# Average Height Of Isogenous Abelian Varieties

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# Introduction

As the title of this thesis suggests we will be studying the average Faltings height formula (which will be referred to as average height formula all throughout this thesis) of elliptic curves and abelian varieties.

More precisely, given an elliptic curve $E/K$ defined over a number field $K$, let $p : X \to B$ be the minimal regular model of $E/K$ with the zero section $s : B \to X$, $B = Spec(R)$, $R$ denoting the ring of integers of the number field $K$, then we define the Faltings height function of the elliptic curve $E$ as:

$$h_F(E) = \frac{1}{[K:\mathbb{Q}]}\left(\log \#(p_*\omega_{X/B}/\omega.R) - \sum_{v\in S_\infty} \epsilon_v \log \|\omega\|_v\right)$$

where $\omega_{X/B}$ is the dualizing sheaf of the arithmetic surface $X$, $\omega$ a non-zero rational section of the line bundle $p_*\omega_{X/B}$ and $S_\infty$ denotes the set of infinite places of $K$ and $\epsilon_v = 1, 2$ according to whether $K_v \cong \mathbb{R}$ or $\mathbb{C}$ respectively.

The Faltings height function is defined analogously for abelian varieties. Let $A_K$ be an abelian variety defined over a number field $K$. Let $\pi : N(A) \to B$ be the Néron model of $A_K$ with zero section $s : B \to N(A)$, where $B = Spec(R)$, where $R$ denotes the ring of integers of the number field $K$. We denote $s^* \wedge^g \Omega_{N(A)/R}$ by $\omega_{A/R}$. Then the Faltings height of $A_K$ is given as

$$h_F(A_K) = \frac{1}{[K:\mathbb{Q}]}\left(\log \#(\omega_{A/R}/R.s) - \sum_{v\in S_\infty} \epsilon_v \log(\|s\|_v)\right)$$

The idea of computing the average height formula has its origin in [Fa 1] where he computes the difference of height of two isogenous semi-abelian varieties. We can state the formula for abelian varieties as:
Let $A_K$, $B_K$ be two abelian varieties defined over a number field $K$, related by an isogeny $f : A_K \to B_K$, let the map extend to $f : N(A) \to N(B)$, then the following formula holds:

$$h_F(B_K) - h_F(A_K) = \frac{1}{2}deg(f) - \frac{1}{[K:\mathbb{Q}]}\log \#(s^*\omega_{ker(f)/R}).$$

This formula will be referred to as the Faltings formula.

Let $E/K$ be a semi-stable elliptic curve defined over a number field $K$. Let $C$ denote all the cyclic subgroups of order $N$ in $E$. Let $E^{'}$ denote the quotient of the elliptic curve $E$ by a cyclic subgroup of order $N$. Then the following formula known as the average height formula for semi-stable elliptic curves holds:

$$\frac{1}{e_N}\sum_{C}\Big(h_F(E^{'})-h_F(E)\Big)=\frac{1}{2}\log N-\lambda_N,$$

where $e_N$ denotes the number of cyclic subgroups of order $N$, and $\lambda_N$ is a constant which depends only on $N$.

Let $A_K$ be a principally polarised abelian variety defined over a number field $K$ with good reduction at a prime $p$. Let $G_i$ denote the isotropic subgroups of order $p^g$ in $A[p]$. Let $A_i$ denote the quotient abelian variety $A/G_i$. Then the following formula which would be referred to as the average height formula for abelian varieties holds:

$$\sum_{C}\left(h_F(A_i)-h_F(A)\right)=\Big(\frac{eg}{2}-m(g,p)\Big)\log p.$$

where the sum runs over all isotropic subgroups of order $p^g$ in $A[p]$, $e$ denotes the number of isotropic subgroups of order $p^g$ in $A[p]$, and $m(g,p)$ denotes a constant which depends on $g$, $p$.

We will start with the proof of Faltings formula for isogenous abelian varieties in chapter one. We will first study the classical theory of height functions in section one and two of chapter one and then proceed to study Faltings height function in section three. In section four we first study the concept of Néron model and why the Faltings stable height remains an invariant under field extension. Finally using all the concepts studied till then we prove Faltings formula for isogenous abelian varieties.

There are at least three ways of approaching the average height formula of elliptic curves. One approach involving basic concepts of Arakelov intersection theory due to Robin de Jong [Ro], one due to Autissier [Au] which involves the concepts of calculating the height on the moduli space of elliptic curves, and another approach which involves concepts from both the methods. We develop Arakelov intersection theory in section one and two of chapter two. In section three we give the proof of average height formula using Robin de Jong's approach. In section four we give an outline of the approach adopted by Autissier, and give the third proof which makes use of results from both the approaches.

In chapter three we prove the average height formula for abelian varieties. We have only one way of proving the average height formula for abelian varieties. It involves the properties of $A_{g,n}^{p}$, the moduli space of principally polarised abelian varieties with good reduction at $p$ and of type $(g;n)$. We introduce the basic notions, like the definitions of abelian schemes, good reduction of abelian varieties, standard sympletic pairing in section one. In section two we study arithmetic intersection theory of arithmetic varieties. In section three we look at another reformulation of Faltings formula for isogenous abelian varieties in terms of Cartier divisors, using the intersection theory developed in section three. In section four we prove the average height formula using all the theory that is developed in the first three sections.

In chapter four we conclude our thesis with a few remarks on the different techniques adopted in proving the average height formula for elliptic curves, and why that only Autissier's approach is naturally generalised to abelian varieties.

# 1 Faltings Formula for Isogenous Abelian Varieties

In this chapter we study the basic notions around height functions and the formula given by Faltings for isogenous abelian varieties.
In section 1 we look at how height functions are defined on projective space. In section 2 we study how height functions are defined on projective varieties. In section 3 we look at how height functions are defined via metrized line bundles and how Faltings height function is defined. In section 4 we look at the proof of Faltings formula for isogenous abelian varieties.

We closely follow Silverman's article 'The Theory of Height Functions' from [AG].

## 1.1 Height on Projective Space

In this section we see how the height function is defined over a projective space. In the next section we extend this definition to projective varieties.

Let us denote the set of places over the number field $K$ by $M_K$, the set of non-archimedean places by $M_K^f$ and the set of archimedean places by $M_K^\infty$. For $v \in M_K^\infty$, $\|a\|_v$ denotes $|\tau(a)|^{\epsilon_v}$ where $\tau : K \to \mathbb{C}$ is an embedding associated to the place $v$ and $\epsilon_v = 1$ or 2 according to whether $\tau$ is a real or a complex embedding respectively for $a \in K$. For $v \in M_K^f$ $\|a\|_v$ denotes $|a|_v^{[K_v:\mathbb{Q}_v]}$, where $|a|_v$ is the usual $v$-adic absolute value on $K, K_v$ denotes the completion of the field $K$ with respect to $v$, for $a \in K$.

**1.1 Definition.** A height function is defined as a function from the points of a projective space $\mathbb{P}^n(K)$ defined over a number field $K$ to the field of real numbers:

$$H_K : \mathbb{P}^n(K) \to \mathbb{R}$$

$$\text{where } H_K(P) = \prod_{v \in M_K} \max\{\|x_\circ\|_v, ............, \|x_n\|_v\}$$

for all $P \in \mathbb{P}^n(K)$.

The height function does not depend on the choice of homogeneous coordinates of the projective space $\mathbb{P}^n(K)$, it is well-defined, but the height function $H_K$ evidently depends on the number field $K$ .

For a finite extension $L/K$, we know that $\sum[L_w : K_v] = [L : K]$, where the sum is over the places $w \in M_L$ lying over a given $v \in M_K$. Hence we can relate the height function $H_K$ defined over $\mathbb{P}^n(K)$ with $H_L$ defined over $\mathbb{P}^n(L)$:

$$H_L(P) = H_K(P)^{[L:K]}.$$

We can always choose homogeneous coordinates for a point $P \in \mathbb{P}^n(K)$ with some $x_i = 1$. Hence $H_K(P)$ which does not depend on the choice of coordinates is always $\geq 1$ for all points $P \in \mathbb{P}^n(K)$.

We saw that our height function $H_K$ depends on the number field $K$. So we now define what is called the absolute height function which is independent of the field of definition.

**1.2 Definition.** The absolute height function is a function from $\mathbb{P}^n(\bar{\mathbb{Q}})$ to the field of real numbers :

$$H : \mathbb{P}^n(\bar{\mathbb{Q}}) \to \mathbb{R},$$

$$H(P) = H_K(P)^{1/[K:\mathbb{Q}]}$$

where $K$ is any number field such that $P \in \mathbb{P}^n(\bar{\mathbb{Q}}.)$.

EXAMPLE: Let $P \in \mathbb{P}^n(\mathbb{Q})$. Let $P = (x_\circ : x_1 : .... : x_n)$ where $x_i \in \mathbb{Z}$ and $gcd(x_\circ, x_1, ..., x_n)$=1. Then

$$H(P) = \max\{|x_\circ|, |x_1|, ...., |x_n|\}.$$

Then it is easy to see that the set $\{P \in \mathbb{P}^n(\mathbb{Q}) : H(P) \leq C\}$ for some constant $C$ is finite.

If $P = (x_\circ : .... : x_n) \in \mathbb{P}^n(\bar{\mathbb{Q}})$ , then we can define $\mathbb{Q}(P)$ as

$$\mathbb{Q}(P) = \mathbb{Q}\left(x_\circ/x_i : .... : x_n/x_i\right) \text{ for some } x_i \neq 0$$

Now we state a very important property of the absolute height function $H$.

**Theorem 1.3** (Finiteness Theorem) Let $C$ and $d$ be constants. Then

$$S = \left\{P \in \mathbb{P}^n(\bar{\mathbb{Q}}) : H(P) \leq C \quad \text{and} \quad [\mathbb{Q}(P) : \mathbb{Q}] \leq d\right\}$$

is a finite set.

**Proof**. We first prove that the set

$$S^{'} = \left\{P \in \mathbb{P}^1(\bar{\mathbb{Q}}) : H(P) \leq C^{'} \text{ and } [\mathbb{Q}(x) : \mathbb{Q}] = d^{'}\right\}$$

is finite.
Let $P^{'} \in \mathbb{P}^1(\bar{\mathbb{Q}})$ and$[\mathbb{Q}(x):\mathbb{Q}] = d$. Let $x^1, ...., x^d$ be the conjugates of $x$ over $\mathbb{Q}$ and let $1 = s_{\circ}$,.....$s_d$ be the elementary symmetric polynomials in $x^1, ....., x^d$. Then each $s_j$ is in $\mathbb{Q}$,and $x$ is a root of the polynomial.

$$F(X) = \sum_{j=0}^{d}(-1)^j s_j X^{d-j} = \prod_{i=1}^{d}(X - x^{(i)}) \in \mathbb{Q}[X].$$

Now using the triangle inequality , one easily checks that $H([1, s_j]) \leq c_j H([1, x])^j$ for certain constants $c_j$ which do not depend on $x$. Hence we can see that there are finitely many sets of $s_j$'s , hence finitely many possibilities for the polynomial $F(X)$, and so only finitely many possibilities for $x$.

Now choose homogeneous coordinates for $P = (x_{\circ} : ... : x_n)$ with some $x_i = 1$. Let us denote $\mathbb{Q}(P)$ by $K$. Then

$$H_K(P) = \prod_{v \in M_K} \{\max \|x_{\circ}\|_v , ..... \|x_n\|_v\} \geq \max_j H_K(x_j)$$

Hence $H(P) \geq H((1 : x_j))$ for all $j$. Now $P^{'}$=[1,$x_j$] $\in \mathbb{P}^n(\bar{\mathbb{Q}})$ with $H(P^{'}) \leq C$ , and $[\mathbb{Q}(x) : \mathbb{Q}] \leq d$. But from the above argument there are only finitely many possibilities for such an $x_i$. Hence the set $S$ is finite. □

In practice the logarithm of the absolute function is more widely used. Hence

$$h(P) = \log H(P)$$

will be referred to as the 'height function' from now on.

## 1.2 Height on Projective Varieties

In the last section we saw how the height function is defined over projective spaces. In this section we see how height functions are defined on projective varieties defined over $\bar{\mathbb{Q}}$.

In order to define a height function on $V$, a projective variety of dimension $n$ over $\bar{\mathbb{Q}}$, we take a map from $V$ into projective space and use the height function from the previous section.

**2.1 Definition.** Let $F : V \to \mathbb{P}^n$ be a morphism. The (logarithmic) height on $V$ relative to $F$ is defined by

$$h^F : V(\bar{\mathbb{Q}}) \to \mathbb{R},\ h^F(P) = h(F(P)).$$

It is well-known that any morphism from the projective variety $V$, $F : V \to \mathbb{P}^n$ into the projective space $\mathbb{P}^n$ is associated to an invertible sheaf (or line bundle) on $V$, namely the pull-back of the twisting sheaf, $F^*O_{\mathbb{P}}(1)$.
In fact we know that if $X$ is any scheme over $A$, and $\phi : X \to \mathbb{P}^n_A$ an $A$-morphism of $X$ to $\mathbb{P}^n_A$, then $\ell = \phi^*(O(1))$ is an invertible sheaf on $X$, and the global sections $s_\circ$, $s_1$,.....,$s_n$ where $s_i$=$\phi^*(x_i)$ generate the sheaf $\ell$. Conversely any invertible sheaf $\ell$ and global generating sections $s_i$ determine a morphism $\phi : X \to \mathbb{P}^n_A$. This statement has been proved in [Ha] chapter two proposition 7.1. Naturally many different maps give rise to the same sheaf. Hence we can expect the height function to be essentially same for the morphisms which determine the same sheaf. The following proposition says that they are essentially the same.

**2.2 Definition.** Two height functions $h$ and $h'$ defined on a projective variety $V$ are said to be equivalent if $\left|h - h'\right|$ is bounded as $P$ ranges over $V(\bar{\mathbb{Q}})$.

**2.3 Theorem.** Let

$$F : V \to \mathbb{P}^n \, and \, G : V \to \mathbb{P}^m$$

be two maps of $V$ such that $F^*O_{\mathbb{P}^n}(1) \cong G^*O_{\mathbb{P}^m}(1)$. Then $h^F$ and $h^G$ are equivalent.

**Proof.** Let $E$ be a divisor in the linear system of $\ell$. (That is $E \geq 0$ and $\ell \approx \mathcal{O}_V(E)$). Then on the complement of $E$, we can write $F = [f_\circ, f_1, ...f_n]$ and $G = [g_\circ, ...g_n]$ with rational functions $f_i$ and $g_j$ such that
$(f_i)$ and $(g_j)$ such that

$$(f_i) = D_i - E \text{ and } (g_j) = D_j^{'} - E \text{ for divisors } D_i \; D_j^{'} \geq 0.$$

We are guarenteed of such a divisors from arguments from [Ha] II.7.8.1.

