Motivic decomposition of projective homogeneous varieties

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§ 1. Introduction

The term "motive" (or sometimes "motif", due to the French origin of the word) goes back to Grothendieck's idea of a universal, "motivic" cohomology theory for algebraic varieties, which he threw in in the late 1960s. Such a theory meant an "embedding" of the category $\mathbf{Var}(k)$ of smooth projective varieties over a field k into a suitable close-to-abelian category, so that all "sufficiently good" cohomology theories on $\mathbf{Var}(k)$ would factor through this embedding. Manin [16] proposed the following approximation to this construction. He introduced the additive category of correspondences $\mathbf{Corr}(k)$, whose objects are the same as the objects of $\mathbf{Var}(k)$, and the morphisms, called correspondences, between two objects X and Y (for simplicity assume X irreducible) are the elements of the Chow group $\mathrm{CH}_{\dim X}(X \times Y)$, i.e. the cycles of dimension $\dim X$ on $X \times Y$ modulo rational equivalence (see [8]). The pseudo-abelian envelope of $\mathbf{Corr}(k)$ is the category of effective Chow motives $\mathbf{Chow}^{eff}(k)$. It is obtained by adding to $\mathbf{Corr}(k)$ the kernels of all projectors. The image of $X \in \mathrm{Ob}\mathbf{Var}(k)$ under the natural functor

$$\mathbf{Var}(k) \to \mathbf{Corr}(k) \to \mathbf{Chow}^{eff}(k)$$

is called the motive of X and denoted by $\mathcal{M}(X)$.

The category $\mathbf{Chow}^{eff}(k)$ has a rich structure. Our interest is in the additive decompositions of the motives $\mathcal{M}(X)$, $X \in \mathrm{Ob}\mathbf{Var}(k)$. For example, the canonical morphism $\mathbb{P}^1_k \to \mathrm{Spec}\, k$ yields the decomposition

$$\mathcal{M}(\mathbb{P}^1_k) \cong \mathcal{M}(\mathrm{Spec}\, k) \oplus \mathbb{L},$$

where $\mathbb{L}$ is an object of $\mathbf{Chow}^{eff}(k)$ called the Tate motive. This decomposition immediately generalizes as $\mathcal{M}(\mathbb{P}^n_k) = \bigoplus_{i=0}^{n} \mathbb{L}^i$, where $\mathbb{L}^i$ denotes the i-th tensor power of $\mathbb{L}$. It appears that a similar decomposition of a motive can be obtained for any variety X having a filtration by closed subvarieties

$$\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_N = X,$$

where the differences $U_i = X_i \setminus X_{i-1}$ are "sufficiently good", for example, isomorphic to affine spaces $\mathbb{A}^{d_i}_k$. (The most general statement is given in [7, Cor. 66.4], and originates from Karpenko [15].)

Observe that our model example, the projective space $\mathbb{P}^n_k$, is a projective homogeneous variety of the algebraic group $\mathrm{PGL}_{n,k}$, and the natural filtration $\mathrm{Spec}\, k = \mathbb{P}^0_k \subseteq \ldots \subseteq \mathbb{P}^{n-1}_k \subseteq \mathbb{P}^n_k$ is in fact induced by the Bruhat decomposition of $\mathrm{PGL}_{n,k}$. This gives us hope to obtain such a filtration for any homogeneous G-variety, where G is a reductive algebraic group. The main goal of the present manuscript is to give an overview of the recent results in this direction obtained by Köck [17], Chernousov, Gille and Merkurjev [3], and Chernousov, Merkurjev [4].

Let G be a reductive algebraic group over k, and let V be a projective G-homogeneous variety which is isomorphic over an algebraic closure K of k to the (geometric) quotient of G by a subgroup P. Since V is projective, P is necessarily a parabolic subgroup of G. If G is a k-split group, then, in particular, P is defined over k and V is isomorphic to G/P over k. This situation is indeed a complete analogue of $\mathbb{P}^n_k$. Namely, the Bruhat decomposition for G induces on V a structure of a cellular space, with cells isomorphic to affine spaces of known dimensions. This allows to compute the Chow group of V, which is just a free abelian group with generators corresponding to the closures of cells, and to obtain a decomposition of the motive $\mathcal{M}(V)$ into a sum of twisted Tate motives. This result is due to Köck [17]. If G is not k-split, but only

k-isotropic, that is, possesses a non-trivial k-split subtorus, and if P is still defined over k (equivalently, V has a k-point), a certain "gluing" of the above Bruhat cells is possible. This gives a coarser k-filtration of V, and with differences which are no more affine spaces, but affine bundles over some smooth projective varieties. Nevertheless, such a filtration is still subject to the decomposition theorem, and hence provides a motivic decomposition of V. This is the main result of [3]. Finally, the paper [4] generalizes both these results and provides a way to compute $\mathcal{M}(V)$ under the hypothesis that G is k-isotropic and possesses a non-trivial k-defined parabolic subgroup P', but not necessarily coinciding with P. The main idea of [4] is to decompose the motive of a product $V \times V'$, where V and V' are projective homogeneous G-varieties, possibly without any k-points. In case when one of these varieties, say V, has a k-point, we obtain a decomposition of the motive of the other one, V', using pull-back.

The thesis is organized as follows. In § 2 we briefly recall the most basic notions and results pertaining to algebraic varieties and groups. In § 3 we define an abstract root system Φ and prove some technical lemmas which will be used later on. After this we pass to the detailed study of algebraic groups. In § 4 we recall the notion of a geometric quotient of varieties and algebraic groups, and reproduce the classical construction of the quotient of an algebraic group by a closed subgroup (Theorem 4.7) In § 5 we describe the structure of reductive algebraic groups over an algebraically closed field. In particular, we prove the Bruhat decomposition (Theorem 5.12) and the classification of parabolic subgroups (Theorem 5.13). In § 6 we discuss how the results of the previous chapter can be carried over to the case of a group over a non-algebraically closed field. The next chapter, § 7, is devoted to the detailed proof of the results of Köck (Theorem 7.3) and Chernousov-Merkurjev (Theorem 7.7) mentioned above. Finally, in § 8 we use these results to obtain some explicit motivic decompositions.

§ 2. Preliminaries

In the present chapter we introduce the basic notions we will use in this work, and manifest the principal conventions. We also recall some elementary results on algebraic varieties and groups that seem important for further exposition. Our main reference is the classical book by Borel [1].

Throughout the thesis, k denotes a field, K denotes an algebraic closure of k, and k_s denotes the separable closure of k in K.

1. Schemes and Varieties. For any scheme X, we denote by $\mathcal{O}_X$ the structure sheaf of X, and if x is a point of this scheme, we write $\mathcal{O}_{X,x}$ for the local ring at this point, m_x for the maximal ideal of $\mathcal{O}_{X,x}$, and $\kappa(x)$ for the residue field $\mathcal{O}_{X,x}/m_x$. For a morphism $f : X \to X'$ we denote by $f^\sharp$ the morphism of sheaves corresponding to f.

For us a *k-variety* (or a *variety over k*) is a reduced separated scheme of finite type over k. We say that a K-variety V is *defined over* k, if there exists a k-variety W such that $W \times_{\operatorname{Spec} k} \operatorname{Spec} K \cong V$. Such a variety W is not necessarily unique, but whenever we say that a variety V is defined over k, we have in mind that we fix some k-variety of this kind; we will denote it by ${}_kV$. Thus, a k-defined variety V over K is actually a *pair* $(V, {}_kV)$ together with an isomorphism ${}_kV \times_{\operatorname{Spec} k} \operatorname{Spec} K \cong V$.

Let V be a variety over K. We will denote by $V(K)$ the set of K-valued points of V, which also coincides with the set of all closed points of V. Unless explicitly stated otherwise, "x is a point of V" means that x is an element of $V(K)$, that is, a closed point. If V is defined over k, then the embedding $k \hookrightarrow K$ induces an embedding of the set ${}_kV(k)$ of all k-valued points of the variety ${}_kV$ into $V(K)$; we will denote the image of this embedding by $V(k)$. We will sometimes use the fact that $V(k_s)$ is dense in $V(K)$ ([1, Cor. AG.13.3]). For shortness, we will sometimes write $K[V]$ and $k[V]$ instead of $\mathcal{O}_V(V)$ and $\mathcal{O}_{{}_kV}({}_kV)$. We also denote by $K(V)$ the ring of rational functions on V, i.e. the limit $\varinjlim \mathcal{O}_V(U)$, where U runs over all open dense subsets of V.

A *morphism of K-varieties* $f : V \to V'$ is just a morphism of K-schemes. Since the set of closed points is dense in the underlying topological space of a variety, the morphism f is uniquely determined by a continuous map $f : V(K) \to V'(K)$ and by $f^\sharp$. If V and V' are varieties defined over k, a morphism $f : V \to V'$ is said to be *defined over* k, if it comes from a morphism of k-schemes ${}_kf : {}_kV \to {}_kV'$. Observe that when f is an isomorphism, ${}_kf$ is also an isomorphism (of k-schemes) [10, Prop. 2.7.1].

For any variety V, we have the natural notions of open and closed subvarieties of V (note that there is only one closed subvariety with a given underlying topological space). If V is a K-variety defined over k, we will also say that a (closed or open) subvariety W is a *k-defined subvariety*, if W comes from a (closed or open) subvariety of ${}_kV$. This is the same as to say that W is a k-defined variety, and the embedding $W \hookrightarrow V$ is a k-defined morphism. We will occasionally say that a (closed or open) subset $S \subseteq V(K)$ is k-defined, meaning that the corresponding subvariety of V is.

We denote by Γ the Galois group $\mathrm{Gal}\,(k_s/k)$. Let V be a k-defined variety over K. We define the action of $\sigma \in \Gamma$ on V as the morphism of schemes

$$\sigma : V = {}_kV \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K \to {}_kV \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K = V,$$

induced in a natural way by the automorphism $\mathrm{Spec}\, K \to \mathrm{Spec}\, K$ corresponding to the extension to K of $\sigma^{-1} : k_s \to k_s$. This morphism $\sigma : V \to V$ is clearly defined over k_s. It takes a closed (resp. open) subvariety W of V to a closed (resp. open) subvariety $\sigma(W)$. In the affine case $\sigma(W)$ is just the subvariety obtained by applying σ to the coefficients of equations defining W.

Observe that if A is a k-algebra and $B = A \otimes_k k_s$, then A is the set of Γ-fixed points of B, if Γ acts on B through the factor k_s. This implies that in the affine case, and hence in general, a morphism of k-varieties $f : V \to V'$ is defined over k if and only if it is defined over k_s and Γ-invariant. The latter can also be checked on the k_s-valued points of ${}_{k_s}V$ and ${}_{k_s}V'$. Consequently, a subvariety W of V is defined over k if and only if it is defined over k_s and $W(k_s)$ is Γ-invariant (see [1, AG.14.3-14.4]).

Recall that a variety V is called *normal*, if any local ring (or, equivalently, any local ring at a closed point) of V is a normal ring, i.e. is integrally closed in its field of fractions. A dominant morphism of varieties $f : V \to W$ is called *separable*, if for any irreducible components V' of V and W' of W such that W' is the closure of $f(V')$, the induced embedding $K(W') \to K(V')$ is a separable extension of fields. It follows from [5, Exp. 5, Th. 2] that a bijective separable morphism of irreducible normal varieties is an isomorphism.

A variety is called *quasi-projective*, if it is isomorphic to an open subvariety of a projective variety.

By the *dimension* $\dim V$ of a variety V we always mean the topological dimension. However, since our varieties are schemes of finite type over a field, it can be understood as the maximal dimension of a local ring at a closed point. Moreover, most our varieties (e.g. algebraic groups, see below) are smooth, and hence $\dim V$ is also the dimension of a tangent space in following sense.

Let x be a closed point of a K-variety V. The *tangent space* to V at x is defined as

$$T_e V = \operatorname{Der}_K(\mathcal{O}_{V,x}, \kappa(x)),$$

the $\mathcal{O}_{V,x}$-module of all K-linear derivations of $\mathcal{O}_{V,x}$ with values in $\kappa(x) = \mathcal{O}_{V,x}/m_x \cong K$. It is canonically isomorphic to the $\mathcal{O}_{V,x}$-module $(m_x/m_x^2)^* = \operatorname{Hom}_K(m_x/m_x^2, K)$. Therefore, an irreducible K-variety V is smooth if and only if $\dim V = \dim_K T_x V$ for any closed point $x \in V$. If $V = \operatorname{Spec} A$ is an affine variety, we also have $T_x V = \operatorname{Der}_K(A, \kappa(x))$, where $\kappa(x)$ becomes an A-module via the localization map $A = O_V(V) \to \mathcal{O}_{V,x}$. If $f : V \to V'$ is a morphism of varieties, the corresponding map $f^\sharp : \mathcal{O}_{V',f(x)} \to \mathcal{O}_{V,x}$ induces a natural morphism

$$(df)_x : T_x V \to T_{f(x)} V',$$

which we call the *tangent morphism* at x. We will sometimes use the fact that a morphism of smooth varieties is separable if and only if every irreducible component of V contains a closed point x such that $(df)_x$ is surjective ([1, Th. AG.17.3]).

For any $n \geq 0$, we write $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ for the n-dimensional affine space over k and the k-dimensional projective space over k respectively.

2. Algebraic groups. An *algebraic group G over k* (or an *algebraic k-group*) is a k-variety G endowed with three structure morphisms: $m : G \times G \to G$ (the multiplication), $i : G \to G$ (the inverse), $e : \operatorname{Spec} k \to G$ (the unit element), which are morphisms of k-varieties and satisfy the usual group axioms. A *morphism* of algebraic groups is a morphism of varieties which is also a homomorphism of groups, i.e. respects the structure morphisms. In the present thesis all algebraic groups are supposed to be *affine*. By an element of a group G we mean an element of $G(k)$, which is a group in the abstract sense.

We say that an algebraic group G over K is *defined over k*, if G is defined over k as a variety and the structure morphisms of G are k-defined morphisms. The notion of a *k-defined morphism* of algebraic groups is analogous.

The basic examples of algebraic groups include the "additive" group $\mathbb{G}_{a,k} = \operatorname{Spec} k[x]$, the "multiplicative" group $\mathbb{G}_{m,k} = \operatorname{Spec} k[x, x^{-1}]$, the general linear group

$$\mathrm{GL}_{n,k} = \operatorname{Spec} k[x_{ij},\ 1 \leq i, j \leq n;\ 1/\det(x_{ij})], \quad n \geq 1$$

(in fact, $\mathbb{G}_{m,k} = \mathrm{GL}_{1,k}$). The groups $\mathbb{G}_{a,K}$ and $\mathbb{G}_{m,K}$ are the only connected algebraic K-groups of dimension 1 ([1, Th. 10.9]).

Let V be a k-vector space. For a k-defined algebraic group G, a morphism of algebraic groups $G \to \mathrm{GL}(V \otimes_k K) \cong \mathrm{GL}_{n,K}$, induced by a k-morphism ${}_kG \to \mathrm{GL}(V) \cong \mathrm{GL}_{n,k}$, is called a *$k$-representation* of G. When we discuss k-representations, we sometimes say that a K-vector subspace W of a K-vector space $V \otimes_k K \cong K^n$ is defined over k; this means that W is is generated by $W \cap k^n$. If W is G-invariant, this allows us to define the induced k-representation $G \to \mathrm{GL}(W)$.

From now on, let G be an algebraic K-group defined over k. Unless explicitly stated otherwise, by a *subgroup* of G we mean a closed algebraic subgroup over K, that is, an algebraic group H, which is a closed subvariety of G such that the closed embedding $H \hookrightarrow G$ commutes with the structure morphisms. The subgroup H is said to be *k-defined subgroup*, if it is k-defined as a subvariety. (Since the structure morphisms of H come from those of G, they are automatically k-defined.)

Observe that a closed subvariety $H \subseteq G$ possesses a structure of an algebraic K-subgroup if and only if $H(K)$ is an abstract subgroup of $G(K)$. Indeed, for example, if $I = \{f \in A \mid f|_H = 0\}$ is the ideal of $A = K[G]$ defining H as a variety, then the invariance of $H(K)$ under $i : G \to G$ means that I is invariant under $i^\sharp$, and hence we have a correctly defined morphism $i^\sharp : A/I \to A/I$, with $A/I = K[H]$. Observe that this structure on H is moreover unique. Consequently, if we are provided with a closed subset $S \subseteq G(K)$ which is also a subgroup, we also have a uniquely determined closed subgroup H of G such that $S = H(K)$.

We define the *kernel* $\ker \varphi$ and the *image* $\operatorname{im} \varphi$ of a K-morphism $\varphi : G \to G'$ as the closed subgroups corresponding to $\ker \varphi(K) \subseteq G(K)$ and $\operatorname{im} \varphi(K) \subseteq G'(K)$ (see [1, Cor. 1.4]).

We denote by G° the connected component of the point e in G. It is a closed normal k-defined subgroup of finite index, whose cosets are both the connected and the irreducible components of G [1, Prop. 1.2].

For any subset $S \subseteq G(K)$, the group-theoretic centralizer $C_{G(K)}(S)$ is always a closed subset of $G(K)$. Therefore, we can speak of a closed subgroup $C_G(S)$ of G. In what follows, when we speak of a *centralizer* $C_G(H)$ of a subvariety H of G, we mean that it is a closed subgroup of G constructed in the above way from the set $S = H(K)$. In particular, the *centre* $C(G)$ of G is the closed subgroup corresponding to the group-theoretic centre $C(G(K))$. If $S \subseteq G(K)$ is a closed subset, i.e. corresponds to a closed subvariety of G, then the group-theoretic normalizer $N_{G(K)}(S)$ is also closed, and can be considered as an algebraic subgroup of G, the *normalizer* of S (or of the corresponding subvariety). However, the question of whether $C_G(S)$ or $N_G(S)$ is defined over k, if G and S are, is more subtle (see [1, Prop. 1.7]).

We say that a (closed) subgroup H of G is normal in G, if $N_G(H) = G$.

Other important subgroups of G are the terms of its derived and descending central series. It appears that if H is a closed k-defined normal subgroup of G, then the group-theoretic commutator subgroup $[G(K), H(K)]$ is a closed k-defined subset of $G(K)$ [1, Prop. 2.3], and hence provides a closed k-defined algebraic subgroup $[G, H]$ of G. This allows us to define a *solvable* (resp. *nilpotent*) algebraic group as one which is solvable (resp. nilpotent) as an abstract group.

Since G is an algebraic group, the tangent space $T_eG = \operatorname{Der}_K(A, \kappa(e))$, where $A = K[G]$, possesses a natural structure of a Lie algebra over K (see [1, 3.3–3.5]). Considered with this structure, it is called the *Lie algebra of* G and denoted by $L(G)$. For example, $L(\mathrm{GL}_{n,K}) = \mathfrak{gl}_{n,K}$, the Lie algebra of all matrices $n \times n$ with the Lie bracket $[X, Y] = XY - YX$. For a closed subgroup H of G defined by an ideal $I \subseteq A$, the Lie algebra $L(H)$ is naturally a Lie subalgebra of $L(G)$, defined by

$$L(H) = \{X \in L(G) \mid X|_I = 0\}.$$

We say that an algebraic k-group H *acts* on a k-variety V, if there is a given morphism of k-varieties $\varphi : H \times V \to V$ (which we may abbreviate to $\varphi(g, v) = g \cdot v = gv$, $g \in H$, $v \in V$, if g is a k-valued point of H) satisfying the commutative diagrams

$$\begin{array}{ccc} H \times H \times V & \xrightarrow{m \times \mathrm{id}_V} & H \times V \\ \downarrow{\scriptstyle \mathrm{id}_H \times \varphi} & & \downarrow{\scriptstyle \varphi} \\ H \times V & \xrightarrow{\varphi} & V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \operatorname{Spec} K \times V & \xrightarrow{e \times \mathrm{id}_V} & H \times V \\ \downarrow{\scriptstyle \cong} & \swarrow{\scriptstyle \varphi} & \\ V & & \end{array} .$$

The variety V is called *H-homogeneous*, if the map

$$H \times V \xrightarrow{\varphi \times pr_V} V \times V,$$

where pr_V denotes the projection of $H \times V$ to V, is surjective. If $H = G$ is a K-group, and V is a K-variety, the fact that V is G-homogeneous means that $G(K)$ acts transitively on $V(K)$. The action of G on a V is said to be *defined over k*, if G, V and φ are defined over k. The following result on K-actions is rather simple, but of utmost importance:

Closed orbit lemma ([1, Prop. 1.8]) *Let G be an algebraic K-group acting on a K-variety V. Then each G-orbit is a smooth variety which is open in its closure in V. Its boundary is a union of orbits of strictly lower dimension. In particular, the orbits of minimal dimension are closed.*

The group G acts on itself via conjugation, and via right and left translations. Consider, in particular, the right translation by a (closed) point g of G

$$\begin{array}{ccc} G & \to & G \\ x & \mapsto & xg \end{array}$$

The corresponding map of K-algebras $\rho_g : K[G] \to K[G]$ satisfies $\rho_g f(x) = f(xg)$ for any $f \in K[G]$, $x \in G$. By [1, Prop. 1.9–1.10] we can choose a finite system of generators $\{f_1, \dots, f_n\}$ of the algebra $K[G]$ so that the n-dimensional K-subspace W of $K[G]$ spanned by these elements is invariant under all ρ_g, $g \in G(K)$. The corresponding morphism

$$\rho : G \to \mathrm{GL}(W)$$

provides a closed homomorphic embedding of G into $\mathrm{GL}_{n,K}$ (in other words, a faithful representation). This shows that every algebraic group is in fact a matrix group. Observe that if G is defined over k then all f_i can be chosen in $k[G]$, and the above embedding is moreover defined over k.

The closed embedding $\varphi : G \to \mathrm{GL}_{n,K}$, constructed above, allows us to introduce the notions of a semi-simple and a unipotent element of G. Namely, $g \in G(K)$ is called *semi-simple* (resp. *unipotent*) if its image under φ is a semi-simple (resp. unipotent) matrix in the usual sense. The correctness of this definition, i.e. its independence of the embedding, is proved in [1, Th. 4.4]. We can also define the Jordan decomposition in G. If $\varphi(g) = h_s h_u$ is the (multiplicative) Jordan decomposition of $\varphi(g)$ in $\mathrm{GL}_{n,K}$ with h_s the semi-simple factor of $\varphi(g)$ and h_u the unipotent one, then $h_s, h_u \in \varphi(G)$, and therefore there is a (unique) decomposition $g = g_s g_u$ in G, with g_s semi-simple and g_u unipotent. We denote by G_s and G_u the sets of semi-simple and unipotent elements of G respectively. The group G is called *unipotent*, if $G(K) = G_u$. The set G_u is a closed subset of $G(K)$ ([1, 4.5]), but we usually can say nothing about G_s, and none of them is a subgroup. However, if G is a connected solvable group, then G_u is a subgroup of $G(K)$ [1, Th. 10.6], and hence can be considered as an algebraic subgroup.

We call a morphism of algebraic K-groups $\chi : G \to \mathbb{G}_{m,K}$ a *character* of G. We say that a character χ is k-defined, if it comes from a morphism ${}_kG \to \mathbb{G}_{m,k}$. We denote the set of all characters of G by $X^*(G)$, and the set of all k-defined characters by $X^*(G)_k$. Since $\mathcal{O}_{\mathbb{G}_{m,K}}(\mathbb{G}_{m,K}) = K[x, x^{-1}]$, we can identify each character $\chi \in X^*(G)$ with an element of $K[G]$, namely, with the image of x under $\chi^\sharp : K[x, x^{-1}] \to K[G]$; if $\chi \in X^*(G)_k$, this will be an element of $k[G]$. It is clear that if χ_1, χ_2 are characters, then their pointwise product

$$\chi_1 \cdot \chi_2 : G \to G \times G \xrightarrow{\chi_1 \times \chi_2} \mathbb{G}_{m,K} \times \mathbb{G}_{m,K} \to \mathbb{G}_{m,K},$$

where the first map is the diagonal map and the last map is the product map, is also a character of G. The same is true for the inverse $\chi^{-1}(g) = \chi(g)^{-1}$ of a character, and $\mathbf{1}(g) = 1$, $g \in G$, behaves as a unit, so $X^*(G)$ is an abelian group, and $X(G)^*_k$ is a subgroup of $X^*(G)$.

Analogously, we define a *cocharacter* λ of G as a morphism $\lambda : \mathbb{G}_{m,K} \to G$, and we denote the set of all cocharacters resp. the set of all k-defined cocharacters of G by $X_*(G)$ resp. $X_*(G)_k$. These sets are also abelian groups with respect to the natural product, and we have a $\mathbb{Z}$-pairing

$$\langle \, , \rangle : X^*(G) \times X_*(G) \to \mathbb{Z},$$

defined by $\langle \chi, \lambda \rangle = m$, if $(\chi \circ \lambda)^\sharp : K[x, x^{-1}] \to K[x, x^{-1}]$ sends x to x^m.

