The abc-conjecture and k-free numbers

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Abstract

In his paper [14], A. Granville proved several strong results about the distribution of square-free values of polynomials, under the assumption of the abc-conjecture. In our thesis, we generalize some of Granville's results to k-free values of polynomials (i.e., values of polynomials not divisible by the k-th power of a prime) . Further, we generalize a result of Granville on the gaps between consecutive square-free numbers to gaps between integers, such that the values of a given polynomial f evaluated at them are k-free.
All our results are under assumption of the abc-conjecture.

Contents

Notation

Let $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ be complex valued functions and $h : \mathbb{R} \to \mathbb{R}^+$. We use the following notation:

$$f(X) = g(X) + O(h(X)) \text{ as } X \to \infty$$

if there are constants X_0 and $C > 0$ such that

$$|f(X) - g(X)| \leq Ch(X)$$

for all $X \in \mathbb{R}$ and $X \geq X_0$;

$f(X) = g(X) + o(h(X))$ as $X \to \infty$ iff $\lim_{X\to\infty} \frac{f(X)-g(X)}{h(X)} = 0$;

$f(X) \sim g(X)$ as $X \to \infty$ iff $\lim_{X\to\infty} \frac{f(X)}{g(X)} = 1$.

We write $f(X) \ll g(X)$ or $g(X) \gg f(X)$ to indicate that $f(X) = O\left(g(X)\right)$

We denote by $\gcd\left(a_1, a_2, \dots, a_r\right)$, $\operatorname{lcm}\left(a_1, a_2, \dots, a_r\right)$, the greatest common divisor, and the lowest common multiple, respectively, of the integers $a_1, a_2, \dots, a_r$.

We say that a positive integer n is k-free if n is not divisible by the k-th power of a prime number.

Chapter 1

Introduction

In 1985, Oesterlé and Masser posed the following conjecture:

The *abc*-conjecture. *Fix $\varepsilon > 0$. If a, b, c are coprime positive integers satisfying $a + b = c$ then*

$$c \ll_\varepsilon N(abc)^{1+\varepsilon},$$

where for a given integer m, $N(m)$ denotes the product of the distinct primes dividing m.

In fact, Oesterlé first posed a weaker conjecture, motivated by a conjecture of Szpiro regarding elliptic curves. Then Masser posed the *abc*-conjecture as stated above motivated by a Theorem of Mason, which gives an similar statement for polynomials.
On its own, the *abc*-conjecture merits much admiration. Like the most intriguing problems in Number Theory, the *abc*-conjecture is easy to state but apparently very difficult to prove.The *abc*-conjecture has many fascinating applications; for instance Fermat's last Theorem, Roth's theorem, and the Mordell conjecture, proved by G. Faltings [4] in 1984.
Another consequence is the following result proved by Langevin [22] and Granville [14]:
Assume that the *abc*-conjecture is true. Let $F(X, Y) \in \mathbb{Q}[X, Y]$ be a homogeneous polynomial of degree $d \geq 3$, without any repeated linear factor such that $F(m, n) \in \mathbb{Z}$ for all $m, n \in \mathbb{Z}$. Fix $\varepsilon > 0$. Then, for any coprime integers m and n,

$$N(F(m, n)) \gg \max\{|m|, |n|\}^{d-2-\varepsilon},$$

where the constant implied by $\gg$ depends only on ε and F. With this consequence we generalize some results of Granville [14] on the distribution problem for the square free values of polynomials to the distribution problem for k-free values of polynomials for every $k \geq 2$.

Let $f(X) \in \mathbb{Q}[X]$ be a non-zero polynomial without repeated roots such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.
In his paper, Granville proved, under the *abc*-conjecture assumption, that if $\gcd_{n\in\mathbb{Z}}(f(n))$ is square free, then there are asymptotically $c_f N$ positive integers $n \leq N$ such that $f(n)$ is square free, where c_f is a positive constant depending only on f.
In section 3.1, we generalize this as follows:

Assume the abc-conjecture. Let k be an integer ≥ 2 and suppose that $\gcd_{n\in\mathbb{Z}}(f(n))$ is k-free. Then there is a positive constant $c_{f,k}$ such that:

$$\#\{n \in \mathbb{Z} : n \leq N, \quad f(n) \quad k\text{-free}\} \sim c_{f,k} N \qquad \text{as } N \to \infty$$

If we do not assume the *abc*-conjecture only under much stronger constraints results have been proved. For example Hooley [18] obtained only the following result.
Let $f(X)$ be an irreducible polynomial of degree $d \geq 3$ for which $\gcd_{n\in\mathbb{Z}} f(n)$ is $(d-1)$-free. Then if $S(x)$ is the number of positive integers $\leq x$ for which $f(n)$ is $(d-1)$-free, we have as $x \to \infty$

$$S(x) = x \prod_p \left(1 - \frac{\omega_f(p)}{p^{d-1}}\right) + O\left(\frac{x}{(\log x)^{A/\log\log\log x}}\right),$$

where $\omega_f(p) = \#\{0 \leq n < p^{d-1} : f(n) \equiv 0 \pmod{p^{d-1}}\}$ and A is a positive constant depending only on f.

In section 3.2 we will investigate the problem of finding an $h = h(x)$ as small as possible such that, for x sufficiently large, there is an integer $m \in (x, x+h]$ such that $f(m)$ is k-free, where $f(X) \in \mathbb{Q}[X]$ is irreducible and $f(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$.
This problem has been investigated in the case $f(X) = X$ and $k = 2$ by Roth [26], and Filaseta and Trifonov [10].In particular Filaseta and Trifonov have shown in 1990 that there is a constant $c > 0$ such that, for x sufficiently large, the interval $(x, x+h]$ with $h = cx^{8/37}$ contains a square free number. Using exponential sums, they showed that $8/37$ may be replaced by $3/14$. A few years later, in 1993, the same authors obtained the following improvement: there exists a constant $c > 0$ such that for x sufficiently large the interval $\left(x, x + cx^{1/3}\log x\right]$ contains a square free number. Under the *abc*-conjecture, Granville [14] showed that $h(x) = x^{\varepsilon}$ ($\varepsilon > 0$ arbitrary) can be taken.

Again assuming the *abc*-conjecture we extend this as follows:

For every $\varepsilon > 0$ and every sufficiently large x, there is an integer $m \in (x, x + x^{\varepsilon}]$ such that $f(m)$ is k-free.

Now, let $s_1, s_2, \ldots$ denote the positive integers m in ascending order such that $f(m)$ is k-free.
The main purpose of chapter 4 is to study the average moments of $s_{n+1} - s_n$; that is, the asymptotic behaviour of $\frac{1}{x} \sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^A$ as $x \to \infty$.
It was Erdős [5] who began to study this problem in the case $f(X) = X$. Erdős showed that, if $0 \leq A \leq 2$, then

$$\sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^A \sim \beta_A x \quad \text{as } x \to \infty \tag{1.1}$$

where β_A is a function depending only on A. In 1973 Hooley[19] extended the range of validity of this result to $0 \leq A \leq 3$; and in 1993, Filaseta [9] extended this further to $0 \leq A < 29/9 = 3,222\ldots$
In our case we will allow any $A > 0$ and generalize this result to every irreducible polynomial $f(X) \in \mathbb{Q}[X]$ such that $f(n)$ is an integer for every $n \in \mathbb{Z}$. Before we state our Theorem we recall the result obtained by Beasley and Filaseta [1] without the assumption of the *abc*-conjecture.
Let $d = \deg(f) \geq 2$, and let $k \geq (\sqrt{2} - 1/2)d$. Let

$$\phi_1 = \frac{(2s+d)(k-s) - d(d-1)}{(2s+d)(k-s) + d(2s+1)},$$

where

$$s = \begin{cases} 1 & \text{if } 2 \leq d \leq 4 \\ \left[\left(\sqrt{2} - 1\right) d/2\right] & \text{if } d \geq 5 \end{cases}$$

