The topology of isolated singularities on complex hypersurfaces

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Masters thesis

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Contents

Chapter 1

Introduction

In this masters thesis we will be concerned with the topology of isolated singular points on complex hypersurfaces. As the name suggests, this is a meeting point of two different areas of mathematics, namely algebraic topology and algebraic geometry.

On the side of algebraic geometry there are plane algebraic curves, objects which have been studied for centuries. These studies include the classification of the possible singular points on an algebraic curve. One can distinguish between local concepts, such as the order of the singular point and the number of distinct tangents to it on the one hand, and global concepts, such as the maximum number of singular points of a certain type. We will only be interested in the local questions.

The introduction of topological methods to study this algebraic situation dates from the beginning of the 20th century. In the historical overview given in [Dur99] it is explained that at the time there was interest in studying complex algebraic surfaces. In the easier case of complex algebraic curves, or Riemann surfaces, one standard device is to present them as coverings of the Riemann sphere, ramified over a few isolated points, called the discriminant locus of the covering. In the neighbourhood of an element of the discriminant locus, the situation is easy to understand and visualize: locally the projection is just $z \mapsto z^n$ for some integer n.

One can carry out a similar construction for algebraic surfaces by describing them as coverings of the complex plane. Then the discriminant locus becomes a complex curve. In the neighbourhood of the smooth points of this curve, the situation was understood, so the attention turned to the singular points of this discriminant curve. Brauner (see [Bra28]) studied this situation by considering the intersection of the discriminant curve with a small 3-sphere in $\mathbb{C}^2$ centered around a singular point on the discriminant curve. This intersection is a knot in S^3, and studying the fundamental group of the complement of this knot gives information about the local branching of the surface under consideration above the singular point.

It thus became important to understand the possible knots that can arise in this way. In his paper, Brauner carried out the construction described in Chapter 4, showing how to construct the iterated torus knots when given the equation of the curve. This used the recently developed methods of algebraic topology to calculate the fundamental groups of the complements of the knots.

Shortly thereafter, Kähler ([Käh29]) gave a much simpler way to derive the same results, by using a polydisk instead of a sphere. The benefits of using this approach is described in Chapter 4. The final result stated in that chapter, about the equivalence of the Puiseux pairs and the corresponding knot, follows from subsequent work done by Zariski [Zar32] and Burau [Bur32].

The case described in Chapter 2 is a generalization of the above. There are of course many ways to generalize. Instead of algebraic curves one considers algebraic sets of arbitrary dimension. One considers only the case where the algebraic set is a hypersurface, in other words, given by a single equation. Finally, one restricts to the case of an isolated singular point.

For this case, Milnor ([Mil68]) introduced the Milnor fibration. One considers a small sphere $S_\epsilon \cong S^{2n-1}$ in $\mathbb{C}^n$ centered around the critical point, and then intersect this sphere with the hypersurface to find a link K in S^{2n-1}. The way that K is embedded in S^{2n-1} gives information about the singular point. To study this, Milnor constructed a fibration of $S^{2n-1} \setminus K$ over S^1. In Chapter 2 we explain this construction, relying heavily on Milnor's book [Mil68].

The fiber of this fibration is called the Milnor fiber of the singularity. Invariants of the singularity can be computed by just considering the topology of the Milnor fiber. The simplest invariant is called the Milnor number of the singularity, and can be defined as the middle Betti number of the Milnor fiber. In Chapter 3 we derive some topological properties of this fiber, and define some of the associated invariants.

The method used is that of applying a perturbation to a function, i.e. changing it a little bit so that the complicated critical point splits up into many simple ones. It is important to point out that we do not make any mention of the most common method of studying singularities, namely that of finding a resolution of the singularity. We also do not go into the details of the analytic geometric background, which is necessary if one is to make a more algebraic study of singularities. This theory can be found in [BK86] for example.

The material described in this thesis is quite established by now. More recently, people have considered the cases where some of the hypotheses are relaxed. For instance, see [Loo84] for an investigation into the case of isolated complete intersection singularities (i.e. no longer just on hypersurfaces). Also see the work of Siersma (for example [Sie01]) for the case of non-isolated singularities.

Chapter 2

The Milnor fibration

2.1 Some definitions

Before starting, let us recall some basic definitions of objects that will be used throughout. We will use the concepts of algebraic sets (real or complex) as well as smooth manifolds, as well as functions defined on such objects.

For smooth manifolds, it will be important to distinguish between regular and critical points of real-valued functions. A critical, or singular point is simply a point where the differential of the function is not surjective. A regular point is a point which is not critical. And a critical value is simply the image of a critical point. In the same way we can define regular and critical points and values of complex analytic functions on complex manifolds.

For algebraic sets, let's start by letting K denote either $\mathbb{R}$ or $\mathbb{C}$. Then an algebraic set $V \subset K^n$ is the vanishing set of a finite set of polynomials $f_1, \dots, f_r \in K[x_1, \dots, x_n]$. We can assume that these polynomials form a set of generators for the ideal $I(V)$, where $I(V)$ denotes the set of polynomials in $K[x_1, \dots, x_n]$ which vanish on V. Now consider the matrix $(\frac{\partial f_i}{\partial x_j})$ evaluated at any point of V. Let ρ be the maximal rank attained by this matrix on V. Then a point x of V is said to be regular if the matrix $(\frac{\partial f_i}{\partial x_j})$ attains this maximal rank at x, otherwise the point is said to be singular.

2.2 Introducing the main object of study

Our starting point will be a complex hypersurface. This is the vanishing set $V(f) \subset \mathbb{C}^n$ of a polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$. We can and will assume f to be square-free throughout the thesis. We are interested in what $V(f)$ looks like in the neighbourhood of a point $x \in V(f)$. We consider not only the topology of $V(f)$ in a neighbourhood of x, but also the way in which it is embedded into $\mathbb{C}^n$. Let $g \in \mathbb{C}[z_1, \dots, z_n]$, and define $x \in V(f)$ and $y \in V(g)$ to be topologically equivalent, if there exists neighbourhoods U and W of x and y respectively, and a homeomorphism $\phi : U \to W$ such that $\phi(V(f) \cap U) = V(g) \cap W$. In other words, if the pair $(U, V(f) \cap U)$ is homeomorphic to $(W, V(g) \cap W)$.

If x is a regular point, then the situation is simple: By the implicit function theorem x is topologically equivalent to any point on the hypersurface in $\mathbb{C}^n$ defined by the equation $z_1 = 0$.

Thus the interesting case is when x is a singular point. We will restrict our attention to the case where x is an *isolated* singular point, although many of the theorems will be valid for the more general case.

2.3 The conical structure

In order to facilitate comparing different singular points, we restrict the neighbourhood U of x to always be an open ball B_ϵ of radius ϵ, centred at x. For simplicity, we take x to be the origin from now on. Then it will follow from theorem 2.3.4 proved below, that for a given singular point (at the origin), there exists some $\epsilon > 0$ such that $(B_\epsilon, V(f) \cap B_\epsilon)$ is homeomorphic to $(B_{\epsilon'}, V(f) \cap B_{\epsilon'})$ for any $0 < \epsilon' < \epsilon$.

We will then make a further simplification, and show that we can restrict ourselves to the boundary of the ball, $S_\epsilon = \delta B_\epsilon$. The aim of this section is then to prove that two singular points are topologically equivalent precisely if the pairs $(S_\epsilon, V(f) \cap S_\epsilon)$ and $(S_\mu, V(g) \cap S_\mu)$ are homeomorphic for $\epsilon, \mu > 0$ small enough.

First, some basic facts about algebraic sets will be needed recalled (for the proofs, see [Mil68]): Let V be a non-empty algebraic set in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Lemma 2.3.1. The set V_s of singular points of V is also an algebraic set which is strictly contained in V.

Theorem 2.3.2. The set $V \setminus V_s$ of non-singular points of V forms an analytic manifold.

Theorem 2.3.3. Let $x \in V$ be a non-singular point, or an isolated singular point. Then there exists an $\epsilon > 0$ such that $S_{\epsilon'}$ intersects V transversally, and hence in a smooth manifold for all $0 < \epsilon' \leq \epsilon$.

Now for the theorem about the conical structure around the singular point.

Theorem 2.3.4. Let V be a real or complex algebraic set as before, and let $x \in V$ be a non-singular point or an isolated singular point. Then there exists an $\epsilon > 0$ such that the pair $(B_\epsilon, V \cap B_\epsilon)$ is homeomorphic to the pair $(C(S_\epsilon), C(V \cap S_\epsilon))$, where $C(X)$ denotes the cone over X.

Proof. Start by choosing $\epsilon > 0$ small enough so that $V \cap B_\epsilon$ contains no singular points other than (possibly) x, and such that $S_{\epsilon'}$ intersects V transversally for all $\epsilon' \leq \epsilon$. Let r be the polynomial function defined by $r(y) = ||y - x||^2$. Let $y \in S_\epsilon$. Then $\ker(dr(y)) = T_y S_\epsilon$ (since S_ϵ is a level-set of r) and thus the transversality condition means precisely that y is a regular point of r restricted to V. This will be used later on.

Recall that if X is a topological space, then the cone over X, denoted by $C(X)$, is constructed by forming the product $X \times [0, 1]$ and identifying $X \times \{0\}$ to a point. In the present context it has a concrete interpretation: If we denote $V \cap S_\epsilon$ by K, then $C(K)$ is simply the set of lines joining x with points of K. Thus $C(S_\epsilon)$ is simply B_ϵ.

The idea of the proof is to construct a smooth vector field v on $B_\epsilon \setminus \{x\}$ such that the lines in the cone just referred to will correspond to the solution curves of v and each solution curve will have exactly one point on S_ϵ. The vector field will have the following two properties: The vector $v(y)$ will point away from x for all y, in the sense that the inner product $\langle v(y), y-x\rangle$ will always be strictly positive. Furthermore, letting $V_r := V \setminus V_s$ denote the regular part of V, then $v(y)$ will be tangent to V_r whenever y is in V_r.

First we construct v locally. Given any point $y_\alpha \in B_\epsilon \setminus \{x\}$, we will construct a vector field v_α in a neighbourhood U_α of y_α satisfying the above two properties.

If $y_\alpha \notin V$, then let U_α be a neighbourhood of y_α such that $U_\alpha \cap V = \emptyset$, and let $v_\alpha(y) = y - x$.

If y_α does belong to V, and hence to V_r, then choose local coordinates $u_1, \ldots, u_n$ around y_α so that V_r corresponds to $u_1 = \cdots = u_k = 0$ around y_α. This implies that the tangent vectors $\frac{\partial}{\partial u_{k+1}}(y_\alpha), \ldots, \frac{\partial}{\partial u_n}(y_\alpha)$ span the tangent space of V_r at y_α. But y_α being a regular point of $r|_{V_r}$ implies that at least one of these tangent vectors, say $\frac{\partial}{\partial u_h}(y_\alpha)$ does not lie in the kernel of $dr(y_\alpha)$, i.e. $\frac{\partial r}{\partial u_h}$ is not zero at y_α. Now let U_α be a small connected neighbourhood of y_α in which $\frac{\partial r}{\partial u_h}$ is non-zero. For $y \in U_\alpha$, define $v_\alpha(y) = \pm\frac{\partial}{\partial u_h}(y) = \pm(\frac{\partial y_1}{\partial u_h}, \ldots, \frac{\partial y_n}{\partial u_h})$, choosing $+$ if $\frac{\partial r}{\partial u_h}$ is positive, otherwise choosing $-$. If $y \in U_\alpha \cap V$ then since $v_\alpha(y)$ is a tangent vector to the coordinate curve u_h lying entirely in V_r, it follows that $v_\alpha(y)$ must be tangent to V_r.

For the other condition, note that $r(y) = (y_1 - x_1)^2 + \cdots + (y_n - x_n)^2$, hence $\frac{\partial r}{\partial y_i} = 2(y_i - x_i)$. So

$$\begin{aligned} 2\langle v_\alpha(y), y-x\rangle &= \sum_{i=1}^n 2(y_i - x_i)(v_\alpha)_i \\ &= \sum_{i=1}^n \frac{\partial r}{\partial y_i}\left(\pm\frac{\partial y_i}{\partial u_h}\right) \\ &= \pm\frac{\partial r}{\partial u_h} \end{aligned}$$

which is positive for all $y \in U_\alpha$.

Now let $\{\lambda_\alpha\}$ be a partion of unity on B_ϵ, such that the support of λ_α is contained in U_α. Then define the vector field v on $B_\epsilon \setminus \{x\}$ by

$$v(y) = \sum_\alpha \lambda_\alpha(y) v_\alpha(y).$$

The vector field v has the two required properties, but we would like solution curves of the vector field to all take the same time in moving from x to a point on the boundary of B_ϵ. This can be done by normalizing the vector field:

$$w(y) = \frac{v(y)}{\langle 2(y-x), v(y)\rangle}.$$

Now look for solutions $p(t)$ of the differential equation $\frac{dy}{dt} = w(y)$. Such solutions must exists locally (see [Hur58]) and are unique: If $y_0 \in B_\epsilon \setminus \{x\}$, then there is some interval (α, β) in $\mathbb{R}$ containing t_0 and a solution $p(t)$ defined on this interval with $p(t_0) = y_0$. (To avoid problems if y is on the boundary of B_ϵ, we can assume that the original vector field was constructed on a slightly larger ball, say $B_{\epsilon'}$ with $\epsilon' > \epsilon$.)

Consider the function $r(p(t))$. The normalization was chosen in order to give:

$$\begin{aligned}\frac{dr}{dt} &= \sum_{i=1}^{n} \frac{\partial r}{\partial y_i}\frac{dy_i}{dt} \\ &= \sum_{i=1}^{n} 2(y_i - x_i)w_i(y) \\ &= \langle 2(y-x), w(y)\rangle \\ &= 1,\end{aligned}$$

where $p(t) = y$. Thus $r(p(t)) = t + c$, for some constant c. By translating the interval of definition, we can assume that $c = 0$. So the solution $p(t)$ is defined on some interval (α, β) contained in the interval $(0, \epsilon'^2)$ (since $0 < r(p(t)) < \epsilon'^2$). We would like to show that the solution can actually be extended to be defined on an interval $(0, \beta')$, where $\beta' > \epsilon^2$.

Using Zorn's lemma, the solution can be extended to a maximum open interval (α', β'). Suppose $\alpha' > 0$. Let $(t_n)_{n\geq 1}$ be a sequence in (α', β') converging to α'. Then the corresponding sequence $(p(t_n))_{n\geq 1}$ will have a limit point in the compact set $\overline{B}_{\epsilon'^2}$, call it y_0. Since $r(p(t)) = t$, we have $r(y_0) = \alpha' > 0$. Hence $y_0 \in B_{\epsilon'^2} \setminus \{x\}$, and w is a smooth vector field around y_0.

Then (see [Mil68, p.21]) there exists a neighbourhood W of y_0 and an interval I around α' such that for any $y_1 \in W$ and $t_1 \in I$, there exists a unique solution $q(t)$ to the differential equation satisfying the initial condition $q(t_1) = y_1$. Now pick a point $t_2 \in I \cap (\alpha', \beta')$, let $y_2 = p(t_2)$, and let $q(t)$ be the unique solution satisfying $q(t_2) = y_2$. Then by uniqueness, $p(t)$ and $q(t)$ must coincide on the interval $I \cap (\alpha', \beta')$. Hence we can extend $p(t)$ to be defined on the larger interval $I \cup (\alpha', \beta')$, contradicting the maximality of (α', β'). Thus $\alpha' = 0$. Similarly it can be shown that $\beta' = \epsilon'^2 > \epsilon^2$.

Note that the solution $p(t)$ on $(0, \epsilon^2]$ is uniquely determined by its value $p(\epsilon^2) \in S_\epsilon$. So we can define a function $P(t, y)$ on $(0, \epsilon^2] \times S_\epsilon$ by setting $P(t, y) = p(t)$ where $p(t)$ is the unique solution curve starting at $y \in S_\epsilon$. Then P maps $(0, \epsilon^2] \times S_\epsilon$ diffeomorphically onto $B_\epsilon \setminus \{x\}$. (The fact that the mapping is smooth follows for example from [Lan72, p.80], and by reversing the vector field one sees that the inverse of the mapping is also smooth, hence it is a diffeomorphism.) Furthermore, since solution curves which start on V, remain on V, P maps $(0, \epsilon^2] \times K$ onto $V \cap (B_\epsilon \setminus \{x\})$ (recall that $K = V \cap S_\epsilon$)).

The function $P(t, y)$ tends uniformly to x as t tends to 0 since $r(P(t, y)) = t$. Thus we can extend P to a mapping from $C(S_\epsilon)$ to B_ϵ, which restricts to a mapping of $C(K)$ to $V \cap B_\epsilon$. □

2.4 The curve selection lemma

Before we can prove the theorem about the Milnor fibration, we need a rather technical lemma, called the curve selection lemma by Milnor (see [Mil68]).

Lemma 2.4.1. Let $V \subset \mathbb{R}^n$ be a real algebraic set, and let $U \subset \mathbb{R}^n$ be an open set defined by finitely many polynomial inequalities:

$$U = \{x \in \mathbb{R}^n \mid g_1(x) > 0, \ldots, g_m(x) > 0\}.$$

If the origin of $\mathbb{R}^n$ is contained in the closure of $U \cap V$, then there exists a real analytic curve $\gamma : [0, \delta) \to \mathbb{R}^n$ with $\gamma(0) = 0$ and $\gamma(t) \in U \cap V$ for $t > 0$.

Remark: the origin is contained in V. We can also assume it is not contained in U, since otherwise the constant curve $\gamma(t) = 0$ will satisfy the requirements of the theorem. It follows that 0 is not an isolated point of V.

Proof. The proof starts by induction on the dimension of V to reduce to the case where the dimension of V is 1. We will repeatedly make use of the fact that an algebraic set which is properly contained in an irreducible algebraic set (i.e. a variety) of dimension say $n - l$, has dimension strictly lower than $n - l$. The idea is then to replace V with an algebraic set of lower dimension which still satisfies the hypotheses of the lemma.

Firstly we note that we can assume V to be irreducible, since if it is not, then it can be replaced by an irreducible component passing through the origin.

Next, if we let V_s denote the set of singular points of V, then we can assume that the origin does not lie in the closure of $V_s \cap U$. Because if it does, then we can replace V by V_s, which is a proper algebraic subset of V and hence has lower dimension. Hence there is a neighbourhood of the origin in which U does not contain any singular points of V.

Denote the dimension of V by $n - l$, and let the polynomials $f_1, \ldots, f_k$ be generators of the ideal $I(V)$. We can consider the one-forms $df_i \in T^*\mathbb{R}^n$ and specifically the co-tangent vectors $df_i(x) \in T_x^*\mathbb{R}^n$. For a point $x \in V$, x is a singular point precisely if

$$\text{rank}(\text{span}\{df_1(x), \ldots, df_k(x)\}) < l.$$

If W is a subspace of

$$T_x^*\mathbb{R}^n \cong (T_x\mathbb{R}^n)^*$$

then let $W^\perp$ denote the subspace of $\mathbb{R}^n$ on which all the elements of W vanish.

Define the following functions:

$$r(x) = ||x||,$$
$$g(x) = g_1(x)g_2(x)\cdots g_m(x)$$

and let $V' \subset V$ be the set of x for which

$$\text{rank}(\text{span}\{df_1(x), \ldots, df_k(x), dr(x), dg(x)\}) \leq l + 1.$$

Now to apply the idea explained at the beginning of the proof of replacing V with V', we just have to verify that V' satisfies the hypotheses.

The first step is to show that 0 is in the closure of $U \cap V'$. Since 0 is in the closure of $U \cap V$, we can choose an arbitrary small $\epsilon > 0$ such that S_ϵ contains points of $U \cap V$. Then consider the compact set K consisting of $x \in V \cap S_\epsilon$ such that $g_1(x) \geq 0, \ldots, g_m(x) \geq 0$. Since the function g is continuous, it must attain a maximum value at some point of K, say at x'. Now for any point y in the non-empty set $U \cap V \cap S_\epsilon$, we have $g_i(y) > 0$ for all i, hence $y \in K$, and $g(y) > 0$. Thus $g(x') \geq g(y) > 0$, so $x' \in U$.