We know that the $F$ has no base points on $V$ which means that the $D_i$'s have no point in common. Let $K$ be a common field of definition for $V$, $f_\circ$, $f_1$,...,$f_n$ and $g_\circ$,....,$g_m$ . Now pick any $j$ and look at the ideal $\mathcal{I} = (f_\circ/g_j, ......f_n/g_j)$ in the ring $\mathcal{R} = K\,[f_\circ/g_j, ......f_n/g_j]$. Since $(f_i/g_j)$=$D_i$-$D_j^{'}$ and the $D_i$'s have no point in common, it follows that I is the unit ideal. Suppose not, then there will be a maximal ideal $\mathcal{M}$ of $\mathcal{R}$ containing $\mathcal{I}$. Since $Spec(R)$ is isomorphic to an open subset of $V$ containing the complement of $D_j^{'}$, $\mathcal{M}$ will correspond to a point $P$ of $V$ not in $D_j^{'}$ such that $(f_i/g_j)(P) = 0$ for all $i$. But then $P$ will lie in the support of all $D_i$ ,yielding a contradiction.

Hence we can find a polynomial $\phi_j(T_\circ, ......T_n) \in K\,[T_\circ, ...T_n]$ having no constant term such that

$$\phi_j(f_\circ \circ /g_j, ......f_n/g_j) = 1$$

Taking the $v$-adic absolute value and using the triangle inequality, one easily finds a constant $C_1$=$C_1(v, F, G, \phi_j) \geq 0$ such that for all $P$ in the complement of $D_j^{'}$ ,

$$max\left\{\left|f_\circ/g_j(P)\right|_v, \ldots.. \left|f_n/g_j(P)\right|_v\right\} \geq C_1$$

We can choose $C_1 = 1$ for all but finitely many $v$, independent of $P$. (Note that it may be necessary to extend $K$ so that $P(K)$.)
Next multiply through by $|g_j(P)|_v$ . Then the equality also holds for $g_j(P) = 0$, so taking the maximum over $j$ yields:

$$max\left\{|f_\circ|_v, \ldots.. |f_n|_v\right\} \geq C_2 max\left\{|g_\circ(P)|_v, \ldots.. |g_m(P)|_v\right\}$$

for a constant $C_2$=$C_2$(v,F,G)$\geq$ 0, where $P$ ranges over the complement of $E$ and $C_2$=1 for all but finitely many $v$. Now raise to the $[K_v : \mathbb{Q}_v]$ power, multiply over all $v \in M_k$ and take the $[K : \mathbb{Q}]$ th root. This gives

$$H(F(P)) \geq C_3 H(G(P)),$$

with $C_3$=$C_3(F, G) > 0$ , as $P$ ranges over the complement of $E$ in $V$.
Next , since $\ell$ has no base points , we can choose finitely many divisors $E_1$,...,$E_r$ in the linear system for $\ell$ so that the $E_i$'s have trivial intersection. In this way we obtain the above inequality on all of $V$. Taking logarithms gives one of the desired bounds, and the others followed by symmetry. □

Now that we have the desired equivalence we try to relate $Pic(V)$, the group of equivalence classes of invertible sheaves and the group of height functions modulo constant functions .
Let us denote the group of functions $\{h : V \to \mathbb{R}\}$ mod $O(1)$ by $\mathcal{H}(V)$.

**2.4 Definition.** Let $\ell$ be a sheaf without base points on $V$. The height function associated to $\ell$ is the class of functions $h^\ell \in \mathcal{H}(V)$ obtained by taking the height function $h^F$ for any map $F$ associated to $\ell$. (From theorem 1.2 $h^\ell$ is well defined.)
**Proposition 2.5** Let $\ell$ and $\mathcal{M}$ be base point-free sheaves on $V$. Then $h^{\ell\otimes\mathcal{M}}$ and $\left(h^\ell + h^{\mathcal{M}}\right)$ are equivalent.
**Proof**. Let $F = [f_\circ, ....f_n]$ and $[g_\circ, ....g_m]$ be maps associated to $\ell$ and $\mathcal{M}$ respectively. Then

$$T = [...., f_i g_j ....]_{0 \leq i \leq n, 0 \leq j \leq m} : V \to \mathbb{P}^{nm+n+m}.$$

is the map associated to the invertible sheaf $\ell \otimes \mathcal{M}$. This map is obtained by composing the diagonal morphism $D : V \to V \times V$ with the composition of the map $F$:V×V $\to$ $\mathbb{P}^n \times \mathbb{P}^m$ with the Segre embedding $S : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$:

$$T = S \circ (F \times G) \circ D$$

$$\text{Since } \max\left\{.., |f_i g_j|_v ..\right\} = \max\left\{.., |f_i|_v ..\right\} \max\left\{.., |g_j|_v ..\right\}$$

we have

$$h^T(p) = h^{F(P)} + h^G(P) + O(1) \text{ for all } P \text{ in } V \qquad \square$$

We know that the set of equivalence classes of invertible sheaves $Pic(V)$ on the projective variety $V$ is a group under the operation $\otimes$. Hence it is very natural to associate to the invertible sheaf $\ell = \ell_1 \otimes \ell_2^{-1}$ the height function $h^{\ell_1} - h^{\ell_2}$.

In fact from proposition 2.5 we can check that $h^{\ell \otimes \ell^{-1}} = h^{\ell} - h^{\ell} = 0$. So we are now all set to relate the group of invertible sheaves under equivalence classes $Pic(V)$ with $\mathcal{H}(V)$. In fact we can formulate all that we have seen into the following theorem:

**2.6 Theorem.** (a) There exists a unique homomorphism

$$\phi : Pic(V) \to \mathcal{H}(V),$$

$$\ell \to h^{\ell}$$

with the property that if $\ell$ has no base points and $F : V \to \mathbb{P}^n$ is a morphism associated to $\ell$, then

$$h^{\ell} = h^{F} + O(1).$$

(b) If $f : V \to W$ is a morphism of smooth varieties, and $\ell$ is an invertible sheaf on $W$, then

$$h^{f^*\ell} = h^{\ell} \circ f + O(1)$$

[That is, the homomorphism in (a) is functorial with respect to morphisms of smooth varieties.]

**Proof.** (a) The map is well defined: for any two functions on $V$ to be equivalent they have to be associated to the same invertible sheaf. The map $\phi$ preserves the group structure as well. The height function associated to $\ell \otimes \mathcal{M}$, $h^{\ell \otimes \mathcal{M}}$ is equal to $h^{\ell} + h^{\mathcal{M}}$ from proposition 2.5. Also the height function corresponding to $O_V$ is a constant function. Hence $\phi$ is a group homomorphism.
(b) If $\ell$ has no base points and $F : W \to \mathbb{P}^n$ is associated to $\ell$, $F \circ f : V \to \mathbb{P}^n$ is the morphism associated to $f^*\ell$. But $h^{F \circ f} = h^F \circ f$ .Hence $h^F \circ f$ is equivalent to $h^{f^*\ell}$. $\square$

**2.7 Corollary** (Finiteness). If $\ell$ is an ample sheaf on $V$, then for all constants $C$ and $d$, the set

$$S = \left\{ P \in V(\bar{\mathbb{Q}}) : h^{\ell}(P) \leq C \, and \, [\mathbb{Q}(P) : \mathbb{Q}] \leq d \right\}$$

is finite.

**Proof.** (From [Ha] theorem 7.5 chapter 2 ) Since $\ell$ is ample, $\ell^{\otimes m}$ is very ample for some integer $m \geq 0$. From proposition 2.6 we have $h^{\ell^{\otimes m}} = mh^{\ell}$. Let $F : V \to \mathbb{P}^n$ be an embedding associated to $\ell^{\otimes m}$. Let $P \in S$, then we have

$$C \geq h^{\ell}(P) = (1/m)h^{\ell^{\otimes m}}(P) + O(1) = h(F(P)) + O(1).$$

From proposition 1.3 we know that the number of $F(P)$ satisfying $h(F(P)) \leq C$ and $[\mathbb{Q}(P) : \mathbb{Q}] \geq d$ is finite. Hence $S$ is finite. □

## 1.3 Heights and Metrized Line Bundles

In this section we look at how height functions can be defined via metrized line bundles on $Spec(R)$, where $R$ is the ring of a integers of a number field $K$.

**3.1 Definition.** Let $X$ be a scheme. A line bundle over $X$ is a locally free $O_X$ module of rank 1.

**3.2 Definition.** A metrized line bundle on $Spec(R)$ is a pair $(\ell, |.|)$, where $\ell$ is a line bundle on $Spec(R)$, and for each $v \in M_K^\infty$ $|.|_v$ is a $v$-adic norm (metric) on the one-dimensional $K_v$-vector space $\ell \otimes_R K_v$.

**3.3 Definition.** The degree of a metrized line bundle $(\ell, |.|)$ is defined as

$$deg(\ell, |.|) \;=\; \log \#(\ell/Rt) \;-\; \sum_{v \in M_K^\infty} \epsilon_v \log |t|_v$$

where $\epsilon_v = [K_v : \mathbb{Q}_v]$ for any $t \in \ell$ such that $t \neq 0$.

The degree of a line bundle is independent of the choice of $t$. That is because : Consider any other section $s \in \ell$ , $s = at$ for some $a \in K^*$, then

$$\begin{aligned}\log \#(\ell/Rs) - \sum_{v \in M_K^\infty} \epsilon_v \log |s|_v &= \log \#(\ell/Rat) - \sum_{v \in M_K^\infty} \epsilon_v \log |at|_v \\ &= \log \# \left((\ell/Rt)(Rt/Rat)\right) \;-\; \sum_{v \in M_K^\infty} \epsilon_v \log |at|_v\end{aligned}$$

Now consider

$$\log \#(Rt/Rat) = \log(\left|N_{K/\mathbb{Q}}(a)\right|) = \sum_{\tau} \log |\tau(a)|$$

where $\tau : K \to \mathbb{C}$ are the complex embeddings of $K$,

$$\sum_{\tau} \log |\tau(a)| = \sum_{v \in M_K^\infty} \epsilon_v \log |a|_v \, .$$

Hence we have

$$\log \#(\ell/Rs) - \sum_{v \in M_K^\infty} \epsilon_v \log |s|_v = \log \#(\ell/Rt) \;-\; \sum_{v \in M_K^\infty} \epsilon_v \log |t|_v \, .$$

Example: If $R$ is a P.I.D., then $\ell$ is free, so we can choose a $t \in \ell$ so that $\ell = Rt$. Then

$$deg(\ell, |.|) = - \sum_{v \in M_K^\infty} \log \|t\|_v .$$

In particular, if $R = \mathbb{Z}$ , then up to + or - 1 there is a unique generator $t$ for $\ell$ and then $deg(\ell, |.|) \ = -\log |t|_\infty$.

**Metrized Line Bundles on Varieties**

In this subsection we define the height function associated to a metrized line bundle $\ell$ of a smooth projective variety $V$ defined over a number field $K$.

**3.4 Definition.** Let $v \in M_K$. A $v$-adic metric on $\ell$ a line bundle defined on the projective variety over $K$ consists of a (non-trivial) $v$-adic norm $|.|_v$ on each fiber $\ell_P \otimes K_v$ such that norms vary continuously with $P \in V(K_v)$. That is if $f \in H^\circ(U, \ell)$ is a section on some open set $U$, and if $U(K_v)$ is given the $v$-adic topology, then the map

$$U(K_v) \to [0, \infty), \ P \to |f_P|_v$$

is continuous.

Let $f$ be a non-zero section of $\Gamma(U, \ell)$, and $D = (f)$ be the divisor of $f$. Then

$$|f_P|_v = 0 \iff f_P = 0 \iff P \in Supp(D)$$

From the fact that the norms vary continuously and the above equation we can interpret $|f_P|_v$ as the $v$-adic distance from $P$ to $D$.

**3.5 Lemma** Let $v \in M_K^\infty$, and suppose that $|.|_v$ , $|.|_v^{'}$ are two $v$-adic metrics on $\ell$. Then there exist constants $c_1$ and $c_2$ such that

$$c_1 \, |.|_v \, \leq \, |.|_v^{'} \leq c_2 \, |.|_v \quad \text{on } V(K_v)$$

**Proof**. In this proof we use the fact $V(K_v)$ is compact. $V(K_v)$ is compact as $V$ is projective. Let us now choose an $f_P \neq 0 \in \ell_P$ for each $P \in$ V$(K_v)$. Then $|f_P|_v \,/\, |f_P|_{v'}$ is independent of choice of $f_P$. Now consider the map

$$F : \, V(K_v) \to (0, \infty), \, P \to \ |f_P|_v \,/\, |f_p|_v^{'} \, .$$

Since $F$ is continuous and $V(K_v)$ is compact, the image of $F$ in the set $(0, \infty)$ is compact as well. Hence there exist constants $c_1$, $c_2$ such that $c_1 \leq \, F(P) \, \leq c_2$ for all $P \in V(K_v)$, which is the desired result. $\square$

Let us consider a very ample line bundle $\ell$ defined on the projective variety $V$. There exists an embedding $F : V \to \mathbb{P}^n_K$ corresponding to the very ample line bundle $\ell$. (i.e. $\ell \approx \mathcal{O}_{\mathbb{P}^n}(1)$ ). Once we fix an embedding of $V$ in $\mathbb{P}^n_K$ any point $P \in V(K)$ extends uniquely to a point in $\mathbb{P}^n_{\mathbb{Z}}(R)$. Or in other words it extends to a map

$$P : Spec(R) \to \mathbb{P}^n_{\mathbb{Z}}.$$

Hence if we are given $v$-adic metrics on $\mathcal{O}_{\mathbb{P}^n}(1)$ for each $v \in M_K^\infty$, then the pull back $P^*\mathcal{O}_{\mathbb{P}^n}(1)$ becomes a metrized line bundle on $Spec(R)$. The following proposition relates the degree of the line bundle $P^*\mathcal{O}_{\mathbb{P}^n}(1)$ with the height function associated to $\ell$.

**3.6 Proposition.** With hypothesis as above, fix $v$-adic metrics on $\mathcal{O}_{\mathbb{P}^n}(1)$ for each $v \in M_K^\infty$ . Then

$$deg\, P^*\mathcal{O}_{\mathbb{P}^n}(1) = [K : \mathbb{Q}]\, h_\ell(P) + O(1)$$

(where $O(1)$ represents a bounded function )

**Proof.** First observe that from lemma 3.5 and the definition of the degree of a metrized line bundle, if the metrics on $\mathcal{O}_{\mathbb{P}^n}(1)$ are changed, then the degree of $P^*\mathcal{O}_{\mathbb{P}^n}(1)$ changes only by a bounded function. Hence it suffices to prove the proposition for any choice of metrics on $\mathcal{O}_{\mathbb{P}^n}(1)$.