Let

$$\phi_2 = \begin{cases} \frac{8d(d-1)}{(2k+d)^2 - 4} & \text{if } \left(\sqrt{2} - 1/2\right) \leq k \leq d \\ \frac{d}{(2k-d+r)} & \text{if } k \geq d+1, \end{cases}$$

where r is the largest positive integer such that $r(r-1) < 2d$. Then $\phi_1 > 0$, $\phi_2 > 0$,
and if

$$0 \leq A < \min\left\{\frac{1}{\phi_2}, 1 + \frac{\phi_1}{\phi_2}, k\right\},$$

then for every irreducible polynomial $f(X) \in \mathbb{Z}[X]$ of degree d such that $\gcd_{n \in \mathbb{Z}} f(n)$ is k-free,

$$\sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^A \sim \beta_A x \quad \text{as } x \to \infty$$

for some constant β_A depending only on A, $f(x)$, and k.
Assuming the *abc*-conjecture we establish the following result, which was

proved by Granville [14] in the special case $f(X) = X, k = 2$:

Let k be an integer $\geq \min(3, \deg(f))$. Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial without any repeated root such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $\gcd_{n\in\mathbb{Z}} f(n)$ is k-free. Suppose the abc-conjecture is true. Then for every real $A > 0$ there exists a constant $\beta_A > 0$ such that:

$$\sum_{s_n \leq x} (s_{n+1} - s_n)^A \sim \beta_A x \quad \textit{as } x \to \infty.$$

Chapter 2

The *abc*-conjecture and some consequences

2.1 The abc-conjecture

We recall the *abc*-conjecture.
The *abc*-conjecture [Oesterlé,Masser,Szpiro].
Fix $\varepsilon > 0$. If a, b, c are coprime positive integers satisfying $a + b = c$ then

$$c \ll_{\varepsilon} N(abc)^{1+\varepsilon},$$

where for a given integer m, $N(m)$ denotes the product of the distinct primes dividing m.

2.2 Consequences of the *abc*-conjecture

Now we state a consequence of the *abc*-conjecture, obtained independently by Granville [14] and Langevin [22] [23], on which all our results will rely.

Theorem 2.1. *Assume that the abc-conjecture is true. Let $F(X, Y) \in \mathbb{Q}[X, Y]$ be a homogeneous polynomial of degree $d \geq 3$, without any repeated linear factor such that $F(m, n) \in \mathbb{Z}$ for all $m, n \in \mathbb{Z}$. Fix $\varepsilon > 0$. Then, for any coprime integers m and n,*

$$N\left(F(m, n)\right) \gg \max\{|m|, |n|\}^{d-2-\varepsilon},$$

where the constant implied by $\gg$ depends only on ε and F.

The proof of this Theorem depends on some Lemmas which we state after giving some definitions.

Let $\varphi(z) = \frac{f(z)}{g(z)}$ a rational function, where $f(z), g(z) \in \mathbb{C}[z]$ are coprime polynomials. We define $\deg(\varphi) = \max(\deg(f), \deg(g))$.

φ defines a map from $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ to $\mathbb{P}^1(\mathbb{C})$ by defining:

(i) $\varphi(z) = \infty$ if $z \neq \infty$, $g(z) = 0$;

(ii) $\varphi(\infty) = \infty$ if $\deg(f) > \deg(g)$;

(iii) $\varphi(\infty) = 0$ if $\deg(f) < \deg(g)$;

(iv) $\varphi(\infty) = \mathrm{lc}(f)/\mathrm{lc}(g)$ if $\deg(f) = \deg(g)$,

where $\mathrm{lc}(f)$ denotes the leading coefficients of a polynomial f.
We define the multiplicity, $\mathrm{mult}_{z_o}(\varphi)$ of φ at $z_0 \in \mathbb{P}^1(\mathbb{C})$ as follows:

- if $z_0 \neq \infty$, $\varphi(z_0) \neq \infty$ we define $\mathrm{mult}_{z_0}(\varphi)$ to be the integer n such that $\varphi(z) - \varphi(z_0) = c\,(z - z_0)^n + (\text{higher power of } (z - z_0))$ and $c \neq 0$;

- if $z_0 \neq \infty$, $\varphi(z_0) = \infty$, define $\mathrm{mult}_{z_0}(\varphi) = \mathrm{mult}_{z_0}\left(\frac{1}{\varphi}\right)$;

- if $z_0 = \infty$, define $\mathrm{mult}_{z_0}(\varphi) = \mathrm{mult}_{z_0}(\varphi^*)$ where $\varphi^*(z) = \varphi\left(\frac{1}{z}\right)$.

We say that φ is ramified at z_0 if $\mathrm{mult}_{z_0}(\varphi) > 1$.
We say that φ is ramified over w_0 if there is $z_0 \in \mathbb{P}^1(\mathbb{C})$ with $\varphi(z_0) = w_0$ such that φ is ramified at z_0.
In general we have $\sum\limits_{z_0 \in \varphi^{-1}(w_0)} \mathrm{mult}_{z_0}(\varphi) = \deg(\varphi)$ for $w_0 \in \mathbb{P}^1(\mathbb{C})$.
The following is a special case of the Riemann-Hurwitz formula:

Lemma 2.2. *Let $\varphi \in \mathbb{C}(z)$ be a rational function. Then:*

$$2\deg(\varphi) - 2 = \sum_{z_0 \in \mathbb{P}^1(\mathbb{C})} \left(\mathrm{mult}_{z_0}(\varphi) - 1\right),$$

Proof. For a statement and proof of the general Riemann-Hurwitz formula, see [24] or [29]. □

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Lemma 2.3 (Belyi[2]). *For any finite subset S of $\mathbb{P}^1\left(\overline{\mathbb{Q}}\right)$, there exists a rational function $\phi(X) \in \mathbb{Q}(X)$, ramified only over $\{0, 1, \infty\}$, such that $\phi(S) \subset \{0, 1, \infty\}$.*

Proof. This useful Lemma is proved, for instance, by Serre as Theorem B on page 71 of [28] (for variations, see Belyi [2], Elkies [4], Langevin [22], [23], or Granville [16]). □

Lemma 2.4. *Let $F(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ be any non-zero homogeneous polynomial. Then we can determine a positive integer D, and homogeneous polynomials $a(X,Y), b(X,Y), c(X,Y) \in \mathbb{Z}[X,Y]$ all of degree D, without common factors such that:*

(i) *$a(X,Y)b(X,Y)c(X,Y)$ has exactly $D+2$ non-proportional linear factors, including the factors of F;*

(ii) *$a(X,Y) + b(X,Y) = c(X,Y)$.*

Proof. We apply Lemma 2.3 with $S = \{(\alpha,\beta) \in \mathbb{P}^1 : F(\alpha,\beta) = 0\}$. Let $\phi(X)$ be the rational function from Lemma 2.3, and write $\phi(X/Y) = a(X,Y)/c(X,Y)$, where $a(X,Y), c(X,Y) \in \mathbb{Z}[X,Y]$ are homogeneous forms, of the same degree as ϕ, (call it D) and without common factors. Let $b(x,y) = c(x,y) - a(x,y)$. Note that:

$$\begin{array}{lll} \phi(x/y) = 0 & \text{if and only if} & a(x,y) = 0; \\ \phi(x/y) = 1 & \text{if and only if} & b(x,y) = 0; \\ \phi(x/y) = \infty & \text{if and only if} & c(x,y) = 0. \end{array}$$