It remains to be shown that $x' \in V'$. By the result of the previous section, we can assume that S_ϵ intersects $U \cap V$ transversally. This means that

$$T_x V \not\subseteq T_x S_\epsilon$$

for any x in $U \cap V \cap S_\epsilon$, using the fact that around such a point $U \cap V$ is a smooth manifold. But

$$T_x V = \cap_i \ker df_i(x) = (\mathrm{span}\{df_1(x), \ldots, df_k(x)\})^\perp \quad \text{and}$$
$$T_x S_\epsilon = \ker dr(x),$$

hence

$$(\mathrm{span}\{df_1(x), \ldots, df_k(x)\})^\perp \not\subseteq \ker dr(x)$$

or equivalently,

$$dr(x) \notin \mathrm{span}\{df_1(x), \ldots, df_k(x)\}.$$

It follows that

$$\mathrm{rank}(\mathrm{span}\{df_1(x), \ldots, df_k(x), dr(x)\}) = l + 1.$$

Next we consider the function g restricted to the smooth manifold $U \cap V \cap S_\epsilon$. The function $g|_{U \cap V \cap S_\epsilon}$ attains a maximum at x', hence it has a critical point at x'. This means precisely that

$$T_{x'}(V \cap S_\epsilon) \subset \ker dg(x').$$

Following the same reasoning as above, it follows that

$$dg(x') \in \mathrm{span}\{df_1(x), \ldots, df_k(x), dr(x)\}$$

and hence

$$\mathrm{rank}(\mathrm{span}\{df_1(x'), \ldots, df_k(x'), dr(x'), dg(x')\}) = l + 1$$

which means that $x' \in V'$.

Thus now we can replace V by V'. But this will lower the dimension only if $V' \neq V$. In the case where $V' = V$ we can repeat the above construction using the function $x_i g(x)$ in the place of $g(x)$, where $x = (x_1, \ldots, x_n)$. Let V_i' be the set of all $x \in V$ such that

$$\mathrm{rank}(\mathrm{span}\{df_1(x), \ldots, df_k(x), dr(x), d(x_i g)(x)\}) \leq l + 1.$$

Then a similar argument as above shows that $0 \in \overline{U \cap V_i'}$. Thus we can replace V by one of the V_i''s. This will work unless $V = V' = V_1' = \cdots = V_n'$. The claim is that this can only happen if the dimension of V is 1.

Using what was done above, we can choose a point x' in $U \cap V$ such that

$$\mathrm{rank}(\mathrm{span}\{df_1(x'), \ldots, df_k(x'), dr(x'), dg(x')\}) = l + 1.$$

If $V = V'$, then $x' \in V'$, hence $dg(x')$ must belong to the $l+1$-dimensional vector space $\text{span}\{df_1(x'), \ldots, df_k(x'), dr(x'), dg(x')\}$. Similarly, if $V = V_i'$, then $d(x_i g)(x')$ must belong to this vector space. Using the identity $d(x_i g) = (dx_i)g + x_i(dg)$ and the fact that $g(x') \neq 0$, it follows that $dx_i(x')$ also belongs to this $l+1$-dimensional vector space. But the $dx_i(x')$'s form a basis for $T^*_{x'}\mathbb{R}^n$. Thus $l + 1 = n$, so V is $n - l = 1$ dimensional.

Now we assume the local description of one-dimensional real varieties. (See [Mil68, p.28].) Since 0 is a non-isolated point of the one-dimensional variety V, a small neighbourhood of V will consist of a union of finitely many branches which intersect only at 0. Each branch is homeomorphic to an open interval of real numbers under a homeomorphism $x = \gamma(t)$ which is given by a power series

$$\gamma(t) = a_1 t + a_2 t^2 + a_3 t^3 + \ldots$$

which converges for $|t| < \epsilon$.

Since $0 \in \overline{U \cap V}$, one of the finitely many branches of V passing through 0 must contain points of U arbitrarily close to 0. Let $x = \gamma(t)$ with $|t| < \epsilon$ be an analytic parametrization of this branch. Then for each g_i, we must have either $g_i(\gamma(t)) > 0$ or $g_i(\gamma(t)) \leq 0$ for all t in some interval $(0, \epsilon')$ (since $g_i(\gamma(t))$ is analytic). Thus the half-branch $\gamma((0, \epsilon'))$ is either contained in U, or disjoint from U, and similarly for the half-branch $\gamma((-\epsilon', 0))$. But by assumption $\gamma((-\epsilon', \epsilon'))$ contains points of U arbitrarily close to 0. Hence one of the half-branches must be contained in U. Thus either $\gamma(t)$ or $\gamma(-t)$ gives the required curve to complete the proof of the lemma. □

2.5 The fibration theorem

From now on V will denote a complex hypersurface. Thus far we have seen that the topological type of an isolated singular point $x \in V$ is determined by its intersection $K = V \cap S_\epsilon$ with a small sphere S_ϵ centered at x. The topological type of x is then determined by the pair (S_ϵ, K). The main result of this section is that $S_\epsilon \setminus K$ is a smooth fiber bundle over the circle S^1. This fiber bundle is uniquely determined by giving a single fiber as well as an automorphism of this fiber corresponding to a generator of the fundamental group of S^1. Much of this thesis will be about describing this fiber in certain special cases, as well as the corresponding automorphism. Several invariants can be computed from this construction. They are all topological invariants of the singular point, but not necessarily complete topological invariants.

Note that throughout this section it is not necessary to assume that the critical point is isolated. This becomes necessary in the next section however. The fibration described in this section was first described by Milnor ([Mil68]) and we will be following his exposition closely.

Getting back to the purpose of this section, let's start by defining the objects concerned. Let f be a complex polynomial in n variables and let $V \subset \mathbb{C}^n$ be its vanishing set. We assume that the origin is contained in V. For any given $\epsilon > 0$ we let K denote the intersection of V with a sphere S_ϵ centred at the origin. Then define a map $\phi : S_\epsilon \setminus K \to S^1$ by setting

$$\phi(z) = \frac{f(z)}{|f(z)|}.$$

The set $S_\epsilon \setminus K$ is an open subset of S_ϵ, hence it is a $2n-1$-dimensional real manifold. The first step towards showing that ϕ is a fibration over S^1 is to show that its fibers are manifolds. So we must show that it is regular everywhere, i.e. that its differential is surjective everywhere. This will be done by giving a simple criterion for determining whether a given point $z \in S_\epsilon \setminus K$ is a critical point of ϕ, and then showing that ϵ can be chosen small enough so that this criterion will never be satisfied.

First a definition. For a given analytical function $g(z_1, \ldots, z_n)$ of n complex variables, define the gradient of g to be the vector

$$\operatorname{grad} g = \left(\overline{\frac{\partial g}{\partial z_1}}, \ldots, \overline{\frac{\partial g}{\partial z_n}}\right).$$

The reason for taking the complex conjugates in this definition is that it allows us to give a simple expression for the directional derivative of g in the direction of a vector v at a point z. If $p(t)$ is a path such that $p(0) = z$ and $p'(0) = v$, then this directional derivative is given by

$$\begin{aligned}\frac{dg(p(t))}{dt}\bigg|_{t=0} &= \langle \frac{dp}{dt}\bigg|_{t=0}, \operatorname{grad} g(z)\rangle \\ &= \langle v, \operatorname{grad} g(z)\rangle,\end{aligned}$$

where $\langle a, b\rangle = \sum_j a_j \overline{b_j}$ is the hermitian inner product on $\mathbb{C}^n$.

In what follows we will often be needing the vector grad $\log f(z)$ for a given $z \in \mathbb{C}^n$ such that $f(z) \neq 0$. Taking the gradient of a function at a point only requires knowing the value of the function in a small neighbourhood of that point. So in this case we can restrict z' to a neighbourhood of z in $\mathbb{C}^n$ such that the image of this neighbourhood under f is contained in a simply connected neighbourhood of $f(z)$ which avoids 0. On such a simply connected set we can choose a branch of the multivalued function $\log f$ and then apply the formula for the gradient given above. Since the values on two specific different branches differ by the constant $2\pi i n$ for some fixed integer n, the gradient does not depend on which branch is chosen. Note that we can write

$$\begin{aligned}\operatorname{grad}\, \log f(z) &= \left(\overline{\frac{\partial \log f}{\partial z_1}(z)}, \ldots, \overline{\frac{\partial \log f}{\partial z_n}(z)}\right) \\ &= \frac{1}{\overline{f(z)}}\left(\overline{\frac{\partial f}{\partial z_1}(z)}, \ldots, \overline{\frac{\partial f}{\partial z_n}(z)}\right) \\ &= \frac{\operatorname{grad} f(z)}{\overline{f(z)}}.\end{aligned}$$

Lemma 2.5.1. The function $\phi : S_\epsilon \setminus K \to S^1$ has a critical point at z precisely if the vector $i \operatorname{grad}\, \log f(z)$ is a real multiple of z.

Proof. Consider the multi-valued real function $\theta(z)$ defined by

$$\frac{f(z)}{|f(z)|} = e^{i\theta(z)}$$

As with the logarithm function, we can show that the derivative of $\theta(z)$ is well defined for any z by choosing a single-valued branch around a sufficiently small neighbourhood of the desired point z.

Taking the logarithm of both sides of the above equation gives

$$i\theta(z) = \log(f(z)) - \log(|f(z)|).$$

Multiplying both sides by $-i$ and taking the real part gives

$$\theta(z) = \mathrm{Re}(-i\log(f(z))).$$

For a given vector v, let $p(t)$ be a path such that $p(0) = z$ and $p'(0) = v$. Then the directional derivative of θ in the direction of v is given by

$$\begin{aligned}\frac{d\theta(p(t))}{dt}\Big|_{t=0} &= \mathrm{Re}\left(\frac{d(-i\log f(p(t)))}{dt}\Big|_{t=0}\right)\\ &= \mathrm{Re}\langle v, \mathrm{grad}\,(-i\log f(z))\rangle\\ &= \mathrm{Re}\langle v, i\,\mathrm{grad}\,\log f(z)\rangle.\end{aligned}$$

At any point of $z \in \mathbb{C}^n$, the tangent space is $\mathbb{C}^n$, equipped with the hermitian inner product. This tangent space can be identified with $\mathbb{R}^{2n}$ in the standard way ($(x_1 + iy_1, \ldots, x_n + iy_n)$ gets mapped to $(x_1, y_1, \ldots, x_n, y_n)$). Under this identification, the standard inner product on $\mathbb{R}^{2n}$ corresponds to the real part of the hermitian inner product, as can be seen in the case $n = 1$ from

$$\mathrm{Re}\langle x + iy, v + iw\rangle = xv + yw.$$

Now since the vector z seen as an element of $T_z\mathbb{R}^{2n}$ is normal to S_ϵ at $z \in S_\epsilon$, we can write $T_z = \mathbb{R}z \perp T_zS_\epsilon$, where $\perp$ denotes the direct sum of two vector spaces which are mutually orthogonal. Thus $(T_zS_\epsilon)^\perp = \mathbb{R}z$.

Consider the differential $d\theta(z) : T_z\mathbb{R}^{2n} \to \mathbb{R}$. The point z is a critical point of $\theta|_{S_\epsilon}$ precisely if $d\theta(z)$ vanishes on T_zS_ϵ. But we have just seen that $d\theta(z)$ vanishes on $\mathbb{R}(i\,\mathrm{grad}\,\log f(z))^\perp$. Hence z is critical point precisely if $T_zS_\epsilon \subset \mathbb{R}(i\,\mathrm{grad}\,\log f(z))^\perp$. Since both vector spaces have dimension $2n-1$, this is equivalent to $T_zS_\epsilon = \mathbb{R}(i\,\mathrm{grad}\,\log f(z))^\perp$, which is equivalent to $(T_zS_\epsilon)^\perp = \mathbb{R}(i\,\mathrm{grad}\,\log f(z))$.

Thus z is a critical point precisely if $\mathbb{R}(i\,\mathrm{grad}\,\log f(z)) = \mathbb{R}z$, that is if $i\,\mathrm{grad}\,\log f(z)$ is a real multiple of z. □

Thus now we have a simple criterion for deciding whether a point $z \in S_\epsilon \setminus K$ is a critical point of $\theta|_{S_\epsilon}$. Next we must show that ϵ can be chosen small enough so that this criterion is never satisfied.

We will prove something slightly stronger:

Lemma 2.5.2. There exists an $\epsilon > 0$ such that for all $z \in \mathbb{C}^n \setminus V$ with $\|z\| \le \epsilon$, the two vectors z and $\mathrm{grad}\,\log f(z)$ are either linearly independent over $\mathbb{C}$ or $\mathrm{grad}\,\log f(z) = \lambda z$ where $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg\lambda| < \frac{\pi}{4}$.

Note that if i grad $\log f(z)$ is a real multiple of z, then grad $\log f(z) = -icz$ where c is a positive real constant, so $\lambda = -ic$ for which the absolute value of the argument is $\frac{\pi}{2} > \frac{\pi}{4}$. Thus this point z cannot be a critical point of θ restricted to $S_{\|z\|}$.

The method for proving this lemma is to assume that there are points arbitrarily close to the origin not satisfying the requirement, then using the curve selection lemma to find an analytic curve starting at the origin consisting of such points, and then using the following lemma to arrive at a contradiction:

Lemma 2.5.3. Let $p : [0, \epsilon) \to \mathbb{C}^n$ be a real analytic path with $p(0) = 0$ and $p(t) \in \mathbb{C}^n \setminus V$ for $t > 0$, i.e. $f(p(t)) \neq 0$ for $t > 0$. Furthermore, suppose that for $t > 0$ we have

$$\text{grad } \log f(p(t)) = \lambda(t)p(t)$$

where $\lambda(t) \in \mathbb{C}$. Then $\lambda(t) \neq 0$ for t small, and the argument of $\lambda(t)$ tends to 0 as $t \to 0$.

Proof. Consider the Taylor expansions

$$\begin{aligned} p(t) &= at^\alpha + a_1 t^{\alpha+1} + a_2 t^{\alpha+2} + \cdots, \\ f(p(t)) &= bt^\beta + b_1 t^{\beta+1} + b_2 t^{\beta+2} + \cdots, \\ \text{grad } f(p(t)) &= ct^\gamma + c_1 t^{\gamma+1} + c_2 t^{\gamma+2} + \cdots, \end{aligned}$$

where the leading coefficients a, b, c are non-zero. Note that $a, a_i, c, c_i \in \mathbb{C}^n$, while $b, b_i \in \mathbb{C}$. These series are all convergent for $|t| < \epsilon'$ for some ϵ'.

For each $t > 0$ we have

$$\text{grad } \log f(p(t)) = \lambda(t)p(t),$$

or equivalently,

$$\text{grad } f(p(t)) = \lambda(t)p(t)\overline{f(p(t))}.$$

Substituting the above Taylor expansions gives

$$(ct^\gamma + \cdots) = \lambda(t)(a\bar{b}t^{\alpha+\beta} + \cdots).$$

It follows that $\lambda(t)$ is a quotient of real analytic functions, and hence has a Laurent expansion of the form

$$\lambda(t) = \lambda_0 t^{\gamma-\alpha-\beta}(1 + k_1 t + k_2 t^2 + \cdots).$$

Furthermore, comparing leading coefficients, we get $c = \lambda_0 a\bar{b}$. Substituting this into the power series expansion of the identity $\frac{df}{dt} = \langle \frac{dp}{dt}, \text{grad } f \rangle$ gives

$$\begin{aligned} (\beta b t^{\beta-1} + \cdots) &= \langle \alpha a t^{\alpha-1} + \cdots, \lambda_0 a\bar{b}t^\gamma + \cdots \rangle \\ &= \alpha ||a||^2 \overline{\lambda_0} b t^{\alpha-1+\gamma} + \cdots. \end{aligned}$$

Comparing leading coefficients gives $\beta = \alpha ||a||^2 \overline{\lambda_0}$, which proves that λ_0 is a positive real number. Thus the argument of $\lambda(t)$ tends to 0 as t tends to 0. Note that $\lambda(t) \neq 0$ for $t > 0$, t small enough. □

Proof of Lemma 2.5.2. There are two cases to consider. First suppose that $\mathbb{C}^n \setminus V$ contains points z arbitrarily close to the origin for which

$$\text{grad} \log f(z) = \lambda z \neq 0,$$

and with the argument of λ having absolute value greater than $\frac{\pi}{4}$. In other words, assume that λ lies in the open half-plane where $\text{Re}((1+i)\lambda) < 0$ or where $\text{Re}((1-i)\lambda) < 0$.

The idea is now to express these conditions by polynomial equalities and inequalities in order to use the curve selection lemma.

Let W be the set of all z in $\mathbb{C}^n$ for which the vectors $\text{grad} f(z)$ and z are linearly dependent. So $z \in W$ if and only if the equations

$$z_j \left(\overline{\frac{\partial f}{\partial z_k}} \right) = z_k \left(\overline{\frac{\partial f}{\partial z_j}} \right)$$

are satisfied for all j and k. Setting $z_j = x_j + iy_j$, and taking real and imaginary parts, we obtain a collection of real polynomial equations in the real variables x_j and y_j. It follows that $W \subset \mathbb{C}^n$ is a real algebraic set.

Note that a point $z \in \mathbb{C}^n \setminus V$ belongs to W if and only if

$$\frac{\text{grad} f(z)}{\overline{f(z)}} = \lambda z$$

for some complex number λ (because $\overline{f(z)} \neq 0$ outside V). Now multiplying with $\overline{f(z)}$ and taking the inner product with $\overline{f(z)}z$ gives

$$\langle \text{grad} f(z), \overline{f(z)}z \rangle = \lambda ||\overline{f(z)}z||^2.$$

So if we define the function λ' on $\mathbb{C}^n$ by

$$\lambda'(z) = \langle \text{grad} f(z), \overline{f(z)}z \rangle,$$

then for $z \in W \setminus V$ for which $\lambda(z) \neq 0$ we have that $\lambda'(z)$ is also non-zero, and the quotient $\frac{\lambda'(z)}{\lambda(z)}$ is a positive real number. Hence in this case $\lambda(z)$ and $\lambda'(z)$ have the same argument.

Note that $\lambda'(z)$ is a polynomial function of the real variables x_j and y_j.

Now let U_+ (respectively U_-) be the open set consisting of $z \in \mathbb{C}^n$ satisfying the real polynomial inequality $\text{Re}((1+i)\lambda'(z)) < 0$ (respectively $\text{Re}((1-i)\lambda'(z)) < 0$).

Note that a point z is in $W \cap (U_+ \cup U_-)$ precisely if the conditions given at the beginning of the proof are satisfied. This is because if $z \in U_+ \cup U_-$ then $\lambda'(z) \neq 0$, hence $\text{grad} \log f(z) \neq 0$. And $z \in W$ guarantees that $\text{grad} \log f(z) = \lambda z$. Finally, using again that $z \in U_+ \cup U_-$ implies that the argument of λ', and hence of λ, has absolute value strictly greater than $\frac{\pi}{4}$. The converse is clear.

Thus our assumption can be translated as saying that there are points z arbitrarily close to the origin with $z \in W \cap (U_+ \cup U_-)$. This implies that $0 \in \overline{W \cap U_+}$ or $0 \in \overline{W \cap U_-}$. Applying the curve selection lemma gives an analytic path $p : [0, \epsilon) \to \mathbb{C}^n$ with $p(0) = 0$ and with either $p(t) \in W \cap U_+$ for all $t > 0$ or $p(t) \in W \cap U_-$ for all $t > 0$. In either case, for each $t > 0$ we get

$$\text{grad} \log f(p(t)) = \lambda(t)p(t)$$

with

$$|\text{argument } \lambda(t)| > \frac{\pi}{4},$$

contradicting Lemma 2.5.3.

It remains to consider the case where there are points $z \in W \setminus (V \cap W)$ arbitrarily close to the origin for which $\lambda'(z) = 0$ or $|\arg \lambda'(z)| = \frac{\pi}{4}$. But then we consider the algebraic subset $W' \subset W$ defined by imposing the additional equation

$$\text{Re}((1+i)\lambda'(z))\text{Re}((1-i)\lambda'(z)) = 0,$$

and let U' be the set of z for which $|f(z)|^2 > 0$. The above condition then means that there are points arbitrarily close to the origin in $W' \cap U'$. Thus we can apply the curve selection lemma again to find a path $p : [0, \epsilon) \to \mathbb{C}^n$ for which $p(0) = 0$ and $p(t) \in W' \cap U'$ for $t > 0$. Then for some $\epsilon' > 0$ either $p(t) = 0$ for all $t \in [0, \epsilon')$, or $p(t) \neq 0$ and $|\arg \lambda'(t)| = \frac{\pi}{4}$. The first possibility would contradict one of the conclusions of the lemma (namely, that $\lambda(t) \neq 0$ for $t > 0$), while the other would also contradict the lemma. This completes this last case, and the proof of the lemma. □

Thus we can prove the following corollary:

Corollary 2.5.4. There exists an $\epsilon' > 0$ such that for any $0 < \epsilon \leq \epsilon'$, the map $\phi : S_\epsilon \setminus K \to S^1$ has no critical points.

