Diagonalization and maximal torus reduction

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Introduction

Any Hermitian matrix H can be diagonalized by unitary matrices, i.e. there exists a unitary matrix A, such that AHA^{-1} is a diagonal matrix with real entries on its main diagonal. Now one can ask themselves the following question. Can any family of Hermitian matrices parametrized by a differentiable manifold be simultaneously diagonalized? That is, given a differentiable manifold M and a differentiable map $f : M \to \mathrm{Herm}(n, \mathbb{C})$, is there any differentiable map $g : M \to U(n, \mathbb{C})$, such that $g(p)f(p)(g(p))^{-1}$ is a diagonal matrix for every $p \in M$? The following example answers this question. Let S^1 be the unit circle and

$$f : S^1 \to \mathrm{Herm}(2, \mathbb{C}),$$

be the map sending $(\cos\varphi, \sin\varphi) \in S^1$ to

$$\begin{pmatrix} \cos(\frac{\varphi}{4}) & \sin(\frac{\varphi}{4}) \\ -\sin(\frac{\varphi}{4}) & \cos(\frac{\varphi}{4}) \end{pmatrix} \begin{pmatrix} 2+\sin(\frac{\varphi}{2}) & 0 \\ 0 & 2+\sin(\frac{\varphi}{2}+\pi) \end{pmatrix} \begin{pmatrix} \cos(\frac{\varphi}{4}) & -\sin(\frac{\varphi}{4}) \\ \sin(\frac{\varphi}{4}) & \cos(\frac{\varphi}{4}) \end{pmatrix}.$$

Define

$$g^* : (a, 2\pi + a) \to U(2, \mathbb{C}),$$

$$\varphi \mapsto \begin{pmatrix} \cos(\frac{\varphi}{4}) & -\sin(\frac{\varphi}{4}) \\ \sin(\frac{\varphi}{4}) & \cos(\frac{\varphi}{4}) \end{pmatrix}$$

and define $h : (a, 2\pi + a) \to S^1 \setminus \{(\cos(a), \sin(a))\}, \varphi \mapsto (\cos(\varphi), \sin(\varphi))$. Let $g : S^1 \setminus \{(\cos(a), \sin(a))\} \to U(n, \mathbb{C})$ be defined by $g^* \circ h^{-1}$. Then it is clear from the definition of f that $g(p)f(p)(g(p))^{-1}$ is a diagonal matrix for every $p \in S^1$. So f is locally diagonalizable, i.e. for every $p \in M$ there exists a neighborhood U such that $f\mid_U$ is simultaneously diagonalizable. Note that f is not simultaneously diagonalizable on the whole S^1. To observe this let us assume the contrary. Let

$$g : S^1 \to U(n, \mathbb{C}),$$

$$(\cos\varphi, \sin\varphi) \mapsto \begin{pmatrix} a_{11}(\varphi) & a_{12}(\varphi) \\ a_{21}(\varphi) & a_{22}(\varphi) \end{pmatrix}$$

be a smooth map such that $g(p)f(p)(g(p))^{-1}$ is a diagonal matrix for every $p \in M$. Note that the only possible diagonalizations for $f(p)$ are

$$\begin{pmatrix} 2+\sin(\frac{\varphi}{2}) & 0 \\ 0 & 2+\sin(\frac{\varphi}{2}+\pi) \end{pmatrix} \text{ and } \begin{pmatrix} 2+\sin(\frac{\varphi}{2}+\pi) & 0 \\ 0 & 2+\sin(\frac{\varphi}{2}) \end{pmatrix}.$$

Since both f and g are smooth maps, $g(p)f(p)(g(p))^{-1}$ varies smoothly with p. So it should be constantly equal to either

$$\begin{pmatrix} 2+\sin(\frac{\varphi}{2}) & 0 \\ 0 & 2+\sin(\frac{\varphi}{2}+\pi) \end{pmatrix} \text{ or } \begin{pmatrix} 2+\sin(\frac{\varphi}{2}+\pi) & 0 \\ 0 & 2+\sin(\frac{\varphi}{2}) \end{pmatrix}.$$

Without loss of generality we can assume that $g(p)f(p)(g(p))^{-1}$ is equal to

$$\begin{pmatrix} 2+\sin(\frac{\varphi}{2}) & 0 \\ 0 & 2+\sin(\frac{\varphi}{2}+\pi) \end{pmatrix}$$

for every $p \in S^1$. Since eigenspaces of the eigenvalues $2 + \sin(\frac{\varphi}{2})$ and $2 + \sin(\frac{\varphi}{2} + \pi)$ are one-dimensional and columns of a unitary matrix are orthonormal to each other, we have

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = e^{\lambda_1(\varphi)} \begin{pmatrix} \cos(\frac{\varphi}{4}) \\ \sin(\frac{\varphi}{4}) \end{pmatrix} \text{ and } \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = e^{\lambda_2(\varphi)} \begin{pmatrix} -\sin(\frac{\varphi}{4}) \\ \cos(\frac{\varphi}{4}) \end{pmatrix}$$

for any $\phi \in (0, 2\pi)$. Here $\lambda_i : (0, 2\pi) \to \mathbb{R}$ is a smooth map for every $i \in \{1, 2\}$. Since

$$\lim_{\varphi \to 0^+} e^{\lambda_1(\varphi)} \begin{pmatrix} \cos(\frac{\varphi}{4}) \\ \sin(\frac{\varphi}{4}) \end{pmatrix} = e^{(\lim_{\varphi \to 0^+} \lambda_1(\varphi))} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\lim_{\varphi \to 2\pi^-} e^{\lambda_1(\varphi)} \begin{pmatrix} \cos(\frac{\varphi}{4}) \\ \sin(\frac{\varphi}{4}) \end{pmatrix} = e^{(\lim_{\varphi \to 2\pi^-} \lambda_1(\varphi))} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

there is a contradiction. Thus f is not diagonalizable on S^1.

The above example motivates a new question. Is any family of Hermitian matrices parametrized by a smooth manifold locally diagonalizable? Our answer to this question is again negative. We illustrate this with an example.

Let $F : \mathbb{R}^2 \to \mathrm{Herm}(2, \mathbb{C})$ be the map defined by $(r \cos\varphi, r \sin\varphi) \mapsto r^2 f(\cos\varphi, \sin\varphi)$. Here f is the map we defined earlier. One can show with exactly the same argument as above that the restriction of F to any arbitrary circle centered at the origin is not diagonalizable. So F is not locally diagonalizable because every open neiborhood of the origin contains a circle around the origin. Do note that $F |_{\mathbb{R}^2 \setminus \{(0,0)\}}$ is locally diagonalizable. Now, we modify our question for the last time and ask ourselves the following. Let $f : M \to \mathrm{Herm}(n, \mathbb{C})$ be a family of Hermitian matrices parametrized by a smooth manifold M. Is there any open dense subset $U \subset M$ such that $f |_U$ is locally diagonalizable? We will give a positive answer to this question in Chapter 1 Proposition 1.5.9. In fact Proposition 1.5.9 has a much stronger statement than this. It says that for any vector bundle $\pi : E \to M$ and any Hermitian morphism $f \in \mathrm{Herm}^+(\mathrm{E})$, there exists an open dense subset W such that f can locally be diagonalized on W. This Proposition in fact endows us with a powerful tool for doing computations on Hermitian morphisms on a vector bundle. We illustrate the value of this Proposition with an example. Since connections behave like differentials in some aspect one may naively think that $D_E(\log f)$ is equal to $f^{-1} \circ D_E(f)$. But, as we will see, Proposition 1.5.9 implies that this is not true.

In Chapter 2 we continue with principal fiber bundles and representations of Lie groups. Finally we prove the main theorem (Theorem 2.9.5). Let P be a principal fiber bundle with K a connected compact Lie group as its fiber, T be a fixed maximal torus in K and C be a fixed closed Weyl chamber in $\mathfrak{t}$. Then for any $f \in \Gamma(ad(P))$ there exists an open dense subset $W \subset B$ such that for any $x \in W$ there exists an open face of the closed Weyl chamber C, denoted by c_x, an open neighborhood of x in W, denoted by U_x, and a T-reduction $\Pi_x \subset P |_{U_x}$ of the restriction $P |_{U_x}$ such that the restriction of $f |_{U_x}$ to Π_x can be given by a smooth map

$$\lambda \in C^\infty(U_x, c_x) \subset C^\infty(U_x, \mathfrak{t}) = \Gamma(ad(\Pi_x)).$$

At first sight they may seem unrelated, but we will prove at the end of Chapter 2 that Proposition 1.5.9 can be deduced from Theorem 2.9.5. Proposition 1.5.9 and Theorem 2.9.5 are from the books [LT1] and [LT2] respectively. The proofs have been expanded in more detail and some mistakes in the original proof given in [LT1] and [LT2] are corrected.

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1 Diagonalization

1.1 Vector bundles

Definition 1.1.1 *Let E,F and M be manifolds, and $\pi : E \to M$ be a differentiable map such that it satisfies the condition of local triviality, i.e. there exists a cover $\{\mathcal{U}_i : i \in I\}$ for M and diffeomorphisms*

$$\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times F,$$

such that $p_1 \circ \theta_i = \pi \mid_{\pi^{-1}(\mathcal{U}_i)}$ where $p_1 : \mathcal{U}_i \times F \to \mathcal{U}_i$ is the projection on the first component. Then $\pi : E \to M$ is called a **fiber bundle** *over M with typical fiber F. The set $\{\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times F\}$ is called* **local trivializations** *for $\pi : E \to M$, and the map π is called the projection of the fiber bundle. A* **trivial bundle** *is a fiber bundle which admits a global trivialization*

$$\theta : E \to M \times F.$$

A **homomorphism** *between two fiber bundles $\pi_E : E \to M$ and $\pi_F : F \to N$ consists of two differentiable maps $f : E \to F$ and $f' : M \to N$ such that $f' \circ \pi_E = \pi_F \circ f$. If $M = N$, $f' : M \to M$ is the identity map and f a diffeomorphism, then f is called an* **isomorphism** *between the fiber bundles $\pi_E : E \to M$ and $\pi_F : F \to M$.*
A **section** *of a fiber bundle $\pi : E \to M$ is a differentiable map $\sigma : M \to E$ such that $\pi \circ \sigma = id_M$, i.e. $\sigma(p) \in E_p$ for every $p \in M$, where E_p is $\pi^{-1}(p)$, which is called the* **fiber** *of E over the point p. We denote the set of all section of a fiber bundle $\pi : E \to M$ by $\Gamma(E)$.*

Definition 1.1.2 *A fiber bundle $\pi : E \to M$ with typical fiber $\mathbb{R}^n$, and local trivializations $\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n$ such that for all $i, j \in I$ and all $p \in \mathcal{U}_{ij}$ the map*

$$\mathbb{R}^n \overset{\simeq}{\to} \{p\} \times \mathbb{R}^n \overset{\theta_i \circ \theta_j^{-1} |\{p\} \times \mathbb{R}^n}{\longrightarrow} \{p\} \times \mathbb{R}^n \overset{\simeq}{\to} \mathbb{R}^n$$

is linear, is called a **(real) vector bundle** *of rank n, where $\mathbb{R}^n \simeq \{p\} \times \mathbb{R}^n$ is defined by $v \mapsto (p, v)$.*

Note that we can endow E_p with a linear structure by requiring the map

$$\theta_{ip} : E_p \overset{\theta_i | E_p}{\longrightarrow} \{p\} \times \mathbb{R}^n \overset{\simeq}{\to} \mathbb{R}^n$$

to be an isomorphism of vector spaces for some $i \in I$ with $p \in \mathcal{U}_i$. It is well-defined because if $j \in I$ be another element of I with $p \in \mathcal{U}_j$ then

$$\theta_j \mid_{E_p} = \theta_i \mid_{E_p} \circ (\theta_j \circ \theta_i^{-1} \mid_{\{p\} \times R^n}),$$

and since $\theta_j \circ \theta_i^{-1} \mid_{\{p\} \times R^n}$ is a linear isomorphism the induced linear structures by θ_j and θ_i are the same.
If E is a vector bundle, $\Gamma(E)$ has a natural vector space structure by addition and scalar multiplication defined on each fiber. Note that every vector bundle $\pi : E \to M$ has a zero section $\sigma : M \to E, \quad p \mapsto 0 \in E_p$, which is the zero vector of the vector space $\Gamma(E)$.

Definition 1.1.3 *A complex vector bundle is a vector bundle $\pi : E \to M$ whose fiber bundle $\pi^{-1}(x)$ is a complex vector space. It is not necessarily a complex manifold.*

Definition 1.1.4 *Let $\pi : E \to M$ be a vector bundle of rank n, and $U \subset M$ be an open subset. Then we define a* **local frame** *over U to be a tuple $s = (s_1, \ldots, s_n)$, where $s_i \in \Gamma(E\mid_U)$, for every $i \in \{1, \ldots, n\}$, such that $(s_1(x), \ldots, s_n(x))$ is a basis for E_x for every $x \in U$.*

Definition 1.1.5 *A* **vector bundle homomorphism** *between two vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$ over the same base manifold M is a differentiable map $f : E \to F$ such that $\pi_F \circ f = \pi_E$ and $f\mid_{E_p} : E_p \to F_p$ is linear. If $f : E \to F$ is a diffeomorphism then f is called an* **isomorphism**. *We denote the set of all vector bundle homomorphisms $E \to F$ by* $\mathrm{Hom}(E, F)$

Lemma 1.1.6 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles and $f : E \to F$ be a vector bundle homomorphism such that $f\mid_{E_p} : E_p \to F_p$ is a linear isomorphism for every $p \in M$, then f is a vector bundle isomorphism.*

For its proof see [Hu].

Let $\pi : E \to M$ be a vector bundle and $\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n$ be a local trivialization for it with respect to a cover $\{\mathcal{U}_i : i \in I\}$. Then we define the **transition functions** of the vector bundle $\pi : E \to M$ with respect to the cover $\{\mathcal{U}_i : i \in I\}$, to be $\{\theta_{ij} : \mathcal{U}_i \to \mathrm{GL}(n, \mathbb{R})\}$ where $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ and θ_{ij} maps $p \in U_{ij}$ to the matrix of the linear isomorphism $\theta_{ip} \circ \theta_{jp}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. Note that they are smooth and that they satisfy the cocycle condition

$$\theta_{ij}\theta_{jk} = \theta_{ik}$$

where multiplication is in $\mathrm{GL}(n, \mathbb{R})$.

Proposition 1.1.7 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles over the same manifold M and $\{\theta_i^E : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n : i \in I\}$, $\{\theta_i^F : \pi_F^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^m : i \in I\}$ be local trivializations of E and F respectively with respect to the open cover $\{\mathcal{U}_i : i \in I\}$. Then any set of smooth maps $\{f_i : \mathcal{U}_i \to \mathrm{M}(n \times m) : i \in I\}$ which satisfies $\theta_{ij}^F f_j = f_i \theta_{ij}^E$ on $\mathcal{U}_{ij}$, induces a homomorphism $f : E \to F$ such that $\theta_i^F \circ f\mid_{\pi_E^{-1}(\mathcal{U}_i)} \circ (\theta_i^E)^{-1}(p, v) = (p, f_i v)$.*

Proof. For every $v \in E$, choose $i \in I$ with $p = \pi(v) \in \mathcal{U}_i$., and define $f(v) := (\theta_{ip}^F)\left(f_i(p)(\theta_{ip}^E(v))\right)$. It is well-defined since for another $j \in I$ such that $p = \pi(v) \in \mathcal{U}_j$. we have

$$\begin{aligned} f(v) : &= (\theta_{jp}^F)\left(f_j(p)(\theta_{jp}^E(v))\right) \\ &= (\theta_{jp}^F)\left((\theta_{ji}^F)^{-1} f_i(p)\theta_{ij}^E(\theta_{jp}^E(v))\right) \\ &= (\theta_{ip}^F)\left(f_i(p)(\theta_{ip}^E(v))\right). \end{aligned}$$

to see that f is differentiable one need just to note that each trivialization constitutes an atlas for the vector bundle. Therefore it suffices to show, $\theta_i^F \circ f\mid_{\pi_E^{-1}(\mathcal{U}_i)} \circ (\theta_i^E)^{-1}$ is differentiable for every $i \in I$. But it is easy to see that $\theta_i^F \circ f\mid_{\pi_E^{-1}(\mathcal{U}_i)} \circ (\theta_i^E)^{-1}(p, v) = (p, f_i v)$. Thus f is differentiable. ■

Remark 1.1.8 *In the above Proposition, if E and F both have the same rank n and the image of the maps $\{f_i : \mathcal{U}_i \to \mathrm{M}(n \times n) : i \in I\}$ lie in $\mathrm{GL}(n, \mathbb{R})$, then the maps $\{f_i^{-1} : \mathcal{U}_i \to \mathrm{GL}(n, \mathbb{R}) : i \in I\}$ satisfy $\theta_{ij}^E f_j^{-1} = f_i^{-1}\theta_{ij}^F$ on $\mathcal{U}_{ij}$. So they induce a vector bundle homomorphism $f' : F \to E$. It is easy to see that f and f' are inverse of each other, so in particular E and F are isomorphic vector bundles.*

Proposition 1.1.9 *Let M be a manifold and $\{\mathcal{U}_i : i \in I\}$ be an open cover for M. Suppose a set of smooth maps $\{\theta_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R})\}$ is given which satisfies the cocycle condition. Then there exists a unique (up to isomorphism) vector bundle of rank n with the $\{\theta_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R})\}$ as its transition functions with respect to a system of local trivializations.*

Proof. We define an equivalence relation in the set $\bigcup_{i\in I} \mathcal{U}_i \times \mathbb{R}^n$ by saying that $(i, p, x) \sim (j, q, y)$ when $p = q$ and $x = \theta_{ij}(p)y$. Since $\{\theta_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R})\}$ satisfies the cocycle condition, this equivalence relation is well-defined. So we can define

$$E := \bigcup_{i\in I} \mathcal{U}_i \times \mathbb{R}^n / \sim,$$

the map

$$\pi_E : E \to M, \quad (i, p, x)/\sim \mapsto p$$

and the bijection

$$\theta_i : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n, \quad (i, p, x)/\sim \mapsto (p, x)$$

for all $i \in I$. We can endow E with a differential structure, by requiring that θ_i is a diffeomorphism for all $i \in I$. It makes sense since the gluing maps are the smooth maps

$$\mathcal{U}_{ij} \times \mathbb{R}^n \to \mathcal{U}_{ij} \times \mathbb{R}^n, \quad (p, x) \mapsto (p, \theta_{ij}(p)x).$$

Then $\pi_E : E \to M$ becomes a vector bundle, with the $\{\theta_{ij}\}$ as transition functions. Now suppose $\pi_F : F \to M$ is another n dimensional vector bundle with a local trivializations $\theta_i : \pi_F(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n$ such that its transition function are θ_{ij}. Then, according to Remark 1.1.8, $\{f_i : \mathcal{U}_i \to \mathrm{GL}(n, \mathbb{R}), p \mapsto I_n\}$ induces an isomorphism $E \to F$. ■

Remark 1.1.10 *Let $\pi : E \to M$ be a vector bundle, $\{\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n : i \in I\}$ be a trivialization with respect to an open cover $\{\mathcal{U}_i : i \in I\}$ and $\{\mathcal{V}_j : j \in J\}$ be a refinement of $\{\mathcal{U}_i : i \in I\}$. Then we can construct a trivialization of E with respect to $\{\mathcal{V}_j : j \in J\}$ by defining $\theta_j := \theta_i |_{\mathcal{V}_j}$ where $i \in I$ such that $\mathcal{V}_j \subset \mathcal{U}_i$. Note that here we are applying the axiom of choice and different choices of i for each j yields different trivializations.*

Example 1.1.11 *Let $\pi : E \to M$ be a vector bundle with transition functions $\{\theta_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R})\}$ with respect to a cover $\mathcal{U}_i : i \in I$ of M. Define*

$$\theta^*_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R}), \quad p \mapsto (\theta_{ij}(p)^t)^{-1}.$$

The $\{\theta^*_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R})\}$ satisfy the cocycle condition, therefore by proposition 1.1.9 the exist a vector bundle over M which has the $\{\theta^*\}$ as its transition functions. We call this vector bundle the dual bundle of E and denote it by E^*. Note that the construction of E^* does not depend to the set of transition functions defining E, because if $\theta' : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R})$ is another set of transition functions for E,(note that here we assumed that the cover to be the same because otherwise by regarding of Remark 1.1.10 we can consider a common refinement of the covers) then we can define $\{f_i : \mathcal{U}_i \to \mathrm{GL}(n, \mathbb{R}), p \to (\theta_{ip} \circ (\theta'_{ip})^{-1} : i \in I\}$, where $\{\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n : i \in I\}$ and $\{\theta'_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n : i \in I\}$ are some systems of local trivializations for E, inducing the transition functions $\{\theta_{ij}\}$ and $\{\theta'_{ij}\}$ respectively. One can see easily that $\theta_{ij} f_j = f_i \theta'_{ij}$ and so $\theta^*_{ij}(f^t_j)^{-1} = (f^t_i)^{-1}{\theta'_{ij}}^*$. Therefore according to Proposition 1.1.9 and its ensuing Remark $\{\theta^*_{ij}\}$ and $\{{\theta'_{ij}}^*\}$ determine the same vector bundle E^* up to an isomorphism. Note that there exists a canonical isomorphism $(E^*)_p \simeq (E_p)^*$.

