasque lattice.

The following lemma, of primary importance, was already used by Lenstra in [20] (in the proof of Prop. 1.2).

Lemma 4.3.3. *Every* Γ*-lattice has a flasque resolution and a coflasque resolution.*

Proof. By duality, we need only consider the existence of a coflasque resolution. We may assume Γ is finite. For a surjective homomorphism of Γ-lattices $R \to M$ with R a permutation lattice, its kernel Q is coflasque if and only if for any subgroup Γ'

of Γ, the induced $R^{\Gamma'} \longrightarrow M^{\Gamma'}$ is again surjective. For each subgroup Γ', let Γ act on $\mathbb{Z}[\Gamma/\Gamma'] \otimes_{\mathbb{Z}} M^{\Gamma'}$ by

$$\sigma \cdot (\overline{\gamma} \otimes m) := \overline{\sigma\gamma} \otimes m \quad \forall\, \sigma \in \Gamma,\ \overline{\gamma} \in \Gamma/\Gamma',\ m \in M^{\Gamma'}\,.$$

We have a natural map $\mathbb{Z}[\Gamma/\Gamma'] \otimes_{\mathbb{Z}} M^{\Gamma'} \longrightarrow M\,;\, \overline{\gamma} \otimes m \mapsto \gamma \cdot m$. To obtain a coflasque resolution of M, it is therefore sufficient to take $R \to M$ to be the direct sum, over all subgroups Γ' of Γ, of the maps $\mathbb{Z}[\Gamma/\Gamma'] \otimes_{\mathbb{Z}} M^{\Gamma'} \to M$. $\square$

We say two Γ-lattices M and N are ***equivalent*** if there are permutation Γ-lattices P and Q such that $M \oplus P \cong N \oplus Q$. The flasque resolution of a given lattice M is unique in the following sense ([7, p.181, Lemme 5]): for any flasque resolution

$$0 \longrightarrow M \longrightarrow P \longrightarrow F \longrightarrow 0$$

the equivalence class of F depends only on M, which we will denote by $\rho(M)$.

Now we take $\Gamma = \mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$. Let T be an algebraic k-torus. We denote by $\hat{T}$ its group of characters, i.e. $\hat{T} := \mathrm{Hom}(\overline{T}\,,\, \mathbb{G}_{m,\,\bar{k}})$. The functor $T \mapsto \hat{T}$ establishes an anti-equivalence between the category of algebraic k-tori with the category of $\mathfrak{g}$-lattices.

Definition 4.3.4. Let T be an algebraic k-torus. We say T is

(1) ***trivial***, or ***split*** ("déployé" in French) over k, if $\hat{T}$ is a trivial $\mathfrak{g}$-lattice (this is equivalent to saying that T is k-isomorphic to a product $\mathbb{G}_{m,\,k} \times \cdots \times \mathbb{G}_{m,\,k}$;

(2) ***quasi-trivial***, or ***quasi-split***, if $\hat{T}$ is a permutation $\mathfrak{g}$-lattice;

(3) ***flasque*** (resp. ***coflasque***) if $\hat{T}$ is a flasque (resp. coflasque) $\mathfrak{g}$-lattice.

Note that a quasi-trivial k-torus T is an open subset of an affine space $\mathbb{A}^n_k$, hence is k-rational. Indeed, the decomposition of a permutation basis of $\hat{T}$ gives rise to a decomposition $T = T_1 \times \cdots \times T_r$ in which each $\hat{T}_i \cong \mathbb{Z}[\mathfrak{g}/\mathfrak{h}_i]$ for some open subgroup $\mathfrak{h}_i$ of $\mathfrak{g}$. If $K_i := \overline{k}^{\mathfrak{h}_i}$ is the subfield of $\bar{k}$ consisting of invariants under $\mathfrak{h}_i$, we have $T_i \cong R_{K_i/k}\mathbb{G}_{m\,,\,K_i}$, where $R_{K_i/k}$ denotes the Weil restriction of scalars. So a quasi-trivial torus T is isomorphic to a finite product of tori of the form $R_{K/k}\mathbb{G}_{m,\,K}$, where K/k is a finite separable extension K/k. As a variety, $R_{K/k}\mathbb{G}_{m,\,K}$ is an open subset of $R_{K/k}\mathbb{A}^1_K \cong \mathbb{A}^d_k$ with $d = [K : k]$. Hence, a quasi-trivial torus T is k-isomorphic to an open subset of some $\mathbb{A}^n_k$.

A ***flasque resolution*** of a k-torus T is a short exact sequence of k-tori

$$1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1$$

with P quasi-trivial and S flasque. The anti-equivalence between the k-tori and $\mathfrak{g}$-lattices establishes a bijective correspondence between flasque resolutions of T and those of the $\mathfrak{g}$-lattice $\hat{T}$. Lemma 4.3.3 tells us that every k-torus has a flasque resolution. The equivalence class $\rho(\hat{T})$ will be also denoted by $\rho(T)$.

Lemma 4.3.5. *Let P be a k-torus. If there is a torus P' such that $P \times P'$ is quasi-trivial, then $H^1(k,\, P) = 0$.*

Proof. We may assume P is quasi-trivial and is of the form $R_{K/k}\mathbb{G}_m$. Then

$$H^1(k,\, P) = H^1(k,\, R_{K/k}\mathbb{G}_m) \cong H^1(K,\, \mathbb{G}_m) = 0$$

by Shapiro's lemma and Hilbert's theorem 90. $\square$

If $1 \to S \to P \to T \to 1$ and $1 \to S' \to P' \to T$ are two flasque resolutions of T, then there exist quasi-trivial tori Q, Q' such that $S \times Q \cong S' \times Q'$. Lemma 4.3.5 implies that, for any flasque resolution

$$1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1\,,$$

the group $H^1(k, S)$ depends only on $\rho(T)$.

The class $\rho(T)$ characterizes an important invariant of the torus T. Here we need the notion of ***stably rationality***. Two integral k-varieties X and Y are said to be stably k-rationally equivalent if $X \times_k \mathbb{A}_k^n \cong Y \times_k \mathbb{A}_k^m$ for some $n, m \geq 1$. An integral k-variety X is said to be ***stably rational*** if it is stably rationally equivalent to some affine space $\mathbb{A}_k^n$.

Note that for any k-variety X, the absolute Galois group $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ acts on $\overline{X}$, hence also on various objects attached to $\overline{X}$, e.g. the Chow groups and so on.

Theorem 4.3.6.

(i) *Let T_1, T_2 be k-tori. Then $\rho(T_1) = \rho(T_2)$ if and only if T_1 and T_2 are stably k-rationally equivalent.*

(ii) *Suppose X is a smooth compactification of a k-torus T. There is a flasque resolution of T*

$$1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1 \tag{4.3}$$

in which $\hat{S} = \mathrm{CH}^1(\overline{X})$ and $\hat{P}$ is the permutation $\mathfrak{g}$-lattice with $\mathbb{Z}$-basis the set of all irreducible components of $\overline{X} \setminus \overline{T}$.

Proof. See [7, pp.189–190] or [3, pp.19–21]. □

Let X be a functor from the category $\mathfrak{Alg}_{/k}$ of k-algebras to the category $\mathfrak{Set}$ of sets, for example $\mathsf{X} = H^1_{ét}(-, G)$ where G is an algebraic group. For a field extension L/k, we can define the ***R-equivalence*** on $\mathsf{X}(L)$ as follows. Let $\mathcal{O}_L$ be the semi-local ring of rational functions $h(t) = f(t)/g(t) \in L(t)$ such that $g(0)g(1) \neq 0$. For $i = 0$ or 1, there is a natural map $\mathcal{O}_L \to L$ sending $h(t)$ to $h(i)$. We say two points $x, y \in \mathsf{X}(L)$ are ***directly R-equivalent*** if there is an $\alpha(t) \in \mathsf{X}(\mathcal{O}_L)$ such that $\alpha(0) = x$ and $\alpha(1) = y$. The R-equivalence on $\mathsf{X}(L)$ is defined to be the equivalence relation generated by the direct R-equivalence. If X is the functor of points $X(-)$ defined by a k-scheme X, this definition of R-equivalence coincides with the old one.

Let $1 \to S \to P \to T$ be an exact sequence of tori flasque with P quasi-trivial. There is an étale k-algebra E such that $P \cong R_{E/k}\mathbb{G}_m$. For any semi-local ring A in $\mathfrak{Alg}_{/k}$, one has

$$H^1_{ét}(A, P) \cong H^1_{ét}(A \otimes_k E, \mathbb{G}_m) = 0$$

by Shapiro's lemma and Hilbert's theorem 90. So for any field extension L/k we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} P(\mathcal{O}_L) & \longrightarrow & T(\mathcal{O}_L) & \longrightarrow & H^1_{ét}(\mathcal{O}_L, S) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ P(L) & \longrightarrow & T(L) & \longrightarrow & H^1(L, S) & \longrightarrow & 0 \end{array}$$

where the vertical maps can be the evaluation at 0 or 1. It follows that there is an induced exact sequence

$$P(L)/R \longrightarrow T(L)/R \longrightarrow H^1(L, S)/R \longrightarrow 0\,.$$

Since the quasi-trivial torus P is rational, we have $P(L)/R = 0$. Hence $T(L)/R \cong H^1(L, S)/R$.

The following result, due to Colliot-Thélène and Sansuc and fundamental for our later use, reveals an important link between flasque resolutions and the R-equivalence for tori.

Theorem 4.3.7 ([7, p.199, Thm. 2])**.** *Let T be a k-tori. Any flasque resolution $1 \to S \to P \to T \to 1$ of T induces an isomorphism of groups*

$$T(k)/R \cong H^1(k, S)\,.$$

Corollary 4.3.8. *For algebraic tori the group $T(k)/R$ is birationally invariant.*

Proof. The group $H^1(k,\, S)$ depends only on $\rho(T)$. The result follows immediately from Thm. 4.3.6 and Thm. 4.3.7. The group $T(k)/R$ is even invariant under stably rational equivalence. □

4.4 Functors related to R-Equivalence and Zero-Cycles

In this section, let T be an algebraic k-torus and let X be a smooth compactification of T.

For each field extension L/k, there is a well-defined map

$$\varphi_L : T(L)/R \longrightarrow A_0(X_L)\,; \quad [t] \mapsto [t] - [1]\,. \tag{4.4}$$

Proposition 4.4.1. *With notation as above, the property that φ_L is bijective (or injective, or surjective) does not depend on the choice of the smooth compactification X.*

Proof. We may assume $L = k$. Let $T \to X$ and $T \to X'$ be two smooth compactifications. There are mutually inverse birational maps $X \dashrightarrow X'$ and $X' \dashrightarrow X$ that are identical on T. Their graphs define two correspondences $\alpha \in \mathbf{Cor}(X,\, X') = \mathrm{CH}_n(X \times X')$ and $\beta \in \mathbf{Cor}(X',\, X) = \mathrm{CH}_n(X' \times X)$. There are induced homomorphisms $\alpha_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(X')$ and $\beta_* : \mathrm{CH}_0(X') \to \mathrm{CH}_0(X)$, mutually inverse, that are compatible with the map φ (cf. [11, §16.1]). □

Questions we are interested are: is φ a group homomorphism and is it an isomorphism? The two questions are easily solved for the cases $\dim T = 1$ or 2, because all tori of dimension ≤ 2 are rational ([33, §4.9]), and the group $A_0(X)$ is birationally invariant for smooth projective varieties ([11, Example 16.1.11], there the hypothesis that the ground field is algebraically closed is useless). So if $\dim T \leq 2$, we have $T(k)/R = 0$ and $A_0(X) \cong A_0(\mathbb{P}^n) = 0$ where $n = 1$ or 2.

The main result discussed in this thesis is that in the 3-dimensional case it can also be proved that φ is a group isomorphism (cf. Thm. 4.6.4).

Let $\mathfrak{Field}_{/k}$ be the category of field extensions of k and let $\mathfrak{Set}$ be the category of sets. We may regard $L \mapsto T(L)/R$ and $L \mapsto A_0(X_L)$ as functors from $\mathfrak{Field}_{/k}$ to $\mathfrak{Set}$. Then $\varphi_L : T(L)/R \to A_0(X_L)$ can be viewed as a morphism of functors.

Consider the flasque resolution in (4.3):

$$1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1\,.$$

The S_L-torsor P_L over T_L can be extended to an S_L-torsor $q : U \to X_L$ over X_L by [7, p.194, Prop. 9]. For a closed point x_L of X_L, the fiber U_{x_L} of q over x_L is a torsor under $S_{\kappa(x_L)}$ over $\kappa(x_L)$. Denote by $[U_{x_L}]$ its class in $H^1(\kappa(x_L),\, S)$. By [7, p.198, Prop. 12], the map

$$\psi_L : \mathrm{CH}_0(X_L) \longrightarrow H^1(L,\, S) = T(L)/R\,; \quad x_L \mapsto N_{\kappa(x_L)/L}([U_{x_L}])\,, \tag{4.5}$$

where the norm map $N_{\kappa(x_L)/L}$ is the usual one in Galois cohomology (it is called the "corestriction" in [29]), extends to a well-defined group homomorphism. Let $\tilde{\psi}_L = \psi_L|_{A_0(X_L)}$ be its restriction to $A_0(X_L)$. It is obvious that $\tilde{\psi}_L \circ \varphi_L = \mathrm{Id}$. It follows that the map φ_L is injective, and when φ_L is bijective, it is an isomorphism of groups since $\tilde{\psi}_L$ is a group homomorphism.