Proof. Let ϵ' be a number found by Lemma 2.5.2. Then the corollary follows directly from Lemma 2.5.1. □

It follows that for each $e^{i\theta} \in S^1$, the fiber

$$F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\epsilon \setminus K$$

is a smooth $2n-2$-dimensional manifold.

Next we want to show that ϕ is a locally trivial fibration. The idea is to construct a vector field v on $S_\epsilon \setminus K$ such that solution curves of the differential equation $\frac{dz}{dt} = v(z)$ carry fibers of ϕ onto other fibers of ϕ. So if we denote $\phi(z) = \frac{f(z)}{|f(z)|}$ by $e^{i\theta(z)}$ as in the beginning of the section, then the first requirement on the vector field v is that the directional derivative of $\theta(z)$ in the direction of $v(z)$ must be constant along a fiber of ϕ, for simplicity say it must be 1 everywhere. Using the expression for this directional derivative found in the proof of Lemma 2.5.1 this translates into requiring that

$$\text{Re}\langle v(z), i \,\text{grad}\, \log f(z)\rangle = 1$$

for all z in $S_\epsilon \setminus K$. It will be seen that a further requirement is necessary to enable us to extend a local solution of the differential equation to a sufficiently large interval.

This is all by way of motivating the following lemma.

Lemma 2.5.5. Let ϵ' be as in Corollary 2.5.4. If $\epsilon < \epsilon'$ then there exists a smooth tangential vector field v on $S_\epsilon \setminus K$ such that for each $z \in S_\epsilon \setminus K$ the complex inner product

$$\langle v(z), i \,\mathrm{grad}\, \log f(z) \rangle$$

is non-zero, and its argument is strictly less than $\frac{\pi}{4}$ in absolute value.

Proof. As in the proof of Lemma 2.3.4 about the conical structure around a singular point, it suffices to construct such a vector field locally around any point $z_\alpha \in S_\epsilon \setminus K$.

There are two possible cases. Firstly, if the vectors z_α and $\mathrm{grad} \log f(z_\alpha)$ are linearly independent over $\mathbb{C}$, then the linear equations

$$\langle v(z_\alpha), z_\alpha \rangle = 0$$
$$\langle v(z_\alpha), i \,\mathrm{grad}\, \log f(z_\alpha) \rangle = 1$$

have a simultaneous solution for $v(z_\alpha)$. The first equation implies in particular that

$$\mathrm{Re}\langle v(z_\alpha), z_\alpha \rangle = 0,$$

so $v(z_\alpha)$ is tangent to S_ϵ at z_α.

Otherwise, if $\mathrm{grad} \log f(z_\alpha) = \lambda z_\alpha$ for some non-zero complex number λ, then set $v(z_\alpha) = iz_\alpha$. Then $\mathrm{Re}\langle iz_\alpha, z_\alpha \rangle = 0$, so again $v(z_\alpha)$ is tangent to S_ϵ at z_α. Furthermore, it follows from Lemma 2.5.2 that λ has argument less than $\frac{\pi}{4}$ in absolute value, hence

$$\begin{aligned} \langle iz_\alpha, i \,\mathrm{grad}\, \log f(z_\alpha) \rangle &= iz_\alpha(-i)\overline{\lambda z_\alpha} \\ &= \overline{\lambda} \|z_\alpha\|^2 \end{aligned}$$

has argument less than $\frac{\pi}{4}$ in absolute value.

In either case one can construct a vector field v_α on $S_\epsilon \setminus K$ in a neighbourhood of z_α which is tangent to S_ϵ and such that $v_\alpha(z_\alpha) = v(z_\alpha)$. The condition

$$|\arg\langle v_\alpha(z), i \,\mathrm{grad}\, \log f(z) \rangle| < \frac{\pi}{4}$$

is an open condition, and hence it will hold in a neighbourhood of z_α where v_α is defined. Using a partition of unity, we obtain a global vector field v satisfying the same properties, and completing the proof of the lemma. □

Now create a vector field w on $S_\epsilon \setminus K$ by normalizing:

$$w(z) = \frac{v(z)}{\mathrm{Re}\langle v(z), i \,\mathrm{grad}\, \log f(z) \rangle}.$$

The vector field w then satisfies two properties. Firstly,

$$\mathrm{Re}\langle w(z), i \,\mathrm{grad}\, \log f(z) \rangle = 1$$

for all z, and since v and w have the same argument, it follows that

$$|\mathrm{Im}\langle w(z), i \,\mathrm{grad}\, \log f(z) \rangle| < 1.$$

Now consider the solutions of the differential equation $\frac{dz}{dt} = w(z)$. The following lemma shows that a local solution can be extended to be defined on the whole of $\mathbb{R}$.

Lemma 2.5.6. Given any $z_0 \in S_\epsilon \setminus K$, there exists a unique smooth path $p : \mathbb{R} \to S_\epsilon \setminus K$ satisfying the differential equation $\frac{dp(t)}{dt} = w(p(t))$ with initial condition $p(0) = z_0$.

Proof. Such a solution exists locally, and can be extended over some maximum interval of real numbers. We just need to verify that $p(t)$ does not tend to K as t tends to some finite limit t_0. This is equivalent to showing that $f(p(t))$ does not tend to 0, or also equivalently, that $\mathrm{Re}(\log f(p(t)))$ does not tend to $-\infty$ as t tends to some t_0.

But the derivative

$$\begin{aligned}\frac{d}{dt}(\mathrm{Re} \log f(p(t))) &= \mathrm{Re}(\frac{d}{dt} \log f(p(t))) \\ &= \mathrm{Re}\langle \frac{dp(t)}{dt}, \mathrm{grad}\ \log f(p(t))\rangle \\ &= -\mathrm{Im}\langle \frac{dp(t)}{dt}, i\ \mathrm{grad}\ \log f(p(t))\rangle\end{aligned}$$

has absolute value less than 1. Hence $|f(p(t))|$ is bounded away from 0 as t tends to any finite limit. □

Now setting $\phi(z) = e^{i\theta(z)}$ as before, we find that

$$\begin{aligned}\frac{d\theta(p(t))}{dt} &= \mathrm{Re}\langle \frac{dp}{dt}, i\,\mathrm{grad}\ \log f(p(t))\rangle \\ &= 1.\end{aligned}$$

Hence

$$\theta(p(t)) = t + c$$

where c is a constant. The point $p(t)$ is a smooth function both of t and of the initial value $z_0 = p(0)$. For a proof of this result, see [War83, p.37] as well as the reference there to [Hur58, p.27]. Thus for every t we can define a function h_t from $S_\epsilon \setminus K$ to itself by setting $h_t(z_0) = p(t)$, where p is the solution of the differential equation $\frac{dp}{dt} = w$ with $p(0) = z_0$. Then h_t is a diffeomorphism from $S_\epsilon \setminus K$ to itself taking each fiber $F_\theta = \phi^{-1}(e^{i\theta})$ onto the fiber $F_{\theta+t}$.

The fibration theorem follows:

Theorem 2.5.7. For $\epsilon < \epsilon'$, the space $S_\epsilon \setminus K$ is a smooth fiber bundle over S^1, with projection mapping given by $\phi(z) = \frac{f(z)}{|f(z)|}$.

Proof. For a given $e^{i\theta} \in S^1$, define a neighbourhood U as

$$U = \{e^{i(t+\theta)} \in S^1 \mid |t| < \frac{\pi}{2}\}$$

Then the trivialization function $\lambda : U \times F_\theta \to \phi^{-1}(U)$ given by $\lambda(e^{i(t+\theta)}, z) \to h_t(z)$ is a diffeomorphism showing that ϕ is locally trivial. □

One could ask in what sense this fibration depends upon ϵ. To compare different fibrations, we use the following definition:

Definition 2.5.8. Let $f : E \to B$ and $f' : E' \to B'$ be smooth fibrations. Then f and f' are equivalent if there exist diffeomorphisms $\Phi : E \to E'$ and $\phi : B \to B'$ such that $\phi \circ f = f' \circ \Phi$.

Let $\epsilon_1, \epsilon_2 < \epsilon'$. Then by the above theorem, both $S_{\epsilon_1} \setminus K$ and $S_{\epsilon_2} \setminus K$ can be seen as smooth fiber bundles over S^1, with the same projection mapping given by $\phi(z) = \frac{f(z)}{|f(z)|}$. It is possible to show that these fibrations are equivalent, by constructing a fiber-preserving diffeomorphism from S_{ϵ_1} to S_{ϵ_2}. We will not give the details here, since they are very similar to those of Lemma 2.5.5 and Proposition 2.6.5.

2.6 An equivalent fibration

There are two fibrations in the literature commonly referred to as *the Milnor fibration*. The one originally constructed by Milnor was described in the previous section. In this section we describe the other fibration, and show that the two are equivalent.

Let f be a complex polynomial with an isolated singularity at 0. Then Theorem 2.3.3 states that for a sufficiently small $\epsilon > 0$, the fiber $f^{-1}(0)$ intersects the sphere S_ϵ transversally, i.e. $f^{-1}(0) \pitchfork S_\epsilon$.

Lemma 2.6.1. There exists a $\delta > 0$ such that $f^{-1}(t) \pitchfork S_\epsilon$ for all $t \in D_\delta$, where D_δ denotes the open disk around $0 \in \mathbb{C}$ with radius δ.

Proof. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way. Let $z_\alpha \in f^{-1}(0) \cap S_\epsilon$. Then since z_α is a regular value of f as a complex polynomial, it must also be a regular value of $f = (f_1, f_2)$ as a map from $\mathbb{R}^{2n}$ to $\mathbb{R}^2$. (To see this, recall that the determinant of the Jacobian matrix of a holomorphic function on $\mathbb{C}$ is given by $|f'|^2$. So if the holomorphic function is regular, i.e. $f' \neq 0$, then the associated smooth function will also be regular.) Thus $\ker df(z_\alpha)$ has dimension $2n-2$. Let g denote the map f restricted to S_ϵ. Then

$$\begin{aligned} & f^{-1}(0) \pitchfork S_\epsilon \text{ at } z_\alpha \\ \Leftrightarrow\ & \ker df(z_\alpha) \nsubseteq T_{z_\alpha} S_\epsilon \\ \Leftrightarrow\ & \ker dg(z_\alpha) \ \ (= \ker df(\alpha) \cap T_{z_\alpha} S_\epsilon) \text{ has dimension } 2n-3 \\ \Leftrightarrow\ & g \text{ is regular at } z_\alpha. \end{aligned}$$

Now we show that g is regular in a neighbourhood of z_α. Choose local coordinates $x_1, \ldots, x_{2n}$ around z_α such that S_ϵ corresponds to the hyperplane $\{x_{2n} = 0\}$ and z_α corresponds to 0. Then $x_1, \ldots, x_{2n-1}$ forms a set of local coordinates for S_ϵ and so the condition that g is regular at z_α translates into the statement that the matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i \le 2; 1 \le j \le 2n-1}$$

is nonsingular at 0. That means that it contains some 2×2 submatrix with non-zero determinant. But since the determinant function is continuous and $\mathbb{R} \setminus \{0\}$ is open, this same submatrix will have non-zero determinant in a neighbourhood of 0. Hence g is regular in a neighbourhood of z_α, say $U_\alpha \cap S_\epsilon$. This implies that for any $z \in U_\alpha \cap S_\epsilon$, $f^{-1}(t) \pitchfork S_\epsilon$ at z, where $t = f(z)$.

Now for any $z_\alpha \in f^{-1}(0) \cap S_\epsilon$ we construct such an open neighbourhood U_α.

Let $U = \bigcup_\alpha U_\alpha$. We would like to find an open disk D_δ around $0 \in \mathbb{R}^2$ such that

$$f^{-1}(D_\delta) \cap S_\epsilon \subseteq U \cap S_\epsilon.$$

Suppose this cannot be done. In other words, suppose that there exists a sequence $(t_n)_{n\geq 1}$ in $\mathbb{R}^2$ converging to 0 such that $f^{-1}(t_n) \cap S_\epsilon \nsubseteq U \cap S_\epsilon$ for every $n \geq 1$. Then let $(z_n)_{n\geq 1}$ be a sequence in $S_\epsilon \setminus U$, such that $f(z_n) = t_n$. Since $S_\epsilon \setminus U$ is compact, we can, by taking a convergent subsequence, assume that $(z_n)_{n\geq 1}$ converges to a point $z \in S_\epsilon \setminus U$. Since f is continuous, it follows that $(t_n)_{n\geq 1}$ converges to $f(z)$. But since limits are unique, $f(z) = 0$. Hence $z \in f^{-1}(0) \cap S_\epsilon \subseteq U \cap S_\epsilon$, which is a contradiction. Thus there exists a $\delta > 0$ with the required property. □

This lemma will enable us to apply the following proposition to our case:

Proposition 2.6.2 (Ehresmann's fibration theorem)**.** Let $f : E \to B$ be a proper submersion between the manifolds-with-boundary E and B. Then f is a locally trivial fibration, i.e. for any $b \in B$ there exists a neighbourhood U of b and a diffeomorphism (called the trivialization) from $f^{-1}(U)$ to $U \times f^{-1}(b)$ preserving the projection to U. Moreover, if A is a closed subset of E such that f restricted to A is still a submersion, then the trivialization can be chosen to map $f^{-1}(U) \cap A$ onto $U \times (f^{-1}(b) \cap A)$. In this case we say that f is a locally trivial fibration of the pair (E, A) over B.

For the proof see [Ehr47] and [Lam81]. We remark that throughout this thesis, the closed subset $A \subset E$ will always be the boundary of E, i.e. ∂E.

Proposition 2.6.3. Choose $\epsilon > 0$ and $\delta > 0$ in such a way as to satisfy the conditions of Lemma 2.6.1. Let $E = \overline{B}_\epsilon \cap f^{-1}(D_\delta \setminus \{0\})$ and $B = D_\delta \setminus \{0\}$. Denote the restriction of f to E by $\tilde{\psi}$, and consider the closed subset $\partial E = S_\epsilon \cap f^{-1}(D_\delta \setminus \{0\}) \subset E$. Then $\tilde{\psi}$ is a locally trivial fibration of the pair $(E, \partial E)$ over B.

Proof. By Ehresmann's fibration theorem, we just have to show that $\tilde{\psi}$ is a proper submersion and $\tilde{\psi}$ restricted to ∂E is still a submersion.

Let $V \subset D_\delta \setminus \{0\}$ be compact. Then it is closed, hence $\tilde{\psi}^{-1}(V)$ is closed. But $\tilde{\psi}^{-1}(V) \subset \overline{B}_\epsilon$ is also bounded, hence it is compact. Thus $\tilde{\psi}$ is proper.

Let $x \in E \setminus \partial E$. Then in a neighbourhood of x, $\tilde{\psi} = f$. But the function f is a submersion everywhere except at the origin, hence $\tilde{\psi}$ is a submersion at x. Now let

$$x \in \partial E = f^{-1}(D_\delta \setminus \{0\}) \cap S_\epsilon.$$

But if $y = f(x)$, then by the previous lemma, $f^{-1}(y) \pitchfork S_\epsilon$ at x, hence f (and also $\tilde{\psi}$) is still a submersion when restricted to ∂M. □

Note that replacing δ by a smaller positive value, say $\delta' > 0$, would yield an equivalent fibration. The equivalence could be established by constructing a vector field on $D_\delta \setminus \{0\}$ whose flow creates a diffeomorphism from $D_\delta \setminus \{0\}$ to $D_{\delta'} \setminus \{0\}$, and then lifting this vector field horizontally to E.

The alternative description of the Milnor fibration, denoted by ψ, is found by restricting $\tilde{\psi}$ to the pre-image of a circle, say $\partial \overline{D}_{\delta'} = S_{\delta'}$, where $0 < \delta' < \delta$, and leaving out the boundary $\partial \overline{B}_\epsilon$. This is still a locally trivial fibration:

$$\psi : B_\epsilon \cap f^{-1}(S_{\delta'}) \to S_{\delta'} = S^1.$$

We replace δ' by δ in what follows.

Now we want to show that this fibration is equivalent to the fibration $\phi : S_\epsilon \setminus f^{-1}(0) \to S^1$, where $\phi(x) = \frac{f(x)}{|f(x)|}$. This will be done in two steps. First we will show that the fibration ψ is equivalent to ϕ', which is ϕ restricted to $S_\epsilon \setminus f^{-1}(\overline{D}_\delta)$. Then we show that ϕ' is equivalent to ϕ.

Lemma 2.6.4. There exists a smooth vector field v on $\overline{B}_\epsilon \setminus V$ such that the inner product

$$\langle v(z), \text{ grad } \log f(z) \rangle$$

is real and positive for all $z \in \overline{B}_\epsilon \setminus V$, and such that the inner product $\langle v(z), z \rangle$ has constant real part.

Proof. The proof is similar to the proof of Lemma 2.5.5. It is enough to construct such a vector field in the neighbourhood of any point $z_\alpha \in \overline{B}_\epsilon \setminus V$. There are two cases to consider. Firstly, if the vectors z_α and grad $\log f(z_\alpha)$ are linearly independent over $\mathbb{C}$, then the linear equations

$$\langle v(z_\alpha), \text{ grad } \log f(z_\alpha) \rangle = 1 \qquad \text{and}$$
$$\langle v(z_\alpha), z_\alpha \rangle = 1$$

have a simultaneous solution for $v(z_\alpha)$.

The second case is if $z_\alpha = \lambda$ grad $\log f(z_\alpha)$ for some $\lambda \in \mathbb{C}$. Then let $v(z_\alpha) = \overline{\lambda} z_\alpha$. From Lemma 2.5.2 it follows that $\text{Re}(\lambda) > 0$. Then

$$\langle v(z_\alpha), \text{ grad } \log f(z_\alpha) \rangle = \|z_\alpha\|^2 \qquad \text{and}$$
$$\text{Re}\langle v(z_\alpha), z_\alpha \rangle = \text{Re}(\lambda) \|z_\alpha\|^2 > 0.$$

So in either case we get a vector $v(z_\alpha)$ satisfying all the conditions of the lemma except that the inner product $\langle v(z_\alpha), z_\alpha \rangle$ has positive real part, but not necessarily constant real part. We can extend this vector to a vector field in a neighbourhood of z_α satisfying the same conditions. Using a partition of unity as before gives a global vector field v. To meet the final requirement, we normalize this vector field by replacing $v(z)$ with

$$\frac{v(z)}{\text{Re}(\langle v(z), z \rangle)}$$

for any $z \in \overline{B}_\epsilon \setminus V$. Note that the denominator is always non-zero, and that $\text{Re}\langle v(z), z \rangle = 1$ for the resulting vector field v. $\square$

Using this lemma, we can prove the following:

Proposition 2.6.5. There is a diffeomorphism from $B_\epsilon \cap f^{-1}(S_\delta)$ to $S_\epsilon \setminus f^{-1}(\overline{D}_\delta)$ which restricts to a diffeomorphism between the fibers $\psi^{-1}(c) = B_\epsilon \cap f^{-1}(c)$ and $\phi'^{-1}(\frac{c}{|c|})$, where ϕ' is as defined above and $|c| = \delta$.

Proof. Let $z_0 \in B_\epsilon \cap f^{-1}(S_\delta)$. Let v be as in Lemma 2.6.4 and let $p(t)$ be the solution of the differential equation

$$\frac{dp(t)}{dt} = v(p(t)) \tag{2.1}$$

with initial condition $p(0) = z_0$. Since

$$\langle v(p(t)),\ \text{grad}\ \log f(p(t))\rangle$$

is real, it follows that

$$\langle v(p(t)), i\ \text{grad}\ \log f(p(t))\rangle$$

is purely imaginary, and hence has real part 0. But we have shown before that this real part is the derivative of the argument of $f(p(t))$. Hence the argument of $f(p(t))$ is constant. It also follows that $|f(p(t))|$ is a strictly-increasing function of t, by performing the following calculation:

$$\begin{aligned}
\langle\frac{dp(t)}{dt},\ \text{grad}\ \log f(p(t))\rangle &= \langle\frac{dp(t)}{dt}, \frac{\text{grad}\ f(p(t))}{\overline{f(p(t))}}\rangle \\
&= \frac{1}{f(p(t))}\langle\frac{dp(t)}{dt}, \text{grad}\ f(p(t))\rangle \\
&= \frac{1}{f(p(t))}\frac{d}{dt}f(p(t)) \\
&= \frac{1}{f(p(t))}\frac{d}{dt}|f(p(t))|\left(\frac{f(p(t))}{|f(p(t))|}\right) \\
&= \frac{1}{|f(p(t))|}\frac{d}{dt}|f(p(t))| > 0,
\end{aligned}$$

hence

$$\frac{d}{dt}|f(p(t))| > 0.$$

Using the other condition gives

$$\begin{aligned}
\frac{d\|p(t)\|^2}{dt} &= \frac{d}{dt}(\text{Re}\langle p(t), p(t)\rangle) \\
&= 2\text{Re}\langle\frac{dp(t)}{dt}, p(t)\rangle > 0.
\end{aligned}$$

Hence $\|p(t)\|$ is a strictly increasing function of t.

