**Beyond the stars: Crossed products of Banach algebras**
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# Beyond the Stars: Crossed Products of Banach Algebras

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# 1 Introduction

The theory of crossed products of $C^*$-algebras provides a connection between two fields of mathematics, on the one hand the study of dynamical systems and on the other the theory of $C^*$-algebras. We shall consider *$C^*$-dynamical systems*, which are triples $(A, G, \alpha)$ consisting of a $C^*$-algebra $A$, a locally compact topological group $G$ and a strongly continuous action $\alpha$ of $G$ on $A$. We can study these systems by looking at its *covariant $*$-representations* on Hilbert spaces. In the theory of $C^*$-dynamical systems there is powerful tool to study these covariant representations. A fundamental theorem asserts that given a $C^*$-dynamical system $(A, G, \alpha)$, there exists a $C^*$-algebra $A \rtimes_\alpha G$ such that the (nondegenerate) covariant $*$-representations of the dynamical system correspond bijectively to the (nondegenerate) $*$-representations of $A \rtimes_\alpha G$. The $C^*$-algebra $A \rtimes_\alpha G$ is called the *crossed product* of $A$ by $G$.

The study of crossed products arises naturally from the study of topological dynamical systems. Suppose we are given a locally compact Hausdorff space $X$ and a homeomorphism $T$ of $X$. The iterates of the map $T$ on $X$ (which we can view as an action of $\mathbb{Z}$ on $X$) yield a *discrete (or classical) dynamical system* $(X, T)$. In fact, iteration of the map $T$ also produces a dynamical system with the $C^*$-algebra $C_0(X)$ as its state space, where $C_0(X)$ is the space of continuous, $\mathbb{C}$-valued funtions vanishing at infinity, equipped with the supremum norm and with its involution defined by complex conjugation. Hence we obtain a special case of a $C^*$-dynamical system $(C_0(X), \mathbb{Z}, \alpha)$ by defining the action $\alpha : \mathbb{Z} \to \mathrm{Aut}(C_0(X))$ by

$$\alpha_n(f) = f \circ T^{-n}$$

The crossed product $C_0(X) \rtimes_\alpha \mathbb{Z}$ is well-studied and some interesting results have been obtained. A theorem that deserves mentioning is one by S.C. Powers, stating that, if the space $X$ is infinite, (the action of) a classical dynamical system is minimal if and only if $C_0(X) \rtimes_\alpha \mathbb{Z}$ is simple (see for example Tomiyama (1987) or Davidson (1996)).

This construction can be generalized in two directions. First, we can consider actions of more general (locally compact) topological groups $G$ on $C_0(X)$. For example, we can study the action of $\mathbb{R}$ on $C_0(X)$ through the homeomorphism $T$ (with the action $\alpha$ defined as above), which is called *flow* in the literature. Second, we can take a more general space $A$ for the dynamical system. By a theorem of Gelfand (see Murphy (1990), theorem 2.1.10), every abelian $C^*$-algebra $A$ is isometrically $*$-isomorphic to $C_0(X)$ for a certain locally compact Hausdorff space $X$. It is therefore natural to consider $C^*$-algebras as the space $A$ for the dynamical system. Due to Gelfand's result, the dynamical systems $(A, G, \alpha)$ are sometimes nicknamed *noncommutative dynamical systems*.

We can also see $C^*$-crossed products as a generalization of *group $C^*$-algebras*, which are used to study unitary representations of a locally compact group on Hilbert spaces. Viewed from the theory of crossed products, the group $C^*$-algebra of a locally compact group is a special case of a crossed product obtained by taking $A = \mathbb{C}$ and $\alpha$ equal to the trivial action.

The theory of $C^*$-crossed products is extensive and well-developed, see for example the beautiful monograph Williams (2007) for a survey of the theory. $C^*$-crossed products have been succesfully applied to study group $C^*$-algebras and to construct $C^*$-algebras with certain desired properties, see Davidson (1996) and Fillmore (1996) for examples.

In this thesis we will attempt to generalize the crossed product yet a little bit further, by removing the involutive structure on the space $A$ of the dynamical system. We will consider algebraic dynamical systems $(A, G, \alpha)$ as above, but with $A$ an arbitrary Banach algebra. The question we try to answer is the following: Given a dynamical system $(A, G, \alpha)$, is there a Banach algebra $A \rtimes_\alpha G$ such that (a subset of) the covariant representations of the dynamical system correspond bijectively with (a subset of) the representations of $A \rtimes_\alpha G$? The hope is, of course, that we can construct the same powerful tool that is available in the study of $C^*$-dynamical systems and $C^*$-algebras. As one might expect, the original construction of the crossed product makes use of many basic properties of $C^*$-algebras, so there are some difficulties to overcome.

Basically the construction of the crossed product boils down to completing the algebra $C_c(G, A)$ equipped with the twisted convolution product with respect to a suitable norm. In the first part of this thesis we will consider several candidate norms to use in our definition of a crossed product. In the second part we try to prove the above mentioned bijection for our new crossed product. The final part discusses the many remaining open questions and gives suggestions for further research.

Our main result states the following partial positive answer to the above question: Given a dynamical system $(A, G, \alpha)$ with $A$ an approximate unital Banach algebra, there exists, for every *faithful* collection of contractive covariant representations $S$ of $(A, G, \alpha)$, an approximate unital Banach algebra $(A \rtimes_\alpha G)_S$ such that there is an injection of the nondegenerate contractive covariant representations of $S$ into the set of (nondegenerate) contractive representations of $(A \rtimes_\alpha G)_S$. Also, we show that given a nondegenerate *contractively extendable* representation of $(A \rtimes_\alpha G)_S$ there exists a nondegenerate contractive covariant representation which is mapped by the injection to the given representation. We obtain the mentioned result for $C^*$-dynamical systems as a special case, by taking $S$ equal to the collection of covariant $*$-representations on Hilbert spaces.

# 2 Crossed products of Banach algebras

In this section we will define, for every *faithful collection of contractive covariant representations*, a crossed product for dynamical systems $(A, G, \alpha)$ with $A$ a Banach algebra. Our exposition will largely parallel that of Williams (2007), which has been our primary source for the theory on $C^*$-crossed products. In contrast to the historical development of the theory, we postpone the discussion of the crossed product for $C^*$-algebras to the end of this section. As will become apparent during the course of this section, we can view the $C^*$-crossed product as one of the newly defined $S$-crossed products for Banach algebraic dynamical systems.

## 2.1 Definition of the crossed product

We do not expect that the crossed product will give a useful tool for studying the dynamical system of *any* Banach algebra. Several of the following results are aimed at determining which restrictions on the Banach algebra $A$ and action $\alpha$ are natural to impose. Furthermore, it will be investigated what the 'right' notion of a crossed product is in the new setting.

We will now first introduce several basic definitions. Throughout this thesis we will also implicitly use many results from the theory of topological groups, Haar integration and integration of Banach space valued functions. We refer the reader to the appendices for the most important theorems and for further references. We will always use $G$ to denote a locally compact topological group and $\mu$ for a fixed left Haar measure on this group.

**Definition 2.1** A *normed algebra* $A$ is an algebra over $\mathbb{C}$ equipped with a norm such that for all $a, b \in A$

$$||ab|| \leq ||a|| \, ||b||.$$

A normed algebra is called a *Banach algebra* if it is complete with respect to its norm.

**Definition 2.2** A *representation* $\pi$ of a normed algebra $A$ on a Banach space $X$ is a homomorphism $\pi : A \to B(X)$ where $B(X)$ is the Banach algebra of bounded linear operators on $X$. It is called

- *continuous* if it is bounded, *contractive* if it is norm decreasing, *isometric* if it is norm-preserving;
- *faithful* if it is injective;
- *non-degenerate* if the linear span of $\{\pi(a)x;\ a \in A, x \in X\}$ is dense in $X$;
- *algebraically cyclic* if there is a vector $y \in X$, called an *algebraic cyclic vector*, such that $X = \pi(A)y = \{\pi(a)y : a \in A\}$.
- *topologically cyclic* if there is a vector $y \in X$, called a *topological cyclic vector*, such that $\pi(A)y$ is dense in $X$.
- *algebraically irreducible* if $\pi$ is non-trivial (i.e. $\pi(A) \neq 0$) and its only invariant subspaces are 0 and $X$.
- *topologically irreducible* if $\pi$ is non-trivial and its only closed invariant subspaces are 0 and $X$.

The following proposition gives a characterization of irreducible representations in terms of its cyclic vectors. The proof of the two statements can be found in Palmer (1994), theorem 4.1.3 and in Dixmier (1977), proposition 2.3.1, respectively.

**Proposition 2.3** *Let $A$ be a normed algebra and let $\pi$ be a representation of $A$ on a Banach space $X$. Then $\pi$ is algebraically irreducible if and only if every non-zero vector in $X$ is an algebraic cyclic vector for $\pi$ and $X \neq 0$. The reprsentation $\pi$ is topologically irreducible if and only if every non-zero vector $x$ in $X$ is a topological cyclic vector for $\pi$ or $X = \mathbb{C}x$.*

We will be mostly interested in representations of a Banach algebra on a Banach space. As the following proposition shows, for such representations it is not necessary to define the seemingly weaker concept of a strongly continuous representation. The proof is an application of the Banach-Steinhaus theorem and can be found in Palmer (1994), proposition 4.2.2.

**Proposition 2.4** *Every strongly continuous representation of a Banach algebra on a Banach space is continuous.*

Every Banach algebra $A$ has a natural representation and anti-representation on itself called the *left* and *right regular representation*, respectively, given by $a \mapsto L_a$ and $a \mapsto R_a$, where

$$L_a(b) = ab, \;\; R_a(b) = ba \;\; (b \in A).$$

**Definition 2.5** Let $A$ be a Banach algebra. A bijective, continuous, multiplicative linear map on $A$ is called an *automorphism* of $A$. The group of automorphisms of $A$ is denoted by $\mathrm{Aut}(A)$.

**Definition 2.6** A *Banach algebraic dynamical system*, or *dynamical system* for short, is a triple $(A, G, \alpha)$, where $A$ is a Banach algebra, $G$ is a locally compact topological group and $\alpha : G \to \mathrm{Aut}(A)$ is a strongly continuous action of $G$ on $A$ (i.e. $s \mapsto \alpha_s(a)$ is continuous for every $a \in A$). The action $\alpha$ is called *uniformly bounded by $M$* if, for some constant $M \geq 1$,

$$\frac{1}{M}||a|| \leq ||\alpha_r(a)|| \leq M||a|| \;\; (r \in G, a \in A).$$

A dynamical system will be called *isometric* if $\alpha : G \to \mathrm{Aut}(A)$ has its image in the subgroup of isometric automorphisms of $A$.

The following lemma shows that a strongly continuous action is uniformly bounded when restricted to a compact subset of $G$. This is a direct consequence of the Banach-Steinhaus theorem and will prove very useful in what follows.

**Lemma 2.7** *Let $(A, G, \alpha)$ be a dynamical system. Then for every compact set $K \subset G$ there exists a constant $M_K \geq 1$ such that*

$$\frac{1}{M_K}||a|| \leq ||\alpha_s(a)|| \leq M_K||a|| \;\; (s \in K, a \in A).$$

**Proof**. For a fixed $a \in A$, the map $s \mapsto \alpha_s(a)$ is continuous, so the set $\{\alpha_s(a) : s \in K\}$ is compact in $A$ and hence bounded. We conclude that $\{\alpha_s : s \in K\}$ forms a pointwise bounded collection in $B(A)$ and, since $A$ is Banach, the second inequality readily follows from the Banach-Steinhaus theorem. The first inequality follows by taking $a = \alpha_{s^{-1}}(b)$ $(b \in A)$ in the second inequality. ✓

The basic building block for our definition of the crossed product is the function algebra $C_c(G, A)$, which we define as follows:

**Definition 2.8** Let $(A, G, \alpha)$ be a dynamical system. The function algebra $C_c(G, A)$ is defined to be the set of continuous, compactly supported functions on $G$ with values in $A$, equipped with the *twisted convolution product*

$$f * g(s) = \int_G f(r)\alpha_r(g(r^{-1}s)) \; d\mu(r) \;\; (f, g \in C_c(G, A)).$$

The support of $f \in C_c(G, A)$ is denoted by $\text{supp}(f)$.

Using lemma 2.7 and lemma 2.10 below, we see that $f * g$ is indeed in $C_c(G, A)$ for $f, g \in C_c(G, A)$. Also, using Fubini's theorem it is not difficult to check that the twisted convolution product is an associative operation.
We shall consider several norms on $C_c(G, A)$, namely the supremum norm $|| \cdot ||_\infty$, the $L^1$-norm $|| \cdot ||_1$ and the $S$-crossed product norms $|| \cdot ||_S$, which will be defined later on (in fact, the $S$-crossed product will be defined as the completion of $C_c(G, A)$ with respect to the $|| \cdot ||_S$-norm).
Occasionally we use the Banach space $L^1(G, A)$. The theory of integration for Banach space valued functions is somewhat involved and most authors on crossed products prefer to simply define $L^1(G, A)$ as the completion of $C_c(G, A)$ with respect to the $|| \cdot ||_1$-norm. Although $L^1(G, A)$ is indeed the completion of $C_c(G, A)$, we need slightly more than that (e.g. Fubini's theorem and the Dominated Convergence theorem), see appendix B for a discussion and the main results.
The next lemma will provide a very useful tool in the following. It is taken from Williams (2007), lemma 1.87.

**Lemma 2.9** *Let $A_0$ be a dense subset of $A$ and define*

$$C_c(G) \odot A_0 = \text{span}\{z \otimes a \ : \ z \in C_c(G), a \in A_0\},$$

*where $z \otimes a(s) := z(s)a$. Then for every $f \in C_c(G, A)$ there exists a sequence $\{f_n\}_{n=1}^\infty$ in $C_c(G) \odot A_0$ such that $f_n \to f$ uniformly on $G$ and for some $N > 0$ and compact set $K$ we have* $\text{supp}(f_n) \subset K$ *for all $n \in \mathbb{N}$. In particular, $C_c(G) \odot A_0$ is dense in $C_c(G, A)$ with the inductive limit topology and in $C_c(G, A)$ with the topology induced by the $L^1$-norm.*

We shall not need the inductive limit topology in what follows. For a definition and discussion we refer to Conway (1985).
The proof of the lemma depends on the following result, which we will use over and over again. It is lemma 1.88 from Williams (2007).

**Lemma 2.10** *Let $f \in C_c(G, A)$ and $\varepsilon > 0$. Then there is a neighborhood $V$ of $e$ in $G$ such that either $sr^{-1} \in V$ or $s^{-1}r \in V$ implies*

$$||f(s) - f(r)|| < \varepsilon.$$

The following definition establishes a notion of representation for dynamical systems.

**Definition 2.11** Let $(A, G, \alpha)$ be a dynamical system and $X$ a Banach space. Then a pair $(\pi, U)$ is called a *covariant representation* of $(A, G, \alpha)$ on $X$ if $\pi : A \to B(X)$ is a representation of $A$ on $X$ and $U : G \to B(X)$ is a strongly continuous homomorphism into the group of surjective linear isometries on $X$ (an *isometric representation* of $G$ on $X$) which satisfy

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1}.$$

A covariant representation $(\pi, U)$ is called *continuous, contractive, faithful, isometric, nondegenerate, (algebraically/topologically) cyclic* or *(algebraically/topologically) irreducible* if $\pi$ is continuous, contractive, faithful, isometric, nondegenerate, (algebraically/topologically) cyclic or (algebraically/topologically) irreducible, respectively.

**Remark**. The restriction of covariant representations in which $U$ is isometric is a natural generalization of the covariant representations considered in the theory of

$C^*-$dynamical systems, where $U$ is restricted to be a unitary representation of $G$ on a Hilbert space $H$.

From the definition it is not immediately clear that non-zero covariant representations exist. Fortunately in the case that $(A, G, \alpha)$ has a uniformly bounded action, given a continuous representation $\pi$ of $A$ on a Banach space $X$ (for example the left regular representation) we can always construct a covariant representation $(\tilde{\pi}, U)$ of $(A, G, \alpha)$ on $L^1(G, X)$ called the *regular covariant representation associated to* $\pi$.

**Proposition 2.12** *Let $(A, G, \alpha)$ be dynamical system, suppose that $\alpha$ is uniformly bounded by $M \geq 1$ and let $\pi : A \to B(X)$ be a continuous representation. Let $1 \leq p < \infty$ be given. Define the pair of representations $(\tilde{\pi}, U)$ of $(A, G, \alpha)$ on $L^p(G, X)$ by*

$$\tilde{\pi}(a)h(r) = \pi(\alpha_r^{-1}(a))(h(r)) \ , \ U_s h(r) = h(s^{-1}r).$$

*Then $(\tilde{\pi}, U)$ is a continuous covariant representation and $||\tilde{\pi}|| \leq M||\pi||$. If $p = 1$, then $(\tilde{\pi}, U)$ is called the* regular covariant representation associated to $\pi$.

**Proof**. It is easy to check that $\tilde{\pi}$ and $U$ are homomorphisms into $B(L^p(G, X))$ and $U_s$ is an invertible isometry for every $s \in G$. It is not immediately obvious that the map $r \mapsto \pi(\alpha_r^{-1}(a))(h(r))$ is measurable. Fix $a \in A$ and $h \in L^p(G, X)$, then $h$ vanishes off a $\sigma$-finite subset of $G$ and hence there is a sequence of simple funtions $\{h_n\}$ and a $\mu$-null set $N$ such that

$$||h_n(r)|| \leq ||h(r)|| \text{ and } h_n(r) \to h(r) \text{ (for all } r \in G - N).$$

Now each $h_n$ is of the form

$$h_n(r) = \sum_{i=1}^{k_n} x_i^n 1_{G_i^n}(r) \ (r \in G),$$

where $k_n \in \mathbb{N}$, $x_i^n \in X$ and $G_i^n \subset G$ is measurable and of finite measure. As $r \mapsto \pi(\alpha_r^{-1}(a))x$ is measurable for every $x \in X$ by continuity of $\pi$ and strong continuity of $\alpha$, we see that

$$r \mapsto \pi(\alpha_r^{-1}(a))h_n(r) = \sum_{i=1}^{k_n} \pi(\alpha_r^{-1}(a))x_i^n 1_{G_i^n}(r)$$

is measurable on $G$. Moreover, for any $r \in G - N$,

$$\begin{aligned} ||\pi(\alpha_r^{-1}(a))h_n(r) - \pi(\alpha_r^{-1}(a))h(r)|| &\leq ||\pi(\alpha_r^{-1}(a))|| \, ||h_n(r) - h(r)|| \\ &\leq M||\pi|| \, ||a|| \, ||h_n(r) - h(r)|| \to 0. \end{aligned}$$

Thus $\tilde{\pi}(a)h$ is $\mu$-almost everywhere the pointwise limit of a sequence of measurable functions and therefore measurable. Also,

$$||\tilde{\pi}(a)h||_p \leq (\int_G ||\pi(\alpha_r^{-1}(a))||^p \, ||h(r)||^p \, d\mu(r))^{1/p} \leq M||\pi|| \, ||a|| \, ||h||_p.$$

We conclude that $\tilde{\pi}$ is a continuous linear operator on $L^p(G, X)$ and $||\tilde{\pi}|| \leq M||\pi||$. To show that $U$ is strongly continuous, let $h \in L^p(G, X)$ and $s^*$ be given. Fix $\varepsilon > 0$. Since $C_c(G, X)$ is dense in $L^p(G, X)$, we can find $\tilde{h} \in C_c(G, X)$ such that $||h - \tilde{h}||_p < \frac{\varepsilon}{3}$ and hence $||U_r h - U_r \tilde{h}||_p < \frac{\varepsilon}{3}$ for every $r \in G$. By lemma 2.10 we can find a neighborhood $V$ of $e$ in $G$ such that if either $sr^{-1} \in V$ or $s^{-1}r \in V$ implies

$$||\tilde{h}(r) - \tilde{h}(s)|| < \frac{\varepsilon}{3(\mu(\text{supp}(\tilde{h})))^{1/p}}.$$

So if we take $s \in s^*V$ then for any $r \in G$ we have $(s^*)^{-1}r(s^{-1}r)^{-1} \in V$ and hence,

$$||U_s\tilde{h} - U_{s^*}\tilde{h}||_p = (\int_G ||\tilde{h}(s^{-1}r) - \tilde{h}((s^*)^{-1}r)||^p \; d\mu(r))^{1/p} < \frac{\varepsilon}{3}.$$

We obtain for $s \in s^*V$,

$$||U_s h - U_{s^*}h||_p \leq ||U_s h - U_s\tilde{h}||_p + ||U_s\tilde{h} - U_{s^*}\tilde{h}||_p + ||U_{s^*}\tilde{h} - U_{s^*}h||_p < \varepsilon,$$

$U$ is strongly continuous.
It only remains to verify the covariance condition:

$$\begin{aligned} U_s\tilde{\pi}(a)U_s^{-1}h(r) &= \tilde{\pi}(a)U_s^{-1}h(s^{-1}r) \\ &= \pi(\alpha_{s^{-1}r}^{-1}(a))(U_s^{-1}h(s^{-1}r)) \\ &= \pi(\alpha_r^{-1}(\alpha_s(a)))(h(r)) \\ &= \tilde{\pi}(\alpha_s(a))h(r). \end{aligned}$$

✓

**Remark**. If $(A, G, \alpha)$ is isometric, another easy contractive covariant representation $(\pi, U)$ is obtained by taking $\pi$ equal to the left regular representation and defining $U$ by $U_s := \alpha_s$ for $s \in G$.

We will now define an equivalence relation on the set of covariant representations of a dynamical system.

**Definition 2.13** Let $(A, G, \alpha)$ be a dynamical system and let $(\pi, U)$ and $(\rho, V)$ be covariant representations of $(A, G, \alpha)$ on Banach spaces $X$ and $Y$, respectively. An *intertwining operator* for $(\pi, U)$ with respect to $(\rho, V)$ is a bounded, invertible linear operator $\Phi : X \to Y$ which satisfies

$$\rho(a)\Phi = \Phi\pi(a) \ , \ V_s\Phi = \Phi U_s \ (a \in A, s \in G).$$

If an intertwining operator exists, we call $(\pi, U)$ *intertwined with* $(\rho, V)$.