Therefore $F(x,y)$ divides $a(x,y)b(x,y)c(x,y)$. If we write $\#\phi^{-1}(u)$ for the number of distinct $t \in \mathbb{P}^1(\mathbb{Q})$ for which $\phi(t) = u$, then $\#\phi^{-1}(0) + \#\phi^{-1}(1) + \#\phi^{-1}(\infty)$ equals the number of distinct linear factors of $a(x,y)b(x,y)c(x,y)$, by the observation immediately above. On the other hand, applying the Riemann-Hurwitz formula to the map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, and the fact that ϕ is ramified only over $\{0,1,\infty\}$ we get:

$$\begin{aligned} 2D &= 2 + \sum_{u \in \phi^{-1}(\{0,1,\infty\})} (\mathrm{mult}_u(\phi) - 1) \\ &= 2 + \sum_{u \in \{0,1,\infty\}} D - \sum_{u \in \phi^{-1}\{0,1,\infty\}} 1 \\ &= 2 + \sum_{u \in \{0,1,\infty\}} D + \sum_{u \in \{0,1,\infty\}} \#\phi^{-1}(u) \\ &= 2 + \sum_{u \in \{0,1,\infty\}} \{D - \#\phi^{-1}(u)\}. \end{aligned}$$

Thus $\#\phi^{-1}(0) + \#\phi^{-1}(1) + \#\phi^{-1}(\infty) = D+2$ which concludes the proof. $\square$

Here we give the definition of discriminant, resultant, and some of their properties.

Definition 2.5. *Let, $g(X) = b\prod_{i=1}^{r}(X - \beta_i) \in \mathbb{Q}[X]$ then we define the discriminant of g by:*

$$\Delta(g) = b^{2r-2} \prod_{1 \le i < j \le r} (\beta_i - \beta_j)^2 .$$

Definition 2.6. *The resultant of two non-zero polynomials*

$$f(X) = b\prod_{i=1}^{s}(X-\beta_i),\ g(X) = c\prod_{j=1}^{r}(X-\gamma_j) \in \mathbb{Q}[X]$$

is defined by:

$$R(f,g) = b^r c^s \prod_{i=1}^{s}\prod_{j=1}^{r}(\beta_i-\gamma_j).$$

We easily deduce from these definitions the following properties:

(R1) $R(f,g) = (-1)^{rs}R(g,f)$;

(R2) $R(f,g) = b^r \prod_{i=1}^{s} g(\beta_i)$;

(R3) $\Delta(f) = (-1)^{s(s-1)/2}\, b^{-1}R(f,f')$;

(R4) If $f(X), g(X) \in \mathbb{Z}[X]$, there exist two polynomials $a\,(X)\,, b\,(X) \in \mathbb{Z}[X]$ with $\deg(a) \leq r-1$, $\deg(b) \leq s-1$ such that:

$$a(X)f(X) + b(X)g(X) = R(f,g).$$

For this last remark see [21] .

Definition 2.7. *Let* $F(X,Y) = \sum_{i=0}^{s} a_i X^{s-i}Y^i$, $G(X,Y) = \sum_{j=0}^{r} b_j X^{r-j}Y^j$ *be two binary homogeneous polynomials in* $\mathbb{Z}[X,Y]$ *such that* $a_0 \neq 0$, $b_0 \neq 0$. *Then we define the resultant of* F *and* G, $R(F,G)$, *by:* $R(F,G) = R(f,g)$, *where* $f(X) = F(X,1)$ *and* $g(X) = G(X,1)$.

Lemma 2.8. *Let* $F, G \in \mathbb{Z}[X,Y]$ *be two binary homogeneous polynomials, without common factor. Let* $m, n \in \mathbb{Z}$ *with* $\gcd(m,n) = 1$. *Then:*

$$\gcd\left(F(m,n), G(m,n)\right)|R(F,G).$$

Proof. Let $F(X,Y) = Y^s f\left(\frac{X}{Y}\right)$ and $G(X,Y) = Y^r g\left(\frac{X}{Y}\right)$ then by $(R4)$ there are two polynomials $a\,(X)\,, b\,(X) \in \mathbb{Z}[X]$ such that $a(X)f(X)+b(X)g(X) = R(f,g)$. Now put $A(X,Y) = Y^{r-1}a\left(\frac{X}{Y}\right)$, $B(X,Y) = Y^{s-1}b\left(\frac{X}{Y}\right)$. Then

$$A(X,Y)F(X,Y) + B(X,Y)G(X,Y) = Y^{r+s-1}R(F,G).$$

So

$$\gcd\left(F(m,n), G(m,n)\right)|n^{r+s-1}R(F,G).$$

By interchanging m and n we get:

$$\gcd\left(F(m,n),G(m,n)\right)|m^{r+s-1}R(F,G),$$

since $\gcd(m,n)=1$. Thus,

$$\gcd\left(F(m,n),G(m,n)\right)|R(F,G).$$

$\square$

For more details see [21] or [25].

Proof of Theorem 2.1. There is no loss of generality to assume that $F(X,Y)\in\mathbb{Z}[X,Y]$. Let $d=\deg(F)$ and let $a(x,y),b(x,y),c(x,y)$ be the homogeneous polynomials from Lemma 2.4. By multiplying together the irreducible factors of $a(x,y)b(x,y)c(x,y)$, we obtain a new polynomial $F(x,y)G(x,y)$ of degree $D+2$.
Let $m,n\in\mathbb{Z}$ with $\gcd(m,n)=1$ and put $r=gcd(a(m,n),b(m,n))$. r is bounded since it divides $R(a,b)$ which is a non-zero integer. Now using this remark we apply the *abc*-conjecture directly to the equation $\frac{a(m,n)}{r}+\frac{b(m,n)}{r}=\frac{c(m,n)}{r}$ to get

$$\max\left\{|a(m,n)|,|b(m,n)|\right\}\ll\left(\prod_{p|abc}p\right)^{1+\varepsilon/D},$$

where here and below constants implied by $\ll$ depend on F and ε. This implies:

$$\max\left\{|a(m,n)|,|b(m,n)|\right\}^{1-\varepsilon/D}\ll\left(\prod_{p|abc}p\right)^{1-\varepsilon^2/D^2}\leq\left(\prod_{p|abc}p\right);$$

hence

$$\max\left\{|a(m,n)|,|b(m,n)|\right\}^{1-\varepsilon/D}\ll\left(\prod_{p|FG}p\right)\ll G(m,n)\left(\prod_{p|F(m,n)}p\right).$$

Now to finish our proof it remains to find an upper bound and a lower bound respectively for $|G(m,n)|=\sum_{i=0}^{D+2-d}g_im^in^{D+2-d-i}$ and $\max\{|a(m,n)|,|b(m,n)|\}$.
Write $H(m,n)=\max\{|m|,|n|\}$, thus $|G(m,n)|=|\sum_{i=0}^{D+2-d}g_im^in^{D+2-d}|\ll$