Example 1.1.12 *Let* $\pi^E : E \to M$ *and* $\pi^F : F \to M$ *be vector bundles and* $\{\theta_i^E : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n : i \in I\}$ *and* $\{\theta_i^F : \pi_F^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^m : i \in I\}$ *be two local trivializations of* E *and* F *with transition functions* $\{\theta_{ij}^E : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R})\}$ *and* $\{\theta_{ij}^F : \mathcal{U}_{ij} \to \mathrm{GL}(m, \mathbb{R})\}$ *with respect to a common cover* $\{\mathcal{U}_i : i \in I\}$ *of* M*. Now define*

$$\theta_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(nm, \mathbb{R}), \quad p \mapsto \theta_{ij}^E(p) \otimes \theta_{ij}^F(p).$$

They satisfy the cocycle condition therefore, according to Proposition 1.1.9 there exists a unique bundle (up to isomorphism) with $\{\theta_{ij}\}$ *as its transition functions. We call this bundle the tensor product of* E *and* F*, and denote it by* $E \otimes F$*. For every* $p \in M$*, we have* $(E \otimes F)_p \simeq E_p \otimes F_p$*.*

Example 1.1.13 *Let* $\pi^E : E \to M$ *be a vector bundle,* $r \leq n$*,* $\{\theta_i^E : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n : i \in I\}$ *be a set of trivializations of* E *with respect to a cover* $\{\mathcal{U}_i : i \in I\}$ *and transition functions* $\{\theta_{ij}^E : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{R})\}$*. Then define*

$$\theta_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(\begin{pmatrix} n \\ r \end{pmatrix}, \mathbb{R}), \quad p \mapsto \Lambda^r(\theta_{ij}(p)).$$

The $\{\theta_{ij}\}$ *satisfy the cocycle condition, therefore according to Proposition 1.1.9 there exists a unique bundle (up to isomorphism) with the* $\{\theta_{ij}\}$ *as transition functions. We call this bundle the* r*-th exterior power of* E*, and denote it by* $\Lambda^r E$*. One can show in a similar way as Example 1.1.11 that the construction of* $\Lambda^r E$ *does not depend on the choice of the transition function defining* E*.*
For every $p \in M$*, we have* $(\Lambda^r E)_p \simeq \Lambda^r E_p$*. To see this, apply the universal factorization property of the exterior power to the* r*-linear alternating map*

$$E_p \times \ldots \times E_p \to (\Lambda^r E)_p$$

$$((i, p, x_1)/\sim, \ldots, (i, p, x_r)/\sim) \mapsto (i, p, \Lambda_r(x_1, \ldots, x_r))/\sim,$$

where the same notation is used as Proposition 1.1.9.

Example 1.1.14 *Let* $\pi_E : E \to M$ *and* $\pi_F : F \to M$ *be two vector bundles. By applying Example 1.1.11 and Example 1.1.12, we can define the vector bundle* $E^* \otimes F$ *over* M*, and we have*

$$(E^* \otimes F)_p = E_P^* \otimes F_p = \mathrm{Hom}(E_p, F_p)$$

This gives a correspondence between sections of $E^* \times F$ *and the vector bundle homomorphisms from* E *to* F*, and this correspondence is actually a vector space isomorphism*

$$\Gamma(E^* \otimes F) \simeq \mathrm{Hom}(E, F).$$

Example 1.1.15 *Let* $\pi_E : E \to M$ *and* $\pi_F : F \to M$ *be two vector bundles. By applying Example 1.1.11 and Example 1.1.13, we can define the vector bundle* $\Lambda^r E^*$ *over* M*. We call this bundle the bundle of* r*-forms on* E*. We have*

$$(\Lambda^r E^*)_p \simeq \Lambda^r(E_p^*).$$

Therefore any section of $\Lambda^r E^*$ *can be seen as* r*-linear alternating form at each fiber* E_p *of* E*, varying smoothly with* p*. In particular* $\Gamma(\Lambda^r \mathrm{T}M^*) = \mathcal{A}^r M$*.*

Lemma 1.1.16 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles on M and let $\lambda : \Gamma(E) \to \Gamma(F)$ be a $\mathcal{C}^\infty(M)$-linear map. Then for every $\sigma \in \Gamma(E)$ and for every $p \in M$, the value of $\lambda(\sigma)(p)$ just depends on $\sigma(p)$.*

Proof. We want to show that if $\sigma, \sigma' \in \Gamma(E)$ be such that $\sigma(p) = \sigma'(p)$, then $\lambda(\sigma)(p) = \lambda(\sigma)(p)$. Or equally we can show that if $\sigma(p) = 0$, then $\lambda(\sigma(p)) = 0$. First we show that λ is local operator i.e. if we have $\sigma, \sigma' \in \Gamma(E)$ such that $\sigma(q) = \sigma'(q)$ for every $q \in V$, where V is an open neighborhood of p, then $\lambda(\sigma)(p) = \lambda(\sigma')(p)$. To see this, let $\psi \in \mathcal{C}^\infty(M)$ with $\mathrm{Supp}(\psi) \subset V$ and $\psi(p) = 1$. Then $\psi\sigma = \psi\sigma'$, and $\psi\lambda(\sigma) = \lambda(\psi\sigma) = \lambda(\psi\sigma') = \psi\lambda(\sigma')$, in particular $\lambda(\sigma)(p) = \lambda(\sigma)(p)$.
Now let σ be section in $\Gamma(E)$ such that $\sigma(p) = 0$, and let $\{u_i\}_{1\leq i\leq n}$ be a local frame on U, an open neighborhood of p. Then $\sigma \mid_U = \sum_{i=1}^n \sigma_i u_i$, where σ_i's are smooth functions on U. Now we take $\psi \in \mathcal{C}^\infty(M)$ such that $\mathrm{Supp}(\psi) \subset U$ and $\psi \mid_{U\setminus V} = 0$, where V is a neighborhood of p in U. Let $\sigma' := \sum_{i=1}^n \sigma'_i u'_i$, where $\sigma'_i \mid_U := \psi\sigma_i \mid_U$, $\sigma'_i \mid_{M\setminus U} := 0$ and $u'_i \mid_U := \psi u_i \mid_U$, $u'_i \mid_{M\setminus U} := 0$. There for $\lambda(\sigma)(p) = \lambda(\sigma')(p) = \sum_{i=1}^n \sigma'_i(p)\lambda(u'_i)(p) = \sum_{i=1}^n \sigma_i(p)\lambda(u_i)(p) = 0$ ∎

Lemma 1.1.17 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles.Then we have:*

$$\begin{aligned}\Gamma(E^* \otimes F) &\simeq \mathrm{Hom}(E, F) \simeq \{\lambda : \Gamma(E) \to \Gamma(F) \text{ linear over } \mathcal{C}^\infty(M)\}\\ &\simeq \Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)\end{aligned}$$

where "$\simeq$" means "isomorphic as $\mathcal{C}^\infty(M)$-module".
In particular, $\Gamma(E \otimes F) \simeq \Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$.

Proof. We have already seen in Example 1.1.14 that $\Gamma(E^* \otimes F) \simeq \mathrm{Hom}(E, F)$. For $\mathrm{Hom}(E, F) \simeq \{\lambda : \Gamma(E) \to \Gamma(F) \text{ linear over } \mathcal{C}^\infty(M)\}$, consider $\varphi \in \mathrm{Hom}(E, F)$ and define

$$\lambda_\varphi : \Gamma(E) \to \Gamma(F), \quad \sigma \mapsto (p \mapsto \varphi(\sigma(p))).$$

Then λ_φ is a $\mathcal{C}^\infty(M)$-linear and $\varphi \mapsto \lambda_\varphi$ is a $\mathcal{C}^\infty(M)$-module homomorphism. Conversely, let $\lambda \in \{\lambda : \Gamma(E) \to \Gamma(F) \text{ linear over } \mathcal{C}^\infty(M)\}$ and define

$$\varphi_\lambda : E \to F, \quad v \mapsto \lambda(\sigma_v)(\pi_E(v)),$$

where σ_v is a section of E such that $\sigma_v(\pi_E(v)) = v$. By Lemma 1.1.16 φ_λ is well-defined. Note that φ_λ is smooth and therefore $\varphi_\lambda \in \mathrm{Hom}(E, F)$, and that $\lambda_{\varphi_\lambda} = \lambda$ and $\varphi_{\lambda_\varphi} = \varphi$. Thus $\varphi \mapsto \lambda_\varphi$ is an $\mathcal{C}^\infty(M)$-module isomorphism.
For proving that $\{\lambda : \Gamma(E^*) \to \Gamma(F) \text{ linear over } \mathcal{C}^\infty(M)\}$ and $\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$ are isomorphic, let $\xi = \sum_{i\in I} \tau_i^* \otimes \eta_i \in \Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$, and define

$$\lambda_\xi : \Gamma(E) \to \Gamma(F), \quad \sigma \mapsto (p \mapsto \sum_{i\in I} [\tau_i^*(p)(\sigma(p))].(\eta_i(p))).$$

Then λ_ξ is linear over $\mathcal{C}^\infty(M)$ and $\xi \mapsto \lambda_\xi$ is a homomorphism of $\mathcal{C}^\infty(M)$. Conversely let $\lambda : \Gamma(E) \to \Gamma(F)$ be linear over $\mathcal{C}^\infty(M)$. Then we define $\xi_\lambda \in \Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$ as follows. Let $\{\mathcal{U}_i : i \in I\}$ be a cover of M and for $i \in I$ let $u_1^i, \ldots, u_n^i$ be a local frame for $\mathcal{U}_i$. Since λ is a local operator as it is shown at Lemma 1.1.16, it induces a $\mathcal{C}^\infty(M)$-linear map $\lambda_i : \gamma(E \mid_{\mathcal{U}_i}) \to \Gamma(F \mid_{\mathcal{U}_i})$, for every $i \in I$. We define $\xi_\lambda^i := \sum_{\alpha=1}^n (u_\alpha^i)^* \otimes \lambda_i(u_\alpha^i) \in \Gamma(E^* \mid_{\mathcal{U}_i}) \otimes_{\mathcal{C}^\infty(\mathcal{U}_i)} \Gamma(F \mid_{\mathcal{U}_i})$. Then $\lambda_{\xi_\lambda^i} = \lambda_i$ and $\xi_{\lambda_\zeta}^i = \zeta$ for every $\zeta \in \Gamma(E^* \mid_{\mathcal{U}_i}) \otimes_{\mathcal{C}^\infty(\mathcal{U}_i)} \Gamma(F \mid_{\mathcal{U}_i})$, so $\xi \mapsto$

λ_ξ gives an isomorphism of $\mathcal{C}^\infty(\mathcal{U}_i)$-modules between $\{\lambda : \Gamma(E) \to \gamma(E)$ linear over $\mathcal{C}^\infty(\mathcal{U}_i)\}$ and $\Gamma(E^* \mid_{\mathcal{U}_i}) \otimes_{\mathcal{C}^\infty(\mathcal{U}_i)} \Gamma(F \mid_{\mathcal{U}_i})$. In particular it follows that ξ_λ^i and ξ_λ^j coincide on $\mathcal{U}_{ij}$. So with using a partition of unity of M we can glue $\{\xi_\lambda^i : i \in I\}$ together and get ξ_λ an element of $\Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$, such that for every $i \in I$ and for every $p \in \mathcal{U}_i$, it holds $\xi_\lambda(p) = \xi_\lambda^i(p)$. Therefore $\lambda_{\xi_\lambda} = \lambda$ for every $\lambda : \Gamma(E) \to \Gamma(F)$ linear over $\mathcal{C}^\infty(M)$, and $\xi_{\lambda_\xi}(p) = \xi$ for every $\xi \in \Gamma(E^* \otimes_{\mathcal{C}^\infty(M)} \Gamma(F))$. Thus $\xi \mapsto \lambda_\xi$ is an isomorphism of $\mathcal{C}^\infty(M)$-modules. ■

Remark 1.1.18 *We can treat complex bundles in the same manner as real bundles. Everything we did in this section for real bundles can also be done for complex vector bundles. All definitions, examples and results (except Example 1.1.13) can be carried out for the complex case just by substituting "$\mathbb{R}$" with "$\mathbb{C}$" and "real" with "complex" throughout the section.*

1.2 Real connections

Definition 1.2.1 *Let $\pi : E \to M$ be a real vector bundle and let $r \in \mathbb{N}$. Define*

$$\begin{aligned} \mathcal{A}^r(E) &:= \Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \mathcal{A}^r(M) = \Gamma(E \otimes \Lambda^r \mathrm{T}^*\mathrm{M}) \\ &= \{\lambda : \Gamma(\mathcal{A}^r(M)) \to \Gamma(E) \text{ linear over } \mathcal{C}^\infty(M)\} \end{aligned}$$

In particular, $\mathcal{A}^0(E) = \Gamma(E)$. Elements of $\mathcal{A}^r(E)$ are called (smooth) r-forms on M with values in E. Note that a $\lambda : \mathcal{A}^r(M) \to \Gamma(E)$ linear over $\mathcal{C}^\infty(M)$, induces a $\mathcal{C}^\infty(M)$-multilinear alternating maps

$$\Gamma(\mathrm{TM}) \times \ldots \times \Gamma(\mathrm{TM}) \to \Gamma(E)$$

Definition 1.2.2 *Let $\pi : E \to M$ be a vector bundle. Then a connection on E is an $\mathbb{R}$-linear map $D : \mathcal{A}^0(E) \to A^1(E)$ which satisfies the Leibnitz rule, i.e. for every $f \in \mathcal{C}^\infty(M)$ and $\sigma \in \Gamma(E)$ it must hold:*

$$D(f\sigma) = \sigma \otimes df + fD(\sigma). \tag{1}$$

Connections exist on every vector bundle, as can be proved using a partition of unity on the base space. See [MS], Lemma 2 of Appendix C.

Let $\pi : E \to M$ be a vector bundle with a connection $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$. Then, we can extend D to a $\mathbb{R}$-linear operator $D : \mathcal{A}^r(E) :\to \mathcal{A}^{r+1}(E)$ for any $r \in \mathbb{N}$ by defining

$$D(\sigma \otimes \omega) = \sigma \otimes d\omega + D(\sigma) \wedge \omega$$

for $\sigma \in \Gamma(E)$ and $\omega \in \mathcal{A}^r(M)$ and extend it by linearity. The exterior product $\mathcal{A}^s(E) \times \mathcal{A}^r(M) \to \mathcal{A}^{s+r}(E)$ on the right hand side is defined as follows: for $\xi = \sigma \otimes \omega \in \mathcal{A}^s(E)$ and $\omega' \in \mathcal{A}^r(M)$, $\xi \wedge \omega' := \sigma \otimes (\omega \wedge \omega')$.
Note that a connection is a local operator i.e. if σ and σ' be two sections of E such that $\sigma \mid_U = \sigma' \mid_U d$ for U an open subset of M, then $D(\sigma)(p) = D(\sigma')(p)$ for every $p \in U$. For seeing that consider $f \in \mathcal{C}^\infty(M)$, such that $\mathrm{supp}(f) \subset U$ and $f(p) = 1$. Then $f\sigma = f\sigma'$. Therefore

$$D(f\sigma) = \sigma \otimes df + fD(\sigma) = \sigma' \otimes df + fD(\sigma') = D(f\sigma')$$

and it implies that $D(\sigma)(p) = D(\sigma')(p)$. Thus a connection $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ induces a connection $D \mid_U: \mathcal{A}^0(E \mid_U) \to \mathcal{A}^1(E \mid_U)$ on each open $U \subset M$, in such a way that $D \mid_U (\sigma \mid_U) = (D(\sigma)) \mid_U$ for every $\sigma \in \mathcal{A}^0(E)$. The induced map $D \mid_U$ will also be denoted by

D.