An algebraic K-group T is called an *n-dimensional torus*, if it is isomorphic to $(\mathbb{G}_{m,K})^n$ for some $n \geq 0$. If moreover T is defined over k and ${}_kT$ is isomorphic to $(\mathbb{G}_{m,k})^n$, then T is called a *k-split torus*. If T is an n-dimensional torus, then both $X^*(T)$ and $X_*(T)$ are isomorphic to $\mathbb{Z}^n$, and the above pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is a perfect pairing. It is easy to see that a k-defined torus T is k-split if and only if $X^*(T) = X^*(T)_k$.

The Galois group $\Gamma = \mathrm{Gal}\,(k_s/k)$ acts on $X^*(G)$ and $X_*(G)$ in a natural way, taking f to $\sigma \circ f \circ \sigma^{-1}$ for any $\sigma \in \Gamma$ (we consider ${}_k(\mathbb{G}_{m,K}) = \mathbb{G}_{m,k}$). For characters this action coincides with the one induced from the Galois action on $K[G]$. A character or a cocharacter is defined over k if and only if it is Γ-invariant.

Let T be a torus acting on an algebraic group G via the morphism $\varphi : T \times G \to G$. Then the corresponding tangent maps $d\varphi_t : L(G) \to L(G)$ provide a representation $T \to \mathrm{GL}(L(G))$. In general, if $T \to \mathrm{GL}_{n,K}$ is a representation of a torus T, then the image of T is conjugate to a subgroup of the group D_n of all diagonal matrices in $\mathrm{GL}_{n,K}$ [1, Prop. 8.2]. Hence we can write

$$L(G) = \bigoplus_{\chi \in X^*(T)} L(G)_\chi,$$

where $L(G)_\chi = \{v \in L(G) \mid t \cdot v = \chi(t)v \;\; \forall \; t \in T\}$. The set of non-zero characters $\chi \in X^*(T)$ such that $L(G)_\chi \neq 0$ is denoted by $\Phi(T, G)$ and called *the set of roots of G with respect to T*. (This should not be confused with the notion of an abstract root system, § 3; cf. Theorem 5.10.) Observe that if H is a T-invariant subgroup of G, then $L(H) \subseteq L(G)$, and hence $\Phi(T, H) \subseteq \Phi(T, G)$.

§ 3. Abstract root systems and Weyl groups

Let V be a finite dimensional vector space over $\mathbb{Q}$. We call an element of $\mathrm{GL}(V)$ a *reflection*, if it has order 2 and induces the identity on a subspace of codimension 1. We say that w is a reflection with respect to $\alpha \in V$, if $w(\alpha) = -\alpha$.

Let $V^* = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$, and denote by $\langle \; , \; \rangle$ the natural pairing of V and V^*. Then for any reflection w with respect to $\alpha \in V$ there exists a unique $\lambda = \lambda_w \in V^*$ such that $w(x) = x - \langle x, \lambda \rangle \, \alpha$ for any $x \in V$.

A *abstract root system* (or just a *root system* for shortness) is a pair (V, Φ), where V is a finite dimensional $\mathbb{Q}$-vector space, and Φ is subset of V, satisfying:

(1) Φ is finite, does not contain 0, and spans V.

(2) If $\alpha, \beta \in \Phi$ are linearly dependent, then $\alpha = \beta$ or $\alpha = -\beta$.
(3) For each $\alpha \in \Phi$ there is a reflection w_α with respect to α which preserves Φ (such a reflection is necessarily unique).
(4) For any $\alpha, \beta \in \Phi$ one has $w_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$ with $n_{\beta,\alpha} \in \mathbb{Z}$.

(These numbers $n_{\alpha,\beta}$ are, actually, the products $\langle \beta, \lambda_\alpha \rangle$, where $\lambda_\alpha = \lambda_{w_\alpha}$ is the corresponding element of V^*.)

Two root systems (V, Φ) and (V', Φ') are called *isomorphic*, if there exists a vector space isomorphism $\varphi : V \to V'$ such that $\varphi(\Phi) = \Phi'$ and φ preserves the integers $n_{\beta,\alpha}$ from the definition of a root system.

The number $\dim V$ is called the *rank* of the root system Φ.

We denote the subgroup of $\mathrm{GL}(V)$ generated by all w_α, $\alpha \in \Phi$, by W_Φ and call it the *Weyl group* of the root system Φ. Since Φ is finite and generates V, it is a finite group.

A subset $\Pi \subseteq \Phi$ is called a *system of simple roots* (or a *system of fundamental roots*, or a *basis*) for Φ if Π is a basis of V, and any root $\beta \in \Phi$ can be represented as a sum $\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha$, with m_α being integral coefficients, all non-negative or all non-positive.

We call an element $\lambda \in V^*$ *regular* (with respect to Φ), if $\langle \alpha, \lambda \rangle \neq 0$ for any $\alpha \in \Phi$. Clearly, regular elements exist.

Theorem 3.1. *Let (V, Φ) be an abstract root system.*

(1) *For any regular element $\lambda \in V^*$, there exists one and only one system of simple roots Π in Φ such that $\langle \alpha, \lambda \rangle > 0$ for any $\alpha \in \Pi$, and conversely, for any system of simple roots Π there is such a λ.*
(2) *The Weyl group $W = W_\Phi$ acts simply transitively on the set of systems of simple roots in Φ.*
(3) *For any system of simple roots $\Pi \subseteq \Phi$, the Weyl group W is generated by w_α, $\alpha \in \Pi$.*

Proof. See [14, Th. 10.1, Th. 10.3]. □

From now on, we fix a system of simple roots Π in an abstract root system Φ.

We will denote by $\Phi^+ = \Phi^+(\Pi)$ (resp. $\Phi^- = \Phi^-(\Pi)$) the set of roots which are decomposed into a linear combination of elements of Π with non-negative (resp. non-positive) coefficients. The elements of Φ^+ (resp. Φ^-) are called the *positive* (resp. *negative*) roots with respect to Π. Clearly, $\Phi = \Phi^+ \amalg \Phi^-$. The definition of a root system implies also that $\Phi^- = -\Phi^+$.

Let $W = W_\Phi$. We set

$$R = R(\Pi) = \{w_\alpha \mid \alpha \in \Pi\}.$$

By Theorem 3.1, any element $w \in W$ can be represented as a product $w = w_1 \dots w_m$ with $w_i \in R$. If the number of factors m is the minimal possible, this decomposition is said to be *reduced*; then we set $l(w) = m$ and call it the *length* of w (with respect to Π).

Lemma 3.2. *Let $w = v_1 \dots v_m$, $v_i \in R$, be a reduced decomposition of $w \in W$ and let $v \in R$.*

(1) *There are only two possibilities for $l(vw)$:*
a) *$l(vw) = l(w) - 1$, and then there exists $1 \leq i \leq m$ such that $vw = v_1 \dots v_{i-1} v_{i+1} \dots v_m$ is a reduced decomposition of vw;*
b) *$l(vw) = l(w) + 1$, and then $vw = vv_1 \dots v_m$ is a reduced decomposition of vw.*

(2) *If* $w = v'_1 \dots v'_t$ *is any decomposition of* w *with* $v'_i \in R$*, then there exist* $1 \leq s_1 < \dots < s_m \leq t$ *such that* $w = v'_{s_1} \dots v'_{s_m}$ *is a reduced decomposition of* w*.*

Proof. For (1) see [2, Ch. IV, §1, Prop. 4]. To prove (2), observe that if the decomposition $w = v'_1 \dots v'_t$ is not reduced, then there is $2 \leq i \leq t$ such that $w' = v'_i \dots v'_t$ is a reduced decomposition, and $l(v'_{i-1}w') \leq l(w')$. Then (1) implies that $l(v'_{i-1}w') = l(w') - 1$ and there exists $i \leq j \leq t$ such that $v'_{i-1}w' = v'_i \dots v'_{j-1}v'_{j+1} \dots v'_t$. The claim now follows by induction on t. □

For any $w \in W$ we set

$$\Phi^+_w = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^+\} \quad \text{and} \quad \Phi'_w = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\}.$$

Lemma 3.3. *Let* $w \in W$*,* $\alpha \in \Pi$*. Then*

(1) $l(ww_\alpha) = l(w) + 1$ *if and only if* $w(\alpha) \in \Phi^+$*;*
(2) $l(w_\alpha w) = l(w) + 1$ *if and only if* $w^{-1}(\alpha) \in \Phi^+$*;*
(3) $l(w) = |\Phi'_w| = |\Phi'_{w^{-1}}|$*.*

Proof. The claim of (1) is proved in [14, 10.2, Lemma C]. Since $l(w) = l(w^{-1})$, the claim (2) follows from (1), as well as the equality $l(w) = |\Phi'_w|$ from $l(w) = |\Phi'_{w^{-1}}|$. We prove $l(w) = |\Phi'_{w^{-1}}|$ by induction on $l(w)$.

Let $\beta \in \Phi^+$. Then $\beta = \sum m_\gamma \gamma$, $\gamma \in \Pi$, where all m_γ are non-negative. If $\beta \neq \alpha$, then $m_\gamma > 0$ for at least one $\gamma \neq \alpha$, since Φ^+ does not contain proportional roots. Since the coefficient near γ in $w_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$ also equals m_γ, the root β is in Φ^+. This shows that $\Phi'_{w_\alpha^{-1}} = \{\alpha\}$, and hence $l(w_\alpha) = |\Phi'_{w_\alpha^{-1}}|$.

Let $w = w_{\alpha_1} \dots w_{\alpha_m}$, $\alpha_i \in \Pi$, be a reduced decomposition of w, that is, $m = l(w)$. Set $\alpha = \alpha_m$ and $w' = ww_\alpha$. Then $l(w') = l(w) - 1$ and $l(w'w_\alpha) = l(w') + 1$. By the induction hypothesis

$$l(w') = |\Phi'_{w'^{-1}}| = |\{\alpha \in \Phi^+ \mid w'(\alpha) \in \Phi^-\}|$$

and $w'(\alpha) \in \Phi^+$. Then $\Phi'_{w'^{-1}} \subseteq \Phi^+ \setminus \{\alpha\}$, and since w_α takes to Φ^- only one positive root α, we have

$$|\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}| = |\{\alpha \in \Phi^+ \mid w'(\alpha) \in \Phi^-\}| + 1.$$

Then $l(w) = l(w') + 1 = |\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}| = |\Phi'_{w^{-1}}|$. □

Let $I \subseteq \Pi$. We will denote by Δ_I the subset of Φ spanned by I, and by W_I the subgroup of W generated by all w_α, $\alpha \in I$. By [2, Ch. VI, §1.7, Cor. 4] Δ_I is a root system, and, clearly, $W_I = W(\Delta_I)$. We write $\Delta^+_I = \Phi^+ \cap \Delta_I$ and $\Delta^-_I = \Phi^- \cap \Delta_I$.

Lemma 3.4. *Let* $I, J \subseteq \Pi$ *and* $w \in W$*. The double coset* $W_I w W_J$ *contains a unique element* w_0 *of minimal length, and any element* $w' \in W_I w W_J$ *can be written in the form* $w' = aw_0b$*, where* $a \in W_I$*,* $b \in W_J$ *and* $l(w') = l(a) + l(w_0) + l(b)$*.*

Proof. Let w_0 be any element of minimal length in $W_I w W_J$. We can write $w' = cw_0d$ for some $c \in W_I$, $d \in W_J$. By Lemma 3.2 (2) there is a reduced decomposition of w' which is obtained from the product of reduced decompositions of c, w_0, and d by erasing some factors. Since $c \in W_I$, $d \in W_J$, by Lemma 3.2 (2) they possess reduced decompositions with all factors in W_I and W_J respectively, and we take these decompositions. Let a and b be the products left from c and d, respectively. Then $a \in W_I$ and $b \in W_J$. Since w_0 was an element of minimal length in $W_I w W_J$, we have erased no factors from the reduced decomposition of w_0. Then $w' = aw_0b$, and since the decomposition of w' is reduced, the decompositions we have obtained for a, w_0, b

are reduced as well. Hence $l(w') = l(a) + l(w_0) + l(b)$. This also implies the uniqueness of w_0. □

The elements of minimal length in the double cosets of $W_I w W_J$, $w \in W$, form a complete system of representatives for $W_I \backslash W / W_J$, which we denote by $W^{I,J}$. They are also characterized as follows.

Lemma 3.5. *For any $w \in W$ we have $w \in W^{I,J}$ if and only if $w(\Delta_J^+) \subseteq \Phi^+$ and $w^{-1}(\Delta_I^+) \subseteq \Phi^+$.*

Proof. If $w \in W^{I,J}$, by Lemma 3.4 we have $l(ww_\alpha) > l(w)$ for any $\alpha \in J$ and $l(w_\alpha w) > l(w)$ for any $\alpha \in I$. Then by Lemma 3.3 we have $w(J) \subseteq \Phi^+$ and $w^{-1}(I) \subseteq \Phi^+$. Since w is additive and the sum of positive roots is a positive root, the result follows. Conversely, if $w \in W$ satisfies $w(\Delta_J^+) \subseteq \Phi^+$ and $w^{-1}(\Delta_I^+) \subseteq \Phi^+$, then by Lemma 3.3 it satisfies $l(ww_\alpha) > l(w)$ for any $\alpha \in J$ and $l(w_\alpha w) > l(w)$ for any $\alpha \in I$. Let w_0 be the element of the smallest length in the coset $W_I w W_J$, and write $w = a w_0 b$, $a \in W_I$, $b \in W_J$. Then $l(w) = l(a) + l(w_0) + l(b)$ by Lemma 3.4, and hence $l(a^{-1} w b^{-1}) = l(w) - l(b^{-1}) - l(a^{-1})$. Suppose that one of a, b, say, a, is non-trivial. Then since each multiplication by an element of R changes the length by ± 1 only, we must have $l(w_\alpha w) < l(w)$, where w_α is the first from the right element in the reduced decomposition of a^{-1}. Since a^{-1} has a reduced decomposition with terms in W_I, this is a contradiction. □

§ 4. Quotients of varieties and algebraic groups

In the present chapter we reproduce the classical construction of the (geometric) quotient of an algebraic group G by a closed subgroup H (Theorem 4.7). The idea is to define an action of G on a projective space $\mathbb{P}^n_K$ so that H is precisely the stabilizer of a certain (closed) point x, and to identify G/H with the G-orbit of x. This is made possible by the fundamental theorem of Chevalley (Theorem 4.2).

1. Chevalley theorem. In order to prove the existence of a quotient of an algebraic group G over k by a k-subgroup H, we need to construct a certain representation of G, which behaves well with respect to H. This representation is provided by the action of G on its affine algebra $K[G]$ with the help of the "exterior powers" construction.

Let V be a finite dimensional vector space over a field k. Recall that both the group $\mathrm{GL}(V)$ and the corresponding Lie algebra $\mathfrak{gl}(V)$ act in a natural way on the exterior powers $\Lambda^m(V)$, $m \geq 0$, of V. More precisely, we have a homomorphism of algebraic groups $\wedge^m : \mathrm{GL}(V) \to \mathrm{GL}(\Lambda^m(V))$, given by

$$\wedge^m g(v_1 \wedge \ldots \wedge v_m) = g(v_1) \wedge \ldots \wedge g(v_m)$$

for any $g \in \mathrm{GL}(V)$. The corresponding tangent morphism $d\wedge^m : \mathfrak{gl}(V) \to \mathfrak{gl}(\Lambda^m(V))$, gives also the action of $\mathfrak{gl}(V)$, in the way

$$d\wedge^m X(v_1 \wedge \ldots \wedge v_m) = \sum_{i=1}^m v_1 \wedge \ldots \wedge v_{i-1} \wedge X v_i \wedge v_{i+1} \wedge \ldots \wedge v_m$$

for any $X \in \mathfrak{gl}(V)$. These actions satisfy the following

Lemma 4.1. *Let U be a d-dimensional subspace of an n-dimensional vector space V over k, and let $g \in \mathrm{GL}(V)$, $X \in \mathfrak{gl}(V)$. Then*

$$\wedge^d g(\Lambda^d U) = \Lambda^d U \iff g(U) = U$$
$$d\wedge^d X(\Lambda^d U) \subseteq \Lambda^d U \iff X(U) \subseteq U.$$

Proof. In both cases the implication $\Leftarrow$ is clear. To prove the inverse ones, we first choose a basis $e_1, \dots, e_n$ of V in such a way that $e_1, \dots, e_d$ span U, and $e_m, \dots, e_{m+d}$ span $g(U)$. Then $\Lambda^d U$ is generated by $e_1 \wedge \dots \wedge e_d$, and $\wedge^d g(\Lambda^d U)$ is generated by $e_m \wedge \dots \wedge e_{m+d}$. These elements of $\Lambda^d V$ are collinear if and only if $m = 1$, that is, if and only if $g(U) = U$.

To prove the second equivalence, we observe that it is linear in X, and not affected by substituting $X - Y$ instead of X, provided $Y(U) \subseteq U$. Denote by W the subspace of U, consisting of all elements whose images under X are in U. Denote by p a projection map $V \to W$, and set $Y = X \circ p$. Then $Y(U) \subseteq U$, and it is easy to see that $(X - Y)(U)$ does not intersect U. So, it is enough to prove our claim in case when $X(U) \cap U = 0$. In this case we can choose such a basis $e_1, \dots, e_n$ of V that $e_1, \dots, e_d$ span U, $e_{d+1} = X(e_1), \dots, e_{d+m} = X(e_m)$ span $X(U)$, and $e_{m+1}, \dots e_d$ span $\ker X \cap U$. Then

$$\begin{aligned} d\wedge^d X(e_1 \wedge \dots \wedge e_d) &= \sum_{i=1}^{d} e_1 \wedge \dots \wedge e_{i-1} \wedge X e_i \wedge e_{i+1} \wedge \dots \wedge e_d \\ &= \sum_{i=1}^{m} e_1 \wedge \dots \wedge e_{i-1} \wedge e_{i+m} \wedge e_{i+1} \wedge \dots \wedge e_d. \end{aligned}$$

The latter sum cannot be collinear to $e_1 \wedge \dots \wedge e_d$ unless $m = 0$, that is, $X(U) = 0 \subseteq U$. □

Theorem 4.2 (Chevalley). *Let G be an algebraic group defined over k, with Lie algebra $\mathfrak{g}$. Let H be a closed k-defined subgroup of G with Lie algebra $\mathfrak{h}$. Then there is a k-representation $\varphi : G \to \mathrm{GL}_{n,K}$, which is a closed embedding, and a k-defined line $L \subseteq K^n$ such that*

$$\begin{aligned} H &= \{g \in G \mid \varphi(g)L = L\}, \\ \mathfrak{h} &= \{X \in \mathfrak{g} \mid d\varphi(X)L \subseteq L\}. \end{aligned}$$

Proof. Denote $K[G]$ by A and $k[G]$ by A_k. Let I denote the ideal of A, corresponding to H. Since H is defined over k, I is generated by $I_k = I \cap A_k$. For any finite set S of generators of the ideal $I_k \subseteq A_k$, by [1, Prop. 1.19] we can find a finite-dimensional k-defined (that is, generated by its intersection with A_k) subspace W of A, containing S, which is invariant under all translation maps ρ_g, $g \in G$ (see § 2). Set $M = I \cap W$. Since both I and W are defined over k, then M also is. Clearly, the ideal I is also generated by M_k. Further, both I and W are invariant under all ρ_h, $h \in H$, so M also is. Since ρ_h are invertible, this means $\rho_h(M) = M$ for any $h \in H$. By [1, Cor. 3.12] we also have $X(M) \subseteq M$ for any $X \in \mathfrak{h}$.

Conversely, if $\rho_g(M) = M$ for some $g \in G$, then $\rho_g(I) = I$, because M generates I and ρ_g is an algebra automorphism. Since $\rho_g(f)(x) = f(gx) = 0$ for any $f \in I$, $x \in H$, by the definition of I we get $gx \in H$, so $g \in H$. This means that

$$H = \{g \in G \mid \rho_g(M) = M\}.$$

Analogously, if $X(M) \subseteq M$ for some $X \in \mathfrak{g}$, then $X(I) \subseteq I$, since M generates the ideal I and X is a derivation. And hence $X \in \mathfrak{h}$ by [1, Prop. 3.8], which proves that

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid X(M) \subseteq M\}.$$

Now we set $V = \Lambda^d W$, where $d = \dim M$, and let $L = \Lambda^d M$. Observe that $V \cong \Lambda^d(W \cap A_k) \otimes_k K$, and $L \cong \Lambda^d(M \cap A_k) \otimes_k K$, where $W \cap A_k$ and $M \cap A_k$

are considered as k-vector spaces. Now the representation $\rho : G \to \mathrm{GL}(W)$ induces a k-representation $\varphi : G \to \mathrm{GL}(V) \cong \mathrm{GL}_{n,K}$ and Lemma 4.1 implies that

$$H = \{g \in G \mid \varphi(g)L = L\} \quad \text{and} \quad \mathfrak{h} = \{X \in \mathfrak{g} \mid d\varphi(X)L \subseteq L\}.$$

If this representation is not an embedding, we should replace φ by the sum $\varphi \oplus \varphi'$, where $\varphi' : G \to \mathrm{GL}_{m,K}$ is any k-representation of G which is a closed embedding. □

Using the theorem above, we can prove even more in case when the subgroup H of G is a normal subgroup.

Theorem 4.3. *Let G be an algebraic group defined over k, and let N be a normal k-defined subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{n}$ denote the Lie algebras of G and N respectively. Then there is a linear k-representation $\psi : G \to \mathrm{GL}_{n,K}$ such that*

$$N = \ker \psi \quad \textit{and} \quad \mathfrak{n} = \ker(d\psi).$$

Proof. Let $A = K[G]$. By Theorem 4.2 there exists a k-representation $\varphi : G \to \mathrm{GL}(V) \cong \mathrm{GL}_{n,K}$, such that N is the stabilizer of a line $L = \langle v \rangle \subseteq V$, also defined over k. Set $\chi_0(g) = \frac{g \cdot v}{v}$, $g \in N$. It is a character of the group N, defined over k. Indeed, if we choose a basis of V with the first vector in L, we see that the representation map

$$\varphi^\sharp : K\left[x_{ij},\ 1 \le i, j \le n;\ 1/\det(x_{ij})_{i,j=1}^n\right] \to A$$

factors through the canonical projection

$$K\left[x_{ij},\ 1/\det(x_{ij})\right] \to K\left[x_{ij},\ 1/\det(x_{ij})\right]\big/(x_{i1} = 0,\ 1 < i \le n).$$

Then we can define the K-algebra homorphism

$$(\chi_0)^\sharp : K[x, x^{-1}] \to A$$

so that it takes x to the image of x_{11}. Since φ is a k-representation and L is k-defined, we see that $\chi_0 \in X^*(N)_k$, because it is invariant under the action of $\Gamma = \mathrm{Gal}\,(k_s/k)$. We assign to any character $\chi \in X^*(N)$ the subspace

$$V_\chi = \{v \in V \mid g \cdot v = \chi(g)v \text{ for any } g \in N\}.$$

Clearly, each V_χ is a N-invariant subspace of V, and all non-zero subspaces V_χ, $\chi \in X(G)$, are linearly independent. Set

$$F = \bigoplus_{\chi \in X(N)_{k_s}} V_\chi.$$

This subspace is invariant under G, since for any $\chi \in X(N)_{k_s}$, $x \in V_\chi$, $g \in G$ and $h \in N$ we have

$$\varphi(h)\varphi(g)(x) = \varphi(g)\varphi(g^{-1}hg)(x) = \chi(g^{-1}hg)g(x),$$

and, clearly, $g \cdot \chi : N \to \mathbb{G}_m$ defined by $(g \cdot \chi)(h) = \chi(g^{-1}hg)$ is also a character of N over k_s, because the conjugation by g is an algebraic k-group automorphism of N. Note that with the above notation $\varphi(g)V_\chi = V_{g \cdot \chi}$, since $g \cdot : X(N)_{k_s} \to X(N)_{k_s}$ is invertible; so, $\varphi(g)$ acts as a permutation of the spaces V_χ.

Moreover, since the Galois group $\mathrm{Gal}\,(k_s/k)$ acts on $X(N)_{k_s}$ in a natural way, and φ is defined over k, the space F is $\mathrm{Gal}\,(k_s/k)$-invariant, and, consequently, also defined over k. Since, finally, $L \subseteq F$, we can assume that $V = F$ without any loss of generality.

Now consider

$$W = \{x \in \mathfrak{gl}(V) \mid x(V_\chi) \subseteq V_\chi \text{ for any } \chi \in X(N)_{k_s}\}.$$

Clearly, $W \cong \bigoplus \mathfrak{gl}(V_\chi)$. The adjoint representation $\mathrm{Ad} : \mathrm{GL}(V) \to \mathrm{GL}(\mathfrak{gl}(V))$, which is defined as $\mathrm{Ad}(x)(y) = x^{-1}yx$, $x \in \mathrm{GL}(V)$, $y \in \mathfrak{gl}(V)$, induces the action of $\varphi(G)$ on $\mathfrak{gl}(V)$. Since an element $y \in W$ preserves all V_χ, and $\varphi(g) \in GL(V)$ permutes them, $\varphi(g)^{-1}y\varphi(g) \in W$, so W is $\varphi(G)$-invariant. This allows to define a representation

$$\psi : G \to \mathrm{GL}(W), \quad \psi(g) = \mathrm{Ad} \circ \varphi(g)|_W.$$

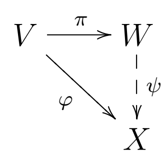
Observe that ψ is defined over k, since Ad, φ and W are (the latter because $\mathrm{Gal}\,(k_s/k)$ permutes the spaces V_χ).