H^{D+2-d}. Note that for every fixed real α, $|m-\alpha n| \ll H$. Moreover, for every real α and β with $\alpha \neq \beta$ we have $(m-\alpha n)-(m-\beta n) = -(\alpha-\beta)n$, and $\alpha(m-\beta n)-\beta(m-\alpha n) = (\alpha-\beta)m$. Thus, we deduce that $\max\{|m-\alpha n|, |m-\beta n|\} \gg H$. So, since $a(x,y), b(x,y)$ have no common factors, $\max\{|a(m,n)|, |b(m,n)|\} \gg H^D$. Substituting these two estimates into the equation above we get:

$$\prod_{primes\, p|F(m,n)} p \gg \frac{\max\{a(m,n), b(m,n)\}^{1-\varepsilon/D}}{G(m,n)} \gg \max\{|m|,|n|\}^{deg(F)-2-\varepsilon}.$$

□

If we wish to consider $f(X) \in \mathbb{Z}[X]$, then we can obtain a stronger consequence of Theorem 2.1 than comes from simply setting $n=1$. If $f(X)$ has degree d then we let $F(X,Y) = Y^{d+1}f(X/Y)$; thus $f(X) = F(X,1)$, but $\deg(F) = \deg(f)+1$. So now, applying Theorem 2.1,

$$\prod_{primes\, p|f(m)} p = \prod_{primes\, p|F(m,1)} p \gg \max\{|m|,|1|\}^{deg(F)-2-\varepsilon} = |m|^{deg(f)-1-\varepsilon}.$$

This yields

Corollary 2.9. *Assume that the abc-conjecture is true. Suppose that $f(X) \in \mathbb{Z}[X]$, has no repeated roots. Fix $\varepsilon > 0$. Then*

$$\prod_{primes\ p|f(m)} p \gg |m|^{\deg(f)-1-\varepsilon}.$$

Where the constant implied by $\gg$ depends on f and ε.

The next result, although an immediate corollary of the Theorem 2.1, will be stated like a Theorem because it will play an important role in what follows.

Theorem 2.10. *Let k be an integer ≥ 2. Assume that the abc-conjecture is true. Suppose that $F(X,Y) \in \mathbb{Z}[X,Y]$ is homogeneous, without any repeated linear factors. Fix $\varepsilon > 0$. If there exists an integer q such that q^k divides $F(m,n)$ for some coprime integers m and n then $q \ll \max\{|m|,|n|\}^{(2+\varepsilon)/(k-1)}$. Also, if $f(X) \in \mathbb{Z}[X]$ has no repeated roots and q^k divides $f(m)$, then $q \ll |m|^{(1+\varepsilon)/(k-1)}$.*

Here the constants implied by $\ll$ depend on ε, and F, f respectively.

Proof. By Theorem 2.1 we have

$$\prod_{primes\, p|F(m,n)} p \gg \max\{|m|,|n|\}^{deg(F)-2-\varepsilon}.$$

This is equivalent to

$$\max\{|m|,|n|\}^{2+\varepsilon} \cdot \prod_{primes\, p|F(m,n)} p \gg \max\{|m|,|n|\}^{\deg(F)}.$$

This implies that

$$|F(m,n)| \ll \max\{|m|,|n|\}^{2+\varepsilon} \cdot \prod_{primes\, p|F(m,n)} p.$$

Since clearly

$$q^{k-1} \prod_{primes\, p|F(m,n)} p \ll |F(m,n)|,$$

we obtain

$$q \ll \max\{|m|,|n|\}^{(2+\varepsilon)/(k-1)}$$

as required.

In the case $f(X) \in \mathbb{Z}[X]$ the proof is similar. $\square$

Chapter 3

Asymptotic estimate for the density of integers n for which $f(n)$ is k-free

Let k be an integer ≥ 2; let $f(X) \in \mathbb{Q}[X]$ be a polynomial such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $\gcd_{n\in\mathbb{Z}} f(n)$ is k-free. Now we will use the previous chapters to derive an asymptotic estimate for the number of positive integers $n \leq N$ such that $f(n)$ is k-free. Further we prove that for every $\varepsilon > 0$ and every sufficiently large z there is an integer $m \in [z, z + z^{\varepsilon})$, for which $f(m)$ is k-free. Both results are proved assuming the abc-conjecture.

3.1 Asymptotic estimate of integers n for which $f(n)$ is k-free

Let k be an integer ≥ 2 and $f(X)$ a polynomial in $\mathbb{Q}[X]$ of degree d without any repeated roots. We assume that $f(m) \in \mathbb{Z}$ for all $m \in \mathbb{Z}$ and $\gcd_{m\in\mathbb{Z}}(f(m))$ is k-free. Under these conditions, we expect that there are infinitely many integers m for which $f(m)$ is k-free but unconditionally this is far from being established.

The following result is an extension of a result of Granville [14] from square-free values to k-free values of polynomials.

Theorem 3.1. *Assume that the abc-conjecture is true. Then, as $N \to \infty$, there are $\sim c_{f,k}N$ positive integers $n \leq N$ for which $f(n)$ is k-free, with:*

$$c_{f,k} := \prod_{p\,prime} \left(1 - \frac{\omega_{f,k}(p)}{p^k}\right)$$

where, for each prime p, $\omega_{f,k}(p)$ denotes the number of integers a in the range $1 \leq a \leq p^k$ for which $f(a) \equiv 0 \pmod{p^k}$.

We first give a definition.

Definition 3.2. *For a polynomial $f(X) \in \mathbb{Q}[X]$, we define $L(f) := \operatorname{lcm}(b, \Delta(bf))$, where b is the smallest positive integer such that $bf(X) \in \mathbb{Z}[X]$.*

In the prove of this Theorem we need some auxiliary results.

Lemma 3.3 (Hensel's lemma)**.** *Let $f(x)$ be a polynomial with integer coefficients of degree d, and let $a_0 \in \mathbb{Z}$ be such that $f(a_0) \equiv 0 \pmod{p}$, $f'(a_0) \not\equiv 0 \pmod{p}$. Then for every $k \geq 1$ there is precisely one congruence class $a \pmod{p^k}$ such that*

$$f(a) \equiv 0 \pmod{p^k}, \ a \equiv a_0 \pmod{p}.$$

Proof. For this proof see also [20]. □

Remark 3.4. *If p does not divide the discriminant of f, and $f(r) \equiv 0 \pmod{p}$, then $f'(r) \not\equiv 0 \pmod{p}$.*

Corollary 3.5. *Let $f(X) \in \mathbb{Q}[X]$ be a polynomial of degree d, such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and let p be a prime such that p does not divide $L(f)$. Then:*

$$\omega_{f,k}(p) = |\{a \pmod{p^k} : f(a) \equiv 0 \pmod{p^k}\}| \leq d.$$

Proof. Let $f(X) = a_0X^d + a_1X^{d-1} + \ldots + a_d$. Let b be as in the Definition 3.2 and let $g(X) = bf(X)$. Then $g(X) = b_0X^d + b_1X^{d-1} + \ldots + b_d \in \mathbb{Z}[X]$ with $b_i = ba_i$ $(i = 0, 1, \ldots, d)$.
Now $f(a) \equiv 0 \pmod{p^k}$ is equivalent to $g(a) \equiv 0 \pmod{p^k}$ since p does not divide b.
The congruence $g(X) \equiv 0 \pmod{p}$ has at most d solutions modulo p (since $g(X) = 0 \pmod{p}$ has at most d zeros in $\mathbb{F}_p$).
Let $x_1, x_2, \ldots, x_r \pmod{p}$ be the solutions to $g(X) \equiv 0 \pmod{p}$.
We have $L(f) = \operatorname{lcm}(b, \Delta(g))$, so by assumption, p does not divide $\Delta(g)$. Further,

$$\Delta(g) = \pm b_0 R(g, g').$$

Now if there is an integer a such that $p|g(a)$, $p|g'(a)$ then $p|R(g, g')$. That is, $p|\Delta(g)$. But this is against our assumption.
So if $g(a) \equiv 0 \pmod{p}$, then $g'(a) \not\equiv 0 \pmod{p}$.
Now let $a \pmod{p^k}$ be a solution to $f(x) \equiv 0 \pmod{p^k}$. Then $g(a) \equiv 0 \pmod{p^k}$, so $g(a) \equiv 0 \pmod{p}$. Hence $a \equiv x_i \pmod{p}$ for some $i \in \{1, 2, \ldots, r\}$. But the residue class $a \pmod{p^k}$ such that $g(a) \equiv 0 \pmod{p^k}$ and $a \equiv x_i \pmod{p}$ is unique, by Lemma 3.3. □

In what follows, we assume that $f(X) \in \mathbb{Q}[X]$, $f(m) \in \mathbb{Z}$ for all $m \in \mathbb{Z}$ and $\gcd_{m\in\mathbb{Z}} f(m)$ is k-free.