Let $\pi : E \to M$ be a vector bundle and $u = (u_1, \ldots, u_n)$ be a local frame over an open subset U of M and let D be a connection on E. Then we can write

$$D(u_\alpha) = \sum_{\beta=1}^{n} u_\beta \otimes (\omega_u)_{\beta\alpha} \tag{2}$$

where $\omega = ((\omega_u)_{\alpha\beta})$ is a matrix of 1-forms on U called the **connection form** of D with respect to the local frame u. If $\sigma = \sum_{\alpha=1}^{n} v_\alpha u_\alpha$ is a section of E over U, then from (1) and (2) we get:

$$D(\sigma) = \sum_{\alpha=1}^{n} u_\alpha \otimes (dv_\alpha + \sum_{\beta=1}^{n} (\omega_u)_{\alpha\beta} v_\beta). \tag{3}$$

Definition 1.2.3 *Let $\pi : E \to M$ be a vector bundle and $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ be a connection on E. Then we define a connection on E^* as follows: for every $\sigma \in \mathcal{A}^0(E)$ and $\eta^* \in \mathcal{A}^0(E^*)$, we define*

$$D^*(\eta^*)(\sigma) := d(\eta^*(\sigma)) - \eta^*(D(\sigma)).$$

It is easy to see that the Leibnitz rule is satisfied.

Not let $u = (u_1, \ldots, u_n)$ be a local frame for E over an open subset U of M and let $u^* = (u_1^*, \ldots, u_n^*)$ be the dual frame of the frame u over U. Then the connection form of D^* with respect to u^* is given by $\omega^*_{u^*} = -(\omega_u)^t$.

Definition 1.2.4 *Let $\pi_E : E \to F$ and $\pi_F : E \to F$ be vector bundles, and let D_E andD_F be connections on E and F respectively. Then we define a connection $D_{E\otimes F}$ as follows: for every $\sigma \in \mathcal{A}^0(E)$ and for every $\eta \in \mathcal{A}^0(F)$ we define*

$$D_{E\otimes F}(\sigma \otimes \eta) := D_E(\sigma) \otimes \eta + \sigma \otimes D_F(\eta).$$

It is easy to see that the Leibnitz rule is satisfied.

Let $u^E = (u_1^E, \ldots, u_n^E)$ and $u^F = (u_1^F, \ldots, u_n^F)$ be local frames of E and F over an open subset $U \subset M$. Consider the local frame

$$u^E \otimes u^F = \{u_i^E \otimes u_j^F : i = 1 \ldots, n, j = 1, \ldots, m\}$$

of $E \otimes F$ over U, where the $\{u_i^E \otimes u_j^F\}$ are ordered lexicographically. Let $\omega_{E\otimes F}$, ω_E and ω_F be the connections forms of $D_{E\otimes F}$, D_E and D_F, with respect to $u^{E\otimes F}$, u^E and u^F respectively. Then

$$\omega_{E\otimes F} = \omega_E \otimes I_m + I_n \otimes \omega_F.$$

Remark 1.2.5 *One can define $D_{E^*\otimes F}$ by composing Definition 1.2.3 and Definition 1.2.4 as follows:*

$$\begin{aligned}
D_{E^*\otimes F}(\sigma^* \otimes \eta)(\alpha) &:= \left(D_{E^*}(\sigma^*) \otimes \eta + \sigma^* \otimes D_F(\eta)\right)(\alpha) \\
&= \left(D_{E^*}(\sigma^*)(\alpha)\right) \otimes \eta + \sigma^*(\alpha).D_F(\eta) \\
&= d(\sigma^*(\alpha)) \otimes \eta - \sigma^*(D_E(\alpha)) \otimes \eta + \sigma^*(\alpha) \otimes D_F(\eta) \\
&= \left(d(\sigma^*(\alpha)) \otimes \eta + \sigma^* \otimes D_F(\eta)\right) - \sigma^*(D_E(\alpha)) \otimes \eta \\
&= D_F(\sigma^*(\alpha).\eta) - \sigma^*(D_E(\alpha)) \otimes \eta \\
&= D_F\left((\sigma^* \otimes \eta)(\alpha)\right) - \left((\sigma^* \otimes \eta)(D_E)\right)(\alpha)
\end{aligned}$$

for $\sigma^* \otimes \eta \in \mathcal{A}^0(E^* \otimes F)$ *and* $\alpha \in \mathcal{A}^0(E)$.
Let $u^E = (u_1^E, \dots, u_n^E)$ *and* $u^F = (u_1^F, \dots, u_n^F)$ *be local frames of* E *and* F *over an open subset* $U \subset M$. *Consider the local frame*

$$(u^E)^* \otimes u^F = \{(u^E)_i^* \otimes u_j^F : i = 1\dots, n, j = 1, \dots, m\}$$

of $E^* \otimes F$ *over* U, *where the* $\{(u^E)_i^* \otimes u_j^F\}$ *are ordered lexicographically. Let* $\omega_{E^*\otimes F}$, ω_E *and* ω_F *be the connections forms of* $D_{E\otimes F}$, D_E *and* D_F, *with respect to* $u^{E\otimes F}$, u^E *and* u^F *respectively. Then*

$$\omega_{E^*\otimes F} = (-\omega_E)^t \otimes I_m + I_n \otimes \omega_F.$$

1.3 Complex connections over real manifolds

Definition 1.3.1 *Let* V *be an* n*-dimensional real vector space and consider the* $2n$*-dimensional real vector space*

$$V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}.$$

Define

$$\mathbb{C} \times V^{\mathbb{C}} \to V^{\mathbb{C}}, \quad \mu(\sum_{i\in I} v_i \otimes \lambda_i) \mapsto \sum_{i\in I} v_i \otimes \mu\lambda_i,$$

where I *is a finite set,* $v_i \in V$ *and* $\mu, \lambda_i \in \mathbb{C}$. *Then* $V^{\mathbb{C}}$ *becomes an* n*-dimensional complex vector space, called the* **complexification** *of* V. *We define the* **conjugation** *map*

$$g : V^{\mathbb{C}} \to V^{\mathbb{C}}, \quad \sum_{i\in I} v_i \otimes \lambda_i \mapsto \sum_{i\in I} v_i \otimes \overline{\lambda}_i.$$

For every $\alpha \in V^{\mathbb{C}}$, $g(\alpha)$ *is called the* **conjugate** *element of* α *in* $V^{\mathbb{C}}$, *and it is denoted by* $\overline{\alpha}$.

Definition 1.3.2 *Let* $\pi : E \to M$ *be a complex vector bundle over a real manifold. An* r*-form on* M *with values in* E *is an element of* $\mathcal{A}^r(E)^{\mathbb{C}} := \Gamma(E) \otimes_{\mathcal{C}^\infty(M,\mathbb{C})} \mathcal{A}^r(M)^{\mathbb{C}}$. *A connection on* E *is a* $\mathbb{C}$*-linear map* $D : \mathcal{A}^0(E)^{\mathbb{C}} \to \mathcal{A}^1(E)^{\mathbb{C}}$ *which satisfies the Leibnitz rule.*

Note that similarly to the real case one can show that a complex connection is a local operator. Therefore a connection $D : \mathcal{A}^0(E)^{\mathbb{C}} \to \mathcal{A}^1(E)^{\mathbb{C}}$ induces a connection $D\mid_U : \mathcal{A}^0(E\mid_U)^{\mathbb{C}} \to \mathcal{A}^1(E\mid_U)^{\mathbb{C}}$ on each open $U \subset M$, in such a way that $D\mid_U (\sigma\mid_U) = (D(\sigma))\mid_U$ for every $\sigma \in \mathcal{A}^0(E)^{\mathbb{C}}$. The induced map $D\mid_U$ will also be denoted by D.
Let $\pi : E \to M$ be a complex vector bundle of rank n and $u = (u_1, \dots, u_n)$ be a local frame over an open subset U of M and let D be a connection on E. Then we can write

$$D(u_\alpha) = \sum_{\beta=1}^n u_\beta \otimes (\omega_u)_{\beta\alpha} \tag{4}$$

where $\omega = ((\omega_u)_{\alpha\beta})$ is a matrix of 1-forms on U i.e. $(\omega_u)_{ij} \in \mathcal{A}^1(U)^{\mathbb{C}}$ for every $i, j \in \{1, \dots, n\}$. ω is called the **connection form** of D with respect to the local frame u. If $\sigma = \sum_{\alpha=1}^n v_\alpha u_\alpha$ is a section of E over U then, by applying Leibnitz rule and using (4), we get:

$$D(\sigma) = \sum_{\alpha=1}^n u_\alpha \otimes (dv_\alpha + \sum_{\beta=1}^n (\omega_u)_{\alpha\beta} v_\beta). \tag{5}$$

Example 1.3.3 *Let $\pi : E \to M$ be a complex vector bundle with transition functions $\{\theta_{ij} : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{C})\}$ with respect to a cover $\{\mathcal{U}_i : i \in I\}$ of M. The functions $\{\overline{\theta_{ij}} : \mathcal{U}_{ij} \to \mathrm{GL}(n, \mathbb{C})\}$ satisfy the cocycle condition, thus by Proposition 1.1.9 (see Remark 1.1.18) they are the transition functions of a complex vector bundle $\overline{E}$ over M, called the* **conjugate bundle** *of E. We have $(\overline{E})_p \simeq \overline{(E_p)}$ in a canonical way. The isomorphism is given by the well-defined $\mathbb{C}$-linear map $(i, p, x)/\sim \mapsto (i, p, \overline{x})/\sim$ (notation as in the proof of Proposition 1.1.9).*

Definition 1.3.4 *Let $\pi : E \to M$ be a complex vector bundle. Combining Examples 1.3.3, 1.1.11 and 1.1.12, we can define the complex vector bundle $E^* \otimes \overline{E}^*$. Since $(E^* \otimes \overline{E}^*)_p = \overline{(E_p)}^*$ for all $p \in M$, we see that a section of $E^* \otimes \overline{E}^*$ gives a sesquilinear map $E_p \times E_p \to \mathbb{C}$ on each fiber of E, varying smoothly with p. An* **Hermitian metric** *on $\pi : E \to M$ is a section h of $E^* \otimes \overline{E}^*$ such that $h(p)$ is an Hermitian inner product on E_p, for all $p \in M$.*

Let $\pi : E \to M$ be a complex vector bundle over a real manifold M, and h be a Hermitian metric in E. Then h induces a map

$$\mathcal{A}^r(E)^{\mathbb{C}} \times \mathcal{A}^s(E)^{\mathbb{C}} \to \mathcal{A}^{r+s}(M)^{\mathbb{C}},$$

by sending

$$(\tau \otimes \upsilon, \kappa \otimes \nu) \mapsto h(\tau, \kappa) \cdot \upsilon \wedge \overline{\nu}$$

and extending linearly, where $r, s \in \mathbb{N}$, $\tau, \kappa \in \Gamma(E)$, $\upsilon \in \mathcal{A}^r(M)^{\mathbb{C}}$ and $\nu \in \mathcal{A}^s(M)^{\mathbb{C}}$.

1.4 Hermitian and Skew-Hermitian

Definition 1.4.1 *Let V be a finite dimensional complex vector space and h a Hermitian inner product over V. Then for any $f \in End(V)$ there exists a unique $f^* \in End(V)$ such that*

$$h(f(v), w) = h(v, f^*(w)),$$

for all $v, w \in V$. We call f^* the adjoint of f. A linear transformation $f \in End(V)$ is called **Hermitian** *(resp. skew-Hermitian) if $f = f^*$ (resp. $f = -f^*$).*

Definition 1.4.2 *A matrix $A \in M(n, \mathbb{C})$ is called* **Hermitian** *(resp. skew-Hermitian) if*

$$A = A^\dagger \quad (resp.\ A = -A^\dagger)$$

Definition 1.4.3 *Let $\pi : E \to B$ a complex vector bundle and h be a Hermitian metric over E i.e. h is a section of $E^* \otimes E^*$ such that $h(p)$ is a Hermitian metric in E_p, for all $p \in M$. Then an endomorphism f of E is said to be a Hermitian endomorphism, (resp. skew-Hermitian endomorphism) if $f = f^*$ (resp. $f = -f^*$), Where f^* is defined by the adjoint transformation of f_p on E_p with respect to h_p. for every $p \in B$. We denote by $Herm(E, h)$ or when h is known by $HermE$, the set of all Hermitian endomorphisms of E. We use Herm^+E to denote the set of all positive definite Hermitian endomorphism of E.*

Let f be Hermitian endomorphism (resp. skew-Hermitian endomorphism) then if (resp. $-if$) is a skew-Hermitian (resp. Hermitian) endomorphism, because $h(if(v), w) = ih(f(v), w) = ih(v, f(w)) = h(v, -if(w))$ (resp. $h(-if(v), w) = -ih(f(v), w) = -ih(v, -f(w)) = h(v, -if(w))$). So there exists a one-to-one correspondence between Hermitian endomorphisms and skew-Hermitian endomorphisms.

1.5 Local diagonalization

Lemma 1.5.1 *Let X be a topological space, and*

$$X = G_r \supset G_{r-1} \supset \dots \supset G_2 \supset G_1 \supset G_0 = \varnothing$$

be a filtration of X by closed subsets, then

$$W := \bigcup_{k=1}^{r} F_k^\circ$$

is an open dense subset of X, where $F_i := G_i \setminus G_{i-1}$ for every $i \in \{1, \dots, r\}$, and $^\circ$ denotes the interior.

Proof. W is an open subset of X, since W is a union of some open subsets. To observe that W is a dense subset, first note that if Y is a dense subset of X and Z is a dense subset of Y, then Z is a dense subset of X. The case $r = 0$ is trivial because then $X = \varnothing$. For $r = 1$, we have

$$W = F_1^\circ = (G_1 \setminus G_0)^\circ = G_1^\circ = G_1,$$

the latter equality holds because $G_1 = X$. Now assume that for every $r < k$ the assertion is true, then we show it also holds for k. Define $G_i' := G_i \cap G_{k-1}^\circ$, for $i \in \{1, \dots, k-1\}$, then

$$G_{k-1}^\circ = G_{k-1}' \supset G_{k-2}' \supset \dots \supset G_2' \supset G_1' \supset G_0' = \varnothing,$$

is a filtration of G_{k-1}° by closed subsets. So according to the assumption, $\cup_{i=1}^{k-1}(F_i')^\circ$ is dense subset of G_{k-1}°, where $F_i' := G_i' \setminus G_{i-1}'$, for every $i \in \{1, \dots, K-1\}$. On the other hand

$$F_k^\circ \cup G_{k-1}^\circ = (G_k \setminus G_{k-1})^\circ \cup G_{k-1}^\circ = G_k \setminus G_{k-1} \cup G_{k-1}^\circ$$

is a dense subset of X. So $F_k^\circ \bigcup (\cup_{i=1}^{k-1}(F_i')^\circ)$ is dense in X, because $F_k^\circ \cup (\cup_{i=1}^{k-1}(F_i')^\circ)$ is dense in $F_k^\circ \cup G_{k-1}^\circ$ and $F_k^\circ \cup G_{k-1}^\circ$ is dense in X. Since we have $(F_i')^\circ \subset F_i^\circ$, for every $i \in \{1, \dots, k-1\}$,

$$F_k^\circ \cup (\cup_{i=1}^{k-1}(F_i')^\circ) \subset \cup_{i=1}^{k} F_i^\circ,$$

thus W is also a dense subset of X. ■

Lemma 1.5.2 *Let X, Y be two topological spaces, and Y be Hausdorff and locally compact, then every continuous and proper map $f : X \to Y$ is closed.*

Proof. Assume that C is a closed subset of X. For every $a \in \overline{f(C)}$ there exists an open subset $U \subset X$, such that $a \in U$ and $\overline{U}$ is a compact subset of Y. Consider $U \cap f(C)$, clearly $U \cap f(C) \neq \varnothing$ and $a \in \overline{U \cap f(C)}$. So there exists a net $(x_\alpha)_{\alpha \in J}$ in $U \cap f(C)$ such that $(x_\alpha)_{\alpha \in J}$ converges to a. Since f is proper, $f^{-1}(\overline{U \cap f(C)})$ is a compact subset of X, therefore $D := f^{-1}(\overline{U \cap f(C)}) \cap C$ is also compact. On the other hand, since $f^{-1}(x_\alpha) \cap D \neq \varnothing$ for all $\alpha \in J$, by the axiom of choice there exists a net $(y_\alpha)_{\alpha \in J}$ such that $y_\alpha \in D$ and $f(y_\alpha) = x_\alpha$ for every $\alpha \in J$. Since D is compact, the net $(y_\alpha)_{\alpha \in J}$ has a convergent subnet $(y_\beta)_{\beta \in K} \to b$. Now note that f maps $(y_\beta)_{\beta \in K}$ to $(x_\beta)_{\beta \in K}$, therefore $(x_\beta)_{\beta_K} \to f(b)$. On the other hand $(x_\beta)_{\beta \in K}$ is a subnet of $(x_\alpha)_{\alpha \in J}$, which implies that $(x_\beta)_{\beta \in K} \to a$. Using the Hausdorff property of Y, we conclude that $a = f(b)$. So $b \in D \subset C$, because D is a closed subset and $(y_\beta)_{\beta \in K} \to b$. Thus $a \in f(C)$, which means $f(C)$ is a closed set. ■

Lemma 1.5.3 *The map*

$$\pi : \mathbb{C}^r \to \mathbb{C}^r, \quad (z_1, \dots, z_n) \mapsto (\sigma_1(z_1, \dots, z_r), \dots, \sigma_r(z_1, \dots, z_r))$$

is proper, where σ_i is the elementary symmetric function of degree i in the variables $z_1, \dots, z_r$, for every $i \in \{1, \dots, r\}$.