Let us prove that ψ satisfies the claim of the theorem. Clearly, $N \subseteq \ker(\psi)$, because it is mapped to the scalar matrices in each $\mathfrak{gl}(V_\chi)$, and therefore commutes with W. Conversely, if $g \in G$ is in $\ker(\psi)$, it means that $\varphi(g)$ commutes with any $w \in W$, in particular, $\varphi|_{V_\chi}$ commutes with the whole $\mathfrak{gl}(V_\chi)$, so it is a scalar. It implies that $\varphi(g)$ leaves L stable, so $g \in N$. Hence indeed $N = \ker(\psi)$.

The above also shows that $\mathfrak{n} \subseteq \ker(d\psi)$, since $d\psi$ takes $\mathfrak{n}$ into the Lie algebra of $\{e\}$, which is isomorphic to K and has only one derivation, the trivial one. To prove the converse inclusion, we recall that $d(\mathrm{Ad}) = \mathrm{ad} : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathfrak{gl}(V))$ acts as $\mathrm{ad}(X)(Y) = XY - YX$ for any $X, Y \in \mathfrak{gl}(V)$. Hence, if $X \in \mathfrak{g}$ is in the kernel of $d\psi$, its image $(d\varphi)(X) \in \mathfrak{gl}(V)$ commutes with all $\mathfrak{gl}(V_\chi)$, hence acts on V_χ as a scalar (maybe zero), hence takes L into L, and therefore, $X \in \mathfrak{n}$. □

2. Quotient morphisms. Let $\pi : V \to W$ be a k-defined morphism of k-defined varieties over K. We say that π is a (*geometric*) *quotient morphism defined over* k, if π is surjective and open, and for any open subset $U \subseteq V$ the map $\pi^\sharp$ induces an isomorphism from $\mathcal{O}_W(\pi(U))$ onto the set of $f \in \mathcal{O}_V(U)$ which are constant (as functions) on the set-theoretic fibers of $\pi|_U$.

Theorem 4.4 (Universal Property). *Let $\pi : V \to W$ be a quotient morphism defined over k. If $\varphi : V \to X$ is a morphism of K-varieties constant on the fibers of π, then there exists a unique morphism $\psi : W \to X$ making the diagram*



commutative. If φ is a k-defined morphism, then so is ψ.

Proof. It is clear that we can define a unique map of sets $\psi : W \to X$ such that $\varphi = \psi \circ \pi$. Since π is open and φ is continuous, this map is also continuous. Further, for any $U \subseteq X$ open, its inverse image $U' = \varphi^{-1}(U)$ is also open, so

$$\pi^\sharp : \mathcal{O}_W(\pi(U')) \to \{f \in \mathcal{O}_V(U') \mid f \text{ is constant on fibers of } \pi|_{U'}\} \subseteq \mathcal{O}_V(U')$$

is an isomorphism of K-algebras. Since φ is constant on fibers of π, all elements of $\varphi^\sharp(\mathcal{O}_X(U)) \subseteq \mathcal{O}_V(U')$ also are, so we can define a map of K-algebras

$$\psi^\sharp = (\pi^\sharp)^{-1} \circ \varphi^\sharp : \mathcal{O}_V(U) \to \mathcal{O}_W(\pi(U')).$$

If $\varphi^\sharp$ and $\pi^\sharp$ (that is, φ and π) are defined over k, then this map is defined over k as well, since it takes $\mathcal{O}_{{}_kX}({}_kU)$ into $\mathcal{O}_{{}_kW}\big({}_k(\pi(U'))\big) = \mathcal{O}_{{}_kW}({}_k\pi({}_kU'))$. It makes ψ into a (k-defined) morphism of varieties, because for any two open sets $U_1 \subseteq U_2 \subseteq X$ the map $\varphi^\sharp$ commutes with restriction maps

$$\begin{array}{ccc} \mathcal{O}_X(U_2) & \longrightarrow & \mathcal{O}_X(U_1) \\ {\scriptstyle\varphi^\sharp}\big\downarrow & & {\scriptstyle\varphi^\sharp}\big\downarrow \\ \mathcal{O}_V(U_2') & \longrightarrow & \mathcal{O}_V(U_1') \end{array}$$

where $U_2' = \varphi^{-1}(U_2)$ and $U_1' = \varphi^{-1}(U_1)$, and

$$(\pi^\sharp)^{-1} : \varphi^\sharp(\mathcal{O}_X(U_1)) \to \mathcal{O}_W(\pi(U_1')), \quad (\pi^\sharp)^{-1} : \varphi^\sharp(\mathcal{O}_X(U_2)) \to \mathcal{O}_W(\pi(U_2'))$$

are isomorphisms, so also commute with restriction maps. □

In certain important cases we can distinguish quotient morphisms using the following criterion.

Lemma 4.5. *Let $\pi : V \to W$ be a surjective open separable morphism of irreducible K-varieties, and assume W is normal. Then π is a quotient morphism.*

Proof. We need to verify only that for any open subset $U \subseteq V$, $\pi^\sharp$ is an isomorphism from $\mathcal{O}_W(\pi(U))$ onto the set of $f \in \mathcal{O}_V(U)$ which are constant on the fibers of $\pi|_U$. Since V is irreducible, U also is; since $\pi(U)$ is open and W is irreducible, $\pi(U)$ is irreducible as well. Since both U and $\pi(U)$ are open dense subsets of V and W respectively, we have the equality of the fields of rational functions $K(U) = K(V)$ and $K(\pi(U)) = K(W)$. Hence, the morphism $\pi|_U : U \to \pi(U)$ is separable, if π is. Finally, since normality is a local property, $\pi(U)$ is normal, if W is. This shows that it is enough to consider the case $U = V$, $\pi(U) = W$.

Since all $K[V] \to K(V)$, $K[W] \to K(W)$ and $\pi^\sharp : K(W) \to K(V)$ are injective, we can identify $K[V]$, $K[W]$ and $K(W)$ with subalgebras of $K(V)$. We need to prove that every $f \in K[V]$ constant on the fibers of π lies in the subring $\pi^\sharp(K[W]) = K[W]$ of $K(V)$. By [1, Prop. AG.18.2], any such f is purely inseparable over $K(W)$, so by the separability of π we have $f \in K(W)$. If $f \notin K[W]$, that is, f is not defined in $x = \pi(y) \in W$, then by [1, Lemma AG.18.3] there is a point $x' = \pi(y') \in W$ such that $1/f$ is defined at x' and $(1/f)(x') = 0$. But this means that $1/f$ considered as an element of $K(V)$ is defined and vanishes at $y' \in V$, which is impossible, since $f \in K[V]$ is defined everywhere. □

3. **Quotients of varieties by groups.** Throughout this subsection we suppose that G is an (affine) k-defined algebraic K-group, V is a k-defined K-variety, and G acts on V with a k-defined action.

We call a surjective morphism $\pi : V \to W$ of K-varieties an *orbit map*, provided the fibers of π are the orbits of G in V. A *(geometric) quotient of V by G defined over k* is an orbit map $\pi : V \to W$ which is a k-defined quotient morphism in the sense of the previous subsection. In particular, it satisfies the following universal property:

Universal property. *If (W, π) is a k-defined quotient of V by G, and $\varphi : V \to X$ is a morphism of varieties constant on the G-orbits in V, then there exists a unique morphism $\psi : W \to X$ making the diagram*

$$\begin{array}{ccc} V & \xrightarrow{\ \pi\ } & W \\ & {\scriptstyle\varphi}\searrow & \big\downarrow{\scriptstyle\psi} \\ & & X \end{array}$$

commutative. If φ is a k-defined morphism, then so is ψ.

Hence, the k-defined quotient of V by G over k, if it exists, is unique up to a unique k-defined isomorphism. We will denote it by V/G.

Lemma 4.6. *Suppose $\pi : V \to W$ is an open separable G-orbit map, and assume that W is a normal variety and that all irreducible components of V are open. Then (W, π) is a quotient of V by G.*

Proof. Since π is surjective and open, it maps each irreducible component V' of V onto an irreducible component W' of W. Since W is normal, for any $x \in W$ its local ring $\mathcal{O}_{W,x}$ is integrally closed, hence x lies in a unique irreducible component; this means that all irreducible components of W are disjoint. Since π is an orbit map, the set $U = \pi^{-1}(W')$, is G-invariant. Therefore it is enough to prove the claim for the case W irreducible.

Observe that G acts transitively on the set $\{V_1, \ldots, V_n\}$ of irreducible components of V. Indeed, for any irreducible component V_i of V and any $g \in G$, we have that gV_i is also an irreducible component; and $GV_i = \pi^{-1}(\pi(V_i)) = \pi^{-1}(W) = V$. Further, by Lemma 4.5, each $\pi|_{V_i} : V_i \to W$ is a quotient of V_i by the stabilizer group H_i of V_i in G. Now for any open $U \subseteq V$, if $f \in \mathcal{O}_V(U)$ is stable on the fibers of $\pi|_U$, we can represent it as $f = \sum_i f_i$, where $f_i \in \mathcal{O}_V\big(\pi^{-1}(\pi(U)) \cap V_i\big)$. Then each f_i is stable on the fibers of $\pi|_{V_i}$, intersected with $U_i = \pi^{-1}(\pi(U)) \cap V_i$, hence $f_i = (\pi|_{V_i})^\sharp(g_i)$ for some $g_i \in \mathcal{O}_W\big(\pi(U_i)\big)$. Since f is constant on the fibers of π, the functions g_i coincide on intersections $\pi(U_i) \cap \pi(U_j)$, and there exists $g \in \mathcal{O}_W\big(\pi(U)\big) = \mathcal{O}_W\big(\bigcup_i \pi(U_i)\big)$ such that $g|_{\pi(U_i)} = g_i$ for any i. Then $f = \pi^\sharp(g)$, and thus lies in $\pi^\sharp(\mathcal{O}_V(U))$. This proves that $\pi : V \to W$ is a quotient of V by G. □

Theorem 4.7. *Let G be a k-defined algebraic group over K and let H be a closed k-defined subgroup of G. Then there exists a k-defined quotient $\pi : G \to G/H$, and both G/H and ${}_k(G/H)$ are smooth quasi-projective varieties. If H is a normal subgroup of G, then G/H is an k-defined algebraic group and π is a morphism of groups.*

Proof. By Theorem 4.2 we have a k-representation $\varphi : G \to \mathrm{GL}(V)$ and a line $L \subseteq V$ defined over k such that

$$H = \{g \in G \mid \varphi(g)L = L\} \quad \text{and} \quad \mathfrak{h} = \{X \in \mathfrak{g} \mid d\varphi(X)L \subseteq L\},$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively. Let $\dim V = n$, and let $q : V \setminus \{0\} \to \mathbb{P}(V) \cong \mathbb{P}_K^{n-1}$ denote the projection onto the projective space of lines in V. Let $x = q(L \setminus \{0\}) \in \mathbb{P}_K^{n-1}(k)$. The group G acts on $\mathbb{P}_K^{n-1}$ via $g \cdot y = q(\varphi(g)y)$, $y \in \mathbb{P}_K^{n-1}(K)$, and this action is k-defined. The variety $Gx \subseteq \mathbb{P}_K^{n-1}$ is quasi-projective, since, being an orbit of G, it is an open subset of its closure by the closed orbit lemma. The map

$$\begin{aligned} \pi : G &\to Gx \\ g &\mapsto g \cdot x \end{aligned}$$

is an orbit map with respect to the action of H on G by right multiplication, since H is the stabilizer of x. It is also defined over k, since x is a k-defined point. The variety ${}_k(Gx)$ is clearly defined over k; it is an open subset of its closure, since the canonical projection $\mathbb{P}_K^{n-1} \to \mathbb{P}_k^{n-1}$ is both open and closed, and hence ${}_k(G/P)$ is also quasi-projective. The smoothness of Gx implies that of ${}_k(Gx)$ by [12, Prop. 17.7.1].

It leaves to prove only that $\ker(d\pi) = \mathfrak{h}$. Indeed, suppose that it is true. Then since $\dim G = \dim H + \dim Gx$ (this follows from the fact that $G/H \cong Gx$ as a topological space) and Gx, as an orbit of G, is smooth, we have

$$\begin{aligned}\dim T_y(Gx) &= \dim Gx\\ &= \dim G - \dim H\\ &= \dim \mathfrak{g} - \dim \mathfrak{h}\\ &= \dim \mathfrak{g} - \dim\ker(d\pi)\\ &= \dim T_{\pi^{-1}(y)}G - \dim\ker(d\pi)_{\pi^{-1}(y)},\end{aligned}$$

which implies by [1, Th. AG.17.3] that π is separable. Since any smooth variety is normal, we have that Gx is normal. Further, all fibers of π are isomorphic to H as varieties, and all irreducible components of H are cosets of H° in H, so also isomorphic; this shows that all irreducible components of fibers of π have the same dimension, hence by [1, Cor. to Prop. AG.18.4] π is open. Summing up, $\pi : G \to Gx$ is an open separable orbit map to a normal variety, and all irreducible components of G are, clearly, open, so Lemma 4.6 says that π is a quotient map. Consequently, we need to prove only that $\ker(d\pi) = \mathfrak{h}$.

Choose a non-zero element v of L and define

$$\begin{aligned}\lambda : G &\to V \setminus \{0\}\\ g &\mapsto \varphi(g)v\end{aligned}$$

so that $\pi = q \circ \lambda$ and $(d\lambda)_e(X) = d\varphi(X)v$ for any $X \in \mathfrak{g}$; here we identify V with $T_eV = T_e(V \setminus \{0\})$. Now since the kernel of $(dq)_v$ is equal to L, we have that for $X \in \mathfrak{g}$

$$(d\pi)_e(X) = 0 \iff d\varphi(X)L \subseteq L,$$

and the statement on the right is equivalent to $X \in \mathfrak{h}$.

Now suppose that H is a normal subgroup of G. In this case Theorem 4.3 permits to choose $\varphi : G \to \mathrm{GL}(V)$ so that $H = \ker\varphi$ and $\mathfrak{h} = \ker(d\varphi)$. Since φ is a k-defined morphism of algebraic groups, $\varphi(G)$ is a closed k-defined subgroup of $\mathrm{GL}(V)$, and hence a k-defined algebraic group. Since H is precisely the stabilizer of $e \in GL(V)$ with respect to the left multiplication by G (via φ), that is, $\varphi(G) = Ge$, we can prove as above that $\pi = \varphi : G \to \varphi(G)$ is a quotient map. ☐

§ 5. Reductive groups over an algebraically closed field

In the present chapter we discuss the structure of a connected algebraic group G over K, first in a general situation, and then in the case when G is a reductive algebraic group (see the definition below; the basic properties are summarized in Theorem 5.7). In particular, we obtain the Bruhat decomposition for G (Theorem 5.12), and deduce the classification and the main properties of parabolic subgroups of G (subsection **5**). Here we do not touch upon the questions of rationality (i.e. of being defined over a smaller field k) of our objects; these are considered in § 6. Thus throughout this chapter all algebraic groups, varieties etc. are over K, and we tend to omit K from our notation.

1. Borel subgroups. Let G be a connected algebraic group (over K). A subgroup B of G is called a *Borel subgroup*, if it is a maximal connected solvable subgroup of G. An overgroup of a Borel subgroup is called a *parabolic subgroup*.

We summarize the main properties of parabolic and Borel subgroups in Theorem 5.1 below. In particular, we will prove that for any parabolic subgroup P, the quotient variety G/P is not only quasi-projective, as it was shown in Theorem 4.7, but even projective. To obtain this result, we need the notion of a flag variety.

Let V be an n-dimensional K-vector space. For any $d \geq 0$, consider the Grassmannian $\mathrm{Gr}_d(V)$, which is the set of all d-dimensional subspaces of V. It admits an embedding (cf. Lemma 4.1) into the projective space $\mathbb{P}(\Lambda^d V)$, namely, the one sending a subspace W to the line $\Lambda^d W$. (This map is also known as the Plücker embedding.) It is well-known (for example, [1, 10.3]) that the image of $\mathrm{Gr}_d(V)$ is a closed algebraic subset of $\mathbb{P}(\Lambda^d V)$. Hence we can introduce on $\mathrm{Gr}_d(V)$ the structure of a projective variety. Further, if $W \in \mathrm{Gr}_d(V)$ and $W' \in \mathrm{Gr}_{d'}(V)$ are two subspaces of V, the fact $W \subseteq W'$ is also expressed by algebraic equations on the coordinates in $\Lambda^d V \times \Lambda^{d'} V$. This allows us to define the *flag variety* of V as the set

$$\mathcal{F}(V) = \{(V_1, \ldots, V_n) \in \mathrm{Gr}_1(V) \times \ldots \times \mathrm{Gr}_n(V) \mid V_i \subseteq V_{i+1},\ 1 \leq i < n\},$$

with the structure of a projective variety induced from $\mathbb{P}(\Lambda^1 V \times \ldots \times \Lambda^n V)$.

Theorem 5.1. *Let G be a connected algebraic group.*

1. *All Borel subgroups are conjugate in G.*
2. *A subgroup $P \subseteq G$ is parabolic if an only if G/P is a projective variety.*
3. *If an automorphism of G fixes all elements of a Borel subgroup B, then it is the identity.*
4. *If P is a parabolic subgroup of G, then P is connected and $P = N_G(P)$.*

Proof. Let B be a Borel subgroup of maximal dimension in G. By Theorem 4.2 there is a faithful representation $\pi : G \to \mathrm{GL}(V)$ with a line $V_1 \subseteq V$ such that

$$B = \{g \in G \mid \varphi(g)V_1 = V_1\} \quad \text{and} \quad L(B) = \{X \in \mathfrak{g} \mid d\varphi(X)V_1 \subseteq V_1\}.$$

Then B also acts on V/V_1, and by the Lie-Kolchin theorem ([1, Th. 10.5]) there is a flag $F = (V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n = V)$ in V, stabilized by B. G acts on the whole flag variety $\mathcal{F}(V)$ of V via π. Since B is the stabilizer of V_1, it is also the stabilizer of F. As in the proof of Theorem 4.7, we have $G/B \cong G \cdot F \subseteq \mathcal{F}(V)$. If some other flag $F' \in \mathcal{F}(V)$ has the stabilizer B' in G, then B' too is a solvable subgroup of G, and the maximality of $\dim B$ implies $\dim B' \leq \dim B$. Hence also $\dim G/B \leq \dim G/B'$, so $G \cdot F$ is an orbit of minimal dimension for G, and hence closed. Then G/B is a projective variety, since the variety of flags is projective. Further, any other Borel subgroup D of G acts on G/B in a natural way, and hence by [1, Th. 10.4] it has there a fixed point. This means that $DxB \subseteq xB$ for some $x \in G$, or $x^{-1}Dx \subseteq B$. By the maximality of a Borel subgroup $x^{-1}Dx = B$, so all Borel subgroups are conjugate.

Let P be a parabolic subgroup of G, and B a Borel subgroup contained in P. Then $G/B \to G/P$ is a surjective map from a complete variety, hence G/P is complete and, consequently, projective. Conversely, if a subgroup P of G is such that G/P is projective, then any Borel subgroup B has a fixed point in G/P, and, as above, its conjugate lies in P. This proves the second statement of the theorem.

The completeness of the variety G/B implies that any automorphism φ of G which is identical on B, is also identical on the whole G. Indeed, the morphism $\varphi' : G \to G$ defined by $\varphi'(g) = \varphi(g)g^{-1}$ factors through the quotient map $G \to G/B$, hence its image is both complete and affine, and thus a point.

Now let us prove (4). First we show that if the claim is true for all Borel subgroups, then it is true for any parabolic subgroup. Indeed, let $n \in N_G(P)$ and let B be a Borel subgroup contained in P. Then nBn^{-1} is also a Borel subgroup, hence by the conjugacy of Borel subgroups of P we have $pnBn^{-1}p^{-1} = B$ for some $p \in P$. By our assumption, $pn \in B$, but then $n \in P$. To see that P is connected, observe that the identity component P° is also a parabolic subgroup, and P is in its normalizer.

It remains to show that every Borel subgroup B of G satisfies $B = N_G(B)$. We argue by induction on $\dim G$. If $\dim G \le 2$, then $G = B$, because if $\dim B \le 1$, then B is abelian, hence by (3) any $x \in B$ acts on G trivially, so B is central in G, and then G itself is abelian. In the general case, denote $N_G(B)$ by N, and let T be a maximal torus of B. The conjugacy of all maximal tori of B (for example, [1, Th. 10.6]) implies $N \subseteq B \cdot N_G(T)$, so it is enough to prove that $N_G(T) \cap N \subseteq B$. Let $S = C_T(n)$ for some element $n \in N \cap N_G(T)$. We have three possibilities.

(i) $\dim S > 0$, $S^\circ \subseteq C(G)$. In this case we easily get $n \in B$, applying induction to G/S°.

(ii) $\dim S > 0$, $S^\circ \nsubseteq C(G)$. Then $C_G(S^\circ) \neq G$ is a connected algebraic group, as it is the centralizer of a torus [1, Cor. 11.12], and $n \in C_G(S^\circ)$. By [1, Prop. 11.15] the intersection $B \cap C_G(S^\circ)$ is a Borel subgroup of $C_G(S^\circ)$, hence the induction applied to $C_G(S^\circ)$ gives $n \in B$.

(iii) $\dim S = 0$, that is, S is finite. Consider the map $f : T \to T$, $f(t) = [n, t]$. Since $\ker f = S$ has dimension 0, f is surjective. This means $T \subseteq [N, N]$. By Theorem 4.2 there is a faithful representation $\pi : G \to \mathrm{GL}(V)$ with a line $L \subseteq V$ such that N is the stabilizer of L. Then N acts on L via a character $\chi \in X^*(N)$. Since the image of χ is abelian and consists of semi-simple elements, we have that both $[N, N]$ and the unipotent part B_u of B are in $\ker \chi$. Then also B is in $\ker \chi$. But this means that for any $x \in L$ the orbit map $G \to V$, $g \mapsto g \cdot x$, factors through G/B and so has a complete affine image, that is, is constant. Then $G = N = B$ and everything is proved. □

Recall that a torus over K is an algebraic group isomorphic to $(\mathbb{G}_{m,K})^n$ for some $n \ge 0$.

Corollary 5.1.1. *All maximal tori of G are conjugate.*

Proof. Since a torus is a solvable subgroup, it is contained in a Borel subgroup. By the theorem all Borel subgroups are conjugate, and inside a solvable group all maximal tori are conjugate by [1, Th. 10.6]. □

Let $\mathcal{B} = \mathcal{B}(G)$ denote the set of all Borel subgroups of G. Since G acts on $\mathcal{B}$ transitively, and $N_G(B) = B$, the map

$$\begin{array}{rcl} G/B & \to & \mathcal{B} \\ gB & \mapsto & gBg^{-1} \end{array}$$

is correctly defined and bijective. This allows us to consider $\mathcal{B}$ as a variety, with the structure induced from G/B. Observe that it is a projective variety.

Subgroups of the form $C_G(T)$, where T is a maximal torus of G, are called *Cartan subgroups* of G. Each subgroup $C_G(T)$ is connected, since by [1, Cor. 11.12] all centralizers of tori are connected. We also have $N_G(T)^\circ = C_G(T)^\circ$ ([1, Cor. 8.10]). Hence the quotient group

$$W = W(T, G) = N_G(T)/C_G(T)$$

is finite. It is called the *Weyl group* of G corresponding to T. Observe that since all maximal tori of G are conjugate, the groups $W(T, G)$ corresponding to different T are isomorphic.

For any maximal torus T of G, the set of T-invariant Borel subgroups is precisely the set of all $B \in \mathcal{B}$ such that $T \subseteq B$. We will denote this set by $\mathcal{B}^T$.

Lemma 5.2. *The group $W = W(T, G)$ acts on $\mathcal{B}^T$, and the action is simply transitive. In particular, $|\mathcal{B}^T|$ is finite.*

Proof. Observe that $C_G(T)$ is a nilpotent algebraic group. Indeed, the quotient $C_G(T)/T$ contains no semi-simple elements, since T is the maximal torus of $C_G(T)$, and thus $C_G(T)/T$ is unipotent. Hence both T and $C_G(T)/T$ are nilpotent, so $C_G(T)$ also is. Now the nilpotency of $C_G(T)$ implies that it is contained in a Borel subgroup $B \in \mathcal{B}^T$. The conjugacy of Borel subgroups (Theorem 5.1) and of maximal tori inside a Borel subgroup imply that $C_G(T)$ is contained in any Borel subgroup from $\mathcal{B}^T$. Hence W acts on $\mathcal{B}^T$. The transitivity of the action is implied by the transitivity of the action of G and the conjugacy of tori inside a Borel subgroup. Suppose that $n \in N_G(T)$ satisfies $nBn^{-1} = B$. Then $n \in B \cap N_G(T) = N_B(T)$ by Theorem 5.1. But since B is a solvable group, we have $N_B(T) = C_B(T)$ ([1, Th. 10.6]), so $n \in C_G(T)$. This proves that the action is simply transitive. □

Theorem 5.3. *Let $\alpha : G \to G'$ be a surjective morphism of algebraic groups. Then for any maximal torus T of G, its image $\alpha(T) = T'$ is a maximal torus of G', and α induces surjective maps*

$$\begin{aligned}&\mathcal{B}(G) \to \mathcal{B}(G'),\\ &\mathcal{B}(G)^T \to \mathcal{B}(G')^{T'},\\ &W(T, G) \to W(T', G').\end{aligned}$$

If $\ker\alpha \subseteq \bigcap_{B\in\mathcal{B}(G)} B$, *all these maps are bijective.*

Proof. Let B be a Borel subgroup of G. Then α induces a surjective morphism $G/B \to G'/\alpha(B)$, and hence $G'/\alpha(B)$ is complete, since G/B is. Then $G'/\alpha(B)$ is projective, and $\alpha(B)$ is parabolic by Theorem 5.1. But it is also connected and solvable, so it is a Borel subgroup of G'. The conjugacy of Borel subgroups and the surjectivity of α implies that we get all Borel subgroups of G'.