Proposition 3.6. *Let α be a fixed real number ≥ 1.*
Then uniformly for $u \geq 0$, the number of integers $n \in (u, u+N]$ for which $f(n)$ is not divisible by the k-th power of a prime $p \leq \alpha N$ is $\sim c_{f,k}N$ as $N \to \infty$.

Remark 3.7. *By this we mean the following: for every $\varepsilon > 0$ there is $N_0 > 0$ such that for every $N \geq N_0$ and every $u \geq 0$ we have:*

$$|S(u, N) - c_{f,k}N| < \varepsilon N,$$

where $S(u, N)$ is the number of integers $n \in (u, u+N]$ such that $f(n)$ is not divisible by the k-th power of a prime $p \leq \alpha N$.

Proof. Let $z = \frac{1}{k+1}\log N$ and choose N large enough such that $z > L(f)$; let $M = \prod\limits_{p\leq z} p^k = \exp\left(k\sum\limits_{p\leq z}\log p\right) = e^{k\theta(z)}$. By the prime number theorem $\theta(z) = z + o(z)$, and so $M = e^{\frac{k}{k+1}\log N(1+o(1))} = N^{\frac{k}{k+1}+o(1)}$ as $N \to \infty$.
For every prime $p \leq z$ and every number $x \geq 0$, there are $\frac{M}{p^k}\omega_{f,k}(p)$ integers $n \in (x, x+M]$ such that $f(n) \equiv 0 \pmod{p^k}$. Hence there are $M\left(1 - \frac{\omega_{f,k}(p)}{p^k}\right)$ integers $n \in (x, x+M]$ such that $f(n)$ is not divisible by p^k. So, by the Chinese Remainder Theorem, there are exactly $M\prod\limits_{p\leq z}\left(1 - \frac{\omega_{f,k}(p)}{p^k}\right)$ integers n in any interval $(x, x+M]$, for which $f(n)$ is not divisible by the k-th power of a prime $p \leq z$. Thus there are

$$M\left(\frac{N}{M} + O(1)\right)\prod_{p\leq z}\left(1 - \frac{\omega_{f,k}(p)}{p^k}\right) = N\left(1 + O\left(\frac{M}{N}\right)\right)\prod_{p\leq z}\left(1 - \frac{\omega_{f,k}(p)}{p^k}\right)$$

integers $n \in (u, u+N]$ for which $f(n)$ is not divisible by the k-th power of a prime $p \leq z$. Notice that the constant implied by O does not depend on u. Now, if a prime p does not divide $L(f)$ then by Corollary 3.4, $\omega_{f,k}(p) \leq d$. Hence

$$\sum_{p>z}\frac{\omega_{f,k}(p)}{p^k} \leq d\sum_{p>z}\frac{1}{p^k} \leq \sum_{n>z}\frac{1}{n^k} \ll \frac{1}{z^{k-1}}.$$

This yields, that $c_{f,k}/\prod\limits_{p\leq z}\left(1 - \frac{\omega_{f,k}(p)}{p^k}\right) = 1 + O\left(\frac{1}{z^{k-1}}\right)$, and so we have proved that, uniformly in u, there are $\sim c_{f,k}N$, as $N \to \infty$, integers n in the interval $(u, u+N]$ for which $f(n)$ is not divisible by the k-th power of a prime $p \leq z$.

As we have shown above there are $\omega_{f,k}(p)\{N/p^k + O(1)\}$ integers in the interval $(u, u+N]$ for which $f(n) \equiv 0 \pmod{p^k}$, for any given prime p. If $p > z$ then this number is, by Corollary 3.4, $\leq dN/p^k + O(d)$. Therefore the number of integers $n \in (u, u+N]$ such that there is a prime $p \in (z, \alpha N]$ for which $f(n) \equiv 0 \pmod{p^k}$ is

$$\ll_d \sum_{z<p\leq\alpha N} \left(\frac{N}{p^k} + 1\right) \ll \frac{N}{z^{k-1}} + \frac{N}{\log N} = o(N).$$

Then the number of integers $n \in (u, u+N]$ such that $f(n)$ is not divisible by the k-th power of a prime $p \leq z$ but $f(n) \equiv 0 \pmod{p^k}$ for some prime $p \in (z, \alpha N]$ is equal to $o(N)$ hence the number of integer $n \in (u, u+N]$ for which $f(n)$ is not divisible by the k-th power of a prime $p \leq \alpha N$ is $\sim c_{f,k}N$ uniformly in u as $N \to \infty$. $\square$

We complete the proof of Theorem 3.1 by showing that, for any fixed $\varepsilon > 0$, there are $O(\varepsilon N)$ integers $n \leq N$ for which $f(n)$ is divisible by the square of a prime $> N$. Observe that this result is true for $f(X)$ it is true for all irreducible factors of $f(X)$; thus we will assume that $f(X)$ is irreducible. Hence it is sufficient to prove the following:

Theorem 3.8. *Assume that the abc-conjecture is true. Suppose that $f(X) \in \mathbb{Q}[X]$ is irreducible of degree $d \geq 2$, with $f(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}$. Then for every $\varepsilon > 0$ there are $O(\varepsilon N)$ integers $n \leq N$ such that $f(n)$ is divisible by the square of a prime $p > N$.*

Remark 3.9. *We may assume $d \geq 2$ since the square of any prime $p > N$ is $\gg N^2$ and so, if N is sufficiently large, cannot divide a non-zero value of a linear polynomial.*

Proof. Consider the new polynomial,

$$F(X) = f(X)f(X+1)f(X+2)\cdots f(X+l-1),$$

where l is an integer to be chosen later.
We claim that this polynomial has no repeated factors. Indeed, suppose that $F(X)$ has repeated factors. Then, $f(X+i) = f(X+j)$ for certain integers i, j with $i \neq j$, since f is irreducible. By substituting X for $X+i$ we obtain $f(X) = f(X+n)$ where $n = j - i \neq 0$.
Taking $X = 0, n, 2n, \ldots$,etc we obtain $f(n) = f(0)$, $f(2n) = f(n) = f(0)$, $f(3n) = f(0), \ldots$, i.e. the polynomial $f(X) - f(0)$ has zeros $0, n, 2n, \ldots$ This is impossible since f is not constant.
For every $n < N$, write $n = jl + i$, where $0 \leq i < l$ and $0 \leq j < [N/l]$. Note

that if there exist a prime $q > N$ such that q^2 divides $f(n)$, then $q \prod_{p|f(n)} p \leq |f(n)| \ll N^{\deg(f)}$ hence $\prod_{p|f(n)} p \ll N^{\deg(f)-1}$. Thus if two of the $f(n+i)$ were divisible by squares of primes $> N$, we would have $\prod_{p|F(n)} p \ll N^{\deg(F)-2}$, contradicting Corollary 2.9. This implies that there is at most one number $f(n+i), 0 \leq i < l$, which is divisible by the square of a prime $> N$. Thus, in total there are $O(N/l)$ integers $n \leq N$ such that $f(n)$ is divisible by the square of a prime $> N$. Selecting $l = [1/\varepsilon]$ the result follows. $\square$

Remark 3.10. *If $k \geq 3$ Theorem* 3.1 *follows directly from Proposition* 3.6 *and Theorem* 2.10.