Thus for some value of t, say t_1, we must have $p(t_1) \in S_\epsilon$. Define a function Θ from $B_\epsilon \cap f^{-1}(S_\delta)$ to S_ϵ by setting $\Theta(z_0) = p(t_1)$. Since the argument remains constant, Θ preserves the fibration. The image of the function Θ is contained in $S_\epsilon \setminus f^{-1}(\overline{D}_\delta)$, since $|f(p(0))| = \delta$, and

$|f(p(t))|$ is strictly increasing, hence $|f(p(t_1))| > \delta$. But this is the entire image, because we can also start with a point in $S_\epsilon \setminus f^{-1}(\overline{D}_\delta)$ and take a solution which goes the other way, and will eventually arrive at a point of $B_\epsilon \cap f^{-1}(S_\delta)$.

Thus the fibrations ψ and ϕ' are equivalent. □

Now for the second part:

Proposition 2.6.6. The fibrations ϕ' and ϕ are equivalent.

Proof. To show this, we need to construct a smooth vector field on $S_\epsilon \setminus f^{-1}(0)$ such that flowing along this vector field gives a fiber-preserving diffeomorphism from $S_\epsilon \setminus f^{-1}(0)$ to $S_\epsilon \setminus f^{-1}(\overline{D}_\delta)$. To do this, we consider the real-valued, positive function $|f|$ on $S_\epsilon \setminus f^{-1}(0)$.

First we use the curve selection lemma to show that for any $\theta \in \mathbb{R}$, δ can be chosen small enough so that $|f|$ restricted to the fiber of ϕ above $e^{i\theta} \in S^1$ has no critical points on the part of the fiber lying in $S_\epsilon \cap f^{-1}(D_\delta \setminus \{0\})$. Suppose this were not true. Then there would exist critical points z of the restriction of $|f|$ with $|f(z)|$ arbitrarily close to 0. This set of critical points would have a limit point z_0 on the compact set S_ϵ and by continuity $f(z_0) = 0$. Then with the intention of applying the curve selection lemma (Lemma 2.4.1), we let $U \subset \mathbb{R}^{2n}$ be the open set where $|f|^2 > 0$, and V the set of critical points of $|f|^2$ restricted to the fiber of ϕ above $e^{i\theta} \in S^1$. Note that U and V are given by polynomial inequalities and equalities respectively. Then we can use the curve selection lemma to conclude the existence of a smooth curve

$$p : (0, \epsilon') \to S_\epsilon \setminus f^{-1}(0)$$

consisting entirely of critical points, with $p(t)$ tending to z_0 as t tends to 0. But then $|f(p(t))|$ must be constant along this path, and hence cannot tend to $|f(p(0))| = 0$.

Thus δ can be chosen small enough so that $|f|$ restricted to the fiber of ϕ above $e^{i\theta} \in S^1$ has no critical points on the part of the fiber lying in $S_\epsilon \cap f^{-1}(D_\delta \setminus \{0\})$, as desired. By possibly choosing δ slightly smaller, we can assume that $|f|$ also has no critical points when restricted to nearby fibers.

Using the compactness of S^1, we can find a single δ such that the restriction of $|f|$ to any of the fibers of ϕ has no critical points in $S_\epsilon \cap f^{-1}(D_\delta \setminus \{0\})$.

Next we construct a vector field on $S_\epsilon \setminus f^{-1}(0)$. This is done in several steps. Fix a δ' with $0 < \delta' \ll \delta$. Then by replacing δ by $\delta - \delta'$ in the above argument, we can assume that the restriction of $|f|$ to the fibers of ϕ has no critical points in $S_\epsilon \cap f^{-1}(D_{\delta+\delta'} \setminus \{0\})$.

Let $z \in S_\epsilon \cap f^{-1}(D_{\delta+\delta'} \setminus \{0\})$, and let θ denote the argument of $f(z)$, i.e. $\phi(z) = e^{i\theta}$. Now let $|f|_\theta$ denote the restriction of $|f|$ to the fiber above $e^{i\theta} \in S^1$. Define

$$v(z) = \frac{\nabla |f|_\theta(z)}{\| \nabla |f|_\theta(z) \|},$$

where ∇ denotes the usual real gradient.

In this way we constuct a vector field v on $S_\epsilon \cap f^{-1}(D_{\delta+\delta'} \setminus \{0\})$ which is clearly tangent to the fibers. To show it is smooth, note that in local coordinates, ϕ is a projection from $\mathbb{R}^{2n-1}$ to $\mathbb{R}$, mapping $(x_1, \ldots, x_{2n-1})$ to x_1 (using the fact that ϕ has no critical points by Lemma 2.5.2).

And in these local coordinates, v is simply $(0, \frac{\partial|f|}{\partial x_2}, \ldots, \frac{\partial|f|}{\partial x_{2n-1}})$ divided by its norm, which is clearly smooth.

Note that if $p(t)$ is a local solution of the differential equation $\frac{dp(t)}{dt} = v(p(t))$, then

$$\frac{d}{dt}|f(p(t))| = 1.$$

Let g be a smooth function on $(0, \infty)$ such that

$$g(t) = 1 \text{ for } t \leq \delta,$$
$$g(t) \in [0,1] \text{ for } t \in (\delta, \delta + \delta') \text{ and}$$
$$g(t) = 0 \text{ for } t \geq \delta + \delta'.$$

Now let w be the vector field on $S_\epsilon \setminus f^{-1}(0)$ defined by

$$w(z) = v(z)g(|f(z)|) \quad \text{if} \quad z \in S_\epsilon \cap f^{-1}(D_{\delta+\delta'} \setminus \{0\}) \quad \text{and}$$
$$w(z) = 0 \text{ otherwise.}$$

Since g is smooth, the vector field w is smooth.

Finally we can define Θ from $S_\epsilon \setminus f^{-1}(0)$ to $S_\epsilon \setminus f^{-1}(\overline{D}_\delta)$ as follows: for a given z, let $p(t)$ be the solution of the differential equation $\frac{dp(t)}{dt} = w(p(t))$ defined over a maximum open interval with $p(0) = z$. Then let $\Theta(z) = p(\delta)$. The function Θ is then a fiber-preserving diffeomorphism, completing the proof of the second part. Thus the two descriptions of the Milnor fibration are equivalent. $\square$

Chapter 3

The Milnor fiber

The goal of this chapter to to study the topology of the Milnor fiber. Our first goal is to determine its homology groups.

3.1 Geometric monodromy

In the previous chapter we showed that the function

$$F : \overline{B}_\epsilon \cap f^{-1}(\partial \overline{D}_\delta) \to \partial \overline{D}_\delta$$

given by $F(z) = f(z)$, is a locally trivial fibration (it follows from Proposition 2.6.3). As before, f is a polynomial in n complex variables with an isolated singular point at the origin. Denote the total space of the fibration by $E := \overline{B}_\epsilon \cap f^{-1}(\partial \overline{D}_\delta)$ and the base space by $B := \partial \overline{D}_\delta$. We will denote the fiber above a point $z \in B$ by F_z and refer to it as the *Milnor fiber*. Let $\mathrm{Diff}(F_\delta)$ denote the group of diffeomorphisms of F_δ, and $\mathrm{Diff}_0(F_\delta)$ the normal subgroup of diffeomorphisms isotopic to the identity. Then the mapping class group of F_δ is defined as

$$\Gamma(F_\delta) := \mathrm{Diff}(F_\delta)/\mathrm{Diff}_0(F_\delta).$$

We will show that there is a natural group homomorphism from $\pi_1(\partial \overline{D}_\delta, \delta)$ to $\Gamma(F_\delta)$. Since $\pi_1(\partial \overline{D}_\delta, \delta) \cong \mathbb{Z}$ is cyclic, it is enough to describe this map for a generator. This generator will be the path γ, where

$$\gamma : [0,1] \to \partial \overline{D}_\delta$$

is given by $\gamma(t) = \delta e^{2\pi i t}$. The corresponding diffeomorphism of the fiber F_δ is found as follows: We construct a vector field on the base space such that γ is the flow along this vector field. Then this vector field is lifted to a vector field on the total space. The flow along this vector field then gives a diffeomorphism of the fiber, and it remains to show that it is defined up to isotopy. This diffeomorphism will be called the *geometric monodromy* and will be denoted by h.

Before we carry out the construction explicitly, note that given simply the fiber F_δ and the geometric monodromy, we can reconstruct the entire fibration F. The total space is the quotient of $F_\delta \times [0,1]$ by the equivalence relation which identifies $(z,0)$ with $(h(z),1)$, and then the mapping to S^1 is simply given by the projection onto the second factor.

Let w be the vector field on $\partial\overline{D}_\delta$ given by $w(\delta e^{2\pi it}) = 2\pi i\delta e^{2\pi it}$. Let γ be the flow along this vector field such that $\gamma(0) = \delta$. Then γ is the extension of the path γ defined above, to $\mathbb{R}$, defined in the same way. Let $\{U_\alpha\}$ be a collection of open sets covering E such that F restricted to any one of these open sets is given as a projection in suitable coordinates. In other words, for a given α, $F|_{U_\alpha}$ is of the form $(x_1,\ldots,x_{2n-1}) \mapsto x_1$. Furthermore, if $U_\alpha \cap \partial E \neq \emptyset$ we can choose the coordinates in such a way that $U_\alpha \cap \partial E$ corresponds to the part where $x_{2n-1} = 0$.

Let y be a local coordinate for the base space in this case. Define a vector field v_α on U_α by setting $v_\alpha(x_1,\ldots,x_{2n-1}) = w_1(x_1)\frac{\partial}{\partial x_1}$, where $w(x_1) = w_1(x_1)\frac{\partial}{\partial y}$. Then v_α has the property that $dF_z(v_\alpha) = w(F(z))$ for any $z \in U_\alpha$. Furthermore, if $z \in \partial E$, then $v_\alpha(z) \in T_z\partial E \subset T_zE$.

By using a partition of unity subordinate to the cover $\{U_\alpha\}$, we can, as in Theorem 2.3.4 for example, create a global vector field v with the properties that $T_zF(v_\alpha) = w(F(z))$ for any $z \in E$, and $v(z) \in T_z\partial E \subset T_zE$ for $z \in \partial E$.

Now for any $t \in \mathbb{R}$ construct a diffeomorphism h_t from F_δ to $F_{\gamma(t)}$ as follows: Let $z_0 \in F_\delta$, and let $p(t)$ be the solution of the differential equation $\frac{dz}{dt} = v(z)$ with $p(0) = z_0$, defined over a maximal interval. We claim that p can be defined on the whole of $\mathbb{R}$. To see this, suppose that p is defined on an open interval (a,b) with $b < \infty$. Then by the compactness of E, we can find a point $z_0 \in E$ such that $p(t)$ tends to z_0 as t tends to b from below. There exists a unique solution q of the same differential equation such that $q(b) = z_0$ on a small neighbourhood $(b-\delta, b+\delta)$ of b. If $z_0 \notin \partial E$, then this is clear. Otherwise, if $z_0 \in \partial E$, then we use the fact that $v(z_0)$ is tangent to ∂E to find a local solution contained in ∂E and hence in E. By uniqueness the two solutions coincide on $(b-\delta, b)$, and thus p can be defined on the larger interval $(a, b+\delta)$. A similar argument shows that p can be defined on $(a-\delta, b)$ for some $\delta > 0$ if $a > -\infty$. Thus, using Zorn's lemma, we can conclude that p is defined on $\mathbb{R}$.

Then define $h_t(z_0)$ as $p(t)$. Since v is a lifting of w, and $F(p(0)) = \gamma(0)$, it follows that $F(p(t)) = \gamma(t)$ and hence $h_t(z_0) \in F_{\gamma(t)}$.

Since $\gamma(0) = \gamma(1) = \delta$, we see that h_1 is a diffeomorphism of F_δ.

Note that we could have chosen a different lifting of the vector field w. We want to show that this would not change the diffeomorphism h_1 in an essential way.

Proposition 3.1.1. Let v' be a vector field on E which is another lifting of w (i.e. $dF_z(v') = w(F(z))$ for any $z \in E$, and $v'(z) \in T_z\partial E \subset T_zE$ for $z \in \partial E$), and let h' be the associated family of diffeomorphisms. Then h_1 and h'_1 are isotopic.

Proof. Let $\mathcal{E}$ denote the space $E \times [0,1]$. Define a vector field V on $\mathcal{E}$ by setting

$$V(z,s) = ((1-s)v(z) + sv'(z), 0).$$

The vector field V is smooth. As in the above argument, we can use solutions of the differential equation $\frac{dz}{dt} = V(z)$ to construct a diffeomorphism H from $F_\delta \times [0,1] \subset \mathcal{E}$ to itself by flowing along these solution curves for one unit of time t. Composing with the projection to the first factor gives the homotopy (also denoted H)

$$H : F_\delta \times [0,1] \to F_\delta$$

such that $H(z,0) = h(z)$ and $H(z,1) = h'(z)$ for all $z \in F_\delta$. Furthermore, for any fixed $s \in [0,1]$, the function $H(z,s)$ is a diffeomorphism from F_δ to itself, hence H is in fact an isotopy between h and h'. □

Definition 3.1.2. The diffeomorphism $h := h_1$ from F_δ to itself is called the *geometric monodromy.* It is defined up to isotopy. The induced homomorphism h_* on the homology groups $H_*(F_\delta, \mathbb{Z})$ is called the *algebraic monodromy.*

One last fact about the diffeomorhism h:

Lemma 3.1.3. The diffeomorphism h can be taken to be the identity on the boundary

$$\partial F_\delta = \partial \overline{B}_\epsilon \cap f^{-1}(\delta).$$

Proof. Consider the fibration given by the restriction of f:

$$(\overline{B}_\epsilon \setminus \overline{B}_{\epsilon/2}) \cap f^{-1}(D_{2\delta}) \to D_{2\delta}$$

where δ is chosen small enough. By Ehresmann's fibration theorem applied to

$$(\overline{B}_\epsilon \setminus B_{\epsilon/2}) \cap f^{-1}(D_{2\delta}) \to D_{2\delta},$$

this fibration is locally trivial. But since $D_{2\delta}$ is contractible, the fibration is in fact trivial. Thus its restriction

$$(\overline{B}_\epsilon \setminus \overline{B}_{\epsilon/2}) \cap f^{-1}(\partial \overline{D}_\delta) \to \partial \overline{D}_\delta$$

is also trivial. Note that the total space of this last fibration is an open subset, say U_0, of E, and contains ∂E. Using this open set, we can modify the construction of the vector field v of E given above in the following way:

Let U_0 be one of the U_α's used to cover E, and assume that none of the other U_α's intersect the boundary ∂E. For these other U_α's, we construct the local lifting v_α as before, but for the vector field v_0 on U_0, we use the trivialization of $U_0 \cong (\overline{B}_\epsilon \setminus \overline{B}_{\epsilon/2}) \cap f^{-1}(\delta) \times S^1$ to construct a vector field whose flow leaves the first factor invariant, and which is a lifting of w on S^1. Thus in particular, flowing along this vector field leaves the boundary ∂E invariant. This is still valid for the global vector field v obtained as before by a partition of unity, since the other U_α's do not intersect the boundary. □

3.2 Homology groups of Milnor fiber

In order to study the Milnor fiber of f, one method is to apply a small perturbation to the function f such that all its critical points become non-degenerate. Let us make these notions more precise.

Definition 3.2.1. A complex analytic function g defined on an n-dimensional complex manifold with boundary M, is said to be *Morse* if all its critical points are contained in $M \setminus \partial M$, if the Hessian $\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{1\le i,j\le n}$ is non-degenerate at each critical point, and if all the critical values are distinct.

It can be shown that a critical point of g is non-degenerate precisely if there are local coordinates around it such that g can be written as $z_1^2 + \cdots + z_n^2$. For the proof, see [Mil63].

In our case, the function g will always be defined on the whole of $\mathbb{C}^n$, but conceptually it is better to allow for the case where g is only defined in a neighbourhood of the origin.

Definition 3.2.2. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, where $0 < \mu_i \le \infty$ for every i. Then the set

$$P_\mu = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| < \mu_i\}$$

is called a *polydisk* of multiradius μ.

Using this, we have the following definition.

Definition 3.2.3. Let $P_{\eta_1} \subset \mathbb{C}^n$ and $P_{\eta_2} \subset \mathbb{C}^k$ be two polydisks. Let g be a complex analytic function in n variables defined on P_{η_1}, and let $F : P_{\eta_1} \times P_{\eta_2} \to \mathbb{C}$ be a complex analytic function such that $F(z_1, \ldots, z_n, 0, \ldots, 0) = g(z_1, \ldots, z_n)$ for all $(z_1, \ldots, z_n) \in P_{\eta_1}$. Then F is said to be a *deformation* of g. In the case where $k = 1$, the deformation is given by a family of functions $g_t : P_{\eta_1} \to \mathbb{C}$, where $g_t(z) = F(z, t)$, and $g_0 = g$.

Henceforth, one can assume that $P_{\eta_1} = \mathbb{C}^n$ and $P_{\eta_2} = \mathbb{C}^k$.

We want to show that we can find a one-parameter deformation (i.e. with $k = 1$) of the polynomial f such that f_1 is Morse, and such that $f_t^{-1}(s)$ is transversal to the sphere S_ϵ for $t \in [0, 1]$ and $s \in D_\delta$. For any $t \in [0, 1]$, f_t will be a polynomial.

For any $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, let

$$l_a : \mathbb{C}^n \to \mathbb{C}$$

be the linear function mapping $(z_1, \ldots, z_n) \in \mathbb{C}^n$ to $\sum a_i z_i$. Also let $f_a := f - l_a$, so that $f_0 = f$.

Lemma 3.2.4. There exist $\delta, \eta > 0$, such that $f_a^{-1}(s) \pitchfork S_\epsilon$ for $|s| < \delta$ and $\|a\| < \eta$, and such that the critical values of f_a on $\overline{B}_\epsilon$ are all contained in the open disk D_δ.

Proof. The proof is similar to that of Lemma 2.6.1. Define a function

$$G : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n$$

which maps (z, a) to $(f_a(z), a)$. Let $A := S_\epsilon \times \mathbb{C}^n \subset \mathbb{C}^n \times \mathbb{C}^n$. We know that $f_0^{-1}(0) \pitchfork S_\epsilon$. It follows that $G^{-1}(0) \pitchfork A$.

Let $z_\alpha \in G^{-1}(0) \cap A$. Then using a similar argument as in Lemma 2.6.1 but with G instead of f, it can be shown that there exists an open neighbourhood U_α of z_α such that for any

$z \in U_\alpha \cap A$, we have $G^{-1}(w) \pitchfork A$, where $G(z) = w$. Writing $w = (s, a)$, this tranversality condition is equivalent to $f_a^{-1}(s) \pitchfork S_\epsilon$.

Repeating this for every $z_\alpha \in G^{-1}(0) \cap A$, we get a collection of open sets $\{U_\alpha\}$. Now we just have to show that it is possible to find a neighbourhood V of $(0,0) \in \mathbb{C} \times \mathbb{C}^n$ such that $G^{-1}(V) \cap A$ is covered by $U := \cup_\alpha U_\alpha$. So suppose it is not possible. Then there exists a sequence $(s_n, a_n) \in \mathbb{C} \times \mathbb{C}^n$ converging to $(0,0)$ such that $G^{-1}(s_n, a_n) \cap A \nsubseteq \cup_\alpha U_\alpha$. Note that $G^{-1}(s_n, a_n) = f_{a_n}^{-1}(s_n) \times \{a_n\}$.

Then we can find a sequence $(z_n, a_n) \in A \subset \mathbb{C}^n \times \mathbb{C}^n$ such that $G(z_n, a_n) = (s_n, a_n)$, and such that $(z_n, a_n) \notin U$. From the definition of A we see that z_n is bounded by ϵ. Since a_n converges to 0, it is also bounded, hence the sequence (z_n, a_n) is bounded, so we can assume it converges to (z, a), say, by taking a convergent subsequence. But then $G(z, a) = (0, 0)$, hence $(z, a) \in G^{-1}(0) \cap A \subset U$. But this contradicts the fact that every term (z_n, a_n) lies outside U.