**Proposition 2.14** *Intertwinedness defines an equivalence relation on the set of covariant representations of a dynamical system.*

**Proof**. Obviously, any covariant representation $(\pi, U)$ is intertwined with itself, through the identity operator (on $X$). Also, if $(\pi, U)$ is intertwined with $(\rho, V)$ through $\Phi$, then $\Phi$ has a bounded inverse and $(\rho, V)$ is intertwined with $(\pi, U)$ through $\Phi^{-1}$. Finally, if $(\pi, U)$ is intertwined with $(\rho, V)$ through $\Phi$ and $(\rho, V)$ is intertwined with $(\sigma, W)$ through $\Psi$, then $\Psi\Phi$ is an intertwining operator for $(\pi, U)$ with $(\sigma, W)$. ✓

**Proposition 2.15** *A dynamical system* $(A, G, \alpha)$ *is isometric if and only if there exists an isometric covariant representation of* $(A, G, \alpha)$.

**Proof**. Suppose that $(\pi, U)$ is an isometric covariant representation of $(A, G, \alpha)$. Then by covariance

$$\pi(\alpha_s(a)) = U_s\pi(a)U_s^{-1} \ (s \in G, a \in A).$$

Moreover, as $U_s$ is isometric for any $s \in G$ and $\pi : A \to X$ is isometric by assumption, we have, for all $s \in G$ and $a \in A$,

$$\begin{aligned}
||a|| &= ||\pi(a)|| \\
&= \sup_{||x|| \leq 1} ||\pi(a)x|| \\
&= \sup_{||x|| \leq 1} ||U_s^{-1}\pi(\alpha_s(a))U_s x|| \\
&= \sup_{||x|| \leq 1} ||\pi(\alpha_s(a))U_s x|| \\
&= \sup_{||y|| \leq 1} ||\pi(\alpha_s(a))y|| \\
&= ||\pi(\alpha_s(a))|| \\
&= ||\alpha_s(a)||,
\end{aligned}$$

where we used that $U_s$ is a permutation of the closed unit ball in $X$. We conclude that $(A, G, \alpha)$ is isometric.
Conversely, suppose that $\alpha_s$ is an isometric automorphism of $A$ for every $s$ in $G$. Let $A^1$ be the Banach algebra obtained by the adjunction of an identity and define $\pi : A \to B(A^1)$ to be the extended left regular representation of $A$:

$$\pi(a)(\lambda + b) = \lambda a + ab \ (a \in A, \lambda + b \in A^1).$$

Then $\pi$ is an isometric representation of $A$ on $A^1$. Let $(\tilde{\pi}, U)$ be the regular covariant representation associated to $\pi$. We will show that $\tilde{\pi}$ is isometric. Notice that it is sufficient to show that $||\tilde{\pi}(a)|| = 1$ for any $a \in A$ with $||a|| = 1$. So let $a$ be any element in $A$ of unit norm, then we have for any $f \in L^1(G, A^1)$,

$$\begin{aligned}
||\tilde{\pi}(a)f||_1 &= \int_G ||\pi(\alpha_r^{-1}(a))(f(r))|| \, d\mu(r) \\
&\leq \int_G ||\pi(\alpha_r^{-1}(a))|| \, ||f(r)|| \, d\mu(r) \\
&= \int_G ||\alpha_r^{-1}(a)|| \, ||f(r)|| \, d\mu(r) \\
&= \int_G ||a|| \, ||f(r)|| \, d\mu(r) \\
&= ||f||_1,
\end{aligned}$$

so $||\tilde{\pi}(a)|| \leq 1$.
Fix $\varepsilon > 0$. Since $||\pi(a)|| = ||a|| = 1$, we can pick $x \in A^1$ of norm 1 such that $||\pi(a)x|| > 1 - \frac{\varepsilon}{2}$. By strong continuity of the action $\alpha$ we can also find an open neighborhood $V$ of $e$ in $G$ such that for $r \in V$

$$||\alpha_r^{-1}(a) - a|| < \frac{\varepsilon}{2}.$$

Define $g_V \in C_c(G, A^1)$ by $g_V = \tilde{g}_V \otimes x$, where $\tilde{g}_V \in C_c(G)$ is such that $||\tilde{g}_V||_1 = 1$ and $\mathrm{supp}(\tilde{g}_V) \subset V$. Then

$$\begin{aligned}
||\tilde{\pi}(a)g_V||_1 &= \int_G ||\pi(\alpha_r^{-1}(a))(g_V(r))|| \; d\mu(r) \\
&= \int_G ||\pi(\alpha_r^{-1}(a))(\tilde{g}_V(r)x)|| \; d\mu(r) \\
&\geq \int_G |\tilde{g}_V(r)| \; (||\pi(a)x|| - ||\pi(\alpha_r^{-1}(a))x - \pi(a)x||) \; d\mu(r) \\
&> \int_G |\tilde{g}_V(r)| \; (1-\varepsilon) \; d\mu(r) \\
&= 1 - \varepsilon,
\end{aligned}$$

where we have used that for $r \in V$,

$$||\pi(\alpha_r^{-1}(a))x - \pi(a)x|| \leq ||\alpha_r^{-1}(a) - a|| < \frac{\varepsilon}{2}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $||\tilde{\pi}(a)|| = 1$. ✓

Our later efforts will concentrate on dynamical systems $(A, G, \alpha)$ with $A$ *approximate unital*, i.e. where $A$ contains a two-sided, bounded approximate unit.

**Definition 2.16** Let $A$ be a normed algebra. Then an *$M$-bounded left approximate unit* for $A$ is a net $\{u_i\}_{i\in J}$ in $A$ such that for some $M > 0$ we have $||u_i|| \leq M$ for all $i \in J$ and for any $a \in A$

$$\lim ||u_i a - a|| = 0.$$

An *$M$-bounded right approximate unit* for $A$ is a net $\{v_i\}_{i\in J}$ in $A$ such that for some $M > 0$ we have $||v_i|| \leq M$ for all $i \in J$ and for any $a \in A$

$$\lim ||a v_i - a|| = 0.$$

An *$M$-bounded approximate unit* for $A$ is a net $\{w_i\}_{i\in J}$ in $A$ which is both an $M$-bounded left and an $M$-bounded right approximate unit for $A$.

As is already apparent from proposition 2.15, isometric dynamical systems enjoy some special properties. For dynamical systems $(A, G, \alpha)$ where $A$ has a left approximate unit contained in its closed unit ball, the isometry property is characterized by submultiplicativity of the $L^1$-norm on $C_c(G) \odot A$. We extract this statement as a corollary from the following proposition.

**Proposition 2.17** *Let $(A, G, \alpha)$ be a dynamical system. Then*

$$||f * g||_1 \leq ||f||_1 \; ||g||_1 \;\; (f, g \in C_c(G) \odot A)$$

*if and only if $||b\alpha_r(a)|| \leq ||b|| \; ||a||$ for any $r \in G$ and $a, b \in A$.*

**Proof**. The 'if' part is easy.

$$\begin{aligned}
||f * g||_1 &= \int_G || \int_G f(r)\alpha_r(g(r^{-1}s)) \; d\mu(r)|| \; d\mu(s) \\
&\leq \int_G \int_G ||f(r)\alpha_r(g(r^{-1}s))|| \; d\mu(r) \; d\mu(s) \\
&\leq \int_G \int_G ||f(r)|| \; ||g(r^{-1}s)|| \; d\mu(r) \; d\mu(s) \\
&= \int_G \int_G ||f(r)|| \; ||g(r^{-1}s)|| \; d\mu(s) \; d\mu(r) \\
&= \int_G ||f(r)|| \int_G ||g(s)|| \; d\mu(s) \; d\mu(r) \\
&= ||f||_1 \; ||g||_1.
\end{aligned}$$

For the other direction, fix $a \in A$ and $r^* \in G$. Define $g \in C_c(G) \odot A$ by

$$g(s) = \tilde{g} \otimes a(s) = \tilde{g}(s)a,$$

where $\tilde{g} \in C_c(G)$ is nonnegative and has integral equal to 1. Notice that

$$||g||_1 = \int_G ||\tilde{g} \otimes a(s)|| \ d\mu(s) = ||a|| \int_G \tilde{g}(s) \ d\mu(s) = ||a||.$$

Fix $\varepsilon > 0$. Then, since $\tilde{g} \in C_c(G)$, there is some symmetric neighborhood $V_1$ of $e$ in $G$ such that $sr^{-1} \in V_1$ or $s^{-1}r \in V_1$ implies

$$|\tilde{g}(s) - \tilde{g}(r)| < \varepsilon$$

(See lemma 2.10). Also, by continuity of $r \mapsto \alpha_r(a)$ at $r^*$ we can find a symmetric neighborhood $V_2$ of $e$ in $G$ such that $r(r^*)^{-1} \in V_2$ or $r^{-1}r^* \in V_2$ implies

$$||\alpha_r(a) - \alpha_{r^*}(a)|| < \varepsilon.$$

Define $V$ to be the neighborhood $r^*(V_1 \cap V_2)$ of $r^*$. Then for $r \in V$ we have $r(r^*)^{-1} \in V_1 \cap V_2$. Pick $\tilde{f}_V \in C_c(G)$ such that $\mathrm{supp}(\tilde{f}_V) \subset V$, $||\tilde{f}_V||_1 = 1$ and $\tilde{f}_V$ is nonnegative. Fix an arbitrary $b \in A$ and let $f_V \in C_c(G) \odot A$ be defined by

$$f_V(s) = \tilde{f}_V \otimes b(s) = \tilde{f}_V(s)b.$$

Notice that $||f_V||_1 \leq ||b||$.
Now, for any $s \in G$,

$$\begin{aligned}
|| \int_G f_V(r)\alpha_r(g(r^{-1}s)) \ d\mu(r) \quad &- \quad b\alpha_{r^*}(a)\tilde{g}((r^*)^{-1}s)|| \\
&= \quad || \int_G f_V(r)\alpha_r(g(r^{-1}s)) \ d\mu(r) - f_V(r)\alpha_{r^*}(a)\tilde{g}((r^*)^{-1}s) \ d\mu(r)|| \\
&\leq \quad ||b|| \int_G \tilde{f}_V(r)||\alpha_r(g(r^{-1}s)) \ d\mu(r) - \alpha_{r^*}(a)\tilde{g}((r^*)^{-1}s)|| \ d\mu(r) \\
&\leq \quad ||b|| \int_G \tilde{f}_V(r)(||\alpha_r(a) - \alpha_{r^*}(a)||\tilde{g}(r^{-1}s) \\
&\qquad + ||\alpha_{r^*}(a)|| \ |\tilde{g}(r^{-1}s) - \tilde{g}((r^*)^{-1}s)|) \ d\mu(r) \\
&< \quad ||b|| \int_G \tilde{f}_V(r)\varepsilon(||\tilde{g}||_\infty + ||\alpha_{r^*}(a)||) \ d\mu(r) \\
&= \quad \varepsilon||b||(||\tilde{g}||_\infty + ||\alpha_{r^*}(a)||).
\end{aligned}$$

In the last inequality we used that $r \in V$ implies that $r = r^*v_1$ for some $v_1 \in V_1$ and hence

$$r^{-1}s((r^*)^{-1}s)^{-1} = r^{-1}r^* = (r^*v_1)^{-1}r^* = v_1^{-1} \in V_1$$

as $V_1$ is symmetric. Since $s \in G$ was arbitrary and moreover $\varepsilon > 0$ was arbitrary we have obtained a net $\{f_V\}$ in $C_c(G) \odot A$ such that

$$\int_G f_V(r)\alpha_r(a)\tilde{g}(r^{-1}s) \ d\mu(r) \to b\alpha_{r^*}(a)\tilde{g}((r^*)^{-1}s)$$

uniformly and, since the supports of $\int_G f_V(r)\alpha_r(a)\tilde{g}(r^{-1}s)\ d\mu(r)$ are eventually contained in a fixed compact set, this convergence holds in $L^1(G,A)$ as well. Hence,

$$\begin{aligned} ||f_V * g||_1 &= \int_G || \int_G f_V(r)\alpha_r(g(r^{-1}s))\ d\mu(r)||\ d\mu(s) \\ &= \int_G || \int_G f_V(r)\tilde{g}(r^{-1}s)\alpha_r(a)\ d\mu(r)||\ d\mu(s) \\ &\to \int_G ||b\alpha_{r^*}(a)\tilde{g}((r^*)^{-1}s)||\ d\mu(s) \\ &= ||b\alpha_{r^*}(a)|| \int_G \tilde{g}((r^*)^{-1}s)\ d\mu(s) \\ &= ||b\alpha_{r^*}(a)||. \end{aligned}$$

So $||f_V * g||_1 \leq ||f_V||_1\ ||g||_1 \leq ||b||\ ||a||$ implies $||b\alpha_{r^*}(a)|| \leq ||b||\ ||a||$. Since $r^* \in G$ and $a, b \in A$ were arbitrary, our proof is complete. ✓

**Remark**. If $||b\alpha_r(a)|| \leq ||b||\ ||a||$ for all $r \in G$ and $a, b \in A$, then the inequality

$$||f * g||_1 \leq ||f||_1\ ||g||_1$$

holds for $f, g \in L^1(G,A)$, as can be easily seen from the first part of the proof.

**Corollary 2.18** *Let $(A,G,\alpha)$ be a dynamical system and suppose that $A$ contains a left approximate unit in its closed unit ball. Then*

$$||f * g||_1 \leq ||f||_1\ ||g||_1\ (f, g \in C_c(G) \odot A)$$

*if and only if $\alpha_r$ is an isometry for all $r$ in $G$, i.e. if and only if $(A,G,\alpha)$ is isometric.*

**Proof**. Suppose that $(A,G,\alpha)$ is isometric. Then, for any $r \in G$ and $a, b \in A$,

$$||b\alpha_r(a)|| \leq ||b||\ ||\alpha_r(a)|| \leq ||b||\ ||a||,$$

so by proposition 2.17 $||f * g||_1 \leq ||f||_1\ ||g||_1$ for all $f, g \in C_c(G) \odot A$.
Conversely, suppose $||\cdot||_1$ is submultiplicative on $C_c(G) \odot A$ and let $\{u_i\}_{i\in J}$ be a left approximate unit for $A$ which is bounded by 1. Then by proposition 2.17 we have $||u_i\alpha_r(a)|| \leq ||a||$ for any $i \in J$, $r \in G$ and $a \in A$. By taking limits, for fixed $r \in G$ and $a \in A$, we obtain $||\alpha_r(a)|| \leq ||a||$ for all $r \in G$ and $a \in A$. Taking $a = \alpha_{r^{-1}}(b)$, we also see that $||b|| \leq ||\alpha_{r^{-1}}(b)||$ $(r \in G,\ b \in A)$. We conclude that $\alpha_r$ is an isometry for all $r \in G$. ✓

**Corollary 2.19** *Let $(A,G,\alpha)$ be a dynamical system and suppose that $A$ contains a left approximate unit which is bounded by $M \geq 1$. Then $\alpha$ is uniformly bounded by $M$ if $||\cdot||_1$ is submultiplicative on $C_c(G) \odot A$.*

As a consequence of proposition 2.17 above, the Banach space $L^1(G,A)$ forms a Banach algebra under the twisted convolution product if and only if the action of the dynamical system $(A,G,\alpha)$ satisfies

$$||b\alpha_r(a)|| \leq ||b||\ ||a||\ (\text{for all } r \in G \text{ and } a, b \in A).$$

If this is the case, we will always implicitly assume that $L^1(G,A)$ is equipped with this product.
In the next lemma we make the first step in obtaining a bounded approximate unit for the $S$-crossed products from a bounded approximate unit in the Banach algebra $A$ of a dynamical system $(A,G,\alpha)$. The subsequent theorem gives an application of this technical lemma.

**Lemma 2.20** *Let $(A, G, \alpha)$ be a dynamical system and suppose that $A$ has an $M$-bounded left approximate unit $\{u_i\}$. Let $\mathcal{V}$ be the collection of all neighborhoods of $e$ in $G$, ordered by reverse inclusion. For every $V \in \mathcal{V}$ define an element $f_V \in C_c(G)$ which is nonnegative, has integral equal to 1 and has its support contained in $V$. Define $f_{(V,i)} = f_V \otimes u_i$, then $\{f_{(V,i)}\}$ defines a net in $C_c(G) \odot A$ with the following properties:*

1. $||f_{(V,i)}||_1 \leq M$;
2. *There exists a compact set $K \subset G$ and $(\tilde{V}, \tilde{i})$ such that* $\mathrm{supp}(f_{(V,i)}) \subset K$ *if* $(\tilde{V}, \tilde{i}) \preceq (V, i)$;
3. *For any $f \in C_c(G, A)$ we have*
$$\lim ||f_{(V,i)} * f - f||_\infty = 0.$$

*If $A$ has an $M$-bounded right approximate unit $\{u_i\}$, then the above holds with the final property of $\{f_{(V,i)}\}$ replaced by $f * f_{(V,i)} \to f$ uniformly on $G$, for any $f \in C_c(G) \odot A$.*

**Proof**. Notice first that $\mathcal{V}$ is a directed set and if we define the partial order
$$(V, i) \preceq (U, j) \text{ if } (V \preceq U \text{ and } i \preceq j)$$
then the set $\{(V, i) : V \in \mathcal{V}, i \in J\}$ becomes a directed set. We will show that the net $\{f_{(V,i)}\}$ satisfies the required properties.
The first property is easily shown,
$$||f_{(V,i)}||_1 = \int_G ||f_V(s)u_i|| \; d\mu(s) \leq M \int_G |f_V(s)| \; d\mu(s) = M.$$
Also, as $G$ is locally compact, we can subsequently find a compact neighborhood $K$ of $e$ and $\hat{V} \in \mathcal{V}$ such that $\mathrm{supp}(f_V) \subset K$ if $\hat{V} \subset V$. Since $\mathrm{supp}(f_{(V,i)}) = \mathrm{supp}(f_V)$ the second property readily follows.
Let $f \in C_c(G, A)$ and let $\varepsilon > 0$. We may assume that $f \neq 0$. Now, for any $s \in G$,
$$\begin{aligned} ||f_{(V,i)} * f(s) - u_i f(s)|| &= ||u_i \int_G f_V(r)(\alpha_r(f(r^{-1}s)) - f(s)) \; d\mu(r)|| \\ &\leq M \int_G f_V(r) ||i_G(r)f(s) - f(s)|| \; d\mu(r), \end{aligned}$$
where for $r \in G$ the operator $i_G(r)$ on $C_c(G, A)$ is defined by $i_G(r)f(s) := \alpha_r(f(r^{-1}s))$. Peeking ahead to the proof of proposition 3.16, we see that there is some neighborhood $\tilde{V}$ of $e$ in $G$ such that $\tilde{V} \preceq V$ implies
$$||i_G(r)f(s) - f(s)|| < \frac{\varepsilon}{2M} \text{ for all } s \in G \text{ and } r \in V.$$
Let $j \in J$ be such that $j \preceq i$ implies $||u_i a - a|| < \frac{\varepsilon}{2||f||_\infty}$. Then if $(\tilde{V}, j) \preceq (V, i)$,
$$\begin{aligned} ||f_{(V,i)} * f(s) - f(s)|| &\leq ||f_{(V,i)} * f(s) - u_i f(s)|| + ||u_i f(s) - f(s)|| \\ &< \frac{\varepsilon}{2} + ||f||_\infty ||u_i a - a|| \\ &< \varepsilon. \end{aligned}$$
This estimate is uniform in $s \in G$. Since $\varepsilon > 0$ was arbitrary the third property follows.
Suppose now that $A$ has an $M$-bounded right approximate unit, which we again

denote by $\{u_i\}$. Let $f \in C_c(G) \odot A$ be given by $f = \tilde{f} \otimes a$, with $\tilde{f} \in C_c(G)$ and $a \in A$. We may assume that $\tilde{f} \neq 0$. Fix $\varepsilon > 0$. We need to show that $f * f_{(V,i)} \to f$ uniformly on $G$.
Since the supports of the $f_{(V,i)}$ are eventually contained in a fixed compact set, we can pick a compact set $\tilde{K}$ in $G$ and $(\tilde{V}, \tilde{i})$ such that the support of $s \mapsto f * f_{(V,i)}(s) - f(s)\alpha_s(u_i)$ is contained in $\tilde{K}$ if $(\tilde{V}, \tilde{i}) \preceq (V, i)$. Define the compact set $K$ by $K = \tilde{K} \cup \mathrm{supp}(f)$ and pick a constant $M_K \geq 1$ as in lemma 2.7.
We have, for any $s \in \mathrm{supp}(f)$,

$$\begin{aligned} ||f(s)\alpha_s(u_i) - f(s)|| &\leq ||\tilde{f}||_\infty ||a\alpha_s(u_i) - a|| \\ &= M_K ||\tilde{f}||_\infty ||\alpha_{s^{-1}}(a)u_i - \alpha_{s^{-1}}(a)||. \end{aligned}$$

Notice that $||f(s)\alpha_s(u_i) - f(s)||$ is zero for $s \in G - \mathrm{supp} f$. Since the map $s \mapsto s^{-1}$ is by definition continuous, $s \mapsto \alpha_{s^{-1}}(a)$ is continuous. For every $s \in K$ there exists an open neighborhood $W_s$ of $s$ such that for any $r \in W_s$

$$||\alpha_{r^{-1}}(a) - \alpha_{s^{-1}}(a)|| < \min\{\frac{\varepsilon}{6M_K M ||\tilde{f}||_\infty}, \frac{\varepsilon}{6M_K ||\tilde{f}||_\infty}\}.$$