Proof. Since $\mathbb{C}^r \simeq \mathbb{R}^{2r}$, the compact subsets of $\mathbb{C}^r$ are precisely the closed and bounded subsets. Let $U \subset \mathbb{C}^r$ be a compact subset, then $\pi^{-1}(U)$ is closed because π is continuous. It suffices to show that $\pi^{-1}(U)$ is bounded. Let $B(0, a)$ be a ball around the origin with radius $a \geq 1$ such that $U \subset B(0, a)$. We know that

$$(z - z_1)(z - z_2) \dots (z - z_r) = z^n + \sum_{i=1}^{r} (-1)^r \sigma_i(z_1, \dots, z_r) z^{n-i}.$$

If we show that a polynomial $f(z) = z^n + b_n z^{n-1} + \dots b_1$, with $b_i < a$ for every $1 \leq i \leq n$, does not have any roots z_0 such that $\mid z_0 \mid \geq 2a$ then we can conclude that $\pi^{-1}(U) \subset B(0, 2a)$. Assume z_0 is a root of f such that $\mid z_0 \mid \geq 2a$, then since $z_0^n = -(b_n z_0^{n-1} + \dots + b_1)$ we have

$$\begin{aligned}
\mid z_0^n \mid &= \mid z_0 \mid^n \\
&= \mid b_n z_0^{n-1} + \dots + b_1 \mid \\
&\leq \mid b_n \mid\mid z_0 \mid^{n-1} + \mid b_{n-1} \mid\mid z_0 \mid^{n-2} + \dots + \mid b_1 \mid \\
&< \frac{\mid z_0 \mid}{2} \mid z_0 \mid^{n-1} + \frac{\mid z_0 \mid}{2} \mid z_0 \mid^{n-2} + \dots + \frac{\mid z_0 \mid}{2} \\
&= \frac{1}{2} \sum_{i=1}^{n} \mid z_0 \mid^i \\
&\leq \mid z_0 \mid,
\end{aligned}$$

which is a contradiction, since $\mid z_0 \mid \geq 2a > 1$. So $f^{-1}(U) \subset B(0, 2a)$, therefore it is compact. ■

Corollary 1.5.4 $\pi : \mathbb{C}^r \to \mathbb{C}^r, (z_1, z_2, ..., z_r) \mapsto (\sigma_1(z_1, ..., z_r), ..., \sigma_r(z_1, ..., z_r))$ *is a closed map.*

Now let us denote by $\mathbb{R}[t]^r \subset \mathbb{R}[t]$ the subspace of all monic real polynomials of degree r. Note that we can endow $\mathbb{R}[t]^r$ with a topological (differentiable) structure by requiring the bijective map

$$\varphi : \mathbb{R}[t]^r \to \mathbb{R}^r, \quad t^r + \sum_{i=1}^{r} a_i t^{r-i} \to (a_1, a_2, ..., a_r)$$

to be a homeomorphism (a diffeomorphism resp.). Now let us denote by $p^r \subset \mathbb{R}[t]^r$ the set of all monic real polynomials of degree r which are products of linear factors, and define $p_k^r \subset p^r$, for $k = 1, ..., r$ to be the subset of polynomials with at most k distinct roots.

Lemma 1.5.5 *p_k^r is a closed subset of p^r.*

Proof. Let $\mathbb{C}[t]^r$ be the space of all complex monic polynomials of degree r, and $\mathbb{C}[t]^r_k$ the subsets of polynomials with at most k different roots. We define a topology on $\mathbb{C}[t]^r$ by saying that $\varphi : \mathbb{C}[t]^r \to \mathbb{C}^r$ be homeomorphism to its image. Since $p^r_k = \mathbb{C}[t]^r_k \cap p^r$, it is sufficient to show that $\mathbb{C}[t]^r_k$ is closed in $\mathbb{C}[t]^r$. According to the lemma above

$$\pi : \mathbb{C}^r \to \mathbb{C}^r, \quad (z_1, ..., z_r) \to (\sigma_1(z_1, ..., z_r), ..., \sigma_r(z_1, ..., z_r))$$

is a closed map. So in particular the image of a diagonal subspace of $\mathbb{C}^r$ under π is closed and since $\mathbb{C}[t]^k_r$ is a union of preimages under φ of finitely many such sets, $\mathbb{C}[t]^r_k$ is closed. ∎

Let

$$B^r_k := \left\{ \nu = (\nu_1, \ldots, \nu_k) \in \mathbb{N}^k : \sum_{i=1}^{k} \nu_i = r \right\},$$

then for every $k \in \{1, \ldots, r\}$ and every $\nu \in B^r_k$, we define $A^r_{k,\nu} \subset p^r$ to be the set of all polynomials in p^r with exactly k distinct roots $\mu_1 < \ldots < \mu_k$, with multiplicities $\nu_1, ..., \nu_k$ respectively.

Remark 1.5.6 *$p^r_k \setminus p^r_{k-1}$ is an open subset of p^r_k, consisting of all polynomials in p^r with precisely k different roots. So $p^r_k \setminus p^r_{k-1}$ is the disjoint union of $A^r_{k,\nu}$ with $\nu \in B^r_k$.*

Lemma 1.5.7 *$A^r_{k,\nu} \subset p^r$ is a $k-$dimensional submanifold of $\mathbb{R}[t]^r$, moreover the map*

$$\pi_{k,\nu} : S_k \to A^r_{k,\nu} \subset \mathbb{R}[t]^r, \quad (x_1, ..., x_k) \mapsto \prod_{l=1}^{k} (t - x_l)^{\nu_l}$$

is a diffeomorphism, where $S_k = \{(x_1, ..., x_k) \mid x_1 < ... < x_k\} \subset \mathbb{R}^k$.

Proof. Let $(x_1, ..., x_k), (x'_1, ..., x'_k) \in S_k$ such that $\prod_{l=1}^{k}(t - x_l)^{\nu_i} = \prod_{l=1}^{k}(t - x'_l)^{\nu_l}$, Now according to the fundamental theorem of algebra we conclude that $x_i = x'_i$ for every $i \in \{1, ..., k\}$, thus $\pi_{k,\nu}$ is bijective. Since the polynomials

$$\{-\nu_l(t - x_l)^{\nu_l - 1}.\prod_{l \neq j}(t - x_j)^{\nu_j} \mid l = 1, ..., k\}$$

are linearly independent, and the coefficients of

$$-\nu_l(t - x_l)^{\nu_l - 1}.\prod_{l \neq j}(t - x_j)^{\nu_j}$$

are the lth row of $D\pi_{k,\nu}(x_1, ..., x_k)$ over the standard basis in both S_k and $R[t]^r$, so $\pi_{k,\nu}$ is an immersion of S_k into $\mathbb{R}[t]^r$. On the other hand according to Corollary 1.5.4 $\pi_{k,\nu}$ is a closed map. Since every injective and closed immersion is a diffeomorphism into its image, $\pi_{k,\nu}$ is a diffeomorphism between S_k and A_k. ∎

Lemma 1.5.8 *Let X be a manifold, and $f : X \to p^r$ a differentiable map, then there exists an open dense subset $W \subset X$ with the following property:*

Every $x \in W$ has an open neighborhood U, such that there exists $k \in \mathbb{N}$, $\nu \in B_k^r$ and differentiable functions $\mu_l : U \to \mathbb{R}$, for every $l \in \{1, ..., k\}$ with $\mu_1 < \mu_2 < ... < \mu_k$, such that

$$f(y)(t) = \prod_{l=1}^{k} (t - \mu_l(y))^{\nu_l}$$

for all $y \in U$.

Proof. Define $G_k := f^{-1}(p_k^r)$; according to Lemma 1.5.1

$$W := \bigcup_{k=1}^{r} F_k^\circ = \bigcup_{k=1}^{r} (G_k \setminus G_{k-1})^\circ$$

is an open dense subset of X. For an arbitrary element x in W, there exists $k_x \in \{1, \ldots, r\}$ such that

$$x \in F_k^\circ = (G_{k_x} \setminus G_{k_x - 1})^\circ.$$

Now consider

$$f \mid_{F_{k_x}^\circ} : F_{k_x}^\circ \to p_{k_x}^r \setminus p_{k_x-1}^r = \cup_{\nu \in B_k^r} A_{k_x,\nu}^r,$$

so there exists a unique $\nu' = (\nu_1, \ldots, \nu_k) \in B_k^r$ such that $F \mid_{F_{k_x}^\circ} (x) \in A_{\nu'}^{k_x}$. Then we define U to be

$$U := f \mid_{F_k^\circ}^{-1} (A_{k,\nu'}),$$

U is an open subset of W, because $f \mid_{F_k^\circ}$ is a continuous map, F_k° is an open subset of W and $A_{k,\nu'}$ is an open subset of $p_k^r \setminus p_{k-1}^r$. On the other hand according to Lemma 1.5.7 $\pi_{k,\nu'} : S_k \to A_{k,\nu'}$ is a diffeomorphism, therefore $\pi_{k,\nu'}^{-1} \circ f \mid_U : U \to S_k$ is a differentiable map, so it can be written in the following form

$$\pi_{k,\nu'}^{-1} \circ f \mid_U = (\mu_1, ..., \mu_k),$$

where $\mu_i : U \to \mathbb{R}$ are differentiable maps such that $\mu_1 < \mu_2 < ... < \mu_k$, and for all $y \in U$, we have

$$f(y)(t) = \prod_{i=1}^{k} (t - \mu_i(y))^{\nu_i}.$$

■

Proposition 1.5.9 *For any $f \in \mathrm{Herm}^+ E$ there is an open dense subset $W \subset X$ such that for every $x \in W$ the following holds: there exists an open neighborhood U of x in W, a unitary basis of differentiable sections $u_1, u_2, ..., u_r$ for E defined over U, and functions $\lambda_1, \lambda_2, ..., \lambda_r \in \mathcal{C}^\infty(U, \mathbb{R})$ such that*

$$f(y) = \sum_{i=1}^{r} e^{\lambda_i(y)} . u_i(y) \otimes u_i^*(y)$$

for all $y \in U$, where $u_1^*, ..., u_r^*$ denotes the dual of the basis $u_1, u_2, ..., u_r$.

Proof. Define the map

$$F : X \to p^r, \quad x \to (-1)^r.\chi_{f(x)},$$

where $\chi_{f(x)}$ is the characteristic polynomial of $f(x)$. Note that differentiability of F follows from differentiability of the determinant. Let $W \subset X$ be the open dense subset in X as in lemma 1.5.8. For $y \in W$ let U be a neighborhood and $\mu_1, ..., \mu_k$ be differentiable functions as there was in the lemma. Define on U subbundles

$$E_l = ker(f - \mu_i id), \quad i = 1, ..., k$$

E is equal to the orthogonal direct sum of E_l's, since $f \in \text{Herm}^+$. By taking U small enough we may assume that there are unitary bases for the E_l on U, which together give a unitary basis $u_1, ..., u_r$ for E on U. Since the μ_l are positive, we are done by defining $\lambda_i := \log(\mu_l)$ for $u_i \in E_l$. ■

Definition 1.5.10 *Let $F \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R})$, where $\mathbb{R}^+ := \{x \in \mathbb{R} : 0 < x\}$. We say that $\tilde{F} : \text{Herm}^+E \to \text{Herm}E$ is an* **extension** *of F, if it is differentiable and the following property holds: For every $f \in \text{Herm}^+E$ and $x \in X$ and every unitary basis $u = (u_1, \ldots, u_r)$ of E_x consisting of eigenvectors of f with eigenvalues $e^{\lambda_1}, \ldots, e^{\lambda_r}$ with $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, it holds*

$$\tilde{F}(f)(x) = \sum_{i=1}^{r} F(e^{\lambda_i}).u_i \otimes u_i^*.$$

Proposition 1.5.11 *Let $f \in \text{Herm}^+E$, $F \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R})$ a function with an extension $\tilde{F}$: $\text{Herm}^+(E) \to \text{Herm}(E)$ and $D_E : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ be a connection on E. Then there is an open dense subset $W \subset X$, such that for every $x \in W$ there exists an open neighborhood U of x in W such that*

$$D_{E^* \otimes E}(\tilde{F}(f)) = \sum_{i=1}^{r} F'(e^{\lambda_i}).e^{\lambda_i}.(d\lambda_i) \otimes u_i \otimes u_i^* + \sum_{i \neq j} \Big(F(e^{\lambda_j}) - F(\lambda_i)\Big).(\omega_u^E)_{ji} \otimes u_i \otimes u_j^*$$

Proof. According to Proposition 1.5.9, there exists an open dense subset $W \subset X$ such that for every $x \in W$ there exists an open neighborhood and a local unitary frame $u = (u_1, \ldots, u_r)$ on U such that

$$f(y) = \sum_{i=1}^{r} e^{\lambda_i(y)}.u_i(y) \otimes u_i^*(y).$$

So for every $y \in U$, we have

$$\tilde{F}(f)(y) = \sum_{i=1}^{r} F(e^{\lambda_i(y)}).u_i(y) \otimes u_i^*(y).$$

By applying the Leibnitz rule, we get

$$D_{E^* \otimes E}(\tilde{F}(f)(y)) = \sum_{i=1}^{r} F'(e^{\lambda_i}).e^{\lambda_i}.(d\lambda) \otimes u_i \otimes u_i^* + \sum_{i=1}^{r} F(e^{\lambda_i}) D_{E^* \otimes E}(u_i \otimes u_i^*).$$

By Remark 1.2.4

$$\omega_u^{E^* \otimes E} = (-\omega_u^E)^t \otimes I_n + I_n \otimes (\omega_u^E),$$

and substituting by (2) and regrouping the sums, we get

$$D_{E^*\otimes E}(\tilde{F}(f)(y)) = \sum_{i=1}^{r} F'(e^{\lambda_i}).e^{\lambda_i}.(d\lambda)\otimes u_i\otimes u_i^* + \sum_{i\neq j}\Big(F(e^{\lambda_j}) - F(\lambda_i)\Big).(\omega_u^E)_{ji}\otimes u_i\otimes u_j^*.$$

■

Proposition 1.5.9 and Proposition 1.5.11 are important tools in analysis of hermitian endomorphisms on a complex vector bundles; we try to illustrate this point in the following examples.

Example 1.5.12 *Define*

$$\log : Herm^+E \to HermE$$

by sending $f(x) = \sum_{i=1}^r e^{\lambda_i}\cdot u_i\otimes u_i^*$ *to* $\log f = \sum_{i=1}^r \lambda_i\cdot u_i\otimes u_i^*$ *for every* $f\in \mathrm{Herm}^+E$ *and* $x\in X$*, where* $u_1,\ldots,u_r$ *is an orthonormal basis of* E_x *consisting of eigenvectors of* f *with eigenvalues* $e^{\lambda_1},\ldots,e^{\lambda_r}$*,* $\lambda_1\ldots,\lambda_r\in\mathbb{R}$*. It is clearly well-define and one can show its differentiability. So it is an extension of* $\log\in\mathcal{C}^\infty(\mathbb{R}^+,\mathbb{R})$*.*
Now let $f\in\mathrm{Herm^+E}$*, then according to Proposition 1.5.9 there exists an open dense subset* W *of* X*, such that for every* $x\in W$ *there exists an open subset* $U\subset W$ *and a local frame* $u_1,\ldots,u_r$*, such that*

$$f = \sum_{i=1}^r e^{\lambda_i}\cdot u_i\otimes u_i^*$$

on U*. Therefore*

$$\log f = \sum_{i=1}^r \lambda_i\cdot u_i\otimes u_i^*.$$

Then using Proposition 1.5.11, we get

$$D_{E^*\otimes E}(\log f) = \sum_{i=1}^r d\lambda_i\otimes u_i\otimes u_i^* + \sum_{i\neq j}(\lambda_j-\lambda_i)\cdot(\omega_u)^{ji}\otimes u_i\otimes u_j^*.$$

But one can see, it is not the same as $f^{-1}\circ D_{E^*\otimes E}(f) = \sum_{i=1}^r d\lambda_i\otimes u_i\otimes u_i^* + \sum_{i\neq j}(e^{\lambda_j-\lambda_i}-1)\cdot(\omega_u)_{ji}u_i\otimes u_j^*$*, as one may naively think, since connections are in some aspect a generalization of the differential.*

Example 1.5.13 *Let* $\pi : E\to X$ *be a vector bundle,* h *be a Hermitian metric on* E *and* h_0 *be the induced Hermitian metric by* h *on* $E^*\otimes E$ *(i.e. if* $u=(u_1,\ldots,u_r)$ *is an orthonormal basis of* E_p *with respect to* h*, then* h_0 *is defined by saying that the basis* $u^*\otimes u$ *is an orthonormal basis for* h_0*),* $f\in\mathrm{Herm}^+(E)$ *and*

$$D_E : \mathcal{A}^0(E)\to\mathcal{A}^1(E).$$

Then there is an open dense subset $W\in X$ *such that, for every* $x\in W$ *there exists an open subset* $U\in W$*, such that the followings hold on* U*:*

$$h_0(f^{-1}\circ D_{E^*\otimes E}(f), D_{E^*\otimes E}(f^\sigma)) = \sum_{i=1}^r \sigma e^{\sigma\lambda_i}.d\lambda_i\wedge\overline{d\lambda_i}$$
$$+\sum_{i\neq j}(e^{\lambda_j-\lambda_i}-1).(e^{\lambda_j}-e^{\lambda_i}).(\omega_u)^{ji}\wedge\overline{(\omega_u^E)}_{ji}$$

and

$$
\begin{aligned}
&h_0(f^{-\frac{\sigma}{2}} \circ D_{E^*\otimes E}(f^\sigma), f^{-\frac{\sigma}{2}} \circ D_{E^*\otimes E}(f^\sigma)) \\
&= \sum_{i=1}^{r} \sigma^2.e^{\sigma\lambda_i}.d\lambda_i \wedge \overline{d\lambda}_i + \sum_{i\neq j} e^{\sigma\lambda_i}.(e^{\sigma\lambda_j} - e^{\sigma\lambda_i})^2.(\omega_u^E)_{ji} \wedge \overline{(\omega_u^E)}_{ji},
\end{aligned}
$$

where $u = u_1, \ldots, u_r$ is a local orthonormal frame on U, ω_u^E is a connection form of E with respect to u and f^σ denotes $\tilde{F}(f)$, the extension of F where $F \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R})$ is the map $x \mapsto x^\sigma$ for some $\sigma \in \mathbb{R} \setminus \{0\}$. For seeing the former equality first note that by Proposition 1.5.11 there is an open dense subset $W \subset X$, such that for every $x \in W$ there exists a neighborhood U of x in X such that

$$
D_{E^*\otimes E}(f^\sigma) = \sum_{i=1}^{r} \sigma.(e^{\lambda_i})^\sigma.d\lambda_i \otimes u_i \otimes u_i^* + \sum_{i\not{j}} (e^{\lambda_j} - e^{\lambda_i})(\omega_u^E)_{ji} \otimes u_i \otimes u_j^*
$$

holds on U. On the other hand, we have $f^{-1} = \sum_{i=1}^{r} e^{-\lambda_i}.u_i \otimes u_i^*$, therefore

$$
f^{-1} \circ D_{E^*\otimes E}(f) = \sum_{i=1}^{r} d\lambda_i \otimes u_i \otimes u_i^* + \sum_{i\neq j} (e^{\lambda_j - \lambda_i} - 1)(\omega_u^E)_{ji} \otimes u_i \otimes u_j^*.
$$