We may regard the assignments $L \mapsto \mathrm{CH}_0(X_L)$, $L \mapsto H^1(L,\, S)$ as functors from $\mathfrak{Field}_{/k}$ to $\mathfrak{Set}$, and ψ_L as a morphism of functors. We now start the study of φ_L and ψ_L as morphisms of functors.

Definition 4.4.2. Let $\mathsf{A} : \mathfrak{Field}_{/k} \longrightarrow \mathfrak{Set}$ be a functor. We say

(1) A is a ***functor with norms***, if for any finite extension E/F with $E, F \in \mathfrak{Field}_{/k}$, there is a given ***norm map*** $N_{E/F} : \mathsf{A}(E) \longrightarrow \mathsf{A}(F)$ such that (i) $N_{E/E} = \mathrm{Id}$ and (ii) if L/E is another finite extension, one has $N_{L/F} = N_{E/F} \circ N_{L/E}$;

(2) A is a ***functor with specializations***, if for any DVR over k of ***geometric type*** (i.e., a DVR that is a localization of a finitely generated k-algebra) O, with quotient field L and residue field K, there is a given map $s_{\mathsf{A},\,O} : \mathsf{A}(L) \longrightarrow \mathsf{A}(K)$ called a ***specialization map***.

Definition 4.4.3. Let $\mathsf{A}, \mathsf{B} : \mathfrak{Field}_{/k} \to \mathfrak{Set}$ be two functors and let $\alpha : \mathsf{A} \to \mathsf{B}$ be a morphism.

(1) Assume that A and B have norms. We say α ***commutes with norms*** if for any finite extension E/F in $\mathfrak{Field}_{/k}$, the following diagram

$$\begin{array}{ccc} \mathsf{A}(E) & \xrightarrow{\alpha_E} & \mathsf{B}(E) \\ {\scriptstyle N_{E/F}}\downarrow & & \downarrow{\scriptstyle N_{E/F}} \\ \mathsf{A}(F) & \xrightarrow{\alpha_F} & \mathsf{B}(F) \end{array}$$

is commutative.

(2) Assume that A and B have specializations. We say α ***commutes with specializations*** if for any DVR O of geometric type with quotient field L and residue field K, the following diagram

$$\begin{array}{ccc} \mathsf{A}(L) & \xrightarrow{\alpha_L} & \mathsf{B}(L) \\ {\scriptstyle s_{\mathsf{A},\,O}}\downarrow & & \downarrow{\scriptstyle s_{\mathsf{B},\,O}} \\ \mathsf{A}(K) & \xrightarrow{\alpha_K} & \mathsf{B}(K) \end{array}$$

is commutative.

Examples 4.4.4. (1) The usual norm (or corestriction) map in Galois cohomology endows the functor $H^1(L, S)$ with norms.

(2) Let E/F be a finite extension in $\mathfrak{Field}_{/k}$. We have

$$\begin{aligned} T(E) &= T(E \otimes_k \bar{k})^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{Alg}_{/k}}(A,\, E \otimes_k \bar{k})^{\mathfrak{g}} \\ &= \mathrm{Hom}_{\mathfrak{Alg}_{/\bar{k}}}(A \otimes_k \bar{k},\, E \otimes_k \bar{k})^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{Alg}_{/\bar{k}}}(\bar{k}\langle \hat{T} \rangle,\, E \otimes_k \bar{k})^{\mathfrak{g}} \\ &= \mathrm{Hom}_{\mathbb{Z}}(\hat{T},\, \overline{E}^*)^{\mathfrak{g}} = (\hat{T}^0 \otimes_{\mathbb{Z}} \overline{E}^*)^{\mathfrak{g}} \end{aligned}$$

where A is the coordinate ring of the torus T, $\overline{E} = E \otimes_k \bar{k}$ and $\bar{k}\langle \hat{T} \rangle$ is the group algebra based on $\hat{T}$. There is a natural norm map

$$N_{E/F} : T(E) = (\hat{T}^0 \otimes \overline{E}^*)^{\mathfrak{g}} \longrightarrow T(F) = (\hat{T}^0 \otimes \overline{F}^*)^{\mathfrak{g}}$$

induced by the usual norm for the field extension E/F. Thus the functor $L \mapsto T(L)$ is equipped with norms. The functor $L \mapsto T(L)/R$ therefore inherits norms from $L \mapsto T(L)$.

The isomorphism of functors $T(L)/R \xrightarrow{\sim} H^1(L, S)$ induced by the natural maps $T(L) \to H^1(L, S)$ clearly commutes with norms.

(3) For a finite extension E/F in $\mathfrak{Field}_{/k}$, the natural morphism $\pi : X_E \to X_F$ is proper. We take the norm map $N_{E/F} : \mathrm{CH}_0(X_E) \to \mathrm{CH}_0(X_F)$ to be the proper push-forward π_*. In this way the functor $L \mapsto \mathrm{CH}_0(X_L)$ is endowed with norms. There is a natural way to equip the functor $L \mapsto A_0(X_L)$ with norms such that the inclusion $A_0(X_L) \hookrightarrow \mathrm{CH}_0(X_L)$ commutes with norms.

(4) Let $1 \to S \to P \to T \to 1$ be any flasque resolution of the torus T. Let α be an endomorphism of the torus S. It induces an endomorphism of functors

$$T(L)/R = H^1(L, S) \longrightarrow T(L)/R = H^1(L, S)\,; \quad t \mapsto \alpha(t)$$

that commutes with norms.

Example 4.4.5. † Let O be a DVR of geometric type over k with quotient field L and residue field K. The specialization map $s : \mathrm{CH}_0(X_L) \to \mathrm{CH}_0(X_K)$ is defined as follows (cf. [11, §20.3]). There is an exact sequence (a variant of Prop. 1.3.6 in the relative case)

$$\mathrm{CH}_1(X_K) \xrightarrow{i_*} \mathrm{CH}_1(X_O) \xrightarrow{j^*} \mathrm{CH}_0(X_L) \longrightarrow 0$$

where i_* and j^* denote the push-forward and pull-back induced by the natural morphisms $X_K \to X_O$ and $X_L \to X_O$. For an element $\alpha \in \mathrm{CH}_0(X_L)$, we pick $\alpha' \in \mathrm{CH}_1(X_O)$ such that $j^*(\alpha') = \alpha$. Then define $s(\alpha) = i^!(\alpha')$, where $i^! : \mathrm{CH}_1(X_O) \to \mathrm{CH}_0(X_K)$ is the Gysin homomorphism (cf. [11, §6.2]). The map s is well-defined since $i^! \circ i_* = 0$ ([11, Thm. 6.3]). Thus, the functor $F \mapsto \mathrm{CH}_0(X_F)$ can be equipped with specializations. The subfunctor $F \mapsto A_0(X_F)$ inherits specializations such that the inclusion $A_0(X_F) \hookrightarrow \mathrm{CH}_0(X_F)$ commutes with specializations.

Example 4.4.6. Let O be a DVR of geometric type over k with quotient field L and residue field K. Let $1 \to S \to P \to T \to 1$ be any flasque resolution of the torus T. By [8, Coro. 4.2], the natural homomorphism $H^1_{ét}(O, S) \to H^1(L, S)$ is an isomorphism since S is flasque. The compostion

$$s : T(L)/R \cong H^1(L, S) \cong H^1_{ét}(O, S) \longrightarrow H^1(K, S) \cong T(K)/R$$

gives a specialization map. The group $H^1(L, S)$ is uniquely determined by the class $\rho(T)$ (Lemma 4.3.5), so the specialization map s is independent of the choice of a flasque resolution. Further, the natural map $T(O) \to H^1_{ét}(O, S)$ is surjective since $H^1_{ét}(O, P) \cong H^1(L, P) = 0$. We have thus the following commutative diagram

$$\begin{array}{ccccccccc}
T(L)/R & \twoheadleftarrow & T(L) & \longleftarrow & T(O) & \longrightarrow & T(K) & \twoheadrightarrow & T(K)/R \\
\wr\downarrow & & & & \downarrow & & & & \downarrow\wr \\
H^1(L, S) & & \xleftarrow{\quad\sim\quad} & & H^1_{ét}(O, S) & & \longrightarrow & & H^1(K, S)
\end{array}$$

which implies in particular that the composition $T(O) \to T(L) \to T(L)/R$ is surjective. Let $p \in T(L)/R$ and let $q \in T(O)$ be a lift of p. It follows easily from the definition that $s(p)$ is the image of q under the composition $T(O) \to T(K) \to T(K)/R$.

Lemma 4.4.7. *Let T be an algebraic k-torus. Let $t, t' \in T$ be two elements such that t' specializes to t (i.e. $t \in \overline{\{t'\}}$). Suppose that the local ring $O = \mathscr{O}_{t',t}$ is a DVR. Let $s = s_O : T(\kappa(t'))/R \to T(\kappa(t))/R$ be the specialization map with respect to O. Then $s(t') = t$.*

Proof. Let $A = k[T]$ be the coordinate ring of T and let $\mathfrak{p}$ and $\mathfrak{p}'$ be prime ideals of A corresponding to t and t' respectively. Then $O = A_{\mathfrak{p}}/\mathfrak{p}'A_{\mathfrak{p}}$,

$$\kappa(t) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \mathrm{Frac}(A/\mathfrak{p}) \quad \text{and} \quad \kappa(t') = \mathrm{Frac}(A_{\mathfrak{p}}/\mathfrak{p}'A_{\mathfrak{p}}) = \mathrm{Frac}(A/\mathfrak{p}')\,.$$

Let $\tilde{t} \in T(O) = \mathrm{Hom}_{\mathfrak{Alg}/k}(A, O)$ be the point given by the natural homomorphism $A \to O$. The images of $\tilde{t}$ under the maps $T(O) \to T(\kappa(t))$ and $T(O) \to T(\kappa(t'))$ coincide with t and t' respectively. The result then follows from Example 4.4.6. □

†This example involves intersection theory about ***general schemes*** rather than algebraic schemes over a field. For details, see [11, Chapt.20]. Also, be careful that our notations here use absolute dimensions.

Now look at the morphisms of functors $\varphi : T(-)/R \to A_0(X_{(-)})$ and $\psi : \mathrm{CH}_0(X_{(-)}) \to H^1(-, S)$.

Proposition 4.4.8. *Let T be a k-torus and X a smooth compactifiaction of T. Then the morphism ψ in (4.5) commutes with norms.*

Proof. Let E/F be a finite extension in $\mathfrak{Field}_{/k}$, x_E a closed point of X_E and x_F the projection of x_E to X_F. We have

$$N_{E/F}([x_E]) = m[x_F] \text{ in } \mathrm{CH}_0(X_F) \quad \text{with} \quad m = [\kappa(x_E) : \kappa(x_F)] .$$

The torsor U_{x_E} over $\kappa(x_E)$ in the definition of ψ can be obtained by the base extension of the torsor U_{x_F} over $\kappa(x_F)$ to $\kappa(x_E)$. The class $[U_{x_E}]$ is the image of $[U_{x_F}]$ under the restriction map $H^1(\kappa(x_F), S) \longrightarrow H^1(\kappa(x_E), S)$ of cohomology groups. By [29, p.119, Prop. 6], we have

$$N_{\kappa(x_E)/\kappa(x_F)}([U_{x_E}]) = m[U_{x_F}] .$$

Hence,

$$\begin{aligned} N_{E/F}(\psi_E[x_E]) &= N_{E/F} N_{\kappa(x_E)/E}([U_{x_E}]) = N_{\kappa(x_E)/F}([U_{x_E}]) \\ &= N_{\kappa(x_F)/F} N_{\kappa(x_E)/\kappa(x_F)}([U_{x_E}]) = N_{\kappa(x_F)/F}(m[U_{x_F}]) \\ &= \psi_F(m[U_{x_F}]) = \psi_F(N_{E/F}([x_E])) . \end{aligned}$$

This finishes the proof. □

Proposition 4.4.9. *Let T be a k-torus and X a smooth compactification of T. Then the functor φ in (4.4) commutes with specializations.*

Proof. Let O be a DVR of geometric type over k with quotient field L and residue field K. For a point $p \in T(O)$, let $[p]$ denote the element in $\mathrm{CH}_1(X_O)$ determined by the graph of the morphism $p : \mathrm{Spec}\, O \to T \to X$. We need prove that the following diagram is commutative

$$\begin{array}{ccccc} T(L) & \longleftarrow & T(O) & \longrightarrow & T(K) \\ \varphi_L \downarrow & & \varphi_O \downarrow & & \varphi_K \downarrow \\ \mathrm{CH}_0(X_L) & \xleftarrow{j^*} & \mathrm{CH}_1(X_O) & \xrightarrow{i^!} & \mathrm{CH}_0(X_K) \end{array} \tag{4.6}$$

where the map φ_O is defined by $\varphi_O(p) := [p] - [1]$, and the maps $i^!$, j^* on the bottom are as in Example 4.4.5.