Now we construct a metric on $\mathcal{O}_{\mathbb{P}^n}(1)$. For each $v \in M_K^\infty$, we define a $v$-adic metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ as follows. Let $x \in H^\circ(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ be a global section. Then we define $|x(P)|_v$ for each $P \in V(K_v)$ as

$$|x(P)|_v \;=\; \min_{\substack{0 \le i \le n \\ x_i(P) \neq 0}} \left\{ |(x/x_i)(P)|_v \right\}.$$

Let $x_\circ, ....x_n$ be the global sections which generate $\mathcal{O}_{\mathbb{P}^n}(1)$ and $P^*(x_i) = x_i(P)$ for each $P \in \mathbb{P}^n(K)$.

Let $v \in M_K^\infty$ be an archimedean valuation.

$$|x_\circ(P)|_v \;=\; \min_{0 \le i \le \le n} \{ |(x_\circ/x_i)|_v \} \text{ (from the definition of the metric on).}\, \mathcal{O}_{\mathbb{P}^n}(1)$$

On the other hand,

$$\begin{aligned} P^*\mathcal{O}_{\mathbb{P}^n}(1)/Rx_\circ(P) &\approx \left( \sum_{i=0}^{n} Rx_i(P) \right) /Rx_\circ(P) \\ &\approx \left( \sum_{i=0}^{n} R(x_i/x_\circ(P) \right) /R; \end{aligned}$$

So

$$\begin{aligned}
\#(P^*\mathcal{O}_{\mathbb{P}^n}(1)/Rx_\circ(P)) &= \left|\mathbf{N}_{K/\mathbb{Q}}\left(\sum_{i=0}^{n} R(x_i/x_\circ)\right)\right|^{-1} \\
&= \prod_{v\in M_K^\circ} \max_{0\le i\le n} \|(x_i/x_\circ)(P)\|_v \,. \\
&\qquad \text{(here we view the } x_i\text{'s as functions on } \mathbb{P}^n(K))
\end{aligned}$$

Hence for this choice of metrics on $\mathcal{O}_{\mathbb{P}^n}(1)$,

$$\begin{aligned}
deg\, P^*\mathcal{O}_{\mathbb{P}^n}(1) &= \#\log(P^*\mathcal{O}_{\mathbb{P}^n}(1)/Rx_\circ(P)) \;-\; \sum_{v\in M_K} \log\|x_\circ(P)\|_v \\
&= \log\left(\prod_{v\in M_K^\circ} \max_{0\le i\le n}\|(x_i/x_\circ)(P)\|_v\right) \;-\; \sum_{v\in M_K} \log \min_{0\le i\le n}\|(x_\circ/x_i(P)\|_v \\
&= \sum_{v\in M_K} \log \max_{0\le i\le n}\|(x_i/x_\circ)(P)\|_v \quad \text{(by considering the } x_i\text{'s functions over } \mathbb{P}^n(K)\text{ )} \\
&= [K:\mathbb{Q}]h([1, x_1/x_\circ(P), ...., x_n/x_\circ(P)]) \text{ ( by definition of height function )} \\
&= [K:\mathbb{Q}]h_\ell(P) \,+\, O(1).
\end{aligned}$$

□

**3.7 Remark.** One could define the height function associated to a line bundle $\ell$ defined on a projective variety (defined over a number field $K$) as $h_\ell = \frac{1}{[K:\mathbb{Q}]}\deg(P^*\mathcal{O}_{\mathbb{P}^n}(1))$ where $P$ is a point in $\mathbb{P}^n(K)$. We have just seen in proposition 3.6 that this way of defining height a function is the same as the way we did before.

## 1.4 Faltings Formula for Abelian Varieties

We have seen the classical theory of height functions in the preceding sections. Proposition 3.6 realizes the height function as the degree of a metrized line bundle. But we cannot expect the height function associated to any arbitrary chosen line bundle to be invariant under field extensions. So we will try to find a metrized line bundle, for which the height function associated to it would remain invariant under a field extension. In this section we will see that the Faltings height function associated to a specific metrized line bundle, namely $\omega_{A/R}$ where $A_K$ is an abelian variety defined over a number field $K$ and $R$ its ring of integers, comes very close to our requirement of being an invariant under field extensions.

In this section we define Faltings height function and then prove the Faltings formula for isogenous abelian varieties. Before setting out to prove the formula we look at a few definitions and review what we have done in the last sections. In fact we find that Faltings formula follows trivially by writing out all the definitions and concepts we have seen so far.

**4.1 Definition.** A group variety over $K$ is a variety over $K$ together with morphisms

$$m : V \times V \to V \ \text{(multiplication)},$$
$$inv : V \to V \ \text{(inverse)},$$

and an element $\epsilon \in V(K)$ such that the structure on $V(\bar{K})$ defined by $m$ and $inv$ is that of a group and with identity element $\epsilon$.

**4.2 Definition.** A complete group variety is called an abelian variety.
Remark: Abelian varieties are projective and commutative.

**4.3 Definition.** Let $f : A \to B$ be a homomorphism of abelian varieties. The kernel $N$ of $f$ is a closed subgroup scheme of $A$ of finite type over $K$. If $f$ is surjective and has finite kernel then it is called an *isogeny*.

In order to define a height function one first needs a metrized line bundle on $Spec(R)$. But we assume that our Abelian variety is defined over a number field. So now we see what a Néron model is , and how it is useful in our scheme of finding a metrized line bundle over $Spec(R)$.

**4.4 Definition.** Let $S$ be a Dedekind scheme of dimension 1 with function field $K = K(S)$. Let $A_K$ be an abelian variety over $K$. The Néron model $N(A)$ of $A_K$ over $S$ is a scheme $\pi : N(A) \to S$ which is smooth, separated and of finite type, with generic fiber isomorphic to $A_K$, and that verifies the following universal property:
For any smooth scheme $X$ over $S$, the canonical map

$$Mor_S(X, N(A)) \to Mor_K(X_K, A_K)$$

is bijective.
Let $A_K$ be an abelian variety defined over a number field $K$, then we denote $s^* \wedge^g \Omega^1_{N(A)/R}$ by $\omega_{A/R}$ where $s$ is the zero section $s : Spec(R) \to N(A)$, of the Néron model $N(A)$. This is a metrized line bundle over $Spec(R)$. We will see very soon how the metrics are defined on this line bundle.

**4.6 Definition.** We define the Faltings height function of the Abelian variety $A_K$ defined over a number field $K$ as

$$h_F(A_K) = \frac{1}{[K : \mathbb{Q}]} deg(\omega_{A/R})$$

If $\pi : N(A) \to Spec(R)$ is proper, then $\pi_* \omega_{A/K} \cong s^* \wedge^g \Omega^1_{N(A)/R}$ , and then $h_F(A_K)$ an be defined as $\frac{1}{[K:\mathbb{Q}]} deg(\pi_* \wedge^g \Omega^1_{N(A)/R})$. We need to introduce the concept of semi-stable and good reduction to see whether $h_F(A_K)$ remains invariant under a field extension of $K$.

**4.7 Definition.** Let $A_K$ be an abelian variety defined over the function field $K$ of the Dedekind scheme $S = Spec(R)$ where $R$ is a Dedekind domain. We say that $A_K$ has good reduction at $v$ a prime in $R$ if and only if there exists a scheme $\pi : X \to Spec(R_v)$, which is smooth and proper, such that the generic fiber of this morphism is $A_K$.
By the universal property of the Néron model we can conclude that $N(A)$ is proper at $v$, if and only if $A_K$ has good reduction at a prime $v$ in $S$. If $A_K$ has good reduction at all primes belonging to $S$, then $A_K$ is said to have good reduction.

**4.8 Definition.** Let the hypothesis be the same as the one in above definition. If $N(A)^\circ$ is the open subgroup scheme whose fibers are the connected components of a Néron model $N(A)$, then the abelian variety $A_K$ is said to be semi-stable at $v$ a prime in $R$, if $N(A)^\circ_{k(v)}$ is an extension of an abelian variety by a torus, where $k(v)$ is the residue class field of $R_v$. If $A_K$ has semi-stable reduction for all primes $v$ in $R$, then $A_K$ is said to have semi-stable reduction.

**4.9 Theorem.** $h_F(A_K)$ is invariant under the extension of ground field for abelian varieties with semi-stable reduction, where $A_K$ is an abelian variety defined over a number field $K$.
**Proof.** We only outline the main idea of the proof. The main idea involved is as follows. If $A_K$ has semi-stable reduction over $K$, then it is known that $N(A_K)^\circ \times_{Spec(O_K)} Spec(O_L)$ is canonically isomorphic to $N(A_L)^\circ$, where $N(A)^\circ$ denotes the open subgroup scheme whose fibers are connected components of the Néron model $N(A)$ (for proof one can look at [CL] ). Hence it follows that $h_F(A_K) = h_F(A_L)$ □

But if $A_K$ does not have a semi-stable reduction over $K$ then its Néron model would not commute with base change of number fields, so we cannot expect our $h_F(A_K)$ to be invariant under base change. Hence we define another height function $h_{geom}$ which remains invariant under base change.
Every abelian variety $X_K$ becomes semi-stable over a field extension $L$ over $K$ by the very well known semi-stable reduction theorem. We define the geometric height $h_{geom}(A_K) = h_F(A_L)$. If $A_K$ has semi-stable reduction over $K$, then $h_{geom}(A_K) = h_F(A_K)$.

**4.10 Definition.** Let $f : A \to B$ be a homomorphism of abelian varieties. If $f$ is surjective and the kernel of $f$ is finite then $f$ is called an isogeny.

Before we embark on the task of proving the Faltings formula we look at how the $v$-adic metrics are defined the projective rank 1 $R$-module $\omega_{A/R}$.
$\omega_{A/R} \otimes_R K$ is canonically isomorphic to $\Gamma(A_K, \wedge^g \Omega_{A_K/K})$.
For each infinite place $v$, we put a metric on $\omega_{A/R} \otimes K_v$ by:

$$\|\omega\|_v^2 = 2^{-g} \int_{A(\bar{K}_v)} |\omega \wedge \bar{\omega}|$$

for all $\omega \in \omega_{A/R}$.
In fact this metric comes from the hermitian inner product on $\omega_{A/R} \otimes K_v$ :

$$< \omega, \mu >_v = 2^{-g} \int_{A(\bar{K}_v)} |\omega \wedge \bar{\mu}|$$

Now we finally state and prove Faltings formula for isogenous abelian varieties.

**4.11 Proposition. (Faltings Formula).** Let $f_K : A_K \to B_K$ be an isogeny of abelian varieties $A_K$, $B_K$ defined over a number field $K$ with ring of integers $R$. Let $N(A)$, $N(B)$ be the Néron models of $A$ and $B$. By the universal property of Néron models $f_K$ extends to a morphism $f : N(A) \to N(B)$. Let $G$ be the kernel of the morphism $f$. Then

$$h_F(B_K) - h_F(A_K) = \frac{1}{2} \log deg(f_K) - \frac{1}{[K:\mathbb{Q}]} \log(\#(s^*\Omega_{G/R}))$$

**Proof.**

$$\begin{aligned} h_F(B_K) - h_F(A_K) &= \frac{1}{[K:\mathbb{Q}]} \big(deg(\omega_{B/R}) - deg(\omega_{A/R})\big) \\ &= \frac{1}{[K:\mathbb{Q}]} \log \#(\omega_{B/R}/R.s) - \frac{1}{[K:\mathbb{Q}]} \log(\omega_{A/R}/R.t) \\ &+ \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K^\infty} \epsilon_v \log |t|_v - \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K^\infty} \epsilon_v \log(|s|_v) \end{aligned}$$

for $s$, $t$ non-zero elements of $\omega_{B/R}$, $\omega_{A/R}$ respectively.
We have an injection $f^*(\omega_{B/R}) \to \omega_{A/R}$ , we can choose $t$ to be $f^*(s)$.

$$\begin{aligned} \int_{A(\bar{K}_v)} \left|f^*(s) \wedge \bar{f}^*(s)\right| &= deg(f_K) \int_{B(\bar{K}_v)} |s \wedge \bar{s}| \\ \Rightarrow |f^*(s)|_v &= \sqrt{d}\, |s|_v \end{aligned}$$

$$\text{Hence} \sum_{v \in M_K^\infty} \epsilon_v \log(|f^*(s)|_v) = \sum_{v \in M_K^\infty} \epsilon_v \log(\sqrt{deg(f_K)}\, |s|_v)$$

$$\begin{aligned} \text{Hence } \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K^\infty} \epsilon_v \log |f^*(s)|_v - \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K^\infty} \epsilon_v \log(|s|) &= \sum_{v \in M_K^\infty} \sqrt{deg(f_K)} \\ &= \frac{1}{2} \log deg(f_K) \qquad (1.1) \end{aligned}$$

Now consider

$$\begin{aligned} \log \#(\omega_{B/R}/R.s) - \log \#(\omega_{A/R}/R.f^*(s)) &= \log \#(\omega_{B/R}/R.(s)) \\ &- \log \#(\omega_{A/R}/R.f^*\omega_{B/R})(f^*\omega_{B/R}/R.f^*(s)) \\ &= - \log \#(\omega_{A/R}/f^*\omega_{B/R}) \ \text{ (as } f^* \text{ is injective)} \end{aligned}$$

We have the following exact sequence :

$$0 \to f^*\Omega_{N(B)/R} \to \Omega_{N(A)/R} \to \Omega_{N(A)/N(B)} \to 0$$

$$\text{So } \ \omega_{A/R} \cong \omega_{B/R} \otimes s^*\Omega_{G/R}$$

Hence we have

$$\log \#(\omega_{A/R}/f^*\omega_{B/R}) = \# \log s^*(\Omega_{G/R}) \tag{1.2}$$

From equation 1.1 and 1.2 we have the desired relation:

$$h_F(B_K) - h_F(A_K) = \frac{1}{2}\log deg(f_K) - \frac{1}{[K:\mathbb{Q}]}\log(\#s^*(\Omega_{G/R})) \quad \square$$

# 2 Average Height of Quotients of a Semi-Stable Elliptic Curve

In the last chapter we have seen the Faltings height formula for abelian varieties. In this chapter we look at the average Faltings height of quotients of a semi-stable elliptic curve by its cyclic subgroups of a fixed order.
Autissier first proved this formula in [Au1] as mentioned in the introduction. The approach opted by Autissier in [Au1] involves the the measure of arithmetic complexity of modular curve $X_\circ(N)$ and also a result due to K*ü*hn on the height of the modular curve $X_\circ(1)$. In this chapter we first look at a more elementary approach due to Robin de Jong in [Ro], then briefly outline Autissier's method in [Au1], and then look at another approach which involves both these approaches. Robin de Jong's approach involves the basic concepts of Arakelov intersection theory from [Ar] and [Fa 2].