Thus it is possible to find the required neighbourhood V of $(0,0)$, which in its turn contains all points (s, a) with $|s| < \delta$ and $\|a\| < \eta$ for some δ and η. Pick any such (s, a), and let $z \in f_a^{-1}(s) \cap S_\epsilon$. This is equivalent to requiring that $(z, a) \in G^{-1}(s, a) \cap A$. Thus $(z, a) \in U_\alpha$ for some α. Hence $f_a^{-1}(s) \pitchfork S_\epsilon$ at z, as required.

Regarding the final statement of the lemma: since the critical values of f_0 on $\overline{B}_\epsilon$ (this is just 0) are all contained in the disk D_δ, we can, by further restricting η and using continuity, ensure that the critical values of f_a on $\overline{B}_\epsilon$ are also contained in the disk for $\|a\| < \eta$. □

Now we want to find some a with $\|a\| < \eta$ such that f_a is a Morse function. This can be done by the following lemma:

Lemma 3.2.5. For almost all $a \in \mathbb{C}^n$, the function $f_a = f - l_a$ is a Morse function.

Proof. Consider the function $df : \mathbb{C}^n \to \mathbb{C}^n$ with component functions $(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$. By Sard's lemma, almost all $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ are regular values for df. This implies that we can find such an a with $\|a\| < \eta$. We now show that for such an a, the function f_a has only non-degenerate critical points.

So suppose that a is a regular value for df. Note that f_a has a critical point at z precisely if $\frac{\partial f}{\partial z_i}(z) = a_i$ for all i, that is, if z lies in $(df)^{-1}(a)$. But since a is a regular value of df, the Jacobian of df at z must have non-zero determinant. This Jacobian is

$$\left(\frac{\partial^2 f}{\partial z_i \partial z_j} \right)_{ij}.$$

But this is also the expression for the Hessian of f_a at z, which shows that z is a non-degenerate critical point of f_a. Thus f_a has only non-degenerate critical points.

This means that f_a is almost a Morse function: the critical values also need to be distinct. This can be achieved by adding an arbitrarily small linear function, because the set of linear functions for which it will not work is a set of measure 0. Since the set of regular values of df is open, then the new f_a would still have only non-degenerate critical points, and hence it is a Morse function. □

Thus we have found an a (call it a_0) with $\|a_0\| < \eta$ and f_{a_0} Morse. But recall that we wanted a perturbation such that f_1 satisfies these properties. This can be done by reparametrizing the perturbation. To be precise, for any $t \in \mathbb{C}$, we henceforth denote by f_t the function which was denoted by f_{ta_0} up to now. Then f_t is a one-parameter deformation of f with f_1 being a Morse function. This should be kept in mind during what follows to avoid confusion.

The strategy for studying the Milnor fiber of the function $f = f_0$ is to first show that there are homeomorphisms:

$$\overline{B}_\epsilon \cap f_0^{-1}(\overline{D}_\delta) \cong \overline{B}_\epsilon \cap f_1^{-1}(\overline{D}_\delta) \text{ and}$$
$$\overline{B}_\epsilon \cap f_0^{-1}(\delta) \cong \overline{B}_\epsilon \cap f_1^{-1}(\delta).$$

Since $\overline{B}_\epsilon \cap f_0^{-1}(\delta)$ is the Milnor fiber of f, this means that we can instead study $\overline{B}_\epsilon \cap f_1^{-1}(\delta)$, which is simpler since f_1 is Morse.

Lemma 3.2.6. There is a diffeomorphism between $\overline{B}_\epsilon \cap f_0^{-1}(\overline{D}_\delta)$ and $\overline{B}_\epsilon \cap f_1^{-1}(\overline{D}_\delta)$.

Proof. The proof uses Ehresmann's fibration theorem. Let $G : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ be the function mapping (z, t) to $(f_t(z), t)$. Let M denote the set $\overline{B}_\epsilon \times \mathbb{C} \cap G^{-1}(\overline{D}_\delta \times D_\eta)$, and denote the restiction of G to this domain also by G. Note that the image is of course contained in $\overline{D}_\delta \times D_\eta$. Now consider the composition of G with the projection onto the second factor, D_η. In other words, the function $G' : M \to D_\eta$ where $G' = \pi_2 \circ G$. This function is clearly a submersion everywhere, since it maps (z, t) to t. Thus by Ehresmann's fibration theorem it is locally trivial. But because D_η is contractible, it is in fact a trivial fibration. Hence the fibers above 0 and 1 are diffeomorphic, as required. □

Lemma 3.2.7. The space $\overline{B}_\epsilon \cap f^{-1}(\overline{D}_\delta)$ is contractible.

Proof. The proof is from [AGZV88].

The first (and largest) step is to create a deformation retraction from $\overline{B}_\epsilon \cap f^{-1}(\overline{D}_\delta)$ to $\overline{B}_\epsilon \cap f^{-1}(0)$.

Let $\epsilon_i = \frac{\epsilon}{2^i}$ for $i \geq 0$. This gives a monotonically decreasing sequence converging to 0 and starting at $\epsilon_0 = \epsilon$. Define a sequence $(\delta_i)_{i\geq 0}$ inductively, by letting $\delta_0 = \delta$, and for each $i \geq 1$ choosing $\delta_i < \delta_{i-1}$ such that the level set $f^{-1}(w)$ is transverse to the sphere S_{ϵ_i} for any $w \in \overline{D}_{\delta_i}$.

The function f determines locally trivial, and hence also trivial, fibrations

$$E_i = f^{-1}(\overline{D}_{\delta_i}) \cap (\overline{B}_{\epsilon_0} \setminus B_{\epsilon_i}) \to \overline{D}_{\delta_i}.$$

The trivializations of these fibrations can be chosen so that they coincide on the intersections

$$E_i \cap E_{i-1} = f^{-1}(\overline{D}_{\delta_i}) \cap (\overline{B}_{\epsilon_0} \setminus B_{\epsilon_{i-1}}).$$

Now consider the deformation g_t of the disk $\overline{D}_{\delta_0}$, defined for $0 \leq t \leq \delta_0$ and given by

$$g_t(z) = \begin{cases} \dfrac{tz}{\|z\|} & \text{for } \|z\| \geq t, \\ z & \text{for } \|z\| \leq t. \end{cases}$$

The mapping g_t maps the disk $\overline{D}_{\delta_0}$ of radius δ_0 into the disk of radius t, keeping the latter fixed. The mapping g_0 is a deformation retraction of the disk $\overline{D}_{\delta_0}$ into the point 0. Since the function f defines the locally trivial fibration

$$f^{-1}(\overline{D}_{\delta_0} \setminus 0) \cap \overline{B}_{\epsilon_0} \to \overline{D}_{\delta_0} \setminus 0,$$

there exists a family G_t $(0 < t \leq \delta_0)$ of mappings of the set $f^{-1}(\overline{D}_{\delta_0}) \cap \overline{B}_{\epsilon_0}$ into itself lifting the homotopy g_t. This family can be chosen in accordance with the structure of the direct product on the sets

$$E_i = f^{-1}(\overline{D}_{\delta_i}) \cap (\overline{B}_{\epsilon_0} \setminus B_{\epsilon_i})$$

for $t \leq \epsilon_i$. The family can then be extended to a family G_t $(0 \leq t \leq \delta_0)$ in which the mapping G_0 is a deformation retraction of the set $f^{-1}(\overline{D}_{\delta_0}) \cap \overline{B}_{\epsilon_0}$ into the set $f^{-1}(0) \cap \overline{B}_{\epsilon_0}$. Thus these two spaces are homotopic.

It remains to show that $f^{-1}(0) \cap \overline{B}_{\epsilon_0}$ is contractible. But in the previous chapter we showed that it is homeomorphic to the cone over $f^{-1}(0) \cap S_{\epsilon_0}$, and thus it is contractible. □

Combining the previous two results, we conclude that $\overline{B}_\epsilon \cap f_1^{-1}(\overline{D}_\delta)$ is contractible. The next lemma allows us to restrict our attention to f_1 from now on.

Lemma 3.2.8. There is a diffeomorphism between $\overline{B}_\epsilon \cap f_0^{-1}(\delta)$ and $\overline{B}_\epsilon \cap f_1^{-1}(\delta)$.

Proof. Let $G : \overline{B}_\epsilon \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n$ be the function which maps (z, a) to $(f_a(z), a)$. Use the injective mapping $\lambda : [0, 1] \to \mathbb{C} \times \mathbb{C}^n$, $\lambda(t) = (\delta, ta_0)$ to identify $[0, 1]$ with its image L under λ. Denote the restriction of G to $G^{-1}(L)$ by G'. The function G' is a submersion and proper, hence by Ehresmann's fibration theorem, it is a locally trivial fibration which is trivial since $[0, 1]$ is contractible. Thus all the fibers are diffeomorphic, including $G'^{-1}(0) \cong \overline{B}_\epsilon \cap f_0^{-1}(\delta)$ and $G'^{-1}(1) \cong \overline{B}_\epsilon \cap f_1^{-1}(\delta)$. □

From now on, let $Y_t := \overline{B}_\epsilon \cap f_t^{-1}(\overline{D}_\delta)$ and let $X_{f_t} := \overline{B}_\epsilon \cap f_t^{-1}(\delta)$. Then what we have seen above is that Y_1 is contractible, and $X_{f_1} \subset Y_1$ is diffeomorphic to the Milnor fiber $X_{f_0} \subset Y_0$ which we want to study. Now consider the long exact homology sequence of the pair (Y_1, X_{f_1}):

$$\to \tilde{H}_k(X_{f_1}) \to \tilde{H}_k(Y_1) \to H_k(Y_1, X_{f_1}) \to \tilde{H}_{k-1}(X_{f_1}) \to$$

Since Y_1 is contractible, $\tilde{H}_k(Y_1) = 0$ for all $k \geq 0$. It follows that there is a group isomorphism

$$H_k(Y_1, X_{f_1}) \cong \tilde{H}_{k-1}(X_{f_1})$$

for $k \geq 1$.

Denote the critical values of f_1 on $\overline{B}_\epsilon$ by $z_1, \ldots, z_\mu$. Recall that f_1 was chosen so that they all lie in the disk D_δ. Around each one of them we can choose a Milnor disk D_{δ_i} and a corresponding Milnor ball B_{ϵ_i} around the corresponding critical point on $f_1^{-1}(z_i)$. The disks are chosen to be small enough to be contained in D_δ and mutually disjoint, and similarly the balls must be contained in B_ϵ and mutually disjoint.

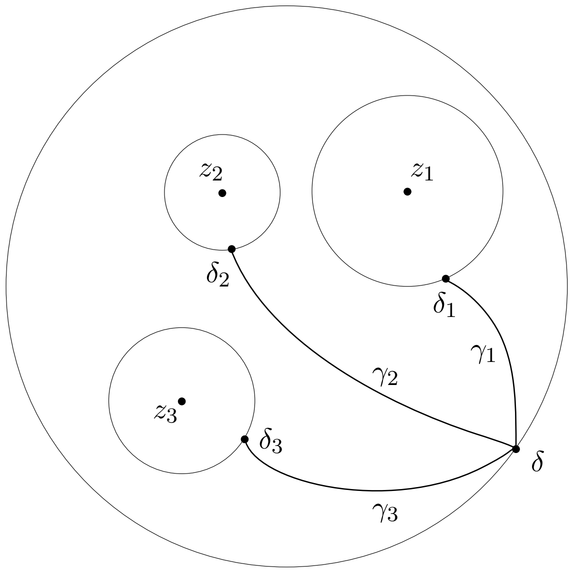

Figure 3.1: *The system of paths in $\overline{D}_\delta$ for $\mu = 3$*

Next, we construct a system of paths, γ_i, each from δ to a fixed point δ_i on the boundary of D_{δ_i} (see Figure 3.1). The paths are chosen so that they only coincide at δ. Denote by Γ the union of all the paths, and let $\overline{\Gamma}$ be the set

$$\Gamma \cup \bigcup_{i=1}^{\mu} \overline{D}_{\delta_i}.$$

The set $\overline{\Gamma}$ is a deformation retract of the disk D_δ. The mapping f_1 is a locally trivial fibration over the complement of $\overline{\Gamma}$ (and thus a trivial fibration), hence this deformation retraction can be lifted to a deformation retraction of Y_1 to $f_1^{-1}(\overline{\Gamma}) \cap \overline{B}_\epsilon$.

The set Γ deformation retracts onto the point δ. The mapping f_1 is a locally trivial fibration over Γ, hence it is trivial, so $f_1^{-1}(\Gamma) \cap \overline{B}_\epsilon$ is homotopic to X_{f_1}.

For every k we get the following group isomorphisms:

$$\begin{aligned} H_k(Y_1, X_{f_1}) &\cong \tilde{H}_{k-1}(X_{f_1}) \\ &\cong \tilde{H}_{k-1}(f_1^{-1}(\Gamma) \cap \overline{B}_\epsilon) \\ &\cong H_k(f_1^{-1}(\overline{\Gamma}) \cap \overline{B}_\epsilon, f_1^{-1}(\Gamma) \cap \overline{B}_\epsilon). \end{aligned}$$

The first isomorphism was proven above. The last isomorphism is proved in a similar way by considering the long exact sequence of the pair $(f_1^{-1}(\overline{\Gamma}) \cap \overline{B}_\epsilon, f_1^{-1}(\Gamma))$ and using the contractibility of $f_1^{-1}(\overline{\Gamma}) \cap \overline{B}_\epsilon$ (since it is homotopic to Y_1). The middle isomorphism follows from the homotopy equivalence between X_{f_1} and $f_1^{-1}(\Gamma) \cap \overline{B}_\epsilon$.

Applying excision to remove $f_1^{-1}(\delta) \cap \overline{B}_\epsilon$ gives the first isomorphism in the following:

$$\begin{aligned}
&H_k(f_1^{-1}(\overline{\Gamma}) \cap \overline{B}_\epsilon, f_1^{-1}(\Gamma) \cap \overline{B}_\epsilon) \\
&\qquad \cong H_k(f_1^{-1}(\overline{\Gamma}) \cap \overline{B}_\epsilon \setminus f_1^{-1}(\delta) \cap \overline{B}_\epsilon, f_1^{-1}(\Gamma) \cap \overline{B}_\epsilon \setminus f_1^{-1}(\delta) \cap \overline{B}_\epsilon) \\
&\qquad \cong H_k\left(\bigcup_{i=1}^{\mu} f_1^{-1}(\overline{D}_{\delta_i}) \cap \overline{B}_\epsilon, \bigcup_{i=1}^{\mu} f_1^{-1}(\delta_i) \cap \overline{B}_\epsilon\right) \\
&\qquad \cong H_k\left(\bigcup_{i=1}^{\mu}(f_1^{-1}(\overline{D}_{\delta_i}) \cap \overline{B}_{\epsilon_i}), \bigcup_{i=1}^{\mu}(f_1^{-1}(\delta_i) \cap \overline{B}_{\epsilon_i})\right) \\
&\qquad \cong \bigoplus_{i=1}^{\mu} H_k\left(f_1^{-1}(\overline{D}_{\delta_i}) \cap \overline{B}_{\epsilon_i}, f_1^{-1}(\delta_i) \cap \overline{B}_{\epsilon_i}\right) \\
&\qquad \cong \bigoplus_{i=1}^{\mu} \tilde{H}_{k-1}(\text{Milnor fiber of a Morse point}).
\end{aligned}$$

The second and third isomorphisms follow by retraction.

It remains to determine the reduced homology groups of the Milnor fiber of a Morse point.

Lemma 3.2.9. The Milnor fiber of a Morse point is diffeomorphic to the disk bundle of the tangent bundle of the sphere S^{n-1}.

Proof. The following proof comes from [AGZV88, p.23].

Let $g : \mathbb{C}^n \to \mathbb{C}$ be the Morse function given by $g(z_1, \ldots, z_n) = \sum_{i=1}^n z_i^2$. First we show that the radii of the Morse ball and Morse disk can be taken to be 2 and 4 respectively. In other words, we have to show that $g^{-1}(w) \pitchfork S_2$ for $|w| < 4$.

Let $z \in g^{-1}(w) \cap S_2$, and suppose that $g^{-1}(w)$ is not transversal to the sphere S_2 at z. Then $dr^2(z)$ is linearly dependent on $dg(z)$ and $d\overline{g}(z)$, that is $dr^2(z) = \alpha dg(z) + \beta d\overline{g}(z)$, where $\alpha, \beta \in \mathbb{C}$. We have

$$\begin{aligned}
dg(z) &= 2\sum z_j dz_j \\
d\overline{g}(z) &= 2\sum \overline{z}_j d\overline{z}_j \\
dr^2 &= \sum \overline{z}_j dz_j + \sum z_j d\overline{z}_j
\end{aligned}$$

from which it follows that

$$\overline{z}_j = 2\alpha z_j \quad \text{for } j = 1, \ldots, n.$$

But $z_j \neq 0$ for some j. Thus $|2\alpha| = 1$, and so

$$r^2(z) = \sum z_j \overline{z}_j = 2\alpha \sum z_j^2 = 2\alpha g(z).$$

Finally,

$$|g(z)| = r^2(z) = 4$$

which contradicts the assumption that $|g(z)| < 4$, thus proving the claim.

Thus the Milnor fiber can be given in this case by $g^{-1}(1) \cap \overline{B}_2$. Let $z_j = x_j + iy_j$. Thus the Milnor fiber as a subspace of $\mathbb{R}^{2n}$ is given as the vanishing set of the following equations:

$$\begin{aligned} &\sum x_j^2 - \sum y_j^2 = 1, \\ &\sum x_j y_j = 0 \text{ and} \\ &\sum x_j^2 + \sum y_j^2 \leq 4. \end{aligned}$$

The disc bundle of the tangent bundle of the sphere S^{n-1} can be given in $\mathbb{R}^{2n}$ with coordinates $\tilde{x}_j$, $\tilde{y}_j$ by the equations:

$$\begin{aligned} &\sum \tilde{x}_j^2 = 1, \\ &\sum \tilde{x}_j \tilde{y}_j = 0 \text{ and} \\ &\sum \tilde{y}_j^2 \leq \rho^2, \end{aligned}$$

where ρ is the radius of the discs in the bundle. A diffeomorphism between these two spaces (with $\rho = \sqrt{\frac{3}{2}}$) is given by the mapping

$$\begin{aligned} \tilde{x}_j &= \frac{x_j}{\sqrt{\sum x_j^2}}, \\ \tilde{y}_j &= y_j. \end{aligned}$$

□

Thus the Milnor fiber of g is homotopic to the sphere S^{n-1}. It follows that $\tilde{H}_k$ of the Milnor fiber of g is 0 for $k \neq n-1$, and $\mathbb{Z}$ for $k = n-1$.

Explicitly, we can consider the sphere S^{n-1} as the subset of the Milnor fiber of the Morse point given by the equations (using the notation of the above lemma)

$$\begin{aligned} &\sum x_j^2 = 1 \text{ and} \\ &y_j = 0 \text{ for all } j. \end{aligned}$$

Then the Milnor fiber deformation retracts onto this sphere, and thus it can be seen as a generator for the $n-1$'th homology group.

Putting it all together yields the expression for the homology groups of X_{f_1}:

$$\tilde{H}_{k-1}(X_{f_1}) = \begin{cases} 0 \text{ if } k \neq n \\ \mathbb{Z}^\mu \text{ if } k = n \end{cases}$$

As a consequence we have

Corollary 3.2.10. The number μ is invariant of the chosen perturbation.

And thus we can make the following definition

Definition 3.2.11. The number μ is an invariant of the singularity called the *Milnor number.*

3.3 The intersection form and monodromy

In the first section of this chapter we defined the geometric monodromy as a diffeomorphism of the Milnor fiber (more precisely, as an isotopy class of diffeomorphisms). We then defined the algebraic monodromy as the automorphisms of the homology groups of the Milnor fiber induced by this diffeomorphism. In the previous section we saw that only the $n-1$'th homology group is non-trivial. Thus, henceforth, when we speak about the *monodromy*, we will be referring to the automophism of this group, $H_{n-1}(F_\delta)$.

First a remark about the number μ that was defined in the previous section as the number of critical points into which the original critical point splits up when undergoing a perturbation to make it a Morse function. This number can also be defined in an algebraic way,

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{z_1, \dots, z_n\}/I,$$

where $\mathbb{C}\{z_1, \dots, z_n\}$ is the ring of convergent power series in n complex variables, and

$$I = \left\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right\rangle.$$

It can be shown that the two definitions coincide. See [Loo84] for details.