3.2 On gaps between integers at which a given polynomial assumes k-free values

In this section we investigate the problem of finding an as small as possible function $h = h(z)$ such that for a given polynomial f and for every sufficiently large z, there is an integer $m \in (z, z+h]$ such that $f(m)$ is k-free.

The following result was proved by Granville [14] in the case $f(X) = X$, $k = 2$.

Theorem 3.11. *Let $k \geq 2$. Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree $d \geq 1$. Assume again that $f(m) \in \mathbb{Z}$ for $m \in \mathbb{Z}$ and that $\gcd_{m \in \mathbb{Z} f(m)}$ is k-free. If the abc-conjecture is true then for every $\varepsilon > 0$ and for every sufficiently large z there is an integer $m \in (z, z+z^\varepsilon]$ such that $f(m)$ is k-free.*

Proof. Choose c such that $c_{f,k} < 1 - c < 1$, and $l := [5/c\varepsilon]$. Define $g(X) = f(X+1)f(X+2)\cdots f(X+l)$.
By proposition 3.6, there is z_0 depending only on f, l, k, ε such that for every $z > z_0$, there are $< (1-c)z^\varepsilon$ integers $m \in (z, z+z^\varepsilon]$ such that $f(m)$ is not divisible by the k-th power of a prime $\leq z^\varepsilon$. Suppose that there is no integer $m \in (z, z+z^\varepsilon]$ such that $f(m)$ is k-free, thus there are a least cz^ε integers $m \in (z, z+z^\varepsilon]$ such that $f(m)$ is divisible by p^k for some prime $p > z^\varepsilon$.
Assuming z_0 is sufficiently large, $z \geq z_0$, we claim that there is an integer $m_0 \in (z, z+z^\varepsilon]$ such that at least $\frac{c}{2}$ of the integers $f(m_0+1), f(m_0+2), \ldots, f(m_0+l)$ are divisible by the k-th power of a prime $> z^\varepsilon$. Thus $g(m)$ is divisible by the square of an integer $> (z^\varepsilon)^{\frac{cl}{2}}$. Hence $g(m)$ is divisible by the square of an integer $> m^2$ and this last statement contradicts Theorem 2.10. $\square$

Proof of the claim: Assume z_0 is large enough such that $z_0^\varepsilon > l$. Let a be the largest integer at most z and r the largest integer such that $a + rl \leq z + z^\varepsilon$. Suppose that none of the sets $\{a+1, \dots, a+l\}$, $\{a+l+1, \dots, a+2l\}, \dots,$ $\{a+(r-1)+1, \dots, a+rl\}$ contains more than $(c/2)l$ integers m for which $f(m)$ is divisible by the k-th power of a prime $p > z^\varepsilon$. Then $(z, z+z^\varepsilon]$ contains altogether at most

$$\begin{aligned} \frac{c}{2}rl + l &\leq \frac{c}{2}z^\varepsilon + l \\ &\leq \frac{c}{2}z^\varepsilon + [\frac{5}{c\varepsilon}] \\ &< cz^\varepsilon \end{aligned}$$

such integers, assuming z is sufficiently large, contradicting our assumption. $\square$

Chapter 4

The average moments of $s_{n+1} - s_n$

In this chapter we will state the most important result of our thesis.
Let k be an integer and let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree d such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $\gcd_{n\in\mathbb{Z}} f(n)$ is k-free.
Let $\{s_n\}_{n=1}^{\infty}$ be the ordered sequence of positive integers m such that $f(m)$ is k-free. Suppose that $k \geq \min(3, d+1)$.

The following result was proved by Granville [14] in the case $f(X) = X$, $k = 2$.

Theorem 4.1. *Suppose the abc-conjecture is true. Then for every real $A > 0$ there exists a constant $\beta_A > 0$ such that:*

$$\sum_{s_n \leq x} (s_{n+1} - s_n)^A \sim \beta_A x \text{ as } x \to \infty.$$

We start with a Lemma.

Lemma 4.2. *Assume the abc-conjecture. Let $a_1, a_2, \ldots, a_l$ be fixed integers. Then there is a number $\gamma_{\underline{a}} = \gamma_{\{a_1,a_2,\ldots,a_l\}}$ such that the number of integers $m \leq x$ such that $f(m), f(m+a_1), \ldots, f(m+a_l)$ are all k-free is $\sim \gamma_{\underline{a}} x$ as $x \to \infty$.*

Proof. As we have seen in the proof of Theorem 3.8, since f is irreducible, no two among the polynomial $f(X), f(X+a_1), \ldots, f(X+a_l)$ have a common factor. So for $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$, the resultant $R_{i,j}$ of $f(X+a_i)$ and $f(X+a_j)$ is $\neq 0$. Let $y = \max\{|R_{i,j}| : 1 \leq i, j \leq l, i \neq j\}$, then if p is a prime with $p > y$ then p divides at most one of the polynomials $f(m), f(m+a_1), \ldots, f(m+a_l)$.

Now let $M = \Big(\prod_{p \le y} p\Big)^k$, and let $\mathcal{A}$ be the set of integers $a \in [0, M-1)$ such that none of $f(a), f(a+a_1), \dots, f(a+a_l)$ is divisible by the k-th power of a prime $p \le y$. Hence for every integer m with $0 \le m \le x$ we have:
$f(m), f(m+a_1), \dots, f(m+a_l)$ all k-free is equivalent to $m = a \pmod M$ for some $a \in \mathcal{A}$ and $f(m), f(m+a_1), \dots, f(m+a_l)$ not divisible by p^k for some prime $p > y$.
Writing $m = m'M + a$ with $a \in \mathcal{A}$ we obtain:
$f(m), f(m+a_1), \dots, f(m+a_l)$ k-free is equivalent to $m = a \pmod M$ for some $a \in \mathcal{A}$ and $g_a(m')$ k-free, where $g_a(X) = f(a+MX)f(a_1+a+MX)\dots f(a_l+a+MX)$.
Now according to Theorem 3.1 assuming the *abc*-conjecture, there is $c_a \ge 0$ such that

$$\#\left\{m' \le x' : g_a(m') \text{ is } k\text{-free}\right\} \sim c_a x' \qquad \text{as } x' \to \infty.$$

So

$$\begin{aligned} |\{m \le x : f(m), f(m+a_1), \dots, & \\ f(m+a_l), \quad \text{are } k\text{-free}\}| &= \sum_{a \in \mathcal{A}} \#\left\{m' \le \frac{x-a}{M} : g_a(m')\, k\text{-free}\right\} \\ &\sim \left(\sum_{a \in \mathcal{A}} \frac{c_a}{M}\right) x \qquad \text{as } x \to \infty. \end{aligned}$$

□

Proof of Theorem 4.1. We introduce some new definitions to simplify our proof:

First, let $S(x;t)$ be the number of integers n such that $s_n \le x$ and $s_{n+1} - s_n = t$.