As stated above we prove the following formula for a semi-stable elliptic curve in this chapter:

$$\frac{1}{e_N}\sum_{C}\Big(h_F(E^{'}) - h_F(E)\Big) = \frac{1}{2}\log N - \lambda_N \tag{2.1}$$

where $e_N$ is the number of cyclic subgroups of $E$ of order $N$, $C$ runs over the cyclic subgroups of order $N$ and $\lambda_N$ is a constant dependent only on $N$.

In section 1 of this chapter we look at the Arakelov Green function defined on a compact and connected Riemann surface of genus $g > 0$. In section 2 we look at the Arakelov intersection theory on arithmetic surfaces. In section 3 we prove 2.1 as a consequence of all the theory developed in the first 2 sections. In the final section we briefly outline the approach of 2.1 using Autissier's method, which actually carries onto the abelian varieties case. We then look at another method which uses results of both the approaches.

## 2.1 Arakelov Green Function

Let $X$ be a compact connected Riemann surface of genus $g > 0$.

**1.1 Definition.** Let $X$ be a connected, compact Riemann surface of genus $g > 0$. Then $H^\circ\left(X, \Omega_X^1\right)$ is equipped with a Hermitian inner product

$$(\omega, \eta) \rightarrow \frac{i}{2} \int_X \omega \wedge \bar{\eta}.$$

Let $(\omega_1, ..., \omega_g)$ be an orthonormal basis with respect to this inner product . We then define the Arakelov $(1,1)$-form $\mu$ to be $\mu = \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \bar{\omega_k}$.

It is easy to check that $\int_X \mu = 1$ and $\mu$ is independent of choice of basis. Using this Arakelov (1,1) form we now define the Arakelov Green Function.

**1.2 Definition.** The Arakelov-Green function $G$ is the unique function $G : X \times X \rightarrow \mathbb{R}_{\geq 0}$ such that the following properties holds:
(1) for all $P \in X$ the function $\log G(P,Q)$ is $\mathbb{C}^\infty$ for $Q \neq P$;
(2) for all $P \in X$ we can write $\log G(P,Q) = \log |z_P(Q)| + f(Q)$ locally about $P$, where $z_P$ is a local coordinate about $P$ and where $f$ is $\mathbb{C}^\infty$ about $P$;
(3) for all $P \in X$ we have

$$\partial_Q \bar{\partial}_Q \log G(P,Q)^2 = 2\pi i \mu(Q)$$

for $Q \neq P$
(4) for all $P \in X$ we have $\int_X \log G(P,Q)\mu(Q) = 0$
The existence of such a function is proved by Arakelov in [Ar]. Properties 1,2,3 determine the Green function up to a constant, and then condition 4 determines a unique Green function. Symmetry of Green function $G(P,Q) = G(Q,P)$ follows from application of Stokes's theorem and the conditions 1,2,3 and 4.

The Arakelov Green function determines a smooth hermitian metric on the line bundles $O_X(D)$ ,where $D$ is a divisor on $X$. In our case it suffices to consider just points. Let $s$ be a canonical generating section of the line bundle $O_X(P)$. The metric determined by the Arakelov Green function $\|.\|_{O_X(P)}$ on $O_X(P)$ is given by:

$$\|s\|_{O_X(P)} = G(P,Q) \text{ for any } Q \in X.$$

**1.3 Definition.** Let $\Omega_X^1$ be the sheaf of holomorphic 1-forms of $X$.
Let $\Delta : X \rightarrow X \times X$ be the diagonal embedding and $O_{X \times X}(-\Delta)$ the sheaf of holomorphic functions vanishing on the diagonal with zeros of order 1. Then there exists an isomorphism called the adjunction isomorphism

$$O_{X \times X}(-\Delta) \cong \Omega_X^1.$$

$O_{X \times X}(-\Delta)$ carries the hermitian metric defined by $\|s\|(P,Q) = G(P,Q)$ with $s$ the canonical generating section of the line bundle $O_{X \times X}(-\Delta)$ .The unique metric on $\Omega_X^1$ that makes the adjunction isomorphism an isometry is called the Arakelov metric $\|.\|_{Ar}$.

**1.4 Definition.** Let $X$ be a connected Riemann surface, and $\ell$ be a holomorphic line bundle on $X$, equipped with a $C^\infty$ hermitian metric $\|.\|$. The curvature form of $(\ell, \|.\|)$ is a (1,1)-form defined locally as

$$curv_{\|\|} = \frac{-1}{2\pi i}\partial_Q\bar{\partial}_Q \log(\|s\|^2)$$

where $s$ is the local generating section of $\ell$ and it satisfies the condition

$$\int_X \frac{-1}{2\pi i}\partial_Q\bar{\partial}_Q \log \|.\|^2 = deg\ell$$

This is because every hermitian line bundle can be built up from $O_X(P)$, which comes with the hermitian metric $\|s\|_{O_X(P)} = G(P,Q)$. Hence from property 3 of the Green function, we can conclude that the curvature form is independent of the generating section $s$ that we choose.

**1.5 Definition.** A smooth Hermitian metric $\|.\|$ on a line bundle $\ell$ on $X$ is called admissible if its curvature form is a multiple of $\mu$. Arakelov has proved in [Ar] that $\|.\|_{Ar}$ is an admissible metric on $\Omega^1_X$.

**1.6 Proposition.** Let $P$ be a point on $X$ and let $z$ be a local coordinate about $P$. Then the norm of $dz$ in $\Omega^1_X$ $\|dz\|_{Ar}$ is given by the formula
$\|dz\|_{Ar}(P) = \lim_{Q\to P} |z(P) - z(Q)| / G(P,Q)$.

**Proof.** Proposition 2.5 of [Ro].

We are now ready to state the complex projection formula for divisors on Riemann surfaces. Later in the next section we see that this formula comes to our aid in proving an Arakelov projection formula.

**1.7 Proposition. (Complex Projection Formula)** Let $X$ and $X'$ be Riemann surfaces of genus 1 and $G_X$ and $G_{X'}$ be the Arakelov Green functions of $X$ and $X'$ respectively. Suppose we have a non constant holomorphic map $f : X \to X'$. Let $D$ be a divisor on $X'$. Then the canonical isomorphism of line bundles

$$f^*O_{X'}(D) \cong O_X(f^*D)$$

is an isometry. In particular we have a projection formula:
for any $P \in X$ the formula

$$G_X(f^*D, P) = G_{X'}(D, f(P))$$

holds.
**Proof.** Proposition 3.2 of [Ro].

We now prove a formula for what is known as the "*Energy of an Isogeny*". Later in section 3 while we prove 2.1 the corollary of this proposition helps us a great deal. But before we look at the proposition we need a notation.

Let $\omega$ be a holomorphic differential of norm 1 in $H^\circ\left(X, \Omega^1_X\right)$. Then we denote $\|\omega\|_{Ar}$ by $A(X)$. This is an invariant of $X$.

**1.8 Proposition. (Energy of an Isogeny)**. Let $X$ and $X^{'}$ be 1-dimensional complex tori related by an isogeny $f : X \to X^{'}$ of degree $N$. Then we have the following formula:

$$\prod_{P \in ker(f), P \neq 0} G(0, P) = \frac{\sqrt{N}.A(X)}{A(X^{'})}$$

**Proof.** The idea of the proof is that we compute the norm of the map $f^*(\Omega^1_{X^{'}}) \to \Omega^1_X$ and show that it is equal to

$$\prod_{P \in ker(f), P \neq 0} G(0, P).$$

Given the isogeny $f : X \to Y$ of degree $N$. We then have the isomorphism between

$$f^*(\Omega^1_{X^{'}}) \to \Omega^1_X \tag{2.2}$$

(as $f$ is an isogeny and $X,\ X^{'}$ can be viewed as complex elliptic curves).

We know from the arguments in the proof of the Faltings formula(proposition 4.11) in last chapter that the norm of the isomorphism

$$f^* : H^\circ\left(X^{'}, \Omega^1_{X^{'}}\right) \to H^\circ\left(X, \Omega^1_X\right).$$

is $\sqrt{N}$. So consider an $\omega^{'} \in H^\circ\left(X^{'}, \Omega^1_{X^{'}}\right)$ of norm 1. We have the Arakelov metric on $\Omega^1_{X^{'}}$ so $\omega^{'}$ has norm $A(X^{'})$ in $\Omega^1_{X^{'}}$ .

Now consider $f^*(\omega^{'})$. It has norm $\sqrt{N}$ in $H^\circ\left(X, \Omega^1_X\right)$. Hence the norm in $\Omega^1_X$ of $f^*(\omega^{'})$ is $\sqrt{N}.A(X)$ as an element of the line bundle $\Omega^1_X$. Hence we can now compute c , the norm of $f^*$ when viewed as an isomorphism of line bundles:

$$c = \frac{\sqrt{N}.A(X)}{A(X^{'})}.$$

From Proposition 1.7 we have:

$$f^*(O_{X^{'}}(0)) \cong O_X(Ker(f)) \tag{2.3}$$

and that $f^*$ to be an isometry on $O_{X^{'}}(0)$ . We tensor 2.2 and 2.3. Since $\Omega^1_{X^{'}} \otimes O_{X^{'}}(0) = \Omega^1_{X^{'}}(0)$ and $\Omega^1_X \otimes O_X(ker(f)) = \bigotimes_{P \in ker(f),\, P \neq 0} O_X(P)$, we have the following isomorphism:

$$f^*(\Omega^1_{X^{'}}(0)) \cong \Omega^1_{X^{'}}(0) \otimes \bigotimes_{P \in ker(f),\, P \neq 0} O_X(P)$$

which has norm $c$. $c$ can again be calculated by looking at how $f^*$ acts on an element belonging to $\Omega^1_{X'}(0)$.

$$f^*(\frac{dz}{z}) \rightarrow \frac{dz}{z} \otimes s,$$

where $s$ is the canonical section of $\bigotimes_{P\in ker(f),\, P\neq 0} O_X(P)$. $\frac{dz}{z}$ has norm 1 in $\Omega^1_{X'}(0))$.
The line bundle $O_X(P)$ is equipped with the smooth hermitian metric $\|.\|_{O_X}(P)$ such that $\|t\|_{O_X(P)} = G(P,Q)$ for any $Q \in X$, and $t$ the canonical generating section of $O_X(P)$.
So the norm of $s$;

$$\|s\| = \prod_{P\in ker(f), P\neq 0} \|s\|_{O_X(P)}(0)$$

is by definition of $\|s\|_{O_X(P)}$ equal to

$$\prod_{P\in ker(f)\,, P\neq 0} G(0,P).$$

Hence we have the norm of $f^*$ to be equal to

$$c = \prod_{P\in\, ker(f), P\neq 0} G(0,P) = \frac{\sqrt{N}.A(X)}{A(X')}. \qquad \square$$

We conclude this section with a nice corollary to proposition 1.8.

**Corollary 1.9.** Let $X$ be a 1-dimensional complex torus. Let $X[N]$ be the kernel of the multiplication by $N$ map $N : X \rightarrow X$. Then we have the formula

$$\prod_{P\in X[N], P\neq 0} G(0,P) = N$$

**Proof.** $X$ is an elliptic curve and multiplication by $N$ is an isogeny of degree $N^2$ , i.e. $\#X[N] = N^2$. Hence the formula follows. $\square$

## 2.2 Arakelov Intersection Theory

In this section we look at the basic notions of Arakelov intersection theory on the minimal regular model of a semi-stable elliptic curve over a number field $K$. Arakelov intersection theory is a very powerful tool to study the arithmetic complexities of algebraic curves and varieties. We start off with the notion of a minimal regular model for an elliptic curve, Arakelov divisors and then state the well known Arakelov projection formula.

**2.1 Definition.** An arithmetic surface is a proper flat morphism $p : X \to B$ of schemes with $X$ regular and with $B$ the spectrum of ring of integers of a number field $K$, such that the generic fiber is a geometrically connected curve of genus 1, and a section $O : B \to X$ of $p$ is given , we then call $p : X \to B$ an elliptic arithmetic surface.

**2.2 Definition.** Minimal Arithmetic Surface: We say that an arithmetic surface $p : X \to B$ with generic fiber of positive genus is called minimal if every proper $B$-morphism $X \to X^{'}$ with $p^{'} : X^{'} \to B$ an arithmetic surface, is an isomorphism.

**2.3 Definition.** Let $E$ be an elliptic curve over a number field $K$. If the generic fiber of the minimal arithmetic surface
$p : X \to B$ is isomorphic to $E$ , then we call $p : X \to B$ the minimal regular model of $E$ over $K$.

The existence of such a minimal model for every elliptic curve $E$ over $K$ is known. (chapter 9, proposition 3.19 and corollary 3.24 [QL]) In general we say a model $p : X \to B$ with the generic fiber isomorphic to the given elliptic curve verifies a property $T$ if $p : X \to B$ verifies $T$.

**2.4 Remark.** Every elliptic curve is indeed an abelian variety of dimension 1. And it is also known that the minimal model $X$ of every elliptic curve $E$ contains the Néron model as a dense open sub scheme. By the universal property of the Néron model any isogeny $f : E \to E^{'}$ extends over a dense open sub-scheme $U$ of $X$ to give $B$-morphism $U \to X^{'}$ and hence a rational map $f : X \to X^{'}$. We also have the following theorem from [QL] (chapter 9,theorem 2.7).

**2.5 Theorem.** Let $X \to S$ be a regular fibered surface. Let $\phi : X \to Z$ be a rational map from $X$ to a projective $S$-scheme $Z$. Then there exists a projective birational morphism $f : \tilde{X} \to X$ made up of a finite sequence of blowing ups of closed points

$$\tilde{X} = X_n \to ..... \to X_\circ = X,$$

and a morphism $g : \tilde{X} \to Z$ such that $\phi \circ f = g$.

Hence we have a proper birational morphism $\pi : \tilde{X} \to X$ made up of finite sequence of blowing ups of singular points, and a morphism

$$\tilde{f} : \tilde{X} \to X^{'}$$

such that $\tilde{f} = f.\pi$ .

All throughout this chapter $p : X \to B$ is an arithmetic surface with the generic fiber $X_K$ geometrically connected. Here $B$ is the ring of integers of a number field $K$. Let $S_f$ denote the set of primes of $O_K$.
$S_\infty$ denote the infinite places . So $S = S_f \cup S_\infty$ denotes all the places of $K$ .
Now we define what an Arakelov divisor is.