Let E be either L or K. Let p be a point in $T(O)$ and let $q \in T(E)$ be its image. We also regard p and q as morphisms $p : \mathrm{Spec}\, O \to X_O$ and $q : \mathrm{Spec}\, E \to X_E$. By (a generalization of) [11, Thm. 6.2], the following diagram is commutative

$$\begin{array}{ccc} \mathrm{CH}_1(\mathrm{Spec}\, O) & \xrightarrow{f} & \mathrm{CH}_0(\mathrm{Spec}\, E) \\ p_* \downarrow & & \downarrow q_* \\ \mathrm{CH}_1(X_O) & \xrightarrow{g} & \mathrm{CH}_0(X_E) \end{array}$$

where f is the map induced by the natural morphism $\mathrm{Spec}\, E \to \mathrm{Spec}\, O$ and $g = j^*$ if $E = L$ or $g = i^!$ if $E = K$. It follows that

$$[q] = q_*(1_E) = q_* f(1_O) = g p_*(1_O) = g([p])$$

proving the commutativity of the diagram (4.6). □

Lemma 4.4.10. *Let A be a Noetherian regular ring. Let $\mathfrak{p}$ be a prime ideal with height* $\operatorname{ht}\mathfrak{p} \geq 1$. *Then there exists a prime ideal $\mathfrak{p}' \subseteq \mathfrak{p}$ such that $A_{\mathfrak{p}}/\mathfrak{p}'A_{\mathfrak{p}}$ is a* 1*-dimensional regular local ring* (*or equivalently, a DVR*).

Proof. The ring $A_{\mathfrak{p}}$ is a regular local ring of dimension $n = \operatorname{ht}\mathfrak{p} \geq 1$. Choose elements $x_1, \dots, x_{n-1}$ from the maxmial ideal $\mathfrak{m} := \mathfrak{p}A_{\mathfrak{p}}$ such that their images in $\mathfrak{m}/\mathfrak{m}^2$ is linearly independent over the residue field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Then $A_{\mathfrak{p}}/(x_1, \dots, x_{n-1})$ is a regular local ring of dimension 1 ([30, p.79, Prop. 22]). There exists a corresponding prime ideal $\mathfrak{p}'$ of A satisfying the required properties. □

Proposition 4.4.11. *Let T be a k-torus, $\mathsf{B} : \mathfrak{Field}_{/k} \to \mathfrak{Set}$ a functor with specializations, and $\theta, \theta' : T(-)/R \to \mathsf{B}$ two morphisms of functors commuting with specializations. If the two maps $\theta_{k(T)}$ and $\theta'_{k(T)}$ coincide at the generic point of T, then $\theta = \theta'$.*

Proof. Let L/k be a field extension and $p \in T(L)$. We want to show $\theta_L(p) = \theta'_L(p)$. Let $t \in T$ the the element that lies in the image of the morphism $p : \operatorname{Spec} L \to T$. We view t as an element of $T(\kappa(t))$. Then the natural map $T(\kappa(t)) \to T(L)$ sends $t \in T(\kappa(t))$ to $p \in T(L)$. It suffices to show that $\theta_{\kappa(t)}(t) = \theta'_{\kappa(t)}(t)$.

We use induction on the codimension of t in T. By assumption, the result holds if t the generic point of T. Now suppose $\operatorname{codim}_t T > 0$. By Lemma 4.4.10, there exists a $t' \in T$ which specializes to t such that the local ring $\mathscr{O}_{t',t}$ is a DVR. Since θ and θ' commute with specializations, it follows from Lemma 4.4.7 and the induction hypothesis that

$$\theta_{\kappa(t)}(t) = \theta_{\kappa(t)}(s(t')) = s_{\mathsf{B}}(\theta_{\kappa(t')}(t')) = s_{\mathsf{B}}(\theta'_{\kappa(t')}(t')) = \theta'_{\kappa(t)}(s(t')) = \theta'_{\kappa(t)}(t)\,.$$

The proposition is thus proved. □

4.5 K-Theory of Toric Models

Let k be a field and $\bar{k}$ a fixed separable closure of k.

Definition 4.5.1. Let T be an algebraic k-torus. A ***smooth projective toric model*** of T is a smooth compactification $T \to X$ such that the translation action of T on itself extends to an action on X.

It is known that any torus admits a smooth projective toric model in arbitrary characteristic (cf. [6]).

Let T be a k-torus and X a smooth projective toric model of T. We are going to use K-theory of schemes. A survey of main results on this topic is given in Appendix A. Recall that the K-groups and K'-groups make no distinction for X since it is smooth.

For each K-group $K_n(X)$, the i-th term of the topological filtration on $K_n(X)$ will be denoted by $K_n(X)^{(i)}$. The quotient $K_n(X)^{(i)}/K_n(X)^{(i+1)}$ will be denoted by $K_n(X)^{(i/i+1)}$ or $\operatorname{gr}_i K_n(X)$. The absolute Galois group $\mathfrak{g} = \operatorname{Gal}(\bar{k}/k)$ acts naturally on the K-groups $K_n(\overline{X})$ and the subgroups $K_n(X)^{(i)}$ are stable under $\mathfrak{g}$-action. We have the BGQ-spectral sequence

$$E_1^{pq}(X) \Longrightarrow K_{-p-q}(X)$$

which converges to the K-groups of X with the topological filtration. The E_2-terms E_2^{pq} are canonically isomorphic to the K-cohomology groups $H^p(X, \mathscr{K}_{-q})$, which may be computed via the Gersten resolution. Further, we have

$$E_2^{pq} = 0\,, \quad \text{if } p < 0, \quad \text{or } \ p + q > 0\,, \quad \text{or } \ p > \dim X$$

and $E_2^{p,\,-p} = H^p(X,\, \mathscr{K}_p) = \mathrm{CH}^p(X)$. So the E_2-terms are as follows:

$$
\begin{array}{ccccc}
\mathrm{CH}^0(X) & 0 & 0 & 0 & \cdots \\
 & & & & \\
H^0(X,\, \mathscr{K}_1) & \mathrm{CH}^1(X) & 0 & 0 & \ddots \\
 & & & & \\
H^0(X,\, \mathscr{K}_2) & H^1(X,\, \mathscr{K}_2) & \mathrm{CH}^2(X) & 0 & \ddots \\
 & & & & \\
H^0(X,\, \mathscr{K}_3) & H^1(X,\, \mathscr{K}_3) & H^2(X,\, \mathscr{K}_3) & \mathrm{CH}^3(X) & \ddots
\end{array}
$$

Useful information we can draw from the above spectral sequence includes the following.

Firstly, by Thm. A.3.12, the differential map

$$d_1 : E_1^{0,\,-1} = \bigoplus_{x \in X^{(0)}} K_1(\kappa(x)) = k(X)^* \longrightarrow E_1^{1,\,-1} = \bigoplus_{x \in X^{(1)}} K_0(\kappa(x)) = Z^1(X)$$

is given by $f \in k(X)^* \mapsto [\mathrm{div}(f)] \in Z^1(X)$. It follows that

$$H^0(X,\, \mathscr{K}_1) = E_2^{0,\,-1} = \mathrm{Ker}\left(k(X)^* \xrightarrow{d_1} Z^1(X)\right) = k^*\,.$$

Secondly, we have

$$\mathrm{CH}^1(X) = E_2^{1,\,-1} \cong E_\infty^{1,\,-1} = K_0(X)^{(1/2)}\,. \tag{4.7}$$

Further, we have on the one hand a pull-back homomorphism $\alpha : k^* = K_1(k) \to K_1(X)$ induced by the structural morphism $X \to \mathrm{Spec}\ k$, and on the other hand an edge homomorphism $\beta : K_1(X) \to H^0(X,\, \mathscr{K}_1) = k^*$. As a matter of fact, the composition $\beta \circ \alpha$ is the identity on k^*. Since the edge homomorphism β factors as

$$K_1(X) = K_1(X)^{(0)} \twoheadrightarrow E_\infty^{0,\,-1} = K_1(X)^{(0/1)} \rightarrowtail E_2^{0,\,-1}$$

which is a surjection followed by an injection, it follows that the map $E_\infty^{0,\,-1} \rightarrowtail E_2^{0,\,-1}$ is also surjective, hence bijective. This implies that all the differentials starting at $E_r^{0,\,-1}$, $r \geq 2$ are zero maps. From this we see immediately that

$$\mathrm{CH}^2(X) = E_2^{2,\,-2} \cong E_\infty^{2,\,-2} = K_0(X)^{(2/3)}\,. \tag{4.8}$$

Proposition 4.5.2. *Let X be a smooth projective toric model of a k-torus T. Then the Chow motive $\overline{X} \in \mathfrak{CM}_{\bar{k}}$ is split.*

Proof. By [19, Prop. 3, Coro. 2], $\overline{X}$ satisfies the conditions (i) and (ii) of Prop. 3.2.8. The result then follows. □

By Lemma 3.2.7, we get the following corollary.

Corollary 4.5.3. *Let X be a smooth projective toric model of a k-torus T. Then the product map*

$$\mathrm{CH}^p(\overline{X}) \otimes K_q(\bar{k}) \longrightarrow H^p(\overline{X},\, \mathscr{K}_{p+q})$$

is an isomorphism for all p, q.

In the following theorem, we collect main results on K-theory of toric models that are useful for later discussion.

Theorem 4.5.4 ([24] and [23, §2.1]). *Let X be a smooth projective toric model of a k-torus. Let $d = \dim X$.*

(i) $K_0(\overline{X})$ *and* $K_0(\overline{X})^{(1)}$ *are invertible $\mathfrak{g}$-lattices.*

(ii) $K_0(\overline{X})^{(d)}$ *is isomorphic to the trivial $\mathfrak{g}$-lattice $\mathbb{Z}$, with a generator $[\mathscr{O}_x]$ where x is a rational point of $\overline{X}$. The natural inclusion $K_0(\overline{X})^{(d)} \to K_0(\overline{X})$ is a split homomorphism of $\mathfrak{g}$-lattices.*

(iii) *The natural maps*

$$K_i(X) \longrightarrow K_i(\overline{X})^{\mathfrak{g}} \quad \text{and} \quad K_i(X)^{(1)} \longrightarrow \left(K_i(\overline{X})^{(1)}\right)^{\mathfrak{g}}$$

are isomorphisms for $i \le 1$.

(iv) *The product maps*

$$K_0(\overline{X}) \otimes K_1(\bar{k}) \longrightarrow K_1(\overline{X}) \quad \text{and} \quad K_0(\overline{X})^{(1)} \otimes K_1(\bar{k}) \longrightarrow K_1(\overline{X})^{(1)}$$

are isomorphisms.

(v) *The natural map $H^1(X, \mathscr{K}_2) \to H^1(\overline{X}, \mathscr{K}_2)^{\mathfrak{g}}$ is an isomorphism.*

4.6 Zero-Cycles on 3-dimensional Toric Models

In this section, let T be a 3-dimensional k-torus and X a smooth projective toric model of T.

The E_2^{pq} being 0 for all $p > 3$, the BGQ-spectral sequence

$$E_2^{pq}(X) = H^p(X, \mathscr{K}_{-q}) \Longrightarrow K_{-p-q}(X)$$

yields an exact sequence (cf. [2, p.328, Thm. 5.11])

$$K_1(X)^{(1)} \longrightarrow H^1(X, \mathscr{K}_2) \longrightarrow \mathrm{CH}^3(X) \xrightarrow{\ g\ } K_0(X) \tag{4.9}$$

where g is the edge homomorphism.

Consider the flasque resolution of T in (4.3)

$$1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1$$

where $\hat{P}$ is the permutation $\mathfrak{g}$-lattice with $\mathbb{Z}$-basis the set of irreducible components of $\overline{X} \setminus \overline{T}$ and $\hat{S} = \mathrm{CH}^1(\overline{X})$. The pairing

$$\mathrm{CH}^1(\overline{X}) \otimes \mathrm{CH}^2(\overline{X}) \longrightarrow \mathbb{Z}\,; \quad \alpha \otimes \beta \mapsto \deg(\alpha \cdot \beta)$$

induces a perfect duality of $\mathfrak{g}$-lattices (Prop. 4.5.2 and Prop. 3.2.8). Hence, $\hat{S}^0 \cong \mathrm{CH}^2(\overline{X})$. Applying (4.7) and (4.8) to $\overline{X}$, we get isomorphisms

$$K_0(\overline{X})^{(1/2)} \cong \mathrm{CH}^1(\overline{X}) = \hat{S}, \quad K_0(\overline{X})^{(2/3)} \cong \mathrm{CH}^2(\overline{X}).$$

Thus, the exact sequence

$$0 \longrightarrow K_0(\overline{X})^{(2)} \longrightarrow K_0(\overline{X})^{(1)} \longrightarrow K_0(\overline{X})^{(1/2)} \longrightarrow 0$$

gives rise to an exact sequence of tori

$$1 \longrightarrow S_1 \longrightarrow Q \longrightarrow S^{\circ} \longrightarrow 1 \tag{4.10}$$

where S° is the dual torus of S, $\hat{S}_1^0 = K_0(\overline{X})^{(2)}$ and $\hat{Q}^0 = K_0(\overline{X})^{(1)}$. By Thm. 4.5.4 (ii), we have isomorphisms

$$\hat{S}_1^0 = K_0(\overline{X})^{(2)} \cong K_0(\overline{X})^{(2/3)} \oplus \mathbb{Z} \cong \mathrm{CH}^2(\overline{X}) \oplus \mathbb{Z} \cong \hat{S}^0 \oplus \mathbb{Z}\,.$$