These open neighborhoods form an open cover of $K$, so we can extract a finite subcover $W_{s_1}, \ldots, W_{s_n}$.
Pick $i_j \in J$ such that $i_j \preceq i$ implies

$$||\alpha_{s_j^{-1}}(a)u_i - \alpha_{s_j^{-1}}(a)|| < \min\{\frac{\varepsilon}{6M_K ||\tilde{f}||_\infty}, \frac{\varepsilon}{30 M_K M_{\hat{V}}^2 ||\tilde{f}||_\infty ||a||}\},$$

where the constant $M_{\hat{V}} \geq 1$ will be defined below.
Now, for any $s \in W_{s_j}$ and $i_j \preceq i$,

$$\begin{aligned} ||\alpha_{s^{-1}}(a)u_i - \alpha_{s^{-1}}(a)|| &\leq ||\alpha_{s^{-1}}(a)u_i - \alpha_{s_j^{-1}}(a)u_i|| + ||\alpha_{s_j^{-1}}(a)u_i - \alpha_{s_j^{-1}}(a)|| \\ &\quad + ||\alpha_{s_j^{-1}}(a) - \alpha_{s^{-1}}(a)|| \\ &< \frac{\varepsilon}{2M_K ||\tilde{f}||_\infty}. \end{aligned}$$

Pick $i^* \in J$ such that $i_j \preceq i^*$ $(1 \leq j \leq n)$ and $\tilde{i} \preceq i^*$, then for $i^* \preceq i$ we have

$$||f(s)\alpha_s(u_i) - f(s)|| < \frac{\varepsilon}{2}$$

uniformly in $s \in G$.
We also have, if $(\tilde{V}, \tilde{i}) \preceq (V, i)$, for any $s \in G$,

$$\begin{aligned}
||f * f_{(V,i)}(s) - f(s)\alpha_s(u_i)|| &= ||\int_G f(r)\alpha_r(f_{(V,i)}(r^{-1}s))\, d\mu(r) - f(s)\alpha_s(u_i)|| \\
&= ||\int_G f(r)f_V(r^{-1}s)\alpha_r(u_i)\, d\mu(r) - f(s)\alpha_s(u_i)|| \\
&= ||\int_G f(sr)f_V(r^{-1})\alpha_{sr}(u_i)\, d\mu(r) - f(s)\alpha_s(u_i)|| \\
&= ||\int_G f(sr^{-1})f_V(r)\Delta(r^{-1})\alpha_{sr^{-1}}(u_i)\, d\mu(r) - f(s)\alpha_s(u_i)|| \\
&= ||\int_G f_V(r)(f(sr^{-1})\Delta(r^{-1})\alpha_{sr^{-1}}(u_i) - f(s)\alpha_s(u_i))\, d\mu(r)|| \\
&\leq \int_G f_V(r)||f(sr^{-1})\Delta(r^{-1})\alpha_{sr^{-1}}(u_i) - f(s)\alpha_s(u_i)||\, d\mu(r) \\
&\leq \int_G f_V(r)(|\Delta(r^{-1}) - 1|\ ||f(sr^{-1})\alpha_{sr^{-1}}(u_i)|| \\
&\quad + ||f(sr^{-1})\alpha_{sr^{-1}}(u_i) - f(sr^{-1})\alpha_s(u_i)|| \\
&\quad + M_K M||f(sr^{-1}) - f(s)||)\, d\mu(r),
\end{aligned}$$

where we used in the last step that $||\alpha_s(u_i)|| \leq M_K M$.
Now, if $\hat{V}$ is a symmetric neighborhood contained in a compact neighborhood of $e$ in $G$, then for $r \in \hat{V}$,

$$||f(sr^{-1})\alpha_{sr^{-1}}(u_i)|| \leq ||f(sr^{-1})||\ ||\alpha_{sr^{-1}}(u_i)|| \leq M_K M_{\hat{V}} M||f||_\infty$$

and since $r \mapsto \Delta(r)$ is a continuous homomorphism (so $\Delta(e) = 1$), there is some symmetric neighborhood $V_1$ of $e$ in $G$ such that for $r \in V_1$,

$$|\Delta(r^{-1}) - 1| < \frac{\varepsilon}{6M_K M_{\hat{V}} M||f||_\infty}.$$

For the second term,

$$\begin{aligned}
||f(sr^{-1})\alpha_{sr^{-1}}(u_i) - f(sr^{-1})\alpha_s(u_i)|| &\leq ||\tilde{f}||_\infty ||a\alpha_{sr^{-1}}(u_i) - a\alpha_s(u_i)|| \\
&\leq ||\tilde{f}||_\infty(||a\alpha_{sr^{-1}}(u_i) - a\alpha_{sr^{-1}}(u_{i^*})|| \\
&\quad + ||a\alpha_{sr^{-1}}(u_{i^*}) - a\alpha_s(u_{i^*})|| \\
&\quad + ||a\alpha_s(u_{i^*}) - a\alpha_s(u_i)||).
\end{aligned}$$

But, for $r \in \hat{V}$,

$$\begin{aligned}
||a\alpha_{sr^{-1}}(u_i) - a\alpha_{sr^{-1}}(u_{i^*})|| &\leq M_K M_{\hat{V}}||\alpha_{rs^{-1}}(a)u_i - \alpha_{rs^{-1}}(a)u_{i^*}|| \\
&= M_K M_{\hat{V}}||\alpha_r(a)\alpha_{s^{-1}}(a)u_i - \alpha_r(a)\alpha_{s^{-1}}(a)u_{i^*}|| \\
&\leq M_K M_{\hat{V}}^2||a||\ ||\alpha_{s^{-1}}(a)u_i - \alpha_{s^{-1}}(a)u_{i^*}|| \\
&\leq M_K M_{\hat{V}}^2||a||(||\alpha_{s^{-1}}(a)u_i - \alpha_{s^{-1}}(a)|| \\
&\quad + ||\alpha_{s^{-1}}(a) - \alpha_{s^{-1}}(a)u_{i^*}||),
\end{aligned}$$

and

$$||a\alpha_s(u_{i^*}) - a\alpha_s(u_i)|| \leq M_K(||\alpha_{s^{-1}}(a)u_{i^*} - \alpha_{s^{-1}}(a)|| + ||\alpha_{s^{-1}}(a) - \alpha_{s^{-1}}(a)u_i||).$$

So if we pick a neighborhood $V_2$ of $e$ in $G$ such that $r \in V_2$ implies

$$||\alpha_{sr^{-1}}(u_{i^*}) - \alpha_s(u_{i^*})|| < \frac{\varepsilon}{30||\tilde{f}||_\infty||a||} \text{ (for all } s \in G)$$

then it follows from our work above that for $s \in K$ (and hence in all of $G$)

$$||f(sr^{-1})\alpha_{sr^{-1}}(u_i) - f(sr^{-1})\alpha_s(u_i)|| < \frac{\varepsilon}{6}$$

if $(\tilde{V}, \tilde{i}) \preceq (V, i)$, $(\hat{V}, i^*) \preceq (V, i)$ and $(V_2, i^*) \preceq (V, i)$. Finally, there exists a neighborhood $V_3$ of $e$ in $G$ such that $r \in V_3$ implies (c.f. lemma 2.10)

$$||f(sr^{-1}) - f(s)|| < \frac{\varepsilon}{6M_K M} \text{ (for all } s \in G).$$

So if we pick $V^* \in \mathcal{V}$ such that $V^* \subset \hat{V} \cap \tilde{V} \cap V_1 \cap V_2 \cap V_3$, then for $(V^*, i^*) \preceq (V, i)$

$$\begin{aligned} ||f * f_{(V,i)}(s) - f(s)\alpha_s(u_i)|| &< \frac{\varepsilon}{2} \int_G f_V(r)\, d\mu(r) \\ &= \frac{\varepsilon}{2}, \end{aligned}$$

uniformly in $s \in G$. We obtain, for all $s \in G$,

$$\begin{aligned} ||f * f_{(V,i)}(s) - f(s)|| &\leq ||f * f_{(V,i)}(s) - f(s)\alpha_s(u_i)|| + ||f(s)\alpha_s(u_i) - f(s)|| \\ &< \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, our proof is complete. ✓

**Theorem 2.21** *Let $(A, G, \alpha)$ be a dynamical system and suppose that $A$ has an $M$-bounded left approximate unit. Then $L^1(G, A)$ has an $M$-bounded left approximate unit contained in $C_c(G) \odot A$. If $A$ has an $M$-bounded approximate unit and $\alpha$ satisfies*

$$||b\alpha_r(a)|| \leq ||b||\ ||a|| \text{ (for all } r \in G \text{ and } a, b \in A),$$

*then $L^1(G, A)$ has an $M$-bounded approximate unit contained in $C_c(G) \odot A$. In other words, $L^1(G, A)$ is an approximate unital Banach algebra under the twisted convolution product in this case.*

**Proof**. Pick any $f \in L^1(G, A)$. By lemma 2.9 $C_c(G) \odot A$ is dense in $L^1(G, A)$, so there is some sequence $\{f_n\} \subset C_c(G) \odot A$ such that $\lim ||f_n - f||_1 = 0$.
Suppose first that $A$ has an $M$-bounded left approximate unit and let the net $\{f_{(V,i)}\} \subset C_c(G) \odot A$ be given by lemma 2.20. Then, for any $n \in \mathbb{N}$,

$$||f_{(V,i)} * f - f||_1 \leq ||f_{(V,i)} * f - f_{(V,i)} * f_n||_1 + ||f_{(V,i)} * f_n - f_n||_1 + ||f_n - f||_1.$$

Let a compact set $K \subset G$ and neighborhood $\tilde{V}$ of $e$ in $G$ be such that $\text{supp}(f_{(V,i)}) \subset K$ for $(\tilde{V}, j) \preceq (V, i)$ and let $M_K \geq 1$ be as in lemma 2.7. If $(\tilde{V}, j) \preceq (V, i)$, then

$$\begin{aligned} ||f_{(V,i)} * f - f_{(V,i)} * f_n||_1 &= ||f_{(V,i)} * (f - f_n)||_1 \\ &\leq \int_G \int_G ||f_{(V,i)}(r)||\ ||\alpha_r((f - f_n)(r^{-1}s))||\ d\mu(r)\ d\mu(s) \\ &\leq \int_G \int_G ||f_{(V,i)}(r)||\ M_K||(f - f_n)(r^{-1}s)||\ d\mu(r)\ d\mu(s) \\ &= \int_G ||f_{(V,i)}(r)|| \int_G M_K||(f - f_n)(r^{-1}s)||\ d\mu(s)\ d\mu(r) \\ &= M_K||f_{(V,i)}||_1\ ||f - f_n||_1 = M_K M||f - f_n||_1. \end{aligned}$$

Hence, if $(\tilde{V}, j) \preceq (V, i)$,

$$||f_{(V,i)} * f - f||_1 \leq (M_K M + 1)||f - f_n||_1 + ||f_{(V,i)} * f_n - f_n||_1.$$

Let $\varepsilon > 0$ be given. If we pick $N \in \mathbb{N}$ such that $||f - f_N||_1 < \frac{\varepsilon}{2(M_K M+1)}$ and subsequently pick $(\tilde{V}, j) \preceq (V^*, i^*)$ such that $(V^*, i^*) \preceq (V, i)$ implies $||f_{(V,i)} * f_N - f_N||_1 < \frac{\varepsilon}{2}$ then

$$||f_{(V,i)} * f - f||_1 < \varepsilon \text{ if } (V^*, i^*) \preceq (V, i).$$

Hence $\lim ||f_{(V,i)} * f - f||_1 = 0$.
Suppose now that $(A, G, \alpha)$ satisfies

$$||b\alpha_r(a)|| \leq ||b|| \; ||a|| \text{ (for all } r \in G \text{ and } a, b \in A)$$

and that $A$ has an $M$-bounded approximate unit. Pick any $f \in L^1(G, A)$, let $\{f_n\} \subset C_c(G) \odot A$ be as above and take the net $f_{(V,i)}$ as in lemma 2.20. Thanks to our work in the above, we already know that $\lim ||f_{(V,i)} * f - f||_1 = 0$. By proposition 2.17 the $L^1$-norm is submultiplicative in this case, so for any $n \in \mathbb{N}$,

$$\begin{aligned} ||f * f_{(V,i)} - f||_1 &\leq ||f * f_{(V,i)} - f_n * f_{(V,i)}||_1 + ||f_n * f_{(V,i)} - f_n||_1 + ||f_n - f||_1 \\ &\leq ||f - f_n||_1 ||f_{(V,i)}||_1 + ||f_n * f_{(V,i)} - f_n||_1 + ||f_n - f||_1 \\ &= (M+1)||f - f_n||_1 + ||f_n * f_{(V,i)} - f_n||_1. \end{aligned}$$

We can now proceed as in the first part of the proof and obtain $\lim ||f * f_{(V,i)} - f||_1 = 0$. We conclude that $L^1(G, A)$ is a Banach algebra with an $M$-bounded approximate unit. ✓

**Lemma 2.22** *Let $f \in C_c(G, A)$ and suppose that $\pi : A \to B(X)$ is a representation of $A$ on a Banach space $X$ such that $s \mapsto \pi(f(s))$ is measurable and integrable. Let $s \mapsto U_s$ be an isometric representation of $G$ on $X$. Then the operator*

$$\int_G \pi(f(s)) U_s \, d\mu(s)(x) := \int_G \pi(f(s)) U_s x \, d\mu(s) \; (x \in X)$$

*is well-defined, linear and bounded. Moreover,*

$$||\int_G \pi(f(s)) U_s \, d\mu(s)|| \leq \int_G ||\pi(f(s))|| \, d\mu(s)$$

*and if $L : X \to Y$ is a bounded linear operator to a Banach space $Y$, then*

$$L(\int_G \pi(f(s)) U_s \, d\mu(s)) = \int_G L\pi(f(s)) U_s \, d\mu(s).$$

**Proof**. Since for every $x \in X$, $s \mapsto U_s x$ is continuous and hence measurable, the operator is well-defined and obviously linear by linearity of the integral and the maps in the integrand. We have for $x \in X$ with $||x|| \leq 1$,

$$\begin{aligned} ||\int_G \pi(f(s)) U_s \, d\mu(s)(x)|| &= ||\int_G \pi(f(s)) U_s x \, d\mu(s)|| \\ &\leq \int_G ||\pi(f(s))|| \; ||U_s x|| \, d\mu(s) \\ &\leq \int_G ||\pi(f(s))|| \, d\mu(s). \end{aligned}$$

This proves the first assertion. For the second assertion,

$$\begin{aligned} L \int_G \pi(f(s)) U_s \, d\mu(s)(x) &= L \int_G \pi(f(s)) U_s x \, d\mu(s) \\ &= \int_G L\pi(f(s)) U_s x \, d\mu(s) \\ &= \int_G L\pi(f(s)) U_s \, d\mu(s)(x). \end{aligned}$$

✓

**Definition 2.23** Let $(A, G, \alpha)$ be a dynamical system and let $(\pi, U)$ be a covariant representation of $(A, G, \alpha)$ on a Banach space $X$. If $s \mapsto \pi(f(s))U_s$ is integrable for every $f \in C_c(G, A)$, then we define the representation $\pi \rtimes U : C_c(G, A) \to X$ by

$$\pi \rtimes U(f) = \int_G \pi(f(s))U_s \; d\mu(s).$$

This is called the *integrated form* of $(\pi, U)$.

We only need to check that the integrated form is multiplicative on $C_c(G, A)$. Let $f, g \in C_c(G, A)$, then

$$\begin{aligned}
\pi \rtimes U(f * g) &= \int_G \int_G \pi(f(r)\alpha_r(g(r^{-1}s)))U_s \; d\mu(r) \; d\mu(s) \\
&= \int_G \int_G \pi(f(r))\pi(\alpha_r(g(r^{-1}s)))U_r U_{r^{-1}} U_s \; d\mu(r) \; d\mu(s) \\
&= \int_G \int_G \pi(f(r))U_r \pi(g(r^{-1}s))U_{r^{-1}s} \; d\mu(r) \; d\mu(s) \\
&= \int_G \int_G \pi(f(r))U_r \pi(g(r^{-1}s))U_{r^{-1}s} \; d\mu(s) \; d\mu(r) \\
&= \int_G \int_G \pi(f(r))U_r \pi(g(s))U_s \; d\mu(s) \; d\mu(r) \\
&= \pi \rtimes U(f) \pi \rtimes U(g).
\end{aligned}$$

**Lemma 2.24** *Let $(A, G, \alpha)$ be a dynamical system and let $(\pi, U)$ be a nondegenerate continuous covariant representation of $(A, G, \alpha)$ on $X$. Then the integrated form $\pi \rtimes U$ is nondegenerate as well.*

**Proof**. Let $x \in X$ and fix $\varepsilon > 0$. By assumption, there is some $n \in \mathbb{N}$, $a_i \in A$ and $x_i \in X$ $(1 \le i \le n)$ such that

$$\|\sum_{i=1}^n \pi(a_i)x_i - x\| < \frac{\varepsilon}{2}.$$

Notice that we may assume that $\pi(a_i) \neq 0$ for all $i$. By strong continuity of $U$ there is some neighborhood $V$ of $e$ in $G$ such that for $s \in V$,

$$\|U_s x_i - x_i\| < \frac{\varepsilon}{2n\|\pi(a_i)\|} \; (1 \le i \le n).$$

Let $f_V \in C_c(G)$ be nonnegative and such that its support is contained in $V$ and its integral is equal to 1. Define $f_i = f_V \otimes a_i$ for $1 \le i \le n$. Then

$$\begin{aligned}
\|\sum_{i=1}^n \pi \rtimes U(f_i)x_i - x\| &= \|\int_G \sum_{i=1}^n \pi(f_i(s))U_s x_i \; d\mu(s) - \int_G f_V(s)x \; d\mu(s)\| \\
&\le \int_G f_V(s)\|\sum_{i=1}^n \pi(a_i)U_s x_i - x\| \; d\mu(s) \\
&\le \int_G f_V(s)(\|\sum_{i=1}^n \pi(a_i)U_s x_i - \sum_{i=1}^n \pi(a_i)x_i\| \\
&\quad + \|\sum_{i=1}^n \pi(a_i)x_i - x\|) \; d\mu(s) \\
&< \int_G f_V(s)(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}) \; d\mu(s) = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ and $x \in X$ were arbitrary, $\pi \rtimes U$ is nondegenerate. ✓

**Lemma 2.25** *Let $(A, G, \alpha)$ be a dynamical system. Then any covariant representation $(\pi, U)$ induces a covariant representation $(\hat{\pi}, \hat{U})$ of $(A, G, \alpha)$ which is nondegenerate.*

**Proof**. Let $(\pi, U)$ be a covariant representation of $(A, G, \alpha)$ on a Banach space $X$. Define

$$\tilde{X} = \overline{\mathrm{span}}\{\pi(a)x : a \in A, x \in X\},$$

where $\overline{\mathrm{span}}$ is the closed linear span. Then $\tilde{X}$ is a closed invariant subspace for $\pi$. Indeed, let $y \in \tilde{X}$, then

$$y = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(a_i^n)x_i^n,$$

for certain $a_i^n \in A$ and $x_i^n \in X$. So by continuity,

$$\pi(a)y = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(aa_i^n)x_i^n$$

is in $\tilde{X}$ for every $a \in A$. Moreover,

$$\begin{aligned} U_s y &= \lim_{n\to\infty} \sum_{i=1}^{n} U_s\pi(a_i^n)x_i^n \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi(\alpha_s(a_i^n))U_s x_i^n \end{aligned}$$

is in $\tilde{X}$ for every $s \in G$, so $\tilde{X}$ is invariant for $U$ as well. Now let $\hat{\pi}$ and $\hat{U}$ be the restrictions of $\pi$ and $U$ to $\tilde{X}$, then $(\hat{\pi}, \hat{U})$ is a nondegenerate covariant representation of $(A, G, \alpha)$ on $\tilde{X}$. ✓

**Lemma 2.26** *Let $(A, G, \alpha)$ be a dynamical system and let $(\pi, U)$ be a covariant representation of $(A, G, \alpha)$ such that the integrated form $\pi \rtimes U$ is well-defined. Then $\pi$ is continuous if and only if $\pi \rtimes U$ is continuous as an operator on $C_c(G, A)$ with the $L^1$-norm and in this case $||\pi \rtimes U|| = ||\pi||$.*

**Proof**. Fix $C \geq 0$. We will show that $||\pi \rtimes U|| \leq C$ if and only if $||\pi|| \leq C$, from which the assertion clearly follows. Suppose first that $||\pi|| \leq C$. Then, for $f \in C_c(G, A)$,

$$\begin{aligned} ||\pi \rtimes U(f)|| &= \sup_{||x||\leq 1} ||\int_G \pi(f(s))U_s \, d\mu(s)x|| \\ &\leq \sup_{||x||\leq 1} \int_G ||\pi(f(s))U_s x|| \, d\mu(s) \\ &\leq \int_G ||\pi(f(s))|| \, d\mu(s) \\ &\leq C||f||_1. \end{aligned}$$

Conversely, suppose that $||\pi \rtimes U|| \leq C$ as an operator on $C_c(G, A)$ with the $L^1$-norm. Fix $x^* \in X$ with $||x^*|| \leq 1$ and $\varepsilon > 0$. By strong continuity of $s \mapsto U_s$, there is a symmetric neighborhood $\tilde{V}$ of $e$ such that if $s \in \tilde{V}$,

$$||U_s x^* - x^*|| < \varepsilon,$$

where we use that $U_e$ is the identity map on $X$. Let $\mathcal{V}$ be the collection of neighborhoods of $e$ in $G$, ordered by reverse inclusion. Define the net $\{f_V\}_{V\in\mathcal{V}}$ by

$f_V = \tilde{f}_V \otimes a$, where $a \in A$ and $\tilde{f}_V \in C_c(G)$ is nonnegative and has integral equal to 1. Notice that by assumption $||\pi \rtimes U(f_V)|| \leq C||f_V||_1 = C||a||$ for all $V \in \mathcal{V}$. Now if $V \subset \tilde{V}$ we have,

$$\begin{aligned} ||\pi(a)\int_G \tilde{f}_V(s)U_s x^* \; d\mu(s) - \pi(a)x^*|| &= ||\pi(a)\int_G \tilde{f}_V(s)(U_s x^* - x^*) \; d\mu(s)|| \\ &\leq ||\pi(a)||\int_G \tilde{f}_V(s)||U_s x^* - x^*|| \; d\mu(s) \\ &\leq ||\pi(a)||\varepsilon. \end{aligned}$$