3.3.1 Vanishing cycles

It turns out that certain cycles in the homology group $H_{n-1}(F_\delta)$ are special in some way, and will be called *vanishing cycles*. Futhermore, one can obtain a basis for $H_{n-1}(F_\delta)$ consisting only out of vanishing cycles, and certain such bases are also special in some way, where any such special basis will be called a *distinguished basis of vanishing cycles*. The motivation for this is that it creates new invariants for the singularity, and that it gives a way to calculate the algebraic monodromy using a perturbation.

To be able to make the appropriate definitions, consider a perturbation of f as in the previous section. There we showed that the intersection of the fiber of f_1 above any δ_i with a small ball $\overline{B}_{\epsilon_i}$ is diffeomorphic to the Milnor fiber of the non-degenerate singularity $g = \sum_{i=1}^n z_i^2$. Furthermore, we showed that the $n-1$'th homology group of g is just $\mathbb{Z}$, and we gave an explicit description of a generating cycle. By considering the inclusion

$$f_1^{-1}(\delta_i) \cap \overline{B}_{\epsilon_i} \subset f_1^{-1}(\delta_i) \cap \overline{B}_\epsilon,$$

where the latter set is just the Milnor fiber (by Lemma 3.2.8), we note that we can consider this cycle as an element of $H_{n-1}(F_\delta)$. It will be denoted by Δ_i. If one looks at the equations for this cycle in the fiber of g above a point in $\mathbb{C}$ moving along a path to the critical value 0, then it becomes smaller and smaller, and finally shrinks to a point, since only the point $(0, \dots, 0) \in \mathbb{C}^n$ satisfies the following equations:

$$g(z) = \sum z_i^2 = 0,$$
$$\sum \mathrm{Re}(z_i)^2 = 0 \text{ and}$$
$$\mathrm{Im}(z_i) = 0 \text{ for all } i.$$

For this reason it is referred to as a *vanishing cycle*. In general, a vanishing cycle of the Milnor fiber would be any $n-1$-cycle of the fiber above δ which shrinks to a point along the path joining δ with some critical value z_i under any perturbation.

Corollary 3.3.1. The vanishing cycles $\Delta_1, \dots, \Delta_\mu$ form a basis for the homology group $H_{n-1}(F_\delta)$.

Proof. This is a corollary of the expression given in the previous section of the homology group of f as a direct sum of the homology groups of the Milnor fibers of Morse points. Since each of these direct summands is generated by the corresponding vanishing cycle Δ_i, the homology group $H_{n-1}(F_\delta)$ will be generated by the collection of Δ_i's. □

In general, a basis of vanishing cycles found in the way described above is called a *weakly distinguished basis of vanishing cycles*. If we renumber the paths γ_i (see the previous section) such that moving counter-clockwise along the inner half of a small circle around δ we encounter in an increasing order $\gamma_1, \gamma_2, \gamma_3$ and so on, then this is called a *distinguished basis of vanishing cycles*. Henceforth we will always assume that the paths were chosen in this fashion.

The purpose of requiring the basis to be distinguished, is that it brings us one step closer to calculating the monodromy (i.e. the automorphism of $H_{n-1}(F_\delta)$) using a given perturbation. Specifically, we will now proceed to associate an automorphism h_{i*} of $H_{n-1}(F_\delta)$ to each critical value such that the monodromy h_* is given by the composition $h_{\mu_*} \circ \cdots \circ h_{2*} \circ h_{1*}$.

Define λ_i to be the path in $\overline{D}_\delta$ starting at δ, going along γ_i to δ_i, then going anti-clockwise around the circle $\partial\overline{D}_{\delta_i}$ and then again returning to δ via the path γ_i. This path does not contain any critical values of the fibration, and hence induces a diffeomorphism h_i of the fiber above δ, i.e. the Milnor fiber. As before, this induces an automorphism of the homology group $H_{n-1}(F_\delta)$, which we call h_{i*}.

It is clear that the composition of paths $\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_\mu$ is homotopic to the path around the boundary of the disk $\overline{D}_\delta$. And since this composition of paths translates into composition of automorphisms of the homology group, it verifies the expression for h_* given above.

The reason for decomposing h_* in this way, is that we will soon give an expression for h_{i*} as a matrix in terms of the basis $\Delta_1, \dots, \Delta_\mu$. Composing these matrices will then yield the matrix for h_*. This expression for h_{i*} requires knowing the intersection matrix, so we turn our attention to that now.

3.3.2 Intersection matrix

Associated to a distinguished basis of vanishing cycles $\Delta_1, \dots, \Delta_\mu$ is a $\mu \times \mu$ matrix S, where the i, j'th entry is given by the intersection number $\Delta_i \circ \Delta_j$. To define this we proceed as follows: first, choose an orientation for each cycle in the basis. The Milnor fiber is a complex manifold, and thus has a canonical orientation. The cycles Δ_i and Δ_j can be represented as $n-1$ dimensional manifolds in the $2n-2$ dimensional Milnor fiber. By applying a small perturbation, we can assume that they intersect transversally at all points of intersection. This means that at a given point of intersection, their tangent spaces taken together span the entire tangent space of the Milnor fiber at that point. Then to every point of intersection (there are only finitely many) we associate $+1$ or -1. To choose which one, let $e_1, \ldots, e_{n-1}$ be an oriented basis for the tangent space of Δ_i at the point in question and let $f_1, \ldots, f_{n-1}$ be an oriented basis for the tangent space of Δ_j. Then if the basis $e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}$ of the tangent space of the Milnor fiber agrees with the original chosen orientation, we associate $+1$ to this intersection point, otherwise -1. The intersection number $\Delta_i \circ \Delta_j$ is defined as the sum of all these intersection numbers.

Definition 3.3.2. The matrix $S = (\Delta_i \circ \Delta_j)_{i,j}$ is called the *intersection matrix* of the singularity with respect to the given distinguished basis of vanishing cycles.

3.3.3 Monodromy

To use the intersection matrix S to calculate the monodromy, we need to use the following expression, which is derived in [AGZV88, p.44]:

$$h_{i*}(a) = a + (-1)^{n(n+1)/2}(a \circ \Delta_i)\Delta_i,$$

where $a \in H_{n-1}(F_\delta)$.

To summarize, we start with a perturbation of the function f, which allows us to find a distinguished basis of vanishing cycles of $H_{n-1}(F_\delta)$. Then we need to find the intersection numbers between these vanishing cycles and from those deduce the monodromy h_*.

There is an important condition that the monodromy must satisfy, given by the following theorem (for a proof, see [Loo84]).

Theorem 3.3.3. The eigenvalues of the monodromy

$$h_* : H_{n-1}(F_\delta, \mathbb{C}) \to H_{n-1}(F_\delta, \mathbb{C})$$

are roots of unity.

For a particularly simple and hence better understood class of singularities, namely the Brieskorn singularities, one can predict precisely *which* roots of unity will occur.

To define a Brieskorn polynomial, let $a_1, \ldots, a_n > 2$ be a sequence of integers, and consider the following polynomial

$$f(z_1, \ldots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n}.$$

Again let F_δ denote the Milnor fiber. Then we have the following theorem ([Mil68]).

Theorem 3.3.4 (Brieskorn-Pham)**.** The homology group $H_{n-1}(F_\delta)$ is free abelian of rank

$$\mu = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1).$$

The eigenvalues of the monodromy

$$h_* : H_n(F_\delta, \mathbb{C}) \to H_n(F_\delta, \mathbb{C})$$

are the products $w_1 w_2 \cdots w_n$ where each w_j ranges over all a_j'th roots of unity other than 1.

The motivation behind introducing all this, is that we will work out an example in Chapter 5, and use the previous theorem to verify the roots of unity found by starting with the intersection matrix.

Chapter 4

Topological classification of singular points on plane algebraic curves

At the beginning of the first chapter we referred to the problem of classifying singularities up to topological equivalence. In general, this problem is unsolved, but in the case where the polynomial f has only two variables, there is a very beautiful solution. In short, the topological type of the singular point of such a polynomial f is determined by the so-called *Puiseux pairs* of the polynomial.

In this chapter we will explain how one arrives to the Puiseux pairs and how they determine the topological type of the singularity. We will not give all proofs in detail, but will at least explain the main ideas and give many examples. Most of the material comes from the very clearly written book by Brieskorn ([BK86]).

4.1 Introduction

We start with a polynomial $f \in \mathbb{C}[x, y]$, and we assume that f has a singular point at 0. Then we consider the vanishing set of f, $V = V(f) \subset \mathbb{C}^2$, in a neighbourhood of 0. We recall that if $g \in \mathbb{C}[x, y]$ is another such polynomial with a singular point at 0, then these two singular points are topologically equivalent precisely if there are neighbourhoods U_f and U_g of $0 \in \mathbb{C}^2$ such that the pair $(U_f, U_f \cap V(f))$ is homeomorphic to the pair $(U_g, U_g \cap V(g))$.

Also recall Theorem 2.3.4, which claims the existence of an $\epsilon > 0$ such that the pair $(B_\epsilon, V \cap B_\epsilon)$ is homeomorphic to the pair $(C(S_\epsilon), C(V \cap S_\epsilon))$, where $C(X)$ denotes the cone over X. This means that to every polynomial f, we can associate the pair (S_ϵ, K), where

$$K := V \cap S_\epsilon.$$

And if $K' = V(g) \cap S_{\epsilon'}$ is the corresponding set for the polynomial g, then f and g are topologically equivalent precisely if there is a homeomorphism from S_ϵ to $S_{\epsilon'}$ which restricts to a homeomorphism from K to K'.

Note that in the $n = 2$ case, S_ϵ is homeomorphic to the three-sphere, S^3. Note that K is one-dimensional, because the non-singular part of V is two-dimensional (as a real manifold) and it intersects S_ϵ transversally. The set K is also closed, and hence compact. Thus K is a disjoint union of a finite number of circles in S^3, also known as a *link*. If f is irreducible as an element of the ring of convergent power series $\mathbb{C}\{x, y\}$, then K has only one connected component, and in this case it is a *knot*.

Definition 4.1.1. A knot is a subset $K \subset S^3$ which is homeomorphic to S^1 and with a specified orientation. Two knots K and L are topologically equivalent, if there is a homeomorphism from S^3 to itself which restricts to an orientation-preserving homeomorphism from K to L.

A link is a subset of S^3 which is homeomorphic to the disjoint union of a finite number of copies of S^1. Topological equivalence of links is defined in the same way.

We remark that the term *link* is also sometimes used to describe the set $K \subset S_\epsilon$ in the higher dimensional case (i.e. when f has more than 2 variables).

The problem then reduces to the following: we have to find an appropriate model for S_ϵ, and we have to find a way to rewrite f such that it is easy to construct the link, and to compare different links.

4.2 A model for S_ϵ

The set S_ϵ is defined as the set of points in $\mathbb{R}^4$ for which the distance to the origin is exactly ϵ. By a model for S_ϵ, we mean any space homeomorphic to it. Note that S_ϵ is the boundary of the closed ball $\overline{B}_\epsilon$. Instead of a closed ball, we will use something homeomorphic to it, namely a closed polydisk. Define a closed polydisk with multiradius (δ, η) by

$$\overline{D} = \{(x, y) \in \mathbb{C}^2 \mid |x| \leq \delta, |y| \leq \eta\}.$$

This polydisk can be written as the product of two circular disks

$$\overline{D} = \{x \in \mathbb{C} \mid |x| \leq \delta\} \times \{y \in \mathbb{C} \mid |y| \leq \eta\}.$$

This gives an expression for the boundary

$$\partial\overline{D} = T^+ \cup T^-,$$

where

$$T^+ = \{x \in \mathbb{C} \mid |x| = \delta\} \times \{y \in \mathbb{C} \mid |y| \leq \eta\} \cong S^1 \times D^2 \text{ and}$$
$$T^- = \{x \in \mathbb{C} \mid |x| \leq \delta\} \times \{y \in \mathbb{C} \mid |y| = \eta\} \cong D^2 \times S^1$$

are two solid tori meeting along the two-dimensional torus

$$T^+ \cap T^- = \{(x, y) \in \mathbb{C}^2 \mid |x| = \delta, |y| = \eta\} \cong S^1 \times S^1.$$

To show that S_ϵ and $\partial\overline{D}$ are homeomorphic, we give a decomposition of S_ϵ into two solid tori. Assume that $\epsilon^2 = \delta^2 + \eta^2$, where δ and η are positive constants. Then

$$S_\epsilon = T_+ \cup T_-,$$

where

$$T_+ = \{(x,y) \in S_\epsilon \mid |y| \le \eta\} \text{ and}$$
$$T_- = \{(x,y) \in S_\epsilon \mid |x| \le \delta\}.$$

Then one can define homeomorphisms

$$T_+ \to T^+ \text{ and}$$
$$T_- \to T^-$$

by

$$(x,y) \mapsto \left(\frac{\delta x}{|x|}, y\right) \text{ and}$$
$$(x,y) \mapsto \left(x, \frac{\eta y}{|y|}\right),$$

which combine to give a homeomorphism

$$\psi : S_\epsilon = T_+ \cup T_- \to T^+ \cup T^- = \partial\overline{D}.$$

The idea is now to show that for sufficiently small ϵ, δ and η, there is a homeomorphism between the pairs $(S_\epsilon, S_\epsilon \cap V)$ and $(\Sigma, \Sigma \cap V)$, where $\Sigma := \partial\overline{D}$. This is proved in [BK86]. There it is also shown that by possibly applying a change of variables to f, one can assume that the intersection $\Sigma \cap V$ is completely contained in the interior of one of the two solid tori constituting Σ, i.e. $\Sigma \cap V \subset \text{int}(T^+)$. This shows the advantage of using polydisks: inside T^+ we have $|x| = \delta$. Therefore, to find the set $\Sigma \cap V$, we only need to let x vary along a circle of radius δ, and see which are the corresponding y values satisfying the equation $f(x,y) = 0$. For this it would be useful to solve the equation for y in terms of x. This cannot be done in general, but it can be done if one allows fractional exponents. This is the subject of the next section.

4.3 Puiseux expansions

Consider the equation $f(x,y) = x^p - y^q = 0$. Solving for y, gives $y = x^{\frac{p}{q}}$ as a solution. More generally, let f be a polynomial of the form

$$f(x,y) = \sum_{a+b\mu=c} \alpha_{a,b} x^a y^b,$$

where $\mu = \frac{p}{q}$, $a,b \in \mathbb{Z}_{\ge 0}$ and $c \in \mathbb{Q}$. Such a polynomial is said to be *quasihomogeneous*. In this case one can obtain a solution for the equation $f = 0$ of the form $y = tx^\mu$ by substitution:

$$\begin{aligned} f(x, tx^\mu) &= \sum \alpha_{a,b} x^a t^b x^{b\mu} \\ &= \sum \alpha_{a,b} t^b x^c \\ &= g(t) x^c. \end{aligned}$$

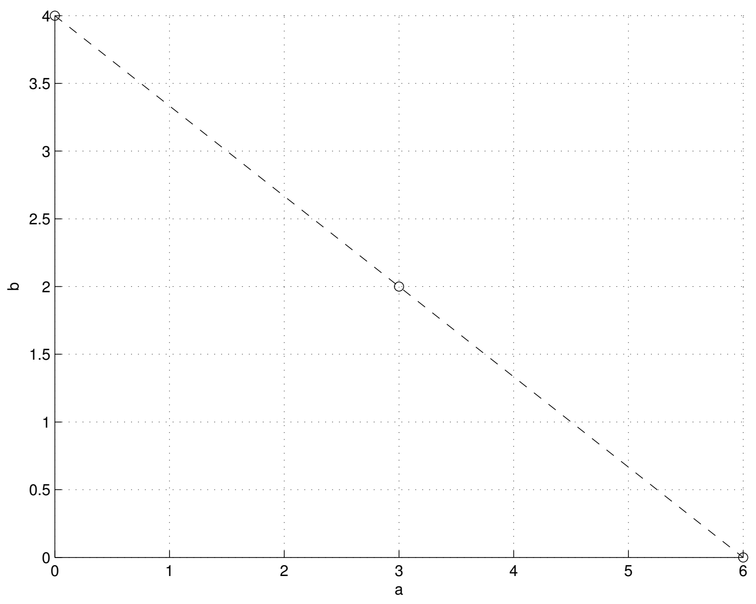

Figure 4.1: *Example 1: Support of* $f(x, y) = y^4 - 2x^3y^2 + x^6$

If t_0 is a zero of the polynomial $g(t)$, then $y = t_0x^{\mu}$ is a solution of the equation $f(x, y) = 0$. As long as f consists of at least two distinct monomials, $g(t)$ will have a zero different from 0, so the solution will not be trivial.

In the general case, one separates f into a quasihomogeneous part called $\tilde{f}$, and a part consisting of *higher order terms* (to be made precise later on). Then one finds the solution to the quasihomogeneous part as above, and uses it as an approximate solution to the equation $f = 0$. Substituting the approximate solution yields a new equation. One can iterate this procedure to obtain succesively more accurate solutions. We will illustrate this process by means of examples. But first, it is necessary to introduce a geometric concept.

Definition 4.3.1. The *support* of a power series $f(x, y) = \sum \alpha_{a,b}x^ay^b \in \mathbb{C}\{x, y\}$ is defined to be the subset of $(\mathbb{Z}_{\geq 0})^2$ consisting of pairs (a, b) for which $\alpha_{a,b} \neq 0$, that is

$$\Delta(f) = \{(a, b) \in (\mathbb{Z}_{\geq 0})^2 \mid \alpha_{a,b} \neq 0\}.$$

For the first example, let $f(x, y) = y^4 - 2x^3y^2 + x^6$. Then the support of f consists of three points corresponding to the three monomials (see Figure 4.1).

The condition for f to be quasihomogeneous, in other words that there must exist rational numbers μ and c such that $a + b\mu = c$ for all $(a, b) \in \Delta(f)$, is clearly equivalent to requiring that all the points in $\Delta(f)$ lie on a straight line. And in this case the gradient of the line will be $-\frac{1}{\mu}$ and it will intersect the x-axis at $(c, 0)$.

As seen in Figure 4.1, all the points lie on a straight line with slope $-\frac{2}{3}$, hence $\mu = \frac{3}{2}$ and we can apply the method described above to find a solution of the form $y = tx^{\frac{3}{2}}$. Substituting this into the equation for f yields

$$f(x, tx^{\frac{3}{2}}) = x^6(t^4 - 2t^2 + 1).$$

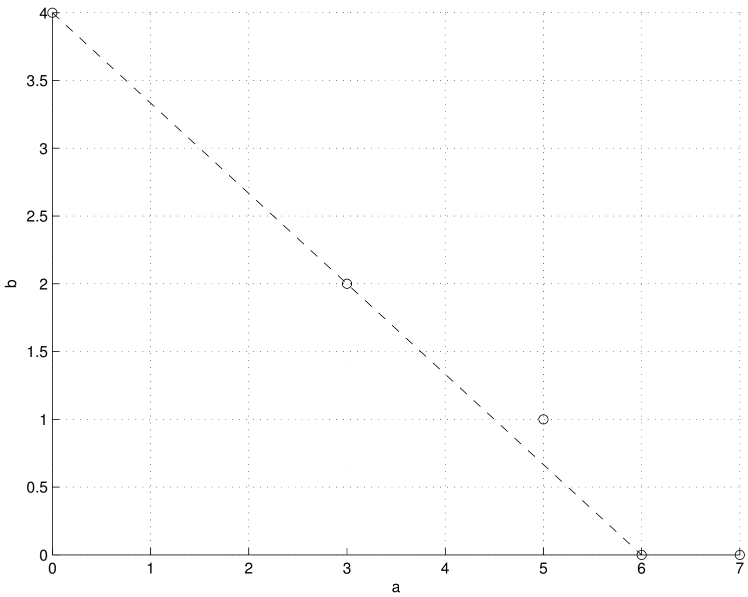

Figure 4.2: *Example 2: Support of* $f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$

The zeros of $g(t) = t^4 - 2t^2 + 1 = (t^2 - 1)^2$ are $t = 1$ and $t = -1$. Thus we get two solutions for the original equation, namely $y = x^{\frac{3}{2}}$ and $y = -x^{\frac{3}{2}}$.

For the second example, let $f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$. Then as can be seen from Figure 4.2, the points do not all lie on a straight line anymore. We can however write f as a sum

$$f = \tilde{f} + h$$

where $\tilde{f} = y^4 - 2x^3y^2 + x^6$ is the quasihomogeneous part which we saw in the previous example, and $h = -4x^5y - x^7$ is an higher order term. The order of a monomial x^ay^b is taken to be $a + \mu b$ in this case instead of the usual $a + b$. Thus all the monomials in $\tilde{f}$ have order 6, while the two monomials of h have orders $6\frac{1}{2}$ and 7 respectively. This is just a different way of saying that all the points corresponding to the monomials of h lie to the right or above of the line going through the points corresponding to the monomials of $\tilde{f}$.