**2.6 Definition.** An Arakelov divisor on $X$ is a formal sum

$$D = D_f + D_\infty$$

where $D_f$ is a Weil divisor on $X$ and $D_\infty$ is a formal linear combination:

$$D_\infty = \sum_{v \in S_\infty} r_v E_v,$$

with real coefficients $r_v$. $E_v$ are formal symbols corresponding to the Riemann surfaces $X(\bar{K_v})$ for an infinite place $v$.
$E_v = p^{-1}\,(Spec(k_v)$ for finite places where $k_v$ is the residue class field at $v$. We denote its order by $q_v$.

To a non zero rational function $f$ on $X$ we associate an Arakelov divisor $(f) = (f)_{fin} + (f)_{inf}$ where $(f)_{fin}$ is then usual divisor of $f$.

$$(f)_{inf} = \sum_{v \in S_\infty} r_v.E_v \quad where \quad r_v = -\int_{X_v} \log \|f\|_v .\mu_v$$

where $X_v$ is the Riemann surface $X(\bar{K_v})$ over $K_v = \mathbb{C}$ and $\mu_v$ is the (1,1)-form associated to the Riemann surface $X_v$ which we have discussed in the last section.
We denote the set of Arakelov divisors on $X$ by $\hat{Div}(X)$. This set carries a group structure. In fact the group admits a decomposition

$$\hat{Div}(X) \cong Div(X) \times \oplus_{v \in S_\infty} X_v$$

**2.7 Definition.** We say two divisors $D$ and $D'$ are linearly equivalent if their difference is an Arakelov divisor $(f)$ for some non zero rational function $f$. We denote group of Arakelov divisors on $X$ modulo their equivalence classes by $\hat{Cl}(X)$.

The following proposition proved by Arakelov in [Ar] is the main theorem which defines the Arakelov intersection for two Arakelov divisors. We are not going into the proof of it, but look at how Arakelov intersection is defined in [Ar].

**2.8 Proposition.** There exists a natural bilinear symmetric intersection pairing $\hat{Div}(X) \times \hat{Div}(X) \to \mathbb{R}$ . This pairing factors through linear equivalence , giving an intersection pairing $\hat{Cl}(X) \times \hat{Cl}(X) \to \mathbb{R}$ This pairing defined above goes by the name of Arakelov intersection pairing.

**2.9 Definition.** Let $P$ and $Q$ be two Arakelov divisors on $X$. If $P$ or $Q$ is a sum of fibers $X_v$ with $v \in S_\infty$ or of components of fibers $E_v$, $v \in S_f$ , then the intersection product is zero. (assuming they have no components in common)
In the most common case when $P$ and $Q$ are sections $(P, Q)$ is

$$(P, Q) = \sum_{v \in S} (P, Q)_v$$

For $v \in S_f$, we define

$$(P, Q)_v = \log q_v.(\text{usual} \quad \text{intersection} \quad \text{multiplicity})$$

This is non zero when $P$, $Q$ intersect $E_v$ at the same point ,and this is seen as logarithm of the $v$- adic distance between $P$, $Q \in X(K) \subseteq X(K_v)$, where $K_v$ is the $v$- adic completion of $K$ .
For infinite places ,we define

$$(P, Q)_v = -\epsilon_v. \log G(P, Q) \quad (v \in S_\infty)$$

where

$$\epsilon = 1, \ \ for \ K_v = \mathbb{R}, \ \epsilon = 2, \ \ for \ K_v = \mathbb{C}$$

$G(P, Q)$ can be viewed as the norm evaluated at $Q$ of the constant section 1 of $O_X(P)$ for the Hermitian metric (which is given by the Arakelov Green function) we discussed in last section.

For any arithmetic surface which is a proper, flat morphism $p : X \to B$ of schemes, with $X$ regular, we define an admissible line bundle $L$ on $X$ to be a line bundle $L$ with a smooth hermitian metric on the restrictions of $L$ to $X_v$. The group of isometry classes of line bundles is denoted by $\hat{Pic}(X)$ .

It is natural to ask whether $\hat{Pic}(X)$ is isomorphic to $\hat{Cl}(X)$ or not. Arakelov has proved that they are indeed isomorphic. We also have the desired isomorphism.[Ar](Proposition 2.2)

**2.10 Proposition.** The group $\hat{Cl}(X)$ is canonically isomorphic to $\hat{Pic}(X)$.

So from now on whenever we write the Arakelov intersection $(P, L)$ for $P$ a section and $L$ an admissible line bundle we actually mean $(P, D)$ where $D$ is the Arakelov divisor corresponding to the admissible line bundle $L$ under the canonical isomorphism.

We have seen in last chapter (1.3), the definition of degree of a metrized line bundle $L$ on $Spec(R)$ to be

$$Deg(L, \|.\|) = \log(\#(L/R.s)) - \sum_{v \in S_\infty} \epsilon_v \log \|s\|_v \tag{2.4}$$

**2.11 Proposition.** Let $L$ be an admissible line bundle on $X$ , let $P : B \to X$ be a section of $p$, and let $s$ be a nonzero rational section of $L$. Then we have

$$(P, L) = \log \#(P^*L/P^*s.O_K) - \sum_{v_\infty} \epsilon_v \log \|P^*s\|_v .$$

The proof follows straight from the definitions and it is all about checking, which we omit. But it is worth observing here that $P^*L$ is a projective module of rank 1 over $B$ together with hermitian metrics at infinite primes where $P : B \to X$ is a section of $X$. Hence $P^*L$ is a metrized line bundle over $B$. Hence proposition 2.11 translates to

$$(P, L) = deg(P^*L)$$

Corresponding to the dualizing sheaf $\omega_{X/B}$ there exists a an admissible line bundle which we again denote by $\omega_{X/B}$. The metrics at infinity are given by Arakelov norm. We have the classical adjunction formula on complex arithmetic surfaces , and it is not very surprising to see the same holds with Arakelov intersection as well. But the proof of the following formula uses the concepts of moduli stacks of elliptic curves, so we skip the proof.

**2.12 Proposition.**(Adjunction Formula) Let $P : B \to X$ be a section of $p$. Then we have the formula

$$(P, P) = -(P, \omega_{X/B})$$

**Proof.** Proposition 7.3 of [*Ro*]. So we have just seen that analogous to the classical adjunction formula we have the Arakelov adjunction formula. So similar to the classical projection formula one can expect the Arakelov projection formula. But before going to see what it could be we need to define pullbacks and push forwards of Arakelov divisors.

**2.13 Definition.** Let $p : X \to B$ and $p^{'} : X^{'} \to B$ be arithmetic surfaces. Suppose there exists a $B$- morphism $f : X \to X^{'}$ . Let $D$ be an Arakelov divisor on $X$ , and write $D = D_{fin} + \sum_{v \in S} r_v E_v$. We define the push forward

$$f_*(D) = f_*(D_{fin}) + N. \sum_{v \in S_\infty} r_v E_v^{'}$$

where $f_*(D_{fin})$ is the usual push forward of Weil Divisor and $N$ is the degree of $f$. Now let $D^{'}$ be an Arakelov divisor on $X^{'}$. The pullback $f^*$ is defined to be:

$$f^*(D^{'}) = f^*(D^{'}_{fin}) + \sum_{v \in S_\infty} r^{'}_v.E^{'}_v$$

on $X$ where $f^*(D_{fin})$ is the usual pull back of a Weil divisor. Now we are ready to state the Arakelov projection formula. The formula holds for Arakelov divisors as well, analogous to Weil Divisors.

**2.14 Proposition.** Let $E$ and $E^{'}$ be elliptic curves over a number field $K$, related by an isogeny $f : E \to E^{'}$. Let $p : X \to B$ and $p^{'} : X^{'} \to B$ be arithmetic surfaces over the ring of integers of $K$ with generic fibers isomorphic to $E$ and $E^{'}$ respectively and suppose that $f$ extends to a $B$-morphism $f : X \to X^{'}$. Then for any Arakelov divisor $D$ on $X$ and any Arakelov divisor $D^{'}$ on $X^{'}$ the following equality holds.

$$(f^* D^{'}, D) = (D^{'}, f_* D)$$

**Proof.** The proof is solely based on the moving lemma from [QL] (corollary 9.1.10), the classical projection formula and proposition 1.8 (complex projection formula). One can look into [Ro] proposition 6.2 for details.

As in the case of Weil divisors the following result for Arakelov divisors follows very easily from the Arakelov projection formula.

**2.15 Corollary.** With the same hypothesis as the proposition above, if we have two Arakelov divisors $D_1^{'}$ and $D_2^{'}$ on $X^{'}$, then the following formula holds true:

$$(f^* D_1^{'}, f^* D_2^{'}) = N.(D_1^{'}, D_2^{'})$$

where $N$ is the degree of the isogeny $f$.

The following lemma is very useful in our scheme of proving eq 2.1 and would be needed in the next section.

**2.16 Lemma.** Let $p : X \to B$ be a minimal arithmetic surface with generic fiber of genus 1 and with relative dualizing sheaf $\omega_{X/B}$ . The canonical homomorphism $p^* p_* \omega_{X/B} \to \omega_{X/B}$ is an isomorphism.
**Proof.** [QL] corollary 9.3.27.

So we see here that the some of classical formulae which hold for Weil divisors do hold for Arakelov divisors. The next proposition links the norm of the minimal discriminant ideal of the elliptic curve with the self Arakelov intersection of a section on the minimal regular model of the elliptic curve. But unlike the proofs of other propositions we have seen so far (which followed more or less from the classical results and the very definitions of Arakelov divisors) the proof of this proposition makes extensive use of properties of the moduli stack of stable curves.
We need to define what a semi-stable elliptic curve is:

**2.17 Definition.** Let $p : X \to B$ be an elliptic arithmetic surface. We call $p$ semi-stable if any fiber of $p$ is either non-singular, or an $n$-gon of projective lines. We call an elliptic curve $E$ over a number field $K$ semi-stable if there exists a semi-stable elliptic arithmetic surface over the ring of integers of $K$ whose generic fiber is isomorphic to $E$.

We know that a semi-stable elliptic surface is always minimal. And given any elliptic curve over a number field $K$, there exists a finite extension $L$ over $K$ such that $E$ become semi-stable over $L$ .

**2.18 Proposition.** Let $E$ be a semi stable elliptic curve over a number field $K$, and let $p : X \to B$ be its regular minimal model over the ring of integers of $K$ . Let $P : B \to X$ be a section of $p$, and let $\triangle(E/K)$ the minimal discriminant ideal of $E$ over $K$. Then the following formula holds:

$$(P, P) = -\frac{1}{12} \log \left| N_{K/\mathbb{Q}}(\triangle(E/K)) \right| .$$

**Proof.** [Ro] Proposition 7.2 or [Sp].

From proposition 7 and adjunction formula one can observe

$$\begin{aligned} -(P, P) &= \frac{1}{12} \log \left| N_{K/\mathbb{Q}}(\triangle(E/K)) \right| \\ &= deg(P^* \omega_{X/B}). \end{aligned}$$

## 2.3 Average Height of Quotients

In this section we will be looking into the first proof of the Average height formula for elliptic curves. We have developed all the machinery that goes into the proof. In next section we will see that the average height formula can also be proved by looking at $X(1)$, the moduli space of elliptic curves. We will find in the next chapter that the method of looking at the moduli space is indispensable in case of higher dimensions. But in case of elliptic curves the Arakelov Green function gives us the intersections at infinity which is no longer the case in higher dimensions.

Let $E/K$ be an elliptic curve defined over a number field $K$. Let $p : X \to B$ be the minimal regular model of $E/K$. We state the Faltings height formula for an elliptic curve which we have already done in the case of abelian varieties.

$$\begin{aligned} h_F(E) = \frac{1}{[K : \mathbb{Q}]} deg(\omega_{E/K}) &= \frac{1}{[K : \mathbb{Q}]} deg(s^* \omega_{X/K}) \\ &= \frac{1}{[K : \mathbb{Q}]} deg(p_* \omega_{X/K}) \end{aligned}$$

Hence we can state Faltings formula (lemma 1.4.11 of chapter 1) for elliptic curves as :

**3.1 Proposition.** Let $E$ and $E'$ be semi stable elliptic curves over a number field $K$, and an isogeny $f : X \to X'$ , of degree $N$, Then

$$h_F(E') - h_F(E) = \frac{1}{2} \log N - \frac{1}{[K : \mathbb{Q}]} \log \#\Omega_{ker(f)/B}$$

**Proof.** Follows straight from lemma 4.11 of chapter 1. $\square$

If $p : X \to B$ is a regular minimal model of $E$ over $O_K$ with $O : B \to X$ the zero section and $\omega$ the dualizing sheaf, then we can always assume $K$ is large enough to make all the $N$-torsion points of $E$ to be $K$-rational. Which in turn implies $E'$ the quotient of $E$ by a cyclic subgroup of order $N$ is an elliptic curve over $K$.

We would be needing the following lemma which is used many a times in the proof of equation 2.1.

**3.2 Lemma.** Let $M$ be a positive integer such that $M \mid N$. Let $X$ be an elliptic curve over an algebraically closed field of characteristic zero. Then each cyclic subgroup of order $M$ is contained in exactly $e_N/e_M$ cyclic subgroups of order $N$.

**Proof.** Let us consider the pairs $(E_M, E_N)$, where $E_N$ denotes a cyclic subgroup of order $N$, and $E_M$ denotes a cyclic subgroup of order $M$ both contained in the $N$-torsion group of the elliptic curve $X$ and $E_M \subset E_N$. Since there exists a unique subroup of order $M$ in $E_N$ the number of such pairs $(E_M, E_N)$ are just $e_N$ in number.

Let a given subgroup of the $N$-torsion group $E_M$ of order $M$ be contained in $e$ subgroups of order $N$ of the $N$-torsion group. The number $e$ remains the same for each subgroup $E_M$ of order $M$ in the $N$-torsion group. We again compute the pairs $(E_M, E_N)$ . We can choose $E_M$ in $e_M$ ways and $E_N$ in $e$ ways.

We then have $e.e_M = e_N$. Hence we can conclude that each cyclic subgroup of order $M$ is contained in exactly $e_N/e_M$ cyclic subgroups of order $N$ $\square$

We restate 2.1 once again and proceed with the proof.

**3.3 Proposition.** Let $E$ be a semi-stable elliptic curve over a number field $K$. For a finite subgroup $C$ of $E$, we denote by $E'$ the quotient of $E$ by $C$. Then we have the formula

$$\frac{1}{e_N} \sum_C \left( h_F(E') - h_F(E) \right) = \frac{1}{2} \log N - \lambda_N$$

where the sum runs over the cyclic subgroups of $E$ of order $N$. We have many ways of writing the Faltings height function of which we use the one which is most convenient to prove our average height formula. We look for the most convenient reformulation of height function. The following lemma happens to be the one.