Thus,

$$\lim ||\pi(a)\int_G \tilde{f}_V(s)U_s x^* d\mu(s)|| = ||\pi(a)x^*||.$$

Now,

$$\begin{aligned} ||\pi \rtimes U(f_V)|| &= ||\int_G \pi(f_V(s))U_s \; d\mu(s)|| \\ &= ||\int_G \pi(a)\tilde{f}_V(s)U_s \; d\mu(s)|| \\ &= ||\pi(a)\int_G \tilde{f}_V(s)U_s \; d\mu(s)|| \\ &= \sup_{||x||\leq 1} ||\pi(a)\int_G \tilde{f}_V(s)U_s x \; d\mu(s)|| \\ &\geq ||\pi(a)\int_G \tilde{f}_V(s)U_s x^* \; d\mu(s)||. \end{aligned}$$

Hence,

$$||\pi(a)\int_G \tilde{f}_V(s)U_s x^* \; d\mu(s)|| \leq C||a||.$$

We obtain by taking limits that $||\pi(a)x^*|| \leq C||a||$ and since $x^* \in X$ with $||x^*|| \leq 1$ was arbitrary we can take the supremum over such $x^*$ to obtain $||\pi(a)|| \leq C||a||$. This completes the proof, as $a \in A$ was arbitrary. ✓

**Lemma 2.27** *Let $(A, G, \alpha)$ be a dynamical system and suppose that $(\pi, U)$ is a covariant representation on $X$ such that $\pi \rtimes U$ is faithful on $C_c(G) \odot A$. Then $\pi : A \to X$ is faithful.*

**Proof**. Suppose that $\pi(a_1) = \pi(a_2)$ for some $a_1, a_2 \in A$. Take $f \in C_c(G)$ such that $f(s^*) = 1$ for some $s^* \in G$. Define $f_1 = f \otimes a_1$ and $f_2 = f \otimes a_2$, then

$$\begin{aligned} \pi \rtimes U(f_1) &= \int_G \pi(f_1(s))U_s \; d\mu(s) \\ &= \pi(a_1)\int_G f(s)U_s \; d\mu(s) \\ &= \pi(a_2)\int_G f(s)U_s \; d\mu(s) \\ &= \pi \rtimes U(f_2). \end{aligned}$$

By assumption $f_1 = f_2$ and by evaluation at $s^*$ we obtain $a_1 = a_2$, $\pi$ is faithful. ✓

As a partial converse of the above lemma we have the following:

**Lemma 2.28** *Let $(A, G, \alpha)$ be dynamical system, suppose that $\pi$ is a faithful, continuous representation of $A$ on a Banach space $X$ and let $1 \leq p < \infty$. If the covariant representation $(\tilde{\pi}, U)$ of $(A, G, \alpha)$ on $L^p(G, X)$ of proposition 2.12 is well-defined, then $\tilde{\pi} \rtimes U$ is faithful as well.*

Let $f \in C_c(G, A)$ be nonzero and pick $r \in G$ such that $f(r) \neq 0$. We will show that $\tilde{\pi} \rtimes U(f) \neq 0$. Since

$$||\tilde{\pi} \rtimes U(f)|| = ||\tilde{\pi} \rtimes U(i_G(r^{-1})f)||$$

we may assume that $r = e$ by replacing $f$ by $i_G(r^{-1})f$ if necessary (see the proof of proposition 3.16). Let $K$ be a compact set such that $\mathrm{supp}(f) \subset K$ and pick $M_K \geq 1$ as in lemma 2.7. Since $\pi$ is faithful, there is some $x \in X$ with $||x|| = 1$ such that $\pi(f(e))x \neq 0$. Fix $\varepsilon > 0$ such that $||\pi(f(e))x|| > \varepsilon$. Now,

$$\begin{aligned} ||\pi(\alpha_r^{-1}(f(s)))x - \pi(f(e))x|| &\leq ||\pi|| \; ||\alpha_r^{-1}(f(s)) - f(e)|| \; ||x|| \\ &\leq ||\pi||(||\alpha_r^{-1}(f(s) - f(e))|| + ||\alpha_r^{-1}(f(e)) - f(e)||) \\ &\leq M_K||\pi||(||f(s) - f(e)|| + ||\alpha_r^{-1}(f(e)) - f(e)||) \\ &< \varepsilon \end{aligned}$$

for all $s, r$ in a (sufficiently small) neighborhood V of $e$ in $G$, by continuity of $f$ at $e$ and $\alpha_r^{-1}$ at $f(e)$.
We now wish to find $\xi \in L^p(G, X)$ such that the $X$-valued function

$$r \mapsto \tilde{\pi} \rtimes U(f)\xi(r) = \int_G \pi(\alpha_r^{-1}(f(s)))\xi(s^{-1}r) \; d\mu(s)$$

is nonzero on a set of positive measure.
Let $W$ be an open symmetric neighborhood of $e$ such that $W^2 \subset V$. Since every non-empty open set of $G$ has strictly positive Haar measure, $\mu(W) > 0$. Define $\xi \in L^p(G, X)$ by $\xi(r) = 1_V(r)x$. Then,

$$\begin{aligned} \int_G \int_G ||\pi(\alpha_r^{-1}(f(s)))1_V(s^{-1}r)x|| \quad d\mu(s) \quad d\mu(r) &\geq \int_W \int_W ||\pi(\alpha_r^{-1}(f(s)))x|| 1_V(s^{-1}r) \; d\mu(s) \; d\mu(r) \\ &\geq \int_W \int_W | \; ||\pi(f(e))x|| - ||\pi(\alpha_r^{-1}(f(s))) - \pi(f(e))x|| \; | \\ &\quad\; 1_V(s^{-1}r) \; d\mu(s) \; d\mu(r) \\ &> \int_W \int_W (||\pi(f(e))x|| - \varepsilon) 1_V(s^{-1}r) \; d\mu(s) \; d\mu(r) \\ &= (||\pi(f(e))x|| - \varepsilon) \int_W \int_W 1_V(s^{-1}r) \; d\mu(s) \; d\mu(r) \\ &= (||\pi(f(e))x|| - \varepsilon)(\mu(W))^2 > 0. \end{aligned}$$

Hence,

$$r \mapsto \int_G ||\pi(\alpha_r^{-1}(f(s)))1_V(s^{-1}r)x|| \; d\mu(s)$$

is strictly positive on a set of positive $\mu$-measure and thus

$$r \mapsto \int_G \pi(\alpha_r^{-1}(f(s)))1_V(s^{-1}r)x \; d\mu(s)$$

is nonzero on a set of positive $\mu$-measure, as required. ✓

**Remark**. An additional condition on $(A, G, \alpha)$ is needed to ensure that $(\tilde{\pi}, U)$ is a well-defined covariant representation of $(A, G, \alpha)$ on $L^p(G, X)$. It is for example sufficient to assume that the action $\alpha$ is uniformly bounded.

**Theorem 2.29** *Let $(A, G, \alpha)$ be a dynamical system. Define the set of contractive covariant representations $S_c$ by*

$$S_c = \{(\pi, U) : (\pi, U) \text{ is a contractive covariant representation of } (A, G, \alpha)\}.$$

*Let $S \subset S_c$ be non-empty and define for $f \in C_c(G,A)$*

$$||f||_S := \sup\{||\pi \rtimes U(f)|| : (\pi, U) \in S\}.$$

*Then $||\cdot||_S$ defines a submultiplicative semi-norm on $C_c(G,A)$ and $||f||_S \leq ||f||_1$ ($f \in C_c(G,A)$). If moreover*

$$\bigcap_{(\pi,U)\in S} \ker(\pi \rtimes U) = \{0\},$$

*then $||\cdot||_S$ defines a norm on $C_c(G,A)$.*

**Proof**. Let $S$ denote any non-empty set of contractive covariant representations of $(A, G, \alpha)$ and let $f, g \in C_c(G,A)$. Then for any $(\pi, U) \in S$,

$$\begin{aligned} ||\pi \rtimes U(\alpha f)|| &= |\alpha| \; ||\pi \rtimes U(f)||; \\ ||\pi \rtimes U(f+g)|| &\leq ||\pi \rtimes U(f)|| + ||\pi \rtimes U(g)||; \\ ||\pi \rtimes U(f * g)|| &\leq ||\pi \rtimes U(f)|| \; ||\pi \rtimes U(g)||; \\ ||\pi \rtimes U(f)|| &\leq ||f||_1. \end{aligned}$$

Taking the supremum over $S$ in these (in)equalities shows that $||\cdot||_S$ defines a submultiplicative seminorm on $C_c(G,A)$ and $||f||_S \leq ||f||_1$. If $S$ satisfies the above condition, then $||\cdot||_S$ is obviously a norm. ✓

The $S$-norms are attractive for two reasons. First, every $(\pi, U) \in S$ naturally yields a contractive representation $\pi \rtimes U$ of $C_c(G,A)$ with the $||\cdot||_S$-norm and conversely if $\pi \rtimes U$ is contractive on $C_c(G,A)$ with the $||\cdot||_S$-norm then $(\pi, U)$ is contractive, c.f. proposition 2.26. Second, unlike the $||\cdot||_1$-norm, the $S$-norms are submultiplicative on $C_c(G,A)$ for *any* dynamical system $(A, G, \alpha)$. Since $||\cdot||_S \leq ||\cdot||_1$ on $C_c(G,A)$, we can still obtain a bounded approximate unit for the completion of $C_c(G,A)$ with respect to the $S$-norm from a bounded approximate unit for $A$, see theorem 3.1. We are led to the following definition.

**Definition 2.30** Let $(A, G, \alpha)$ be a dynamical system. Suppose that $S$ is a *faithful class of contractive covariant representations* (or *faithful class* for short), i.e. $S$ is non-empty and

$$\bigcap_{(\pi,U)\in S} \ker(\pi \rtimes U) = \{0\}.$$

Then we define the *$S$-crossed product of $A$ by $G$*, denoted by $(A \rtimes_\alpha G)_S$, as the Banach algebra obtained by the completion of $C_c(G,A)$ with respect to the $||\cdot||_S$-norm. If $S = S_c$ is a faithful class we shall simply call this Banach algebra the *crossed product of $A$ by $G$*, denoted by $A \rtimes_\alpha G$, and define the *crossed product norm* by $||\cdot||_c := ||\cdot||_{S_c}$.

Under suitable assumptions on $(A, G, \alpha)$ we can show the existence of a faithful class of contractive covariant representations. If $(A, G, \alpha)$ has a uniformly bounded action, then the integrated form of the regular covariant representation associated to the extended left regular representation is faithful by proposition 2.12 and lemma 2.28. In the following we will simply assume that a faithful class exists.

## 2.2 Comparison with $C^*$-crossed products

The theory presented in the previous section largely follows the lines of the construction of the crossed product for $C^*$-algebras in Williams (2007). In this section we define the $C^*$-crossed product as a particular instance of the $S$-crossed product and refer the reader to Williams for a full exposition. The theorems from the theory of $C^*$-algebras that are used in the following are collected in appendix C.

**Definition 2.31** A $C^*$*-dynamical system* is a triple $(A, G, \alpha)$, where $A$ is a $C^*$-algebra, $G$ is a locally compact topological group and $\alpha : G \to \mathrm{Aut}(A)$ is a strongly continuous action of $G$ on $A$.

A $C^*$-dynamical system is automatically isometric in the sense of definition 2.6, see proposition C.3.
Although the covariant representation of $(A, G, \alpha)$ in definition 2.11 is still well-defined for $C^*$-dynamical systems, we are naturally most interested in covariant representations that preserve the involutive structure of the state space.

**Definition 2.32** Let $(A, G, \alpha)$ be a dynamical system and $H$ a Hilbert space. Then a covariant representation representation $(\pi, U)$ is called a *covariant* $*$*-representation* of $(A, G, \alpha)$ on $H$ if $\pi : A \to B(X)$ is a $*$-representation of $A$ on $H$.

In the literature the group of surjective linear isometries on a Hilbert space $H$ is better known as *the unitary group* and the strongly continuous homomorphism $U : G \to B(H)$ is therefore often called a *unitary representation of* $G$ *on* $H$. Since every unitary $T$ satisfies $T^* = T^{-1}$, the covariance condition for covariant $*$-representations is usually denoted as

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^* \ (a \in A, s \in G).$$

We now wish to define a $C^*$-algebra $A \rtimes_\alpha G$ in the same way as above, by completing $C_c(G, A)$ with respect to a suitable norm. We can define an involution on $C_c(G, A)$ by

$$f^*(s) := \Delta(s^{-1})\alpha_s(f(s^{-1})^*) \ (s \in G)$$

This turns $C_c(G, A)$ into a $*$-algebra. Since a $C^*$-dynamical system is isometric, we obtain from proposition 2.17 that the completion of $C_c(G, A)$ with respect to the $L^1$-norm, $L^1(G, A)$, is a Banach $*$-algebra in this case. In fact, since every $C^*$-algebra has an approximate unit contained in its unit ball and it follows by theorem 2.21 that $L^1(G, A)$ has an approximate unit as well.
Another important consequence of proposition C.3 is that every $*$-representation $\pi$ of $A$ on a Hilbert space is necessarily contractive. Therefore, by lemma 2.26, every covariant $*$-representation of $(A, G, \alpha)$ on a Hilbert space has an integrated form which is $L^1$-norm decreasing. It is also easy to show that the integrated form is $*$-preserving. We can thus define the $C^*$-crossed product as in theorem 2.30, by taking $S$ equal to the set of covariant $*$-representations.

**Definition 2.33** Let $(A, G, \alpha)$ be a $C^*$-dynamical system. Let $S^*$ be the set of covariant $*$-representations of $(A, G, \alpha)$. Then the $C^*$*-crossed product of* $A$ *by* $G$ is defined as the completion of $C_c(G, A)$ in the $C^*$*-crossed product norm* $||\cdot||_* := ||\cdot||_{S^*}$ and is denoted by $(A \rtimes_\alpha G)_{S^*}$.

To make sure that $(A \rtimes_\alpha G)_{S^*}$ is indeed a $C^*$-algebra, there are two things that need to be checked. First, we need to show that $S^*$ is a faithful class and second, we need to show that the involution is isometric with respect to the $||\cdot||_*$-norm and that the $C^*$-rule holds. For the first issue we can use proposition 2.12. Given a $*$-representation $\pi : A \to B(H)$ we obtain a covariant $*$-representation $(\tilde{\pi}, U)$ of $A$ on $L^2(G, H)$, which is a Hilbert space. Furthermore, we can prove that the integrated form $\tilde{\pi} \rtimes U$ is faithful on $C_c(G, A)$ if $\pi$ is a faithful $*$-representation (see lemma 2.28). Hence, we can resolve the issue by showing the existence of a faithful $*$-representation of $A$ on some Hilbert space $H$. This statement is precisely one of the assertions of the famous *Gelfand-Naimark-Segal theorem*.
Once we have established that $||\cdot||_*$ defines a norm on $C_c(G, A)$, it is easy to check that $(A \rtimes_\alpha G)_{S^*}$ satisfies all the properties of a $C^*$-algebra.

# 3 Representation theory of the crossed product

In this section we will focus on dynamical systems $(A, G, \alpha)$ with $A$ a Banach algebra with a bounded approximate unit. Our goal is to establish for every faithful class $S$ an injection of the nondegenerate contractive covariant representations of $S$ into the set of nondegenerate contractive representations of $(A \rtimes_\alpha G)_S$ and to provide sufficient condition for surjectivity of this injective map in case $S = S_c$. The main result of this section is theorem 3.18. At the end of the section we will again compare the results with the theory for $C^*$-crossed products.

## 3.1 Main theorem

An attractive feature of using an $||\cdot||_S$-norm, where $S$ is a faithful class, is that a bounded approximate unit for $A$ gives a bounded approximate unit for the $S$-crossed product. The proof is an easy version of the proof of theorem 2.21. In contrast to the situation for the $L^1$-norm, we do not need additional assumptions on $(A, G, \alpha)$ to ensure that the norm is submultiplicative. We record our important observation in the following theorem.

**Theorem 3.1** *Let $(A, G, \alpha)$ be a dynamical system and suppose that $A$ has an $M$-bounded (left/right) approximate unit. Suppose that $S$ is a faithful class of contractive covariant representations. Then $(A \rtimes_\alpha G)_S$ has an $M$-bounded (left/right) approximate unit as well.*

**Proof**. Suppose $A$ has an $M$-bounded right approximate unit $\{u_i\}$ and let $\{f_{(V,i)}\} \subset C_c(G) \odot A$ be given by lemma 2.20. Fix $\varepsilon > 0$. Let $f \in (A \rtimes_\alpha G)_S$ and pick $\tilde{f} \in C_c(G) \odot A$ such that $||f - \tilde{f}||_S < \min\{\frac{\varepsilon}{3M}, \frac{\varepsilon}{3}\}$. Pick $(\tilde{V}, \tilde{i})$ such that $(\tilde{V}, \tilde{i}) \preceq (V, i)$ implies $||\tilde{f} * f_{(V,i)} - \tilde{f}||_1 < \frac{\varepsilon}{3}$. Then, if $(\tilde{V}, \tilde{i}) \preceq (V, i)$,

$$\begin{aligned} ||f * f_{(V,i)} - f||_S &\leq ||f * f_{(V,i)} - \tilde{f} * f_{(V,i)}||_S + ||\tilde{f} * f_{(V,i)} - \tilde{f}||_S + ||\tilde{f} - f||_S \\ &\leq ||f - \tilde{f}||_S ||f_{(V,i)}||_S + ||\tilde{f} * f_{(V,i)} - \tilde{f}||_1 + ||\tilde{f} - f||_S \\ &< ||f - \tilde{f}||_S ||f_{(V,i)}||_1 + \frac{2}{3}\varepsilon < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\lim ||f * f_{(V,i)} - f||_S = 0$. If $A$ has an $M$-bounded left approximate unit, then we analogously obtain $\lim ||f_{(V,i)} * f - f||_S = 0$ for $f \in (A \rtimes_\alpha G)_S$. ✓

Lemma 3.3 below is instrumental in proving the final assertion of theorem 3.18, that the injection preserves equivalence and irreducibility. First we need the following technical result, which is an adapted version of Dixmier (1977), 2.2.10.

**Lemma 3.2** *Let $A$ be a Banach algebra with a bounded left approximate unit $\{u_i\}$ and let $\pi : A \to B(X)$ be a continuous nondegenerate representation. Then $\pi(u_i) \to I_X$ in the strong operator topology, where $I_X$ is the identity map on $X$.*

**Proof**. Fix $\varepsilon > 0$, let $x \in X$ be arbitrary and let $M \geq 1$ be such that $||u_i|| \leq M$ for all $i \in J$. Since $\pi$ is nondegenerate, there are $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in X$ such that $||x - \sum_{j=1}^n \pi(a_j)x_j|| < \min\{\frac{\varepsilon}{3M||\pi||}, \frac{\varepsilon}{3}\}$. Also, for any $a \in A$ and $y \in X$,

$$||\pi(u_i a)y - \pi(a)y|| \leq ||\pi|| \, ||u_i a - a|| \, ||y|| \to 0.$$

So we can find some $i^*$ such that $i^* \preceq i$ implies

$$||\sum_{j=1}^{n}(\pi(u_i a_j)x_j - \pi(a_j)x_j)|| < \frac{\varepsilon}{3}.$$

Hence, for $i^* \preceq i$,

$$\begin{aligned}||\pi(u_i)x - x|| &\leq ||\pi(u_i)x - \pi(u_i)\sum_{j=1}^{n}\pi(a_j)x_j|| + ||\sum_{j=1}^{n}(\pi(u_ia_j)x_j - \pi(a_j)x_j)|| \\ &\quad + ||\sum_{j=1}^{n}\pi(a_j)x_j - x|| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

where we used that $||\pi(u_i)|| \leq M||\pi||$. ✓

**Lemma 3.3** *Let $(A, G, \alpha)$ be a dynamical system, let $(\pi, U)$ and $(\rho, V)$ be non-degenerate covariant representations on Banach spaces $X$ and $Y$ respectively and suppose that their integrated forms are well-defined. Then $(\pi, U)$ and $(\rho, V)$ are intertwined if and only if $\pi \rtimes U$ and $\rho \rtimes V$ are, and the intertwining operators are the same. Suppose that $A$ has a bounded left approximate unit. Let $S$ be a faithful class and let $(\pi, U) \in S$ be nondegenerate, then $\tilde{X}$ is a closed invariant subspace for $\pi$ and $U$ if and only if $\tilde{X}$ is a closed invariant subspace for $\pi \rtimes U$. In particular, $(\pi, U)$ is topologically irreducible if and only if $\pi \rtimes U$ is.*

**Proof**. Let $\Phi : X \to Y$ be an intertwining operator for $(\pi, U)$ with respect to $(\rho, V)$. Then for any $x \in X$ and $f \in A \rtimes_\alpha G$ we have

$$\begin{aligned}\Phi\pi \rtimes U(f)x &= \Phi\int_G \pi(f(s))U_s x \, d\mu(s) \\ &= \int_G \Phi\pi(f(s))U_s x \, d\mu(s) \\ &= \int_G \rho(f(s))\Phi U_s x \, d\mu(s) \\ &= \int_G \rho(f(s))V_s\Phi x \, d\mu(s) \\ &= \rho \rtimes U(f)\Phi x.\end{aligned}$$

Conversely, if $\Psi : X \to Y$ is an intertwining operator for $\pi \rtimes U$ with respect to $\rho \rtimes V$, then for $x \in X$ and $s \in G$ (c.f. the proof of proposition 3.16)

$$\begin{aligned}\Psi U_s\pi \rtimes U(f)x &= \Psi\pi \rtimes U(i_G(s)f)x \\ &= \rho \rtimes V(i_G(s)f)\Psi x \\ &= V_s\rho \rtimes V(f)\Psi x \\ &= V_s\Psi\pi \rtimes U(f)x.\end{aligned}$$