Now we use the solution $y = x^{\frac{3}{2}}$ of $\tilde{f}(x,y) = 0$ as an approximate solution for $f(x,y) = 0$. We write the true solution as

$$y = x^{\frac{3}{2}}(1 + y_1).$$

Also, in order to avoid working with fractional exponents of x, we replace $x^{\frac{1}{2}}$ by x_1. Substituting all this in $f(x,y)$ gives

$$\begin{aligned} f(x,y) &= f(x_1^2, x_1^3(1 + y_1)) \\ &= x_1^{12} \cdot f_1(x_1, y_1), \end{aligned}$$

where

$$f_1(x_1, y_1) = y_1^4 + 4y_1^3 + 4y_1^2 - 4x_1y_1 - 4x_1 - x_1^2.$$

Again, we consider the support $\Delta(f_1)$. Below we will describe a systematic way of finding the quasihomogeneous part of f_1, but in this case we just note that we can again write f_1 as a sum

$$f_1 = \tilde{f}_1 + g_1$$

where $\tilde{f}_1 = 4y_1^2 - 4x_1$ are the monomials corresponding to points on the line, and $g_1 = y_1^4 + 4y_1^3 - 4x_1y_1 - x_1^2$ are the higher order terms. Solving $\tilde{f}_1(x_1, y_1) = 0$ for y_1 is easy in this case because there are only two terms, giving the solution $y_1 = x_1^{\frac{1}{2}}$. Of course, we know that there should exist a solution of the form $t_1x_1^{\frac{1}{2}}$ since the slope of the line is -2, and we could follow the previous method of substitution to deduce that $t_1 = \pm 1$.

Now following the same path as before, we conclude that $y_1 = x_1^{\frac{1}{2}}$ should give an approximate solution of f_1. But if we do the substitution, then we get

$$f_1(x_1, x_1^{\frac{1}{2}}) = x_1^2 + 4x_1^{\frac{3}{2}} + 4x_1 - 4x_1^{\frac{3}{2}} - 4x_1 - x_1^2 = 0.$$

Thus $y_1 = x_1^{\frac{1}{2}}$ is a solution of the equation $f_1(x_1, y_1) = 0$, and hence

$$y = x^{\frac{3}{2}}(1 + y_1) = x^{\frac{3}{2}}(1 + x_1^{\frac{1}{2}}) = x^{\frac{3}{2}}(1 + x^{\frac{1}{4}}) = x^{\frac{3}{2}} + x^{\frac{7}{4}}$$

is a solution of the original equation $f(x, y) = 0$.

After these examples, we now give a short description of what the general procedure would involve. Firstly one needs to assume that there is at least one point of $\Delta(f)$ lying on the y-axis. This can always be done by applying a coordinate change to f if necessary. One then takes the lowest point on the y-axis, and considers the family of straight lines in the plane passing through this point. Starting with the vertical line in this family, one rotates it anticlockwise until it encounters a point of $\Delta(f)$ not lying on the y-axis. The monomials of f corresponding to the points on this line then form the quasihomogeneous part of f. As in the examples, all the points in the support now lie on the right of or above this line. One then finds a solution for the quasihomogeneous part, uses it as an approximate solution in order to find a new equation, and iterates, as in the second example. In general the process does not end after a finite number of steps. Rather, one finds an infinite series for y in terms of fractional powers of x.

Before making this more precise, we give the following definition.

Definition 4.3.2. Let

$$f = \sum_{\substack{a,b \geq 0 \\ (a,b) \neq (0,0)}} \alpha_{a,b} x^a y^b \in \mathbb{C}\{x, y\}$$

be a convergent power series. Then f is said to be *y-general* if the power series $f(0, y) \in \mathbb{C}\{y\}$ does not equal 0, and f is called y-general of *order m* if $\alpha_{0,m} \neq 0$ and $\alpha_{0,i} = 0$ for $i < m$.

It is not hard to show (see [Fis01, p.105]) that if f has order m, then f can be made to be y-general of order m via a linear change of coordinates.

Then there is the following theorem (see [Fis01, p.137] and [BK86, p.386]):

Theorem 4.3.3. Let $f \in \mathbb{C}\{x, y\}$ be irreducible and y-general of order m. Then there exists an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ there exists a $\delta > 0$ such that if

$$V = \{(x, y) \in \mathbb{C}^2 \mid |x| < \delta, |y| < \epsilon, f(x, y) = 0\}$$

is the vanishing set of f in a neighbourhood of 0, then one can find a converging power series $y(z) \in \mathbb{C}\{z\}$ for which the mapping

$$\begin{aligned} \pi : & D_{\delta^{\frac{1}{m}}} \to \mathbb{C}^2 \\ & z \mapsto (z^m, y(z)) \end{aligned}$$

is a holomorphic surjection onto V, the restriction

$$\pi : D_{\delta^{\frac{1}{m}}} \setminus \{0\} \to V \setminus \{0\}$$

is biholomorphic, and $\pi^{-1}(0) = \{0\}$.

In other words, for z small enough we have $f(z^n, y(z)) = 0$ (where $n = m$ in the theorem). It can be shown that this power series $y(z)$ is unique up to replacing z by $\zeta_n^k z$, where ζ_n is a primitive n-th root of unity, and $k \in \mathbb{Z}$.

By replacing z with $x^{\frac{1}{n}}$ we get the following definition:

Definition 4.3.4. The series $y(x) = \sum_i \alpha_i x^{\frac{i}{n}}$ obtained by replacing z with $x^{\frac{1}{n}}$ is called a *Puiseux expansion* for the curve with equation $f(x, y) = 0$. It satisfies the equation $f(x, y(x)) = 0$ for small values of x.

4.4 Puiseux pairs

For this section we will assume that f is irreducible in the ring $\mathbb{C}\{x, y\}$.

As a result of the previous two sections, we now have a method for constructing the set $\Sigma \cap V \subset \Sigma$. We start by constructing a Puiseux expansion $y(x)$ for $f(x, y) = 0$. For a given value of x, there are up to n corresponding values for y depending on which branch of the multivalued function $x^{\frac{1}{n}}$ we choose. As before, we choose the parameters of the polydisk in such a way that $\Sigma \cap V$ is completely contained in the interior of one of the two solid tori comprising the boundary Σ of the polydisk. Then we let x move along the circle with radius δ, and trace the n or less paths followed by the corresponding values of y in $\mathrm{int}(T^+)$.

As an example, consider the polynomial $f(x, y) = y^2 - x^3$. Let $x(t)$ be a path around the unit circle

$$x(t) = e^{2\pi i t}.$$

The solutions of $f(x(t), y) = 0$ are then

$$\begin{aligned} y_1(t) &= e^{2\pi i (\frac{3}{2}) t} \text{ and} \\ y_2(t) &= -e^{2\pi i (\frac{3}{2}) t}. \end{aligned}$$

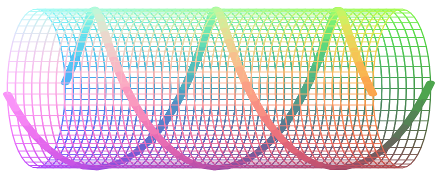

Figure 4.3: *Graph of solutions*

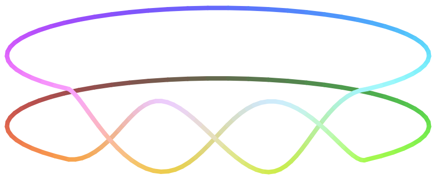

Figure 4.4: *The corresponding knot*

Instead of viewing the domain of t as a circle, we consider it to be the interval $[0,1]$ with the endpoints identified. The the graphs of the two solutions lie on the cylinder

$$[0,1] \times S^1 \subset [0,1] \times \mathbb{C},$$

as can be seen in Figure 4.3.

Now consider the cylinder with the circles on the ends identified, giving a torus. This torus lies inside the solid torus T^+ making up part of Σ. It is clear that the union of the two solution graphs gives a knot inside Σ. By viewing the cylinder in Figure 4.3 from the top and connecting the ends, we see that the knot looks as in Figure 4.4.

In principle this could be carried out for the Puiseux expansion corresponding to any polynomial $f(x,y)$ to give a corresponding link. Furthermore, two polynomials are topologically equivalent precisely if the corresponding links are equivalent. Thus we need to ask which changes of the Puiseux expansion really change the link in an essential way, and which changes just perturb it slightly into something equivalent to the original.

It will be helpful to consider two examples. For the first example, suppose that the Puiseux expansion of f is $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$. [1] Consider first the approximation $y = x^{\frac{3}{2}}$. The knot is then again as in Figure 4.4. Now consider the full expression for y. The radius of the circle along which x moves, i.e. δ, can be taken to be very small, in which case the contribution of the second term $x^{\frac{7}{4}}$ is small compared to that of the first term $x^{\frac{3}{2}}$. Thus in the figure of the knot for the full expression (Figure 4.5), we can see that each string has been replaced by two strings twisting around each other. Without giving a proof, it is plausible that the resulting knot is not equivalent to the knot corresponding to the approximate solution $y = x^{\frac{3}{2}}$. Thus the addition

[1] As a multivalued function one should choose the branches of the different terms to be consistent with each other. To avoid ambiguity one could rewrite the expansion as $y = x_1^6 + x_1^7$, where $x = x_1^4$, which makes it clear that there are only 4 possible branches, and not 8. Similar remarks apply to the expansions which follow.

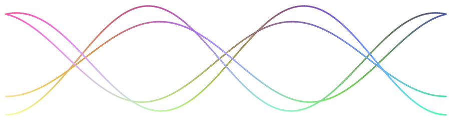

Figure 4.5: *The knot (without the ends joined) of* $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$

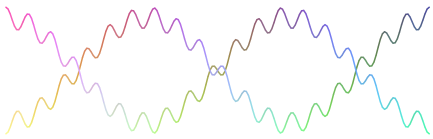

Figure 4.6: *The knot (without the ends joined) of* $y = x^{\frac{3}{2}} + x^{\frac{37}{2}}$

of the second term in the Puiseux expansion produces an essential change in the corresponding knot.

For the second example, suppose that the Puiseux expansion of f is $y = x^{\frac{3}{2}} + x^{\frac{37}{2}}$. Again consider the approximate solution $y = x^{\frac{3}{2}}$, giving the knot from Figure 4.4. This time, the addition of the extra term $x^{\frac{37}{2}}$ does not yield extra strings, as seen in Figure 4.6. Rather, each string now oscillates around its previous position. It is possible to smooth out the strings, showing that the two knots *are* equivalent in this case.

The examples suggest that adding a term in the Puiseux expansion only alters the equivalence class of the knot if the denominator of the exponent of the term increases, since this implies that the number of strings must increase.

Now assume that f is y-general of order m, where m is the order of f. This is equivalent to requiring that the y-axis is not a tangent at the singular point of the curve $V(f)$. It is also equivalent to requiring that the smallest exponent in the Puiseux expansion is at least 1.

We now proceed to define what are called the *Puiseux pairs* of f. Firstly, let us write the Puiseux expansion of f in the form

$$y = \sum \alpha_k x^k,$$

with $k \in \mathbb{Q}$, $k \geq 1$. Then if all k are integers, no Puiseux pairs are defined. Otherwise, there is a smallest k_1 which is not an integer. We can write

$$k_1 = \frac{n_1}{m_1}$$

where $n_1 > m_1$ and n_1 and m_1 are relatively prime. The pair (m_1, n_1) is called the first Puiseux pair of f. Some of the following exponents may be of the form $\frac{q}{m_1}$, but if not all of them are, then we will come to a k_2 which cannot be expressed in that form. We then write k_2 in the form

$$k_2 = \frac{n_2}{m_1 m_2}$$

where $m_2 > 1$ and n_2 and m_2 are relatively prime. The numbers n_2 and m_2 are uniquely determined by these conditions, and we define the pair (m_2, n_2) as the second Puiseux pair.

In general, if the Puiseux pairs $(m_1, n_1), \cdots, (m_j, n_j)$ are already defined, then let k_{j+1} be the smallest exponent for which the preceding exponents can all be expressed in the form

$$k = \frac{q}{m_1 \cdots m_j},$$

while k_{j+1} cannot. Writing

$$k_{j+1} = \frac{n_{j+1}}{m_1 \cdots m_{j+1}}$$

with n_{j+1} and m_{j+1} relatively prime and $m_{j+1} > 1$, we define (m_{j+1}, n_{j+1}) to be the next Puiseux pair.

Eventually this process has to stop (because there is a bound on the size of the denominators of the exponents). Thus we get a finite sequence $(m_1, n_1), \ldots, (m_g, n_g)$ of pairs of integers.

Definition 4.4.1. The pairs $(m_1, n_1), \ldots, (m_g, n_g)$ defined in this way are called the *Puiseux pairs* of f.

For example, if $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$, then the Puiseux pairs are

$$(m_1, n_1) = (2, 3)$$
$$(m_2, n_2) = (2, 7).$$

The above reasoning should make the following theorem plausible ([BK86, p.411]).

Theorem 4.4.2. Puiseux expansions with the same Puiseux pairs give topologically equivalent knots.

4.5 Torus knots

The knots that arise from irreducible polynomials are all of a very special kind, called *iterated torus knots*. In this section we will describe how the Puiseux pairs of a polynomial can be used to define the corresponding knot. This is relevant to being able to construct the Milnor fiber in the next chapter.

Firstly we will consider *torus knots*. These are the knots associated to polynomials of the basic form $f(x, y) = x^p - y^q$ where p and q are relatively prime. As before, we consider the knot $K = S_\epsilon \cap V(f)$, or more explicitly

$$K = \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = \epsilon^2, x^p = y^q\}.$$

The second condition implies that $|x|^p = |y|^q$, and if we solve this for $|y|$ and substitute into the first condition, we find positive constants η and μ such that $|x| = \eta$ and $|y| = \mu$ for all $(x, y) \in K$. Note that these two constants satisfy the equations

$$\eta^2 + \mu^2 = \epsilon^2$$
$$\eta^p = \mu^q.$$

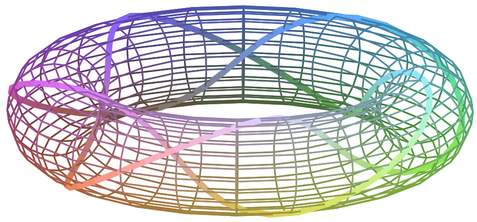

Figure 4.7: *Torus knot of type* (5, 2)

Now recall that we constructed a model for S_ϵ as the union of two solid tori

$$S_\epsilon = T_+ \cup T_-,$$

where

$$T_+ = \{(x, y) \in S_\epsilon \mid |y| \leq \mu\} \text{ and}$$
$$T_- = \{(x, y) \in S_\epsilon \mid |x| \leq \eta\}.$$

The intersection of these two solid tori is a two-dimensional torus consisting of all points in S_ϵ for which $|x| = \eta$ and $|y| = \mu$. Thus we see that K lies inside this torus. This torus can be modelled as $S^1 \times S^1$ by the mapping

$$\psi : S^1 \times S^1 \to T_- \cap T_+ \subset \mathbb{C}^2$$
$$(a, b) \mapsto (\eta a, \mu b)$$

Then K is precisely the image of S^1 after composing ψ with the map $S^1 \to S^1 \times S^1$ taking $t \in S^1$ to (t^q, t^p). Hence we see that K is obtained by moving around the torus $T_- \cap T_+$ p times in the one direction and q times in the other direction. See Figure 4.7 for the case where $p = 2$ and $q = 5$.

Note that the polynomial f in this case has just one Puiseux pair, namely (q, p). Now we want to consider the more general case of an arbitrary irreducible polynomial f (in two variables, with an isolated singular point at 0 as usual). First, it is necessary to explain the concept of *iterated torus knots*.

In the example of the torus knot, we had a torus, and we constructed a knot on this torus depending on two relatively prime integers p and q. It is important to note that we also had a trivialization of the torus, in other words, a map between the torus and $S^1 \times S^1$. Choosing a different trivialization would in general give a different knot.

To construct the torus knot, we could also have started with a simple (untied) knot inside S_ϵ. Then we consider a small tubular neighbourhood of this knot, and consider the surface of this solid torus. If we use the correct trivialization then we will get the same torus knot as before. The idea of the iterated torus knot is to repeat this process. In other words, one then considers a small tubular neighbourhood of the torus knot, and constructs another knot on its surface. This knot is again determined by two integers specifying how many times it must turn around the torus in the one direction and how many times in the other direction. It turns out that these two numbers are exactly the next Puiseux pair.

However, we still have to address the question of choosing the trivialization. This involves choosing a longitude circle and a latitude circle on the torus. These are two non-trivial loops on the torus with intersection number ± 1. Given such circles, one can find a trivialization (that is a mapping from the torus to $S^1 \times S^1$) such that the longitude circle corresponds to $\{t_1\} \times S^1$ and the latitude circle to $S^1 \times \{t_2\}$ for some $t_1, t_2 \in S^1$. The following method to choose these two circles comes from the book of Brieskorn ([BK86]).

We fix a decomposition of $S_\epsilon = T_- \cup T_+$ into two solid tori by writing $\epsilon^2 = \eta^2 + \mu^2$ as previously described. From that description also follows the trivializations

$$T_+ \cong S^1 \times D^2$$
$$T_- \cong D^2 \times S^1.$$

We call a knot K *regularly embedded* when it is contained in

$$\mathrm{int}(T_+) \cong S^1 \times \mathrm{int}(D^2),$$

and the projection $K \to S^1$ is a differentiable orientation-preserving covering. Of course in our cases we will always be able to assume that the knot satisfies this requirement. Then we choose a tubular neighbourhood T of K which is small in the sense that for each $t \in S^1$, the intersection $T \cap (\{t\} \times D^2)$ consists of disjoints disks $D_i(t)$ around the points $p_i(t)$ of $K \cap (\{t\} \times D^2)$. We choose the boundary of one such disk as the longitude circle on $F = \partial T$. As a latitude circle on F, we choose a curve whose points $q_i(t) \in D_i(t)$ are such that $q_i(t) - p_i(t)$ has constant direction in D^2.

We now proceed to define iterated torus knots in a recursive fashion.

Definition 4.5.1. The trivial knot $S^1 \times \{0\} \subset S^1 \times D^2 \subset S^3$ is *the torus knot of order* 0. A torus knot of type (m_1, n_1) on the boundary of a tubular neighbourhood of the trivial knot is *a torus knot of order* 1 *and type* (m_1, n_1). If we take the tubular neighbourhood to be small enough, then this knot is regularly embedded.

Now assume that $K_i \subset S^1 \times D^2 \subset S^3$ is a regularly embedded torus knot of order i and type $(m_1, n_1), \ldots, (m_i, n_i)$, and let K_{i+1} be a torus knot of type (m_{i+1}, n_{i+1}) on a tubular neighbourhood of K_i. Then K_{i+1} is called *a torus knot of order* $i+1$ *and type* $(m_1, n_1), \ldots, (m_{i+1}, n_{i+1})$.

Torus knots of higher order are called *iterated torus knot.*

The point of making these definitions is to be able to state the following proposition (see [BK86, p.438]).

Proposition 4.5.2. The knot corresponding to the Puiseux expansion

$$y = x^{\frac{n_1}{m_1}} + x^{\frac{n_2}{m_1 m_2}} + \cdots + x^{\frac{n_g}{m_1 \cdots m_g}}$$

with Puiseux pairs $(m_1, n_1), \ldots, (m_g, n_g)$ is an iterated torus knot of order g and type

$$(m_1, n_1), \ldots, (m_g, n_g).$$

Furthermore, one can use methods from knot theory to classify the iterated torus knots, yielding the following result.

Proposition 4.5.3. Puiseux expansions with different Puiseux pairs give different iterated torus knots.

Combining this with Theorem 4.4.2 shows that Puiseux expansions have the same Puiseux pairs if and only if they give the same iterated torus knots.

In conclusion, the results of this chapter can be summarized by the following theorem:

Theorem 4.5.4. Let $f \in \mathbb{C}\{x, y\}$ be a convergent power series with an isolated singularity at 0. Then f can be parametrized by a Puiseux expansion with Puiseux pairs $(m_1, n_1), \ldots, (m_g, n_g)$. Furthermore, the intersection of $V(f)$ with a small sphere S_ϵ is an iterated torus knot of type $(m_1, n_1), \ldots, (m_g, n_g)$. Two such isolated singularities are topologically equivalent if and only if their Puiseux pairs coincide.