Now since α preserves the Jordan decomposition, if T is a maximal torus of B, then $T' = \alpha(T)$ is a maximal torus of $\alpha(B)$, and hence of G', because $\alpha(B)$ is a maximal solvable subgroup. Clearly, the map $\mathcal{B}(G)^T \to \mathcal{B}(G')^{T'}$ is also surjective. And the map of Weyl groups is surjective, because they act simply transitively on $\mathcal{B}(G)^T$ and $\mathcal{B}(G')^{T'}$ respectively.

If $\ker\alpha \subseteq \bigcap_{B\in\mathcal{B}(G)} B$, then the map $\mathcal{B}(G) \to \mathcal{B}(G')$ is injective, and this implies the injectivity of all other maps. □

The subgroup $R(G) = \Big(\bigcap_{B\in\mathcal{B}} B\Big)^\circ$ is called the *radical* of G. It is the unique maximal connected solvable normal subgroup of G. Its subgroup $R_u(G) = R(G)_u$ is called the *unipotent radical* of G. It is the unique maximal connected unipotent normal subgroup of G. The group G is called *reductive* (respectively, *semi-simple*) if $R_u(G) = \{e\}$ (respectively, $R(G) = \{e\}$). Theorem 5.3 implies that $G/R_u(G)$ is always reductive, and $G/R(G)$ is always semi-simple.

Lemma 5.4. *If G is a reductive algebraic group, then $R(G) = C(G)^\circ$, and it is a torus.*

Proof. Clearly, $C(G)^\circ \subseteq R(G)$. Since G is reductive, we have $R(G) = R(G)_s$, so it is a torus. The centralizer of a torus is connected ([1, Cor. 11.12]), so $C_G(R(G)) = N_G(R(G))^\circ = G$ and $R(G) \subseteq C(G)^\circ$. □

2. Groups of semi-simple rank 1. The *semi-simple rank* of a connected algebraic group G is the dimension of $T/(T \cap R(G))$, that is, the dimension of a maximal torus in $G/R(G)$.

We are going to study the groups of semi-simple rank 1. The model example of such a group is the projective linear group $\mathrm{PGL}_{2,K} = \mathrm{GL}_{2,K}/C(GL_{2,K})$, whose K-points look as follows. The group

$$C(GL_{2,K})(K) = \left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \mid c \in K^* \right\}$$

is the group of scalar 2×2 matrices. Denote the projection $\mathrm{GL}_{2,K}(K) \to \mathrm{PGL}_{2,K}(K)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The subgroup

$$T(K) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in K^* \right\} = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; a \in K^* \right\}$$

is the group of K-points of the maximal torus T of $\mathrm{PGL}_{2,K}$. Clearly, T has rank 1. The group $\mathrm{PGL}_{2,K}$ is perfect [1, 10.8], hence not solvable, so its semi-simple rank equals 1.

All algebraic groups of semi-simple rank 1 are characterized in the following way.

Lemma 5.5. *Let G be an algebraic group, T a maximal torus of G, and $W = W(T, G)$ the Weyl group of G. The following conditions are equivalent:*

(1) *G has semi-simple rank 1;*
(2) *$|W| = 2$;*
(3) *$G/B \cong \mathbb{P}^1_K$ for any Borel subgroup B of G;*
(4) *there exists a surjective morphism of algebraic groups $\varphi : G \to \mathrm{PGL}_{2,K}$ such that $\ker \varphi = \bigcap\limits_{B \in \mathcal{B}} B$.*

Proof. (1) ⇒ (2) Since G has semi-simple rank 1, it is not solvable, then $\dim(G/B) \geq 1$. As in the proof of Theorem 4.7 we can choose a representation $G \to \mathrm{GL}(V)$ so that G/B is isomorphic to a closed subvariety Gx of $\mathbb{P}(V)$, where V is a K-vector space. Then by [1, Prop. 13.5] T has at least $\dim(G/B) + 1 = 2$ fixed points in G/B, so $\mathcal{B}^T$ has at least two elements. Thus $|W| \geq 2$. On the other hand, by Theorem 5.3 W is isomorphic to the Weyl group of the quotient $G/R(G)$, and the latter is embedded into $\mathrm{Aut}\,(\mathbb{G}_{m,K})$, which consists of two elements; so $|W| \leq 2$.

(2) ⇒ (3) $|W| > 1$ implies that G is not solvable, and hence $\dim(G/B) \geq 1$. As above, T has at least $\dim(G/B) + 1$ fixed points in the projective variety G/B, so $|W| \geq \dim(G/B)+1$. This implies $\dim(G/B) = 1$. Since T acts non-trivially on G/B, we can find a cocharacter $\lambda : \mathbb{G}_{m,K} \to T$ such that $\lambda(\mathbb{G}_{m,K})$ also acts non-trivially. Since G/B is an irreducible variety of dimension 1, for any non-fixed $x \in G/B$ the orbit map $\mathbb{G}_{m,K} \to \lambda(\mathbb{G}_{m,K})x \subseteq G/B$ is dominant. Since G/B is complete, this map can be extended to a dominant morphism $\mathbb{P}^1_K \to G/B$ ([1, AG.18.5]). Then G/B is isomorphic to $\mathbb{P}^1_K$ by [13, Th. 6.3].

(3) ⇒ (4) Since $G/B \cong \mathbb{P}^1_K$, by [1, Prop. 10.8] the action of G on G/B is given by a morphism $\varphi : G \to \mathrm{PGL}_{2,K}$. Clearly, $\ker \varphi = \bigcap\limits_{B' \in \mathcal{B}} B'$. Since G is not solvable, $\varphi(G)$ is not solvable as well. Then $\dim(\varphi(G)) > 2$, as all groups of smaller dimension are solvable, see the proof of Theorem 5.1 (4). Since $\mathrm{PGL}_{2,K}$ is connected and has dimension 3, φ is surjective.

(4) $\Rightarrow$ (1) The existence of φ implies by Theorem 5.3 that the semi-simple rank of G equals that of $\mathrm{PGL}_{2,K}$, and $\mathrm{PGL}_{2,K}$ has semi-simple rank 1. □

The groups of semi-simple rank 1 are actually the "building blocks" for general reductive groups (Theorem 5.7), and their properties summarized in the following theorem are implicitly used in almost every statement that follows.

Theorem 5.6. *Let G be an algebraic group of semi-simple rank 1. Let T be a maximal torus of G, acting on G by conjugation, and let B and B' be the two elements of $\mathcal{B}^T$. Then*

(1) *The unipotent radical B_u of B is isomorphic to $\mathbb{G}_{a,K}$; $\Phi(T,B)$ consists of one element, $\Phi(T,B)=\{\alpha\}$, and $L(B)=L(T)\oplus\mathfrak{g}_\alpha$, where $\dim\mathfrak{g}_\alpha=1$. The subgroup B_u is the unique T-invariant connected subgroup of G with the Lie algebra $\mathfrak{g}_\alpha$. The same holds for B' with $-\alpha$ instead of α.*
(2) *$B\cap B'=T$, and $L(B)\cap L(B')=L(T)$.*
(3) *$L(G)=L(B)+L(B')=L(T)\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}$, and $\Phi(T,G)=\{\alpha,-\alpha\}$.*

Proof. By Lemma 5.5, there is a surjective morphism $\varphi: G\to\mathrm{PGL}_{2,K}$ with the kernel $I=\bigcap_{Q\in\mathcal{B}}Q$. By Theorem 5.3, $\varphi(T)$ is a maximal torus and $\varphi(B)$ a Borel subgroup of $\mathrm{PGL}_{2,K}$. Since all Borel subgroups and all maximal tori are conjugate (Theorem 5.1 and Corollary 5.1.1), composing φ with an automorphism of $\mathrm{PGL}_{2,K}$, we may assume that $\varphi(T)$ is the standard torus $\left\{\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix},\ a,b\in K^*\right\}$ of $\mathrm{PGL}_{2,K}$, and $\varphi(B)$ is the standard Borel subgroup $\left\{\begin{bmatrix} a & c\\ 0 & b\end{bmatrix},\ a,b\in K^*,\ c\in K\right\}$. It is easy to see that the unipotent radical $\varphi(B)_u$ of $\varphi(B)$ coincides with $\left\{\begin{bmatrix} 1 & c\\ 0 & 1\end{bmatrix},\ c\in K\right\}$, and hence $\varphi(B)_u\cong\mathbb{G}_{a,K}$. Since $R_u(G)=\{e\}$, the kernel $\ker(\varphi|_{B_u})$ is finite; then $\dim B_u=1$. But $\mathbb{G}_{a,K}$ is the only connected unipotent K-group of dimension 1, hence $B_u\cong\mathbb{G}_{a,K}$. Being a connected solvable group, B is a semidirect product of its maximal torus T and B_u by [1, 10.6], hence $L(B)=L(T)\oplus L(B_u)$. Since $B_u\cong\mathbb{G}_{a,K}$, it coincides with $L(B_u)$ as a variety, hence $\dim L(B_u)=1$ and T acts on B_u by means of the same unique character $\alpha\in X^*(T)$ as on $L(B_u)$. Since $\varphi(B)$ is not abelian, T acts on B_u non-trivially, so $\Phi(T,B)=\{\alpha\}$.

The matrix $\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}\in PGL_2$ normalizes $\varphi(T)$, so it is a representative of the non-trivial element of the Weyl group $W(\varphi(T),\mathrm{PGL}_{2,K})$ (recall that $|W(\varphi(T),\mathrm{PGL}_{2,K})|=2$). Hence $\varphi(B')$ coincides with $\left\{\begin{bmatrix} a & 0\\ c & b\end{bmatrix},\ a,b\in K^*,\ c\in K\right\}$, and, similarly, $B'_u\cong\mathbb{G}_{a,K}$. It is easy to see that $\varphi(T)$ acts on $\varphi(B)$ and $\varphi(B')$ with two inverse characters; hence the action of T on B'_u is given by $-\alpha$.

Further, $B_u\cap B'_u$ is a T-invariant subgroup of $B_u\cong\mathbb{G}_{a,K}$. Since $B_u\cap B'_u\neq B_u$ and the action of T is a non-trivial linear action, we must have $B_u\cap B'_u=\{e\}$. Consequently, $B\cap B'=T$. By the same token, $L(B)\cap L(B')=L(T)$. Then

$$\dim(L(B)+L(B'))=\dim B+\dim B'_u=\dim B+1,$$

and since by Lemma 5.5 we have $\dim G=\dim B+1$, we obtain

$$L(G)=L(B)+L(B')=L(T)\oplus L(B_u)\oplus L(B'_u).$$

Then also $\Phi(T,G)=\{\alpha,-\alpha\}$.

Now the only thing to prove is that B_u is the unique T-invariant connected subgroup with the Lie algebra $L(B_u)=\mathfrak{g}_\alpha$. Let H be another such subgroup of G. Since

$\dim H = 1$, we have $H \cong \mathbb{G}_{m,K}$ or $H \cong \mathbb{G}_{a,K}$. In the first case T has to centralize H, since $N_G(H)^\circ = C_G(H)^\circ$ by [1, Cor. 8.10], but T acts non-trivially on $L(H)$. Hence $H \cong \mathbb{G}_{a,K}$, in particular, H is unipotent. Then $T \ltimes H$ is a connected solvable subgroup of G containing T, and therefore $T \ltimes H$ coincides with B or B'. Because of the Lie algebra it coincides with B, and the result follows. □

3. **Structure of reductive groups.** From now on, let the group G be a reductive algebraic group. We fix a maximal torus T of G, and set $\Phi = \Phi(T, G)$, the set of roots of G with respect to the conjugation action of T. We will denote by $\mathfrak{g}$ the Lie algebra $L(G)$ of G, and by $\mathfrak{g}_\alpha$ the subspace of $\mathfrak{g} = L(G)$ corresponding to the root $\alpha \in \Phi(T, G)$.

Observe that the Weyl group $W = W(T, G)$ acts on $X^*(T)$ via

$$w(\chi)(t) = \chi(wtw^{-1}), \qquad w \in W,\ \chi \in X^*(T),\ t \in T.$$

Theorem 5.7. *Let G be a reductive algebraic group. Let T be a fixed maximal torus of G, and denote by $\Phi = \Phi(T, G)$ the set of roots of G with respect to the conjugation action of T on G. Then*

(1) *$L(T)$ coincides with the set of T-stable elements of $\mathfrak{g}$, and $\mathfrak{g} = L(T) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$.*
(2) *$R(G) = C(G)^\circ = \Big(\bigcap_{\alpha\in\Phi} T_\alpha\Big)^\circ$, where $T_\alpha = (\ker\alpha)^\circ$.*
(3) *Φ generates a subgroup of finite index in $X^*(T/C(G)^\circ) \subseteq X^*(T)$. If $\alpha, \beta \in \Phi$ are linearly dependent, then $\alpha = \pm\beta$.*
(4) *$G_\alpha = C_G(T_\alpha)$ is a reductive group of semi-simple rank 1, and $L(G_\alpha) = L(T) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$; consequently, $\dim \mathfrak{g}_\alpha = 1$, and $W(T, G_\alpha) \subseteq W(T, G)$ consists of two elements 1 and w_α, where w_α is a reflection on $X^*(T) \otimes \mathbb{Q}$, satisfying $w_\alpha(\alpha) = -\alpha$.*
(5) *For any $\alpha \in \Phi$, there is a unique connected T-invariant subgroup U_α of G such that $L(U_\alpha) = \mathfrak{g}_\alpha$. This subgroup is the unipotent part of a Borel subgroup of G_α containing T; consequently, $U_\alpha \cong \mathbb{G}_{a,K}$ and $G_\alpha = \langle T, U_\alpha, U_{-\alpha}\rangle$.*
(6) *For any $B \in \mathcal{B}^T$ the set of roots $\Phi(B) = \Phi(T, B)$ contains exactly one element of any pair $\{\alpha, -\alpha\}$; hence, $\Phi = \Phi(B) \amalg (-\Phi(B))$. Moreover, there exists $\lambda \in X_*(T)$ such that $\Phi(B) = \{\alpha \in \Phi \mid \langle\alpha, \lambda\rangle > 0\}$.*
(7) *For any $\lambda \in X_*(T)$ such that $\langle\alpha, \lambda\rangle \neq 0$ for any $\alpha \in \Phi$, there exists a unique Borel subgroup $B_\lambda \in \mathcal{B}^T$ such that $\Phi(B_\lambda) = \{\alpha \in \Phi \mid \langle\alpha, \lambda\rangle > 0\}$.*

Proof. See [1, Th. 13.18]; essentially based on Theorem 5.6. □

Corollary 5.7.1. *Let H be a closed connected T-invariant subgroup of G. Then*

$$L(H) = L(T \cap H) \oplus \bigoplus_{\alpha\in\Phi(T,H)} \mathfrak{g}_\alpha,$$

$$H = \langle (T \cap H)^\circ;\ U_\alpha,\ \alpha \in \Phi(T, H)\rangle.$$

Proof. Since the representation of T in $\mathfrak{g}$ is completely reducible, we have

$$L(H) = (L(T) \cap L(H)) \oplus \bigoplus_{\alpha\in\Phi(T,H)} (L(H) \cap \mathfrak{g}_\alpha).$$

Since all $\mathfrak{g}_\alpha$ have dimension 1, the intersection $L(H) \cap \mathfrak{g}_\alpha$ equals $\mathfrak{g}_\alpha$ or 0. By [1, 9.2, Cor.], we also have $L(T \cap H) = L(T) \cap L(H)$, hence the equality for the Lie algebra. To prove the second equality, observe that if $H = C_H(T)$, then $L(H) = L(H \cap T)$ and hence $H = (H \cap T)^\circ$, and the statement is clear. If $H \neq C_H(T)$, by [1, Prop. 9.4] H

is generated by all subgroups $C_H(T_\alpha)^\circ = (H \cap G_\alpha)^\circ$, $\alpha \in \Phi(T,H)$. Hence it is enough to prove the equality for $H \subseteq G_\alpha$. In this case, if H is solvable, then $\alpha \in \Phi(T,H)$ implies $T \ltimes H = T \ltimes U_\alpha$, since $T \ltimes U_\alpha$ and $T \ltimes U_{-\alpha}$ are the only T-invariant Borel subgroups of G_α. If H is not solvable, then it contains unipotent elements of two distinct Borel subgroups, and since H is T-invariant, it contains the whole unipotent radicals of these subgroups. This implies that $T \ltimes H = G_\alpha$, and hence H contains both U_α and $U_{-\alpha}$. □

Corollary 5.7.2. *Let B be a Borel subgroup of G, and let U be its unipotent radical.*

(1) $L(U) = \bigoplus_{\alpha\in\Phi(B)} \mathfrak{g}_\alpha$.

(2) *Denote $L(U)$ by $\mathfrak{u}$. The map*

$$H \mapsto \mathfrak{h} = L(H)$$

is a lattice monomorphism from the lattice Λ of all T-invariant closed subgroups H of U to the lattice of all T-invariant Lie subalgebras $\mathfrak{h}$ of $\mathfrak{u}$.

(3) *For any subgroup $H \in \Lambda$ we have*

$$H = \langle U_\alpha,\ \alpha \in \Phi(T,H)\rangle = \langle U_\alpha,\ \mathfrak{g}_\alpha \subseteq \mathfrak{h}\rangle\,.$$

Moreover, H is connected, and directly spanned by U_α, $\alpha \in \Phi(T,H)$, in the sense that if $\alpha_1,\dots,\alpha_n$ are all roots of $\Phi(T,H)$ in any order, the product morphism

$$U_{\alpha_1} \times \ldots \times U_{\alpha_n} \to H$$

is an isomorphism of varieties.

(4) *If for some $u \in U$ and $H \in \Lambda$ we have $uHu^{-1} \in \Lambda$, then $uHu^{-1} = H$.*

Proof. See [1, Prop. 14.4].

□

Our next goal is to show that $\Phi = \Phi(T,G)$ is an abstract root system in the sense of § 3. For any $\alpha \in \Phi$, let θ_α denote an isomorphism between $\mathbb{G}_{a,K}$ and U_α. We introduce the action of T on $\mathbb{G}_{a,K}$ so that $t\theta_\alpha(x)t^{-1} = \theta_\alpha(\alpha(t)x)$ for any $t \in T$ and $x \in K = \mathbb{G}_{a,K}(K)$.

Lemma 5.8 (Chevalley commutator formula)**.** *Let $\alpha, \beta \in \Phi$ be such that $\alpha \neq \pm\beta$. Then*

$$[\theta_\alpha(x), \theta_\beta(y)] = \prod_{\substack{r,s>0,\\ r\alpha+s\beta\in\Phi}} \theta_{r\alpha+s\beta}(c_{r\alpha+s\beta}x^r y^s)$$

for some constants $c_{r\alpha+s\beta} \in K$ and for all $x, y \in K$.

Proof. For any $\alpha \neq \pm\beta \in \Phi$, since they are linearly independent, we can find a cocharacter $\lambda \in X_*(T)$ such that $\langle\gamma,\lambda\rangle \neq 0$ for any $\gamma \in \Phi$, and both $\langle\alpha,\lambda\rangle$ and $\langle\alpha,\beta\rangle$ are positive. By Theorem 5.7, there is a Borel subgroup $B = B_\lambda \in \mathcal{B}^T$ such that

$$\Phi(T,B) = \Phi(B) = \{\gamma \in \Phi \mid \langle\gamma,\lambda\rangle > 0\}.$$

Then $\alpha, \beta \in \Phi(B)$. Let U be the unipotent radical of B. Then $\Phi(T,B) = \Phi(T,U)$. By Corollary 5.7.2, if $\alpha_1,\dots,\alpha_n$ are all roots of $\Phi(T,U) = \Phi(B)$, the product map

$$U_{\alpha_1} \times \ldots \times U_{\alpha_n} \to U$$

is an isomorphism of varieties. Then $[\theta_\alpha(x), \theta_\beta(y)] = \prod_{1\le i\le n} \theta_{\alpha_i}(P_i(x,y))$ for some polynomials P_i in two variables x, y. Further, for any $t \in T$ we have $t[\theta_\alpha(x), \theta_\beta(y)]t^{-1} =$

$[t\theta_\alpha(x)t^{-1}, t\theta_\beta(y)t^{-1}]$, hence $\alpha_i(t)P_i(x,y) = P_i(\alpha(t)x, \beta(t)y)$. This easily implies that all P_i are monomials, and non-zero only when $\alpha_i = r\alpha + s\beta$. □

Let Ψ be any subset of Φ such that Ψ is closed under addition and $\Psi \subseteq \Phi(B)$ for some Borel subgroup $B \in \mathcal{B}^T$. The subgroup

$$U_\Psi = \langle U_\alpha,\ \alpha \in \Psi \rangle$$

is a T-invariant closed subgroup of U. An easy induction on $|\Psi|$ shows that U_Ψ is the subgroup with Lie algebra

$$L(U_\Psi) = \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha,$$

and hence by Corollary 5.7.2 U_Ψ is directly spanned by U_α, $\alpha \in \Psi$.

Lemma 5.9. *Let $\alpha \in \Phi$, and let $w_\alpha \in W = W(T,G)$ be the generator of the subgroup $W(T, G_\alpha)$. Then for any $\beta \in \Phi$ we have*

$$w_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha,$$

where $n_{\beta,\alpha} \in \mathbb{Z}$. In particular, $n_{\alpha,\alpha} = 2$.

Proof. By Theorem 5.7 $w_\alpha(\alpha) = -\alpha$, hence the case $\alpha = \pm\beta$ is clear. Suppose that $\alpha \neq \beta$. Consider the set

$$\Psi = \{r\alpha + s\beta \in \Phi \mid r, s \in \mathbb{Z},\ s > 0\}.$$

It is, clearly, closed under addition, and we can find a Borel subgroup $B \in \mathcal{B}^T$ such that $\Psi \subseteq \Phi(B)$ as in the proof of Lemma 5.8. Further, for any $\gamma \in \Psi$ we have $i\gamma \pm j\alpha \in \Psi$ for any $i, j > 0$, hence U_α and $U_{-\alpha}$ normalize U_Ψ by Lemma 5.8. Then the subgroup G_α also does. Consider $w_\alpha \in W$ as an element of $N_{G_\alpha}(T)$. Then $w_\alpha U_\beta w_\alpha^{-1} \subseteq U_\Psi$. But $w_\alpha U_\beta w_\alpha^{-1} = U_{w_\alpha(\beta)}$, hence $\mathfrak{g}_{w_\alpha(\beta)} \subseteq L(U_\Psi)$ and $w_\alpha(\beta) \in \Psi$. Then $w_\alpha(\beta) = r\alpha + s\beta$, $r, s \in \mathbb{Z}$. The fact that w_α has order 2 implies that $s = 1$. □

Theorem 5.10. *Let V be the $\mathbb{Q}$-vector space $X^*(T/C(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{Q}$, identified canonically with a subspace of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\Phi = \Phi(T,G)$ is a abstract root system in V with Weyl group $W = W(T,G)$.*

Proof. By the very definition of $\Phi(T,G)$, it is finite and does not contain 0. By Theorem 5.7 $\Phi(T,G)$ generates $V = X^*(T/C(G)^\circ) \otimes \mathbb{Q}$, and posesses the necessary reflections w_α. By Lemma 5.9 these reflections act with integral coefficients. Also by Theorem 5.7 the elements $\pm\alpha\Phi(T,G)$ are the only elements linearly dependent with $\alpha \in \Phi(T,G)$. Hence Φ is a root system.