Since the linear span of $\{\pi \rtimes U(f)x : \ f \in A \rtimes_\alpha G, x \in X\}$ (c.f. lemma 2.24) is dense in $X$, it follows that

$$\Psi U_s = V_s\Psi \ (s \in G).$$

Similarly, using $\pi(a)\pi \rtimes U(f) = \pi \rtimes U(i_A(a)f)$ (see proposition 3.16), we obtain

$$\Psi\pi(a) = \rho(a)\Psi \ (a \in A).$$

This proves the first statement.
Suppose now that $A$ has a bounded left approximate unit and let $S$ be a faithful class. Suppose that $\tilde{X}$ is a closed invariant subspace for $\pi$ and $U$. Then $\pi(f(s))U_s x$

is in $\tilde{X}$ for every $f \in C_c(G,A)$, $x \in \tilde{X}$ and $s \in G$. Fix $x \in \tilde{X}$ and let $f \in C_c(G,A)$. Then there exists a sequence $\{f_n\} \subset L^1(G,X)$ of the form

$$f_n(s) := \sum_{i=1}^{k_n} \pi(f(s_i))U_{s_i} 1_{G_i}(s)x,$$

where $k_n \in \mathbb{N}$, $s_1, \ldots, s_{k_n} \in G$ and $G_i \subset G$ with $\mu(G_i) < \infty$ are such that $f_n(s) \to \pi(f(s))U_s x$, for all $s \in G$. By the dominated convergence theorem we have

$$\int_G f_n(s)\ d\mu(s) \to \pi \rtimes U(f)x,$$

and,

$$\int_G f_n(s)\ d\mu(s) = \sum_{i=1}^{k_n} \pi(f(s_i))U_{s_i}\mu(G_i)x$$

is in $\tilde{X}$ for all $n \in \mathbb{N}$. Since $\tilde{X}$ is closed, we obtain $\pi \rtimes U(f)x \in \tilde{X}$. Now let $g \in (A \rtimes_\alpha G)_S$ and let $\{g_n\} \subset C_c(G,A)$ be such that $||g_n - g||_S \to 0$. Then

$$||\pi \rtimes U(g)x - \pi \rtimes U(g_n)x|| \leq ||\pi \rtimes U(g) - \pi \rtimes U(g_n)||\ ||x|| \leq ||g - g_n||_S\ ||x|| \to 0$$

as $n \to \infty$. Since $\pi \rtimes U(g_n)x \in \tilde{X}$ for all $n$, we obtain $\pi \rtimes U(g)x \in \tilde{X}$. As $x \in \tilde{X}$ was arbitrary, we conclude that $\tilde{X}$ is invariant for $\pi \rtimes U$.
Conversely, suppose that $\tilde{X}$ is invariant for $\pi \rtimes U$. Let $\{f_i\}$ be a bounded left approximate unit for $(A \rtimes_\alpha G)_S$ (c.f. theorem 3.1), then by lemma 3.2 we have $\pi \rtimes U(f_i) \to I_X$ in the strong operator topology, as $\pi \rtimes U$ is nondegenerate. For any $x \in \tilde{X}$,

$$\begin{aligned}\pi(a)x &= \lim \pi(a)\pi \rtimes U(f_i)x \\ &= \lim \pi \rtimes U(i_A(a)f_i)x.\end{aligned}$$

Thus, for all $a \in A$, $\pi(a)x$ is in $\tilde{X}$ as $\tilde{X}$ is closed. Also,

$$\begin{aligned}U_r x &= \lim U_r \pi \rtimes U(f_i)x \\ &= \lim \pi \rtimes U(i_G(r)f_i)x,\end{aligned}$$

so $U_r x$ is in $\tilde{X}$ for all $r \in G$. We conclude that $\tilde{X}$ is invariant for $(\pi, U)$. ✓

**Proposition 3.4** *Let $(A, G, \alpha)$ be a dynamical system and let $(\pi, U)$ be a contractive covariant representation of $(A, G, \alpha)$ such that $\pi \rtimes U$ is isometric on $A \rtimes_\alpha G$. If $\pi$ is not isometric, then $(A, G, \alpha)$ is not isometric.*

**Proof**. Suppose that $||\pi \rtimes U(f)|| = ||f||_c$ for all $f \in A \rtimes_\alpha G$, but $\pi$ is not isometric. Then there is some $\delta > 0$ and $\tilde{a} \in A$ such that $||\pi(\tilde{a})|| < ||\tilde{a}|| - \delta$. Let $\mathcal{V}$ be the collection of neighborhoods of $e$ in $G$. Define, for each $V \in \mathcal{V}$, $f_V \in C_c(G,A)$ by $f_V = \tilde{f}_V \otimes \tilde{a}$, where $\tilde{f}_V \in C_c(G)$ is nonnegative, has its support contained in $V$ and integral equal to 1. Then,

$$\begin{aligned}||\pi \rtimes U(f_V)|| &\leq \sup_{||x|| \leq 1} \int_G ||\pi(f_V(s))U_s x||\ d\mu(s) \\ &\leq \int_G ||\pi(f_V(s))||\ d\mu(s) \\ &= \int_G \tilde{f}_V(s)||\pi(\tilde{a})||\ d\mu(s) \\ &< \int_G \tilde{f}_V(s)(||\tilde{a}|| - \delta)\ d\mu(s) \\ &= (||\tilde{a}|| - \delta).\end{aligned}$$

Suppose now that an isometric covariant representation of $(\rho, \tilde{U})$ of $(A, G, \alpha)$ exists. Then we can pick $\tilde{x}$ such that $||\rho(\tilde{a})\tilde{x}|| > ||\rho(\tilde{a})|| - \frac{\delta}{2}$. Now, by strong continuity of $U$ we can find $\tilde{V} \in \mathcal{V}$ such that $\tilde{V} \preceq V$ implies

$$\begin{aligned} ||\rho \rtimes \tilde{U}(f_V)|| &= ||\rho(\tilde{a}) \int_G \tilde{f}_V(s)\tilde{U}_s\tilde{x} \; d\mu(s)|| \\ &\geq ||\rho(\tilde{a})\tilde{x}|| - \frac{\delta}{2} \\ &> ||\rho(\tilde{a})|| - \delta \\ &= ||\tilde{a}|| - \delta \\ &> ||\pi \rtimes U(f_V)||. \end{aligned}$$

This contradiction shows that no isometric covariant representations exist and hence $(A, G, \alpha)$ is not isometric by proposition 2.15. ✓

**Corollary 3.5** *If $(A, G, \alpha)$ is isometric, then $\pi$ is isometric on $A$ if $\pi \rtimes U$ is isometric on $A \rtimes_\alpha G$.*

In the following we will establish an injection of the nondegenerate contractive covariant representations of $S$ into the set of nondegenerate contractive representations of $(A \rtimes_\alpha G)_S$, when $A$ has a bounded approximate unit and $S$ is a faithful class of contractive covariant representations. This injection will be given by

$$(\pi, U) \mapsto \pi \rtimes U.$$

For $S = S_c$ we will also investigate if this map is surjective. To obtain a covariant representation whose integrated form equals a given representation of $(A \rtimes_\alpha G)_S$ we will need to recover copies of $A$ and $G$ from $(A \rtimes_\alpha G)_S$. Although these are in general not contained in $(A \rtimes_\alpha G)_S$ itself, we can (under suitable conditions) extract copies from the *algebra of double centralizers* of $(A \rtimes_\alpha G)_S$. The following first definition (3.6) and theorem (3.9) are taken from Palmer (1994), definition 1.2.2 and theorem 1.2.4(b).

**Definition 3.6** Let $A$ be a Banach algebra. A *left centralizer* of $A$ is an element $L \in L(A)$ such that

$$L(ab) = L(a)b \text{ (for all } a, b \in A).$$

A *right centralizer* of $A$ is an element $R \in L(A)$ such that

$$R(ab) = aR(b) \text{ (for all } a, b \in A).$$

A *double centralizer* of $A$ is a pair $(L, R)$, where $L$ and $R$ are a left and right centralizer, respectively, satisfying the *double centralizer condition*

$$aL(b) = R(a)b \text{ (for all } a, b \in A).$$

The *algebra $\mathcal{D}(A)$ of double centralizers* of $A$ is the set of double centralizers equipped with pointwise linear operations and multiplication defined by

$$(L_1, R_1)(L_2, R_2) = (L_1L_2, R_2R_1),$$

for any $(L_1, R_1), (L_2, R_2) \in \mathcal{D}(A)$.
The *algebra $\mathcal{D}_B(A)$ of bounded double centralizers* of $A$ is defined by $\mathcal{D}_B(A) = \{(L, R) \in \mathcal{D}(A) : L, R \in B(A)\}$ and given the norm

$$||(L, R)|| = \max\{||L||, ||R||\} \;((L, R) \in \mathcal{D}_B(A)).$$

Notice that $(I, I)$, where $I$ is the identity map on $A$, is the identity element of the (bounded) double centralizer algebra and $||(I, I)|| = 1$.

**Definition 3.7** Let $A$ be a Banach algebra. The *left and right annihilator* of $A$ are defined by:

$$\begin{aligned} \mathcal{A}_A^L &= \{a \in A : ab = 0 \text{ for all } b \in A\} \\ \mathcal{A}_A^R &= \{a \in A : ba = 0 \text{ for all } b \in A\} \end{aligned}$$

If $\mathcal{A}_A^L = \mathcal{A}_A^R = \{0\}$ then $A$ is called *faithful*.

The reason for calling $A$ faithful if the annihilators are zero is that in this case there exists an injective homomorphism of $A$ into $\mathcal{D}_B(A)$, as is shown in the following proposition.

**Proposition 3.8** *Let $A$ be a Banach algebra. Then the map from $A$ into $\mathcal{D}_B(A)$ defined by*

$$a \mapsto (L_a, R_a) \ (a \in A)$$

*with $L_a$ and $R_a$ the left and right regular representation, respectively, defines a homomorphism called the* regular homomorphism*. The regular homomorphism is contractive and maps $A$ onto an ideal in $\mathcal{D}(A)$. It is injective if $A$ is faithful. Suppose that $A$ has an $M$-bounded left or right approximate unit. Then*

$$\frac{||a||}{M} \leq ||(L_a, R_a)|| \leq ||a|| \ (a \in A),$$

*the regular homomorphism is a homeomorphism between $A$ and $\{(L_a, R_a) : a \in A\}$. If $A$ contains a (not necessarily bounded) approximate unit $\{u_i\}$, then $(L_{u_i}, R_{u_i})$ converges to the identity of $\mathcal{D}(A)$ in the strong operator topology.*

**Proof**. Since $L_a$ is a representation and $R_a$ an anti-representation of $A$ on itself, the map $a \mapsto (L_a, R_a)$ is obviously an algebra homomorphism. Clearly, $||L(a)|| \leq ||a||$ and $||R(a)|| \leq ||a||$ $(a \in A)$, so the regular homomorphism is contractive and takes values in $\mathcal{D}_B(A)$. For any $(L, R) \in \mathcal{D}(A)$ and $a, b \in A$ we have,

$$\begin{aligned} L_a L(b) &= aL(b) = R(a)b = L_{R(a)}b \\ LL_a(b) &= L(ab) = L(a)b = L_{L(a)}b \\ R_a R(b) &= R(b)a = bL(a) = R_{L(a)}b \\ RR_a(b) &= R(ba) = bR(a) = R_{R(a)}b. \end{aligned}$$

Hence the regular homomorphism maps $A$ onto an ideal in $\mathcal{D}(A)$. By noticing that the left and right annihilators are exactly the kernels of the left and right regular representation we see that the regular homomorphism is injective if $A$ is faithful. Suppose now that $A$ has an $M$-bounded right approximate unit $\{u_i\}$. Then

$$||L_a|| \geq \frac{||au_i||}{||u_i||} \geq \frac{||au_i||}{M} \ (a \in A).$$

By taking limits we obtain

$$||(L_a, R_a)|| \geq ||L_a|| \geq \frac{||a||}{M} \ (a \in A).$$

Similarly, if $A$ has an $M$-bounded left approximate unit we obtain

$$||(L_a, R_a)|| \geq ||R_a|| \geq \frac{||a||}{M} \ (a \in A).$$

Finally, for any $a \in A$,

$$||(L_{u_i} - I)(a)|| = ||u_i a - a|| \to 0 \ , \ ||(R_{u_i} - I)(a)|| = ||au_i - a|| \to 0$$

so $(L_{u_i}, R_{u_i})$ converges to the identity in the strong operator topology. ✓

**Theorem 3.9** *Let $A$ be a faithful Banach algebra. Then every double centralizer is bounded and $\mathcal{D}(A) = \mathcal{D}_B(A)$ is a Banach algebra under its norm.*

**Proof**. See Palmer (1994), theorem 1.2.4(b). ✓

**Remark**. The class of faithful Banach algebras is fairly broad and includes for example Banach algebras which are semisimple, semiprime or which have an approximate unit (which is not necessarily bounded). If $A$ is a faithful Banach algebra, we will simply denote the algebra of bounded double centralizers on $A$ by $\mathcal{D}(A)$.

**Corollary 3.10** *Let $(A, G, \alpha)$ be a dynamical system, let $S$ be a faithful class of contractive covariant representations and suppose $A$ has a bounded approximate unit. Then $\mathcal{D}((A \rtimes_\alpha G)_S)$ is a unital Banach algebra.*

**Proof**. By theorem 3.1 $(A \rtimes_\alpha G)_S$ has a bounded approximate unit and hence $(A \rtimes_\alpha G)_S$ is a faithful Banach algebra. The statement now follows directly from theorem 3.9. ✓

**Definition 3.11** Let $A$ be a faithful Banach algebra, let $X$ be a Banach space and let $L(X)$ denote the algebra of linear operators on $X$. A representation $\pi$ of $A$ on a Banach space $X$ is called *extendable* if there exists a homomorphism $\overline{\pi} : \mathcal{D}(A) \to L(X)$, called an *extension* of $\pi$, which satisfies

$$\overline{\pi}(L_a, R_a) = \pi(a) \ (a \in A),$$

where $L_a$ and $R_a$ are the left and right regular representation of $a \in A$. An extendable representation $\pi$ is said to be *normed extendable* its extension $\overline{\pi}$ is a representation of the double centralizer algebra $\mathcal{D}(A)$ on $X$, i.e. it takes values in $B(X)$. A normed extendable representation $\pi$ is called *continuously, contractively* or *isometrically extendable* if it has a continuous, contractive or isometric extension to a representation of the double centralizer algebra $\mathcal{D}(A)$ on $X$, respectively.

**Lemma 3.12** *Let $A$ be a faithful Banach algebra and let $\pi$ be an extendable representation of $A$ on a Banach space $X$. Then*

$$\overline{\pi}(L, R)x = \pi(L(c))y \ ((L, R) \in \mathcal{D}(A))$$

*for any $x \in X$ of the form $x = \pi(c)y$ $(c \in A, y \in X)$.*

**Proof**. Suppose $x \in X$ is of the form $x = \pi(c)y$ for some $c \in A$ and $y \in X$. Then for any $(L, R) \in \mathcal{D}(A)$,

$$\begin{aligned} \overline{\pi}(L, R)x &= \overline{\pi}(L, R)\pi(c)y \\ &= \overline{\pi}((L, R)(L_c, R_c))y \\ &= \overline{\pi}(L_{L(c)}, R_{L(c)})y \\ &= \pi(L(c))y. \end{aligned}$$

✓

The following theorem taken from Bonsall & Duncan (1973), theorem 11.10, allows us to deduce some additional results for continuous (normed) extendable representations of approximate unital Banach algebras.

**Theorem 3.13** *Let $A$ be a Banach algebra and let $\pi : A \to B(X)$ be a continuous representation of $A$ on a Banach space $X$. Suppose that there is a net $\{u_i\}$ in $A$ such that $\{\pi(u_i)\}$ is a bounded net that converges to the identity on $X$ in the strong operator topology. Let $x \in X$ and $\delta > 0$. Then there is an $a \in A$ and $y \in X$ such that $x = \pi(a)y$ and $||x - y|| < \delta$.*

Notice that if $A$ is a Banach algebra with a bounded (left) approximate unit and $\pi : A \to B(X)$ is a nondegenerate continuous representation, then the conditions of the theorem are satisfied by lemma 3.2.

**Theorem 3.14** *Let $A$ be a Banach algebra and suppose that $A$ contains a bounded approximate unit. Let $\pi : A \to B(X)$ be a nondegenerate continuous, extendable representation of $A$ on a Banach space $X$. Then its extension is unique, nondegenerate and strongly continuous. If $\pi$ is normed extendable, then its unique extension is continuous.*

**Proof**. Let $\overline{\pi} : \mathcal{D}(A) \to L(X)$ be an extension of $\pi$. By theorem 3.13 every $x \in X$ is of the form $x = \pi(c)y$ for some $c \in A$ and $y \in X$, so by lemma 3.12 the extension $\overline{\pi}$ is unique and it is obviously nondegenerate. Fix $x \in X$ and let $c \in A$ and $y \in X$ be such that $x = \pi(c)y$. Fix $(L, R) \in \mathcal{D}(A)$ and suppose that the sequence $(L_n, R_n)$ converges to $(L, R)$ in $\mathcal{D}(A)$. By lemma 3.12 we have

$$\begin{aligned} ||\overline{\pi}(L,R)x - \overline{\pi}(L_n,R_n)x|| &= ||\pi(L(c))y - \pi(L_n(c))y|| \\ &\leq ||\pi(L(c) - L_n(c))||\;||y|| \\ &\leq ||\pi||\;||(L - L_n)(c)||\;||y|| \\ &\leq ||\pi||\;||L - L_n||\;||c||\;||y|| \\ &\leq ||\pi||\;||(L,R) - (L_n,R_n)||\;||c||\;||y|| \to 0. \end{aligned}$$

Since this holds for every $x \in X$ we conclude that $\overline{\pi}$ is strongly continuous.
Finally, if $\pi$ is normed extendable then its unique extension $\overline{\pi} : \mathcal{D}(A) \to B(X)$ is a strongly continuous representation. By proposition 2.4 it follows that $\overline{\pi}$ is continuous, $\pi$ is continuously extendable. ✓

**Remark**. Notice that the final statement of this theorem does not tell whether or not the operator norm of the representation is preserved, e.g. if the extension of a normed extendable contractive representation is again contractive.

In Palmer (1994), proposition 4.1.16, it is shown that any algebraically irreducible representation of an algebra $A$ has a unique extension to an algebraically irreducible homomorphism of the double centralizer algebra over $A$ into $X$. The following theorem adapts the argument to the situation where $A$ has a bounded approximate to obtain a stronger version of this result for continuous representations.

**Theorem 3.15** *Let $A$ be a Banach algebra with a bounded approximate unit. Then any continuous algebraically cyclic representation $\pi$ of $A$ on $X$ extends uniquely to a strongly continuous algebraically cyclic homomorphism $\overline{\pi} : \mathcal{D}(A) \to L(X)$. In particular, any continuous algebraically irreducible representation $\pi$ of $A$ on $X$ extends uniquely to a strongly continuous algebraically irreducible homomorphism $\overline{\pi} : \mathcal{D}(A) \to L(X)$.*

**Proof**. Let $\pi : A \to B(X)$ be a continuous algebraically cyclic representation and let $\{u_i\}$ be a bounded approximate unit of $A$. For $x, y \in X$ and $b, c \in A$, suppose that $\pi(b)x = \pi(c)y$. Then any $a \in A$ and $(L, R) \in \mathcal{D}(A)$ satisfy

$$\begin{aligned} \pi(a)(\pi(L(b))x - \pi(L(c))y) &= \pi(aL(b))x - \pi(aL(c))y \\ &= \pi(R(a)b)x - \pi(R(a)c)y \\ &= \pi(R(a))(\pi(b)x - \pi(c)y) = 0. \end{aligned}$$

By lemma 3.2 we have (as the above holds for all $a \in A$)

$$0 = \pi(u_i)(\pi(L(b))x - \pi(L(c))y) \to (\pi(L(b))x - \pi(L(c))y).$$

So $\pi(L(b))x = \pi(L(c))y$ for all $(L,R) \in \mathcal{D}(A)$. We can therefore define $\overline{\pi} : \mathcal{D}(A) \to L(X)$ by

$$\overline{\pi}(L,R)x = \pi(L(c))y,$$

where $c \in A$ and $y \in X$ are such that $x = \pi(c)y$. By assumption, there is a fixed vector $y$ such that every $x \in X$ is of this form (for some $c \in A$). Hence, $\overline{\pi}(L,R)$ is well-defined on all of $X$ and it easy to check that $\overline{\pi}$ is a homomorphism of $\mathcal{D}(A)$ into $L(X)$. Since for $a \in A$ and $x \in X$ we have

$$\overline{\pi}(L_a, R_a)x = \pi(L_a(c))y = \pi(a)\pi(c)y = \pi(a)x,$$

$\overline{\pi}$ is an extension of $\pi$. Clearly $y$ is an algebraically cyclic vector for $\overline{\pi}$. The uniqueness and strong continuity of $\overline{\pi}$ follow from theorem 3.14.
The final statement follows from the above, as every non-zero vector of $X$ is an algebraic cyclic vector for an algebraically irreducible representation. ✓

**Remark**. It is tempting to think that the argument in theorem 3.15 together with theorem 3.13 gives an extension of every nondegenerate continuous representation of an algebra $A$ with a bounded approximate unit on a Banach space $X$. Note, however, that although every $x \in X$ has the form $x = \pi(c)y$, the definition $\overline{\pi}(L,R)x = \pi(L(c))y$ does not necessarily give a map that is additive or multiplicative. In the theorem above this a consequence of the fact that $x = \pi(c)y$ for a *fixed* vector $y \in X$.
Also notice that in the theorem above we cannot conclude that the extension is continuous, since $\overline{\pi}(L,R)$ is not necessarily bounded.