**3.4 Lemma.** Let $E$ be a semi-stable elliptic curve over a number field $K$. Let $p : X \to B$ be the minimal regular model for $E$ over $O_K$ . Let $O : B \to X$ be the zero section and $\omega$ be the relative dualizing sheaf of $p$, then we have

$$\frac{(O, \omega)}{[K : \mathbb{Q}]} = h_F(E) - \frac{1}{[K : \mathbb{Q}]} \sum_{v \in S_\infty} \epsilon_v \log \|A\| (X_v).$$

**Proof.** From lemma 2.16 we have

$$p^* p_* \omega \cong \omega$$

Hence we have $\phi : p_*\omega \cong O^*\omega$.
Now consider $(O, \omega)$:

$$(O, \omega) = deg(O^*\omega) \text{ (from proposition 2.11, )}$$

$$deg(O^*\omega) = \frac{1}{[K:\mathbb{Q}]}\left(\log \#(O^*\omega/O^*s.O_K) - \sum_{v \in S_\infty} \epsilon_v \log \|s\|_{v,O^*\omega}\right)$$

where $s$ is a section of $\omega$. From $\phi : p_*\omega \cong O^*\omega$ we have

$$\|s\|_{v,p_*\omega} = \|s\|_{v,O^*\omega} \cdot \|A\|(X_v)$$

Hence $\#(p_*\omega/p_*s.O_K) = \#(O^*\omega/O^*s.O_K)$. Hence we have

$$(O, \omega) = deg(O^*\omega),$$

$$deg(O^*\omega) = \frac{1}{[K:\mathbb{Q}]}\left(\log \#(p_*\omega/s.O_K) - \sum_{v\,\in\, S_\infty} \epsilon_v \log \|\omega\|_{v,p_*\omega} - \sum_{v\,\in\, S_\infty} \epsilon_v \log \|A\|(X_v)\right)$$

$$= h_F(E) - \frac{1}{[K:\mathbb{Q}]}\sum_{v\,\in\, S_\infty} \epsilon_v \log \|A\|(X_v)$$

hence the lemma is proved. $\square$.

It is known that the number of cyclic subgroups of order $N$ of an elliptic curve over $\mathbb{C}$ denoted by $e_N$ is given by

$$e_N = N \prod_{p|N}\left(1 + \frac{1}{p}\right).$$

The following 2 lemmas together with the above lemma prove our 2.1.
We denote by $C$ a cyclic subgroup of order $N$ , by $E^{'}$ quotient of $E$ by $C$, and $p : X^{'} \to B$ the regular minimal model of $E^{'}$ over $B$, $O^{'}$ the zero section, $\omega^{'}$ the dualizing sheaf and $f_C : E \to E^{'}$ the isogeny between $E$ and $E^{'}$.

**3.5 Lemma.** We have

$$\sum_C \left((O, \omega) - (O^{'}, \omega^{'})\right) = 0$$

where the sum runs over cyclic subgroups of order $N$.

**Proof.** The proof of the lemma makes extensive use of the Arakelov projection formula, adjunction formula and a Möbius inversion argument. We can extend the $N$-torsion points of $E$ over the regular minimal model $X$ of $E$ over $K$.
Let us fix an $M$ such that $M \mid N$. We consider $E[M]$ the set of sections corresponding to $M$-torsion points on $E$. Hence $E[M]$ is a subgroup of $E[N]$. Now consider $\bar{E}[M]$ the set of sections corresponding to points on $E$ of exact order $M$.

Consider $E^{'}$ quotient elliptic curve given by $E/E[M]$. We have an isogeny $M : E \rightarrow E$ (multiplication by $M$) with kernel $E[M]$. (Here $E[M]$ is indeed the set of $M$-torsion points , which is not to be confused with $\bar{E}[M]$ defined in last paragraph.)
To proceed further we need to apply the Arakelov projection formula . But we have seen that in order to apply the formula our map $M$ should extend to a $B$ morphism $X \rightarrow X$, but from the remark 1, we know that we can work with a cover $\tilde{X}$ of $X$ . But this is not bad as it would give only exceptional curves at singular points on the fibers and such curves have empty intersection with sections of $\tilde{X} \rightarrow B$.
Now

$$\begin{aligned} \sum_{Q \in E[M]\, Q \neq 0} (Q,O) + (O,O) &= (M^{*}O,O) \\ &= (O,O) \\ \sum_{Q \in E[M]\, Q \neq 0} (Q,O) &= 0 \quad from\ (2.5) \end{aligned}$$

Now consider,

$$\sum_{Q \in E[N]\ Q \neq 0} (Q,O) = \sum_{M|N} \sum_{Q \in \bar{E}[M]\, Q \neq 0} (Q,O)$$

$$\text{Denote} \sum_{Q \in E[N]\ Q \neq 0} (Q,O)\ \ by\ \ g(N)$$

$$\text{Denote} \sum_{Q \in \bar{E}[M]\, Q \neq 0} (Q,O)\ \ by\ \ f(M)$$

Then we have

$$g(N) = \sum_{M|N} f(M)$$

hence by Möbius inversion argument we have:

$$f(N) = \sum_{M|N} \mu(N/M) g(M) \text{ where } \mu \text{ is the Möbius function}$$

but we know that

$$g(M) = \sum_{Q \in E[M]\ Q \neq 0} (Q,O) = 0.$$

Hence

$$f(M) = \sum_{Q \in \bar{E}[M]\, Q \neq 0} (Q,O) = 0\ for\ all\ M \mid N. \tag{2.5}$$

Now consider

$$\begin{aligned} \sum_{C} \Big( (O,\omega) - (O^{'},\omega^{'}) \Big) &= \sum_{C} \Big( (O^{'},O^{'}) - (O,O) \Big) \quad \text{(By adjunction formula)} \\ &= \sum_{C} \sum_{Q \in C, Q \neq 0} (Q,O) \quad \text{(By the projection formula)} \end{aligned}$$

As

$$\sum_C \sum_{Q\in C, Q\neq 0} (Q,O) + (O,O) = (f_C^* O', O)$$
$$= (O', O')$$

Hence

$$\sum_C \sum_{Q\in C, Q\neq 0} (Q,O) = \sum_{M|N} \frac{e_N}{e_M} \sum_{Q\in \bar{E}[M]} (Q,O) \quad \text{(By the Lemma 3.2)}$$
$$= 0 \quad \text{(By equation 2.5 )}$$

The following lemma coupled with the two lemmas 3.4 and 3.5 proved above we have the proof of 2.1.

**3.6 Lemma.** Let $X$ be a one dimensional complex torus and $G$ the Arakelov Green function on $X$ .Then

$$\frac{1}{e_N} \sum_C \sum_{Q\in C, Q\neq 0} \log G(O,Q) = \lambda_N$$

where the first sum runs over the cyclic subgroups of $X$ of order $N$ , and the second sum runs over non zero points of $C$.

**Proof.** For a positive integer $M \mid N$ let us denote the set of $M$ torsion points on $X$ by $X[M]$, and the set of $M$ torsion points of exact order $M$ by $\bar{X}$[M]. By corollary 1.10 in section 1, we have

$$\sum_{Q\in X[M], Q\neq 0} \log G(0,Q) = \log M$$

We now use the multiplicative version of Möbius inversion formula:

$$\prod_{Q\in X[N], Q\neq 0} G(0,Q) = \prod_{M|N} \prod_{Q\in \bar{X}[M]} G(0,Q)$$

Denote

$$\prod_{Q\in X[N], Q\neq 0} G(0,Q) \quad \text{by } j(N)$$
$$\prod_{Q\in \bar{X}[M]} G(0,Q) \quad \text{by } h(M)$$

$$j(N) = \prod_{M|N} h(M)$$

Then by the Möbius inversion argument we have

$$h(N) = \prod_{M|N} (f(M))^{\mu(d)}$$

Hence we have

$$\begin{aligned}\sum_{Q\in\bar{X}[M]} \log G(0,Q) &= \sum_{d|M}\sum_{Q\in X[M/d]} \mu(d)\log G(0,Q)\\ &= -\sum_{Q\in X[p^{r-1}]} \log G(0,Q) + \sum_{Q\in X[p^r]} \log G(0,Q) \text{ (When } M=p^r \text{ for some prime } p)\\ &= \log p \text{ (When} M=p^r \text{ for some prime } p)\end{aligned}$$

$$\sum_{Q\in\bar{X}[M]} \log G(0,Q) = 0 (\text{if M is not of the form } p^r \text{ for some prime P})$$

That is because: it suffices to prove for $M = st$ , where $s,t$ are primes and $s \neq t$ .

$$\sum_{Q\in\bar{X}[M]} \log G(0,P) = -\sum_{Q\in X[s]} \log G(0,Q) - \sum_{Q\in X[t]} \log G(0,Q) + \sum_{Q\in X[st]} \log G(0,Q) = 0$$

Now by lemma 2.2 we have

$$\frac{1}{e_N}\sum_{C}\sum_{Q\in C,Q\neq 0} \log G(0,Q) = \frac{1}{e_N}\sum_{M|N,M>1}\frac{e_N}{e_M}\sum_{Q\in\bar{X}[M]} \log G(0,Q)$$

Hence we have

$$\frac{1}{e_N}\sum_{C}\sum_{Q\in C,Q\neq 0} \log G(0,Q) = \sum_{p|N,p\|N}\left(\frac{1}{e_p}+...+\frac{1}{e_{p^r}}\right)\log p$$

Now consider

$$\left(\frac{1}{e_p}+...+\frac{1}{e_{p^r}}\right) = \frac{1}{p(1+\frac{1}{p})}+....+\frac{1}{p^r(1+\frac{1}{p})}$$

And,

$$\begin{aligned}\frac{1}{p(1+\frac{1}{p})}+....+\frac{1}{p^r(1+\frac{1}{p})} &= \frac{1}{p+1}\left(\frac{1-\frac{1}{p^r}}{1-\frac{1}{p}}\right)\\ &= \frac{p^r-1}{p^{r-1}(p^2-1)}\end{aligned}$$

Hence we have

$$\frac{1}{e_N}\sum_{C}\sum_{Q\in C,Q\neq 0} \log G(O,Q) = \lambda_N$$

Now we can prove proposition 3.3 (or eq 2.1) by gluing all these 3 lemmas.

**Proof of proposition3.3.** We glue lemmas 3.4, 3.5 and 3.6 and prove the average height formula for elliptic curves.

$$\begin{aligned}
\frac{1}{e_N}\sum_C\Big(h_F(E^{'})-h_F(E)\Big) &= \frac{1}{e_N}\frac{1}{[K:\mathbb{Q}]}\sum_C\Big((O^{'},\omega^{'})-(O,\omega)\Big)+\frac{1}{e_N}\frac{1}{[K:\mathbb{Q}]}\sum_C\sum_v\epsilon_v\log\|A\|(X_v^{'})\\
&-\frac{1}{e_N}\frac{1}{[K:\mathbb{Q}]}\sum_C\sum_v\epsilon_v\log\|A\|(X_v)\ \text{(from lemma 3.4 )}\\
&=\frac{1}{e_N}\frac{1}{[K:\mathbb{Q}]}\sum_C\sum_v\epsilon_v\log\frac{\|A\|(X_v^{'})}{\|A\|(X_v)}\ \text{(from lemma 3.5)}\\
&=\frac{1}{e_N}\frac{1}{[K:\mathbb{Q}]}\sum_C\sum_v\epsilon_v\left(\log\sqrt{N}\right)\\
&-\frac{1}{e_N}\frac{1}{[K:\mathbb{Q}]}\sum_C\sum_v\epsilon_v\log\prod_{\substack{P\in C\\P\neq 0}}G(0,P^v)\ \text{(from proposition 1.9)}\\
&=\frac{1}{2}\log N-\frac{1}{e_N}\frac{1}{[K:\mathbb{Q}]}\sum_v\epsilon_v\lambda_N\ \text{(from 3.6 )}\\
&=\frac{1}{2}\log N-\lambda_N \qquad \square
\end{aligned}$$

## 2.4 Autissier's Proof

In this section we look at Autissier's way of proving 2.1, and then give a proof of the average height formula using both Autissier's approach and Robin de Jong's approach. His method involves the moduli spaces $X(1)$ and $X_0(N)$. As stated earlier we would only give an outline of the proof. We will see in complete detail in the next chapter that the same method carries on to Abelian varieties. In fact it happens to be the most convenient the way of proving the average height formula for abelian varieties.

We denote the equivalence classes of isomorphic elliptic curves, the moduli space of elliptic curves by $X(1)$.
We denote by $X_\circ(N)$ the Deligne-Rapoport compactification of the moduli space of cyclic isogenies of elliptic curves of degree $N$.

**4.1 Definition.** The modular group denoted by $\Gamma(1)$, is the quotient group

$$\Gamma(1)=SL_2(\mathbb{Z})/\left\{\pm Id\right\}$$

where $Id$ is the identity matrix.

**4.2 Definition.** We define $\Gamma_0(N)$ the subgroup of $\Gamma(1)$ as

$$\Gamma_0(N)=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Gamma(1):c\equiv 0(\text{Mod}\quad \text{N})\right\}$$

**4.3 Definition.** The extended upper half plane $\mathbf{H}^*$ is the union of the upper half plane $\mathbf{H}$ and the $\mathbb{Q}$-rational points of the projective line,

$$\mathbf{H}^* = \mathbf{H} \cup \mathbb{P}^1(\mathbb{Q}) = \mathbf{H} \cup \mathbb{Q} \cup \{\infty\}$$

$\Gamma(1)$ defines a natural action on $\mathbf{H}^*$ as follows

$$\text{for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1), \quad \tau \in \mathbf{H}, \qquad \gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$

$$\text{for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tau = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{P}^1(\mathbb{Q}), \qquad \gamma(\tau) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

**4.4 Theorem.** There exists a one- one correspondence between the following sets :

$$\frac{\{elliptic\ curves\ defined\ over\ \mathbb{C}\}}{\mathbb{C} - isomorphisms} \longleftrightarrow \{\Lambda_\tau\} \longleftrightarrow \Gamma(1) \setminus \mathbf{H}$$

where $\{\Lambda_\tau\}$ denotes the equivalence classes of homothetic lattices.
**Proof.** The proof can be found in [Si] proposition 4.4 chapter 1.