Let W_Φ be the Weyl group of Φ as an abstract root system. Clearly, $W_\Phi \subseteq W$. Recall that W acts simply transitively on the set of Borel subgroups $B \in B^T$, hence $|W| = |\mathcal{B}^T|$. On the other hand, by Theorem 5.7, for each Borel subgroup $B \in \mathcal{B}^T$ there is an element $\lambda \in X_*(T)$ such that $\Phi(B) = \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle > 0\}$. Since $\Phi = \Phi(B) \amalg (-\Phi(B))$, this element λ is regular in the sense of Theorem 3.1, and therefore $\Phi(B)$ contains a system of simple roots. Hence each B defines a system of simple roots in Φ, and these systems are distinct, since the Lie algebras of Borel subgroups are. The Weyl group W_Φ acts simply transitively on the systems of simple roots (Theorem 3.1), so $|W_\Phi| \geq |\mathcal{B}^T|$. This means that $W_\Phi = W$. □

4. The Bruhat decomposition. As above, we fix a maximal torus T of G and set $\Phi = \Phi(T, G)$. Let us fix also a Borel subgroup $B \supseteq T$. By Theorem 5.10 Φ is an abstract root system, hence by Theorem 5.7 combined with Theorem 3.1 we can choose a (unique) system of simple roots in Φ such that $\Phi(B) = \Phi^+$, the set of positive roots with respect to Π. Theorem 5.7 also implies the existence of a unique Borel subgroup B^- of G such that $\Phi(B^-) = \Phi^- = -\Phi(B)$, the set of negative roots. We denote the unipotent radicals of B and B^- by U and U^- respectively.

Recall that an element $w \in W$ is actually a coset of $N_G(T)$ modulo $C_G(T)$. We allow ourselves to write w instead of its representative $n \in N_G(T)$ in those formulas that do not depend on the choice of such a representative.

We set

$$U_w = U \cap wUw^{-1} \quad \text{and} \quad U'_w = U \cap wU^-w^{-1}.$$

By Corollary 5.7.2 these subgroups are generated by all subgroups U_α such that α is in the set

$$\Phi(U_w) = \Phi^+_w = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^+\} \quad \text{or} \quad \Phi(U'_w) = \Phi'_w = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\}$$

respectively.

Lemma 5.11. *If $w, w' \in W$ and $\Phi^+_w = \Phi^+_{w'}$, then $w = w'$.*

Proof. See [2]. □

Theorem 5.12. (1) **Bruhat decomposition of G.** *The group G is the disjoint union of double cosets BwB, $w \in W$:*

$$G = \coprod_{w \in W} BwB.$$

For any $w \in W$ the morphism

$$\begin{array}{rcl} U'_w \times B & \to & BwB \\ (x, y) & \mapsto & xwy \end{array}$$

is an isomorphism of varieties.

(2) **Cellular decomposition of G/B.** *The variety G/B is the disjoint union of the U-orbits Uwx_0, $w \in W$:*

$$G/B = \coprod_{w \in W} Uwx_0,$$

where $x_0 \in G/B$ is the image of B under the projection $G \to G/B$. For any $w \in W$ the morphism

$$\begin{array}{rcl} U'_w & \to & Uwx_0 \\ u & \mapsto & uwx_0 \end{array}$$

is an isomorphism of varieties.

Proof. Observe that since $B = T \ltimes U$ and W normalizes T, we have $BwB = UwB$ and $Bwx_0 = Uwx_0$. Hence (1) and (2) are essentially equivalent, and it's enough to prove that

1) if $w, w' \in W$ and $Uwx_0 = Uw'x_0$, then $w = w'$;
2) $G = BWB$;
3) the map of varieties $U'_w \times B \to BwB$, defined in the theorem, is an isomorphism.

Proof of 1). The equality $Uwx_0 = Uw'x_0$ means that there is an $u \in U$ such that $w'x_0 = uwx_0$. But this implies that

$$\begin{aligned} U_{w'} = U \cap w'Uw'^{-1} &= U \cap w'Bw'^{-1} \\ &= U \cap uwBw^{-1}u^{-1} \\ &= u(U \cap wBw^{-1})u^{-1} \\ &= uU_w u^{-1}. \end{aligned}$$

Then $uU_wu^{-1} = U_{w'}$ is a T-invariant subgroup of U, hence $U_{w'} = U_w$ by Corollary 5.7.2. This also implies $\Phi(U_w) = \Phi(U_{w'})$, and then $w' = w$ by Lemma 5.11.

Proof of 2). First we see that 2) holds if G has semi-simple rank 1. In this case $|W| = 2$, and by 1) it's enough to show that U has only 2 orbits on G/B. Clearly, U acts non-trivially on the point $x \in G/B$ corresponding to the opposite Borel subgroup B^-. Since G/B is a complete variety, if we identify $U \cong \mathbb{G}_{a,K}$ with $\mathbb{P}^1_K$ minus a point, the orbit map $U \to Ux \subseteq G/B$ extends to a map $\mathbb{P}^1_K \to G/B$ by [1, AG.18.5]. By Lemma 5.5, we have $G/B \cong \mathbb{P}^1_K$; since the image of the above map should be complete and one-dimensional, it is equal to G/B. Thus G/B is a union of Ux and the unique point, which is nothing but x_0.

For a general G, this implies that $G_\alpha x = (U_\alpha x) \cup (U_\alpha w_\alpha x)$ for any $\alpha \in \Phi$, $x \in \mathcal{B}^T$. Indeed, if we consider the Borel subgroup $B_x \cap G_\alpha$ of G_α, with B_x being the Borel subgroup of G, corresponding to x, it is either the subgroup with the unipotent radical U_α, and then it is just the above statement, or the subgroup with the unipotent radical $U_{-\alpha} = w_\alpha U_\alpha w_\alpha$. In the latter case the above statement gives $G_\alpha x = (w_\alpha U_\alpha w_\alpha x) \cup (w_\alpha U_\alpha x)$, which implies $G_\alpha x = (U_\alpha w_\alpha x) \cup (U_\alpha x)$.

Further, we note that for any $\alpha \in \Pi$ the set of roots $\Psi = \Phi^+ \setminus \{\alpha\}$ is closed; the corresponding group U_Ψ is normalized by T, and by U_α, $U_{-\alpha}$, since for any $\gamma \in \Psi$ the combination $r\alpha + s\gamma$, $s > 0$, if it is a root, is a positive root, different from α. Hence U_Ψ is normalized by G_α by Lemma 5.8. We have $B = UT = U_\alpha U_\Psi T$, which implies that

$$\begin{aligned} G_\alpha Bx &= G_\alpha U_\alpha U_\Psi Tx = G_\alpha U_\Psi Tx \\ &= G_\alpha U_\Psi x = U_\Psi G_\alpha x \\ &= U_\Psi(U_\alpha w_\alpha x) \cup U_\Psi(U_\alpha x) \\ &= Uw_\alpha x \cup Ux \end{aligned}$$

for any $x \in \mathcal{B}^T$. Then we have

$$G_\alpha BwB = (Uw_\alpha wB) \cup (UwB) \subseteq BWB.$$

But the subgroups G_α, $\alpha \in \Pi$, generate G, hence $G = G \cdot BWB \subseteq BWB$, and 2) is proved.

Proof of 3). We prove that the map

$$\begin{aligned} f : U'_w \times B &\to BwB \\ (x, y) &\mapsto xwy \end{aligned}$$

is an isomorphism. Since $U_w = U \cap wUw^{-1}$, we have $U_w w \subseteq wU$. Analogously, $U'_w w \subseteq wU^-$. Then

$$BwB = UwB = (U'_w U_w)wB = U'_w wB,$$

so f is surjective. And f is injective, since $U'_w w \subseteq wU^- \cap Uw$ implies $U'_w w \cap B = \{e\}$.

Now if f is separable, it is an isomorphism by [1, Th. AG.18.2]. On the other hand, by [1, Th. AG.17.3] f is separable, if

$$(df)_{(e,e)} : T_{(e,e)}(U'_w \times B) \to T_w(BwB)$$

is surjective. But since $\dim(BwB) \le \dim(U'_w \times B)$, the latter is equivalent to the injectivity of $(df)_{(e,e)}$. So let us prove that $(df)_{(e,e)}$ is injective. We have $T_{(e,e)}(U'_w \times B) = T_eU'_w \oplus T_eB$ and

$$(df)_{(e,e)}(X,Y) = d(f\circ i_1)_e X + d(f \circ i_2)_e Y \quad \text{for any } X \in T_eU'_w,\ Y \in T_eB,$$

where i_1 and i_2 denote the natural embeddings

$$\begin{array}{rcl} i_1 : U'_w & \to & U'_w \times B \\ x & \mapsto & (x,e) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} i_2 : B & \to & U'_w \times B \\ x & \mapsto & (e,x) \end{array}.$$

But the maps $f \circ i_1$ and $f\circ i_2$ are just the right and the left multiplication by w inside G, hence $(df)_{(e,e)}$ is the sum of two isomorphisms

$$d(f\circ i_1)_e : T_eU'_w \to T_w(U'_w w) \quad \text{and} \quad d(f\circ i_2)_e : T_eB \to T_w(wB).$$

Since $\Phi(T,U'_w) = \Phi'_w = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\}$ does not intersect $\Phi(T, wBw^{-1}) = w(\Phi^+)$, we have $T_w(U'_w w) \cap T_w(wB) = 0$. Hence $\ker(df)_{(e,e)} = 0$. □

5. **Parabolic subgroups.** We keep the notation of the previous subsection. In particular, T is a fixed maximal torus, and $B \subseteq T$ a fixed Borel subgroup of the reductive group G.

A parabolic subgroup P of G is called *standard*, if it contains B. Since all Borel subgroups of G are conjugate, any parabolic subgroup is conjugate to a standard parabolic subgroup. This standard parabolic subgroup is moreover unique, since if parabolic subgroups P and $Q = gPg^{-1}$ both contain B, then $B = p(gBg^{-1})p^{-1}$ for some $p \in P$, since all Borel subgroups of P are conjugate by Theorem 5.1. By the same theorem this means $pg \in B$, and hence $P = Q$.

For any subset $I \subseteq \Pi$ we will denote by W_I the subgroup of $W = W(\Phi)$, generated by all w_α, $\alpha \in I$. This subgroup is the Weyl group of the root subsystem Δ_I of Φ, generated by I (see § 3).

Theorem 5.13. *Let G be a reductive algebraic group, T a maximal torus of G, B a Borel subgroup containing T.*

(1) *The map $I \mapsto BW_IB$ is a bijection of the set of all subsets of Π onto the set of all standard parabolic subgroups of G.*
(2) *Let P, Q be two standard parabolic subgoups of G corresponding to the subsets $I, J \subseteq \Pi$. Then*

$$G = \coprod_{w\in\tilde{W}} QwP$$

for any set $\tilde{W}$ of representatives of double cosets $W_J\backslash W/W_I$.

Lemma 5.14. *For any w_α, $\alpha \in \Pi$, and any $w \in W$ we have*

(1) $w_\alpha BwB \subseteq Bw_\alpha wB \cup BwB$ *and* $BwBw_\alpha \subseteq Bww_\alpha B \cup BwB$;
(2) *if* $l(w_\alpha w) = l(w) + 1$, *then* $w_\alpha BwB \subseteq Bw_\alpha wB$;
(3) *if* $l(w_\alpha w) = l(w) - 1$, *then* $w_\alpha BwB \cap BwB \neq \emptyset$.

Proof. The first inclusion in (1) follows from the equality $G_\alpha BwB = (Uw_\alpha wB) \cup (UwB)$ appearing in the proof of Theorem 5.12. The second one follows from the first if we invert both sides.

To prove (2), we use the induction on $l(w)$. Considering any reduced decomposition of w, we can write $w = w'w_\beta$, where $w' \in W$, $\beta \in \Pi$ and $l(w') = l(w) - 1$. Suppose that $w_\alpha BwB \not\subseteq Bw_\alpha wB$; then $w_\alpha BwB \cap BwB \neq \emptyset$, and hence $w_\alpha Bw \cap BwB \neq \emptyset$. Then also $w_\alpha Bww_\beta \cap BwBw_\beta \neq \emptyset$. But

$$l(w_\alpha w') \geq l(w_\alpha w'w_\beta) - 1 = l(w_\alpha w) - 1 \geq l(w) - 1 = l(w'),$$

so by the hypothesis $w_\alpha Bww_\beta B = w_\alpha Bw'B \subseteq Bw_\alpha w'B$, which implies $BwBw_\beta \cap Bw_\alpha w'B \neq \emptyset$. By (1) we have

$$wBw_\beta B \subseteq Bww_\beta B \cup BwB = Bw'B \cup BwB,$$

hence $Bw_\alpha w'B$ intersects one of $Bw'B$, BwB and hence coincides with it by Theorem 5.12. But $w_\alpha w' = w'$ is impossible, and $w_\alpha w' = w$ is impossible as well, because $l(w') < l(w) \leq l(w_\alpha w)$. This proves the claim.

It leaves to prove (3). Observe that $w_\alpha Bw_\alpha \not\subseteq B$, since $w_\alpha U_\alpha w_\alpha = U_{-\alpha} \not\subseteq B$. Since by (1) we have $w_\alpha Bw_\alpha B \subseteq B \cup Bw_\alpha B$, this implies $w_\alpha Bw_\alpha \cap Bw_\alpha B \neq \emptyset$. Muptiplying this by $w_\alpha w$, we get $w_\alpha Bw \cap Bw_\alpha Bw_\alpha w \neq \emptyset$. If $l(w_\alpha^2 w) = l(w) \geq l(w_\alpha w)$, by (2) $Bw_\alpha Bw_\alpha w \subseteq BwB$. Hence $w_\alpha Bw \cap BwB \neq \emptyset$. □

Proof of Theorem 5.13. (1) Lemma 5.14 easily implies that each BW_IB is a subgroup of G. It is a closed subgroup, since it is generated by B and by G_α, $\alpha \in I$, which are all irreducible subvarieties of G containing 1, hence generate a closed subgroup by [1, Prop. 2.2]. So each BW_IB is a standard parabolic subgroup of G. Conversely, let P be any standard parabolic subgroup of G. Set $W_P = P \cap W$. Since $B \subseteq P$, by Theorem 5.12 we have $P = \coprod_{w \in W_P} BwB$. Set $I = \{\alpha \in \Pi \mid w_\alpha \in W_P\}$. Obviously, $BW_IB \subseteq P$. To prove the inverse inclusion, consider any $w \in W_P$. Let $w = w_{\alpha_1} \dots w_{\alpha_r}$ be a reduced decomposition of w with respect to Π. Since $l(w_{\alpha_1} w) < l(w)$, by Lemma 5.14 we have $w_{\alpha_1} BwB \cap BwB \neq \emptyset$, which implies $w_{\alpha_1} \in P$. Proceeding by induction, we see that all w_{α_i}, $1 \leq i \leq r$, are in P, and hence in W_I. Then $w \in BW_IB$, and also $P = \coprod_{w \in W_P} BwB \subseteq BW_IB$.

(2) By (1) we know that $Q = BW_JB$ and $P = BW_IB$. Since $W_J\tilde{W}W_I = W$, the Bruhat decomposition for G implies that $G = \bigcup_{w \in \tilde{W}} QwP$. Suppose that $QwP = Qw'P$ for some $w, w' \in \tilde{W}$, that is, $w' \in BW_JBwBW_IB$. By Lemma 5.14 we have $W_JBwBW_I \subseteq BW_JwW_IB$, hence $w' \in BW_JwW_IB$. Then the Bruhat decomposition implies that $w' \in W_JwW_I$. □

We say that a parabolic subgroup P of G is a parabolic subgroup *of type* I, if P is conjugate to the standard parabolic subgroup BW_IB.

For a subset $I \subseteq \Pi$, we set

$$T_I = \Big(\bigcap_{\alpha \in I} \ker \alpha\Big)^\circ.$$

Theorem 5.15. *Let $P_I = BW_IB$ be a standard parabolic subgroup of a reductive group G, corresponding to the subset $I \subseteq \Pi$. Set $\Sigma_I = \Phi^+ \setminus \Delta_I$. Then*

(1) *$C_G(T_I)$ is a reductive algebraic group, $\Phi(T, C_G(T_I)) = \Delta_I$.*

(2) $P_I = C_G(T_I) \ltimes R_u(P_I)$, *and the unipotent radical* $R_u(P_I)$ *of* P_I *is equal to* $U_{\Sigma_I} = \langle U_\alpha,\ \alpha \in \Sigma_I \rangle$.
(3) *We have* $T_I = C(C_G(T_I))^\circ = C_G(T_I) \cap R(P_I)$, *it is a maximal torus and a Cartan subgroup of the radical* $R(P_I)$.

Proof. Denote $C_G(T_I)$ by H. It is is a closed connected subgroup of G by [1, Cor. 11.12], and it contais T. Therefore $H = \langle T;\ U_\alpha,\ \alpha \in \Phi(T,H)\rangle$ by Corollary 5.7.1. If for some $\alpha \in \Phi$ the group U_α centralizes T_I, then $U_{-\alpha}$ also does. This means that H is in fact generated by G_α, $\alpha \in \Phi(T,H)$. Now if $R_u(H)$ is non-trivial, being a closed connected T-invariant subgroup of G, it contains some U_α, $\alpha \in \Phi(T,H)$. But then $U_\alpha \subseteq (R_u(H) \cap G_\alpha)^\circ \subseteq R_u(G_\alpha)$, and $R_u(G_\alpha) = \{e\}$ by Theorem 5.7. This proves that H is reductive.

Clearly, all U_α, $\alpha \in \Delta_I$, centralize T_I. Hence $\mathfrak{g}_\alpha \subseteq L(H)$, $\alpha \in \Delta_I$, which implies $\Delta_I \subseteq \Phi(T,H)$. Conversely, if for some $\beta \in \Phi$ the group U_β centralizes T_I, this means that $T_I \subseteq \ker\beta$. But if $\beta \notin \Delta_I$, the roots in I and β are linearly independent. This means that there exists $\lambda \in X_*(T)$ such that $\langle \alpha, \lambda\rangle = 0$ for all $\alpha \in I$ and $\langle \beta, \lambda\rangle \neq 0$. Then $\lambda(G_m) \subseteq T$ is contained in T_I, but not in $\ker\beta$. Hence $\Phi(H) = \Delta_I$, and (1) is proved.

Further, by the definition of Σ_I and by Lemma 5.8 the group H normalizes U_{Σ_I}. Then $U_{\Sigma_I} \cap H$ is a normal unipotent subgroup of H, and hence $U_{\Sigma_I} \cap H$ is finite, because H is reductive. But then $U_{\Sigma_I} \cap H$ is central in H, since H is connected and acts on $U_{\Sigma_I} \cap H$. But $C(H)$ is contained in all Borel subgroups of H, and hence in T, which means that $U_{\Sigma_I} \cap H$ consists of semi-simple elements. Since it is unipotent, we have $U_{\Sigma_I} \cap H = \{e\}$. Since the Lie algebras $L(U_\sigma)$ and $L(H)$ do not intersect as well, the product $X = H \cdot U_{\Sigma_I}$ inside G is the semi-direct product $H \ltimes U_{\Sigma_I}$. Clearly, $R_u(X) = U_{\Sigma_I}$. Since $B \subseteq X$, it is a standard parabolic subgroup. Since both H and U_{Σ_I} are contained in P_I, we have $P_I = X$. This proves (2).

By Theorem 5.7 applied to the reductive group H, we have $T_I = C(H)^\circ = R(H)$, since $\Phi(T,H) = \Delta_I$. Moreover, $T_I \ltimes U_{\Sigma_I}$ is normal in P_I and solvable, hence contained in $R(P_I)$. Then $T_I \subseteq H \cap R(P_I)$. Since T_I is the radical of H, we have $T_I = H \cap R(P_I)$. This also shows that $R(P_I) = T_I \ltimes U_{\Sigma_I}$, and so T_I is a maximal torus of $R(P_I)$. Finally, T_I coincides with its centralizer in $R(P_I)$, since $C_G(T_I) = H$ by the definition of H. This finishes the proof of (3). □

§ 6. Split and non-split reductive algebraic groups

Let G be a reductive algebraic K-group defined over k. We are interested in whether the subgroups of G considered in § 5, such as maximal tori, Borel subgroups, parabolic subgroups etc., are in fact k-defined subgroups, and under what conditions. Recall that whenever we say that a subvariety resp. a subgroup of a k-defined group G is defined over k, we mean, on one hand, that it is a k-defined variety resp. a k-defined group, and on the other hand, that its embedding into G is defined over k.

We will use the following fundamental result: any k-defined algebraic group G contains a maximal torus T defined over k, and the corresponding Cartan subgroup $C_G(T)$ is also k-defined [6, Exp. XIV, Th. 1.1].

1. Structure of k-defined reductive groups. From now on, let G be a reductive algebraic K-group defined over k. Recall that a torus T defined over k is called k-split, if ${}_kT$ is isomorphic to $(\mathbb{G}_{m,k})^n$ for some $n \geq 0$. The group G is called *isotropic* over k, if it contains a non-trivial k-split torus, and *anisotropic* over k otherwise. Further, G is called *k-split*, if it contains a k-split maximal torus.

By [1, Prop. 8.11] any k-defined torus T splits over a finite separable extension of k; in particular, any algebraic group defined over k is k_s-split.

Theorem 6.1. *Let G be k-split reductive group with a k-split maximal torus T. Then the subgroups U_α, $\alpha \in \Phi = \Phi(T,G)$, are defined over k, and the isomorphisms $\theta_\alpha : \mathbb{G}_{a,K} \to U_\alpha$, $\alpha \in \Phi$, of § 5, subsection* **3**, *can be chosen so that they are extensions to K of k-isomorphisms $\mathbb{G}_{a,k} \to {}_k(U_\alpha)$.*

Proof. [1, Th. 18.7] □

Corollary 6.1.1. *Let G be k-split reductive group with a k-split maximal torus T. Then the Weyl group $W = W(T,G)$ is defined over k in the sense that every coset in $N_G(T)/C_G(T)$ contains an element of $G(k)$.*

Proof. For any $\alpha \in \Phi$, the subgroup $G_\alpha = \langle T, U_\alpha, U_{-\alpha}\rangle$, $\alpha \in \Phi$, is defined over k by Theorem 6.1. By Theorem 5.7 it is a reductive algebraic group of semi-simple rank 1. Set $B_\alpha = T \ltimes U_\alpha$ and $B_{-\alpha} = T \ltimes U_{-\alpha}$. These are Borel subgroups of G_α, and they are as well defined over k. By [1, Prop. 15.2] the group B_α, acting by translations on the complete k-defined variety $G_\alpha/B_{-\alpha} \cong \mathbb{P}^1_K$, has a fixed point in $(G_\alpha/B_{-\alpha})(k)$. By [1, Cor. 15.7] the restriction $G_\alpha(k) \to (G_\alpha/B_{-\alpha})(k)$ of the natural projection $\pi : G_\alpha \to G_\alpha/B_{-\alpha}$ is surjective, hence there is an element $g \in G_\alpha(k)$ such that $B_\alpha g B_{-\alpha} = gB_{-\alpha}$. Then $g^{-1}B_\alpha g = B_{-\alpha}$. Let $n \in N_{G_\alpha}(T)$ be any representative of the class $w_\alpha \in W$ (recall that $W(T,G_\alpha) \subseteq W$). Then $nB_\alpha n^{-1} = B_{-\alpha}$ as well, hence $gn \in B_\alpha \cap B_{-\alpha} = T$, because B_α and $B_{-\alpha}$ are self-normalizable by Theorem 5.1. This shows that for any $\alpha \in \Phi$ the coset in $N_G(T)/C_G(T)$ corresponding to w_α contains an element of $G(k)$. Since W is generated by w_α, $\alpha \in \Phi$, this means that every coset contains an element of $G(k)$. □

Corollary 6.1.2. *Let G be a k-split reductive group with a maximal k-defined torus T, B a Borel subgroup of G containing T, U the unipotent radical of B. For any T-invariant closed subgroup H of U, if $\alpha_1, \dots, \alpha_n$ are all roots of $\Phi(T,H)$ in any order, then the product morphism*

$$ {}_k(U_{\alpha_1}) \times \ldots \times {}_k(U_{\alpha_n}) \to {}_k(H)$$

is an isomorphism of k-varieties. Consequently, ${}_kH$ is isomorphic to $\mathbb{A}^n_k$ as a variety.

Proof. Follows from Corollary 5.7.2 and the fact that an isomorphism defined over k is a k-isomorphism ([10, Prop. 2.7.1]). □

We see that if G is k-split, all T-invariant connected subgroups of G are k-defined. In the general case the criterion is given by the following lemma.