**Proposition 3.16** *Let $(A, G, \alpha)$ be a dynamical system, let $S$ be a faithful class of contractive covariant representations and suppose that $A$ has a bounded approximate unit. Then there is a contractive injective homomorphism*

$$(i_A, j_A) : A \to \mathcal{D}((A \rtimes_\alpha G)_S)$$

*such that for $f \in C_c(G, A)$, $a \in A$ and $s \in G$,*

$$i_A(a)f(s) = af(s),\ \ j_A(a)f(s) = f(s)\alpha_s(a).$$

*Furthermore, there is a strongly continuous homomorphism*

$$(i_G, j_G) : G \to \mathcal{D}((A \rtimes_\alpha G)_S)$$

*which is injective if $A \neq 0$, takes values in the group of invertible isometric double centralizers and is such that for $f \in C_c(G, A)$ and $r, s \in G$,*

$$i_G(r)f(s) = \alpha_r(f(r^{-1}s)),\ \ j_G(r)f(s) = \Delta(r^{-1})f(sr^{-1}).$$

*The pair $((i_A, j_A), (i_G, j_G))$ is covariant:*

$$(i_A(\alpha_r(a)), j_A(\alpha_r(a))) = (i_G(r), j_G(r))(i_A(a), j_A(a))(i_G(r), j_G(r))^{-1}\ (r \in G, a \in A).$$

*If $(\pi, U) \in S$ is such that $\pi \rtimes U$ is nondegenerate and extendable, then for all $a \in A$ and $s \in G$,*

$$\overline{\pi \rtimes U}(i_A(a), j_A(a)) = \pi(a)\ ,\ \overline{\pi \rtimes U}(i_G(s), j_G(s)) = U_s.$$

**Proof**. We will first show that $(i_A(a), j_A(a))$ is a double centralizer of $C_c(G, A)$ for any $a \in A$. We have for any $f, g \in C_c(G, A)$,

$$\begin{aligned} i_A(a)(f * g)(s) &= i_A(a) \int_G f(r)\alpha_r(g(r^{-1}s))\, d\mu(r) \\ &= \int_G af(r)\alpha_r(g(r^{-1}s))\, d\mu(r) \\ &= (i_A(a)f) * g(s), \end{aligned}$$

$$\begin{aligned} j_A(a)(f*g)(s) &= j_A(a)\int_G f(r)\alpha_r(g(r^{-1}s))\ d\mu(r) \\ &= \int_G f(r)\alpha_r(g(r^{-1}s))\alpha_s(a)\ d\mu(r) \\ &= \int_G f(r)\alpha_r(g(r^{-1}s)\alpha_{r^{-1}s}(a))\ d\mu(r) \\ &= \int_G f(r)\alpha_r(j_A(a)g(r^{-1}s))\ d\mu(r) \\ &= f*(j_A(a)g)(s), \end{aligned}$$

$$\begin{aligned} f*(i_A(a)g)(s) &= \int_G f(r)\alpha_r(i_A(a)g)(r^{-1}s)\ d\mu(r) \\ &= \int_G f(r)\alpha_r(a)\alpha_r(g(r^{-1}s))\ d\mu(r) \\ &= (j_A(a)f)*g(s). \end{aligned}$$

So $(i_A(a), j_A(a))$ is a double centralizer of $C_c(G,A)$. Moreover, $(i_A, j_A)$ is a homomorphism since for $a,b\in A$ and $f\in C_c(G,A)$,

$$i_A(ab)f(s) = abf(s) = ai_A(b)f(s) = i_A(a)i_A(b)f(s)$$

and

$$j_A(ab)f(s) = f(s)\alpha_s(ab) = f(s)\alpha_s(a)\alpha_s(b) = j_A(a)f(s)\alpha_s(b) = j_A(b)j_A(a)f(s).$$

Moreover, $i_A$ and $j_A$ are contractive on $C_c(G,A)$ with respect to the $||\cdot||_S$-norm. Indeed, if $(\pi,U)\in S$ then

$$\begin{aligned} \pi\rtimes U(i_A(a)f) &= \int_G \pi(af(s))U_s\ d\mu(s) \\ &= \pi(a)\pi\rtimes U(f), \end{aligned}$$

and

$$\begin{aligned} \pi\rtimes U(j_A(a)f) &= \int_G \pi(f(s)\alpha_s(a))U_s\ d\mu(s) \\ &= \int_G \pi(f(s))\pi(\alpha_s(a))U_s\ d\mu(s) \\ &= \int_G \pi(f(s))U_s\pi(a)\ d\mu(s) \\ &= \pi\rtimes U(f)\pi(a). \end{aligned}$$

Hence,

$$||\pi\rtimes U(i_A(a)f)|| \leq ||\pi(a)||\ ||\pi\rtimes U(f)|| \leq ||a||\ ||\pi\rtimes U(f)||.$$

Taking the supremum over all $(\pi,U)\in S$ yields

$$||i_A(a)f||_S \leq ||a||\ ||f||_S\ (f\in C_c(G,A)),$$

and similarly

$$||j_A(a)f||_S \leq ||a||\ ||f||_S\ (f\in C_c(G,A)).$$

We can therefore extend $(i_A(a), j_A(a))$ by continuity to a bounded double centralizer of $(A\rtimes_\alpha G)_S$ and the homomorphism $a\mapsto (i_A(a), j_A(a))$ from $A$ into $\mathcal{D}((A\rtimes_\alpha G)_S)$ is obviously contractive.

Let $(\pi, U) \in S$ be such that $\pi \rtimes U$ is nondegenerate and extendable. Notice that $\pi \rtimes U$ is contractive. For any $x \in X$ of the form $x = \pi \rtimes U(f)y$,

$$\begin{aligned}
\overline{\pi \rtimes U}(i_A(a), j_A(a))x &= \pi \rtimes U(i_A(a)f)y \\
&= \pi(a)\pi \rtimes U(f)y \\
&= \pi(a)x.
\end{aligned}$$

But by theorem 3.13, every $x \in X$ is of this form and we obtain

$$\overline{\pi \rtimes U}(i_A(a), j_A(a)) = \pi(a) \ (a \in A).$$

To show injectivity of the homomorphism $a \mapsto (i_A(a), j_A(a))$, suppose that $(i_A(a), j_A(a)) = (i_A(b), j_A(b))$ for some $a, b \in A$. Then $i_A(a) = i_A(b)$ and hence $i_A(a)f = i_A(b)f$ for all $f \in (A \rtimes_\alpha G)_S$. Take $f = \tilde{f} \otimes c$, where $c \in A$ and $\tilde{f} \in C_c(G)$ satisfies $\tilde{f}(e) = 1$, then evaluation at $e$ gives $(a - b)c = 0$. By replacing $c$ with an approximate unit of $A$ and taking limits, we obtain $a = b$, as required.
We will now prove the assertions about $(i_G, j_G)$. It is easy to see that $(i_G(t), j_G(t))$ is a double centralizer of $C_c(G, A)$ for any $t \in G$, since for any $f, g \in C_c(G, A)$,

$$\begin{aligned}
i_G(t)(f * g)(s) &= i_G(t) \int_G f(r)\alpha_r(g(r^{-1}s)) \, d\mu(r) \\
&= \alpha_t(\int_G f(r)\alpha_r(g(r^{-1}t^{-1}s)) \, d\mu(r)) \\
&= \int_G \alpha_t(f(r))\alpha_{tr}(g((tr)^{-1}s)) \, d\mu(r) \\
&= \int_G \alpha_t(f(t^{-1}r))\alpha_r(g(r^{-1}s)) \, d\mu(r) \\
&= (i_G(t)f) * g(s),
\end{aligned}$$

$$\begin{aligned}
j_G(t)(f * g)(s) &= j_G(t) \int_G f(r)\alpha_r(g(r^{-1}s)) \, d\mu(r) \\
&= \int_G \Delta(t^{-1})f(r)\alpha_r(g(r^{-1}st^{-1})) \, d\mu(r) \\
&= \int_G f(r)\alpha_r(\Delta(t^{-1})g(r^{-1}st^{-1})) \, d\mu(r) \\
&= \int_G f(r)\alpha_r(j_G(t)g(r^{-1}s)) \, d\mu(r) \\
&= f * (j_G(t)g)(s),
\end{aligned}$$

$$\begin{aligned}
f * (i_G(t)g)(s) &= \int_G f(r)\alpha_r((i_G(t)g)(r^{-1}s)) \, d\mu(r) \\
&= \int_G f(r)\alpha_r(\alpha_t(g(t^{-1}r^{-1}s))) \, d\mu(r) \\
&= \int_G f(r)\alpha_{rt}(g((rt)^{-1}s)) \, d\mu(r) \\
&= \int_G \Delta(t^{-1})f(rt^{-1})\alpha_r(g(r^{-1}s)) \, d\mu(r) \\
&= \int_G (j_G(t)f)(r)\alpha_r(g(r^{-1}s)) \, d\mu(r) \\
&= (j_G(t)f) * g(s).
\end{aligned}$$

Also, $r \mapsto (i_G(r), j_G(r))$ is a homomorphism, as

$$\begin{aligned} i_G(rs)f(t) = \alpha_{rs}(f((rs)^{-1}t)) &= \alpha_r(\alpha_s(f(s^{-1}r^{-1}t))) \\ &= \alpha_r((i_G(s)f)(r^{-1}t)) \\ &= i_G(r)(i_G(s)f)(t), \end{aligned}$$

and

$$\begin{aligned} j_G(rs)f(t) = \Delta((rs)^{-1})f(t(rs)^{-1}) &= \Delta(s^{-1})\Delta(r^{-1})f(ts^{-1}r^{-1}) \\ &= \Delta(s^{-1})j_G(r)f(ts^{-1}) \\ &= j_G(s)(j_G(r)f)(t). \end{aligned}$$

Note that by the above we have $i_G(r^{-1}) = i_G(r)^{-1}$ and $j_G(r^{-1}) = j_G(r)^{-1}$.
We will now show that every $(i_G(r), j_G(r))$ extends to an isometric double centralizer of $(A \rtimes_\alpha G)_S$. Let $(\pi, U) \in S$, then for any $f \in C_c(G, A)$

$$\begin{aligned} \pi \rtimes U(i_G(r)f) &= \int_G \pi(i_G(r)f(s))U_s \ d\mu(s) \\ &= \int_G \pi(\alpha_r(f(r^{-1}s)))U_s \ d\mu(s) \\ &= \int_G \pi(\alpha_r(f(s)))U_{rs} \ d\mu(s) \\ &= \int_G U_r\pi(f(s))U_s \ d\mu(s) \\ &= U_r\pi \rtimes U(f), \end{aligned}$$

and

$$\begin{aligned} \pi \rtimes U(j_G(r)f) &= \int_G \pi(j_G(r)f(s))U_s \ d\mu(s) \\ &= \int_G \pi(\Delta(r^{-1})f(sr^{-1}))U_s \ d\mu(s) \\ &= \int_G \pi(f(s))U_{sr} \ d\mu(s) \\ &= \pi \rtimes U(f)U_r. \end{aligned}$$

Since $U_r$ is a surjective isometry we obtain

$$||\pi \rtimes U(i_G(r)f)|| = ||\pi \rtimes U(f)||, \ ||\pi \rtimes U(j_G(r)f)|| = ||\pi \rtimes U(f)||.$$

Taking the supremum over all $(\pi, U) \in S$ yields

$$||i_G(r)f||_S = ||f||_S, \ ||j_G(r)f||_S = ||f||_S \ (f \in C_c(G, A)).$$

We can therefore extend $(i_G(r), j_G(r))$ to an isometric double centralizer of $(A \rtimes_\alpha G)_S$.
Let $(\pi, U) \in S$ be such that $\pi \rtimes U$ is nondegenerate and extendable. For any $x \in X$ of the form $x = \pi \rtimes U(f)y$,

$$\begin{aligned} \overline{\pi \rtimes U}(i_G(r), j_G(r))x &= \pi \rtimes U(i_G(r)f)y \\ &= U_r\pi \rtimes U(f)y \\ &= U_r x. \end{aligned}$$

By theorem 3.13, every $x \in X$ is of this form and we obtain

$$\overline{\pi \rtimes U}(i_G(r), j_G(r)) = U_r \ (r \in G).$$

Suppose now that $(i_G(r), j_G(r)) = (i_G(s), j_G(s))$ for some $r, s \in G$. Then $i_G(rs^{-1}) = i_G(e)$ and hence, for all $f \in C_c(G, A)$,

$$\alpha_{rs^{-1}}(f(sr^{-1}t)) = f(t) \ (t \in G).$$

Suppose now that $A \neq 0$ and that $s \neq r$. Then $rs^{-1} \neq e$ and we can find an open neighborhood $V$ of $e$ in $G$ such that $rs^{-1} \notin V$. By Urysohn's lemma there exists a $\tilde{f} \in C_c(G)$ such that $0 \leq \tilde{f}(t) \leq 1$ for all $t \in G$, $\tilde{f}(e) = 1$ and $\tilde{f}(t) = 0$ if $t \notin V$. Let $a \in A$ be given and define $f \in C_c(G) \odot A$ by $f = \tilde{f} \otimes \alpha_{sr^{-1}}(a)$. If we now evaluate the above equation at $t = rs^{-1}$ we obtain $a = 0$. But this holds for any $a \in A$, contrary to our assumption. This contradiction shows that we must have $r = s$ if $A \neq 0$, the homomorphism $r \mapsto (i_G(r), j_G(r))$ is injective in this case.
It will now be shown that it is also strongly continuous, first when the image maps are restricted to $C_c(G, A)$ with the $||\cdot||_S$-norm and a fortiori when they are not restricted (i.e. on all of $(A \rtimes_\alpha G)_S$). Moreover, it suffices to show strong continuity at $e$. Indeed,

$$||i_G(r)f - i_G(s)f||_S = ||i_G(s^{-1})(i_G(r)f - i_G(s)f)||_S = ||i_G(s^{-1}r)f - f||_S;$$

$$||j_G(r)f - j_G(s)f||_S = ||j_G(s^{-1})(j_G(r)f - j_G(s)f)||_S = ||j_G(rs^{-1})f - f||_S.$$

So fix $f \in C_c(G, A)$ and $\varepsilon > 0$. We may assume that $f \neq 0$. Let $W$ be a compact symmetric neighborhood of $e$ in $G$, then $K := W\text{supp}(f)W$ is compact (and hence of finite measure) and if $r \in W$,

$$\text{supp}(i_G(r)f) \subset K, \ \text{supp}(j_G(r)f) \subset K.$$

Pick $M_W \geq 1$ as in lemma 2.7. By uniform continuity of $f$ we can find a neighborhood $V \subset W$ of $e$ in $G$ such that if $r \in V$ then for all $s \in G$

$$||f(r^{-1}s) - f(s)|| < \frac{\varepsilon}{2M_W\mu(K)}, \ ||f(sr^{-1}) - f(s)|| < \frac{\varepsilon}{2\mu(K)}.$$

Shrinking $V$ if necessary, $r \in V$ also implies by continuity of $\Delta$ that

$$|\Delta(r^{-1}) - 1| < \frac{\varepsilon}{2||f||_\infty \mu(K)}$$

and also for all $s \in G$,

$$||\alpha_r(f(s)) - f(s)|| < \frac{\varepsilon}{2\mu(K)}.$$

The last assertion can be seen as follows. We have for any $s \in G$,

$$\begin{aligned} ||\alpha_r(f(s)) - f(s)|| &\leq ||\alpha_r(f(s) - f(s^*))|| + ||\alpha_r(f(s^*)) - f(s^*)|| + ||f(s^*) - f(s)|| \\ &\leq (M_W + 1)||f(s) - f(s^*)|| + ||\alpha_r(f(s^*)) - f(s^*)||. \end{aligned}$$

Since $f$ has compact support, for every $\delta > 0$ there are finitely many $s_j \in G$ such that for every $s \in G$ there is some $s_j$ such that $||f(s) - f(s_j)|| < \delta$. For details see the proof of theorem 2.21.
Now,

$$||i_G(r)f(s) - f(s)|| \leq ||\alpha_r(f(r^{-1}s) - f(s))|| + ||\alpha_r(f(s)) - f(s)||$$

and we obtain $||i_G(r)f - f||_S \leq ||i_G(r)f - f||_1 < \varepsilon$ for $r \in V$.
Similarly, for any $r \in V$

$$\begin{aligned} ||j_G(r)f(s) - f(s)||_S &\leq ||j_G(r)f(s) - f(s)||_1 \\ &= \int_G ||\Delta(r^{-1})f(sr^{-1}) - f(s)|| \, d\mu(s) \\ &\leq \int_K ||f||_\infty |\Delta(r^{-1}) - 1| + ||f(sr^{-1}) - f(s)|| \, d\mu(s) < \varepsilon. \end{aligned}$$

To complete the proof it remains to show that $((i_A, j_A), (i_G, j_G))$ is a covariant pair. We have for any $r \in G, a \in A$ and $f \in C_c(G, A)$

$$\begin{aligned} i_G(r)i_A(a)i_G(r)^{-1}f(s) &= i_G(r)i_A(a)\alpha_{r^{-1}}(f(rs)) \\ &= i_G(r)a\alpha_{r^{-1}}(f(rs)) \\ &= \alpha_r(a)f(s) = i_A(\alpha_r(a))f(s), \end{aligned}$$

and moreover

$$\begin{aligned} j_G(r^{-1})j_A(a)j_G(r)f(s) &= \Delta(r^{-1})j_G(r^{-1})j_A(a)f(sr^{-1}) \\ &= \Delta(r^{-1})j_G(r^{-1})f(sr^{-1})\alpha_s(a) \\ &= \Delta(r)\Delta(r^{-1})f(srr^{-1})\alpha_{sr}(a) \\ &= f(s)\alpha_s(\alpha_r(a)) = j_A(\alpha_r(a))f(s). \end{aligned}$$

Hence,

$$\begin{aligned} (i_G(r), j_G(r))(i_A(a), j_A(a))(i_G(r), j_G(r))^{-1} &= (i_G(r)i_A(a), j_A(a)j_G(r))(i_G(r)^{-1}, j_G(r)^{-1}) \\ &= (i_G(r)i_A(a)i_G(r)^{-1}, j_G(r)^{-1}j_A(a)j_G(r)) \\ &= (i_A(\alpha_r(a)), j_A(\alpha_r(a))). \end{aligned}$$

Since $C_c(G, A)$ is dense in $(A \rtimes_\alpha G)_S$, our proof is complete. ✓

The following lemma provides an explicit expression for a 'copy' of $C_c(G, A)$ in $\mathcal{D}((A \rtimes_\alpha G)_S)$.

**Lemma 3.17** *Let $(A, G, \alpha)$ be a dynamical system, let $S$ be a faithful class of contractive covariant representations and suppose that $A$ has a bounded approximate unit. Let $g \in C_c(G, A)$. Then*

$$\int_G (i_A(g(s)), j_A(g(s)))(i_G(s), j_G(s))\, d\mu(s) = (L_g, R_g).$$

**Proof**. We have

$$\int_G (i_A(g(s)), j_A(g(s)))(i_G(s), j_G(s))\, d\mu(s) = \int_G (i_A(g(s))i_G(s), j_G(s)j_A(g(s)))\, d\mu(s).$$

We will first show that

$$\int_G (i_A(g(s))i_G(s), j_G(s)j_A(g(s)))\, d\mu(s) = (\int_G i_A(g(s))i_G(s)\, d\mu(s), \int_G j_G(s)j_A(g(s))\, d\mu(s)).$$

So let $(f_1, f_2) \in (A \rtimes_\alpha G)_S \oplus (A \rtimes_\alpha G)_S$ be arbitrary, where

$$(A \rtimes_\alpha G)_S \oplus (A \rtimes_\alpha G)_S := \{(f, g) \in (A \rtimes_\alpha G)_S \times (A \rtimes_\alpha G)_S; ||(f, g)|| := \max\{||f||_S, ||g||_S\}\}$$

is the direct sum with the supremum norm. This is a Banach space as $(A \rtimes_\alpha G)_S$ is, see Conway (1985), proposition 4.4.
Notice that the maps

$$s \mapsto i_A(g(s))i_G(s)f_1 \text{ and } s \mapsto j_G(s)j_A(g(s))f_2$$

are continuous and have compact support contained in $\mathrm{supp}(g)$. Hence for every $k \in \mathbb{N}$ there exists a finite number of points $s_1, \ldots, s_{n(k)}$ in $\mathrm{supp}(g)$ and open neighborhoods $G_1, \ldots, G_{n(k)}$ of these points in $G$ which form an open cover of $\mathrm{supp}(g)$ and are such that

$$||i_A(g(s))i_G(s)f_1 - i_A(g(s_i))i_G(s_i)f_1||_S < \frac{1}{k} \text{ for all } s \in G_i$$

and

$$||j_G(s)j_A(g(s))f_2 - j_G(s_i)j_A(g(s_i))f_2||_S < \frac{1}{k} \text{ for all } s \in G_i.$$

Define the step functions $\xi_k, \eta_k \in L^1(G, (A \rtimes_\alpha G)_S)$ by

$$\xi_k(s) = \sum_{i=1}^{n(k)} i_A(g(s_i))i_G(s_i)f_1 1_{G_i}(s),$$

$$\eta_k(s) = \sum_{i=1}^{n(k)} j_G(s_i)j_A(g(s_i))f_2 1_{G_i}(s).$$

Since we can do this for every $k \in \mathbb{N}$, we obtain two sequences $\{\xi_k\}, \{\eta_k\}$ which satisfy

$$\xi_k(s) \to i_A(g(s))i_G(s)f_1 \ , \ \eta_k(s) \to j_G(s)j_A(g(s))f_2,$$

but also,

$$(\xi_k(s), \eta_k(s)) \to (i_A(g(s))i_G(s)f_1, j_G(s)j_A(g(s))f_2)$$

in $(A \rtimes_\alpha G)_S \oplus (A \rtimes_\alpha G)_S$. By the Dominated Convergence theorem we easily obtain

$$\int_G (\xi_k(s), \eta_k(s)) \ d\mu(s) \to \int_G (i_A(g(s))i_G(s), j_G(s)j_A(g(s))) \ d\mu(s),$$

and,

$$(\int_G \xi_k(s) \ d\mu(s), \int_G \eta_k(s) \ d\mu(s)) \to (\int_G i_A(g(s))i_G(s) \ d\mu(s), \int_G j_G(s)j_A(g(s)) \ d\mu(s)).$$