Chapter 5

Constructing the Milnor fiber

As in the previous chapter, we will restrict ourselves to the case where f is a polynomial in two variables, and for simplicity assume it to be irreducible in the ring of convergent power series $\mathbb{C}\{x, y\}$. Let $K \subset S_\epsilon$ be the corresponding knot as described before. Then the associated Milnor fibration is the fibration of $S_\epsilon \setminus K$ over S^1. The knot K is the boundary of the closure of any of the fibers (see [Mil68, p.55]). In this chapter we will refer to the closure of any of the fibers as the Milnor fiber and denote it by F_δ. It is a compact two dimensional manifold-with-boundary, with the boundary consisting of a single connected component homeomorphic to S^1.

In this chapter we will describe two different ways of constructing the Milnor fiber up to homeomorphism. It only depends on the corresponding knot. The first way (from [AGZV88]) will be a very intuitive method which only works for the simplest cases but allows one to find the corresponding vanishing cycles on the Milnor fiber. The second way (from [BK86]) will be valid for any irreducible polynomial f with a single Puiseux pair. For the more general case of an arbitrary number of Puiseux pairs, one can consult [BK86, p.555] for a construction of the Milnor fiber, although it soon becomes very hard to visualize the Milnor fiber together with its embedding into S^3.

5.1 Constructing the Milnor fiber by using a perturbation

In the beginning of [AGZV88] it is shown how to obtain the Milnor fiber, the associated vanishing cycles and the monodromy for a few very simple examples. In this section we will use this method for a slightly more complicated example. Although the procedure does not easily generalize to more complicated examples, it gives one a nice geometric picture and some intuition about the basic objects that were defined in Chapter 3.

The example we will consider is $f(x, y) = y^2 - x^5$. Consider the perturbation

$$f_t(x, y) = y^2 - x^5 + 5t^4 x,$$

where $t \in \mathbb{C}$. For $t \neq 0$, the function f_t has 4 critical points, namely

$$(e^{\frac{k\pi i}{2}} t, 0), \qquad k = 0, 1, 2, 3,$$

or equivalently

$$(\pm t, 0), (\pm it, 0).$$

The corresponding critical values are $\pm 4t^5$ and $\pm i4t^5$. By calculating the second partial derivatives, one can verify that for $t \neq 0$, all the critical points of f_t are non-degenerate, and since the critical values are distinct, it follows that f_t is a Morse function. Hence we can consider t to be a small fixed value, different from 0, and use the resulting function $\tilde{f} := f_t$ to determine the Milnor fiber of f. Take t to be positive and real for simplicity.

It has been shown in a previous chapter that the Milnor fiber of f is given by taking the fiber of $\tilde{f}$ above δ, intersected with a small ball $\overline{B}_\epsilon$ which contains all the critical points. By using Ehresmann's fibration theorem we could show that $\tilde{f}$ gives a locally trivial fibration over the disk $\overline{D}_\delta$ with the critical values removes. It follows that the Milnor fiber of f is also given by taking the fiber of $\tilde{f}$ above 0, intersected with the same small ball $\overline{B}_\epsilon$. In the previous chapter we saw that the small ball can be replaced by a polydisk

$$\overline{D} = \{(x, y) \in \mathbb{C}^2 \mid |x| \leq \delta, |y| \leq \mu\}.$$

By choosing $\delta > t$ and μ large enough, we get the following description of the Milnor fiber in this case:

$$\tilde{f}^{-1}(0) \cap \overline{D} = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^5 - 5t^4 x, |x| \leq \delta\}.$$

By μ large enough, we mean that the condition $|y| \leq \mu$ is automatically satisfied for the points in the above set. This set is part of the Riemann surface corresponding to the equation $y^2 = x^5 - 5t^4 x$. To construct it, we solve for y

$$y = \pm\sqrt{x(x^4 - 5t^4)}$$

and find the branching points:

$$x = 0$$
$$x = 5^{\frac{1}{4}} e^{\frac{k\pi i}{2}} t, \text{ where } k = 0, 1, 2, 3.$$

If we wanted to construct the entire Riemann surface corresponding to the equation, we would have had to include ∞ as a branching point. Then there would be six branching points on the Riemann sphere, which could be joined pairwise by non-intersecting paths. One would then take two concentric Riemann spheres, cut each along these same paths, and join them again in such a way that crossing a path takes one from the inner sphere to the outer sphere, and vice versa.

Since we only want to construct a small part of the Riemann surface, we need to take two closed disks of radius δ, and cut them along appropriate paths joining the branching points. Note that since there are an uneven number of branching points, one of the branching points will be joined to a point on the boundary of the disk. This is actually the path going to the branching point at ∞. Figure 5.1 shows the five branch points and the paths.

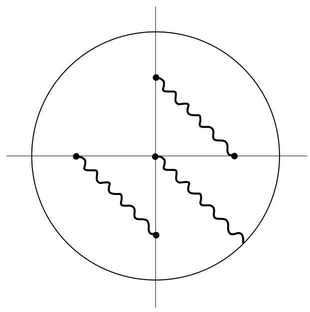

Figure 5.1: *One of two disks used to construct the Milnor fiber*

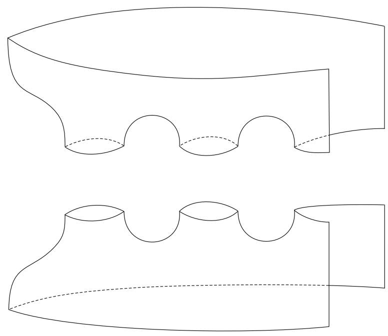

Figure 5.2: *An intermediate step in deforming the Milnor fiber*

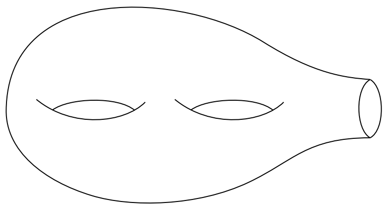

Figure 5.3: *Milnor fiber in a recognizable form*

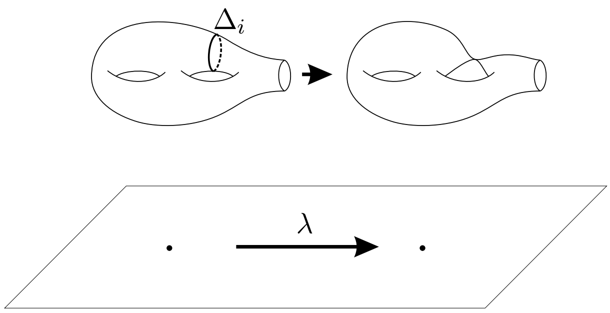

Figure 5.4: *Effect of λ moving to one of the critical values*

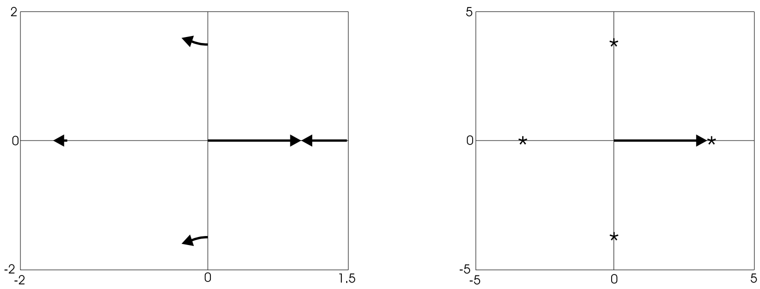

Figure 5.5: *Branching points change as λ moves*

It takes some effort to visualize the process, and to deform the resulting surface into something recognizable. (See Figure 5.2 for an intermediate step.) But having done that, one finds that the Milnor fiber is homeomorphic to the surface-with-boundary in Figure 5.3.

Next, we want to consider the problem of finding the vanishing cycles on the Milnor fiber. From the explanation in the second chapter, we know that we should consider how the fiber of $\tilde{f}$ above a point $\lambda \in \mathbb{C}$ changes as λ approaches one of the critical values. For each critical value there should be one cycle in the first homology group which vanishes as can be seen in Figure 5.4. To find these vanishing cycles, we look at how the branch points move as λ goes from 0 to one of the critical values. Figure 5.5 shows what happens when λ moves from 0 to $4t^5$. The right side of the figure shows the path followed by λ. The left side shows how the branching points change. Two of the branching points move towards each other and become one. The associated vanishing cycle must thus be a cycle which encircles these two branching points. One can draw the cycle on the Milnor fiber as follows: fix a point on the cycle as being on the first of the two closed disks used to construct the Milnor fiber. Then move along the cycle, alternating disks everytime you cross one of the paths between branching points.

On the disks used to construct the Milnor fiber, the cycles look simply as in Figure 5.6. For each cycle, we choose an orientation, and label them as shown in the figure. It is then possible to draw the cycles on the resulting Milnor fiber. Figure 5.7 shows the Milnor fiber as before, but with four oriented cycles generating the first homology group. By working carefully, one can compute expressions for the vanishing cycles in terms of this basis (note that these expressions

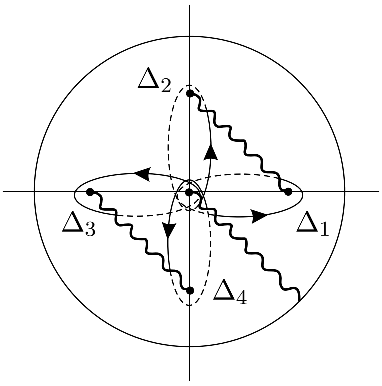

Figure 5.6: *Vanishing cycles on disk*

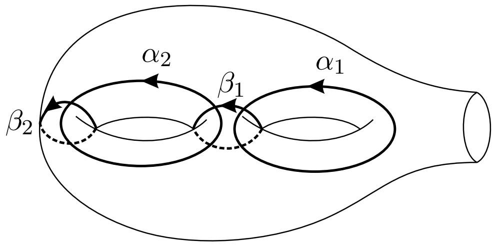

Figure 5.7: *Milnor fiber with generators of first homology group*

depend upon the way in which one goes from Figure 5.2 to Figure 5.3):

$$\Delta_1 = \alpha_1$$
$$\Delta_2 = \alpha_1 - \beta_1$$
$$\Delta_3 = \alpha_1 + \alpha_2 - \beta_1$$
$$\Delta_4 = \alpha_1 + \alpha_2 - \beta_1 - \beta_2$$

It is easy to see that the intersection numbers of the α's and β's are

$$\alpha_1 \circ \beta_1 = -1$$
$$\alpha_2 \circ \beta_1 = 1$$
$$\alpha_2 \circ \beta_2 = -1$$

and 0 for all the rest. From this we can deduce the intersection matrix for the distinguished basis of vanishing cycles:

$$S = (\Delta_i \circ \Delta_j)_{i,j} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

As explained before on page 34, to each vanishing cycle Δ_i, we can associate a diffeomorphism h_i of the Milnor fiber F_δ. This gives an associated homomorphism h_{i*} of the homology group

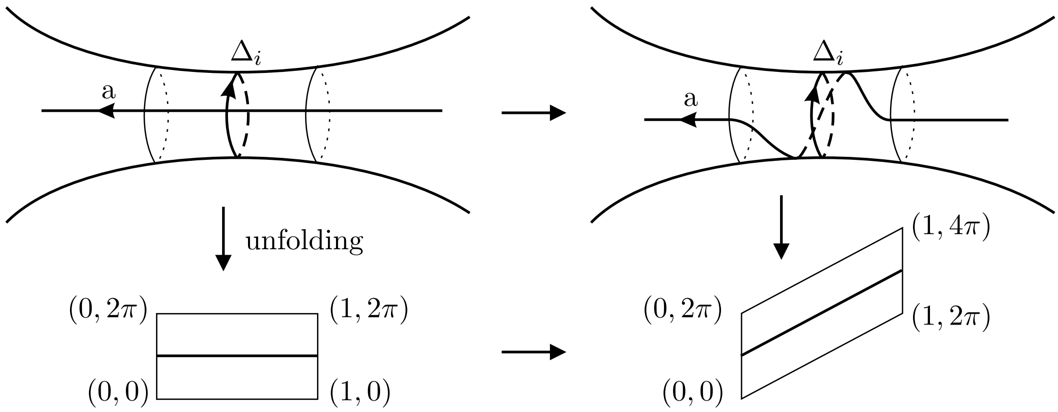

Figure 5.8: *Performing a Dehn twist*

$H_1(F_\delta)$. Since the Δ_i's form a basis for this homology group, we can express h_{i*} in terms of this basis. On page 35 we gave the formula (with $n-1=1$)

$$h_{i*}(a) = a - (a \circ \Delta_i)\Delta_i,$$

where $a \in H_1(F_\delta)$. In the present case where $n = 2$, it is actually not too hard to deduce the formula. The first step would be to show that the diffeomorphism h_i corresponding to the loop Δ_i is simply a Dehn twist around Δ_i. See Figure 5.8 for a description of a Dehn twist. Let $a \in H_1(F_\delta)$. Then near every point of intersection between a and Δ_i, the effect of h_i is to replace the part of a close to the point of intersection with either $a + \Delta_i$ or $a - \Delta_i$. The sign depends upon our convention of which way the Dehn twist goes, and on the intersection number at that point (1 or -1). By taking care of the signs, one arrives at the above expression for $h_{i*}(a)$.

Applying this formula by using the above intersection matrix, gives

$$h_{1*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad h_{2*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$h_{3*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad h_{4*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Now we can calculate the monodromy

$$h_* = h_{4*} \circ h_{3*} \circ h_{2*} \circ h_{1*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

The eigenvalues of h_* are $-\zeta_5, -\zeta_5^2, -\zeta_5^3$ and $-\zeta_5^4$, where $\zeta_5 = e^{\frac{2\pi i}{5}}$ is a fixed primitive 5-th root of unity. Note that these are exactly the roots predicted by Theorem 3.3.4.

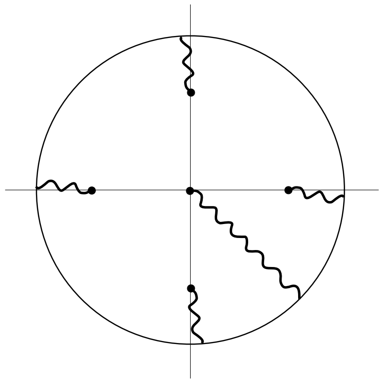

Figure 5.9: *Different branching pattern*

Figure 5.10: *Milnor fiber constructed in different way*

Finally, with the next section in mind, we give an alternative branching pattern (see Figure 5.9), which allows us to express the Milnor fiber in a different format (see Figure 5.10). Again, it requires a bit of thought to convince oneself that the Milnor fiber is indeed of this form. The corresponding vanishing cycles are now easier to visualize: choose one of the strips connecting the upper disk with the lower disk. Then each of the four vanishing cycles can be found by starting with a point on this marked strip, moving up to the upper disk, then across to one of the other four strips, down this strip, and across the lower disk to where the marked strip starts, and then up again to the point on this strip where the loop started. It is now also simpler to deduce the intersection matrix, which turns out to be exactly the same as before.

5.2 Constructing the Milnor fiber as a spanning surface

Consider again the knot $K \subset S^3$ and the Milnor fiber F_δ. As mentioned before, F_δ has K as its boundary. In knot theory it is shown that for any given knot $L \subset S^3$, one can find an orientable surface-with-boundary M, such that $\partial M = L$ (see [Rol90]). Such a surface is called a *spanning surface*. The genus of such a surface is defined as the genus of the compact surface obtained by attaching a disk to the boundary (which is homeomorphic to S^1). Among all such surfaces, there will be some that obtain a certain minimal genus. In such cases the surface is called a *minimal Seifert surface* corresponding to the knot.

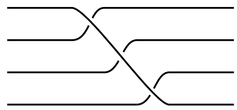

Figure 5.11: *Part of knot*

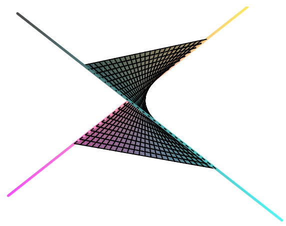

Figure 5.12: *How to attach a rectangle at each crossing*

Two Seifert surfaces are said to be equivalent if there is a homeomorphism of S^3 taking the one surface to the other. In general, a knot may have inequivalent Seifert surfaces, but in our case this does not occur. This is because all the knots we deal with are *fibered knots*. For the details see [BK86, p.542]. It then follows from the Neuwirth-Stallings Theorem (see [Mil68, p.83]) that the Milnor fiber is a Seifert surface for the knot K. There is a general method to find a Seifert surface of a given knot (described in [Rol90] and [BK86]), so thus we can apply this method in our case to find the Milnor fiber using just the knot K. We will describe this for the case we are interested in, namely a torus knot of type (p, q) in S^3.

Consider the knot to be drawn on the surface of a torus in the manner described previously. In other words, twisting p times around in the one direction, and q times in the other direction. Then we project the knot onto a plane which is perpendicular to the axis of rotational symmetry of the torus (i.e. the axis passing through the hole). Then the knot can be seen as constructed by cyclically joining p copies of the q strings shown in Figure 5.11 end-to-end.

To construct the spanning surface, we start by attaching a twisted rectangle to each crossing, in the sense shown in Figure 5.12. Now if we remove the interiors of these rectangles, and the parts of their boundaries lying on the projection of the knot, then what remains is q disjoint circles. We then attach a disk to each of these circles (such that the circle is the boundary of the disk). Then the disks together with the little rectangles make up the desired surface.

To be able to visualize the resulting surface, we move these disks so that they are stacked on top of each other (with some space in between). Then the little rectangles become stretched into long strips with a single twist. Thus two disks which are next to each other are connected by p strips, each twisted once in the same direction. See Figure 5.13 for the case where $p = 4$ and $q = 3$. Note that for $p = 5$ and $q = 2$ we again get the surface obtained in the previous section in Figure 5.10.

Figure 5.13: *Milnor fiber as disks with joining strips*

Bibliography

[AGZV88] V. I. Arnol′d, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. II*, Monographs in Mathematics, vol. 83, Birkhäuser Boston Inc., Boston, MA, 1988, Monodromy and asymptotics of integrals, Translated from the Russian by Hugh Porteous, Translation revised by the authors and James Montaldi.

[BK86] Egbert Brieskorn and Horst Knörrer, *Plane algebraic curves*, Birkhäuser Verlag, Basel, 1986, Translated from the German by John Stillwell.

[Bra28] K. Brauner, *Zur geometrie der funktionen zweier komplexen veränderlichen*, Abh. Math. Sem. Hamburg **6** (1928), 1–54.

[Bur32] W. Burau, *Kennzeichnung der schlauchknoten*, Abh. Math. Sem. Hamburg **9** (1932), 125–133.

[Dur99] Alan H. Durfee, *Singularities*, History of topology, North-Holland, Amsterdam, 1999, pp. 417–434.

[Ehr47] Charles Ehresmann, *Sur les espaces fibrés différentiables*, C. R. Acad. Sci. Paris **224** (1947), 1611–1612.

[Fis01] Gerd Fischer, *Plane algebraic curves*, Student Mathematical Library, vol. 15, American Mathematical Society, Providence, RI, 2001, Translated from the 1994 German original by Leslie Kay.

[Hur58] Witold Hurewicz, *Lectures on ordinary differential equations*, The Technology Press of the Massachusetts Institute of Technology, Cambridge, Mass., 1958.

[Käh29] Erich Kähler, *Über die Verzweigung einer algebraischen Funktion zweier Veränderlichen in der Umgebung einer singulären Stelle*, Math. Z. **30** (1929), no. 1, 188–204.

[Lam81] Klaus Lamotke, *The topology of complex projective varieties after S. Lefschetz*, Topology **20** (1981), no. 1, 15–51.

[Lan72] Serge Lang, *Differential manifolds*, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Don Mills, Ont., 1972.

[Loo84] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Mathematical Society Lecture Note Series, vol. 77, Cambridge University Press, Cambridge, 1984.

[Mil63] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963.

[Mil68] John Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., 1968.

[Rol90] Dale Rolfsen, *Knots and links*, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Houston, TX, 1990, Corrected reprint of the 1976 original.

[Sie01] Dirk Siersma, *The vanishing topology of non isolated singularities*, New developments in singularity theory (Cambridge, 2000), NATO Sci. Ser. II Math. Phys. Chem., vol. 21, Kluwer Acad. Publ., Dordrecht, 2001, pp. 447–472.

[War83] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition.

[Zar32] Oscar Zariski, *On the Topology of Algebroid Singularities*, Amer. J. Math. **54** (1932), no. 3, 453–465.