Lemma 6.2. *Let G be a k-defined reductive group, and let T be a maximal torus of G defined over k. A closed connected T-invariant subgroup H of G is defined over k if and only if both $(H\cap T)^\circ(k_s)$ and $\Phi(T,H)$ are $\Gamma = \mathrm{Gal}\,(k_s/k)$-invariant.*

Proof. The direct implication is clear. To prove the inverse, we observe that the torus T is k_s-split ([1, Prop. 8.11]), and hence the corresponding subgroups U_α, $\alpha \in \Phi(T,G)$, are defined over k_s by Theorem 6.1. On the other hand, $(T\cap H)^\circ$ is defined over k_s, since any closed subgroup of a split torus is defined over the field where it splits. hence the groups $U_\alpha(k_s)$, $\alpha \in \Phi(T,H)$, are permuted by Γ, and $(H\cap T)^\circ(k_s)$ is Γ-invariant. Then the closure of $\langle (H\cap T)^\circ(k_s);\ U_\alpha(k_s),\ \alpha \in \Phi(T,H)\rangle$ is defined over k by [1, Th. AG.14.4]. Since for any K-variety V the set $V(k_s)$ is dense in V, this closure coincides with H by Corollary 5.7.1. □

Theorem 6.3. *Let G be a k-defined reductive algebraic group.*

(1) *If S is a k-split subtorus of G, then there exists a k-defined parabolic subgroup P of G such that $P = C_G(S) \ltimes R_u(P)$.*
(2) *Conversely, let P be a proper k-defined parabolic subgroup of G. Then $R(P)$, $R_u(P)$ are defined over k. If S_0 is a maximal k-defined torus of $R(P)$ and S is a maximal k-split subtorus of $R(P)$ contained in S_0, then $C_G(S_0) = C_G(S)$ is also k-defined and $P \cong C_G(S) \ltimes R_u(P)$ is a k-defined isomorphism.*
(3) *In the above setting, P is a minimal k-defined parabolic subgroup of G if and only if S is a maximal k-split torus of G.*
(4) *All minimal k-defined parabolic subgroups of G are conjugate under $G(k)$.*

Proof. See [1, Prop. 20.4–20.9]. □

Corollary 6.3.1. *If P and P' are two k-defined parabolic subgroups of G conjugate under $G(K)$, then they are conjugate under $G(k)$.*

Proof. Let Q and Q' be two minimal k-defined parabolic subgroups contained in P and P' respectively. By Theorem 6.3 we have $gQg^{-1} = Q'$ for an element $g \in G(k)$. Then $gPg^{-1} \cap P' \supseteq Q'$, hence $gPg^{-1} \cap P'$ contains a Borel subgroup (not necessarily defined over k) of G. Since P' and P are conjugate, this means that $gPg^{-1} = P'$. □

2. **The $*$-action of the Galois group.** From now on, let S be a maximal (k-defined and) k-split torus of a k-defined reductive group G, and let T be a maximal k-defined torus of G containing S. The group G is called *quasi-split* over k, if $C_G(S) = T$. By Theorem 6.3 this is equivalent to the existence of a k-defined Borel subgroup of G.

Let us define what is called the *$*$-action* of $\Gamma = \mathrm{Gal}\,(k_s/k)$. Observe that Γ acts by permutations on the set of all conjugacy classes of parabolic subgroups of G. Clearly, it also acts on its subset consisting of conjugacy classes of maximal proper parabolic subgroups. By Theorem 5.15 these classes are in one-to-one correspondence with the elements of a fixed system of simple roots Π of the root system Φ of G. The corresponding action of Γ on Π is called the $*$-action. There is also an equivalent way to define it, based on the natural action of Γ on the group of characters $X^*(T)$, which contains Φ. For any $\sigma \in \Gamma$, $\sigma(\Pi)$ is a system of simple roots for Φ; then by Theorem 3.1 there is a unique element w_σ of the Weyl group of G such that $w_\sigma(\sigma(\Pi)) = \Pi$. We set $\sigma^* = w_\sigma \circ \sigma$. The action of Γ on Π via σ^*, $\sigma \in \Gamma$, is the same as the one defined above.

We say that the group G over k is of *inner* (resp. *outer*) *type*, if the $*$-action on Π is trivial (resp. non-trivial).

All parabolic subgroups of G conjugate to the standard parabolic subgroup P of type $J \subseteq \Pi$, that is, all parabolic subgroups of type J, are in one-to-one correspondence with all (closed) points of the quotient variety G/P (Theorem 5.1). Recall (Theorem 4.7) that we construct G/P as a G-orbit for an action of G on some projective space $\mathbb{P}^n_K$. Even if P is not k-defined, we can still require that this action is k-defined, i.e. comes from an action of ${}_kG$ on $\mathbb{P}^n_k$ (cf. the proof of Theorem 4.2). We call the variety G/P endowed with such an embedding *the variety of parabolic subgroups of G of type J*, and say that it is *defined over k*, if it is a k-defined subvariety of $\mathbb{P}^n_K$ with respect to ${}_k(\mathbb{P}^n_K) = \mathbb{P}^n_k$.

Lemma 6.4. *Let P be a parabolic subgroup of G of type $J \subseteq \Pi$. The projective variety G/P of parabolic subgroups of type J is defined over k if and only if J is stable under the $*$-action. If it holds, the variety ${}_k(G/P)$ is a smooth projective k-variety.*

Proof. Since G is k_s-split, by Lemma 6.2 any parabolic subgroup P is defined over k_s, and hence G/P also is. Then G/P is defined over k, that is, stable under the action of Γ on $\mathbb{P}^n_K$, if and only if Γ preserves the conjugacy class of P. But this means precisely that the subset $J \subseteq \Pi$, corresponding to this conjugacy class, is stable under the $*$-action. Further, it is clear that ${}_k(G/P)$ is projective, since it is the image of G/P under the natural projection $\mathbb{P}^n_K \to \mathbb{P}^n_k$. The smoothness of ${}_k(G/P)$ follows from the smoothness of G/P by [12, Prop. 17.7.1]. □

Lemma 6.5. *Suppose that the minimal k-defined parabolic subgroups of G are of type $I_0 \subseteq \Pi$. Then G contains a k-defined parabolic subgroup of type I if and only if $I_0 \subseteq I$ and I is stable under the $*$-action.*

Proof. If P is a k-defined parabolic subgroup of type I, then P contains a minimal k-defined parabolic subgroup; assuming it to be standard, we see that $I_0 \subseteq I$ by Theorem 5.13. The variety G/P is defined over k by Theorem 4.7, hence I is stable under the $*$-action by Lemma 6.4.

To prove the converse, let P_0 be a standard minimal k-defined parabolic subgroup of G, containing the k-defined torus T, and let P be a standard parabolic subgroup of type I, containing P_0. Consider $\sigma \in \Gamma$. We have defined an element $w_\sigma \in W = W(T, G)$ such that $\sigma^* = w_\sigma \circ \sigma$. Since P_0 is k-defined, by Lemma 6.2 the set of roots $\Phi(T, P_0)$ is invariant under σ^{-1}. Further, recall that $\sigma^*(\Pi) = \Pi$, and I_0 is stable under σ^*. Then the set $\Phi(T, P_0)$, being the union of Φ^+ and of the root subsystem Δ_{I_0} generated by I_0, is also σ^*-stable. Hence $\Phi(T, P_0)$ is invariant under $w_\sigma = \sigma^* \circ \sigma^{-1}$. Since T is also invariant under the action of w_σ, this means that w_σ normalizes P_0. Since $P_0 = N_G(P_0)$, we have $w_\sigma \in P_0 \subseteq P$, and hence $\Phi(T, P)$ is invariant under w_σ. But $\Phi(T, P)$ is also σ^*-invariant, hence it is invariant under $\sigma = w_\sigma^{-1} \circ \sigma^*$. Thus we have proved that $\Phi(T, P)$ is invariant under all $\sigma \in \Gamma$. Since P contains T, it is therefore k-defined by Lemma 6.2. □

The data consisting of the $*$-action of Γ on the system of simple roots Π, and of the type $I_0 \subseteq \Pi$ of minimal k-defined parabolic subgroups of G, is called *the Tits index* of G (see [20]).

Lemma 6.6. *If G is a quasi-split semi-simple algebraic group defined over k, and the $*$-action on Π is trivial, then G is a k-split algebraic group.*

Proof. We need to prove that G contains a k-split maximal torus. Clearly, we can assume that the standard Borel subgroup B is defined over k. Now let $\alpha \in \Pi$ be a simple root, and let P be the standard parabolic subgroup of type $\{\alpha\} \subseteq \Pi$. By Lemma 6.5 combined with the fact that all minimal k-defined parabolic subgroups are conjugate over k, the subgroup P is defined over k. By Theorem 5.15 the intersection $T_\alpha = T \cap R(P)$ is a maximal torus of $R(P)$, and $G_\alpha = C_G(T_\alpha)$ is a reductive algebraic group with the root system $\Phi(T, G_\alpha) = \{\alpha, -\alpha\}$. By Lemma 6.2 the intersection $T \cap R(P)$ is a k-defined torus of $R(P)$, hence by Theorem 6.3 the group G_α is defined over k. Applying Lemma 6.2 again, we see that $\Phi(T, G_\alpha) = \{\alpha, -\alpha\}$ is Γ-invariant. On the other hand, since B is defined over k, the set $\Phi(T, B)$ is Γ-invariant, and therefore $\{\alpha\} = \Phi(T, B) \cap \{\alpha, -\alpha\}$ is Γ-invariant as well. This shows that all characters $\alpha \in \Pi$ of the torus T are defined over k. Since G is semi-simple, by Theorem 5.10 the set Π spans the $\mathbb{Q}$-vector space $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence all characters of T are defined over k, and T is a k-split torus. □

§ 7. Motivic decompositions of projective homogeneous varieties

Let X be a k-variety, and let G be a reductive algebraic K-group defined over k. We say that X is a *projective homogeneous k-variety of G*, if X is isomorphic to ${}_k(G/P)$, where P is a parabolic subgroup of G, and G/P is a k-defined variety of parabolic subgroups in the sense of § 6. In this case X is indeed a smooth projective k-variety by Lemma 6.4, and homogeneous with respect to the action of ${}_kG$. In the present chapter we study the motives (see the definition below) of such varieties. Our main tool is Theorem 7.1, originating from Karpenko [15], which says that a suitable filtration of X by closed subvarieties will provide an explicit additive decomposition of the motive $\mathcal{M}(X)$. The rest of the chapter is devoted to construction of such filtrations in different cases. The most general result is given in Theorem 7.7, which is [4, Th. 6.3] of Chernousov and Merkurjev.

1. The category of effective Chow motives. Let $\mathbf{Var}(k)$ be the category of smooth projective varieties over k. For any object X of $\mathbf{Var}(k)$, we denote by $\mathrm{CH}_p(X)$ the p-th Chow group of X, that is, the group of cycles of dimension p modulo rational equivalence [7, 8].

The category $\mathbf{Corr}(k)$ of correspondences over k is the category whose objects are the same as the objects of $\mathbf{Var}(k)$, and the morphisms, called *correspondences*, are given by

$$\mathrm{Hom}_{\mathbf{Corr}(k)}(X,Y)=\bigoplus_{l=1}^{n}\mathrm{CH}_{\dim X_l}(X_l\times Y),$$

where $X_1,\dots,X_n$ are the irreducible components of X. The composition $\beta\circ\alpha$ of two correspondences $\alpha\in\mathrm{Hom}_{\mathbf{Corr}(k)}(X,Y)$ and $\beta\in\mathrm{Hom}_{\mathbf{Corr}(k)}(Y,Z)$ is defined as

$$\beta\circ\alpha=(pr_{X\times Z})_*\left(pr^*_{X\times Y}(\alpha)\cdot pr^*_{Y\times Z}(\beta)\right),$$

where $pr_{X\times Z}, pr_{Y\times Z}, pr_{X\times Y}$ denote the projections of $X\times Y\times Z$ to the corresponding factors, upper and lower stars indicate the pull-backs and push-forwards [8, 1.4, 1.7] respectively, and $\cdot$ is the product in the Chow group [8, Ch. 8]. The category $\mathbf{Corr}(k)$ is an additive category with the abelian group structure on $\mathrm{Hom}\,(X,Y)$ given by the addition of cycles, and the coproduct of objects being the usual coproduct of varieties.

Identifying a morphism $f:X\to Y$ of schemes with the correspondence $[\Gamma_f]\in\mathrm{Hom}_{\mathbf{Corr}(k)}(X,Y)$, the class of the graph of f, we obtain a functor

$$\mathbf{Var}(k)\to\mathbf{Corr}(k)$$

(see [15, Prop. 1.4]).

The pseudo-abelian envelope of $\mathbf{Corr}(k)$ is called the category of *effective Chow motives* and denoted by $\mathbf{Chow}^{eff}(k)$. The objects of $\mathbf{Chow}^{eff}(k)$, called *motives*, are pairs (X,p), where X is an object of $\mathbf{Var}(k)$ and $p\in\mathrm{Hom}_{\mathbf{Corr}(k)}(X,X)$ is a projector, that is, satisfies $p\circ p=p$. The morphisms between two objects (X,p) and (X',p') are the compositions $p'\circ f\circ p$, where $f\in\mathrm{Hom}_{\mathbf{Corr}(k)}(X,X')$. The category $\mathbf{Corr}(k)$ is embedded into $\mathbf{Chow}^{eff}(k)$ in a natural way. The composition of functors

$$\mathcal{M}:\mathbf{Var}(k)\to\mathbf{Corr}(k)\to\mathbf{Chow}^{eff}(k)$$

takes a variety X to the pair $\mathcal{M}(X)=(X,[\Delta_X])$, where Δ_X is the image of the diagonal embedding $X\to X\times_{\mathrm{Spec}\,k}X$, i.e. the graph of id_X.

The category $\mathbf{Chow}^{eff}(k)$ inherits the additive structure of $\mathbf{Corr}(k)$. Moreover, it is a symmetric tensor additive category with respect to the tensor product defined by the fiber product of varieties

$$(X, p) \otimes (Y, q) = (X \times_{\operatorname{Spec} k} Y, p \times q).$$

Since $X \times_{\operatorname{Spec} k} \operatorname{Spec} k \cong X$, the motive $\mathcal{M}(\operatorname{Spec} k)$ is a neutral element for tensor multiplication. We will denote it by $\mathbf{1}$.

Let x be a k-point of $\mathbb{P}^1_k$. The motive $(\mathbb{P}^1_k, p)$, where $p = [\mathbb{P}^1_k \times x] \in \mathrm{CH}_1(\mathbb{P}^1_k \times \mathbb{P}^1_k)$, actually does not depend on the choice of x. It is called the *Tate motive* and denoted by $\mathbb{L}$. We have $\mathcal{M}(\mathbb{P}^1_k) \cong \mathbb{L} \oplus \mathbf{1}$.

Let $\mathbb{L}^n$ denote the n-th tensor power of $\mathbb{L}$. For a motive $M \in \mathbf{Chow}^{eff}(k)$ we set $M(n) = M \otimes \mathbb{L}^n$. These objects are called the *twists* of M. For any objects X, Y of $\mathbf{Var}(k)$ and for any $r, s \geq 0$ we have

$$\operatorname{Hom}_{\mathbf{Chow}^{eff}(k)}(\mathcal{M}(X)(r), \mathcal{M}(Y)(s)) = \bigoplus_{i=1}^{n} \mathrm{CH}_{\dim X_i + r - s}(X_i \times Y),$$

where $X_1, \ldots, X_n$ are the irreducible components of X.

Theorem 7.1. *Let X be a smooth projective variety over k. Suppose there is a filtration of X by closed subvarieties*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n = X$$

together with flat morphisms $f_i : X_i \setminus X_{i-1} \to Y_i$ of constant relative dimension a_i for every $i = 0, 2, \ldots, n$, where Y_i are smooth projective varieties over k. Suppose in addition that the fiber of every f_i over any (not necessarily closed) point $y \in Y_i$ is isomorphic to $\mathbb{A}^{a_i}_{\kappa(y)}$. Then there is an isomorphism:

$$\mathcal{M}(X) \cong \bigoplus_{i=1}^{n} \mathcal{M}(Y_i)(a_i).$$

This theorem was originally proved by Karpenko [15, Th. 6.5, Cor. 6.11] for the case when the maps $f_i : X_i \setminus X_{i-1} \to Y_i$ are vector bundle morphisms, using the results of Rost [19]. Later in [3, Th. 7.2] Chernousov, Gille and Merkurjev noticed that the proof actually applies to flat maps with fibers isomorphic to affine spaces, and the current version of the book by Elman, Merkurjev and Karpenko [7, Cor. 66.4] contains this theorem in the above form.

Proof of Theorem 7.1 (a sketch). Let $Z = X_{n-1} \subseteq X$ be a closed subvariety of X and $U = X \setminus Z$ be the corresponding open subscheme. There is a classical exact sequence of Chow groups [8, Prop. 1.8]

$$\mathrm{CH}(Z) \to \mathrm{CH}(X) \to \mathrm{CH}(U) \to 0.$$

It can be extended to the left by means of the "higher Chow groups" $A_*(-, K_*)$ [7] (see also [19]). More precisely, for any $p \geq 0$ we have a long exact sequence [7, 52.D]

$$\ldots \to A_{p+1}(Z, K_{-p}) \to A_{p+1}(X, K_{-p}) \to A_{p+1}(U, K_{-p}) \to \\ \to \mathrm{CH}_p(Z) \to \mathrm{CH}_p(X) \to \mathrm{CH}_p(U) \to 0,$$

with the identification $\mathrm{CH}_p(-) = A_p(-, K_{-p})$. If $U \to Y = Y_n$ is a flat morphism such that its fiber at any point $z \in Y$ is isomorphic to $\mathbb{A}^d_{\kappa(z)}$, the homotopy invariance [7, Th. 52.11] implies that each pull-back homomorphism $A_i(Y, K_{-j}) \to A_{i+d}(U, K_{-d-j})$

is an isomorphism, in particular, $\mathrm{CH}_p(Y) \cong \mathrm{CH}_{p+d}(U)$. On the other hand, it appears that $A_i(Y, K_{-j}) \to A_{i+d}(U, K_{-d-j})$ factors through the map $A_{i+d}(X, K_{-d-j}) \to A_{i+d}(U, K_{-d-j})$, and therefore the latter is split surjective. Hence we get

$$\mathrm{CH}_p(X) \cong \mathrm{CH}_p(Z) \oplus \mathrm{CH}_p(U) \cong \mathrm{CH}_p(Z) \oplus \mathrm{CH}_{p-d}(Y).$$

Since for any smooth projective variety T the set $T \times Z$, $T \times X$, $T \times U$, $T \times Y$ satisfies the same conditions as Z, X, U, Y, we have the equality $\mathrm{CH}_p(T \times X) \cong \mathrm{CH}_p(T \times Z) \oplus \mathrm{CH}_{p-d}(T \times Y)$ for any T.

Recall that $Z = X_{n-1}$. Proceeding by induction, we obtain an isomorphism $\mathrm{CH}_p(T \times X) \cong \bigoplus_{i=0}^n \mathrm{CH}_{p-d_i}(T \times Y_i)$, since $X_{-1} = \emptyset$. If we substitute $p = \dim T$, this gives an isomorphism

$$\mathrm{Hom}\,(\mathcal{M}(T), \mathcal{M}(X)) \cong \bigoplus_{i=0}^{n} \mathrm{Hom}\,(\mathcal{M}(T), \mathcal{M}(Y_i)(d_i)).$$

Then the Yoneda lemma implies that $\mathcal{M}(X) \cong \bigoplus_{i=0}^n \mathcal{M}(Y_i)(d_i)$. □

2. **Motivic decomposition in the split case.** Let G be a k-split reductive algebraic group, T a k-split maximal torus of G, $\Phi = \Phi(T, G)$, $W = W(T, G) = W_\Phi$, Π a system of simple roots for Φ, and B a Borel subgroup of G corresponding to Π.

By Theorem 6.1 all subgroups U_α, $\alpha \in \Phi$, are defined over k. Hence also all standard parabolic subgroups

$$P_I = BW_IB = \langle T;\ U_\alpha,\ \alpha \in \Delta_I \cup \Sigma_I \rangle\,, \quad I \subseteq \Pi,$$

are defined over k, and the projective varieties G/P_I are as well. We will show that the Bruhat decomposition of G implies that all ${}_k(G/P_I)$, $I \subseteq \Pi$, are in fact cellular k-varieties with cells of the form $\mathbb{A}_k^n$, $n \geq 0$. This is the simplest possible case of Theorem 7.1, and we obtain very simple motivic decompositions of ${}_k(G/P_I)$.

Fix $I \subseteq \Pi$ and let $P = P_I$. By Lemma 3.4 the set $W^I = W^{\emptyset, I}$ of the elements of minimal length in cosets modulo W_I is a set of representatives for $\{e\}\backslash W/W_I = W/W_I$. Hence by Theorem 5.13 we have

$$G = \coprod_{w \in W^I} BwP.$$

Let $\pi : G \to G/P$ be the canonical projection. For any $w \in W^I$, we set

$$X_w = \pi(BwP) = BwP/P.$$

Lemma 7.2. *Let G be a k-split reductive algebraic group. Then*

(1) $G/P = \coprod_{w \in W^I} X_w$.

(2) *For any $w \in W^I$, X_w is a k-defined variety, and ${}_k(X_w) \cong \mathbb{A}_k^{l(w)}$. The closure of X_w in G/P is contained in $X_w \amalg \coprod_{w' \in W^I,\ l(w') < l(w)} X_{w'}$.*

(3) *Let $w_1, \ldots, w_n$ be the list of all elements of W^I ordered so that their length increases. Then $V_i = \coprod_{k=0}^{i} X_{w_k}$, $0 \leq i \leq n$, are closed subvarieties of G/P defined over k.*

Proof. The claim of (1) is clear, and (3) follows from (1) and (2). Thus we need to prove (2). Since P is k-defined, the morphism π is also k-defined. By Corollary 6.1.1, the element $w \in W^I$ has a representative in $N_G(T) \cap G(k)$. Then $X_w = \pi(BwP)$ is an orbit of the k-defined point $wP \in G/P$ with respect to the action of B on G/P by left translations. By the closed orbit lemma X_w has a structure of a variety, and since B is k-defined, X_w is also k-defined. To prove that ${}_k(X_w) \cong \mathbb{A}_k^{l(w)}$, we recall Theorem 5.12. It states that in the particular case $P = B$ we have $X_w \cong U'_w$, where U'_w is the closed T-invariant subgroup of $U = B_u$, corresponding to the set of roots Φ'_w. This followed from the fact (see part 3) of the proof of Theorem 5.12) that the morphism

$$\begin{array}{rcl} f : U'_w \times B & \to & BwB \\ (x, y) & \mapsto & xwy \end{array}$$

is an isomorphism. It is easy to see that the proof of this statement carries over to the morphism

$$\begin{array}{rcl} f : U'_w \times P & \to & BwP \\ (x, y) & \mapsto & xwy, \end{array}$$

with the only difference that the final equality $T_w(U'_w w) \cap T_w(wP) = 0$ now follows from the property $w(\Delta_I^-) \subseteq \Phi^-$ (Lemma 3.5) of the element $w \in W^{\emptyset, I}$. It is clear, moreover, that f is defined over k, and hence provides an isomorphism of k-varieties ${}_k(X_w) \cong {}_k(U'_w)$. By Corollary 6.1.2 the variety ${}_k(U'_w)$ is isomorphic to $\mathbb{A}_k^{l(w)}$, since $|\Phi(T, U'_w)| = |\Phi'_w| = l(w)$ by Lemma 3.3. We finish the proof of (2) observing that by the closed orbit lemma the closure of the B-orbit X_w is a union of X_w and of some orbits of lower dimension, that is, of some varieties $X_{w'}$ with $l(w') < l(w)$. □

We can now deduce the theorem of Köck [17, Th. 2.1].

Theorem 7.3. *Let G be a k-split reductive algebraic group, and let $P = P_I$, $I \subseteq \Pi$, be a parabolic subgroup of G. There is an isomorphism*

$$\mathcal{M}({}_k(G/P)) \cong \bigoplus_{w \in W^I} \mathbb{L}^{l(w)}.$$

Proof. The filtration $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n$, where $X_i = {}_k(V_i)$, $0 \leq i \leq n$, constructed in Lemma 7.2, satisfies all conditions of Theorem 7.1, if we take for $f_i : X_i \to Y_i$ the canonical morphisms $X_i \to \operatorname{Spec} k$. Since $\mathcal{M}(\operatorname{Spec} k)(l(w)) = \mathbb{L}^{l(w)}$, the result follows. □

3. **Motivic decomposition in the non-split case.** The results of this subsection are taken from Chernousov, Gille, Merkurjev [3] and Chernousov, Merkurjev [4]. The main theorem, Theorem 7.7 (see [4, Prop. 5.1, Th. 6.3]), gives a decomposition of the motive $\mathcal{M}(X \times X')$, where X and X' are projective homogeneous k-varieties of a k-defined reductive algebraic group G. This is done by constructing a filtration of $X \times X'$, which satisfies Theorem 7.1. In case when *one* of these varieties, say X, has a k-point, we can obtain a decomposition of the motive of the other one, X', using pull-back (Corollary 7.7.1). In the particular case $X = X'$ this is the main result of [3], which gives a decomposition of any projective homogeneous variety with a k-point (Corollary 7.7.2). In the particular case when the reductive group is k-split, we also deduce the main result of the previous subsection, Theorem 7.3 ([17, Th. 2.1]), substituting $X = {}_k(G/B)$, where B is a Borel subgroup of G (Corollary 7.7.3).

Let G be a reductive algebraic K-group defined over k. We fix a k-defined maximal torus T of G, a maximal k-split torus $S \subseteq T$ of G. We consider $\Phi = \Phi(T, G)$, the root system of G, and $W = W(T,G)$, the Weyl group. We also fix a Borel subgroup $B \supseteq T$ and denote by Π the corresponding system of simple roots in Φ, and by Φ^+ (resp. Φ^-) the set of positive (resp. negative) roots with respect to Π.

Recall that we define the $*$-action of $\Gamma = \mathrm{Gal}\,(k_s/k)$ on the root system Φ as

$$\sigma^*(\alpha) = w_\sigma(\sigma(\alpha)), \quad \sigma \in \Gamma,\ \alpha \in \Phi$$

(see § 6). We also consider the corresponding action on W, which is given by $\sigma^*(w) = w_\sigma \sigma(w) w_\sigma^{-1}$, $w \in W$.