Let $S'(x, T)$ denote the number of integers n such that $s_n \le x$, and $T \le s_{n+1} - s_n < 2T$, and such that there are $\ge (5c/6)T$ integers m in the interval (s_n, s_{n+1}) such that $f(m)$ is not divisible by the k-th power of a prime $\le 2T$ or $> T^A$.

Let t be a positive integer. For any subset I of $\{1, 2, \dots, t-1\}$ we denote by S_I the set of integers $n \le x$ for which $f(n), f(n+t)$ and $f(n+a)$ for all $a \in I$ are k-free. Notice that $|S_\emptyset|$ denotes the number of integers $n \le x$ such that $f(n), f(n+t)$ are k-free and without conditions for $f(n+1), f(n+2), \dots, f(n+t-1)$. Then by Lemma 4.2, we have $|S_I| \sim \gamma_{I \cup \{0,1\}} x$ for some

$\gamma_{I\cup\{0,1\}} > 0$ and by the rule of inclusion-exclusion,

$$\begin{aligned} S\left(x,t\right) &= |S_{\emptyset}| - \sum_{i=1}^{t-1} |S_{\{i\}}| + \sum_{1\le i_1<i_2\le t-1} |S_{\{i_1,i_2\}}| - \sum_{1\le i_1<i_2<i_3\le t-1} |S_{\{i_1,i_2,i_3\}}| + \ldots \\ &= \sum_{I} \left(-1\right)^{|I|} S_I \sim \sum_{I} \left(-1\right)^{|I|} \gamma_{I\cup\{0,1\}} x = \delta_t x \end{aligned}$$

as $x \to \infty$.

We claim, that under assumption of the abc-conjecture, we have for every sufficiently large x, and $T > 0$,

$$\sum_{T\le t<2T} S(x,t) \ll_A x/T^{A+1}.$$

Then we have:

$$\begin{aligned} \frac{1}{x}\sum_{t\ge T} S(x,t)t^A &= \frac{1}{x}\sum_{j=0}^{\infty} \sum_{2^jT\le t<2^{j+1}T} S(x,t)t^A \\ &\ll \frac{1}{x}\sum_{j=0}^{\infty} \frac{x}{\left(2^jT\right)^{A+1}} \left(2^{j+1}T\right)^A \\ &\ll \frac{2^A}{T}\sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \\ &\ll \frac{1}{T}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{x}\sum_{s_n\le x} \left(s_{n+1}-s_n\right)^A &= \frac{1}{x}\sum_{t=1}^{\infty} S(x,t)t^A \\ &= \frac{1}{x}\sum_{t=1}^{T} S(x,t)t^A + \frac{1}{x}\sum_{t\ge T} S(x,t)t^A \\ &= \frac{1}{x}\sum_{t=1}^{T} S(x,t)t^A + E(x,T), \text{ with } |E(x,T)| \le \frac{c_1}{T}, \end{aligned}$$

where c_1 is independent of x.

Fixing T and letting $x \to \infty$, we infer, $\frac{1}{x}\sum\limits_{t=1}^{T} S(x,t)t^A \to \sum\limits_{t=1}^{T} \delta_t t^A$.

Hence $\frac{1}{x}\sum\limits_{t=1}^{\infty} S(x,t)t^A$ is bounded as $x \to \infty$, by say c_2.

Now:
$$\frac{1}{x}\sum_{t=1}^{T} S(x,t)t^A \leq \frac{1}{x}\sum_{t=1}^{\infty} S(x,t)t^A + \frac{c_1}{T} \leq c_2 + \frac{c_1}{T}$$
for all x.
This implies $\sum\limits_{t=1}^{T} \delta_t t^A \leq c_2 + \frac{c_1}{T}$; so $\sum\limits_{t=1}^{T} \delta_t t^A$ is bounded independently of T. Thus $\beta_A := \sum\limits_{t=1}^{\infty} \delta_t t^A$ converges.
Let $\delta > 0$ then for every $T > 0$ there is $x_0(\delta, T)$ such that
$$|\frac{1}{x}\sum_{t=1}^{T} S(x,t)t^A - \sum_{t=1}^{T} \delta_t t^A| < \frac{\delta}{3}$$
for all $x \geq x_0(\delta, T)$. There is T_0 such that
$$|\sum_{t=1}^{T} \delta_t t^A - \beta_A| < \frac{\delta}{3}$$
for all $T \geq T_0$.
Take $T \geq \max\left(T_0, \frac{c_2}{3\delta}\right)$ and then $x \geq x_0\left(\delta, T\right)$, thus,
$$\begin{aligned}
|\frac{1}{x}\sum_{s_n \leq x} \left(s_{n+1} - s_n\right)^A - \beta_A| &= |\frac{1}{x}\sum_{t=1}^{\infty} S(x,t)t^A - \beta_A| \\
&\leq |\frac{1}{x}\sum_{t=1}^{\infty} S(x,t)t^A - \frac{1}{x}\sum_{t=1}^{T} S(x,t)t^A| \\
&+ |\frac{1}{x}\sum_{t=1}^{T} S(x,t)t^A - \sum_{t=1}^{T} \delta_t t^A| + |\sum_{t=1}^{T} \delta_t t^A - \beta_A| \\
&\leq \frac{c_1}{T} + \frac{\delta}{3} + \frac{\delta}{3} \\
&\leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.
\end{aligned}$$
So $\frac{1}{x}\sum\limits_{n \leq x} \left(s_{n+1} - s_n\right)^A \to \beta_A$ as $x \to \infty$. $\square$

We can assume that T is sufficiently large. By Theorem 3.11, we know that $S(x,t) = 0$ when $t \geq x^{\varepsilon}$ and x is sufficiently large.We apply this with
$$\varepsilon = \begin{cases} \min\left(\frac{1}{kA(A+1)}, \frac{k-5/2}{A(k-1)^2}\right) & \text{if } k \geq 3, d \geq 2, \\ \frac{1}{kA(A+1)} & \text{if } k \geq 2, d = 1. \end{cases}$$

Thus we will prove the claim assuming that $T < x^{\varepsilon}$ and x is sufficiently large. Let B be the smallest integer $\geq A$.

Proof of the claim: By Proposition 3.6, there are $\geq ct$ integers m, for some constant $c < c_{f,k}$, in any interval of length $t \geq T$, for which $f(m)$ is not divisible by the k-th power of a prime $\leq 2T$. For any $s_n \leq x$ counted by $\sum_{T \leq t < 2T} S(x;t)$ but not by $S'(x,T)$, there must be $> (c/6)T$ integers $m \in (s_n, s_{n+1})$ for which $f(m)$ is divisible by the k-th power of a prime $p > T^A$. Otherwise there would be at most $(c/6)T$ integers $m \in (s_n, s_{n+1})$ for which $f(m)$ is divisible by the k-th power of a prime $p > T^A$, implying that we have $\geq T - (c/6)T > (5c/6)T$ integers $m \in (s_n, s_{n+1})$ for which $f(m)$ is not divisible by the k-th power of a prime $p > T^A$. But this means precisely that $s_n \in S'(x,T)$, contradicting our choice. Therefore

$$\begin{aligned}
\frac{cT}{6}\left(\sum_{T \leq t < 2T} S(x,t) - S'(x,T)\right) &\leq \sum_{\substack{m \leq x \\ \exists\, p > T^A:\, p^k | f(m)}} 1 \\
&\leq \sum_{p > T^A} \sum_{m \leq x,\, p^k | f(m)} 1 \\
&\leq \sum_{p > T^A} \omega_{f,k}(p) \left(\frac{x}{p^k} + 1\right) \\
&\ll_d \sum_{p > T^A} \frac{x}{p^k} + \sum_{\substack{p > T^A \\ \exists\, m \leq x:\, p^k | f(m)}} 1 \\
&\ll_d \frac{x}{T^{A(k-1)}} + \sum_{\substack{p > T^A \\ \exists\, m \leq x:\, p^k | f(m)}} 1.
\end{aligned}$$