But,

$$\begin{aligned}
(\int_G \xi_k(s) \ d\mu(s), \int_G \eta_k(s) \ d\mu(s)) &= (\sum_{i=1}^{n(k)} i_A(g(s_i))i_G(s_i)f_1\mu(G_i), \sum_{i=1}^{n(k)} j_G(s_i)j_A(g(s_i))f_2\mu(G_i)) \\
&= \sum_{i=1}^{n(k)} \mu(G_i)(i_A(g(s_i))i_G(s_i)f_1, j_G(s_i)j_A(g(s_i))f_2) \\
&= \sum_{i=1}^{n(k)} \int_G (i_A(g(s_i))i_G(s_i)f_1, j_G(s_i)j_A(g(s_i))f_2)1_{G_i}(s) \ d\mu(s) \\
&= \int_G (\xi_k(s), \eta_k(s)) \ d\mu(s).
\end{aligned}$$

This proves our claim.
Let any $f \in C_c(G, A)$ be given. Then for any $r \in G$,

$$\int_G i_A(g(s))i_G(s)f(r) \ d\mu(s) = \int_G g(s)\alpha_s(f(s^{-1}r)) \ d\mu(s) = g * f(r),$$

and,

$$\begin{aligned}
\int_G j_G(s)j_A(g(s))f(r) \ d\mu(s) &= \int_G j_G(s)f(r)\alpha_r(g(s)) \ d\mu(s) \\
&= \int_G \Delta(s^{-1})f(rs^{-1})\alpha_{rs^{-1}}(g(s)) \ d\mu(s) \\
&= \int_G f(rs)\alpha_{rs}(g(s^{-1})) \ d\mu(s) \\
&= \int_G f(s)\alpha_s(g(s^{-1}r)) \ d\mu(s) \\
&= f * g(r).
\end{aligned}$$

Therefore,

$$\int_G i_A(g(s))i_G(s)\ d\mu(s)(f) = L_g(f)\ ,\ \int_G j_G(s)j_A(g(s))\ d\mu(s)(f) = R_g(f).$$

This completes the proof. ✓

**Theorem 3.18** *Let $(A, G, \alpha)$ be a dynamical system, let $S$ be a faithful class of contractive covariant representations and suppose that $A$ has a bounded approximate unit. Then the map*

$$(\pi, U) \mapsto \pi \rtimes U$$

*gives an injection of the nondegenerate covariant representations of $S$ into the set of nondegenerate contractive representations of $(A \rtimes_\alpha G)_S$. The injection preserves equivalence and topological irreducibility.*
*Every nondegenerate contractively extendable representation of $(A \rtimes_\alpha G)_S$ is of the form $\pi \rtimes U$ for some nondegenerate contractive covariant representation $(\pi, U)$ of $(A, G, \alpha)$. In particular, if $S_c$, the set of all contractive covariant representations of $(A, G, \alpha)$, is a faithful class, then the injection has all nondegenerate contractively extendable representations of $A \rtimes_\alpha G$ in its image.*

**Proof**. We will first show injectivity. Suppose that $\pi \rtimes U = \tilde{\pi} \rtimes \tilde{U}$ for $(\pi, U), (\tilde{\pi}, \tilde{U}) \in S$ nondegenerate contractive covariant representations of $(A, G, \alpha)$ on the same space $X$. Then for any $a \in A$ and $f \in (A \rtimes_\alpha G)_S$,

$$\begin{aligned}\pi(a)\pi \rtimes U(f) &= \pi \rtimes U(i_A(a)f)\\ &= \tilde{\pi} \rtimes \tilde{U}(i_A(a)f)\\ &= \tilde{\pi}(a)\tilde{\pi} \rtimes \tilde{U}(f)\\ &= \tilde{\pi}(a)\pi \rtimes U(f).\end{aligned}$$

Hence $(\pi(a) - \tilde{\pi}(a))\pi \rtimes U(f) = 0$ for all $f \in (A \rtimes_\alpha G)_S$. By nondegeneracy of $\pi \rtimes U$ we obtain $\pi(a) = \tilde{\pi}(a)$ for all $a \in A$. Analogously we have

$$U_r\pi \rtimes U(f) = \tilde{U}_r\pi \rtimes U(f)\ (r \in G, f \in (A \rtimes_\alpha G)_S),$$

so $U_r = \tilde{U}_r$ for all $r \in G$.
The second assertion follows directly from lemma 3.3.
To prove the final statement, let $T : (A \rtimes_\alpha G)_S \to B(X)$ be any nondegenerate, contractively extendable representation. Supposing for the moment that $T = \pi \rtimes U$, we know by proposition 3.16 that

$$\overline{\pi \rtimes U}(i_A(a), j_A(a)) = \pi(a)\ ,\ \overline{\pi \rtimes U}(i_G(r), j_G(r)) = U_r\ (r \in G, a \in A).$$

So let us define

$$\pi(a) := \overline{T}(i_A(a), j_A(a))\ ,\ U_r := \overline{T}(i_G(r), j_G(r))\ (a \in A, r \in G),$$

where $\overline{T}$ is the unique extension of $T$ to a nondegenerate contractive representation of $\mathcal{D}((A \rtimes_\alpha G)_S)$.
We will first show that $(\pi, U)$ is a nondegenerate contractive covariant representation of $(A, G, \alpha)$ on $X$. We have by proposition 3.16,

$$\begin{aligned}U_r\pi(a)U_r^{-1} &= \overline{T}((i_G(r), j_G(r))(i_A(a), j_A(a))(i_G(r), j_G(r))^{-1})\\ &= \overline{T}(i_A(\alpha_r(a)), j_A(\alpha_r(a)))\\ &= \pi(\alpha_r(a)),\end{aligned}$$

so $(\pi, U)$ is a covariant pair. Moreover, since $i_G : G \to \mathcal{D}((A \rtimes_\alpha G)_S)$ is strongly continuous and $T$ is nondegenerate and contractive, $U$ is strongly continuous. Indeed, let $x \in X$ be of the form $x = T(f)y$ for some $f \in (A \rtimes_\alpha G)_S$ and $y \in X$, then

$$\begin{aligned} ||U_r x - U_s x|| &= ||\overline{T}(i_G(r), j_G(r))x - \overline{T}(i_G(s), j_G(s))x|| \\ &= ||T(i_G(r)f)y - T(i_G(s)(f))y|| \\ &= ||T((i_G(r) - i_G(s))(f))y|| \\ &\leq ||T||\ ||(i_G(r) - i_G(s))(f)||_S\ ||y||. \end{aligned}$$

The final expression can be made arbitrarily small by choosing $r \in G$ in a small enough neighborhood of $s$, by strong continuity of $i_G$. By nondegeneracy of $T$ the linear combinations of the elements $x = T(f)y$ are dense in $X$ and the result readily follows. Obviously, $\pi$ is contractive as $(i_A, j_A)$ and $\overline{T}$ are contractive. It is also clear that $U$ is an isometric representation of $G$ as $\overline{T}$ is contractive.
To prove nondegeneracy of $\pi$, let $\{u_i\}_{i\in J}$ be a bounded approximate unit in A and fix $\varepsilon > 0$. Let $M \geq 1$ be such that $||u_i|| \leq M$ for all $i \in J$ and let $x \in X$ be of the form $x = T(f)y$. Pick $\tilde{f} \in C_c(G) \odot A$ such that $||f - \tilde{f}||_S < \frac{\varepsilon}{3M||y||}$. Subsequently we can find $i^* \in J$ such that $i^* \preceq i$ implies $||u_i\tilde{f} - \tilde{f}||_1 < \frac{\varepsilon}{3||y||}$. Hence if $i^* \preceq i$,

$$\begin{aligned} ||\overline{T}(i_A(u_i), j_A(u_i))x - x|| &= ||T(i_A(u_i)(f))y - T(f)y|| \\ &\leq ||T||\ ||i_A(u_i)f - f||_S\ ||y|| \\ &\leq ||y||(||i_A(u_i)f - i_A(u_i)\tilde{f}||_S + ||i_A(u_i)\tilde{f} - \tilde{f}||_S + ||\tilde{f} - f||_S) \\ &\leq ||y||((M+1)||f - \tilde{f}||_S + ||u_i\tilde{f} - \tilde{f}||_1) < \varepsilon. \end{aligned}$$

Since $T$ is nondegenerate, it follows that $\overline{T}(i_A(u_i), j_A(u_i))$ converges strongly to $I_X$ and we conclude that $\pi$ is nondegenerate.
Finally, we need to show that $T$ coincides with $\pi \rtimes U$. Let $g \in C_c(G, A)$, then by lemma 3.17 we have

$$\begin{aligned} \pi \rtimes U(g) &= \int_G \pi(g(s))U_s\ d\mu(s) \\ &= \int_G \overline{T}(i_A(g(s)), j_A(g(s)))\overline{T}(i_G(s), j_G(s))\ d\mu(s) \\ &= \int_G \overline{T}((i_A(g(s)), j_A(g(s)))(i_G(s), j_G(s)))\ d\mu(s) \\ &= \overline{T}\int_G (i_A(g(s)), j_A(g(s)))(i_G(s), j_G(s))\ d\mu(s) \\ &= \overline{T}(L_g, R_g) \\ &= T(g). \end{aligned}$$

Since $C_c(G, A)$ is dense in $(A \rtimes_\alpha G)_S$ and $T$ and $\pi \rtimes U$ are contractive, it easily follows that $T$ coincides with $\pi \rtimes U$. ✓

**Remark**. To obtain a surjectivity result for a faithful class $S$ we need to show that the contractive covariant representation $(\pi, U)$ that is constructed using $T$ is again in $S$. We could accomplish this by assuming that $S$ is saturated, in the sense that if a contractive covariant representation of $(A, G, \alpha)$ on a certain Banach space $X$ is in $S$, then all contractive covariant representations of $(A, G, \alpha)$ on $X$ are in $S$. Alternatively, we can give a direct proof that $(\pi, U)$ is in $S$ by using the structure of $S$. For example, in the case of a $C^*$-dynamical system we show that a nondegenerate $*$-representation $T$ of the crossed product induces a covariant $*$-representation of the dynamical system.

## 3.2 Comparison with $C^*$-crossed products

We have seen in section 2.2 that we can view the $C^*$-crossed product as a special case of the crossed product defined in theorem 2.30, by taking a $C^*$-algebra $A$ as the dynamical system $(A, G, \alpha)$ and by defining the $C^*$-crossed product norm using the faithful class of representations

$$S^* := \{(\pi, U) : (\pi, U) \text{ is a covariant } *-\text{representation of } (A, G, \alpha)\}.$$

We can now combine the results in the previous section with results from the theory of $C^*$-algebras to obtain a classic version of theorem 3.18 for $C^*$-dynamical systems. Notice first that all requirements of theorem 3.18 are satisfied, $(A, G, \alpha)$ is automatically isometric, $A$ always has an approximate unit contained in its closed unit ball and $S^*$ is a faithful class. By proposition C.3 we also find that every $*$-representation of $(A \rtimes_\alpha G)_{S^*}$ is contractive. It is easy to see that the operation

$$(L, R) \mapsto (R^*, L^*) \ ((L, R) \in \mathcal{D}((A \rtimes_\alpha G)_{S^*}))$$

with $L^*(a) := (L(a^*))^*$, $R^*(a) := (R(a^*))^*$, defines an involution on $\mathcal{D}((A \rtimes_\alpha G)_{S^*})$ which turns it into a $C^*$-algebra (see Murphy (1990), theorem 2.1.5). Furthermore, since $(A \rtimes_\alpha G)_{S^*}$ is a closed (two-sided) ideal in its double centralizer algebra $\mathcal{D}((A \rtimes_\alpha G)_{S^*})$, it follows by theorem C.5 that *every* nondegenerate $*$-representation of $(A \rtimes_\alpha G)_{S^*}$ is contractively extendable to a $*$-representation of $\mathcal{D}((A \rtimes_\alpha G)_{S^*})$. It is easy to check that $\pi = \overline{T}(i_A, j_A)$ (in the notation of the proof of theorem 3.18) is $*$-preserving. We have obtained the following theorem.

**Theorem 3.19** *Let $(A, G, \alpha)$ be a $C^*$-dynamical system. Then*

$$(\pi, U) \leftrightarrow \pi \rtimes U$$

*is a bijective correspondence between the nondegenerate covariant $*$-representations of $(A, G, \alpha)$ and the nondegenerate $*$-representations of $(A \rtimes_\alpha G)_{S^*}$. This bijection preserves equivalence and topological irreducibility.*

# 4 Open Questions

As we conclude this thesis, many questions remain unanswered. This section summarizes the main unresolved issues and provides some topics for further research.
The most pressing question at hand is on theorem 3.18. As it stands we have obtained an injection $(\pi, U) \mapsto \pi \rtimes U$ of the nondegenerate contractive covariant representations in $S$ into the collection of nondegenerate contractive representations of $(A \rtimes_\alpha G)_S$. We would like to obtain the sort of bijection as in theorem 3.19. If we continue investigating the surjectivity of the map along the lines set out in the previous section, the main obstruction is the extendability of representations. There are two approaches we can take to solve this issue. First, we can investigate the theory of extendability in general:

- Given a faithful Banach algebra $A$, which representations are extendable to $\mathcal{D}(A)$? Which are normed extendable? Are there counterexamples of representations that are not extendable?
- If a continuous representation is normed extendable, is its extension then also continuous? When is a representation contractively extendable?
- Are the classes of Banach algebras for which every representation (in a certain class) is normed extendable?
- Can we put restrictions on $A$, $G$ and the action $\alpha$ to ensure that $(A \rtimes_\alpha G)_S$ belongs to such a class? A rather trivial example is the case where $G$ is discrete and $A$ is unital, in which case $(A \rtimes_\alpha G)_S$ is unital (for every faithful class $S$) and coincides with its double centralizer algebra.

We can also take a more hands-on approach by sharpening the proof of theorem 3.18:

- Can we loosen the restrictions on the representation $T$ of $(A \rtimes_\alpha G)_S$ and still obtain the covariant representation $(\pi, U)$? For example, is it sufficient if $T$ has a strongly continuous (not necessarily normed) extension?
- Is it sufficient if $T$ is continuously extendable? We suspect that if $A$ has an approximate unit contained in its closed unit ball, then it is possible to show that $(\pi, U)$ is a contractive covariant representation. We could still show that $T = \pi \rtimes U$ and use arguments along the lines of the proof of lemma 2.26 to show that $\pi$ is contractive and $U$ is isometric.

We could also abandon the approach using the double centralizer algebra:

- Can we obtain a covariant representation $(\pi, U)$ from a representation $T$ of $(A \rtimes_\alpha G)_S$, such that $T = \pi \rtimes U$, without using an extension of $T$ to the double centralizer algebra? In Pedersen (1979) theorem 3.19 of this thesis is proved without using extensions of representations of the crossed product to its double centralizer algebra. There is a strong indication that this approach may work for representations on reflexive Banach spaces. We refer to theorem 7.6.4 and proposition 3.12.3 of this book.

Another open issue is the definition of the crossed product. Currently we have obtained an $S$-crossed product for any faithful class $S$ of contractive covariant representations of $(A, G, \alpha)$. For further development of the theory it is important to settle on one definition of *the* crossed product of $A$ by $G$.

- Given a dynamical system $(A, G, \alpha)$, is there a natural choice for the faithful class of contractive covariant representations $S$? In case $A$ is a $C^*$-algebra, it is natural to consider convariant representations $(\pi, U)$ in which $\pi$ is $*$-preserving, hence the choice $S = S^*$ in the definition.

- What restriction should we put on $S$ to ensure that we obtain a bijection in theorem 3.18 (see the remark following the theorem)?

- In special cases, there may be other Banach algebras to consider as a candidate for the definition for the crossed product. For example: if $(A, G, \alpha)$ satisfies the condition in theorem 2.17, is there a bijection between the nondegenerate contractive covariant representations of $(A, G, \alpha)$ and the nondegenerate contractive representations of $L^1(G, A)$?

- Are there other norms that we can consider for the completion of $C_c(G, A)$? Any candidate must be submultiplicative on $C_c(G, A)$ to ensure that the completion is a Banach algebra. In this thesis we have only considered the $|| \cdot ||_1$ and the $|| \cdot ||_S$ norms on $C_c(G, A)$.

- Once we have settled on a definition of the crossed product for a certain class of dynamical systems $(A, G, \alpha)$, can we prove results which are analogous to or generalizations of ones for $C^*$-crossed products? See Williams (2007) for many desired results.

A related issue is of course the existence of a faithful class $S$ of contractive covariant representations for a given dynamical system $(A, G, \alpha)$.

- Given a dynamical system $(A, G, \alpha)$, is there a faithful class of contractive covariant representations of $(A, G, \alpha)$?

- Is there a contractive covariant representation $(\pi, U)$ such that $\pi \rtimes U$ is faithful?

- In the case that $\alpha$ is not isometric, can we, given a representation $\pi$ of $A$, construct a contractive covariant representation $(\tilde{\pi}, U)$ such that $\tilde{\pi} \rtimes U$ is faithful? If $\alpha$ is uniformly bounded by $M$ it is sufficient to find a faithful representation $\pi$ with $||\pi|| \leq \frac{1}{M}$ (c.f. proposition 2.12 and lemma 2.28).

- If $\alpha$ is not uniformly bounded, it is still uniformly bounded on compact sets. Is this sufficient to prove the existence of a contractive covariant representation? One idea is to 'localize' the approach in proposition 2.12, e.g. to construct a covariant representation on $L^p(H, X)$, where $H$ is a compact normal subgroup of $G$.

Once we have settled on a definition of the crossed product, we can look for a characterization of the crossed product in terms of a universal property.

- Can the crossed product be characterized in terms of a universal property, such as Iain Raeburn's characterization for $C^*$-crossed products (see Williams (2007), theorem 2.61)?

The questions may prove very difficult to answer for a general dynamical system $(A, G, \alpha)$. Here we provide some avenues of research for special choices of $A$, $G$ and/or $\alpha$.

- In Dixmier (1977), proposition 2.7.1, it is shown that the norm on a $C^*$-algebra $A$ can be expressed as

  $$||a|| = \sup_{\pi \in \Pi} ||\pi(a)|| = \sup_{\pi \in \tilde{\Pi}} ||\pi(a)||,$$

  where $\Pi$ is the collection of $*$-representations of $A$ on a Hilbert space and $\tilde{\Pi}$ is the subcollection of topologically irreducible $*$-representations of $A$ on a

Hilbert space. In corollary 2.8.4 it is shown that a $*$-representation is topologically irreducible if and only if it is algebraically irreducible. If $G$ is the trivial group, then the $C^*$-crossed product $A \rtimes_\alpha G$ reduces to the $C^*$-algebra $A$. If $A$ is a Banach $*$-algebra, and we assume that $\Pi$ is a faithful class, then we can complete $A$ in the norm above and obtain a $C^*$-algebra $\tilde{A}$, known as the *enveloping $C^*$-algebra*. It seems there are now two ways to define a $C^*$-crossed products using $A$. We can first form the enveloping $C^*$-algebra $\tilde{A}$ and then form the $C^*$-algebra $(\tilde{A} \rtimes_\alpha G)_{S^*}$. Alternatively, we can form $(A \rtimes_\alpha G)_{S^*}$, which is a $C^*$-algebra. Are $\tilde{A} \rtimes_\alpha G$ and $(A \rtimes_\alpha G)_{S^*}$ isometrically $*$-isomorphic, as one would expect?

- More generally, we can ask the question above for any Banach algebra $A$. If we complete $A$ in a norm

$$||a|| = \sup_{\pi \in \Pi} ||\pi(a)||$$

to obtain a Banach algebra $\tilde{A}$ and subsequently form $(\tilde{A} \rtimes_\alpha G)_S$ for a faithful class $S$ 'similar to $\Pi$', is it isometrically isomorphic to $(A \rtimes_\alpha G)_S$?

- In this thesis, we have only used dynamical systems with a locally compact group $G$. We can simplify the situation by putting restrictions on $G$, e.g. the action $\alpha$ is always uniformly bounded if $G$ is compact. It will be interesting to concentrate on abelian locally compact groups. Abelian harmonic analysis is a well-developed field and its tools may be beneficiary to solving the problems above. More generally, in the theory of $C^*$-crossed products many simplifications occur if $G$ is assumed to be *amenable*. We refer to Davidson (1996) for a definition and proof that every abelian locally compact group is amenable (Corollary VII.2.2).

- In this thesis we only assume that the space $A$ of a dynamical system has a bounded approximate unit. We may impose many other assumptions that are still generalizations of the $C^*$-crossed product theory. For example, we can concentrate on representations on reflexive Banach spaces. Or we can look at pairs of representations (a representation on a Banach space and an 'adjoint representation' on the dual space) instead of looking at single representations.

We now state some technical questions of minor importance.

- We suspect that in order to establish the injection in theorem 3.18 it may be sufficient to know that the double centralizer algebra of the crossed product $A \rtimes_\alpha G$ is a well-defined Banach algebra. Under what conditions on $(A, G, \alpha)$ is $A \rtimes_\alpha G$ a faithful Banach algebra?

- Are there interesting examples of covariant pairs $(\pi, U)$ in which $U$ is not an isometric representation? In the proofs in this thesis it is usually sufficient to assume that $U$ is strongly continuous and uniformly bounded.

- If $(\pi, U)$ is a nondegenerate covariant representation on $X$ and $\tilde{X} \subset X$ is a closed invariant subspace, is the subrepresentation $(\tilde{\pi}, \tilde{U})$ of $(\pi, U)$ on $\tilde{X}$ again nondegenerate?