So we can write

$$
\begin{aligned}
&h_0\left(f^{-1} \circ D_{E^*\otimes E}(f), D_{E^*\otimes E}(f^\sigma)\right) \\
&= \sum_{i=1}^{r}\sum_{i'=1}^{r} h_0\left(d\lambda_i \otimes u_i \otimes u_i^*, \sigma e^{\sigma\lambda_{i'}}.d\lambda_{i'} \otimes u_{i'} \otimes u_{i'}^*\right) \\
&+ \sum_{i\neq j}\sum_{i'\neq j'} h_0\left((e^{\lambda_j-\lambda_i} - 1).(\omega_u^E)_{ji} \otimes u_i \otimes u_j^*, (e^{\lambda_{j'}} - e^{-\lambda_{i'}}).(\omega_u^E)_{j'i'} \otimes u_{i'} \otimes u_{j'}^*\right) \\
&+ \sum_{i=1}^{r}\sum_{i'\neq j'} h_0\left(d\lambda_i \otimes u_i \otimes u_i^*, (e^{\lambda_{j'}} - e^{\lambda_{i'}}).(\omega_u^E)_{j'i'} \otimes u_{i'} \otimes u_{j'}^*\right) \\
&+ \sum_{i\neq j}\sum_{i'=0}^{r} h_0\left((e^{\lambda_j-\lambda_i} - 1).(\omega_u)_{ji} \otimes u_i \otimes u_j^*, \sigma e^{\sigma\lambda_{i'}}.d\lambda_{i'} \otimes u_{i'} \otimes u_{i'}^*\right) \\
&= \sum_{i=1}^{r}\sum_{i'=1}^{r} \overline{\sigma e^{\sigma\lambda}}_{i'} h_0\left(u_i \otimes u_i^*, u_{i'} \otimes u_{i'}^*\right) d\lambda_i \wedge \overline{d\lambda}_{i'} \\
&+ \sum_{i\neq j}\sum_{i'\neq j'} (e^{\lambda_j-\lambda_i} - 1).\overline{(e^{\lambda}_{j'} - e^{\lambda_{i'}})} h_0\left(u_i \otimes u_j^*, (u_{i'} \otimes u_{j'}^*\right) (\omega_u^E)_{ji} \wedge \overline{\omega_u)}_{j'i'} \\
&+ \sum_{i=1}^{r}\sum_{i'\neq j'} \overline{(e^{\lambda_{j'}} - e^{\lambda})}_{i'} h_0\left(u_i \otimes u_i^*, u_{i'} \otimes u_{j'}^*\right) d\lambda_i \wedge \overline{(\omega_u)}_{j'i'} \\
&+ \sum_{i\neq j}\sum_{i'=0}^{r} (e^{\lambda_j-\lambda_i} - 1).\overline{\sigma e^{\sigma\lambda}}_{i'} h_0\left(u_i \otimes u_j^*, u_{i'} \otimes u_{i'}^*\right) (\omega_u^E)_{ji} \wedge \overline{d\lambda}_{i'} \\
&= \sum_{i=1}^{r} \sigma e^{\sigma\lambda_i}.d\lambda_i \wedge \overline{d\lambda}_i
\end{aligned}
$$

$$+\sum_{i\neq j}(e^{\lambda_j-\lambda_i}-1).(e^{\lambda_j}-e^{\lambda_i}).(\omega_u^E)_{ji}\wedge\overline{(\omega_u^E)}_{ji}$$

which proves the former equality. For seeing the latter one, first note that the following holds on U

$$f^{-\frac{\sigma}{2}}\circ D_{E^*\otimes E}(f^\sigma)=\sum_{i=1}^{r}\sigma e^{\sigma\lambda_i}d(\lambda_i)\otimes u_i\otimes u_i^*+\sum_{i\neq j}(e^{\sigma\lambda_j}-e^{\sigma\lambda_i}).e^{-\frac{\sigma}{2}\lambda_i}.\alpha_{ji}\otimes u_i\otimes u_j^*.$$

So we can write

$$\begin{aligned}
&h_0(f^{-\frac{\sigma}{2}}\circ D_{E^*\otimes E}(f^\sigma),f^{-\frac{\sigma}{2}}\circ D_{E^*\otimes E}(f^\sigma))\\
&=\sum_{i=1}^{r}\sum_{i'=1}^{r}h_0\left(\sigma e^{\frac{\sigma}{2}\lambda_i}d\lambda_i\otimes u_i\otimes u_i^*,\sigma e^{\frac{\sigma}{2}\lambda_{i'}}.d\lambda_{i'}\otimes u_{i'}\otimes u_{i'}^*\right)\\
&+\sum_{i\neq j}\sum_{i'\neq j'}h_0\left((e^{\sigma\lambda_j}-e^{\sigma\lambda_i}).e^{-\frac{\sigma}{2}\lambda_i}.(\omega_u^E)_{ji}\otimes u_i\otimes u_j^*,(e^{\sigma\lambda_{j'}}-e^{\sigma\lambda_{i'}}).e^{-\frac{\sigma}{2}\lambda_{i'}}.(\omega_u^E)_{j'i'}\otimes u_{i'}\otimes u_{j'}^*\right)\\
&+\sum_{i=1}^{r}\sum_{i'\neq j'}h_0\left((e^{\sigma\lambda_j}-e^{\sigma\lambda_i}).e^{-\frac{\sigma}{2}\lambda_i}.(\omega_u^E)_{ji}\otimes u_i\otimes u_j^*,(e^{\sigma\lambda_{j'}}-e^{\sigma\lambda_{i'}}).e^{-\frac{\sigma}{2}\lambda_{i'}}.(\omega_u^E)_{j'i'}\otimes u_{i'}\otimes u_{j'}^*\right)\\
&+\sum_{i\neq j}\sum_{i'=0}^{r}h_0\left((e^{\sigma\lambda_j}-e^{\sigma\lambda_i}).e^{-\frac{\sigma}{2}\lambda_i}.(\omega_u^E)_{ji}\otimes u_i\otimes u_j^*,\sigma e^{\frac{\sigma}{2}\lambda_{i'}}.d\lambda_{i'}\otimes u_{i'}\otimes u_{i'}^*\right)\\
&=\sum_{i=1}^{r}\sigma^2.e^{\sigma\lambda_i}.d\lambda_i\wedge\overline{d\lambda_i}+\sum_{i\neq j}e^{\sigma\lambda_i}.(e^{\sigma\lambda_j}-e^{\sigma\lambda_i})^2.(\omega_u^E)_{ji}\wedge\overline{(\omega_u^E)}_{ji}.
\end{aligned}$$

Example 1.5.14 *Consider E,f and D_E be as they were in Example 1.5.13. Then there exists an open dense subset $W\subset X$ such that for every $x\in W$ there exists an open neighborhood $U\subset W$ on which the following holds:*

$$\mathrm{tr}(f^{-1}\circ D_{E^*\otimes E}(f)\circ f^\sigma)=d(\frac{1}{\sigma}.\mathrm{tr}(f^\sigma)).$$

For seeing that first note that, according to Proposition 1.5.11 and proposition 1.5.9, there exists an open dense subset $W\subset X$ such that, for every $x\in W$ there exists an neighborhood of $U\subset W$ such that

$$f^{-1}\circ D_{E^*\otimes E}(f)=\sum_{i=1}^{r}d\lambda_i\otimes u_i\otimes u_i^*+\sum_{i\not j}(e^{\lambda_j-\lambda_i}-1)(\omega_u^E)_{ij}\otimes u_i\otimes u_j^*$$

and

$$f^\sigma=\sum_{i=1}^{r}e^{\sigma\lambda_i}.u_i\otimes u_i^*$$

where $u=(u_1,\dots,u_n)$ is the orthonormal frame. Therefore

$$f^{-1}\circ D_{E^*\otimes E}(f)\circ f^\sigma=\sum_{i=1}^{r}e^{\sigma\lambda_i}(d\lambda_i\otimes e_i+\sum_{j\neq i}(e^{\lambda_i-\lambda_j}-1)(\omega_u)_{ij}\otimes u_j)\otimes u_i^*.$$

So

$$\mathrm{tr}(f^{-1} \circ D_{E^* \otimes E}(f) \circ f^{\sigma}) = \sum_{i=1}^{r} e^{\sigma \lambda_i} d\lambda_i.$$

On the other hand $\mathrm{tr}(f_i^{\sigma}) = \sum_{i=1}^{r} e^{\sigma \lambda_i}$*, therefore*

$$d(\frac{1}{\sigma}.\mathrm{tr}(f_i^{\sigma})) = \sum_{i=1}^{r} \frac{1}{\sigma}.\sigma.d\lambda_i.e^{\sigma \lambda_i} = \sum_{i=1}^{r} d\lambda_i e^{\sigma \lambda_i}.$$

Thus

$$\mathrm{tr}(f^{-1} \circ D_{E^* \otimes E}(f) \circ f^{\sigma}) = d(\frac{1}{\sigma} tr(f^{\sigma})).$$

2 Maximal torus reduction

2.1 The adjoint representation

Proposition 2.1.1 *Let be a Lie group G acting on M from the left smoothly, (i.e. there is a smooth map $\mu : G \times M \to M$, such that $\mu(\tau\rho, m) = \mu(\tau, \mu(\rho, m))$ and $\mu(e, m) = m$,for every $\tau, \rho \in G$ and $m \in M$) and let $m_0 \in M$ be a fixed point, i.e. $\mu_g(m_0) = m_0$ for every $g \in G$. Then*

$$\psi : G \to \mathrm{Aut}(T(M)_{m_0}), \quad g \mapsto d\mu_g \mid_{T(M)_{m_0}}$$

is a representation of G.

Proof. First note that ψ is group homomorphism because

$$\psi(gh) = d\mu_{gh} \mid_{T(M)_{m_0}} = d(\mu_g \circ \mu_h) \mid_{T(M)_{m_0}} = d\mu_g \mid_{T(M)_{m_0}} \circ \mu_h \mid_{T(M)_{m_0}} = \psi(g) \circ \psi(h),$$

for every $g, h \in G$.
It only remains to prove that ψ is a differentiable map. Let $(v_1, \ldots, v_d)$ be a basis for the vector space $T(M)_{m_0}$ with respect to this basis, we can identify automorphisms of $T(M)_{m_0}$ with $\mathrm{GL}(d, \mathbb{R})$. Let us denote this identification by $\sigma : \mathrm{Aut}(T(M)_{m_0}) \to \mathrm{GL}(d, \mathbb{R})$. σ is a chart for $\mathrm{Aut}(T(M)_{m_0})$, so it suffices that $\sigma \circ \psi$ is smooth. One gets the matrix associated with an element of $\mathrm{Aut}(T(M)_{m_0})$ by applying this element to the basis of $T(M)_{m_0}$ and then applying the dual basis. So it suffices to prove that if $v_0 \in T(M)_{m_0}$ and if $\alpha \in TM^*_{m_0}$, then

$$g \mapsto \alpha\left(d\mu_g(v_0)\right)$$

is smooth. Since α is linear and therefore a smooth map from $T(M)_{m_0}$ to $\mathbb{R}$ it suffices to show that

$$g \mapsto d\mu_g(v_0)$$

is a smooth map form G into $\mathrm{T}M_{m_0}$, or equivalently that it is a smooth map from G into $T(M)$. But it is exactly the composition of smooth maps

$$G \to T(G) \times T(M) \to T(G \times M) \stackrel{d\mu}{\to} T(M)$$

in which the first map sends $g \mapsto ((g, 0), (m_0, v_0))$, the second map is the canonical diffeomorphism of $T(G) \times T(M)$ with $T(G \times M)$, and the third map is $d\mu$. Thus ψ is smooth. ■

Definition 2.1.2 *Let* $\mathfrak{g}$ *be a Lie algebra and* V *be a vector space, then a map* $\psi : \mathfrak{g} \to End(V)$ *is called a* **Lie algebra representation** *of* $\mathfrak{g}$*, if* ψ *is a Lie algebra homomorphism.*

Definition 2.1.3 *Let* G *be a Lie group. It acts on itself from the left by conjugation*

$$a : G \times G \to G, \quad (h, g) \mapsto hgh^{-1}$$

We denote $a(h, g)$ *by* $a_h(g)$*. Since the identity is a fixed point of this action by Proposition 2.1.1 the map*

$$h \mapsto da_h \mid_{T(G)_e} \in \mathrm{Aut}(\mathrm{T}(G)_e) = \mathrm{Aut}(\mathfrak{g})$$

is a representation of G *into* $Aut(\mathfrak{g})$*. This is called the* **adjoint representation** *and is denoted by*

$$Ad : G \to \mathrm{Aut}(\mathfrak{g}).$$

We denote the differential of the adjoint representation by ad,

$$d(\mathrm{Ad}) = ad.$$

Proposition 2.1.4 *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$*, and let* $X, Y \in \mathfrak{g}$*. Then*

$$\mathrm{ad}(X)Y = [X, Y].$$

For a proof see [W] page 115.

2.2 Principal fiber bundles

Definition 2.2.1 *Let* G *be a Lie group,* P *be a manifold and* G *acting on* P *from the right such that:*

1. $M := P/G$ *has a manifold structure such that* $\pi : P \to P/G = M$ *is smooth.(i.e.* P/G *as topological space can be endow with a manifolds structure such that* $\pi : P \to P/G$ *is a smooth map.)*

2. *There exists an open cover* $\{U_i : i \in I\}$ *of* M *and diffeomorphisms*

$$\theta_i : \pi^{-1}(U_i) \to G \times U_i, \quad u \mapsto (\pi(u), \varphi_i((u))$$

where $\varphi_i : \pi^{-1}(U_i) \to G$ *satisfies* $\varphi_i(ug) = \varphi_i(u)g$ *for all* $u \in \pi^{-1}(U_i)$ *and* $g \in G$*. Then* $\pi : p \to M$ *is called a* **principal fiber bundle** *over* M*, and this open cover and maps together are called* **local trivializations** *of* P*. In particular* $\pi : P \to M$ *is fiber bundle with typical fiber* G*. We will write* $P(M, G, \pi)$ *or* $P(M, G)$ *(or simply* P*) to denote a principal fiber bundle* $\pi : P \to M$*. The action* $P \times G \to P$ *will be denoted by* $(u, g) \mapsto ug$*.*

Note that from 2. it follows that the action of G on P is differentiable and free.
Clearly any trivial fiber bundle admits a global section. For example one can take $M \to M \times G, m \mapsto (m, e)$. The converse is also true. If a principal fiber bundle admits a global section $\sigma : M \to P$ then it is trivial because one can consider the inverse of the smooth map $M \times G \to p, (p, g) \mapsto \sigma(p)g$ as a global trivialization for P. In particular, condition 2. in Definition 2.2.1 is equivalent to requiring the existence of an open cover $\{U_i, i \in I\}$ of M and local sections $\{\sigma_i : U_i \to P\}$.

Example 2.2.2 *Let $\pi_E : E \to M$ be a vector bundle with local trivializations $\{\theta_i^E : \pi_E^{-1}(U_i) \to U_i \times \mathbb{R}^n\}$. Define $L(E)$ to be the set of all linear frames on $\pi_E : E \to M$ i.e. the set of all $(v_1^p, \ldots, v_n^p)$ where $\{v_1^p \ldots v_n^p\}$ constitutes a basis for the fiber E_p. Define the projection $\pi : L(E) \to M$ to be the map sending a frame at p to p. Let $GL(n, \mathbb{R})$ act on $L(E)$ from the right by $(u, A) \mapsto uA$, where uA is $(u_1 A, \ldots, u_n A)$. Then we have $L(E)/GL(n, \mathbb{R}) = M$. Define sections $\sigma_i : U_i \to L(E)$ by*

$$p \mapsto \mathcal{U}(p) := ((\theta_i^E)^{-1}(p, e_1), \ldots, (\theta_i^E)^{-1}(p, e_n)). \tag{6}$$

The $\{\sigma_i : \mathcal{U}_i \to P\}$ induce bijections $\{\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times GL(n, \mathbb{R}), i \in I\}$ and we can define a differentiable structure on $L(E)$ by requesting these bijections to be diffeomorphisms, this is well-defined because the gluing maps are the smooth map $\mathcal{U}_{ij} \times GL(n, \mathbb{R}) \to \mathcal{U}_{ij} \times GL(n, \mathbb{R}), (p, A) \mapsto \theta_{ij}^E(p)A$ where $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ and θ_{ij} is the transition function. Then $L(E)(M, GL(n, \mathbb{R}))$ becomes a principal fiber bundle over M, it is called the **frame bundle** *of the vector bundle $\pi_E : E \to M$.*

Let $\pi_E : E \to M$ be a complex vector bundle with local trivializations $\{\theta_i^E : \phi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{C}^n\}$ with respect to a cover $\{\mathcal{U}_i : i \in I\}$ of M and h be a Hermitian metric over E. Since $\theta_i^E : \phi_{-1}^E(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{C}^n$ is a diffeomorphism for every $i \in I$ and $\theta_i^E \mid_{E_p}: E_P \to p \times \mathbb{C}^n$ is a linear isomorphism, for every $p \in \mathcal{U}_i$, $h \mid_{\pi_E^{-1}(\mathcal{U}_i)}$ induces a Hermitian metric h' on $\mathcal{U}_i \times \mathbb{C}^n$. For every $p \in \mathcal{U}_i$ the set

$$\{\{p\} \times e_1, \ldots, \{p\} \times e_n\}$$

is an ordered basis for $\{p\} \times \mathbb{C}^n$ so by the Gram-Schmidt process we get an orthonormal ordered basis $\{e_1', \ldots, e_n'\}$ with respect to the Hermitian metric h'. Define $\sigma_p : \{p\} \times \mathbb{C}^n \to \{p\} \times \mathbb{C}^n$ to be the linear isomorphism which maps $\{e_1, \ldots, e_n\}$ to $\{\{p\} \times e_1, \ldots, \{p\} \times e_n\}$. Then define $\sigma_i' : \mathcal{U}_i \times \mathbb{C}^n \to \mathcal{U}_i \times \mathbb{C}_n, (p, v) \to (p, \sigma_p(v))$. One can see that σ' is a diffeomorphism and the induced Hermitian metric h' by this map is the standard Hermitian metric in $\mathcal{U}_i \times \mathbb{C}^n$. Define $\theta_i' : \pi_{-1}^E((U)_i) \to \mathcal{U}_i \times \mathbb{C}^n$ to be $\theta_i' := \theta_i^E \circ \sigma_i'$. Then $\{\theta_i' : \pi_E^{-1}(\mathcal{U}_i \to \mathcal{U}_i \times \mathbb{C}^n : i \in I)\}$ is a set of local trivializations with respect to the cover $\{(U)_i, i \in I\}$, such that h is locally induced by the standard Hermitian metric of $\mathcal{U}_i \times \mathbb{C}_n$. Let us restate the above results as a Remark.