Let $I, J \subseteq \Pi$ be two sets of simple roots, and let P and P' be two standard (i.e., containing B) parabolic subgroups of G of types I and J respectively. Suppose that I and J are invariant under the $*$-action of Γ. Then also the subgroups W_I, W_J of the Weyl group W are $*$-invariant, and therefore there is a $*$-action of Γ on the set of double cosets $W_I\backslash W/W_J$.

Since I, J are $*$-stable, by Lemma 6.4 the projective homogeneous G-varieties G/P and G/P' are defined over k. We set

$$X = {}_k(G/P) \quad \text{and} \quad X' = {}_k(G/P').$$

Consider the diagonal action of G on the k-defined variety

$$(G/P) \times_{\mathrm{Spec}\, K} (G/P') \cong (X \times_{\mathrm{Spec}\, k} X') \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K.$$

Lemma 7.4. (1) *The assignment $w \mapsto (P,\, wP'w^{-1})$ induces a bijection between the set of double cosets $W_I\backslash W/W_J$ and the set of G-orbits in $G/P \times G/P'$.*

(2) *Each G-orbit in $G/P \times G/P'$ is defined over k_s. The above bijection is Γ-equivariant (where Γ acts on W via the $*$-action, and in a natural way on the orbits).*

Proof. (1) Since G acts transitively on G/P and G/P', any G-orbit in $G/P \times G/P'$ contains an element of the form $(P, gP'g^{-1})$. It is easy to see that two pairs $(P, g_1P'g_1^{-1})$ and $(P, g_2P'g_2^{-1})$ lie in the same G-orbit if and only if $Pg_1P' = Pg_2P'$. This means that the map $g \mapsto (P, gP'g^{-1})$ provides a bijection between $P\backslash G/P'$ and the set of all G-orbits. On the other hand, Theorem 5.13 gives a bijection between $W_I\backslash W/W_J$ and $P\backslash G/P'$.

(2) Since G is k_s-split, by Theorem 6.1 the parabolic subgroups P and P', and hence the action of G on $G/P \times G/P'$, are defined over k_s. Since by Corollary 6.1.1 any coset in $W = N_G(T)/C_G(T)$ contains a representative w from $G(k_s)$, the pair $(P,\, wP'w^{-1})$ is a k_s-point of the corresponding orbit; hence all orbits are defined over k_s.

To see that the bijection is Γ-equivariant, observe that for any $\sigma \in \Gamma$ and any $w \in W$ we have $\sigma^*(w) = w_\sigma \sigma(w) w_\sigma^{-1}$, and since $I,\, J$ are $*$-stable, we have

$$\begin{aligned}\big(w_\sigma\sigma(P)w_\sigma^{-1},\ w_\sigma\sigma(wP'w^{-1})w_\sigma^{-1}\big) &= \big(w_\sigma\sigma(P)w_\sigma^{-1},\ \sigma^*(w)w_\sigma\sigma(P')w_\sigma^{-1}\sigma^*(w^{-1})\big)\\ &= (P,\ \sigma^*(w)P'\sigma^*(w)^{-1}).\end{aligned}$$

□

Recall that the set of double cosets $W_I\backslash W/W_J$ has a system of representatives $W^{I,J}$ consisting of the elements of minimal length in the corresponding cosets (Lemma 3.4). By the previous lemma, each G-orbit in $G/P \times G/P'$ contains a unique element of the form $(P, wP'w^{-1})$, where $w \in W^{I,J}$.

Lemma 7.5. *For any $w \in W^{I,J}$, the subgroup $Q_w = (P \cap wP'w^{-1}) \cdot R_u(P)$ of G has the following properties.*

(1) *Q_w is a standard (i.e., containing B) parabolic subgroup of G of type $I_w = I \cap w(\Delta_J^+)$.*
(2) *For any $\sigma \in \Gamma$ we have $w_\sigma \sigma(Q_w) w_\sigma^{-1} = Q_{\sigma^*(w)}$.*
(3) *If the subgroups P and $wP'w^{-1}$ are defined over k, then Q_w also is, and ${}_k\left(Q_w/(P \cap wP'w^{-1})\right)$ is k-isomorphic to $\mathbb{A}_k^{l(w)}$.*

Proof. (1) To prove that Q_w is a standard parabolic subgroup, we need to show that $U_\alpha \subseteq Q_w$ for any $\alpha \in \Phi^+$. Since $R_u(P) \subseteq B_w$, it is enough to consider $\alpha \in \Delta_I^+$. But since $w \in W^{I,J}$, by Lemma 3.5 we have $\Delta_I^+ \subseteq w(\Phi^+)$, hence $U_\alpha \subseteq wP'w^{-1}$, and therefore $U_\alpha \subseteq wP'w^{-1} \cap P$. Now let $R \subseteq \Pi$ be the type of Q_w. It is clear that $I_w \subseteq R$. Conversely, let $\alpha \in R$. Then $\alpha, -\alpha \in \Phi(T, Q_w)$, and hence $\alpha, -\alpha \in \Phi(T, P \cap wP'w^{-1}) = \Phi(T, P) \cap w(\Phi(T, P'))$. This means that $\alpha \in \Delta_I \cap w(\Delta_J)$. Since $\alpha \in \Pi$ and by Lemma 3.5 we have $w(\Delta_J^-) \subseteq \Phi^-$, this implies $\alpha \in I_w$.

(2) The calculations in the proof of Lemma 7.4 show that $P \cap wP'w^{-1} = P \cap \sigma^*(w)P'\sigma^*(w)^{-1}$; on the other hand, $w_\sigma\sigma(R_u(P))w_\sigma^{-1} = R_u(w_\sigma\sigma(P)w_\sigma^{-1})$, since this transformation is an automorphism of G. Hence $w_\sigma\sigma(Q_w)w_\sigma^{-1} = Q_{\sigma^*(w)}$.

(3) The subgroup Q_w is k-defined, since by Lemma 6.2 the intersection $P \cap wP'w^{-1}$ is defined over k, and by Theorem 6.3 the unipotent radical $R_u(P)$ is also k-defined. Also by Theorem 6.3 there is a k-defined decomposition $Q_w = C_G(S') \ltimes R_u(Q)$, where S' is the maximal k-split torus contained in $T \cap R(Q_w) = T_{I_w}$ (cf. Theorem 5.15). By the same theorem, since S' is contained in both parabolic subgroups P and $wP'w^{-1}$, its centralizer $C_G(S')$ also is. This means that we have a k-defined decomposition $P \cap wP'w^{-1} = C_G(S') \ltimes H$, where

$$H = (P \cap wP'w^{-1}) \cap R_u(P) = wP'w^{-1} \cap R_u(P),$$

and therefore

$$Q_w/(P \cap wP'w^{-1}) \cong R_u(P)/H.$$

Observe that both $R_u(P)$ and H are T-invariant k-defined subgroups of $U = R_u(B)$, hence by Corollary 5.7.2 their dimensions are equal to $|\Phi(T, R_u(P))|$ resp. $|\Phi(T, H)|$. Set

$$\Psi = \Phi(T, R_u(P)), \qquad \Psi_1 = \Phi(T, H), \qquad \Psi_2 = \Psi \setminus \Psi_1.$$

Let us prove that

$$\Psi_2 = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\} = \Phi'_w.$$

It is clear that $\Psi = \Phi^+ \setminus \Delta_I^+$ and $\Psi_1 = \Psi \cap w(\Phi^+ \cup \Delta_J)$. Hence

$$\Psi_2 = \Psi \setminus w(\Phi^+ \cup \Delta_J) = \Psi \cap w(\Phi^- \setminus \Delta_J^-).$$

Then the inclusion $\Psi_2 \subseteq \Phi'_w$ is clear. Conversely, for any $\alpha \in \Phi'_w$ we have $\alpha \in \Phi^+ \setminus \Delta_I^+ = \Psi$ by Lemma 3.5. Since α is a positive root, $w^{-1}(\alpha)$ is a negative root, and since $w(\Delta_J^-) \subseteq \Phi^-$ by Lemma 3.5, we also have $\alpha \in w(\Phi^- \setminus \Delta_J^-)$. Hence $\Psi_2 = \Phi'_w$. In particular, $|\Psi_2| = l(w)$ by Lemma 3.3.

Let $H' = \langle U_\alpha,\ \alpha \in \Psi_2 \rangle$. Then, since Ψ_2 is an additively closed set of roots, by Corollary 5.7.2 we have $\Phi(T, H') = \Psi_2$, and by Lemma 6.2 H' is defined over k, since $\Psi_2 = \Psi \setminus \Psi_1$ is Γ-invariant. By Corollary 5.7.2, we have $H \times H' \cong R_u(P)$ as a K-variety, the morphism being the product morphism. This morphism is defined over k, and hence the corresponding k-morphism is also an isomorphism by [10, Prop. 2.7.1]. Then also the natural projection ${}_kH' \to {}_k(R_u(P)/H)$ is a k-isomorphism. By [1, Prop. 8.11] the torus T splits over a finite separable extension k' of k. Then by Theorem 6.1 H' is a k'-solvable group in the sense of [18], that is, possesses a filtration by normal k'-defined subgroups such that all successive quotients are k'-isomorphic to $\mathbb{G}_{a,k'}$. Since k' is a separable extension of k, by [18, Th. 3] a k'-solvable group is also k-solvable, so H' is a k-solvable group. It is also unipotent, hence by [18, Cor. 2 of Th. 1] it is k-isomorphic to the direct product of $\dim H' = |\Psi_2| = l(w)$ copies of $\mathbb{G}_{a,k}$. Hence ${}_k(R_u(P)/H) \cong \mathbb{A}_k^{l(w)}$. □

Let $D \in W_I\backslash W/W_J$ be the double coset with the representative $w \in W^{I,J}$, and let O_D be the corresponding G-orbit in $G/P \times G/P'$. Since the G-stabilizer of the point $(P, wP'w^{-1}) \in O_D$ is equal to $P \cap wP'w^{-1}$, we have an isomorphism

$$O_D \cong G/(P \cap wP'w^{-1}).$$

Let $Z_D \cong G/Q_w$ be the variety of parabolic subgroups of G of type I_w. Consider the natural G-invariant morphism

$$\begin{array}{rccl} \lambda_D : & O_D & \to & Z_D \\ & (Q, Q') & \mapsto & R_u(Q) \cdot (Q \cap Q'). \end{array}$$

Clearly, λ_D is defined over k_s.

Let now δ denote the $*$-orbit of the double coset $D \in W_I\backslash W/W_J$. Since $w = w(D)$ is the element of minimal length in D, for any $\sigma \in \Gamma$ the image $\sigma^*(w)$ of w is the element of the minimal length in $\sigma^*(D)$, and $l(w) = l(\sigma^*(w))$. We set

$$l_\delta = l(w), \quad \text{if } w \in W^{I,J} \text{ and } D = W_I w W_J \text{ for some } D \in \delta.$$

Let $O_\delta = \coprod_{D\in\delta} O_D$. It is a k_s-defined open subvariety of its closure in $G/P \times G/P'$. Since $O_\delta(k_s)$ is Γ-invariant by Lemma 7.4, O_δ is moreover a k-defined variety.

The varieties $Z_D = G/Q_w$, $D = W_I w W_J \in \delta$, need not to be defined over k. However, there are natural k_s-morphisms

$$\sigma : Z_D \to Z_{\sigma^*(D)},$$

induced by the action of $\sigma \in \Gamma$ on the parabolic subgroups of G. Indeed, by Lemma 7.5 (2) the morphism $\sigma : G \to G$ takes a parabolic subgroup of type I_w into a parabolic subgroup of type $I_{\sigma^*(w)}$. Therefore the k_s-defined morphism

$$\begin{array}{rccl} \amalg \lambda_D : & O_\delta = \coprod_{D\in\delta} O_D & \to & \coprod_{D\in\delta} Z_D \\ & (Q, Q') & \mapsto & R_u(Q) \cdot (Q \cap Q') \end{array}$$

is well-defined and Γ-invariant. Since each Z_D is a projective variety, the variety $Z_\delta = \coprod_{D\in\delta} Z_D$ is defined over k, and, clearly, ${}_k(Z_\delta)$ is projective (cf. Lemma 6.4). Being Γ-invariant, the morphism $\lambda_\delta = \amalg \lambda_D$ is also defined over k.

Lemma 7.6. *The morphism ${}_k(\lambda_\delta) : {}_k(O_\delta) \to {}_k(Z_\delta)$ is flat and the fiber ${}_k(\lambda_\delta)^{-1}(z)$ of ${}_k(\lambda_\delta)$ at any (not necessarily closed) point $z \in {}_k(Z_\delta)$ is isomorphic to $\mathbb{A}^{l_\delta}_{\kappa(z)}$, where $\kappa(z)$ is the residue field at z.*

Proof. Denote ${}_k(\lambda_\delta)^{-1}(z)$ by F_z. The statement $F_z \cong \mathbb{A}^{l_\delta}_{\kappa(z)}$ can be, clearly, checked after extending the base to $\kappa(z)$. Denote $\kappa(z)$ by L. Then the scheme ${}_L(Z_\delta) = {}_k(Z_\delta) \times_{\operatorname{Spec} k} \operatorname{Spec} \kappa(z)$, and hence also a scheme Z_D for some $D \in \delta$, contains a (closed) point over L. This means that G contains an L-defined parabolic subgroup Q of type I_w, where $w \in W^{I,J}$ is the representative of D. Our definition of the map $\lambda_D : O_D \to Z_D$, where O_D is just any orbit of G in $G/P \times G/P'$, does not depend on the choice of the maximal k-defined torus T and the Borel subgroup $B \in \mathcal{B}^T$, corresponding to the set of simple roots $\Pi \subseteq \Phi(T, G)$. Hence we can assume that Q

is a standard parabolic subgroup of type I_w, that is, $Q = Q_w$. Then since $Q_w \subseteq P$, by Lemma 6.5 the parabolic subgroup P is defined over L. Further, for any $\sigma \in \Gamma$ we have $w_\sigma \in Q_w$ (see the proof of Lemma 6.5). Then Theorem 5.13 implies that $w_\sigma \in W_{I_w}$, and since $I_w \subseteq w(\Delta_J)$, we have $w_\sigma \in wP'w^{-1}$. Therefore, $wP'w^{-1}$ is also defined over L. The above means that both $O_D \cong G/(P \cap wP'w^{-1})$ and $Z_D \cong G/Q_w$ are defined over L, and the morphism $(\lambda_\delta)_L$ is actually $(\lambda_D)_L$. The fiber of λ_D at z is naturally L-isomorphic to the L-defined variety $Q_w/(P \cap wP'w^{-1})$. Now Lemma 7.5 (3) implies that F_z is isomorphic to $\mathbb{A}_L^{l(w)} = \mathbb{A}_{\kappa(z)}^{l(w)}$.

To see that $f = {}_k(\lambda_\delta)$ is flat, we can assume that $k = K$, since the base extension morphism $k \to K$ is faithfully flat. Observe that Z_D is irreducible; let z_0 be the generic point of Z_D. By the above (applied to the case when $L = \kappa(z_0)$ and K is an algebraic closure of $\kappa(z_0)$), the fiber at z_0 is isomorphic to $\mathbb{A}^{l_\delta}_{\kappa(z_0)}$. Hence we have

$$O_D \times_{Z_D} \operatorname{Spec} \kappa(z_0) \cong \mathbb{A}^{l_\delta}_{\kappa(z_0)} \cong (\mathbb{A}^{l_\delta}_K \times_{\operatorname{Spec} K} Z_D) \times_{Z_D} \operatorname{Spec} \kappa(z_0).$$

Since $\kappa(z_0) = \mathcal{O}_{Z_D,z_0} = \varinjlim \mathcal{O}_{Z_D}(U)$, where U runs over all open affine neighbourhoods of z_0, by [11, Cor. 8.8.2.5] there is a open neighbourhood $U \ni z_0$ such that

$$O_D \times_{Z_D} U \cong (\mathbb{A}^{l_\delta}_K \times_{\operatorname{Spec} K} Z_D) \times_{Z_D} U \cong \mathbb{A}^{l_\delta}_K \times_{\operatorname{Spec} K} U.$$

The open subset U contains a closed point, because the set of closed points is dense in Z_D. Since the group G acts transitively on the closed points of $Z_D \cong G/Q_w$, the topological space Z_D is covered by open subsets $U \subseteq Z_D$ satisfying $O_D \times_{Z_D} U \cong \mathbb{A}^{l_\delta}_K \times_{\operatorname{Spec} K} U$. Since the projection map $\mathbb{A}^{l_\delta}_K \times_{\operatorname{Spec} K} U \to U$ is flat, the map $\lambda_D : O_D \to Z_D$ is flat as well. □

Theorem 7.7. *Let $X = {}_k(G/P)$ and $X' = {}_k(G/P')$ be projective homogeneous varieties of a k-defined reductive algebraic group G. In the above notation, there is an isomorphism*

$$\mathcal{M}(X \times_{\operatorname{Spec} k} X') \cong \bigoplus_{\delta \in \Delta} \mathcal{M}\big({}_k(Z_\delta)\big)(l_\delta),$$

where Δ is the set of all orbits for the $*$-action on $W_I \backslash W / W_J$.

Proof. Recall that by Lemma 7.4 the elements of $W_I \backslash W / W_J$ are in one-to-one correspondence with the G-orbits in $G/P \times G/P'$, with the $*$-action on the former coinciding with the usual Galois action on the latter; here the orbit corresponding to $D \in W_I \backslash W / W_J$ is precisely O_D. For every $j \geq 0$ let V_j' be the union of orbits of dimension at most j. By the closed orbit lemma the closure of an orbit is a union of this orbit and of some orbits of lower dimension. Hence for any $*$-orbit $\delta \in \Delta$ such that for all $D \in \delta$ the orbit O_D has dimension j, the variety

$$V = V_{j-1} \amalg \coprod_{D \in \delta} O_D = V_{j-1} \amalg O_\delta$$

is closed. Since it is defined over k_s and Γ-invariant, it is defined over k. Hence we can construct a filtration

$$\emptyset = W_{-1} \subseteq W_0 \subseteq \ldots \subseteq W_n = G/P \times G/P'$$

consisting of closed k-defined subvarieties of $G/P \times G/P'$ and such that for every $i \geq 0$ the difference $W_i \setminus W_{i-1}$ coincides with O_δ for some $\delta \in D$. This filtration induces the filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n = X \times X'$$

over k, where $X_i = {}_k(W_i)$, $i \geq 0$, with differences of the form ${}_k(O_\delta)$. The morphisms ${}_k(\lambda_\delta) : {}_k(O_\delta) \to {}_k(Z_\delta)$, constructed above, are by Lemma 7.6 flat k-morphisms to projective k-varieties, with fibers isomorphic to $\mathbb{A}^{l(w)}_{\kappa(z)}$ for any $z \in {}_k(Z_\delta)$. The varieties ${}_k(Z_\delta)$ are smooth, since after the extension of the base field they become isomorphic to the smooth varieties G/Q_w (see [12, Prop. 17.7.1]). Then we can apply Theorem 7.1. □

Corollary 7.7.1. *Under the hypothesis of Theorem 7.7.2, suppose moreover that P is k-defined, that is, the variety $X = {}_k(G/P)$ has a k-point. Then there is an isomorphism*

$$\mathcal{M}(X') \cong \bigoplus_{\delta \in \Delta} \mathcal{M}(Y_\delta)(l_\delta),$$

where $Y_\delta \cong {}_k(Z_\delta) \times_X \operatorname{Spec} k$ are smooth projective varieties.

Proof. Let

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n = X \times X'$$

be the filtration for $X \times X' = {}_k(G/P \times G/P')$ constructed in the proof of the theorem, and let $\operatorname{Spec} k \to X$ be the k-point of X. The products $\operatorname{Spec} k \times_X X_i = X'_i$ then define a k-filtration of the k-variety $(X \times X') \times_X \operatorname{Spec} k \cong X'$. Also by the construction, for every $i \geq 0$ the difference $X'_i \setminus X'_{i-1}$ is isomorphic to ${}_k(O_\delta) \times_X \operatorname{Spec} k$ for some $\delta \in \Delta$. The variety $Z_\delta = \coprod_{D \in \delta} Z_D$ possesses a natural morphism to $X = G/P$, since $Z_D \cong G/Q_w$ for a parabolic subgroup $Q_w \subseteq P$. This morphism is clearly defined over k_s and Γ-invariant, hence it is defined over k (if one looks on the quotients as on varieties of parabolic subgroups of corresponding types). Since the morphism $\lambda_\delta : O_\delta \to Z_\delta$ comes from morphisms $\lambda_D : O_D \to Z_D$, taking the pair $(P, wP'w^{-1}) \in O_D$ to Q_w, we have a commutative diagram of k-morphisms

$$\begin{array}{ccc} {}_k(O_\delta) & \hookrightarrow & X \times X' \ . \\ \big\downarrow {\scriptstyle {}_k(\lambda_\delta)} & \searrow & \big\downarrow {\scriptstyle pr_1} \\ {}_k(Z_\delta) & \longrightarrow & X \end{array}$$

Multiplying all varieties by $\times_X \operatorname{Spec} k$, we get therefore a k-defined morphism

$${}_k(\lambda_\delta) \times \mathrm{id}_k : {}_k(O_\delta) \times_X \operatorname{Spec} k \to {}_k(Z_\delta) \times_X \operatorname{Spec} k = Y_\delta.$$

Clearly, the morphism ${}_k(\lambda_\delta) \times \mathrm{id}_k$ is flat. The k-scheme $Y_\delta = {}_k(Z_\delta) \times_X \operatorname{Spec} k$ is precisely the fiber of the morphism ${}_k(Z_\delta) \to {}_k(G/P)$ at the k-point. Hence if k' is a finite separable extension $k \to k'$ such that G is k'-split, ${}_{k'}(Z'_\delta)$ is isomorphic to the disjoint union of smooth projective fibers ${}_{k'}(P/Q_w)$ of the morphisms ${}_{k'}(G/Q_w) \to {}_{k'}(G/P)$ (Lemma 6.4). Therefore ${}_k(Z'_\delta)$ is smooth projective by [12, Prop. 17.7.1] and [9, Cor. 6.6.5]. Hence we can apply Theorem 7.1. □

The following corollary is the main result of [3].

Corollary 7.7.2. *Let G be a k-defined reductive algebraic group, Q a parabolic k-defined subgroup of G of type I. Then there is an isomorphism*

$$\mathcal{M}({}_k(G/Q)) \cong \bigoplus_{\delta \in \Delta} \mathcal{M}(Y_\delta)(l_\delta),$$

where Δ is the set of all orbits for the $*$-action on $W_I \backslash W / W_I$. If k_δ is a finite extension of k stabilizing every coset $D \in \delta$, then $Y_\delta \times_{\operatorname{Spec} k} \operatorname{Spec} k_\delta = \coprod_{D \in \delta} Y_D$, where *the varieties Y_D, $D \in \delta$, are k_δ-isomorphic to homogeneous varieties of the reductive algebraic k-defined group $L_Q \cong Q/R_u(Q)$, having the root system Δ_I.*

Proof. We apply Theorem 7.7 in case $P = P' = Q$. Then, as in Corollary 7.7.1, we have $\mathcal{M}({}_k(G/P)) \cong \bigoplus_{\delta\in\Delta} \mathcal{M}(Y_\delta)(l_\delta)$, where $Y_\delta \cong {}_k(Z_\delta) \times_{{}_k(G/P)} \operatorname{Spec} k$. Since k_δ preserves a coset $D \in \delta$, it also preserves the canonical representative $w \in W^{I,I}$ of D, because it is the unique element of minimal length in D. Hence the subgroup $Q_w = R_u(P) \cdot (P \cap wPw^{-1})$ is defined over k_δ. Since T is a k-defined torus, the torus $T_I = T \cap R(P)$ of Theorem 5.15 is a maximal k-defined torus of $R(P)$. If S' is a maximal k-split torus contained in T_I, by Theorem 6.3 we have a k-defined isomorphism $C_G(S') = C_G(T_I) \cong P/R_u(P)$. By Theorem 5.15, $L_P = C_G(T_I)$ is a reductive algebraic group with $\Phi(T, L_P) = \Delta_I$.

Further, Y_D is precisely the fiber of the morphism ${}_{k_\delta}(G/Q_w) \to {}_{k_\delta}(G/P)$ at the k_δ-point P, and hence is naturally k_δ-isomorphic to

$$ {}_{k_\delta}(P/Q_w) \cong {}_{k_\delta}\left(L_P/(L_P \cap wPw^{-1})\right). $$

Since $\Delta_I^+ \subseteq w(\Phi^+)$ by Lemma 3.5, the group $L_P \cap wPw^{-1}$ is a parabolic k_δ-defined subgroup of L_P, and thus Y_δ is a homogeneous L_P-variety. □

Corollary 7.7.3 (Theorem 7.3)**.** *Let G be a k-split reductive algebraic group, Q a parabolic subgroup of G of type I. There is an isomorphism*

$$ \mathcal{M}(G/Q) \cong \bigoplus_{D\in W/W_I} \mathbb{L}^{l_D}, $$

where each number l_D is the length $l(w)$ of the representative $w \in W^{\emptyset,I}$ of $D \in W/W_I$.