We show that the last sum is $\ll \frac{x}{T^{A(k-1)}}$. First assume that $k \geq 2, d = 1$. Then if $p^k | f(m)$ we have $p \ll |m|^{1/k} \ll x^{1/k}$ hence

$$\sum_{\substack{p > T^A \\ \exists\, m \leq x:\, p^k | f(m)}} 1 \ll x^{1/k} \ll \frac{x}{T^{A(k-1)}}$$

by our assumption $T < x^{\frac{1}{kA(A+1)}}$.
Second assume that $k \geq 3, d \geq 2$. If $p^k | f(m)$ for some integer $m \leq x$,

by Theorem 2.10, $p \ll_\theta |m|^{\frac{1+\theta}{k-1}} \ll x^{\frac{1+\theta}{k-1}}$, for every $\theta > 0$, so in particular $p \le x^{\frac{3/2}{k-1}}$ if x is sufficiently large. Hence

$$\sum_{\substack{p>T^A \\ \exists\, m\le x:\, p^k|f(m)}} 1 < x^{\frac{3/2}{k-1}} < \frac{x}{T^{A(k-1)}},$$

by our assumption $T < x^{\frac{k-5/2}{A(k-1)^2}}$. Thus we conclude that if x is sufficiently large and $T < x^{\varepsilon}$ we have

$$\left(\sum_{T\le t<2T} S(x,t) - S'(x,T)\right) \ll \frac{x}{T^{A(k-1)+1}} \ll \frac{x}{T^{A+1}}.$$

For every s_n counted by $S'(x;T)$ we have $\ge (5c/6)T$ integers in the interval (s_n, s_{n+1}) such that $f(m)$ is divisible by the k-th power of a prime in the range $[2T, T^A]$. We consider B-tuples of such integers

$$s_n < m_1 < m_2 < \ldots < m_B < s_{n+1}.$$

For such a tuple there are primes $p_1, p_2, \ldots, p_B$ with $2T \le p_i < T^A$ for $i \in \{1, 2, \ldots, B\}$ such that

$$f(m_j) \equiv 0 \pmod{p_j^k},$$

and the number of such integers is at least $\binom{[(5c/6)T]}{B}$.
Let $i_1 = 1, q_1 = p_1$; let i_2 be the smallest index $i \in \{2, 3, \ldots, B\}$ such that $p_i \ne p_1$ put $q_2 = p_{i_2}$; let i_3 be the smallest index $i \in \{3, 4, \ldots, B\}$ such that $p_{i_3} \notin \{q_1, q_2\}$; put $q_3 = p_{i_3}$, etc. Consider this sequence, $i_1 = 1 < i_2 < \ldots < i_u \le B$ of indices. Let $d_2 = m_{i_2} - m_1, d_3 = m_{i_3} - m_1, \ldots, d_u = m_{i_u} - m_1$. The number of possibilities for $(d_2, d_3, \ldots, d_u)$ is

$$\le (2T)^{u-1}.$$

Now for any fixed $(d_2, d_3, \ldots, d_u)$ we have

$$\left\{\begin{array}{lll} f(m_1) & \equiv 0 & \pmod{q_1^k} \\ f(m_{i_2}) & \equiv 0 & \pmod{q_2^k} \\ f(m_{i_3}) & \equiv 0 & \pmod{q_3^k} \\ & \vdots & \\ f(m_{i_u}) & \equiv 0 & \pmod{q_u^k} \end{array}\right. \Longleftrightarrow \left\{\begin{array}{lll} f(m_1) & \equiv 0 & \pmod{q_1^k} \\ f(m_1+d_2) & \equiv 0 & \pmod{q_2^k} \\ f(m_1+d_3) & \equiv 0 & \pmod{q_3^k} \\ & \vdots & \\ f(m_1+d_u) & \equiv 0 & \pmod{q_u^k} \end{array}\right.$$

By Corollary 3.4, m_j is congruent to one of $\leq d$ incongruent numbers modulo q_j^k for each j. So by the Chinese Remainder Theorem, m_1 belong to one of at most d^u residue classes modulo $(q_1 q_2 \dots q_u)^k$. Hence for each of these residue classes we have

$$d^u \left(x/(q_1 q_2 \dots q_u)^k + 1 \right)$$

possibilities for m_1; since $(q_1 q_2 \dots q_u)^k \leq T^{Auk} \leq T^{ABk} \leq T^{A(A+1)k} < x$ this gives at most

$$\frac{2x}{(q_1 q_2 \dots q_u)^k} d^u$$

possibilities for m_1.
Taking into account the possibilities for $(d_2, d_3, \dots, d_u)$ we get at most

$$\ll T^{u-1} \left(x/(q_1 q_2 \dots q_u)^k \right)$$

possibilities for $(m_1, m_{i_2}, \dots, m_{i_u})$.
It remains to take into account the m_i with $i \notin \{1, i_2, \dots, i_u\}$.
Let $i \notin \{1, i_2, i_3, \dots, i_u\}$. Then $p_i = q_j$ for some $j \in \{1, 2, \dots, u\}$, hence

$$f(m_i) \equiv f(m_{i_j}) \equiv 0 \pmod{q_j^k}.$$

Let $\omega_1, \omega_2, \dots, \omega_r$ be the solutions of $f(x) \equiv 0 \pmod{q_j}$, $0 \leq x < q_j$. Then by corollary 3.4, $r \leq \deg(f)$. Now since $|m_{i_j} - m_i| \leq 2T < q_j$ we have $m_{i_j} - m_i = \omega_{l_1} - \omega_{l_2}$ for some $l_1, l_2 \in \{1, 2, \dots, r\}$. So given m_{i_j}, there are at most d^2 possibilities for m_i.
This gives altogether at most

$$\left(d^2\right)^{B-u}$$

possibilities for the tuples $(m_i : i \notin \{1, i_2, i_3, \dots, i_u\})$.
Hence for the tuples $(m_1, m_2, \dots, m_B)$ we have at most

$$T^{u-1} \left(x/(q_1 q_2 \dots q_u)^k \right) \left(2d^2\right)^{B-u} \ll T^{u-1} \left(x/(q_1 q_2 \dots q_u)^k \right)$$

possibilities where $q_1, q_2, \dots, q_u$ are the distinct primes among $p_1, p_2, \dots, p_B$. For given $q_1, q_2, \dots, q_u$ there are at most $u^B \leq B^B \ll 1$ possibilities for $p_1, p_2, \ldots, p_B$ so:

$$\begin{aligned}
S'(x,T)T^B &\ll \sum_{u=1}^{B} \sum_{2T<q_1<\ldots<q_u<T^A} T^{u-1} \frac{x}{(q_1 \ldots q_u)^k} \\
&\ll x \sum_{u=1}^{B} T^{u-1} \left(\sum_{q>2T} \frac{1}{q^k} \right)^u \\
&\ll x \sum_{u=1}^{B} T^{u-1} \left(\frac{1}{T^{k-1}} \right)^u \\
&\ll \frac{x}{T}
\end{aligned}$$

Hence

$$S'(x,T) \ll \frac{x}{T^{B+1}} \ll \frac{x}{T^{A+1}},$$

which proves our claim, and completes the proof of Theorem 4.1.

□

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