- Is $(i_A, j_A)$ a homeomorphism between $A$ and its image (see proposition 3.16)?

We end this section with three possible applications of the crossed products for Banach algebraic dynamical systems.

- If $G$ is discrete and $A$ is unital, the map in theorem 3.18 is bijective for $S = S_c$. It would be interesting to take $A$ equal to the disc algebra, i.e. the Banach algebra of functions on the closed unit disc in $\mathbb{C}$ which are holomorphic on the interior and continuous up to the boundary, and the action of $G = \mathbb{Z}$ on $A$ induced by the fractional linear transformations of the disc. We can compare the results to Buske & Peters (1998), where an analogous *semicrossed product* is introduced, using the semigroup $\mathbb{Z}^+$ instead of $\mathbb{Z}$.

- By taking $A = \mathbb{C}$ and $\alpha$ equal to the trivial action, we obtain a generalization of the group $C^*$-algebra. We can use this to study isometric representations of locally compact groups on Banach spaces.

- Let $G$ be a locally compact group acting a locally compact Hausdorff space $X$ on which an invariant Borel measure exists. The action of $G$ on $X$ induces an isometric action $\alpha$ of $G$ on $C_0(X)$. Although $C_0(X)$ is a $C^*$-algebra, there are interesting covariant representations of $(C_0(X), G, \alpha)$ which are not $*$-preserving. For example, we can represent $C_0(X)$ on $L^p(G, X)$ $(1 \leq p < \infty)$ by pointwise multiplication and we also have an isometric representation of $G$ on $L^p(G, X)$, since the borel measure on $X$ is invariant under the action of $G$ on $X$. This is a covariant pair.

# A Topological groups and Haar measure

This appendix collects some elementary definitions and results on topological groups and Haar measure. There are several good books available on these subjects, see for example Hewitt & Ross (1979) and Nachbin (1965). We omit all the proofs.

**Definition A.1** A *topological group* $G$ is a group on which a topology is defined such that the following hold:

1. one-point sets are closed in $G$;
2. the map $(s, r) \mapsto sr$ is continuous from $G \times G$ to $G$;
3. the map $s \mapsto s^{-1}$ is continuous on $G$.

A basic consequence of the definition is that the maps $s \mapsto s^{-1}$, $s \mapsto rs$ and $s \mapsto sr$ (for a fixed $r \in G$) are homeomorphisms of $G$.
Any group with the discrete topology is a topological group (e.g. $\mathbb{Z}$). Other elementary examples are $\mathbb{Z}^n$ with the discrete topology, $\mathbb{R}^n$ with the Euclidean topology and the $n$-sphere $\mathbb{T}^n$ with the subspace topology induced by the Euclidean topology. We record the following elementary properties of a topological group:

**Lemma A.2** *Any topological group $G$ is Hausdorff and regular. $G$ is locally compact if and only if every point in $G$ has a compact neighborhood.*

It is sometimes useful to work with a *symmetric neighborhood* of the identity $e$ in $G$, i.e. a neighborhood $V$ of $e$ such that $s \in V$ if and only if $s^{-1} \in V$. These are easy to obtain. If $W$ is any neighborhood of $e$ in $G$, then $W^{-1} := \{w^{-1} \ : \ w \in W\}$ is a neighborhood of $e$ as well and $V = W \cap W^{-1}$ is a symmetric neighborhood of $e$.
In the main text we work exclusively with locally compact topological groups. Our fascination stems from the fact that on such groups a measure (called *Haar measure*) can be defined which respects the group structure. First, we need some terminology. The *Borel sigma algebra* over $G$ is defined as the sigma algebra generated by the open sets of $G$. Any measure on this sigma algebra is called a *Borel* measure. The following theorem, the *Riesz representation theorem*, shows that we can associate to any positive linear functional $I : C_c(G) \to \mathbb{C}$ a sigma algebra $\Sigma$ containing all the open sets of $G$ and a unique measure $\mu$ on $\Sigma$ which represents the functional $I$. This measure is called a *Radon measure*. For the proof of this theorem, see Rudin (1987), theorem 2.14.

**Theorem A.3** *Let $G$ be a locally compact group and let $I$ be a positive linear functional on $C_c(G)$. Then there exists a sigma algebra $\Sigma$ over $G$ which contains all the open sets in $G$, and there exists a unique measure $\mu$ on $G$ which represents $I$ in the sense that*

1. $I(f) = \int_G f(s) \ d\mu(s) \ (f \in C_c(G))$,

   *and which has the following additional properties:*

2. $\mu(K) < \infty$ *for every compact set* $K \subset G$;
3. $\mu(C) = \inf\{\mu(V) \ : \ C \subset V, \ V \text{ open}\}$ *for all* $C \in \Sigma$.
4. $\mu(V) = \sup\{\mu(K) \ : \ K \subset V, \ K \text{ compact}\}$ *for every open set $V$ and every $V \in \Sigma$ with $\mu(V) < \infty$;*
5. *If $C \in \Sigma$, $B \subset C$, and $\mu(C) = 0$, then $B \in \Sigma$.*

A measure with properties 3 and 4 is called *regular*, a measure with property 5 is called *complete*.

**Definition A.4** Let $G$ be a locally compact topological group. A measure $\mu$ on $G$ is called *left invariant* if

$$\mu(sC) = \mu(C) \text{ for all } s \in G \text{ and measurable } C \subset G.$$

A left invariant Radon measure on $G$ is called *left Haar measure*, or simply *Haar measure*. Similarly, *right Haar measure* is a right invariant Radon measure on $G$.

The fundamental theorem on Haar measure alluded to before is the following existence and uniqueness theorem.

**Theorem A.5** *Every locally compact group has a Haar measure which is unique up to a strictly positive scalar multiple.*

For topological groups of finite measure it is customary to normalize Haar measure to make it completely unique. We simply fix a version $\mu$ of Haar measure once and for all.
We record the following frequently used properties of Haar measure in a separate lemma.

**Lemma A.6** *Let $G$ be a locally compact group. Then every open set has strictly positive Haar measure and every compact set has finite Haar measure.*

We end this section with some frequently used properties of Haar integrals of $L^1$ functions on $G$ (where $L^1(G)$ is the completion of $C_c(G)$ in the $||\cdot||_1$-norm). Although we state them for scalar valued functions, they are easily seen to hold for Banach space-valued functions as well (see appendix B).

**Proposition A.7** *Let $\mu$ be Haar measure on a locally compact group. Then for any $f \in L^1(G)$*

$$\int_G f(r^{-1}s)\ d\mu(s) = \int_G f(s)\ d\mu(s).$$

*Moreover, there exists a continuous homomorphism $\Delta : G \to (0, \infty)$ such that for all $f \in L^1(G)$*

$$\int_G \Delta(r) f(sr)\ d\mu(s) = \int_G f(s)\ d\mu(s).$$

*The homomorphism $\Delta$ is independent of the choice of Haar measure and is called the* modular function.
*Finally, for any $f \in L^1(G)$,*

$$\int_G f(s^{-1})\Delta(s^{-1})\ d\mu(s) = \int_G f(s)\ d\mu(s).$$

**Remark**. You may wonder why we only work with left Haar measure. In fact, the choice for left or right Haar measure is quite arbitrary. If $\mu$ defines a version of left Haar measure on $G$, then $\nu(C) := \mu(C^{-1})$ defines a version of right Haar measure. By the last equation in the proposition above, left and right Haar measure are equivalent (i.e. mutually absolutely continuous), with Radon-Nikodym derivative equal to

$$\frac{d\nu}{d\mu}(s) = \Delta(s^{-1})\ (s \in G).$$

It seems to have become standard in the literature to use left Haar measure and it is therefore easier to work with in practice.

## B Banach space valued integration

This appendix presents a bird's view of the theory of *Banach space valued integration*, also known as *vector valued integration.* Our intention is to shed some light on the Banach space $L^1(G, A)$ and to build a toolkit of results which are implicitly used throughout the main text. Our focus will be on the integration of functions $f : G \to X$, where $G$ is a locally compact group and $X$ a Banach space, with respect to Haar measure $\mu$ on $G$. We note that the results below hold for more general domains and Radon measures (as defined in appendix A). Our sources for the theory of vector valued integration are Dunford & Schwartz (1958) and Williams (2007).
The first order of business is to define a suitable notion of measurability for functions with values in a Banach space. In analogy with Lebesgue's theory for scalar valued functions, we would like every measurable function to be a limit of a sequence of simple functions (see definition B.7). Since the closure of the images of a sequence of simple functions forms a separable subspace of $X$, it is sensible to restrict our attention to functions which are locally separably-valued.

**Definition B.1** Let $X$ be a Banach space. A function $f : G \to X$ is called *essentially separably-valued* on a subset $S$ of $G$ if there is a separable subspace $D$ of $X$ and a $\mu$-null set $N \subset S$ such that $f(x) \in \overline{D}$ for all $x \in S - N$.

In contrast to the scalar-valued case, there are several ways to define measurability for vector-valued functions and each definition has its own charms.

**Definition B.2** Let $X$ be a Banach space. Then a function $f : G \to X$ is *strongly measurable* if

1. $f^{-1}(V)$ is measurable for all open sets $V \in G$;
2. $f$ is essentially separably-valued on every compact subset of $G$.

**Definition B.3** Let $X$ be a Banach space. Then a function $f : G \to X$ is *weakly measurable* if

1. $\phi \circ f : G \to \mathbb{C}$ is measurable for every continuous linear functional $\phi$ on $X$;
2. $f$ is essentially separably-valued on every compact subset of $G$.

**Definition B.4** Let $X$ be a Banach space. Then a function $f : G \to X$ is *C-measurable* if for every compact set $K \subset G$ and $\varepsilon > 0$ there is a compact set $\tilde{K}$ such that $\mu(K - \tilde{K}) < \varepsilon$ and such that the restriction of $f$ to $\tilde{K}$ is continuous.

Fortunately, the three notions above coincide.

**Lemma B.5** *Let $X$ be a Banach space and $f : G \to X$ a function. Then the following are equivalent:*

1. *$f$ is strongly measurable;*
2. *$f$ is weakly measurable;*
3. *$f$ is C-measurable.*

A Banach space valued function satisfying these three equivalent properties is called *measurable.* We have the following familiar result on pointwise limits of measurable functions.

**Lemma B.6** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable Banach space valued functions such that $f_n(s) \to f(s)$ for $\mu$-almost every $s \in G$. Then $f : G \to X$ is measurable.*

**Definition B.7** A measurable function $g : G \to X$ is called *simple* if it takes finitely many values $b_1, \ldots, b_n$ and $\mu(\{s \in G \ : \ g(s) = b_i\}) < \infty$ if $b_i \neq 0$.

As a consequence of our well-chosen definition of measurability, we obtain the following desired result:

**Lemma B.8** *Let $X$ be a Banach space and $f : G \to X$ be a function. Then $f$ is measurable if and only if for each compact set $K \subset G$ there is a sequence of simple functions $\{g_n\}_{n=1}^{\infty}$ such that for almost all $s \in K$*

$$||g_n(s)|| \leq ||f(s)||, \ g_n(s) \to f(s).$$

If we know that a measurable function has its support contained in a $\sigma$-finite subset of $G$, then we have an even stronger statement.

**Lemma B.9** *Let $X$ be a Banach space and let $f : G \to X$ be a measurable function which vanishes off a $\sigma$-finite subset. Then there is a sequence of simple functions $\{g_n\}_{n=1}^{\infty}$ and a $\mu$-null set $N$ such that for all $s \in G - N$*

$$||g_n(s)|| \leq ||f(s)||, \ g_n(s) \to f(s).$$

**Definition B.10** A measurable function $f : G \to X$ is *integrable* if

$$||f||_1 := \int_G ||f(s)|| \ d\mu(s) < \infty.$$

We define an equivalence relation on the set of integrable functions by identifying all integrable functions which are equal $\mu$-almost everywhere. The set of equivalence classes of integrable functions equipped with the $||\cdot||_1$ norm is denoted by $L^1(G, X)$.

The following two propositions state some important properties of the space $L^1(G, X)$ and the integral on $L^1(G, X)$.

**Proposition B.11** *Let $G$ be a locally compact group, $\mu$ Haar measure on $G$ and $X$ a Banach space. Then $L^1(G, X)$ is a Banach space. Moreover, both the collection of simple functions and $C_c(G, A)$ are dense in $L^1(G, X)$.*

**Proposition B.12** *Let $G$ be a locally compact group, $\mu$ Haar measure on $G$ and $X$ a Banach space. Then*

$$f \mapsto \int_G f(s) \ d\mu(s)$$

*is a contractive linear map from $L^1(G, X)$ to $X$. The integral is characterized by the property*

$$\phi(\int_G f(s) \ d\mu(s)) = \int_G \phi(f(s)) \ d\mu(s) \text{ for all } \phi \in X^*,$$

*where $X^*$ is the dual space of $X$. Finally, if $L : X \to Y$ is a bounded linear map into a Banach space $Y$ then*

$$L(\int_G f(s) \ d\mu(s)) = \int_G L(f(s)) \ d\mu(s).$$

We end this section with a generalization of two cornerstones of the theory of integration for scalar-valued funtions, the Dominated Covergence theorem and Fubini's theorem.

**Theorem B.13** *Let $G$ be a locally compact group, $\mu$ Haar measure on $G$ and $X$ a Banach space. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions on $G$ taking values in $X$ such that for some nonnegative, integrable function $g \in L^1(G)$ we have $||f_n(s)|| \leq g(s)$ for $\mu$-almost every $s \in G$ and $f_n(s) \to f(s)$ $\mu$-almost everywhere. Then $f_n \to f$ in $L^1(G, X)$.*

**Theorem B.14** *Let $G_1, G_2$ be locally compact groups, $\mu$ and $\nu$ Haar measure on $G_1$ and $G_2$, respectively, and $X$ a Banach space. If $f \in L^1(G_1 \times G_2, X)$, then the following hold:*

1. *For $\mu$-almost every $s \in G_1$, $r \mapsto f(s, r)$ belongs to $L^1(G_1, X)$;*
2. *For $\nu$-almost every $r \in G_2$, $s \mapsto f(s, r)$ belongs to $L^1(G_2, X)$;*
3. *The map*
$$s \mapsto \int_{G_2} f(s, r)\ d\nu(r)$$
*is defined $\mu$-almost everywhere and moreover defines an element of $L^1(G_1, X)$;*
4. *The map*
$$r \mapsto \int_{G_1} f(s, r)\ d\mu(s)$$
*is defined $\nu$-almost everywhere and moreover defines an element of $L^1(G_2, X)$;*
5. *The double integrals*
$$\int_{G_1} \int_{G_2} f(s, r)\ d\nu(r)\ d\mu(s) \text{ and } \int_{G_2} \int_{G_1} f(s, r)\ d\mu(s)\ d\nu(r)$$
*are equal, with common value*
$$\int_{G_1 \times G_2} f(s, r)\ d(\mu \times \nu)(s, r).$$

In the main text we occasionally use the spaces $L^p(G, X)$ for $1 < p < \infty$, which consist of the (equivalence classes of) measurable functions satisfying

$$||f||_p := (\int_G ||f(s)||^p\ d\mu(s))^{1/p} < \infty.$$

As in the scalar-valued case, $L^p(G, X)$ is a Banach space and both $C_c(G, X)$ and the step functions are dense in $L^p(G, X)$.

## C $C^*$-algebras

This appendix presents some results from the theory of $C^*$-algebras that we use in the development of the $C^*$-crossed product. For a more thorough introduction to the beautiful theory of $C^*$-algebras we warmly recommend Murphy (1990), or the authoritative (but older) volume Dixmier (1977).

**Definition C.1** An *involution* on an algebra $A$ is map $a \mapsto a^*$ such that $(\lambda a)^* = \overline{\lambda} a^*$, $a^{**} = a$ and $(ab)^* = b^* a^*$ for all $\lambda \in \mathbb{C}$ and $a, b \in A$. An algebra endowed with an involution is called an *involutive algebra* or $*$*-algebra*. A *Banach* $*$*-algebra* is a $*$-algebra with a complete submultiplicative norm such that $||a^*|| = ||a||$ for $a \in A$. A $C^*$*-algebra* is a Banach $*$-algebra such that $||a^*a|| = ||a||^2$ for $a \in A$.

**Examples**. Here are three examples of $C^*$-algebras:

- The complex numbers $\mathbb{C}$ becomes a $C^*$-algebra when endowed with the involution $a^* := \overline{a}$ $(a \in \mathbb{C})$, i.e. taking the complex conjugate.
- Let $\Omega$ be a locally compact Hausdorff space. Then $C_0(\Omega)$, the space of complex valued functions vanishing at infinity, is a Banach algebra under the supremum norm and a $C^*$ algebra when equipped with the involution $f^* := \overline{f}$, i.e. under complex conjugation.
- Let $H$ be a Hilbert space, then the space of bounded linear operators on $H$ with the operator norm, $B(H)$, is a $C^*$-algebra when the involution on $B(H)$ is given by taking the adjoint of an operator.

When we study the homomorphisms of algebras with an involutive structure we are especially interested in homomorphisms that not only preserve the linearity and multiplication on the space, but also its involution. Such homomorphisms are called $*$*-preserving* or simply $*$*-homomorphisms*. In the special case of representations of a $C^*$-algebras, our focus will be on representations on Hilbert spaces which preserve the involution.

**Definition C.2** A $*$*-representation* of a $C^*$-algebra $A$ on a Hilbert space $H$ is a representation $\pi : A \to B(H)$ which preserves the involution of $A$, i.e. $\pi(a^*) = \pi(a)^*$ for all $a \in A$.

If $A$ is a $C^*$-algebra we will let $\mathrm{Aut}(A)$ denote the group of involution preserving-automorphisms.
The assumption of the $C^*$-rule ($||a^*a|| = ||a||^2$) on an involutive Banach algebra looks innocent enough, but it in fact has some far-reaching consequences. Much more is known about $C^*$-algebras than about involutive Banach algebras, and about Banach algebras without an involution even less is known. An elementary result in which the $C^*$-rule plays a vital rule is the following.

**Proposition C.3** *A $*$-homomorphism $\phi : A \to B$ from a Banach $*$-algebra $A$ to a $C^*$-algebra $B$ is necessarily norm-decreasing. In particular, every $*$-automorphism of a $C^*$-algebra is necessarily isometric.*

The following two properties of $C^*$-algebras prove to be extremely useful in the analysis of the representation theory of the $C^*$-crossed product.

**Theorem C.4** *Every $C^*$-algebra has an approximate unit contained in its unit ball.*

**Theorem C.5** *Let $A$ be a $C^*$-algebra, $I$ a closed two-sided ideal and $\pi$ a nondegenerate representation of $I$ on a Hilbert space $H$. Then there is a unique representation $\tilde{\pi}$ of $A$ on $H$ which extends $\pi$.*

We end this section with a celebrated result known as the *Gelfand-Naimark-Segal theorem*. It shows that we can think of any $C^*$-algebra as a closed, self-adjoint subalgebra of $B(H)$, for some Hilbert space $H$.

**Theorem C.6** *Every $C^*$-algebra $A$ is isometrically $*$-isomorphic to a closed self-adjoint subalgebra of $B(H)$, for some Hilbert space $H$. In particular, every $C^*$-algebra has a faithful $*$-representation on a Hilbert space $H$.*

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I dedicate my thesis to my nephew/niece who, at the time of this writing, is still to be born. Your proud uncle wishes you a life full of love, friendship, happiness and prosperity.

# References


Bonsall, F.F. and Duncan, J., 1973, Complete normed algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 80, Springer-Verlag New York-Heidelberg.

Buske, D.R. and Peters, J.R., 1998, Semicrossed Products of the Disk Algebra: Contractive Representations and Maximal Ideals, Pacific Journal of Mathematics, vol. 185, No. 1, pp. 97-113.

Conway, J.B., 1985, A Course in Functional Analysis, Graduate texts in mathematics, 96, Springer-Verlag New York Inc.

Dales, H.G., 2000, Banach Algebras and Automatic Continuity, London Mathematical Society Monographs, 24, Oxford University Press.

Davidson, K. and Donsig A., 2002, Real Analysis with Real Applications, Prentice Hall, Inc.

Davidson, K., 1996, $C^*$-Algebras by Example, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI.

Dixmier, J., 1977, $C^*$-Algebras, North-Holland Mathematical Library, vol. 15, North-Holland Publishing Company.

Douglas, R.G., 1998, Banach Algebra Techniques in Operator Theory, Second Edition, Graduate Texts in Mathematics, 179, Springer-Verlag New York, Inc.

Dunford, N. and Schwartz, J.T., 1958, Linear Operators, Part I: General Theory, Interscience Publishers, Inc.

Fillmore, P.A., 1996, A User's Guide to Operator Algebras, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc.

Hewitt, E. and Ross, K.A., 1979, Abstract Harmonic Analysis Vol. I, Second Edition, Grundlehren der Mathematischen Wissenschaften, vol. 115, Springer-Verlag, Berlin.

Palmer, T.W., 1994, Banach Algebras and the General Theory of $*$-Algebras, Volume I: Algebras and Banach Algebras, Encyclopedia of Mathematics and its Applications, vol. 49, Cambridge University Press.

Pedersen, G.K., 1979, $C^*$-Algebras and their Automorphism Groups, London Mathematical Society Monographs, 14, Academic Press Inc.

Munkres, J.R., 2000, Topology, Second Edition, Prentice Hall, Inc.

Nachbin, L., 1965, The Haar Integral, D. Van Nostrand Company, Inc.

Murphy, G.J., 1990, $C^*$-Algebras and Operator Theory, Academic Press.

Rickart, C.E., 1960, Banach Algebras, D. Van Nostrand Company, Inc.

Rudin, W., 1987, Real and Complex Analysis, Third Edition, McGraw-Hill Inc.

Tomiyama, J., 1987, Invitation to $C^*$-Algebras and Topological Dynamics, World Scientific Advanced Series in Dynamical Systems, vol. 3, World Scientific Publishing Co. Pte. Ltd.

Williams, D.P., 2007, Crossed Products of $C^*$-Algebras, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society.