Remark 2.2.3 *Let $\pi_E : E \to M$ be a complex vector bundle and h be a Hermitian metric over E. Then there exists local trivializations $\{\theta_i^E : \phi_{-1}^E(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{C}^n\}$ with respect to a cover $\{\mathcal{U}_i : i \in I\}$ of M such that h is locally induced by the standard Hermitian metric of $\mathcal{U}_i \times \mathbb{C}^n$.*

Example 2.2.4 *Let $\pi_E : E \to M$ be a complex vector bundle and h be a Hermitian metric over E with local trivializations $\{\theta_i^E : \phi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{C}^n\}$ as in Remark 2.2.3. Define $LO(E)$ to be the set of all linear orthogonal frames on $\pi_E : E \to M$, i.e. the set of all $(v_1^p, \ldots, v_n^p)$ where $\{v_1^p \ldots v_n^p\}$ constitutes an orthonormal basis for the fiber E_p. Define the projection $\pi : LO(E) \to M$ to be the map sending a frame at p to p. Let $U(n, \mathbb{C})$ (unitary matrices) act on $LO(E)$ from the right by $(u, A) \mapsto uA$. By the same argument as Example 2.2.2 one can endow $LO(E)$ with a manifold structure and shows that $LO(M)(U(n, \mathbb{C}), M)$ is a principal fiber bundle, which is called the* **orthonormal frame bundle** *of the vector bundle E.*

Definition 2.2.5 *A* **homomorphism** *of a principal fiber bundle $Q(N, H)$ into another principal fiber bundle $P(M, G)$ consists of a smooth map $f : Q \to P$ and a Lie group homomorphism $f' : H \to G$ such that $f(ug) = f(u)f'(g)$ for all $u \in Q$ and $g \in H$.*

Note that a homomorphism (f, f') as in Definition 2.2.5 induces a smooth map $f'' : N \to M$ by sending n to $\pi_P \circ f(u)$ where u is an arbitrary element of Q_n.

Definition 2.2.6 *A homomorphism $(f, f') : Q(N, H) \to P(M, G)$ is an* **embedding** *if the induced $f'' : N \to M$ is an embedding and $f' : H \to G$ a monomorphism (then, in particular, $f : Q \to P$ is an embedding). We call $f(Q)(f''(N), f'(H))$ a* **subbundle** *of $P(M, G)$. If moreover $N = M$ and the induced $f'' : M \to M$ is the identity, then $(f, f') : Q(M, H) \to P(M, G)$ is called a* **reduction** *(relative to $f' : H \to G$) of (the structure group of) $P(M, G)$ to H or an* **extension** *(relative to $f' : H \to G$) of (the structure group of) $Q(M, H)$ to G. The bundle $Q(M, H)$ is called a* **reduced** *bundle of $P(M, G)$. Given $P(M, G)$ and a Lie subgroup H of G, we say that $P(Q, G)$ is* **reducible** *to H if there is a reduced bundle $Q(M, H)$. An* **isomorphism** *between two principal fiber bundles $Q(M, G)$ and $P(M, G)$ is a homomorphism $(f, f') : Q(M, G) \to P(M, G)$ such that f' and the induced f'' are the identity on G and M respectively. In particular, $f : Q \to P$ is a diffeomorphism. An isomorphism $P(M, G) \to P(M, G)$ is called an* **automorphism** *of $P(M, G)$.*

Example 2.2.7 *Let $\pi_E : E \to M$ be a complex vector bundle and h be a Hermitian metric over E with local trivializations $\{\theta_i^E : \phi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{C}^n\}$ as in Remark 2.2.3. Consider $LO(E)(M, U(n, \mathbb{C}))$ and $L(E)(n, GL(n, \mathbb{C}))$ as defined in Example 2.2.4 and Example 2.2.2. Then $(i, i') : LO(E)(M, U(n, \mathbb{C})) \to L(E)(M, GL(n, \mathbb{C}))$ is a reduction of $L(E)(M, GL(n, \mathbb{C}))$ to $U(M, U(n, \mathbb{C}))$ where $i : LO(E) \to L(E)$ and $i' : U(n, \mathbb{C}) \to GL(n, \mathbb{R})$ are the inclusion morphisms.*

Let $P(M, G)$ be a principal fiber bundle and F be a manifold on which G acts differentiably on the left. Then define the following action of G on $P \times F$:

$$(P \times F) \times G \to P \times F, \quad ((u, f), g) \mapsto (ug, g^{-1}f).$$

Define $E = P \times_G F := (P \times F)/G$ and denote by $(u, f)/\sim$ the equivalence class of $(u, f) \in P \times F$ in E. Define $\pi_E : E \to M$ to be the map $(u, f) \mapsto \pi(u)$ where π is the projection of $P \to M$. Let $\{\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G, u \mapsto (\pi(u), \varphi_i(u))\}$ be local trivializations of $P(M, G)$ with respect to a cover $\{\mathcal{U}_i, i \in I\}$ of M and define for all $i \in I$ a bijection

$$\theta_i^E : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times F, \quad (u, f)/\sim \mapsto (\pi(u), \varphi_i(u)f).$$

We can define a differentiable structure on E by requiring the θ_i^E's to be diffeomorphism. This is well-defined because the gluing maps $\theta_{ij}^E : \mathcal{U}_{ij} \times F \to \mathcal{U}_{ij} \times F, (p, f) \mapsto (p, \theta_{ij}(p)f)$ are smooth. Then $E = P \times_G F$ becomes a fiber bundle over M with fiber F.

Definition 2.2.8 *Let $P(M, G)$ be a principal fiber bundle and F be a manifold on which G acts differentiably on the left. Then $\phi : P \to F$ is called equivariant if $\phi(ug) = g^{-1}\phi(u)$ for all $u \in P$ and $g \in G$.*

Proposition 2.2.9 *Let $P(M, G)$ be a principal fiber bundle and F be a manifold on which G acts differentiably on the left. Then*

$$\{\text{ sections of } \mathrm{P} \times_{\mathrm{G}} \mathrm{F}\} \overset{1-1}{\longleftrightarrow} \{\mathrm{G}-\text{equivariant smooth maps } \mathrm{P} \to \mathrm{F}\}$$

Proof. First note that the action of G on each fiber P_a is free and transitive, so for every $u, v \in P_a$ there exists a unique $g_{u,v}$ such that $ug_{u,v} = v$.
Now let $\sigma \in \Gamma(P \times_G F)$. We represent $\sigma(a)$ by $(u_a, f_a)/\sim$ where a is in M and $u_a \in P$ and $f_a \in F$. Define

$$\hat{\sigma} : P \to F, \quad v \mapsto g^{-1}_{u_{\pi(v)},v} f_{\pi(v)},$$

note that $\hat{\sigma}$ is independent of the representation of σ i.e. if $(u_a, f_a) \sim (u', f')$ then $g^{-1}_{u_a,v} f_a = g^{-1}_{u',v} f'$ because

$$g^{-1}_{u_a,v} f_a = (g_{u_a,u'} g_{u',v})^{-1} f_a = g^{-1}_{u',v} g^{-1}_{u_a,u'} f_a = g^{-1}_{u',v} f'.$$

Since

$$\hat{\sigma}(gv) = g^{-1}_{u_{\pi(gv)},gv} f_{\pi(gv)} = g^{-1} g^{-1}_{u_{\pi(v)},v} f_{\pi(v)} = g^{-1}\hat{\sigma}(v),$$

we have that $\hat{\sigma}$ is G-equivarent.
Smoothness can be seen as follows. Observe first that if $\{\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G, i \in I\}$ are local trivializations of $P(M,G)$ with respect to a cover $\{\mathcal{U}_i; i \in I\}$ of M, then for $p \in \mathcal{U}_i$ we can write $\sigma(p) = (\theta_i^{-1}(p, e), f_i(p))/\sim$ for some map $f_i : \mathcal{U}_i \to F$ which is smooth because it is the composition θ_i^E, where θ_i^E is given by (6). Since $\hat{\sigma}$ is locally the composition of the map

$$\mathcal{U}_i \times G \to F, \quad (p, g) \mapsto g^{-1} f_i(p)$$

with the trivialization θ_i, it follows that it is smooth. Conversely, given a smooth, G-equivariant map $\phi : P \to F$, the map

$$\phi_0 : M \to P \times_G F, \quad p \mapsto (u, \phi(u))/\sim$$

is well-defined and smooth, thus it is a section of $P \times_G F$. Clearly, $\hat{\phi}_0 = \phi$ for all G-equivariant maps $P \to F$ and $\hat{\sigma}_0 = \sigma$ for all sections σ of $P \times_G F$. ■

Example 2.2.10 *Let $P(G, M)$ be principal fiber bundle, $\{\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G.i \in I\}$ be local trivalizations of P with respect to the cover $\{\mathcal{U}_i, i \in I\}$ and $\varrho : G \to \mathrm{Aut(V)}$ be a Lie group representation of G on a vector space V. Then $E = P \times_G V$ is a vector bundle over M, where G acts on V from left by $(g, v) \mapsto \varrho(g)(v)$. To see this first note that for every $p \in (U_{ij})$ the map*

$$V \overset{\cong}{\to} \{p\} \times V \overset{\theta_i^E \circ (\theta_j^E)^{-1}|_{\{p\}\times V}}{\longrightarrow} \{p\} \times V \overset{\cong}{\to} V$$

is linear, because it is given by $v \mapsto \varrho(\theta_{ij}(p))(v)$.
Let $\alpha_v : V \to \mathbb{R}^n$ be the identification of V with $\mathbb{R}^n$ with respect to a given basis of $v_1, \ldots, v_n$ (which also induces an isomorphism). Then the local trivializations of E induced by (6) are :

$$\theta_i^E : \pi^{-1}(\mathcal{U}_i \to \mathcal{U}_i \times \mathbb{R}^n, \quad (u, w)/\sim\mapsto \big(\pi(u), \alpha'_v\left(\varrho\left(\varphi_i(u)\right)\alpha_v(w)\right)\big)$$

where $\varphi_i = p_2 \circ \theta_i : \pi^{-1}(\mathcal{U}_i) \to G$.

Proposition 2.2.11 *Let $P(M, G)$ be a principal fiber bundle, H be closed subgroup of G and*

$$\mu : P \to P \times_G (G/H), \quad p \mapsto (p, H)/\sim.$$

Then there exists a one-to-one correspondence between H-reductions of P and global sections of $P \times_G (G/H)$, which corresponds the H-reduction $f : Q \to P$ to the section $\mu \circ f$ and the section $\sigma : M \to P \times_G (G/H)$ to the H-reduction $\{u \in P : (u, H)/\sim= \sigma(\pi(u))\}$

For its proof see [K] page 57.

Remark 2.2.12 *Let σ be a section in $P \times_G (G/H)$ and $\hat{\sigma}$ be its corresponding G-equivariant map $P \to G/H$ as it is defined in Proposition 2.2.9, then*

$$\{u \in P : (u, H)/\sim = \sigma(\pi(u))\} = \{u \in P : \hat{\sigma}(u) = H\}.$$

2.3 Maximal torus

Definition 2.3.1 *A torus T of G is a subgroup of G isomorphic to a product of $U(1, \mathbb{C})$ factors. A* **maximal torus** *T is a torus such that there is no torus H in G such that $T \subset H$ is a proper inclusion.*

Definition 2.3.2 *Given a maximal torus T in a connected Lie group G, we define the* **normalizer** *of T to be the subgroup*

$$N_G(T) = \{g \in G : gT = Tg\} = \{g \in G : gTg^{-1} = T\}$$

of G. T is a normal subgroup of $N_G(T)$ and the quotient group

$$W_G(T) = N_G(T)/T$$

is called the **Weyl group** *of G with respect to T.*

$W_G(T)$ acts on T since $N_G(T)$ acts by conjugation

$$(n, t) \in N_G(T) \times T \to ntn^{-1} \in T$$

and its subgroup T acts trivially.

Theorem 2.3.3 *Let G be a compact connected Lie group and T be a maximal torus of G, then any element of G is conjugate to an element of T.*

For a proof see [S] section 8.1

Corollary 2.3.4 *Any two maximal tori T_1, T_2 of G are conjugate.*

Proof. This can be proved by noting that a torus T always has a generating element, i.e. an element whose powers are dense in T. If θ_i are angular variables labelling the points in the torus, a generating element can be constructed by taking a linear combination of these variables, with coefficients independent over $\mathbb{Q}$.
If t_2 is a generator of T_2, then by the theorem

$$x^{-1} t_2 x = t$$

for some $x \in G$ and $t \in T$. Thus $t_2 \in xTx^{-1}$ and so are all its powers. Since t_2 is a generator, $T_2 \subset xTx^{-1}$. But since T_2 is a maximal torus, we must have $T_2 = xTx^{-1}$. ■

Corollary 2.3.5 *All maximal tori have the same dimension. We will call this common dimension the rank of G.*

Definition 2.3.6 *For a compact Lie group G, a* **Cartan sub-algebra** *of $\mathfrak{g}$ is the Lie algebra of a maximal torus $T \subset G$.*

Theorem 2.3.7 *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and with G simply connected. Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists a unique homomorphism $\varphi : G \to H$ such that $d\varphi = \psi$.*

For a proof see [W] Theorem 3.27

Corollary 2.3.8 *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\varphi : \mathfrak{g} \to$ $\mathrm{End}(V)$ be a representation of Lie algebras. Then there exists a unique representation ψ : $G \to Aut(V)$ such that $d\varphi = \psi$. So in particular we have a one-to-one correspondence between Lie group representations of G and Lie algebra representations of $\mathfrak{g}$.*

2.4 Abstract root systems

Let E be a n-dimensional Euclidean vector space equipped with the inner product $(,)$. For any $\alpha \in E$, $\alpha \neq 0$, denote by L_α the hyperplane in E orthogonal to α and by r_α the reflection with respect L_α. The reflection r_α can be written in the form

$$r_\alpha(\beta) = \beta - < \beta \mid \alpha > \quad (\beta \in E) \cdot \alpha$$

where

$$< \beta \mid \alpha > = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Note that L_α, r_α and the function $< \beta \mid \alpha >$ do not change if the inner product in E is replaced by $c(,)$, where $c > 0$.

Definition 2.4.1 *A subset $\Delta^a \subset E$ is called an* **abstract root system** *in E if:*

1. *Δ^a spans E;*

2. *Δ^a is finite and consists of nonzero vectors;*

3. *$r_\alpha(\Delta^a) = \Delta^a$ for any $\alpha \in \Delta^a$;*

4. *$< \alpha \mid \beta > \in \mathbb{Z}$ for any $\alpha, \beta \in \Delta^a$.*

Let $\alpha \in \Delta^a$. It follows from 3. that $-\alpha \in \Delta^a$, and one can easily deduce from 4. that if $c \in \mathbb{R}$ and $c\alpha \in \Delta^a$, then $c \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. A root system is said to be reduced if the following condition holds:
5. if $\alpha \in \Delta^a$ and $c\alpha \in \Delta^a$ for some $\alpha \in \Delta^a$, then $c = \pm 1$.

Definition 2.4.2 *Let Δ^a be a reduced abstract root system of E. The* **abstract Weyl group** *of Δ^a is the subgroup of the isometry group of E which is generated by r_α's for $\alpha \in \Delta^a$. It is denoted by $W_{\Delta^a}(E)$.*

Definition 2.4.3 *Let Δ^a be a reduced abstract root system of E, then each connected components of the open set*

$$E \setminus \bigcup_{\alpha \in \Delta^a} (L_\alpha),$$

is called an **open Weyl chamber** *of the abstract root system Δ^a*

Lemma 2.4.4 *The action of the abstract Weyl group on E preserves $E \setminus \cup_{\alpha\in\Delta^a} L_\alpha$. Moreover its restriction to $E \setminus \bigcup_{\alpha\in\Delta^a}(L_\alpha)$ is free.*

For a proof see [OV].

Note that since λ is a homeomorphism from $E \setminus \bigcup_{\alpha\in\Delta^a}(L_\alpha)$ to itself it maps each connected component to another. Therefore λ maps each open Weyl chamber to an open Weyl chamber. In fact λ induces a permutation over the set of all the open Weyl chambers. Since the orbit of each point in an open Weyl chamber meets each open Weyl chamber exactly once and according to Lemma 2.4.4 the action of the Weyl group is free, the induced action is also free.

Lemma 2.4.5 *The induced action of the abstract Weyl group on the set of all Weyl chambers is simply transitive.*

Proof. Transitivity of the induced action of the abstract Weyl group actually means that the orbit of any point in $E \setminus \cup_{\alpha\in\Delta^a} L_\alpha$ meets all the Weyl chambers. Let us assume the contrary. Let $p \in E \setminus \cup_{\alpha\in\Delta^a} L_\alpha$ such that its orbit under action of the abstract Weyl group dose not meet all the Weyl chambers, then clearly there exist two Weyl chambers C_1, C_2 which are separated by a hyperplane L_α for $\alpha \in \Delta^a$ such that the orbit of p meets C_1 but it does not meet C_2. Suppose $\lambda(p) \in C_1$ for some $\lambda \in W_{\Delta^a}(E)$, then $r_\alpha \circ \lambda(p)$ is an element of C_2 which is a contradiction. For observing that the induced action of the abstract Weyl group on the set of all Weyl chambers is free, one just needs to note that according to 2.4.4 the action of the abstract Weyl group on E is free and the orbit of each point meets any Weyl chamber only once (see [OV]). ■

Corollary 2.4.6 *All the Weyl chambers of an abstract root system are isometric.*

Definition 2.4.7 *The topological closure $\overline{C}$ of an open Weyl chamber C is called a* **closed Weyl chamber**. *Let us define r-dimensional* **open faces** *of $\overline{C}$ recursively with respect to r. First we define the n-dimensional open face of $\overline{C}$ to be the open Weyl chamber C. Then assume we have defined open faces of the closed Weyl chamber of dimension greater than r then we define a r-dimensional face to be the interior of the intersection of $\overline{C} \setminus \bigcup_{r<i} C_i$ with an r-dimensional linear subspace of E when it is nonempty, where the interior means interior in the r-dimensional linear subspace and C_i is the union of all open faces of dimension i.*

2.5 The root system

Definition 2.5.1 *A* **weight** *is a real irreducible representation of a torus T (i.e. T is a Lie group such that $T = (S^1)^n$, for some $n \in \mathbb{N}$). Now let G be a Lie group, T be a maximal torus in G and $\rho : G \to \mathrm{Aut}(V)$ be a representation of G. Then for a given weight σ, the sum of all invariant subspaces of V under $\rho|_T$, such that restriction of $\rho|_T$ to them is isomorphic to σ, is called the* **weight space** *of σ associated to ρ.*

Any irreducible representation of $T \simeq \mathbb{R}^k/\mathbb{Z}^k$ is either the one-dimension trivial representation or it is isomorphic to $\theta_{\mathbf{n}}$ for some $\mathbf{n} \in \mathbb{N}^k \setminus (0, \dots, 0)$, where $\theta_{\mathbf{n}}$ is defined as follows: First let us define

$$(\theta_{\mathbf{n}})_* : \mathfrak{t} \to \mathbb{R}, \quad (x_1, \dots, x_k) \mapsto n_1 x_1 + \ldots + n_k x_k$$

in $\mathfrak{t}^*$, where $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and $\mathfrak{t}$ is identified with $\mathbb{R}^k$. Then we define

$$\theta_{\mathbf{n}} : T \simeq (\mathbb{R}/\mathbb{Z})^k \to \mathbb{R}^2$$

$$([x_1], \ldots, [x_k]) \in (\mathbb{R}/\mathbb{Z})^k \mapsto \begin{pmatrix} \cos(2\pi(\theta_{\mathbf{n}})_*(x_1, \ldots, x_k)) & -\sin(2\pi(\theta_{\mathbf{n}})_*(x_1, \ldots, x_k)) \\ \sin(2\pi(\theta_{\mathbf{n}})_*(x_1, \ldots, x_k)) & \cos(2\pi(\theta_{\mathbf{n}})_*(x_1, \ldots, x_k)) \end{pmatrix}.$$

See [WI].