Hence $S_1 \cong S \times \mathbb{G}_m$. By Thm. 4.5.4 (i), $\hat{Q}^0$ is an invertible $\mathfrak{g}$-lattice, so there exists a k-torus $\widetilde{Q}$ such that $Q \times \widetilde{Q}$ is a quasi-trivial torus. As a product of flasque tori, $S_1 \times \widetilde{Q}$ is also flasque. The exact sequence

$$1 \longrightarrow S_1 \times \widetilde{Q} \longrightarrow Q \times \widetilde{Q} \longrightarrow S^\circ \longrightarrow 1$$

is thus a flasque resolution of S°. By Thm. 4.3.7 and Lemma 4.3.5, we have

$$S^\circ(L)/R \cong H^1(L,\, S_1 \times \widetilde{Q}) \cong H^1(L,\, S_1) \cong H^1(L,\, S) \cong T(L)/R \tag{4.11}$$

for any field extension L/k. It follows from the exact sequence (4.10) that

$$\mathrm{Coker}\,(Q(k) \to S^\circ(k)) = S^\circ(k)/R\,. \tag{4.12}$$

Remark 4.6.1. We have the following interpretation of R-equivalence on a 3-dimensional torus T: there is a natural isomorphism

$$T(k)/R \cong H^1(k,\, T^\circ)/R$$

where T° is the dual torus of T. Indeed, the dual of the flasque resolution (4.3) gives an isomorphism $S^\circ(k)/R \cong H^1(k,\, T^\circ)/R$ (cf. §4.3, p.55). Using (4.11), we get

$$T(k)/R \cong S^\circ(k)/R \cong H^1(k,\, T^\circ)/R\,.$$

The edge homomorphism $g : \mathrm{CH}^3(X) \to K_0(X)$ factors as

$$\mathrm{CH}^3(X) \xrightarrow{\eta} E_\infty^{3,\,-3} = K_0(X)^{(3)} \hookrightarrow K_0(X)$$

where η sends the class of a closed point x to the class $[\mathscr{O}_x]$, and the composition

$$\mathrm{CH}^3(X) \longrightarrow \mathrm{CH}^3(\overline{X}) \xrightarrow{\overline{\eta}} K_0(\overline{X})^{(3)} \cong \mathbb{Z}$$

is the degree map (cf. [24, §5]). Since the map $K_0(X)^{(3)} \to K_0(\overline{X})^{(3)}$ is injective by Thm. 4.5.4, taking into account the following commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^3(X) & \xrightarrow{\ \eta\ } & K_0(X)^{(3)} \\ \downarrow & & \downarrow \\ \mathrm{CH}^3(\overline{X}) & \xrightarrow{\ \overline{\eta}\ } & K_0(\overline{X})^{(3)} \end{array}$$

we conclude that

$$\mathrm{Ker}\,(g) = A_0(X)\,. \tag{4.13}$$

Using Thm. 4.5.4 (iii)–(v) and Coro. 4.5.3, we obtain isomorphisms

$$K_1(X)^{(1)} \cong \Big(K_1(\overline{X})^{(1)}\Big)^{\mathfrak{g}} \cong \Big(K_0(\overline{X})^{(1)} \otimes K_1(\bar{k})\Big)^{\mathfrak{g}} = (\hat{Q}^0 \otimes \bar{k}^*)^{\mathfrak{g}} = Q(k) \tag{4.14}$$

and

$$H^1(X,\, \mathscr{K}_2) \cong H^1(\overline{X},\, \mathscr{K}_2)^{\mathfrak{g}} \cong \Big(\mathrm{CH}^1(\overline{X}) \otimes K_1(\bar{k})\Big)^{\mathfrak{g}} = (\hat{S} \otimes \bar{k}^*)^{\mathfrak{g}} = S^\circ(k)\,. \tag{4.15}$$

It follows from the fact that the BGQ-spectral sequence is compatible with products (cf. [12, §7]) that, the map $K_1(X)^{(1)} \to H^1(X, \mathscr{K}_2)$ in (4.9) coincides with the map $Q(k) \to S^\circ(k)$ given by (4.10) under the identifications (4.14) and (4.15). Hence, by (4.9), (4.12) and (4.13), we have

$$\begin{aligned} S^\circ(k)/R &= \mathrm{Coker}\,(Q(k) \to S^\circ(k)) \\ &= \mathrm{Coker}\,(K_1(X)^{(1)} \to H^1(X, \mathscr{K}_2)) \cong \mathrm{Ker}\,(g) = A_0(X)\,. \end{aligned} \tag{4.16}$$

Combined with (4.11), this yields

$$T(k)/R \cong S^\circ(k)/R \cong A_0(X)\,. \tag{4.17}$$

We have thus proved the following result.

Proposition 4.6.2. *Let T be a 3-dimensional k-torus and X a smooth projective toric model. Then there is an isomorphism of functors from $\mathfrak{Field}_{/k}$ to $\mathfrak{Set}$*

$$\rho : \; T(-)/R \xrightarrow{\sim} A_0(X_{(-)}) \tag{4.18}$$

For each $L \in \mathfrak{Field}_{/k}$, $\rho_L : T(L)/R \xrightarrow{\sim} A_0(X_L)$ is in fact a group isomorphism.

Remark 4.6.3. When the field k is finitely generated over its prime field, or over $\mathbb{C}$, or is a p-adic field, the group $T(k)/R$ is known to be finite (cf. [7] and [5]). In all these cases, the isomorphism (4.17) implies finiteness of the group $A_0(X)$.

Theorem 4.6.4 (***Main Theorem***)**.** *Let T be a k-torus of dimension ≤ 3 and X a smooth compactification of T. Then the map*

$$\varphi : T(k)/R \longrightarrow A_0(X)\,; \quad t \mapsto [t] - [1]$$

is an isomorphism of groups.

First proof of main theorem. As mentioned earlier, we need only prove the theorem for 3-dimensional tori.

The field k is the union of all the subfields k' that are finitely generated over its prime field. There exists a subfield k_0 that is finitely generated over the prime field such that T and X are defined over k_0. If k'/k_0 and k''/k_0 are two subextensions of k/k_0 such that k', k'' are finitely generated over the prime field and $k' \subseteq k''$, then the natural maps $T(k')/R \to T(k'')/R$ and $A_0(X_{k'}) \to A_0(X_{k''})$ are compatible with φ. If $\varphi_{k'} : T(k')/R \to A_0(X_{k'})$ is an isomorphism for every k', then so is the map $\varphi_k : T(k)/R \to A_0(X_k)$. So we may assume that the field k itself is finitely generated over the prime field.

By [7, p.192, Thm. 1], $T(k)/R$ is finite. Then (4.18) shows that $A_0(X)$ is a finite group of the same cardinality. Since $\varphi : T(k)/R \to A_0(X)$ is injective, it is also bijective. It has an inverse $\widetilde{\psi}$ which is known to be a group homomorphism. So φ is an isomorphism of groups. □

The above proof is straightforward and does not use the machinery of norms and specializations of functors. But it is based on deep, albeit classical, arithmetic-geometric result. We are going to give an alternative proof of our main theorem in the next section.

4.7 Second Proof of Main Theorem

Proposition 4.7.1. *Let T be a k-torus of dimension 3 and X a smooth projective toric model of T. Then the isomorphism of functors ρ in (4.18) commutes with norms.*

Proof. In view of Examples 4.4.4 (2), one finds easily that the isomorphism $T(L)/R \cong S^\circ(L)/R$ in (4.11) commutes norms. It is therefore sufficient to prove the isomorphism $S^\circ(L)/R \xrightarrow{\sim} A_0(X_L)$ given by (4.16) commutes with norms.

Let E/F be a finite extension in $\mathfrak{Field}_{/k}$. We need only prove that the following diagram is commutative

$$\begin{array}{ccccc} S^\circ(E) & \xrightarrow{\sim} & H^1(X_E\,,\mathscr{K}_2) & \longrightarrow & \mathrm{CH}^3(X_E)=\mathrm{CH}_0(X_E) \\ {\scriptstyle N_{E/F}}\big\downarrow & & {\scriptstyle N_{E/F}}\big\downarrow & & {\scriptstyle N_{E/F}}\big\downarrow \\ S^\circ(F) & \xrightarrow{\sim} & H^1(X_F\,,\mathscr{K}_2) & \longrightarrow & \mathrm{CH}^3(X_F)=\mathrm{CH}_0(X_F) \end{array} \tag{4.19}$$

The functor $f_* : \mathcal{M}(X_E) \to \mathcal{M}(X_F)$ induced by the natural projection $f : X_E \to X_F$ takes $\mathcal{M}^p(X_E)$ into $\mathcal{M}^p(X_F)$. So f_* induces a morphism of the BGQ-spectral sequences for X_E and X_F. Thus the commutativity of the right square in (4.19) follows from the functoriality of the BGQ-spectral sequences.

By Thm. 4.5.4 (v), the left square in (4.19) injects into the following diagram

$$\begin{array}{ccc} S^\circ(\overline{E}) & \longrightarrow & H^1(X_{\overline{E}}\,,\mathscr{K}_2) \\ {\scriptstyle N_{\overline{E}/\overline{F}}}\big\downarrow & & \big\downarrow{\scriptstyle N_{\overline{E}/\overline{F}}} \\ S^\circ(\overline{F}) & \longrightarrow & H^1(X_{\overline{F}}\,,\mathscr{K}_2) \end{array}$$

Hence, to prove the left square is commutative, we may assume the torus S° splits. Then the square becomes

$$\begin{array}{ccc} \hat{S}\otimes E^* & \longrightarrow & H^1(X_E\,,\mathscr{K}_2) \\ {\scriptstyle N_{E/F}}\big\downarrow & & \big\downarrow{\scriptstyle N_{E/F}} \\ \hat{S}\otimes F^* & \longrightarrow & H^1(X_F\,,\mathscr{K}_2) \end{array}$$

where the horizontal maps are product maps under the identification $\hat{S} = \mathrm{CH}^1(X)$. The commutativity then follows from the projection formula in K-cohomology (cf. [28, §14.5]). □

Proposition 4.7.2. *Let T be a 3-dimensional k-torus and X a smooth projective toric model of T. Then the isomorphism of functors ρ in* (4.18) *commutes with specializations.*

Proof. Note that the isomorphism in (4.11) commutes with specializations. We need only prove that the isomorphism $S^\circ(L)/R \xrightarrow{\sim} A_0(X_L)$ in (4.16) commutes with specializations.

Let O be a DVR of geometric type over k with quotient field L and residue field K. We first consider the diagram

$$\begin{array}{ccccc} H^1(X_K,\mathscr{K}_2) & \longleftarrow & H^1(X_O,\mathscr{K}_2) & \longrightarrow & H^1(X_L,\mathscr{K}_2) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{CH}^3(X_K) & \longleftarrow & \mathrm{CH}^3(X_O) & \longrightarrow & \mathrm{CH}^3(X_L) \end{array} \tag{4.20}$$

where the vertical maps are the differentials in the corresponding BGQ-spectral sequences. The right square is commutative because the natural morphism $g : X_L \to X_O$ is flat, it induces an exact functor $\mathcal{M}(X_O) \to \mathcal{M}(X_L)$ that respects the topological filtration, hence a pull-back homomorphisms of the BGQ-spectral sequences.

Note that X_O is projective over O. X_O admits an ample invertible sheaf. So the natural morphism $f : X_K \to X_O$ induces an exact functor $f^* : \mathcal{M}(X_O, f) \to$

$\mathcal{M}(X_K)$ that respects the topological filtration (cf. §A.3.1, p.76). It follows that f induces a pull-back homomorphism of the BGQ-spectral sequences for X_O and X_K, whence the commutativity of the left square in (4.20).

It remains to show the following diagram is commutative

$$\begin{array}{ccccc} S^\circ(K) & \longleftarrow & S^\circ(O) & \longrightarrow & S^\circ(L) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(X_K,\, \mathscr{K}_2) & \longleftarrow & H^1(X_O,\, \mathscr{K}_2) & \longrightarrow & H^1(X_L,\, \mathscr{K}_2) \end{array}$$

As in the proof of Prop. 4.7.1, we may assume the torus S° splits. Then the vertical maps in the above diagram are the product maps under the identifications $S^\circ(A) = \hat{S} \otimes A^* = \mathrm{CH}^1(X) \otimes A^*$ for $A = K$, O or L. The commutativity is then a consequence of the projection formula in K-cohomology (cf. [28, §14.5]). □

Let T be a k-torus and let $1 \to S \to P \to T$ be a flasque resolution of T. Let $\mathrm{End}_k(S)$ be the group of endomorphism of the k-torus S. An element $\alpha \in \mathrm{End}_k(S)$ determines an endomorphism of the functor $T(-)/R = H^1(-,\, S)$, and this morphism of functors commutes with norms and specializations (cf. Examples 4.4.4 (4) and Example 4.4.5).