So we now see that X(1)($\mathbb{C}$), the moduli space of complex elliptic curves is nothing but $\Gamma(1) \setminus \mathbf{H}^*$. We can also identify $X_0$(N)($\mathbb{C}$) with the moduli space of ordered pairs $(E, C)$ where $E$ is an complex elliptic curve and $C$ a cyclic subgroup of order $N$. Two such pairs $(E, C)$ and $(E^{'}, C^{'})$ are said to be equivalent if some isomorphism from $E$ to $E^{'}$ takes $C$ to $C^{'}$ .

**4.5 Remark.** 1. The $j$-invariant $j : X(1) \overset{\cong}{\to} \mathbb{P}^1_{\mathbb{Z}}$ is an isomorphism .
2. $X_0$(N) is a normal arithmetic surface.

Let us denote $X \times_{\mathbb{Z}}$X by $P$. We have a finite morphism $i_N : X_0(N) \to P$ , which associates to each cyclic isogeny $\alpha : E \to E^{'}$ of degree $N$ to its source and target ($E$, $E^{'}$). Let us denote the image of this morphism $i_N$ by $T_N$. $T_N$ is bi-rational to $X_0$(N). $T_N$ is an integral Cartier divisor on $P$.

Let $[\infty]$ denote the divisor associated to the point $\infty \in X(1)(\mathbb{Z})$. Let us denote $O_{X(1)}([\infty]) = j^* O_{\mathbb{P}^1}(1)$ by $M$.

**4.6 Definition.** Let $X$ be an arithmetic surface, $W$ a closed set of $X(\mathbb{C})$ stable under complex conjugation and $\hat{L} = (L, \| \, \|)$ an invertible sheaf on $X$ with a family of Hermitian norms $\| \, \|$ on $L_{\mathbb{C}}$ such that they vary continuously on $X(\mathbb{C}) - W$, then $\hat{L}$ is said to be singular along $W$. If for a family $\| \, \|^{'}$ (L,$\| \, \|^{'}$) is $C^\infty$ , the function $\log(\| \, \| / \| \, \|^{'})$ is $L^1_2$ and $\hat{L}$ is said to be $L^1_2$ singular along $W$.

Let $\hat{L}$ be an invertible sheaf on $X$ singular along $W$. Let $Z_1$(X) be the group of 1-cycles on $X$. Let us denote the group of 1-cycles $D \in Z_1(X)$ such that the support of $D_{\mathbb{C}} \cup W = \phi$ by $Z_1(X, W)$.

**4.7 Definition.** Let $X$ be an arithmetic variety and $\hat{L}$ be $L^1_2$ singular along a closed set $W \subset$ X($\mathbb{C}$) which is stable under complex conjugation . Let $Y \in Z_1(X, W)$ be a 1-cycle. We then define the height of $Y$ relative to $\hat{L}$ as
1. If $Y$ is a horizontal divisor , then $h_{\hat{L}}$(Y)=deg($\hat{L}_{|Y}$) , where deg denotes the arithmetic degree defined above.
2. If $Y$ is a vertical divisor, that is contains a fiber of a prime number $p$, then $h_{\hat{L}}$(Y)= $deg_{\mathbb{F}_p}(L_{|Y}) \log p$.

**Remark 4.8** The isometry classes of $L^1_2$ singular $\hat{L}$'s form a group denoted by $Pic(X; L^1_2)$. From [Bo] we have the following symmetric bilinear form.

$$< . >: Pic(X; L^1_2) \times Pic(X; L^1_2) \to \mathbb{R}$$

For a $\hat{L} \in$ Pic(X;$L^1_2$) the height of $X$ relative to $\hat{L}$ is given by
$h_{\hat{L}}(X) =< \hat{L}.\hat{L} >$.

Now with all this theory we an attack our problem of trying to define height function on invertible sheaves on our moduli space $X(1)$.
So let us now consider the invertible sheaf $M = j^* O_{\mathbb{P}^1}(1)$ on $X(1)$. Let us consider the metric $\|.\|_m$ given by

$$\|1\|_m = |\Delta(\tau)| \, (im(\tau))^6 \tag{2.6}$$

$\hat{M}$= (M;$\|.\|_m$) is an hermitian ,invertible and $L^1_2$ sheaf along $\infty_{\mathbb{C}}$.
Now let us consider $E$ an elliptic curve over $\bar{\mathbb{Q}}$. This defines a point $x \in X(1)(\bar{\mathbb{Q}})$.Let $Y$ be the point on $X(1)(\bar{\mathbb{Q}})$ corresponding to $E$. The following proposition proved by Faltings in [Fa 3] provides us the relation between Faltings height function and the height function that we have defined here.

**4.9 Proposition.** Let $E$ be a semi-stable elliptic curve defined over a number field $K$ . Let $\Delta$(E/K) be the minimal discriminant ideal of $E$ over $K$. Then the following formula holds as the sum runs over the complex embeddings over $K$.

$$h_F(E) = \frac{1}{[K:\mathbb{Q}]} \left( \frac{1}{12} \log \left| N_{K/\mathbb{Q}}(\Delta(E/K)) \right| - \frac{1}{12} \sum_v \epsilon_v \log((2\pi)^{12} \, \|\Delta\| \, (X_v)) \right)$$

where $\|\Delta\| \, (X)$ is equal to $|\Delta(\tau)| \, (im(\tau))^6$ if $X$ is isomorphic to $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$
**Proof** From lemma 3.4 we have that

$$(O, \omega) = h_F(E) - \frac{1}{[K:\mathbb{Q}]} \sum_{v \in S_\infty} \epsilon_v \log \|A\| \, (X_v) \tag{2.7}$$

But from the adjunction formula (proposition 2.12) and proposition 2.18, we have that

$$(O,\omega) = \frac{1}{12}\log\left|N_{K/\mathbb{Q}}(E/K)\right|.$$

But from [Fa 2] or proposition 4.6 from [Ro] we have that

$$A(X) = \frac{1}{(2\pi)\left\|\Delta\right\|(X)^{1/12}}$$

Hence we have

$$\begin{aligned} h_F(E) &= \frac{1}{[K:\mathbb{Q}]}\left((O,\omega) - \sum_{v\in S_\infty}\log\left\|A\right\|(X_v)\right) \\ &= \frac{1}{[K:\mathbb{Q}]}\left(\frac{1}{12}\log\left|N_{K/\mathbb{Q}}(\Delta(E/K))\right| - \frac{1}{12}\sum_v \epsilon_v \log((2\pi)^{12}\left\|\Delta\right\|(X_v))\right). \quad \square \end{aligned}$$

From proposition 4.9 we can conclude that $h_F(E) = \frac{1}{[K:\mathbb{Q}]}\frac{1}{12}h_{\hat{M}}(Y)$.

Having seen how Faltings height function is related to the height function defined on an invertible sheaf on $X(1)$, we now proceed to see how this theory can be used to prove equation 2.1.
We have the projection maps $pr_1$, $pr_2$ from $P$ to $X(1)$. Let us put $\hat{L} = pr_1^*\hat{M} \otimes pr_2^*\hat{M}$.
We have already stated that $T_N$ the image of $i_N$ is an integral divisor on $P$.
Let $Y$ be a integral Weil divisor on $X(1)$ and $[k(Y){:}\mathbb{Q}] = n$. As $M$ is a very ample invertible sheaf we have a global section $s$ of $M^{\otimes n}$ on $X(1)$ such that $div(s) = Y$.
Let us denote

$$\pi_i = pr_i \circ i_N : X_0(N) \to X(1)$$

Hence we have map $\pi_{2_*} : Z_1(X_0(N)) \to Z_1(X(1))$ . Let us denote $\pi_{2_*}(div(\pi_1^* s))$ by $(T_{N*}Y)$ where $\pi_1^*$ is the inverse image of the invertible sheaf M on $X_0(N)$. $(T_{N*}Y)$ is a 1-cycle on X(1).

The idea now is to compute the height of the divisor $T_N$ and the one cycle $T_{N*}Y$ relative to the hermitian line bundles $\hat{L}$ and $M$. Once $h_{\hat{L}}(T_N)$ and $h_{\hat{M}}(T_{M*})$ are computed we can get the average height formula by applying the relation:

$$\frac{1}{12n}\frac{1}{[K:\mathbb{Q}]}h_{\hat{M}}(T_{N*}Y) = \sum_{i=1}^{e_N} h_F(E_i) \text{ and } h_F(E) = \frac{1}{12}\frac{1}{[K:\mathbb{Q}]}h_{\hat{M}}(Y)$$

In fact all we need is a formula for $h_{\hat{M}}(T_{N*}(Y))$ in terms of $h_{\hat{M}}(Y))$.

Autissier first computes $h_{\hat{L}}(T_N(Y))$ and then uses the relation $h_{\hat{L}_{|Y'}}(Y') = 2h_{\hat{M}}(Y) + 12k_1 n$, to compute $h_{\hat{M}}(T_{N*}(Y))$, where $k_1$ is $12\zeta'(-1) - \log\pi - 12$ and $\zeta$ is the Riemann zeta function.

**4.10 Proposition.** Let $N \geq 1$, $Y$ an integer point of $X(1)$ (i.e. an integral closed sub-scheme) such that $Y_{\mathbb{C}} \cap \infty_{\mathbb{C}} = \phi$. Let $[k(Y) : \mathbb{Q}] = n$. Then we have the following formulae.

$$h_{\hat{L}}(T_N) = 12e_N(\log N - 2\lambda_N + 4k_1), \quad \frac{1}{ne_N}h_{\hat{M}}(T_{N*}Y) = \frac{1}{[K:\mathbb{Q}]}h_{\hat{M}}(Y) + 6\log N - 12\lambda_N. \tag{2.8}$$

**Proof** [Au 1], theorem 3.2.

We now look at an alternate approach of proving the average height formula. Autissier uses the following 2 lemmas in his proof of proposition 4.10. On observing closely these two lemmas coupled with our two lemmas 3.5 and 2.12 give the average height formula. This approach involves basic properties of modular forms and $\Delta(\tau)$ the modular discriminant.

Before we state and prove the two lemmas we look at the basic concepts of modular forms and $\Delta(\tau)$ the modular discriminant.

**4.11 Definition.** Let $k$ be an integer. A meromorphic function $f : \mathbf{H} \to \mathbb{C}$ is weakly modular of weight $k$ if

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \text{ for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \text{ and } \tau \in \mathbf{H}.$$

**4.12 Definition.** A cusp form of weight $k$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that the following conditions hold.

1. $f$ is holomorphic on $\mathbf{H}$
2. $f$ is weakly modular of weight $k$.
3. $f$ is holomorphic at $\infty$.
4. The leading coefficient $a_\circ$ of the fourier expansion of $f$

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n \quad , q = e^{2\pi i\tau}$$

is zero.

It is easy to see that $\lim_{im(\tau)\to\infty} f(\tau) = 0$, as $q^n \to 0$ as $im(\tau) \to \infty$.

Now let us consider the modular discriminant

$$\Delta : \mathbf{H} \to \mathbb{C} \quad \Delta(\tau) = q\prod_{k=1}^{\infty}(1 - q^k)^{24}.$$

$\Delta(\tau)$ is the unique cusp-form of weight 12 on $SL_2(\mathbb{Z})$. The leading coefficient $a_\circ = 0$ and $a_1 = 1$ in the Fourier expansion of $\Delta(\tau)$.

Now consider the set of matrices $C_N$ defined as:

$$C_N = \left\{ \gamma = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in M_2(\mathbb{Z});\ a_\gamma d_\gamma = N,\ a_\gamma \geq 1,\ 0 \leq b_\gamma \leq d_\gamma - 1 \text{ and } gcd(a_\gamma, b_\gamma, d_\gamma) = 1 \right\}$$

$$e_N = \#C_N = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

The following lemma gives us all the cyclic sub lattices $\Lambda_{\tau'}$ of order $N$ of a lattice $\Lambda_\tau$ in the complex plane.

**4.13 Lemma.** Let $\tau \in \mathbf{H}$ and consider the lattice $\Lambda_\tau = [1, \tau]$. Then
1.Consider the sub lattice $\Lambda_\tau \subset [1, \tau]$ of index $N$, there is a unique $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in C_N$ such that $\Lambda_\tau = d[1, \gamma(\tau)]$.
2. Conversely if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in C_N$ , then $d[1, \gamma(\tau)]$ is a cyclic sub lattice of index $N$ in $[1, \tau]$.
**Proof.** The proof can be found in [Co], lemma 11.24 chapter 11.

Let $E/K$ be an elliptic curve defined over a number field $K$. Let $E'$ be a the elliptic curve obtained by quotient of $E$ by a cyclic subgroup of order $N$. Let $\Lambda_\tau$ belong to the isomorphic class of lattices that corresponds to $E$, then from theorem 4.13 and 4.4 we can safely conclude that $\Lambda_{\gamma\tau}$ belongs to the isomorphic classes of lattices that correspond to $E'$ for some $\gamma \in C_N$. Let $p : X \to B$ be the regular minimal model of $E/K$. Then $X_v$ for each infinite place is a 1-dimensional complex torus. Let $X'$ be the regular minimal model of $E'$, then $\Lambda_{\tau'}$ the lattice corresponding to $X_v'$ is a cyclic sub lattice of order $N$ in $\Lambda_\tau$ the lattice corresponding to $X_v$, for each infinite place $v$. Hence $\tau' = \gamma\tau$ for some $\gamma \in C_N$.

The following two lemmas are proved by Autissier in [Au] in proving the average height formula.

**4.14 Lemma.** For $N \geq 2$, and for all $\tau \in \mathbf{H}$ we have

$$\prod_{\gamma \in C_N} \Delta(\gamma(\tau)) = [-\Delta(\tau)]^{e_N}.$$

**Proof.** [Au 2] lemma 2.2 . $\square$

**4.15 Lemma.**

$$\sum_{\gamma \in C_N} \log \frac{d_\gamma}{a_\gamma} = e_N(\log N - 2\lambda_N)$$

**Proof.** We use finite mathematical induction to prove the lemma. Let us denote L.H.S by $S_N$. For $N = 1$, the lemma holds. Let us assume that the lemma holds for all $N < r$. If $r$ is a prime, we have $S_r = (r-1)\log r$ (as when we have $d_\gamma = r$, and $a_\gamma = 1$ $b_\gamma$ can take $r$ values from 0 to $r-1$, when $d_{\gamma=1}$, then $b_\gamma = 0$, and $a_\gamma = 1$)
If $r = st$, where $s$, $t$ are co-prime then one can check that we have the relation

$$S_{st} = [e_t S_s + e_s S_t].$$

Since $s$, $t$ are $< r$, we ca conclude that the lemma holds for $r$.
If $r = p^a$ for some prime $p$, then one can directly calculate as

$$S_r = [p^{r-1}(p+1)r - 2\frac{p^r - 1}{p-1}]\log p$$

Hence by the principle of finite mathematical induction we can conclude that the lemma holds for all positive integers. $\square$

Let us consider $im(\gamma\tau)$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in C_N$. $im(\gamma\tau) = im(\frac{a\tau+b}{d}) = im(\frac{a\tau}{d}) = \frac{a}{d}im(\tau)$.