Proof. We apply Corollary 7.7.1 in the case $P = B$, $P' = Q$. Let $w \in W^I = W^{\emptyset,I}$. Since G is k-split, the $*$-action is trivial, and the subgroups P, P', $wP'w^{-1}$, Q_w are defined over k by Theorem 6.1. Any $*$-orbit $\delta \subseteq W/W_I$ consists of one element, $\delta = \{D\}$. We have $I_w = \emptyset \cap w(\Delta_I^+) = \emptyset$, hence $Q_w = B = P$. Then the k-varieties Y_δ of Corollary 7.7.1 satisfy

$$ Y_\delta \cong {}_k(G/B) \times_{{}_k(G/B)} \operatorname{Spec} k \cong \operatorname{Spec} k. $$

Hence $\mathcal{M}(Y_\delta)(l_\delta) \cong \mathbb{L}^{l_D}$. □

§ 8. Examples of motivic decompositions

In the present chapter we use the results of § 7 to obtain some explicit motivic decompositions. We keep the same notation as above. Namely, G denotes a k-defined reductive algebraic group over K, T a maximal k-defined torus of G, $\Phi = \Phi(T, G)$ the set of roots of G with respect to T, $W = W(T, G)$ the corresponding Weyl group, and Π a fixed system of simple roots in Φ, corresponding to a Borel subgroup B containing T. We write $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, where n is the rank of Φ and the numbering of roots follows Bourbaki [2]. All parabolic subgroups we speak of are standard.

1. Motive of a projective space. Let $G = \mathrm{PGL}_{n+1,K}$, the projective linear group of dimension $n + 1$ over K, and let ${}_kG = \mathrm{PGL}_{n+1,k}$, the same group over k. This is a k-split semi-simple group with the root system $\Phi = A_n$. The Weyl group W is isomorphic to the symmetric group S_{n+1}, with the reflection w_{α_i} corresponding to the transposition $(i, i + 1)$. Let P be the standard parabolic subgroup of G of type $I = \Pi \setminus \alpha_1$. The corresponding variety $X = {}_k(G/P)$ is isomorphic to $\mathbb{P}^n_k$. The Weyl group W_I is identified with the subgroup of S_{n+1} generated by all reflections except for (12), and is isomorphic to S_n, since Δ_n is spanned by $\{\alpha_2, \ldots, \alpha_n\}$ and hence is a root system of type A_{n-1}.

By Corollary 7.7.3, we have $\mathcal{M}(X) \cong \bigoplus\limits_{D \in W/W_I} \mathbb{L}^{l(D)}$. Since $|S_{n+1}|/|S_n| = n+1$, the quotient W/W_I has $n+1$ elements. Consider the elements

$$w_i = (i, i+1)(i-1, i) \ldots (12) = (1, i+1, i, \ldots, 2), \qquad 1 \leq i \leq n.$$

Clearly, $l(w_i) = i$, and for any $1 < j \leq n$ we have $l(w_i \cdot (j, j+1)) = i+1$. Then by Lemma 3.5 these elements are the representatives of minimal length in their cosets modulo W_I, and since they are of different lengths, they form, together with e, the whole set $W^{\emptyset, I}$. Thus our decomposition of X becomes

$$\mathcal{M}(\mathbb{P}^n_k) = \bigoplus_{i=0}^{n} \mathbb{L}^i = \mathbf{1} \oplus \mathbb{L} \oplus \ldots \oplus \mathbb{L}^n,$$

the classical decomposition of [16] (in fact, it follows directly from the definition of $\mathbb{L}$).

2. **Groups of type** F_4**.** Let G be a k-defined semi-simple algebraic group over k such that $\Phi = F_4$. Since the Dynkin diagram (see [2]) of F_4 has no non-trivial automorphism, the $*$-action on $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is always trivial, that is, G is of inner type. The Tits classification [20] states that there are only three possibilities for minimal k-defined parabolic subgroups of G. Namely, either G is k-anisotropic, that is, has no non-trivial k-defined parabolic subgroup, or G is k-quasi-split, and hence k-split by Lemma 6.6, or the minimal k-defined parabolic subgroup of G is of type $\{\alpha_1, \alpha_2, \alpha_3\}$.

The split case. In case when G is k-split, G has a k-defined parabolic subgroup P of any possible type $I \subseteq \Pi$, and the situation is subject to Corollary 7.7.3. For example, in the particular case $P = B$ we obtain the decomposition

$$\mathcal{M}(G/B) \cong \bigoplus_{w \in W} \mathbb{L}^{l(w)}.$$

The isotropic case. Consider now the case when the minimal k-defined parabolic subgroup P of G is of type $I = \{\alpha_1, \alpha_2, \alpha_3\}$. For any parabolic subgroup P' of type J, we obtain a decomposition of the motive $\mathcal{M}\big({}_k(G/P')\big)$ using Corollary 7.7.1. Namely, since the $*$-action is trivial, we have the decomposition

$$\mathcal{M}({}_k(G/P')) \cong \bigoplus_{w \in W^{I,J}} \mathcal{M}(Y_w)(l(w)),$$

where $Y_w \cong {}_k(G/Q_w) \times_{{}_k(G/P)} \operatorname{Spec} k$ for any $w \in W^{I,J}$ (we have ${}_k(Z_\delta) = {}_k(Z_D)$ for any $D \in \delta \subseteq W_I \backslash W / W_J$). Each Y_w is therefore the fiber of the natural k-morphism $\varphi : {}_k(G/Q_w) \to {}_k(G/P)$ at the point corresponding to ${}_kP$. Since P is defined over k, by Theorems 6.3 and 5.15 we have a k-defined isomorphism $P \cong L \ltimes R_u(P)$, where L is a k-defined reductive algebraic group with the root system $\Delta_I = B_3$. The group $L \cap Q_w = L \cap wP'w^{-1}$ is a parabolic subgroup of L of type $I_w \subseteq I$, since $\Delta_I^+ \subseteq w(\Phi^+)$ by Lemma 3.5. The variety $L/L \cap Q_w$ of parabolic subgroups of L of type I_w admits a natural embedding into the variety G/Q_w of parabolic subgroups of G of type I_w, given by

$$\begin{array}{rcl} i : L/L \cap Q_w & \to & G/Q_w \\ Q & \mapsto & Q \ltimes R_u(P). \end{array}$$

Since the $*$-action is trivial, $L/L \cap Q_w$ is also defined over k. Then, clearly, the embedding i is also defined over k, because it is defined over k_s and Γ-invariant. Since over K it is the isomorphism

$$L/L \cap Q_w \cong P/Q_w \cong (G/Q_w) \times_{G/P} \operatorname{Spec} K,$$

it also provides a k-isomorphism ${}_k(L/L \cap Q_w) \cong Y_w$. Summing up, we have the decomposition

$$\mathcal{M}({}_k(G/B)) \cong \bigoplus_{w \in W^{I,J}} \mathcal{M}\left({}_k(L/L \cap Q_w)\right)(l(w)),$$

where $L/L \cap Q_w$ is the variety of parabolic subgroups of type I_w in the reductive k-group L of type $\Delta_I = B_3$.

For example, let $P' = B$, the parabolic subgroup of type $J = \emptyset$. Since $I_w = \emptyset$ for any $w \in W^{I,\emptyset}$, all these k-varieties $L/L \cap Q_w$ are isomorphic the the variety $L/L \cap B$ of Borel subgroups of the reductive k-group L of type $\Delta_I = B_3$. Let us compute $W^{I,\emptyset}$. We have $|W| = 2^7 \cdot 3^2$ and $|W_I| = 2^3 \cdot 3! = 2^4 \cdot 3$. Computer try-out provides the following reduced decompositions of the elements w_i, $1 \le i \le 24$, of $W^{I,\emptyset} = W^{\{\alpha_1,\alpha_2,\alpha_3\},\emptyset}$:

i	w_i	$l(w_i)$
1	e	0
2	w_{α_4}	1
3	$w_{\alpha_4}w_{\alpha_3}$	2
4	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}$	3
5	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}$	4
6	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}$	4
7	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}$	5
8	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}$	5
9	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}$	6
10	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_4}$	6
11	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}$	7
12	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}$	7
13	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}$	8
14	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}w_{\alpha_3}$	8
15	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_3}$	9
16	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}$	9
17	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}$	10
18	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}$	10
19	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}$	11
20	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}$	11
21	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}$	12
22	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}$	13
23	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}$	14
24	$w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}$	15

Thus we have

$$\begin{aligned}\mathcal{M}({}_k(G/B)) &\cong \bigoplus_{i=0}^{15} \mathcal{M}\left({}_k(L/L \cap B)\right)(i) \oplus \bigoplus_{i=4}^{11} \mathcal{M}\left({}_k(L/L \cap B)\right)(i) \\ &\cong \mathcal{M}({}_k(L/L \cap B)) \otimes \mathcal{M}(\mathbb{P}^7) \otimes (\mathcal{M}(\mathbb{P}^8) \oplus \mathbb{L}^4).\end{aligned}$$

Let us compute also the motive $\mathcal{M}({}_k(G/P))$. The set $W^{I,I} = W^{\{\alpha_1,\alpha_2,\alpha_3\},\{\alpha_1,\alpha_2,\alpha_3\}}$ consists of those elements $w \in W^{I,\emptyset}$ which are the representatives of minimal length of their double cosets W_IwW_I. This condition is equivalent by Lemma 3.3 to $w(\alpha) \in \Phi^+$ for each $\alpha \in I$. Testing the elements of $W^{I,\emptyset}$ computed above, we obtain $W^{I,I} = \{w_1 = e, w_2, w_8, w_{13}, w_{24}\}$. Since the $*$-action is trivial, the decomposition looks as

$$\mathcal{M}({}_k(G/P)) \cong \bigoplus_{w\in W^{I,I}} \mathcal{M}({}_k(P/Q_w))(l(w)) \cong \bigoplus_{w\in W^{I,I}} \mathcal{M}({}_k(L/L\cap wPw^{-1}))(l(w)).$$

Here $L \cong P/R_u(P)$ is a k-defined reductive algebraic group with the root system $\Delta_I = B_3$ (containing the system of simple roots $\{\alpha_1,\alpha_2,\alpha_3\}$), the same as occurred in the computation for G/B, and $L/L\cap wPw^{-1}$ is the variety of parabolic subgroups of L of type I_w. The types $I_w = I \cap w(\Delta_I^+)$, $w \in W^{I,I}$, are the following.

$w \in W^{I,I}$	$l(w)$	$I_w = I \cap w(\Delta_I^+)$
w_1	0	$\{\alpha_1,\alpha_2,\alpha_3\}$
w_2	1	$\{\alpha_1,\alpha_2\}$
w_8	5	$\{\alpha_2,\alpha_3\}$
w_{13}	8	$\{\alpha_1,\alpha_2\}$
w_{24}	15	$\{\alpha_1,\alpha_2,\alpha_3\}$

Hence we obtain the decomposition

$$\begin{aligned}\mathcal{M}({}_k(G/P)) &\cong \mathbf{1} \oplus \mathcal{M}({}_k(L/P_{1,2}))(1) \oplus \mathcal{M}({}_k(L/P_{2,3}))(5) \oplus \mathcal{M}({}_k(L/P_{1,2}))(8) \oplus \mathbb{L}^{15} \\ &\cong \mathbf{1} \oplus \mathcal{M}({}_k(L/P_{1,2}))(1) \otimes \big(\mathbf{1} \oplus \mathbb{L}^7\big) \oplus \mathcal{M}({}_k(L/P_{2,3}))(5) \oplus \mathbb{L}^{15},\end{aligned}$$

where $P_{i,j}$ denotes the standard parabolic subgroup of L of type $\{\alpha_i,\alpha_j\}$.

The anisotropic case. If the group G is k-anisotropic, we still can compute the motives of products like $G/P \times G/P'$, using Theorem 7.7 itself. Let us compute, for example, $\mathcal{M}({}_k(G/P \times G/P'))$, where P is a parabolic subgroup of type $\{\alpha_1,\alpha_2,\alpha_3\}$ (the same as above, but not defined over k now) and P' is the parabolic subgroup of G of type $J = \{1, 4\}$. Since the $*$-action is trivial, we have the decomposition

$$\mathcal{M}\big({}_k(G/P \times G/P')\big) \cong \bigoplus_{w\in W^{I,J}} \mathcal{M}({}_k(G/Q_w))(l(w)),$$

where G/Q_w is the variety of parabolic subgroups of G of type I_w. Computer try-out gives the following list for $W^{I,J} \subseteq W^{I,\emptyset}$.

$w \in W^{I,J}$	$l(w)$	$I_w = I \cap w(\Delta_J^+)$
w_1	0	$\{\alpha_1\}$
w_3	2	$\{\alpha_1,\alpha_3\}$
w_4	3	$\{\alpha_3\}$
w_6	4	$\emptyset$
w_9	6	$\{\alpha_2\}$
w_{11}	7	$\{\alpha_2\}$
w_{14}	8	$\{\alpha_2\}$
w_{16}	9	$\emptyset$
w_{20}	11	$\{\alpha_3\}$
w_{22}	13	$\{\alpha_1,\alpha_3\}$
w_{23}	14	$\{\alpha_1\}$

This gives the decomposition

$$\begin{aligned}\mathcal{M}\big({}_k(G/P\times G/P')\big) \;\cong\; & \mathcal{M}({}_k(G/P_{\{\alpha_1\}}))\oplus\mathcal{M}({}_k(G/P_{\{\alpha_1,\alpha_3\}}))(2)\otimes\big(\mathbf{1}\oplus\mathbb{L}^{11}\big)\\ & \oplus\mathcal{M}({}_k(G/P_{\{\alpha_3\}}))(3)\otimes\big(\mathbf{1}\oplus\mathbb{L}^{8}\big)\\ & \oplus\mathcal{M}({}_k(G/B))(4)\otimes\big(\mathbf{1}\oplus\mathbb{L}^{5}\big)\\ & \oplus\mathcal{M}({}_k(G/P_{\{\alpha_2\}}))(6)\otimes\mathcal{M}(\mathbb{P}^2_k).\end{aligned}$$

3. **Groups of type $^3D_4$.** Let G be a k-defined semi-simple algebraic group with $\Phi = D_4$. The group of automorphisms of the Dynkin diagram of Φ is isomorphic to S_3, and the $*$-action of Γ on Π can be non-trivial. We consider the case when Γ acts as the cyclic group of order 3, that is, by three permutations $\{\mathrm{id}, r, r^2\}$, where $r(\alpha_1)=\alpha_3$, $r(\alpha_3)=\alpha_4$, $r(\alpha_4)=\alpha_1$, and $r(\alpha_2)=\alpha_2$.

The only $*$-stable subsets of Π are $\emptyset$, $\{\alpha_2\}$, $\{\alpha_1,\alpha_3,\alpha_4\}$, and Π. Let us compute $\mathcal{M}({}_k(G/P\times G/P'))$, where P is of type $\{\alpha_1,\alpha_3,\alpha_4\}$ and P' is of type $\{\alpha_2\}$. The set of representatives $W^{I,J}$ of the classes $W_I\backslash W/W_J$ consists of the following elements w_i, $1\le i\le 15$.

i	w_i	$l(w_i)$	$I_{w_i} = I\cap w_i(\Delta_J^+)$
1	e	0	$\emptyset$
2	$w_{\alpha_2}w_{\alpha_1}$	2	$\{\alpha_1\}$
3	$w_{\alpha_2}w_{\alpha_3}$	2	$\{\alpha_3\}$
4	$w_{\alpha_2}w_{\alpha_4}$	2	$\{\alpha_4\}$
5	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}$	3	$\emptyset$
6	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_4}$	3	$\emptyset$
7	$w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}$	3	$\emptyset$
8	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_4}$	4	$\emptyset$
9	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}$	5	$\emptyset$
10	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_4}w_{\alpha_2}w_{\alpha_3}$	5	$\emptyset$
11	$w_{\alpha_2}w_{\alpha_3}w_{\alpha_4}w_{\alpha_2}w_{\alpha_1}$	5	$\emptyset$
12	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}w_{\alpha_2}w_{\alpha_1}$	7	$\{\alpha_3\}$
13	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}w_{\alpha_2}w_{\alpha_3}$	7	$\{\alpha_1\}$
14	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_4}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}$	7	$\{\alpha_4\}$
15	$w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}w_{\alpha_2}w_{\alpha_1}w_{\alpha_3}$	8	$\emptyset$

By Theorem 7.7 we obtain the following decomposition:

$$\mathcal{M}({}_k(G/P\times G/P'))\cong\mathcal{M}(Y)(2)\otimes(\mathbf{1}\oplus\mathbb{L}^5)\oplus\mathcal{M}({}_k(G/B))\otimes(\mathbf{1}\oplus\mathbb{L}^3\oplus\mathbb{L}^4\oplus\mathbb{L}^5\oplus\mathbb{L}^8),$$

where Y is a k-variety such that

$$Y\times_{\operatorname{Spec}k}\operatorname{Spec}K\cong G/P_{\{\alpha_1\}}\amalg G/P_{\{\alpha_3\}}\amalg G/P_{\{\alpha_4\}}$$

(three isomorphic projective quadrics of dimension 6).

4. **Odd-dimensional orthogonal groups.** Let G be a k-defined semi-simple algebraic group such that ${}_kG$ is the special orthogonal group $\mathrm{SO}_{2n+1,k}(q)$, where q is a quadratic form over k. Then $\Phi = B_n$, and the minimal k-defined parabolic subgoup of G has type $\Pi\setminus\{\alpha_1,\alpha_2,\dots,\alpha_r\}$, where $r\ge 0$ is the Witt index of q (see [20]). Consequently, if G is isotropic, the parabolic subgroup P of type

$I = \Pi \setminus \{\alpha_1\} = \{\alpha_2, \alpha_3, \dots, \alpha_n\}$ is k-defined. Let us compute the motivic decompositions provided by P. Note that the $*$-action on Π is trivial, since the Dynkin diagram of B_n has no non-trivial automorphisms.

As above, we need to compute the set of representatives $W^{I,\emptyset}$ of $W_I \backslash W$. Consider the element

$$\widetilde{w} = w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_{n-1}} w_{\alpha_n} w_{\alpha_{n-1}} \dots w_{\alpha_2} w_{\alpha_1}.$$

of the Weyl group W.

Lemma 8.1. *The above decomposition of $\widetilde{w}$ is reduced.*

Proof. We show that this decomposition of $\widetilde{w}$ is reduced, or, equivalently, $l(\widetilde{w}) = 2n-1$, by induction on n. The case $n = 2$ can be checked by hand. Suppose further that the decomposition $\widetilde{w}' = w_{\alpha_2} \dots w_{\alpha_{n-1}} w_{\alpha_n} w_{\alpha_{n-1}} \dots w_{\alpha_2}$ is reduced. Then by Lemma 3.3 there are exactly $l(\widetilde{w}')$ roots $\alpha \in \Phi^+$ such that $(\widetilde{w}')^{-1}(\alpha) \in \Phi^-$. Since $\widetilde{w}'$ is also an element of the Weyl group $W_{\Pi\setminus\{\alpha_1\}}$ of the root subsystem $\Delta_{\Pi\setminus\{\alpha_1\}}$ of Φ, all these roots are in $\Delta^+_{\Pi\setminus\{\alpha_1\}}$. Hence $(\widetilde{w}')^{-1}(\alpha_1) \in \Phi^+$, and consequently, $l(w_{\alpha_1}\widetilde{w}') = l(\widetilde{w}')+1 = 2n-2$ by the inductive assumption. To show that $l(\widetilde{w}) = 2n-1 = w_{\alpha_1}\widetilde{w}'+1$, we need moreover to prove that $w_{\alpha_1}\widetilde{w}'(\alpha_1) \in \Phi^+$. Since the length of $\widetilde{w}'$ is also equal to the number of positive roots it sends to Φ^-, the same reasoning as above shows that $\widetilde{w}'(\alpha_1) \in \Phi^+$. Then if $w_{\alpha_1}\widetilde{w}'(\alpha_1) \in \Phi^-$, we must have $\widetilde{w}'(\alpha_1) = \alpha_1$, since it is the only positive root sent to Φ^- by w_{α_1}. On the other hand, since $l(\widetilde{w}'w_{\alpha_2}) = l(\widetilde{w}') - 1$, we have $\widetilde{w}'(\alpha_2) \in \Delta^-_{\Pi\setminus\{\alpha_1\}}$. Now if $\widetilde{w}'(\alpha_1) = \alpha_1$, then the root $\widetilde{w}'(\alpha_1 + \alpha_2)$ has both positive and negative coefficients in its decomposition into a sum of simple roots, which is impossible. Hence $w_{\alpha_1}\widetilde{w}'(\alpha_1) \in \Phi^+$, and $l(\widetilde{w}) = 2n-1$. □

We can prove in a similar fashion that $l(w_{\alpha_i}\widetilde{w}) = l(\widetilde{w})+1$ for any $\alpha_i \in \Pi\setminus\{\alpha_1\} = I$, and therefore $\widetilde{w}$ is the element of minimal length in its coset $W_I\widetilde{w}$, that is, lies in $W^{I,\emptyset}$. Consider the set A of all products of the form $w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_i}$, $1 \le i \le n$, and of the form $w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_n} w_{\alpha_{n-1}} \dots w_{\alpha_{i+1}} w_{\alpha_i}$, $n-1 \ge i \ge 1$ (all continuous subwords of $\widetilde{w} = w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_{n-1}} w_{\alpha_n} w_{\alpha_{n-1}} \dots w_{\alpha_2} w_{\alpha_1}$ starting from the left). For any $\alpha_i \in I$, since $l(w_{\alpha_i}\widetilde{w}) = l(\widetilde{w}) + 1$, we also have $l(w_{\alpha_i} w) = l(w) + 1$ for any $w \in A$; hence all elements $w \in A$ are the elements of minimal length in their respective cosets $W_I w$. Since they are of different lengths, their cosets are distinct. Since there are exactly $2n-1$ of them, and $|W|/|W_I| = 2^n n!/2^{n-1}(n-1)! = 2n$, together with the representative e of the coset P they form the set $W^{I,\emptyset}$.

Now we compute, for example, $\mathcal{M}({}_k(G/P \times G/B))$. Since the $*$-action is trivial, by Theorem 7.7 we have

$$\begin{aligned} \mathcal{M}({}_k(G/P \times G/B)) &\cong \bigoplus_{w\in W^{I,\emptyset}} \mathcal{M}({}_k(G/B))(l(w)) \\ &\cong \bigoplus_{i=0}^{2n-1} \mathcal{M}({}_k(G/B))(i) \cong \mathcal{M}({}_k(G/B)) \otimes \mathbb{P}^{2n-1}. \end{aligned}$$

If G is isotropic, that is, P is k-defined, we can compute $\mathcal{M}({}_k(G/B))$ by Corollary 7.7.1. As in the F_4 example, we obtain

$$\mathcal{M}({}_k(G/B)) \cong \bigoplus_{w\in W^{I,\emptyset}} \mathcal{M}({}_k(L/L\cap B))(l(w)) \cong \mathcal{M}({}_k(L/L\cap B)) \otimes \mathbb{P}^{2n-1},$$

where L is a reductive k-defined subgroup of G with $\Phi(T, L) = \Delta_I = B_{n-1}$. If P is not a minimal k-defined parabolic subgroup, that is, the Witt index r is greater than

1, the group L is also isotropic, and we can continue by induction. Then we have

$$\mathcal{M}({}_k(G/B)) \cong \mathcal{M}({}_k(G'/B')) \otimes \mathbb{P}^{2nr-r^2},$$

where G'/B' is the variety of Borel subgroups of a k-defined anisotropic reductive group G' with the root system B_{n-r}.

Let us also compute $\mathcal{M}({}_k(G/P))$ using Corollary 7.7.2. It is clear that the subset $W^{I,I} \subseteq W^{I,\emptyset}$ consists of three elements $e, w_{\alpha_1}, \widetilde{w}$. We have $I_e = I \cap \Delta_I^+ = I$; $I_{w_{\alpha_1}} = I \cap w_{\alpha_1}(\Delta_I^+) = I \setminus \{\alpha_2\}$, since $w_{\alpha_1}(\alpha_i) = \alpha_i$ for any $i > 2$, and no linear combination of α_1 and α_2 is in Δ_I; finally, one easily checks by induction that $I_{\widetilde{w}} = I \cap \widetilde{w}(\Delta_I^+) = I$. Thus we have the decomposition

$$\begin{aligned}\mathcal{M}({}_k(G/P)) &\cong \mathcal{M}({}_k(L/L \cap P)) \oplus \mathcal{M}({}_k(L/L \cap Q_{w_{\alpha_1}}))(1) \oplus \mathcal{M}({}_k(L/L \cap P))(2n-1) \\ &\cong \mathbf{1} \oplus \mathcal{M}({}_k(L/L \cap P_{\{\alpha_1,\alpha_2\}}))(1) \oplus \mathbb{L}^{2n-1}.\end{aligned}$$

Proceeding by induction, we get

$$\mathcal{M}({}_k(G/P)) \cong \mathbb{P}^{r-1} \oplus \mathcal{M}({}_k(G'/P'))(r) \oplus \mathbb{L}^{2nr-r(r+1)/2},$$

where G' is as above and G'/P' is the corresponding variety of parabolic subgroups of G'.

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