Proposition 4.7.3. *Let T be a k-torus, $1 \to S \to P \to T \to 1$ a flasque resolution and ξ the generic point of T. Then every element of the group $T(k(T))/R$ is of the form $t \cdot \alpha(\xi)$, where $t \in T(k)/R$ and $\alpha \in \mathrm{End}_k(S)$.*

Proof. Consider the exact sequence

$$0 \longrightarrow \bar{k}[\,\overline{T}\,]^* \longrightarrow \bar{k}\big(\,\overline{T}\,\big)^* \longrightarrow \mathrm{Div}(\,\overline{T}\,) \longrightarrow 0\,.$$

The group of units $\bar{k}[\,\overline{T}\,]^*$ of the ring of regular functions $\bar{k}[\,\overline{T}\,]$ on $\overline{T}$ is the direct product of $\bar{k}^*$ with $\hat{T}$ (see, e.g., [3, p.6, Prop. 4.1]). Tensoring with $\hat{S}^0$ yields an exact sequence

$$0 \longrightarrow (\hat{S}^0 \otimes \bar{k}^*) \times (\hat{S}^0 \otimes \hat{T}) \longrightarrow \hat{S}^0 \otimes \bar{k}\big(\,\overline{T}\,\big)^* \longrightarrow \mathrm{Div}(\,\overline{T}\,) \longrightarrow 0\,.$$

Taking Galois cohomology gives an exact sequence

$$H^1(k,\, S) \times H^1(k,\, \hat{S}^0 \otimes \hat{T}) \longrightarrow H^1(k(T),\, S) \longrightarrow H^1(k,\, \hat{S}^0 \otimes \mathrm{Div}(\,\overline{T}\,))\,.$$

Note that $\hat{S}^0$ is a coflasque lattice and $\mathrm{Div}(\,\overline{T}\,)$ is a (usually infinite) direct sum of permutation lattices. So we have $H^1(k,\, \hat{S}^0 \otimes \mathrm{Div}(\,\overline{T}\,)) = 0$ ([7, p.179, Lemme 1 (vi)]), whence a surjection

$$H^1(k,\, S) \times H^1(k,\, \hat{S}^0 \otimes \hat{T}) \longrightarrow H^1(k(T),\, S)\,. \tag{4.21}$$

Similarly, by tensoring the exact sequence

$$0 \longrightarrow \hat{T} \longrightarrow \hat{P} \longrightarrow \hat{S} \longrightarrow 0$$

with $\hat{S}^0$ and taking Galois cohomomogy, we get a surjection

$$\mathrm{End}_k(S) = H^0(k,\, \hat{S}^0 \otimes \hat{S}) \longrightarrow H^1(k,\, \hat{S}^0 \otimes \hat{T}) \tag{4.22}$$

since $H^1(k,\, \hat{S}^0 \otimes \hat{P}) = 0$. Combining the two surjective homomorphisms in (4.21) and (4.22), we obtain a surjective homomorphism

$$\big(T(k)/R\big) \times \mathrm{End}_k(S) \longrightarrow T(k(T))/R\,.$$

After a careful inspection on the construction of this map, we see that the image of an element $(1,\, \alpha) \in \big(T(k)/R\big) \times \mathrm{End}_k(S)$ in $T(k(T))/R$ is equal to $\alpha(\xi)$. The result then follows. □

Corollary 4.7.4. *Let T be a 3-dimensional k-torus, $1 \to S \to P \to T \to 1$ the flasque resolution in* (4.3) *and X a smooth projective toric model of T. There is an $\alpha \in \mathrm{End}_k(S)$ such that $\rho^{-1} \circ \varphi = \alpha$ as endomorphisms of the functor $T(-)/R$.*

Proof. By Prop. 4.7.3, there is an $\alpha \in \mathrm{End}_k(S)$ and a $t_0 \in T(k)/R$ such that $\rho^{-1} \circ \varphi(\xi) = t_0 \cdot \alpha(\xi)$, where ξ is the generic point of T. By Props. 4.4.9 and 4.7.2, the morphism $\rho^{-1} \circ \varphi$ commutes with specializations. So Prop. 4.4.11 implies that $\rho^{-1} \circ \varphi = t_0 \cdot \alpha$ as morphism of functors. Since $\rho^{-1} \circ \varphi(1) = 1 = \alpha(1)$, we get $t_0 = 1$. □

Proposition 4.7.5. *Let T be a 3-dimensional k-torus and X a smooth compactification of T. Then the morphism of functors φ in* (4.4) *is an isomorphism if and only if it commutes with norms.*

Proof. If φ is an isomorphism, then it commutes with norms because its inverse $\tilde{\psi}$ does by Prop. 4.4.8.

Conversely, suppose φ commutes with norms. We may assume X is a toric model. Then Coro. 4.7.4 implies that $\varphi_F : T(F)/R \to A_0(X_F)$ is a group homomorphism for each $F \in \mathfrak{Field}_{/k}$. We already know that φ_F is injective. It remains to show φ_F is surjective. By Prop. 1.5.1, every closed point of X is rationally equivalent to a zero-cycle with support in T. Since φ_F is a group homomorphism, it suffices to show that for any closed point x in T, with $\deg [x] = n$, the element $[x] - n[1] \in A_0(X_F)$ is in the image of φ_F. Let $E = \kappa(x)$ and let $x' \in T(E)$ be the natural point over x. By the commutativity of φ with norms, we get

$$[x] - n[1] = N_{E/F}([x'] - [1]) = N_{E/F}(\varphi_E(x')) = \varphi_F(N_{E/F}(x'))\,.$$

Hence, φ_F is surjective. □

Second proof of main theorem. We may assume X is a smooth projective toric model. By Coro. 4.7.4 and Prop. 4.7.1, the morphism of functors φ commutes with norms. It follows from Prop. 4.7.5 that φ is an isomorphism. □

4.8 Chow Groups of Lower Dimensional Tori

We are interested in the Chow group $\mathrm{CH}_0(T)$ of zero-cycles of a torus T over a field k.

Consider first the easiest case: $\dim T = 1$. If $T = \mathbb{G}_m$, we have $\mathrm{CH}_0(T) = 0$ since $\mathrm{CH}_0(\mathbb{A}^1) = 0$ and as an open subset of $\mathbb{A}^1$ there is a surjection $\mathrm{CH}_0(\mathbb{A}^1) \to \mathrm{CH}_0(\mathbb{G}_m)$. Now suppose $T = R^1_{K/k}\mathbb{G}_m$ for a quadratic extension K/k. For simplicity, assume $\mathrm{char}\, k \neq 2$. Then $K = k(\sqrt{a})$ for some $a \in k^* \setminus k^{*2}$. Using the k-basis $\{ 1, \sqrt{a} \}$ of K, one finds easily that $T \cong \mathrm{Spec}\,\big(k[x, y]/(x^2 - ay^2 - 1)\big)$ as k-varieties. Then the conic X defined by $x^2 - ay^2 = z^2$ in $\mathbb{P}^2_k$ is a smooth compactification of T. Since X has a rational point $(1 : 0 : 1)$, we have $X \cong \mathbb{P}^1_k$ so that $\mathrm{CH}_0(X) = \mathbb{Z} \cdot [1]$, where 1 denotes the neutral element of the group $T(k)$. The complement $Z := X \setminus T$ is isomorphic to $\mathrm{Proj}\,\big(k[x, y, z]/(x^2 - ay^2, z)\big) \cong \mathrm{Proj}\,\big(k[x, y]/(x^2 - ay^2)\big)$. So Z consists of a single point $P = (\sqrt{a} : 1 : 0)$ with residue field $\kappa(P) = K$. The natural map $\mathrm{CH}_0(Z) \longrightarrow \mathrm{CH}_0(X) = \mathbb{Z} \cdot [1]$ sends P to $(\deg P) \cdot [1] = 2 \cdot [1]$. Hence, from the exact sequence

$$\mathrm{CH}_0(Z) \longrightarrow \mathrm{CH}_0(X) = \mathbb{Z} \longrightarrow \mathrm{CH}_0(T) \longrightarrow 0$$

it follows that $\mathrm{CH}_0(T) = \mathbb{Z}/2\mathbb{Z}$.

So $\mathrm{CH}_0(T)$ can be easily computed if $\dim T = 1$. However, it seems that the question is so easy only for 1-dimensional tori. A recent theorem (Thm. 4.8.6), due to Merkurjev, provides a method for computing $\mathrm{CH}_0(T)$ for tori T with $\dim T \leq 3$.

Definition 4.8.1. Let T be a k-torus. We say T is ***anisotropic*** over k if $\hat{T}^{\mathfrak{g}} = 0$, where $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$ is the absolute Galois group of k and $\hat{T}$ is the group of characters of T. If T is not anisotropic, we also say it is ***isotropic***.

Lemma 4.8.2. *Let T be an isotropic k-torus. Then*
(i) *T contains a (closed) subgroup isomorphic to $\mathbb{G}_m$;*
(ii) *T is stably birational to an anisotropic torus T_1 with $\dim T_1 < \dim T$.*

Proof. Let K/k be a finite Galois extension such that T splits over K. Let $G = \mathrm{Gal}(K/k)$ and we may regard $\hat{T}$ as a G-lattice. Define $N = \sum_{s\in G} s$. The image of multiplication by N on $\hat{T}$ is a G-sublattice of $\hat{T}$ with trivial G-action. Its kernel is a lattice L such that $L^G = 0$. So T can be put into an exact sequence of tori

$$1 \longrightarrow T_0 \longrightarrow T \longrightarrow T_1 \longrightarrow 1$$

where T_0 is split and T_1 is anisotropic. By Hilbert's theorem 90, any torsor under $\mathbb{G}_m$ is locally trivial for the Zariski topology. So there is a k-birational equivalence between T and $T_0 \times_k T_1$. T_0 is isomorphic to a product of copies of $\mathbb{G}_m$ and T_1 is anisotropic of dimension less than $\dim T$. This finishes the proof. □

Lemma 4.8.3. *Let T be a k-torus, X a smooth compactification of T and $Z = X\backslash T$. Then T is isotropic over k if and only if $Z(k) \neq \emptyset$.*

Proof. The sufficiency part is a restatement of [7, p.203, Lemme 12] or [33, Prop. 17.3].

Now suppose T is isotropic. By Lemma 4.8.2, T contains a subgroup isomorphic to $\mathbb{G}_m$. The embedding of $\mathbb{G}_m$ into T extends to a morphism $f : \mathbb{P}^1 \to X$. Since f is nonconstant, $f(\mathbb{P}^1) \nsubseteq T$. Thus $f(0)$ or $f(\infty)$ is a rational point of Z. □

Let X be a smooth compactification of a torus T and $Z = X \setminus T$. Define

$$i_T := \gcd\{\, [L : k] \mid L/k \text{ a finite extension such that } T \text{ is isotropic over } L \,\},$$

and

$$n_Z := \gcd\{\, [L : k] \mid L/k \text{ a finite extension such that } Z(L) \neq \emptyset \,\}.$$

Corollary 4.8.4. *With notation as above, the number i_T coincides with n_Z. Therefore, the integer n_Z does not depend on the choice of the smooth compactification.*

Proof. This is immediate from Lemma 4.8.3. □

Proposition 4.8.5. *Let T be a k-torus. The class $[1]$ of $1 \in T(k)$ is an element of order i_T in the group $\mathrm{CH}_0(T)$.*

Proof. If T is isotropic, there is a closed subgroup T_0 of T isomorphic to $\mathbb{G}_m$. As $\mathrm{CH}_0(\mathbb{G}_m) = 0$, we have $[1] = 0$ in $\mathrm{CH}_0(T_0)$ and hence also in $\mathrm{CH}_0(T)$.

In the general case, let L/k be any finite field extension such that T is isotropic over L. Then we have $[1] = 0$ in $\mathrm{CH}_0(T_L)$, and hence applying the norm map $N_{L/k}$ yields $[L : k] \cdot [1] = 0$ in $\mathrm{CH}_0(T)$. So $i_T \cdot [1] = 0$ and $[1] \in \mathrm{CH}_0(T)$ is an element of finite order. Let m be its order in $\mathrm{CH}_0(T)$. Then the cycle $m \cdot [1] \in \mathrm{CH}_0(X)$ lies in the image of the natural push-forward map $\mathrm{CH}_0(Z) \to \mathrm{CH}_0(X)$. In particular, there is a zero-cycle on Z that has degree m. This implies that there exist integers $m_i \in \mathbb{Z}$ and finite extensions L_i/k such that $Z(L_i) \neq \emptyset$ for every i and $m = \sum_i m_i[L_i : k]$. It follows that $n_Z = i_T$ divides m. Hence $i_T = m$ as required. □

Theorem 4.8.6 (Merkurjev)**.** *Let T be a k-torus of dimension at most 3. Then the map*

$$\mu_T : \big(T(k)/R\big) \oplus (\mathbb{Z}/i_T\mathbb{Z}) \longrightarrow \mathrm{CH}_0(T)\,; \quad (t,\, l) \mapsto [t] - [1] + l \cdot [1]$$

is an isomorphism.

Proof. The map μ_T is well-defined according to Prop. 4.8.5.

Since X has a rational point, we have a natural exact sequence

$$0 \longrightarrow A_0(X) \longrightarrow \mathrm{CH}_0(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

which yields $\mathrm{CH}_0(X) = A_0(X) \oplus \mathbb{Z}\cdot[1]$. Let $z \in Z$ be a closed point and let $F = \kappa(z)$ be its residue field. By Lemma 4.8.3, T_F is isotropic. Lemma 4.8.2 shows that T_F is stably birational to a torus of dimension ≤ 2. So T_F is stably rational and thus $T(F)/R = 0$ (Coro. 4.3.8). Using (4.18), we get $A_0(X_F) = 0$. Consider now the following commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0(Z_F) & \longrightarrow & \mathrm{CH}_0(X_F) \\ {\scriptstyle N_{F/k}}\downarrow & & \downarrow{\scriptstyle N_{F/k}} \\ \mathrm{CH}_0(Z) & \longrightarrow & \mathrm{CH}_0(X) \end{array}$$

where the horizontal maps are the natural push-forward maps. Let $z' \in Z_F$ be the canonical rational point lying over $z \in Z$. The image of z' in $\mathrm{CH}_0(X_F) = A_0(X_F) \oplus \mathbb{Z} \cdot [1]$ is equal to $([z'] - [1], [1]) = (0, [1])$. The diagram shows that the image of z in $\mathrm{CH}_0(X) = A_0(X) \oplus \mathbb{Z} \cdot [1]$ is equal to $(0, [F : k] \cdot [1])$. It follows that the image of $\mathrm{CH}_0(Z)$ in $\mathrm{CH}_0(X) = A_0(X) \oplus \mathbb{Z} \cdot [1]$ is equal to $0 \oplus n_Z\mathbb{Z} = 0 \oplus i_T\mathbb{Z}$. Thus, the natural exact sequence

$$\mathrm{CH}_0(Z) \longrightarrow \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(T) \longrightarrow 0$$

together with Thm. 4.6.4 gives the desired result. □

Now let us compute the group $\mathrm{CH}_0(T)$ for some examples of lower dimensional tori, using Merkurjev's theorem.