Definition 2.5.2 *Let G be a Lie group and T a maximal torus is G. Then $\alpha \in \mathfrak{t}^*$ is a* **root** *of G, if there exists $\mathbf{n} \in \mathbb{N}^k \setminus (0, \ldots, 0)$ such that $\alpha = (\theta_{\mathbf{n}})_*$ and $\theta_{\mathbf{n}}$ has non-empty weight space associated to Ad_G, where Ad_G is the adjoint representation of G. We denote the set of all roots of G by Δ.*

Remark 2.5.3 *Let G be a connected compact Lie group and T be a maximal torus in G. Then there exists a nondegenerate invariant symmetric bilinear from $(,)$ in $\mathfrak{g}$, such that its restriction to $\mathfrak{t}$ is positive definite. So $\mathfrak{t}$ is an Euclidean space with restriction of $(,)$ to $\mathfrak{t}$, and note that there exists a natural isomorphism between an Euclidean space and its dual, which is given by sending $t \in \mathfrak{t}$ to $(t, -) \in \mathfrak{t}^*$. Thus we can consider Δ as a subset of $\mathfrak{t}$, and it turns out that Δ is a reduced abstract root system and the abstract Weyl group of this reduced abstract root system coinsides with the Weyl group. We call the Weyl chambers of this root system the Weyl chamber of the adjoint representation with respect to $\mathfrak{t}$.*

see [OV] page 70-71

2.6 Complexification of the Lie algebra

Definition 2.6.1 *Let K be a Lie group with Lie algebra $\mathfrak{k}$. We define the complexification of $\mathfrak{k}$ to be*

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C}.$$

It inherits a Lie algebra structure from $\mathfrak{k}$ by defining the bracket over $\mathfrak{k}^{\mathbb{C}}$ to be

$$[v, w]_{\mathfrak{k}^{\mathbb{C}}} = \sum_{i \in I, j \in J} c_i . c'_j . [v_i, w_j],$$

where $v = \sum_{i \in I} v_i \otimes c_i$ and $w = \sum_{j \in J} w_j \otimes c'_j$ two arbitrary elements of $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C}$ and I, J are two finite sets.

Definition 2.6.2 *A non-zero linear map $\alpha^c : \mathfrak{t}^{\mathbb{C}} \to \mathbb{C}$ is called a complex root of the representation* ad*, if*

$$\mathfrak{k}^{\mathbb{C}}_{\alpha^c} = \{v \in \mathfrak{k}^{\mathbb{C}} : [t, v]_{\mathfrak{k}^{\mathbb{C}}} = \alpha^c(t)v, \text{for all } t \in \mathfrak{t}^{\mathbb{C}}\}$$

is not zero, where $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C} \subset \mathfrak{k} \otimes \mathbb{C}$.

Theorem 2.6.3 *Let K be a Lie group with Lie algebra $\mathfrak{k}$ and R be set of its complex roots then:*

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}}_0 \oplus \left(\bigoplus_{\alpha^c \in R} \mathfrak{k}^{\mathbb{C}}_{\alpha^c} \right)$$

where $\mathfrak{k}_0 = \{v \in \mathfrak{k}^{\mathbb{C}} : [t, v]_{\mathfrak{k}^{\mathbb{C}}} = 0, t \in \mathfrak{t}\}$.

For a proof See [WI].

2.7 Analytic structure

Let X be a topological space. A **functional structure** on X is a map $\mathcal{F}_X$ assigning to each open set U of X a subalgebra $\mathcal{F}_X(U)$ of the algebra of real-valued continuous functions on U such that the following conditions are satisfied.

1. for every U non-empty, $\mathcal{F}_x(U)$ contains all the constant functions on U; $\mathcal{F}_X(\varnothing)$ is the algebra consisting of only 0.

2. for every open subsets U and V such that $U \subset V$, and for every $f \in \mathcal{F}_X(V)$, the restriction $f\mid_U$ belongs to $\mathcal{F}_X(U)$; we agree that $f\mid_\varnothing$ is the only element 0 of $\mathcal{F}_X(\varnothing)$.

3. for every open cover $\{V_i\}_{i\in I}$ of U, and f a real-valued function on U such that $f\mid_{V_i}\in \mathcal{F}_X(V_i)$ for every $i \in I$, then $f \in \mathcal{F}_X(V)$.

A structured space $(X, \mathcal{F}_X)$ is a space X that is equipped with a functional structure $\mathcal{F}_X$. A morphism of structured spaces $(X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is a continuous map $\gamma : X \to Y$ such that, for every open subset V of Y and every $f \in \mathcal{F}_Y(V)$, the composite function $f \circ \gamma$ belongs to $\mathcal{F}_X(\gamma^{-1}(V))$. A morphism of structured spaces is an isomorphism of structured spaces when it is bijective and γ^{-1} is also a morphism of structured spaces.

Definition 2.7.1 *An* **analytic manifold** *M is a manifold, equipped with a functional structure $(F)_M$ such that it admits a atlas $(U_i, h_i, V_i)_{i\in I}$, such that $h_i : (U_i, \mathcal{F}_{U_i}) \to (V_i, \mathcal{A}_{V_i})$ is an isomorphism of structured spaces for every $i \in I$, where $\mathcal{A}_{V_i}$ is the functional structure of analytic funtions. If M and N are analytic manifolds, then a morphism of structured spaces $(M, \mathcal{F}_M) \to (N, \mathcal{F}_N)$ is simply be called an analytic map of X into Y. If such a morphism is an isomorphism then it will simply called an analytic isomorphism.*

Definition 2.7.2 *An analytic group is a group G that is equipped with the structure of an analytic manifold such that the map*

$$G \times G \to G, \quad (x, y) \mapsto xy^{-1}$$

is an analytic map.

Proposition 2.7.3 *For every connected Lie group G, there exists a unique analytic structure which makes G an analytic group.*

For existence see [K] page 43 and for uniqueness see [H] page 88.

Theorem 2.7.4 *Let G be an analytic group, and let H be a closed subgroup of G. We define $\mathcal{F}_{G/H}$ on G/H as follows. Let U be any open subset of G/H. f is in $\mathcal{F}_{G/H}(U)$ iff $f \circ \pi$ is in $\mathcal{F}_G(\pi^{-1}(U))$ where π is the natural map $G \to G/H$. Then the functional structure $\mathcal{F}_{G/H}$ makes G/H into an analytic manifold, and the canonical map*

$$q : G \times G/H \to G/H, \quad (x, yH) \mapsto xyH,$$

is analytic.

For a proof see [H] page 93.

Corollary 2.7.5 *Let G be an analytic group, and let H be a closed subgroup of G. Then for every analytic map f from G into an analytic manifold M which is constant on each coset of H in G, the corresponding map f^π of G/H into M, where $f^\pi \circ \pi = f$, is analytic.*

For a proof see [H] pages 93-94.

2.8 The unitary Lie group

Definition 2.8.1 *A matrix $U \in M(n,\mathbb{C})$ is called to be a* **unitary matrix** *if*

$$UU^* = U^*U = I_n,$$

where the U^* is the conjugate transpose of U. We denote the set of all unitary matrices by $U(n,\mathbb{C})$, which is a Lie subgroup of $GL(n,\mathbb{C})$.

Proposition 2.8.2 *The Lie algebra of the unitary group $\mathfrak{u}(n,\mathbb{C})$ is Lie algebra isomorphic with the space of all skew-Hermitian matrices with standard Lie bracket.*

Proof. $U(n,\mathbb{C})$ is zero set of the map

$$f : M(n,\mathbb{C}) \to M(n,\mathbb{C}), \quad U \mapsto UU^* - I_n.$$

For every $U, A \in M(n,\mathbb{C})$, we have

$$\begin{aligned} Df(U)A &= \frac{d}{dt} f(U+tA) = \frac{d}{dt}\left((U+tA)(U^*+tA^*)\right) \\ &= \frac{d}{dt}\left(UU^* + tUA^* + tAU^* + t^2AA^*\right) \\ &= (UA^* + AU^* + tAA^*) \mid_{t=0} = UA^* + AU^*. \end{aligned}$$

So

$$T(U(n,\mathbb{C})) \mid_{I_n} = \{A \in M(n,\mathbb{C}) : Df(I_n)A = A^* + A = 0\}.$$

Thus

$$\mathfrak{u}(n,\mathbb{C}) \simeq \{A \in M(n,\mathbb{C}) : A = -A^*\}$$

■

A maximal torus for $U(n,\mathbb{C})$ is

$$T = \left\{ \begin{pmatrix} t_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t_n \end{pmatrix} :\mid t_1 \mid = \dots = \mid t_n \mid = 1 \right\}.$$

See [B] page 91. So the associated Cartan subalgebra of T is

$$\mathfrak{t} = \left\{ \begin{pmatrix} ir_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & ir_n \end{pmatrix} \in \mathrm{M}(n,\mathbb{C}) : r_i \in \mathbb{R} \right\}.$$

Now let

$$A = \begin{pmatrix} e^{it_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{it_n} \end{pmatrix} \in T$$

and $B = (a_{lk}) \in \mathrm{U}(n,\mathbb{C})$, then

$$\mathrm{Ad}(A)B = ABA^{-1} = ((e^{it_l}/e^{it_k})a_{lk}) = (e^{i(t_l - t_k)}a_{lk}).$$

Then for every $i, j \in \{1, \ldots, n\}$ W_{ij} is an invariant subspace of $U(n, \mathbb{C})$, where $W_{ij} =< E_{ij}, E_{ji} > \cap U(n, \mathbb{C})$, here E_{ij} is the matrix with 1 at ith row and jth column and 0 elsewhere. Since for $i < j$, restriction of Ad_T to W_{ij} is

$$e^{2\pi i(t_l - t_k)} = \begin{pmatrix} \cos(2\pi(t_l - t_k) & -\sin(2\pi(t_l - t_k)) \\ \sin(2\pi(t_l - t_k)) & \cos(2\pi(t_l - t_k)) \end{pmatrix}$$

the weight space corresponding to $e^{2\pi i(t_l - t_k)}$ is non-empty. So according to the definition of root

$$\alpha_{ij} : \mathfrak{t} \to \mathbb{R}, \begin{pmatrix} it_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & it_n \end{pmatrix} \to t_i - t_j$$

is a root. Note that in the case $i = j$, W_{ij} is one-dimensional and restriction of Ad_T is the trivial representation. On the other hand, since

$$U(n, \mathbb{C}) = \bigoplus_{i \leq j} W_{ij}$$

and the decomposition of a representation into irreducible representations is unique, all the roots are of the above form. So the open Weyl chambers of T are $\{(it_1, \ldots, it_n) : t_{\sigma(1)} < \ldots < t_{\sigma(n)}\}$, for every σ a permutation of $1, \ldots, n$. In particular $\{(it_1, \ldots, it_n) : t_1 < \ldots < t_n\}$, is an open Weyl chamber of T.

2.9 Local maximal torus reductions of a K-bundle

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$ and $T \subset K$ be a fixed maximal torus. We denote by $\mathfrak{k}/K$ the quotient of $\mathfrak{k}$ by K, where K acts on $\mathfrak{k}$ by the adjoint representation.

Proposition 2.9.1 *Let K be a connected compact Lie group and $C \subset \mathfrak{t}$ be a closed Weyl chamber, where $\mathfrak{t}$ is the associated Cartan subalgebra of T. Then the natural map*

$$C \longrightarrow \frac{\mathfrak{k}}{K}$$

is a homeomorphism.

Proof. First note that for any $\varphi \in \mathfrak{k}$ there exists an element ψ in $\mathfrak{t}$ which is conjugate with φ, i.e. there exists $k \in K$ such that $Ad_k(\varphi) = \psi$ (see [OV] page 78). So the natural map

$$\mathfrak{t} \to \frac{\mathfrak{k}}{K}$$

is surjective. Now let $\varphi_1, \varphi_2 \in \mathfrak{t}$ be conjugate i.e. $\mathrm{Ad}_k(\varphi_1) = \varphi_2$ for some $k \in K$, then $\mathrm{Ad}_k(T)$ and T are both maximal tori of K and both are contained in $Z_K(\varphi_2)$ the connected centralizer of φ_2. Therefore they are conjugate in $Z_K(\varphi_2)$, which means there exists $u \in Z_K(\varphi_2)$ such that $\mathrm{Ad}_u(\mathrm{Ad}_k(T)) = T$. So $\kappa := uk$ is in the normalizer $N_K(T)$, therefore φ_1 and φ_2 are conjugate modulo $N_K(T)$. This implies that the natural map

$$\frac{\mathfrak{t}}{N_K(T)} \to \frac{\mathfrak{k}}{K}$$

is injective, but according to the above argument it is also surjective. Since any bijective quotient map is a homeomorphism, the natural map $\mathfrak{t}/N_K(T) \to \mathfrak{k}/K$ is in particular a homeomorphism. So

$$\frac{\mathfrak{k}}{K} = \frac{\mathfrak{t}}{N_K(T)}.$$

Now just note that $T \subset N_K(T)$ and action of T on $\mathfrak{t}$ is trivial, therefore

$$\frac{\mathfrak{t}}{N_K(T)} = \frac{\mathfrak{t}}{N_K(T)/T} = \frac{\mathfrak{t}}{W_K(T)}.$$

So

$$\frac{\mathfrak{k}}{K} = \frac{\mathfrak{t}}{N_K(T)} = \frac{\mathfrak{t}}{W_K(T)}.$$

Now we show that $\mathfrak{t}/W_K(T)$ is isomorphic with a closed Weyl chambers C. For showing that we need to show that two distinct element of a closed Weyl chamber C cannot be conjugate modulo $W_K(T)$. Suppose $x, y \in C$ and $0 \neq w \in W_K(T)$ such that $w(x) = y$. Since the action of $W_K(T)$ on Weyl chambers is free and transitive, $w(C)$ and C are two distinct Weyl chambers and $y = w(x) \in C \cap w(C)$. Let c be the common face of C and $w(C)$ of minimal dimension which contains y. Since C and $w(C)$ have a common face c, one can see there exists a sequence of Weyl chambers $(C_i)_{1\leq i\leq n}$ such that $C_1 = C$, $C_n = w(C)$ and C_i,C_{i+1} have a common face of codimension 1, which contains c. We denote by w_i the reflection with respect to the common face of C_i and C_{i+1} with codimension 1. Consider $w_{n-1} \circ w_{n-2} \circ \ldots \circ w_1$, it maps C to $w(C)$, which implies $w_{n-1} \circ w_{n-2} \circ \ldots \circ w_1 = w$ because action of $W_K(T)$ on Weyl chambers is free and transitive. on the other hand $w_{n-1} \circ w_{n-2} \circ \ldots \circ w_1$ preserves y because for every $i \in 1, \ldots, n-1$, w_i is the reflection with respect to the common face of C_i and C_{i+1} which contains c and therefore it preserves y. Thus $\mathfrak{t}/W_K(T) = C$. ■

Let K be a connected compact Lie group and P a principal K-bundle over an arbitrary manifold B. We fix a closed Weyl chamber $C \subset \mathfrak{t}$. C decomposes as a disjoint union of convex sets, the open faces of C. Each open face is the interior of a face of C.
Let $\varphi \in \Gamma(ad(P))$ ($ad(P) := P \times_K \mathfrak{k}$, where K acts on $\mathfrak{k}$ from the left by $(k, p) \mapsto \mathrm{Ad}(k)p$), by Proposition 2.2.9 we can regard φ as a K-equivariant map $f : P \to \mathfrak{k}$, define

$$[\varphi] : B \to \mathfrak{k}/K, \quad p \mapsto f(u)/\sim,$$

where $u \in P_p$ and $f(u)/\sim$ is used to denote the equivalent class of $f(u)$ in $\mathfrak{k}/K$. Note that it is well-defined i.e. it is independent of the choice of $u \in P_p$, because if v is another element in P_p, there exists $g_{u,v}$ such that $ug_{u,v} = v$ as was mentioned in the proof of Proposition 2.2.9. Therefore $\mathrm{Ad}(g_{u,v}^{-1})f(u) = f(v)$, which means $f(v)$ and $f(u)$ are in the same equivalent class.

Lemma 2.9.2 *Let c be a open face of a close Weyl chamber C of T, then $Z_K(\gamma) = \{k \in K : \mathrm{Ad}(k)\gamma = \gamma\}$ is the same for any $\gamma \in c$, therefore $Z_K(\gamma) = Z_K(c)$ for every $\gamma \in c$.*

Proof. Since K is connected, any centralizer of the form $Z_K(u)$ with $u \in \mathfrak{k}$ is connected, so it suffices to look at the Lie algebras of these centralizers $z_K(u) = \{k \in \mathfrak{k} : [u, k] = 0\}$. According to Theorem 2.6.3 we have

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k}_0^{\mathbb{C}} \oplus \left(\bigoplus_{\alpha_c \in R} \mathfrak{k}_{\alpha^c}^{\mathbb{C}} \right)$$

therefore for every $k \in \mathfrak{k}$ we can write $k = k_0 + \sum_{\alpha_c} k_{\alpha^c}$, where $k_0 \in \mathfrak{k}_0$ and $k_{\alpha^c} \in \mathfrak{k}_{\alpha^c}$ for every $\alpha^c \in R$. So

$$[u,k]_{\mathfrak{k}^{\mathbb{C}}} = [u,k_0]_{\mathfrak{k}^{\mathbb{C}}} + \sum_{\alpha^c}[u,k_{\alpha^c}]_{\mathfrak{k}^{\mathbb{C}}} = \sum_{\alpha^c \in R} \alpha^c(u) k_{\alpha^c}.$$

Since the none-zero $\{k_{\alpha^c}\}_{\alpha \in R}$ are linearly independent, $\sum_{\alpha^c \in R} \alpha^c(u) k_{\alpha^c}$ is zero if and only if $\alpha^c(u)$ is zero for every $\alpha^c \in R$ with $k_{\alpha^c} \neq 0$. Therefore

$$z_k(u) = \{k \in \mathfrak{k}^{\mathbb{C}} : [k,u]_{\mathfrak{k}^{\mathbb{C}}} = 0\} \cap \mathfrak{k} = \left(\left[\bigoplus_{\alpha^c \in R, \alpha^c(u)=0} \mathfrak{k}^{\mathbb{C}}_{\alpha^c} \right] \oplus \mathfrak{k}^{\mathbb{C}}_0 \right) \cap \mathfrak{k}.$$

But the map

$$\gamma \mapsto \{\alpha^c \in R : \alpha^c(\gamma) = 0\} \subset R$$

is constant on the open face c, because

$$c \subset \bigcap_{\alpha^c \in \Lambda} \ker(\alpha^c),$$

where $\Lambda := \{\alpha^c \in \Delta^a : \exists \gamma \in c \text{ such that } \alpha^c(\gamma) = 0\}$. ∎

Lemma 2.9.3 *The map*

$$\omega : K/Z_K(c) \times c \to \mathfrak{k}, \quad ([k],\gamma) \mapsto \mathrm{Ad}(k)\gamma.$$

is an embedding. We denote the image of ω *by* $ad_K(c)$.