Examples 4.8.7.

(1) Let K/k be a quadratic extension and $T = R^1_{K/k}\mathbb{G}_m$. We have seen that if $\mathrm{char}\, k \neq 2$, T is isomorphic to a plane curve given by an equation of the form $x^2 - ay^2 - 1 = 0$, where $a \in k^* \setminus k^{*2}$. The conic X defined by $x^2 - ay^2 = z^2$ in $\mathbb{P}^2_k$ is a smooth compactification of T and $Z = X \setminus T$ may be defined by $x^2 - ay^2 = 0 = z$. Clearly, $n_Z \neq 1$ and $n_Z \,|\, 2$. So $n_Z = 2$. Moreover, T is a rational curve, hence $T(k)/R = 0$. So Merkurjev's theorem gives the same result as it should be.

It is also easy to compute directly the number i_T. In this way, we can even work in arbitrary characteristic. Indeed, let $G = \mathrm{Gal}(K/k)$. $\hat{T}$ can be regarded as a G-module and it may be put into an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \longrightarrow \hat{T} \longrightarrow 0\,.$$

Clearly, $\hat{T}^G = 0$ so that T is isotropic over K but not over k. Hence $i_T = 2$.

(2) Let K/k be a cyclic Galois extension of degree 3 and let $T = R_{K/k}\mathbb{G}_m$. As an open subset of $\mathbb{A}^3_k$, we have $\mathrm{CH}_0(T) = 0$. Now look at what Merkurjev's theorem gives us. We have $T(k)/R = 0$ since T is rational. The character group is $\hat{T} = \mathbb{Z}[G]$ where $G = \mathrm{Gal}(K/k) = \{\, \sigma_0,\, \sigma_1\, \sigma_2 \,\} \cong \mathbb{Z}/3\mathbb{Z}$. Since $\hat{T}^G = \mathbb{Z}(\sigma_0 + \sigma_1 + \sigma_2) \neq 0$, T is isotropic over k and hence $i_T = 1$. So Merkurjev's theorem gives indeed the expected result: $\mathrm{CH}_0(T) = 0$.

(3) Assume $\mathrm{char}\, k = 0$. Let $a \in k^* \setminus k^{*3}$, $K = k(\sqrt[3]{a})$ and $T = R^1_{K/k}\mathbb{G}_m$. Then $\dim T = 2$ and T is rational. Using the k-basis $\{\, 1\,,\, \sqrt[3]{a},\, \sqrt[3]{a^2} \,\}$ of K, one finds that

$$T \cong \mathrm{Spec}\, \left(k[x_1,\, x_2,\, x_3]/(x_1^3 + ax_2^3 + a^2x_3^3 - 3ax_1x_2x_3 - 1)\right)\,.$$

Let

$$X' := \mathrm{Proj}\, \left(k[x_1,\, x_2,\, x_3,\, y]/(x_1^3 + ax_2^3 + a^2x_3^3 - 3ax_1x_2x_3 - y^3)\right)$$

and $Z' := X' \setminus T$. Then $Z'(k) = \emptyset$, because if there was $(\alpha_1 : \alpha_2 : \alpha_3 : 0) \in Z'(k)$, one would get a nonzero element $\alpha = \alpha_1 + \alpha_2 \sqrt[3]{a} + \alpha_3 \sqrt[3]{a^2}$ in K which satisfies $N_{K/k}(\alpha) = 0$, but this is absurd since K is a field.

Let X be a desingularization of X' that respects the open set T and let $Z = X \backslash T$. Then $Z(k) = \emptyset$. Moreover, $Z(K) \neq \emptyset$ since there exists a point $P = (\sqrt[3]{a} : -1 : 0 : 0) \in Z'(K)$ which is nonsingular in X'. It follows that $n_Z \neq 1$ and $n_Z \,|\, 3$, i.e. $n_Z = 3$. So by Merkurjev's theorem, $\mathrm{CH}_0(T) \cong \mathbb{Z}/3\mathbb{Z}$, generated by $[1] \in \mathrm{CH}_0(T)$.

(4) Let K/k be a cyclic Galois extension of degree 3 and let $T = R^1_{K/k}\mathbb{G}_m$. Then $\dim T = 2$, T is rational and therefore $T(k)/R = 0$. The exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\varepsilon^0} \mathbb{Z}[G] \longrightarrow \hat{T} \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\varepsilon^0} \mathbb{Z}[G]^G \longrightarrow \hat{T}^G \longrightarrow 0$$

since $H^1(G, \mathbb{Z}) = 0$. We know that $\mathbb{Z}[G]^G = \varepsilon^0(\mathbb{Z})$, so $\hat{T}^G = 0$. Hence $i_T = 3$ and $\mathrm{CH}_0(T) \cong \mathbb{Z}/3\mathbb{Z}$.

(5) Suppose $K = k(\sqrt{a}\,, \sqrt{b}\,)$ is a Galois extension of k with group $G = \mathrm{Gal}(K/k) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let $T = R^1_{K/k}\mathbb{G}_m$. Write $G = \{\, e_{ij} \,|\, 0 \le i,\, j \le 1 \,\}$ in such a way that $e_{ij} \mapsto (i,\, j) \in (\mathbb{Z}/2\mathbb{Z})^2$ is an isomorphism. Considering the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\varepsilon^0} \mathbb{Z}[G] \longrightarrow \hat{T} \longrightarrow 0$$

we see that $\mathbb{Z}[G]^G = \varepsilon^0(\mathbb{Z})$ which implies $\hat{T}^G = 0$, and that $\mathbb{Z}[G]^{e_{01}} = \mathbb{Z}(e_{00} + e_{01}) + \mathbb{Z}(e_{10} + e_{11}) \neq \varepsilon^0(\mathbb{Z})$ which implies $\hat{T}^{e_{01}} \neq 0$. So we get $i_T = 2$ and thus $\mathrm{CH}_0(T) \cong \big(T(k)/R\big) \oplus \mathbb{Z}/2\mathbb{Z}$. In this case, to know $\mathrm{CH}_0(T)$ is to know $T(k)/R$. By using an exact sequence which involves the "Shafarevich groups", one may prove that if k is a number field, then $T(k)/R \cong (\mathbb{Z}/2\mathbb{Z})^{s-1}$ for some $s \ge 1$.

As a specific example, if $K/k = \mathbb{Q}(\sqrt{-1}\,, \sqrt{2})/\mathbb{Q}$, one has $T(\mathbb{Q})/R = 0$ (cf. [33, p.183, Example 1]). So in that case, we have $\mathrm{CH}_0(T) = \mathbb{Z}/2\mathbb{Z}$.

(6) Let K/k be a cyclic Galois extension of degree 4 and let $T = R^1_{K/k}\mathbb{G}_m$. We have $T(k)/R = 0$ by [33, p.182, Prop. 2 or Thm. 1]. Write $G = \mathrm{Gal}(K/k) = \{\, \sigma_0, \sigma_1\,, \sigma_2\,, \sigma_3 \,\}$. Considering the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\varepsilon^0} \mathbb{Z}[G] \longrightarrow \hat{T} \longrightarrow 0$$

we find that $\mathbb{Z}[G]^G = \varepsilon^0(\mathbb{Z})$ which implies $\hat{T}^G = 0$, and that $\mathbb{Z}[G]^{\sigma_2} = \mathbb{Z}(\sigma_0 + \sigma_2) + \mathbb{Z}(\sigma_1 + \sigma_3) \neq \varepsilon^0(\mathbb{Z})$ which implies $\hat{T}^{\sigma_2} \neq 0$. Therefore, $i_T = 2$ and we get $\mathrm{CH}_0(T) = \mathbb{Z}/2\mathbb{Z}$ by Merkurjev's theorem.

Appendix A

Survey on Higher Algebraic K-Theory

In this appendix, we collect basic facts and results on Quillen's higher algebraic K-theory. Things of greatest interest for us will be the insights to the Chow groups this theory has led to. The basic reference is Quillen's paper [27].

A.1 Classifying Space of a Category

A.1.1 Simplicial Sets

For each nonnegative integer n, let

$$\underline{\mathbf{n}} := \{ 0 < 1 < \cdots < n \}$$

be the ordered set consisting of $0, 1, \ldots, n$. Form a category $\triangle$ by taking the ordered sets $\underline{\mathbf{n}}$ as objects and taking as morphisms the maps $f : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ with the property that $f(i) \leq f(j)$ for all $i < j$.

For each positive integer n, we have $n+1$ morphisms in $\triangle$:

$$\partial_i^n : \underline{\mathbf{n-1}} \longrightarrow \underline{\mathbf{n}} , \quad i = 0 , \ldots , n$$

given by

$$\partial_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

We call these ∂_i^n the ***face maps***. On the other hand, we have n maps

$$s_i^{n-1} : \underline{\mathbf{n}} \longrightarrow \underline{\mathbf{n-1}} , \quad i = 0 , \ldots , n-1$$

given by

$$s_i^{n-1}(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

These are called ***degeneracy maps***.

Definition A.1.1. A ***simplicial object*** in a category $\mathcal{C}$ is a (covariant) functor from $\triangle^{op}$ to $\mathcal{C}$. A ***morphism*** of simpicial objects in $\mathcal{C}$ is a natural transformation of functors. A ***simplicial set*** (resp. ***simplicial space***) is a simplicial object in $\mathfrak{Set}$ (resp. $\mathfrak{Top}$).

Let $F : \triangle^{op} \to \mathfrak{Set}$ be a simplicial set. Then for each nonnegative integer n, the set $F(\underline{\mathbf{n}})$ is called the set of ***n-simplices*** of F. The maps ∂_i^n give rise to $(n+1)$ maps of sets $F(\partial_i^n) : F(\underline{\mathbf{n}}) \to F(\underline{\mathbf{n-1}})$, called the ***face maps***, which associate to each n-simplex in $F(\underline{\mathbf{n}})$ a collection of $(n-1)$-simplices in $F(\underline{\mathbf{n-1}})$. Similarly, the n maps s_i^{n-1} give maps $F(s_i^{n-1}) : F(\underline{\mathbf{n-1}}) \to F(\underline{\mathbf{n}})$, associating to each $(n-1)$-simplex a collection of "degenerate" n-simplices; these maps $F(s_i^{n-1})$ are called ***degeneracies***.

For $\delta \in F(\underline{\mathbf{n}})$, we call $F(\partial_i^n)(\delta) \in F(\underline{\mathbf{n-1}})$ the i-th ***face*** of δ, and $F(s_i^n)(\delta) \in F(\underline{\mathbf{n+1}})$ the i-th ***degenerate simplex*** of δ.

A.1.2 Geometric Realization

To each simplicial set $F : \triangle^{op} \to \mathfrak{Set}$, one can associate a topological space $|F|$, called the ***geometric realization*** of F. It is defined as the quotient space of $\coprod_{n\geq 0} F(\underline{\mathbf{n}}) \times \Delta_n$ modulo a suitable equivalence relation (cf. [31, §3]), where $F(\underline{\mathbf{n}})$ is given the discrete topology and Δ_n is the standard n-simplex in $\mathbb{R}^{n+1}$, i.e.

$$\Delta_n := \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,|\, t_i \geq 0 \text{ and } \sum t_i = 1 \} .$$

This construction of geometric realization is functorial in the sense that if $F \to G$ is a morphism of simplicial sets, there is a corresponding continuous map $|F| \to |G|$ of topological spaces. If $F \times G$ denotes the simplicial set whose n-simplices are $F(\underline{\mathbf{n}}) \times G(\underline{\mathbf{n}})$, with obvious maps, then the natural map $|F \times G| \to |F| \times |G|$ is a continuous bijection. When $|F|$ or $|G|$ is compact, it induces a homeomorphism from $|F \times G|$ to $|F| \times |G|$.

A.1.3 Classifying Space

Let $\mathcal{C}$ be a small category. The ***nerve*** of $\mathcal{C}$, denoted $\mathrm{N}\mathcal{C}$, is the simplicial set defined as follows: an n-simplex of $\mathrm{N}\mathcal{C}$ is a diagram

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} A_n \tag{A.1}$$

with $A_i \in \mathcal{C}$, $f_i \in \mathrm{Mor}_{\mathcal{C}}(A_{i-1}, A_i)$; given a morphism $f : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ in $\triangle$, the corresponding morphism $\mathrm{N}\mathcal{C}(\underline{\mathbf{n}}) \longrightarrow \mathrm{N}\mathcal{C}(\underline{\mathbf{m}})$ maps the above n-simplex to the m-simplex

$$B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2 \xrightarrow{g_3} \cdots \xrightarrow{g_m} B_m$$

where $B_j = A_{f(j)}$ and $g_{j+1} : B_j \to B_{j+1}$ is the composite map $A_{f(j)} \to \cdots \to A_{f(j+1)}$ where if $f(j) = f(j+1)$, let $A_{f(j)} \to A_{f(j+1)}$ be the identity map. In particular, the i-th face of the n-simplex in (A.1) is the $(n-1)$-simplex

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \longrightarrow A_{i-1} \xrightarrow{f_{i+1}\circ f_i} A_{i+1} \longrightarrow \cdots \xrightarrow{f_n} A_n$$

and the i-th degenerate simplex of (A.1) is the $(n+1)$-simplex

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \longrightarrow A_i \xrightarrow{\mathrm{Id}} A_i \xrightarrow{f_{i+1}} A_{i+1} \longrightarrow \cdots \xrightarrow{f_n} A_n \,.$$

The ***classifying space*** of $\mathcal{C}$ is defined to be the geometric realization of $\mathrm{N}\mathcal{C}$ and is denoted by $\mathrm{B}\mathcal{C}$, i.e. $\mathrm{B}\mathcal{C} := |\mathrm{N}\mathcal{C}|$.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor between small categories, then there is an induced map of simpicial sets $\mathrm{N}\mathcal{C} \to \mathrm{N}\mathcal{D}$, and hence an induced continuous map $\mathrm{B}F : \mathrm{B}\mathcal{C} \longrightarrow \mathrm{B}\mathcal{D}$.

Lemma A.1.2 ([27, p.92, Prop. 2])**.** *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between small categories such that there is a natural transformation $F \to G$, then the maps $\mathrm{B}F, \mathrm{B}G : \mathrm{B}\mathcal{C} \to \mathrm{B}\mathcal{D}$ are homotopic.*

Corollary A.1.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. Suppose F has a left adjoint or a right adjoint. Then $\mathrm{B}F$ is a homotopy equivalence of $\mathrm{B}\mathcal{C}$ and $\mathrm{B}\mathcal{D}$. In particular, if $\mathcal{C}$ is a small category with an initial object or a final object, then $\mathrm{B}\mathcal{C}$ is contractible.*

Proof. If F has a left adjoint G, then there are natural transformations $GF \to \mathrm{Id}$ and $\mathrm{Id} \to FG$. The first assertion follows from Lemma A.1.2. If $\mathcal{C}$ has an initial object α, let $\mathcal{D}$ be the small category with only one object α. Then the constant functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint, which is the inclusion $\mathcal{D} \to \mathcal{C}$. Thus the second assertion follows from the first one. □

We say a small category is ***contractible*** if its classifying space is; and a functor F is a ***homotopy equivalence*** if $\mathrm{B}F$ is one.

A.2 Exact Categories and Quillen's Q-Construction

For our purposes, an ***exact category*** is an additive category $\mathcal{C}$ embedded as a full subcategory of an abelian category $\mathcal{A}$, which is closed under extension in the sense that if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{A.2}$$

is an exact sequence in $\mathcal{A}$ with $M', M'' \in \mathcal{C}$, then M is isomorphic to an object in $\mathcal{C}$. An ***exact sequence*** in $\mathcal{C}$ is then defined to be a sequence of the form (A.2) which is exact in $\mathcal{A}$ such that all the terms lie in $\mathcal{C}$. An ***exact functor*** $F : \mathcal{C} \to \mathcal{D}$ between exact categories is an additive functor such that if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence in $\mathcal{C}$, then

$$0 \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0$$

is an exact sequence in $\mathcal{D}$.

Let $\mathcal{C}$ be an exact category. Define a new category $\mathrm{Q}\mathcal{C}$ as follows: let the objects of $\mathrm{Q}\mathcal{C}$ be the objects of $\mathcal{C}$, but a morphism $X \to Y$ in $\mathrm{Q}\mathcal{C}$ is an isomorphism class of diagrams of the form

$$X \overset{q}{\twoheadleftarrow} Z \overset{i}{\rightarrowtail} Y \tag{A.3}$$

where i is an ***admissible monomorphism*** and q is an ***admissible epimorphism***; by definition, this means that there are exact sequences

$$0 \longrightarrow Z \overset{i}{\longrightarrow} Y \longrightarrow Y' \longrightarrow 0$$

and

$$0 \longrightarrow X' \longrightarrow Z \overset{q}{\longrightarrow} X \longrightarrow 0$$

in $\mathcal{C}$; and we say another diagram

$$X \overset{q'}{\twoheadleftarrow} Z' \overset{i'}{\rightarrowtail} Y$$

is isomorphic to the diagram (A.3) if there is an isomorphism $Z \xrightarrow{\sim} Z'$ in $\mathcal{C}$ making the diagram

$$\begin{array}{ccccc} X & \overset{q}{\twoheadleftarrow} & Z & \overset{i}{\rightarrowtail} & Y \\ \| & & \downarrow\wr & & \| \\ X & \overset{q'}{\twoheadleftarrow} & Z' & \overset{i'}{\rightarrowtail} & Y \end{array}$$

commute. Composition of morphisms in Q$\mathcal{C}$ is defined as follows. Given morphisms $X \twoheadleftarrow Z \rightarrowtail Y$ and $Y \twoheadleftarrow V \rightarrowtail T$, form the following diagram in the ambient abelian category $\mathcal{A}$

$$\begin{array}{ccccc} X & & & & \\ \uparrow\!\uparrow & & & & \\ Z & \rightarrowtail & Y & & \\ \uparrow\!\uparrow & & \uparrow\!\uparrow & & \\ Z \times_Y V & \rightarrowtail & V & \rightarrowtail & T \end{array}$$

Since $\mathcal{C}$ is closed under extension, and

$$\mathrm{Ker}\,(Z \times_Y V \to Z) \cong \mathrm{Ker}\,(V \to Y)\,,$$

we have $Z \times_Y V \in \mathcal{C}$, and $Z \times_Y V \twoheadrightarrow X$, $Z \times_Y V \rightarrowtail T$ are respectively admissible epimorphism and monomorphism. Hence the diagram $X \twoheadleftarrow Z \times_Y V \rightarrowtail T$ defines a morphism in Q$\mathcal{C}$ from X to T. One checks that the isomorphism class of this diagram depends only on the isomorphism classes of $X \twoheadleftarrow Z \rightarrowtail Y$ and $Y \twoheadleftarrow V \rightarrowtail T$, so that we have a well-defined composition rule for morphisms. Next, one verifies that the composition is associative. Thus, when the isomorphism classes of diagrams of the form (A.3) always form a set (e.g. if for every object of $\mathcal{C}$, its subobjects form a set), then Q$\mathcal{C}$ is a well-defined category. In particular, for a small exact category $\mathcal{C}$, the category Q$\mathcal{C}$ is defined and is small.

Recall that for an essentially small category $\mathcal{C}$, its ***Grothendieck group*** $K_0(\mathcal{C})$ is defined as the quotient of the free abelian group on isomorphism classes of objects of $\mathcal{C}$, divided by the subgroup generated by elements of the form $[M] - [M'] - [M'']$ for each exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\mathcal{C}$.

Theorem A.2.1 ([27, p.102, Thm. 1])**.** *Let $\mathcal{C}$ be a small exact category. Let* 0 *be a zero object of $\mathcal{C}$. Then the fundamental group* $\pi_1(\mathrm{BQ}\mathcal{C}\,,\,0)$ *is canonically isomorphic to the Grothendieck group* $K_0(\mathcal{C})$*.*

This motivates the following definition of K-groups.

Definition A.2.2. For a small exact category $\mathcal{C}$, the ***K-groups*** are defined as $K_i(\mathcal{C}) := \pi_{i+1}(\mathrm{BQ}\mathcal{C}\,,\,0)\,,\ \ \forall\, i \geq 0$.

This definition for K-groups is in fact independent of the choice of the zero object in $\mathcal{C}$. Furthermore, note that the preceding definition extends to essentially small categories: if $\mathcal{C}$ is an essentially small category, there exists a small subcategory $\mathcal{C}'$ equivalent to $\mathcal{C}$, so we can find define $K_i(\mathcal{C})$ to be $K_i(\mathcal{C}')$, the choice of $\mathcal{C}'$ being irrelevant by Lemma A.1.2. Note that higher homotopy groups of any topological space are all abelian groups (see e.g. [13, p.30, Coro. 7.7]), so the K-groups are all abelian.

From now on, exact categories will always be assumed to be essentially small, except when mentioned otherwise.

If $f : \mathcal{C} \to \mathcal{D}$ is an exact functor between exact categories, there is an induced functor from Q$\mathcal{C}$ to Q$\mathcal{D}$, and hence a homomorphism of K-groups for each i, which will be denoted

$$f_* : K_i(\mathcal{C}) \longrightarrow K_i(\mathcal{D})\,.$$

Proposition A.2.3 ([27, p.106, Coro. 1]). *Let* $0 \longrightarrow F' \to F \to F'' \to 0$ *be an exact sequence of exact functors* $\mathcal{C} \to \mathcal{D}$ *between exact categories (meaning that there are natural exact sequences* $0 \to F'(M) \to F(M) \to F''(M) \to 0$ *in* $\mathcal{D}$ *for all* $M \in \mathcal{C}$ *). Then*

$$F_* = F'_* + F''_* : K_i(\mathcal{C}) \longrightarrow K_i(\mathcal{D})\,,\quad \forall\, i \geq 0\,.$$

Theorem A.2.4 ([27, p.108, Thm. 3], ***Resolution Theorem***). *Let* $\mathcal{P}$ *be a full additive subcategory of an exact category* $\mathcal{M}$ *which is closed under extension in* $\mathcal{M}$*. Then* $\mathcal{P}$ *is an exact category in a natural way such that the inclusion* $\mathcal{P} \hookrightarrow \mathcal{M}$ *is an exact functor.*

Assume further that

(i) *if* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence in* $\mathcal{M}$ *and* $M, M'' \in \mathcal{P}$*, then* $M' \in \mathcal{P}$;

(ii) *for each object* $M \in \mathcal{M}$*, there is a finite resolution*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

with $P_i \in \mathcal{P}$*.*

Then $\mathrm{BQ}\mathcal{P} \to \mathrm{BQ}\mathcal{M}$ *is a homotopy equivalence, and hence* $K_i(\mathcal{P}) \cong K_i(\mathcal{M})$ *for all* $i \geq 0$*.*

A.3 K-Theory of Rings and Schemes

A.3.1 Basic Constructions

Let A be a Noetherian ring. Let $\mathcal{M}(A)$ denote the category of finitely generated A-modules. It is essentially small and can be regarded as a full subcategory of the abelian category of all A-modules which is closed under extension since A is Noetherian. So the notion $K_i(\mathcal{M}(A))$ makes sense. Let $\mathcal{P}(A)$ denote the category of finitely generated projective A-modules, so that $\mathcal{P}(A)$ is an exact category where all exact sequences split. We define $K'_i(A) := K_i(\mathcal{M}(A))$ and $K_i(A) := K_i(\mathcal{P}(A))$. The inclusion $\mathcal{P}(A) \hookrightarrow \mathcal{M}(A)$ induces natural homomorphisms $K_i(A) \to K'_i(A)$.

Recall that a Noetherian ring A is called ***regular*** if every finitely generated A-module has a finite resolution by finitely generated projective A-modules. This definition coincides with the usual one for Noetherian local rings (cf. [30, p.76, Thm. 9]).

Thus, the resolution theorem shows that if A is a regular Noetherian ring, $K_i(A) \cong K'_i(A)$ for all i.

Theorem A.3.1 ([27, p.122, Thm. 8]). *Let* A *be a Noetherian ring. Then there are natural isomorphisms for all* $i \geq 0$,

$$K'_i(A) \cong K'_i(A[t]) \quad \textit{and} \quad K'_i(A[t,\, t^{-1}]) \cong K'_i(A) \oplus K'_{i-1}(A)\,,$$

where for $i = 0$*, we set* $K'_{-1}(A) = 0$*.*

Corollary A.3.2. *If* A *is a regular Noetherian ring, then for all* $i \geq 0$,

$$K_i(A) \cong K_i(A[t]) \quad \textit{and} \quad K_i(A[t,\, t^{-1}]) \cong K_i(A) \oplus K_{i-1}(A)\,,$$

where $K_{-1}(A)$ *is defined to be* 0*.*

Remark A.3.3. An important fact we shall use is that our definition of $K_1(A)$ coincides with the one in Milnor's book [26]. In particular, $K_1(A) = A^*$ for any local ring A.

Now turn to K-theory of schemes.

For the rest of this appendix, we will assume all schemes to be Noetherian and separated, unless explicitly mentioned otherwise.

Let X be a (Noetherian separated) scheme. Let $\mathcal{P}(X)$ be the category of locally free sheaves of finite rank on X, and let $\mathcal{M}(X)$ be the category of coherent sheaves on X. They are essentially small exact categories in the natural way, embedded as full subcategories of the category of quasi-coherent sheaves of $\mathscr{O}_X$-modules. Define $K_i(X) := K_i(\mathcal{P}(X))$ and $K_i'(X) := K_i(\mathcal{M}(X))$. If X is regular, then since X is quasi-compact, every coherent sheaf on X is a quotient of a locally free sheaf of finite rank, and hence has a finite resolution by locally free sheaves of finite rank. So we have in this case $K_i(X) \cong K_i'(X)$ by the resolution theorem. If $X = \operatorname{Spec} A$ is affine, then we have natural equivalence of categories $\mathcal{P}(X) \simeq \mathcal{P}(A)$ and $\mathcal{M}(X) \simeq \mathcal{M}(A)$. Hence $K_i(X) \cong K_i(A)$ and $K_i'(X) \cong K_i'(A)$.