Proof. First of all note that it is a well-defined map, because for every $p \in c$ and $k, k' \in K$, such that $k^{-1}k' = Z_K(c)$ we have $p = \mathrm{Ad}(k^{-1}k')(p) = \mathrm{Ad}(k)^{-1}.\mathrm{Ad}(k')(p)$, therefore $\mathrm{Ad}(k')(p) = \mathrm{Ad}(k)(p)$.
Now we define

$$K \to \mathrm{Hom}(< c >, \mathfrak{k}), \quad k \mapsto \mathrm{Ad}(k) \circ i,$$

where $< c >$ is the linear span space of c and i is the inclusion map $< c > \to \mathfrak{k}$. This map is analytic because it is the composition of $\mathrm{Ad} : K \to \mathrm{Aut}\mathfrak{k}$ and i, which are both analytic. According to the proof of Lemma 2.9.2, $Z_K(\gamma)$ is the same not only for every $\gamma \in c$ but for every $\gamma \in < c >$, since the map

$$\gamma \mapsto \{\alpha^c \in R : \alpha^c(\gamma) = 0\} \subset R$$

is constant on whole $< c >$. So according to Corollary 2.7.5

$$\Upsilon : K/Z_K(c) \to \mathrm{Hom}(< c >, \mathfrak{k})$$

is analytic and in particular differentiable, where we consider $\mathrm{Hom}(< c >, \mathfrak{k})$ with its natural analytic structure as a finite dimensional vector space.
Now one can see easily that $\omega = \Pi \circ (\Upsilon \times I)$, where Π is

$$\Pi : \mathrm{Hom}(< c >, \mathfrak{k}) \times < c > \to \mathfrak{k}, \quad (A,k) \mapsto A(k),$$

and $I : c \rightarrow< c >$ is the inclusion map. So ω is differentiable, since it is a composition of differentiable maps.
For every $k \in K$, the map

$$l_k : K/Z(c) \to K/Z(c), \quad [a] \mapsto [ka]$$

is a diffeomorphism due to Corollary 2.7.5, so to show that ω is an immersion it suffices to check injectivity of the differential just for points of type $([e], p)$, where $p \in c$. Because for an arbitrary point $([k_0], p) \in K/Z(c) \times c$ we have $\omega = \mathrm{Ad}(k_0) \circ \omega \circ (l_{k_0^{-1}} \times \mathrm{id}_c)$ by applying the chain rule we get $D\omega([k_0], p) = D\mathrm{Ad}(k_0) \circ D\omega([e], p) \circ D(l_{k_0^{-1}} \times \mathrm{id}_c)([k_0], p)$, but $D\mathrm{Ad}(k_0) = \mathrm{Ad}(k_0) \in \mathrm{Aut}(\mathfrak{k})$ and $l_{k_0^{-1}} \times \mathrm{id}_c$ is a diffeomorphism, therefore $D\omega(k_0, p)$ is injective if and only if $D\omega(0, p)$ is injective. Let $(\phi(t), \psi(t))$ be a curve in $K/Z_K(c) \times c$ such that $(\phi(0), \psi(0)) = ([e], p)$ and let $\phi'(0) \in \mathfrak{k}/z_K(c)$ and $\psi'(0) \in< c >$ be their derivation at zero. We want to show that when $\phi'(0)$ and $\psi'(0)$ are not simultaneously zero, then $\omega'(\phi(0), \psi(0))(\phi', \psi')$ is not zero, which is by the chain rule equivalent to

$$((\mathrm{Ad})(\phi(0)) \circ i)\,(\psi'(0)) + \big(\mathrm{ad}(\phi'(0)) \circ i\big)\,(\psi(0)) \neq 0.$$

Since $\phi(0) = [e]$ and

$$\big(\mathrm{ad}(\phi'(0)) \circ i\big)\,(\psi(0)) = [\phi'(0), \psi(0)],$$

we have $((\mathrm{Ad})(\phi(0)) \circ i)\,(\psi'(0)) + (\mathrm{ad}(\phi'(0)) \circ i)\,(\psi(0)) = \psi'(0) + [\phi'(0), \psi(0)]$. Now as in the proof of Lemma 2.9.2 we have

$$\begin{aligned} [\phi'(0), \psi(0)] &= -[\psi(0), \phi'(0)]_{\mathfrak{k}^{\mathbb{C}}} \\ &= -[\psi(0), \phi'(0)_0]_{\mathfrak{k}^{\mathbb{C}}} - \sum_{\alpha^c} [\psi(0), \phi'(0)_{\alpha^c}]_{\mathfrak{k}^{\mathbb{C}}} \\ &= -\sum_{\alpha^c \in R} \alpha^c(\psi(0))\phi'(0)_{\alpha_c}. \end{aligned}$$

So $[\phi'(0), \psi(0)]$ does not lay in $< c >\subset \mathfrak{k}$, except when it is zero. One can see easily that for $c = 0$ the lemma holds, so we can always assume $\psi(t) \neq 0$. Therefore $\psi'(0) + [\phi'(0), \psi(0)] \neq 0$ when ψ' and ϕ' are not simultaneously zero.
It remains to show that ω is a homeomorphism onto its image $\mathrm{ad}_K(c)$. First note that ω is injective. Let $\{a_j\}_{j\in\mathbb{N}}$ be a converging sequence in $\mathrm{ad}_K(c)$, and let $\{(b_j, d_j)\}_{j\in\mathbb{N}}$ be the inverse image of $\{a_j\}_{j\in\mathbb{N}}$; it suffices to show that $\{(b_j, d_j)\}_{j\in\mathbb{N}}$ converges in $K/Z_K(c) \times c$. For this we just need to prove that $\{d_j\}_{j\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$, are converging in c and $K/Z(c)$ respectively. Let h be the natural map $\mathfrak{k} \to \mathfrak{k}/K \simeq C$. Then $d_j = h(a_j)$ for every $j \in \mathbb{N}$, since the orbit of each point meets a closed Weyl chamber at exactly one point (see [OV] page 78). So $\{d_j\}_{j\in\mathbb{N}}$ converges in C, because h is continuous and $\{a_i\}_{j\in\mathbb{N}}$ converges in $\mathrm{ad}_K(c)$. Let W be a compact nighborhood of d, where d is the limit of $\{d_j\}_{j\in\mathbb{N}}$. Then there exists $N_0 \in \mathbb{N}$ such that $d_j \in W$ for $N_0 < j$, so (b_j, a_j) lies in $K/Z(c) \times W$. But

$$\omega\,|_{K/Z(c)\times W} : K/Z(c) \times W \to \mathfrak{k}$$

is a homeomorphism to its image, because $K/Z(c) \times W$ is compact and $\mathfrak{k}$ is Hausdorff. Therefore $\{(b_j, d_j)\}_{N_0<j}$ converges and $\omega^{-1} : ad_K(c) \to K/Z(c)$ is continuous. ■

Proposition 2.9.4 *Let $\varphi \in \Gamma(ad(P))$. Suppose that the map*

$$B \xrightarrow{[\varphi]} \mathfrak{k}/K \xrightarrow{\simeq} C$$

takes values in a fixed open face c of C. Then the space $\varphi^{-1}(c)/T \subset P/T$ is a locally trivial fiber bundle with standard fiber $Z_K(c)/T$ over B.

Proof. According to Lemma 2.9.3, ω is an embedding, therefore it is a diffeomorphism between $K/Z(c)\times c$ and $\mathrm{ad}_K(c)$. Let $\tau : \mathrm{ad}_K(c) \to K/Z(c)\times c$ be its inverse. Since $[\varphi](B) \subset c$, the image of φ lies in $ad_k(c)$. So we can define the map $\xi := \pi_1 \circ \tau \circ \varphi : P \to K/Z(c)$, where $\pi_1 : K/Z(c) \times c \to K/Z(c)$ is the projection map on the first component. ξ is a K-equivariant map because for every $u \in P$ and $g \in K$ we have

$$\varphi(ug) = \mathrm{ad}(g^{-1})\varphi(u) = \omega\left(g^{-1}\pi_1\left(\tau\left(\varphi(u)\right)\right), \pi_2\left(\tau\left(\varphi(u)\right)\right)\right),$$

so

$$\tau(\varphi(ug)) = (g^{-1}\pi_1 \circ \tau(\varphi(u)), \pi_2 \circ \tau(\varphi(u))),$$

which means $\xi(ug) = g^{-1}\xi(u)$. So according to the Proposition 2.2.11 and its ensuing remark $\{u \in P : \xi(u) = [e]\}$ is an $Z(c)$-reduction of P, but

$$\begin{aligned}\{u \in P : \xi(u) = [e]\} &= \{u \in P : \tau \circ \varphi(u) = (k,p), k \in Z(c) \text{ and } p \in c\} \\ &= \{u \in P : \varphi(u) \in c\} \\ &= \varphi^{-1}(c).\end{aligned}$$

Therefore $\varphi^{-1}(c)$ is a principal fiber bundle with typical fiber $Z(c)$, thus $\varphi^{-1}(c)/T$ is a locally trivial fiber bundle. ■

Theorem 2.9.5 *Let T be a fixed maximal torus in K and C be a fixed closed Weyl chamber in $\mathfrak{t}$. Then for any $f \in \Gamma(ad(P))$ there exists an open dense subset $W \subset B$ such that for any $x \in W$ there exists c_x, an open face of the closed Weyl chamber C, U_x an open neighborhood of x in W and a T-reduction $\Pi_x \subset P\,|_{U_x}$ of the restriction $P\,|_{U_x}$, such that the restriction of $f\,|_{U_x}$ to Π_x can be given by a smooth map*

$$\lambda \in C^\infty(U_x, c_x) \subset C^\infty(U_x, \mathfrak{t}) = \Gamma(ad(\Pi_x)).$$

Note that "=" in $C^\infty(U_x, \mathfrak{t}) = \Gamma(ad(\Pi_x))$ makes sense, because according to Proposition 2.2.9 there is a natural correspondence between sections of $\Gamma(ad(\Pi_x))$ and T-equivariant maps $\Pi_x \to \mathfrak{t}$, but since T acts on $\mathfrak{t}$ trivially, any T-equivariant map $\Pi_x \to \mathfrak{k}$ is fixed on any fiber of $ad(\Pi_x)$, so it can be given by a smooth map from U_x the basis manifold of Π_x to $\mathfrak{k}$.

Proof. Regard f as a K-equivariant map $f : P \to \mathfrak{k}$ and consider $[f] : B \to \mathfrak{k}/K \simeq C$ as it is defined above. Consider the filtration

$$C = C_{d+1} \supset C_d \supset \ldots \supset C_0$$

of C, where C_i is the union of all open faces of the Weyl chamber with dimension less than i, for all $i \in \{1, \ldots, d+1\}$ and $C_0 = \varnothing$. We get a filtration by closed subsets

$$B_{d+1} \supset B_d \supset \ldots \supset B_0$$

of B, where $B_i := [f]^{-1}(C_i)$. As in Lemma 1.5.1 we define

$$F_i := B_i \setminus B_{i-1} = [f]^{-1}(C_i \setminus C_{i-1}).$$

According to Lemma 1.5.1

$$W := \bigcup_{k=1}^{r} F_k^\circ$$

is an open dense subset of B. For every $x \in W$ there exists $i_x \in \{1, \dots, d, d+1\}$ such that $x \in F_{i_x}^\circ$, so there exists an open contractible neighborhood U_x of x in $F_{i_x}^\circ$. Since U_x is connected, $[f]^{-1}(U_x)$ lies entirely in c_x, a single open face of C, therefore by Lemma 2.9.4 $(f\mid_{\pi^{-1}(U_x)})^{-1}(c_x)/T$ is a trivial bundle; triviality is yielded by the fact that U_x is contractible. Since $(f\mid_{\pi^{-1}(U_x)})^{-1}(c_x)/T$ is trivial we can find a section σ_x. Consider σ_x as section in $(P\mid_{U_x})/T$, we define a T-reduction of $P\mid_{U_x}$ by

$$(g, g') : (U_x \times T)(U_x, T) \to P\mid_{U_x}(U_x, K), \quad g : (u, t) \mapsto \sigma_x(u)t \text{ and } g' = Id.$$

and denote it by Π_x. Note that the restriction of f to Π_x is a smooth c_x-valued map, since $f \circ g(u, t) = f(\sigma_x(u)t) = f(\sigma_x(u)) \in c_x$. ■

Proposition 2.9.6 *Theorem 2.9.5 implies Proposition 1.5.9, therefore it can be seen as a generalization of proposition 1.5.9.*

Proof. Let $\pi_E : E \to B$ be a complex vector bundle of rank d, h a Hermitian metric in E, f a positive definite Hermitian endomorphism of E and $LO(E)(B, U(d, \mathbb{C}))$ the principal orthogonal frame bundle of E as it is defined in Example 2.2.4. Then according to Theorem 2.9.5 for a fixed maximal torus T in $U(d, \mathbb{C})$ and a fixed closed Weyl chamber C in $\mathfrak{t}$, if g is a section in $(B, ad(LO(E))) = LO(E) \times_{ad} \mathfrak{u}(d, \mathbb{C})$, there exists an open dense $W \subset B$ such that for every $x \in W$ there exists an open face $c_x \subset C$, an open neighborhood U_x of x in B and a T-reduction $\Pi_x \subset P\mid_{U_x}$ of the restriction $P\mid_{U_x}$ such that $g\mid_{U_x}$ is defined by a smooth map

$$\lambda \in C^\infty(U_x, c_x) \subset C^\infty(U_x, \mathfrak{t}) = \Gamma(ad(\Pi_x)).$$

So in particular it holds for the maximal torus

$$T = \{\mathrm{diag}(e^{2\pi i\lambda_1}, \dots, e^{2\pi i\lambda_d}) : \lambda_i \in \mathbb{R}\},$$

where

$$diag(e^{2\pi i\lambda_1}, \dots, e^{2\pi i\lambda_d}) = \begin{pmatrix} e^{2\pi\lambda_1} & 0 & \dots & 0 \\ 0 & e^{2\pi i\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{2\pi i\lambda_d} \end{pmatrix}$$

and the closed Weyl chamber

$$\{diag(i\lambda_1, \dots, i\lambda_d) : \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d\}.$$

Now we show that there is a natural correspondence between skew-Hermitian endomorphism of E and $U(d, \mathbb{C})$-equivariant smooth maps $LO(E) \to \mathfrak{u}(d, \mathbb{C})$. Let g be a skew-Hermitian endomorphism of E, we define

$$\theta_g : LO(E) \to \mathfrak{u}(d, \mathbb{C}), \quad u \mapsto M_u^g$$

where M_u^g denotes the matrix of $g \mid_{E_{\pi_u}}$ with respect to the orthogonal basis u (here matrices effect from left instead of right). Since g is a skew-Hermitian endomorphism, M_u^g is in $\mathfrak{u}(d, \mathbb{C})$. Let $u' = uA$ for an arbitrary $A \in U(n, \mathbb{C})$, then $M_{u'}^g = A^{-1} M_u^g A$; therefore θ_g is $U(d, \mathbb{C})$-equivariant and smoothness follows from smoothness of g.
Now let $\theta : LO(E) \to \mathfrak{u}(d, \mathbb{C})$ be a $U(d, \mathbb{C})$-equivariant smooth map, then we define

$$g^\theta : E \to E, \quad v \mapsto v\theta(u),$$

where u is an orthogonal basis for $E_{\pi_E(v)}$ and v is vector of $E_{\pi_E(v)}$ written in the basis u. g^θ is well-defined, because if u' be another orthogonal basis for $E_{\pi_E(v)}$, then $\theta(u') = A_{u,u'}^{-1}\theta(u)A_{u,u'}$, where $A_{u,u'}$ is the basis change matrix of u to u'. Since the matrix of a skew-Hermitian morphism with respect to an orthogonal basis is a skew-Hermitian matrix, g^θ is skew-Hermitian. One can see easily $g^{\theta_g} = g$ and $\theta_{g^\theta} = \theta$. Now consider θ_{if}, then there exists an open dense $W \subset B$ such that for every $x \in W$ there exists an open face $c_x \subset C$, U_x an open neighborhood of x in B and a T-reduction $\Pi_x \subset P \mid_{U_x}$ of the restriction $P \mid_{U_x}$ such that $\theta_{if} \mid_{U_x}$ is defined by a smooth map

$$\lambda \in C^\infty(U_x, c_x) \subset C^\infty(U_x, \mathfrak{t}) = \Gamma(ad(\Pi_x)).$$

Let $\sigma : U' \to \Pi_x$ be local section of Π_x on an open neighborhood of x in U_x, then $\theta_{if} \circ \sigma = \lambda \circ \sigma : U' \to C$. Therefore there exist smooth functions $\mu_i : U' \to \mathbb{R}$, for $i \in 1, \dots, d$ such that $\theta_{if} \circ \sigma(p) = M_{\sigma(p)}^f = diag(i\mu_1, \dots, i\mu_d)$. So $\theta^f \circ \sigma(p) = M_{\sigma(p)}^f = diag(\mu_1, \dots, \mu_d)$, for every $p \in U'$, which means

$$f(p) = \sum_{i=1}^{d} \mu_i . e_i(p) \otimes e^i(p),$$

for every $p \in U'$, where $\sigma(p) =< e_1(p), \dots, e_d(p) >$ and $\{e^i(p)\}_{1 \le i \le d}$ is the dual basis of $\{e_i(p)\}_{1 \le i \le d}$. Since $f \in Herm^+(E)$, for every $i \in 1, \dots, d$ and $p \in U'$, we have $0 < \mu_i$. So we can set $\lambda_i(p) = log(\mu_i(p))$, then we get

$$f(p) = \sum_{i=1}^{d} e^{\lambda_i}(p) . e_i(p) \otimes e^i(p)$$

■

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