Given $\mathscr{E}$ in $\mathcal{P}(X)$, we have an exact functor $\mathscr{E} \otimes - : \mathcal{P}(X) \to \mathcal{P}(X)$ which induces a homomorphism of K-groups $(\mathscr{E} \otimes -)_* : K_i(X) \to K_i(X)$. If $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is an exact sequence in $\mathcal{P}(X)$, then Prop. A.2.3 implies

$$(\mathscr{E} \otimes -)_* = (\mathscr{E}' \otimes -)_* + (\mathscr{E}'' \otimes -)_* \,.$$

Thus we obtain multiplication maps

$$K_0(X) \otimes_{\mathbb{Z}} K_i(X) \longrightarrow K_i(X)\,, \quad [\mathscr{E}] \otimes \eta \mapsto (\mathscr{E} \otimes -)_* \eta\,.$$

Since $K_0(X)$ is a ring with the multiplication induced by tensor products of sheaves, these maps make $K_i(X)$ into modules over the ring $K_0(X)$. Similarly, we can make $K_i'(X)$ into modules over $K_0(X)$.

Remark A.3.4. One can define product maps $K_i(X) \otimes_{\mathbb{Z}} K_j(X) \longrightarrow K_{i+j}(X)$ for all i, j, but this requires more machinery (cf. [31, p.58, Remark 5.7]).

Let $f : X \to Y$ be a morphism of schemes, then the inverse image functor $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ is exact, hence it induces homomorphisms of K-groups $f^* : K_i(Y) \to K_i(X)$. Thus each K_i becomes a contravariant functor from (Noetherian separated) schemes to abelian groups.

If $f : X \to Y$ is flat, then the functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is exact, so there are induced homomorphisms: $K_i'(Y) \to K_i'(X)$. Thus, K_i' is a contravariant functor on the subcategory of schemes and flat morphisms.

Let $f : X \to Y$ be a morphism of schemes. We say f is of ***finite*** Tor***-dimension*** if there is an integer $N > 0$ such that $\mathscr{T}or_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{F}) = 0$ for all $i \geq N$ and all $\mathscr{F} \in \mathcal{M}(Y)$. Let $\mathcal{M}(Y, f) \subseteq \mathcal{M}(Y)$ be the full subcategory consisting of sheaves $\mathscr{F}$ satisfying $\mathscr{T}or_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{F}) = 0$ for all $i > 0$. Then we have $\mathcal{P}(Y) \subseteq \mathcal{M}(Y, f)$. Assuming that every $\mathscr{F} \in \mathcal{M}(Y)$ is a quotient of a member in $\mathcal{M}(Y, f)$, the resolution theorem implies that the inclusion $\mathcal{M}(Y, f) \hookrightarrow \mathcal{M}(Y)$ induces isomorphisms $K_i(\mathcal{M}(Y, f)) \cong K_i'(Y)$. Combining this isomorphism with the homomorphism induced by the exact functor $f^* : \mathcal{M}(Y, f) \to \mathcal{M}(X)$, we obtain a homomorphism $f^* : K_i'(Y) \to K_i'(X)$. The condition that every coherent sheaf on Y is a quotient of an object of $\mathcal{M}(Y, f)$ holds if

(i) f is flat (whence $\mathcal{M}(Y, f) = \mathcal{M}(Y)$);

or

(ii) every coherent sheaf on Y is a quotient of an object of $\mathcal{P}(Y)$ (e.g. if Y has an ample invertible sheaf).

Let $f : X \to Y$ be a proper morphism now. The higher direct image functors $R^i f_*$ carry coherent sheaves on X to coherent sheaves on Y. Let $\mathcal{F}(X, f)$ denote the full subcategory of $\mathcal{M}(X)$ consisting of $\mathscr{F}$ such that $R^i f_* \mathscr{F} = 0$ for all $i > 0$. Assume that every coherent sheaf on X is a subsheaf of a sheaf in $\mathcal{F}(X, f)$. Then there will be an isomorphism $K_i(\mathcal{F}(X, f)) \cong K_i'(X)$ for every i (this is a consequence of the resolution theorem and the fact that there is an isomorphism $Q\mathcal{C} \cong Q\mathcal{C}^{op}$, [31, p.104, Coro. 6.3]). Composing this isomorphism with the homomorphism induced by the exact functor $f_* : \mathcal{F}(X, f) \to \mathcal{M}(Y)$, we obtain a homomorphism $f_* : K_i'(X) \to K_i'(Y)$. The assumption that every coherent sheaf on X can be embedded into an object of $\mathcal{F}(X, f)$ holds if

(i) f is finite (whence $\mathcal{F}(X, f) = \mathcal{M}(X)$);

or

(ii) X has an ample invertible sheaf.

Proposition A.3.5 ([31, p.64, Prop. 5.12], ***Projection formula***)**.** *Suppose $f : X \to Y$ is a proper morphism which is of finite* Tor*-dimension. Assume X and Y have ample invertible sheaves so that the maps*

$$f^* : K_i'(Y) \longrightarrow K_i'(X)\,, \quad \text{and} \quad f_* : K_i'(X) \longrightarrow K_i'(Y)$$

are defined. Then

(i) *there is a well-defined homomorphism $f_* : K_i(X) \longrightarrow K_i(Y)$ giving a commutative diagram*

$$\begin{array}{ccc} K_i(X) & \longrightarrow & K_i'(X) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ K_i(Y) & \longrightarrow & K_i'(Y) \end{array}$$

(ii) *for any $x \in K_0(X)$, $y \in K_i'(Y)$, we have*

$$f_*(x) \cdot y = f_*(x \cdot f^*(y))\,.$$

(iii) *for any $y \in K_0(Y)$, , $x \in K_i'(X)$, we have*

$$f_*(f^*(y) \cdot x) = y \cdot f_*(x)\,.$$

Proposition A.3.6 ([27, p.127, Prop. 3.2])**.** *Let Z be a closed subscheme of X, let $\iota : Z \hookrightarrow X$ be the inclusion, and let $\mathscr{I}$ be the ideal of $\mathscr{O}_X$ defining Z.*

(i) *If $\mathscr{I}$ is nilpotent, then $\iota_* : K_i'(Z) \to K_i'(X)$ is an isomorphism for every i. In particular, $K_i'(X_{\rm red}) \cong K_i'(X)$.*

(ii) *Let U be the complement of Z in X, and let $j : U \to X$ be the inclusion. Then there is a long exact sequence*

$$\cdots \longrightarrow K_{i+1}' \longrightarrow K_i'(Z) \xrightarrow{\iota_*} K_i'(X) \xrightarrow{j^*} K_i'(U) \longrightarrow \cdots$$

The next two results reveal connections of the K'-groups of a scheme and those of affine or projective bundles (cf. §1.4 and §2.1) on it.

Proposition A.3.7 ([27, p.128, Prop. 4.1])**.** *Let $f : P \to X$ be a flat morphism whose fibres are affine spaces. Then $f^* : K_i'(X) \to K_i'(P)$ is an isomorphism for every $i \geq 0$.*

Proposition A.3.8 ([27, p.129, Prop. 4.3])**.** *Let E be a vector bundle of rank $r = e + 1$ on a scheme X. Let $\mathbb{P}(E)$ be the associated projective bundle and let $f : \mathbb{P}(E) \to X$ be the projection. Then for each $i \geq 0$, we have a $K_0(\mathbb{P}(E))$-module isomorphism*

$$K_0(\mathbb{P}(E)) \otimes_{K_0(X)} K_i'(X) \xrightarrow{\sim} K_i'(\mathbb{P}(E))$$

given by

$$y \otimes x \mapsto y \cdot f^*(x)\,.$$

Equivalently, if $z \in K_0(\mathbb{P}(E))$ is the class of the sheaf $\mathscr{O}_E(-1)$ on $\mathbb{P}(E)$, then we have an isomorphism

$$K_i'(X)^{\oplus r} \xrightarrow{\sim} K_i'(\mathbb{P}(E))\,, \quad (x_j)_{0 \leq j \leq e} \mapsto \sum_{j=0}^{e} z^j \cdot f^*(x_j)\,.$$

Concerned with the K-groups of projective bundles, we have the following theorem, as opposed to Prop. A.3.8.

Theorem A.3.9 ([27, p.142, Thm. 2.1]). *Let E be a vector bundle of rank $r = e+1$ on a scheme X and let $\mathbb{P}(E)$ be the associated projective bundle, and $f : \mathbb{P}(E) \to X$ the natural projection. Then there are isomorphisms of $K_0(\mathbb{P}(E))$-modules for all $i \geq 0$,*

$$K_0(\mathbb{P}(E)) \otimes_{K_0(X)} K_i(X) \cong K_i(\mathbb{P}(E)), \quad y \otimes x \mapsto y \cdot f^*(x).$$

Equivalently, there are isomorphisms

$$K_i(X)^{\oplus r} \cong K_i(\mathbb{P}(E)), \quad (\alpha_j)_{0 \leq j \leq e} \mapsto \sum_{j=0}^{e} z^j \cdot f^*(\alpha_j),$$

where $z \in K_0(X)$ is the class of $\mathscr{O}_E(-1)$.

A.3.2 BGQ-Spectral Sequence and Chow Groups

Let X be a (Noetherian separated) scheme. Let $\mathcal{M}^p(X) \subseteq \mathcal{M}(X)$ be the Serre subcategory (meaning a subcategory which is closed under subobjects, quotients and extensions) consisting of those coherent sheaves $\mathscr{F}$ whose support has codimension $\geq p$ in X. Define a decreasing filtration on $K_i'(X) = K_i(\mathcal{M}(X))$ by

$$F^p K_i'(X) := \text{ image of } (K_i(\mathcal{M}^p(X)) \longrightarrow K_i'(X)).$$

This is called the ***filtration by codimension of support*** or the ***topological filtration***. This filtration is finite provided that $\dim X$ is finite.

There is a spectral sequence, called the Brown–Gersten–Quillen (BGQ) spectral sequence, relating the K-groups of points on X with the K'-groups of the scheme X.

Theorem A.3.10 ([27, p.131, Thm. 5.4], ***BGQ-Spectral Sequence***). *Let X be a Noetherian separated scheme. Let $X^{(p)}$ denote the set of $x \in X$ such that* $\operatorname{codim}(\overline{\{x\}}, X) = p$. *Then there is a spectral sequence*

$$E_1^{pq} = E_1^{pq}(X) = \bigoplus_{x \in X^{(p)}} K_{-p-q}(\kappa(x)) \Longrightarrow K'_{-p-q}(X)$$

which is convergent when X has finite dimension. Here we set $K_n = K_n' = 0$ for $n < 0$.

The E_2-terms of the BGQ-spectral sequence is computable thanks to Gersten's resolution, which is stated in the following proposition.

Proposition A.3.11 ([27, pp.132–133, Props. 5.6, 5.8 and Coro. 5.10]). *Let X be a regular separated scheme of finite type over a field.*

(i) *For every $p \geq 0$, there is an exact sequence*

$$0 \longrightarrow K_p(X) \xrightarrow{\varepsilon} E_1^{0,-p}(X) \xrightarrow{d} \cdots \longrightarrow E_1^{p-1,-p}(X) \xrightarrow{d} E_1^{p,-p}(X) \longrightarrow 0$$

where the map

$$\varepsilon : \quad K_p(X) \longrightarrow E_1^{0,-p}(X) = \bigoplus_{x \in X^{(0)}} K_p(\kappa(x))$$

is induced by the pull-backs by the canonical morphisms ι_x : $\operatorname{Spec} \kappa(x) \to X$, *and the maps d are differential maps of the* BGQ*-spectral sequence.*

(ii) *Let $\mathscr{K}_p$ denote the sheaf associated to the presheaf $U \mapsto K_p(U)$, and let*

$$\mathscr{E}_1^{pq} := \bigoplus_{x \in X^{(p)}} \iota_{x*}\big(K_{-p-q}(\kappa(x))\big)$$

where $K_{-p-q}(\kappa(x))$ is regarded as the constant sheaf on Spec $\kappa(x)$ *associated to the group $K_{-p-q}(\kappa(x))$. Then for every p there is a flasque resolution of $\mathscr{K}_p$:*

$$0 \longrightarrow \mathscr{K}_p \xrightarrow{\ \varepsilon\ } \mathscr{E}_1^{0,\,-p} \xrightarrow{\ d\ } \cdots \longrightarrow \mathscr{E}_1^{p-1,\,-p} \xrightarrow{\ d\ } \mathscr{E}_1^{p,\,-p} \longrightarrow 0\,.$$

(iii) *There is a canonical isomorphism*

$$E_2^{pq}(X) \cong H^p(X,\, \mathscr{K}_{-q})\,.$$

for every p, q.

Note that for any $x \in X^{(p-1)}$, $K_1(\kappa(x)) = \kappa(x)^*$ (cf. Remark A.3.3) may be identified with $k(W)^*$, where $k(W)$ is the function field of $W := \overline{\{\, x \,\}}$. Also, for any $x \in X^{(p)}$, $K_0(\kappa(x))$ can be identified with $\mathbb{Z} \cdot [V]$ with $V = \overline{\{\, x \,\}}$, so $E_1^{p,\,-p}(X)$ can be identified with the group $Z^p(X)$ of cycles of codimension p (cf. Chapt. 1).

Theorem A.3.12 ([27, p.137, Thm. 5.19])**.** *For a regular separated scheme X of finite type over a field k, the differential map*

$$d : E_1^{p-1,\,-p}(X) \longrightarrow E_1^{p\,,\,-p}(X)$$

in the BGQ*-spectral sequence coincides with the map*

$$\bigoplus_{\operatorname{codim} W = p-1} k(W)^* \longrightarrow Z^p(X)\,; \quad \oplus f_i \mapsto \sum [\operatorname{div}(f_i)]\,.$$

Consequently, there are canonical isomorphisms

$$E_2^{p\,,\,-p}(X) \cong \mathrm{CH}^p(X)\,, \quad \forall\, p \geq 0\,.$$

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