**Proof of eq 2.1.** Now we can easily prove the average height formula for elliptic curves by combining lemma 4.13, 4.14 and 4.9 of this section , and lemmas 3.5 and 2.12 of section 3, 2 respectively of this chapter.

$$\begin{aligned}
\frac{1}{e_N}\sum_C \left(h_F(E^{'}) - h_F(E)\right) &= \frac{1}{[K:\mathbb{Q}]}\frac{1}{e_N}\sum_C \frac{1}{12}\left(\log\left|N_{K/\mathbb{Q}}(\Delta(E^{'}/K))\right| - \log\left|N_{K/\mathbb{Q}}(\Delta(E/K))\right|\right) \\
&+ \frac{1}{[K:\mathbb{Q}]}\frac{1}{e_N}\sum_C \frac{1}{12}\sum_v \epsilon_v\left(\log(2\pi)^{12}\left\|\Delta\right\|(X_v) - \log(2\pi)^{12}\left\|\Delta\right\|(X_v^{'})\right) \\
&= \frac{1}{[K:\mathbb{Q}]}\frac{1}{e_N}\sum_C \left((O,O) - (O^{'},O^{'})\right) \text{ proposition 4.9 and lemma 2.16} \\
&+ \frac{1}{[K:\mathbb{Q}]}\frac{1}{e_N}\frac{1}{12}\sum_v \epsilon_v \sum_C \left(\log\left\|\Delta\right\|(X_v) - \log\left\|\Delta\right\|(X_v^{'})\right) \\
&= 0 + \frac{1}{[K:\mathbb{Q}]}\frac{1}{e_N}\frac{1}{12}\sum_v \epsilon_v \sum_{\gamma\in C_N} \left(\log|\Delta(\tau)| - \log|\Delta(\gamma\tau)|\right) \text{ from lemma 3.5} \\
&+ \frac{1}{[K:\mathbb{Q}]}\frac{1}{e_N}\frac{1}{12}\sum_v \epsilon_v \sum_{\gamma\in C_N} \left(\log\left|(im(\tau))^6\right| - \log\left|(im(\gamma\tau))^6\right|\right) \\
&= 0 + \frac{1}{[K:\mathbb{Q}]}\frac{1}{e_N}\frac{1}{12}\sum_v \epsilon_v \sum_{\gamma\in C_N} \left(\log(\frac{d_\gamma}{a_\gamma})^6\right) \text{ from lemma 4.14} \\
&= \frac{1}{2}\log N - \lambda_N \text{ from lemma 4.15} \quad \square
\end{aligned}$$

# 3 Average Faltings Height of Isogenous Abelian Varieties

We have seen in the last chapter a proof of the formula for the Faltings height of the quotients of a semi-stable elliptic curve by its cyclic subgroups of fixed order. In this chapter we look at the formula for the average height of the quotients of a principally polarised abelian variety of dimension $g$, with good reduction at a prime $p$ by isotropic subgroups of order $p^g$.

In the last chapter we have seen that the formula for the elliptic curve case could be derived in three ways. One due to Robin de Jong which involved Arakelov intersection theory, one due to Autissier which involved advanced concepts like the height of the modular curve and another which involved concepts from both the approaches. As we have stated before, in the case of abelian varieties to proceed with elementary methods would be a very tedious job.

We have 3 sections in this chapter. In the first section we look at the basic definitions and the basic notion of intersection of Cartier divisors on arithmetic varieties.
In section 2 we discuss at an alternate way of looking at the Faltings Formula (lemma 4.11 of chapter 1 of this thesis) in terms of the arithmetic intersection product of a Cartier divisor and the basic concepts of a moduli scheme $A^p_{g,n}$. In section 3 we finally prove the average height formula for isogenous $p$-ordinary principally polarised abelian varieties.

## 3.1 Basic Notions

**1.1 Definition.** A group functor $F$ over a fixed scheme $S$ is a co-functor from the category of schemes over $S$ to the category of groups. If the group functor $F$ is representable (by a scheme over $S$) and if $G/S$ is the representative object, we call $G$ a group scheme over $S$.

**1.2 Definition.** An abelian scheme $\pi : A \to S$ of relative dimension $g$ is a proper, smooth group scheme over a Noetherian base scheme $S$ whose geometric fibers are connected and of dimension $g$. The fibers of an abelian scheme are abelian varieties.
It is well known that an abelian scheme is a commutative group scheme.

**1.3 Definition.** Let $\pi : A \to S$ be an abelian scheme with unit section $e : S \to A$.
1. For any invertible sheaf $\ell$ on $A$ a rigidification of $\ell$ is an isomorphism $\epsilon : O_S \to e^*\ell$.
2. The relative Picard functor $Pic(A/S)$ is given by the isomorphism classes of invertible sheaves $\ell$ on $A \times_S T$ with rigidification along $e_T = e_S \times T$.
3. The sub-functor $Pic^\circ(A/S)$ of $Pic(A/S)$ is given by the isomorphism classes of invertible sheaves $\ell$ on $A \times_S T$ with rigidification along $e_T$ such that for all $t \in T$, $\ell \otimes k(t)$ is algebraically equivalent to zero on $A_t$.
4. The dual abelian scheme denoted by $A^t$ is the algebraic space representing the relative Picard functor $Pic^\circ(A/S)$.
It is well known that $\mathrm{A}^t$ is an abelian scheme over $S$.

**1.4 Definition.** A polarization of $A/S$ is a homomorphism $\lambda : A \to A^t$ such that for each geometric point $\bar{s}$ of $S$, $\lambda_{\bar{s}} = \lambda(\ell_{\bar{s}})$ for some ample invertible sheaf $\ell_{\bar{s}}$ on $A_{\bar{s}}$. Here $\lambda(\ell_{\bar{s}})$ is a map $A \to A^t$ such that $a \to t_a^*\ell \otimes \ell^{-1}$, where $t_a$ is the translation-by-$a$-map. We say that $A$ is principally polarized if $\lambda$ is an isomorphism. The degree of the polarization is its degree as an isogeny from $A$ to $A^t$.
We will be considering only principally polarised abelian varieties in this chapter, with more adjectives attached to them, which will be defined in this section.
Let $g$ and $n \geq 0$, and $p$ a prime number. Then we denote $\mathbb{Z}/n\mathbb{Z}$ by $C_n$ and let $C_n \times \mu_n$ be denoted by $G_n$. Let $G_n^g$ denote $(\mathbb{Z}/n\mathbb{Z})^g \times \mu_n^g$. $C_n$ and $\mu_n$ are $\mathbb{Z}$-schemes.
Let us consider the $\mathbb{Z}$-morphism

$$\phi_m : G_n^g \to G_n^g$$

$$\text{defined by} \quad \phi_m(a_1, \zeta_1, ...., a_g, \zeta_g) = (ma_1, \zeta_1, ..., ma_g, \zeta_g).$$

**1.5 Definition.** Let $A$ abelian variety of genus $g$ over a field $K$ of characteristic $p$. If $A$ admits a polarization whose degree of polarization is prime to $p$, then $A$ is said to be $p$-ordinary if the $p$-kernel of $A$ $A[p](\bar{K}) \cong (\mathbb{Z}/p\mathbb{Z})^g$ i.e. $A_{\bar{K}}[p]$ is isomorphic as a group scheme to $(\mathbb{Z}/p\mathbb{Z})^g \times \mu_p^g$.

**1.6 Definition.** An abelian scheme $A$ over $S$ is said to be $p$-ordinary when all the geometric fibers of $\pi : A \to S$ in characteristic $p$ are ordinary abelian varieties.

**1.7 Definition.** An abelian variety $A$ of dimension $g$ over $\bar{\mathbb{Q}}$ is said to have a good reduction at $p$, when there exists a finite, flat and integral $\mathbb{Z}_{(p)}$-algebra $R \subset \bar{\mathbb{Q}}$ and a $p$-ordinary $R$-abelian scheme $A'$ such that $A'_{\bar{\mathbb{Q}}} = A$.

Consider the isogeny $n_A : A \to A$. We have the dual isogeny $n_{A^t} : A^t \to A^t$. We then have a canonical skew symmetric non degenerate pairing given by:

$$\overline{e_n} : A[n] \times A^t[n] \to \mu_n$$

Let $a$, $a^{'} \in A[n]$, $A^t[n]$ respectively. Let $D$ be the divisor on $A$ corresponding to $a^{'}$. Then we have

$$\bar{e}_n(a, a^{'}) = g/g \circ t_a.$$

where $(g) = n_A^{-1} D$ and $n_A$ is the multiplication by the integer $n$ map. This pairing is called the Weil pairing.
If $\lambda$ is a principal polarization then we have a canonical non-degenerate skew symmetric pairing

$$A[n] \times A[n] \to \mu_n$$

also called the Weil pairing. We will be referring to this pairing in all our work.

We will also recall what is called the symplectic pairing: Let $S$ be a base scheme and $n \in \mathbb{Z}$ be $\geq 1$. We have a standard sympletic pairing

$$e : (\mathbb{Z}/n\mathbb{Z})^g \times (\mu_{n,S})^g \times (\mathbb{Z}/n\mathbb{Z})^g \times (\mu_{n,S})^g \to \mu_{m,S},$$

$$e\left((a_1, \zeta_1, ...., a_g, \zeta_g), (b_1, \zeta_1, .., b_g, \zeta_g)\right) = \prod_{i=1}^{g} (e^{2\pi i \frac{a_i}{n}} \zeta_i e^{-2\pi i \frac{b_i}{n}} \zeta_i^{-1}).$$

We shall be denoting a principally polarised abelian scheme of dimension $g$ by $(A/S; \lambda)$ where $\lambda$ is a principal polarization.

**1.8 Definition.** A symplectic level $n$ structure on an abelian scheme $(A/S; \lambda)$ is an isomorphism

$$\alpha : G^g_{n;S} : \tilde{\to} A[n]$$

which identifies the standard symplectic pairing with the Weil pairing on $A[n]$. When such an isomorphism exists for some $n$, then we say that the abelian scheme $(A; \lambda; \alpha)$ is of type $(g, n)$.
Here one can observe that if $p \mid n$, and if $(A/S; \lambda)$ admits a symplectic level $n$ structure then $A$ is $p$-ordinary.
Let $r$ be an integer $\geq 1$. We assume that $p \not| n$ for the rest of the section.

**1.9 Definition.** Let $(A_1; \lambda_1)$ and $(A_2; \lambda_2)$ be principally polarised abelian schemes of dimension $g$. Let $f : A_2 \to A_1$ be an isogeny of rank $p^{rg}$. $f$ is said to be of **type** $P$ when

$$f^t \circ \lambda_2 \circ f = [p^r] \circ \lambda_1.$$

**1.10 Definition.** Let $(A_1/S; \lambda_1; \alpha_1)$ and $(A_2; \lambda_2; \alpha_2)$ be abelian schemes of type $(g, n)$. Let $f : A_1 \to A_2$ be an isogeny of rank $p^{rg}$. $f$ is said to be **of type** $PN$ when

$$f \circ \alpha_1 = \alpha_2 \circ \phi_{p^r} \quad \text{and } f \text{ is of type } P.$$

**1.11 Remark.** Let $(A_1/S; \lambda_1; \alpha_1)$ be an abelian scheme of type $(g, n)$ (i.e. it is principally polarised of dimension $g$ and equipped with a symplectic level $n$ structure) and $(A_2/S; \lambda_2)$ a principally polarised abelian scheme. Let $f : A_1 \to A_2$ be an isogeny of rank $p^{rg}$ and of type $P$. Then there exists a unique symplectic level $n$ structure $\alpha_2$ on $(A_2, \lambda)$ which makes $f$ an isogeny of type $PN$.

Let us now consider an algebraically closed field $K$ $(K = \bar{K})$ . Let $(A_\circ, \lambda_\circ)$ be a principally polarised abelian variety of dimension $g$ over $K$. Then we define

**1.12 Definition.** A closed subgroup $G$ of $A_\circ[p^r]$ of rank $p^{rg}$ is said to be isotropic if the restriction of the Weil pairing to $G$ i.e. $\overline{e_m} : G \times G \to \mu_n$ is trivial.

**1.13 Theorem.** Consider $G$ a subgroup of $A_\circ[p^r]$ of rank $p^{rg}$. Then $G$ is isotropic iff there exists a unique principal polarization of $A_\circ/G$ such that the isogeny $A_\circ \to A_\circ/G$ is of type $P$.
**Proof.** The proof can be found in [Mu 1], which we skip .

## 3.2 Intersection Theory for Arithmetic Varieties

In the last chapter we have seen the Arakelov intersection theory on the regular minimal model of an elliptic curve. In this section we study arithmetic intersection theory defined for higher dimensional arithmetic varieties. We will study the basic notions of arithmetic intersection theory for metrics induced by finite primes.
We first define what an arithmetic variety is: Let $p$ be a prime number, then $\mathbb{Z}_{(p)}$ the localisation of $\mathbb{Z}$ at the prime ideal $p$. Denote $Spec(\mathbb{Z}_{(p)})$ by $A_\circ$.

**2.1 Definition.** An arithmetic variety $X$ is a quasi-projective,reduced and flat scheme over $\mathbb{Z}_{(p)}$. We say a point $x \in X$ is an integer point if it is closed and integral as a sub scheme of $X$ which is finite and flat over $\mathbb{Z}_{(p)}$.

**2.2 Definition.** A Cartier divisor $D$ on $X$ is said to be vertical when $D_{\mathbb{Q}}$, the restriction of $D$ to $X_{\mathbb{Q}}$ is zero. Let $D_V(X)$ denote the group of vertical Cartier divisors on $X$.
Let us denote the group of $\mathbb{Z}$-linear combinations of integer points of $X$ by $Z_1^h(X)$ and the group of 0-cycles on $X$ by $Z_\circ(X)$.
Let $D$ be a vertical Cartier divisor on $X$ and $E$ an integral point of $X$. We denote by $< D, E >$ the Weil divisor associated to $D_{|E}$ on $E$. This pairing extends to a bilinear form which determines the intersection product:

$$<;>: D_V(X) \times Z_1^h \to Z_\circ(X)$$

where the action is given by:

$$< D; \sum \lambda_i E_i > \to \sum \lambda_i < D; E_i > .$$

When $E$ is a closed point of $X$ lets put $deg_p(x) = [k(E) : \mathbb{F}_p]$. We have a homomorphism

$$deg_p : Z_\circ(X) \to \mathbb{Z}.$$

Now we have a bilinear form by composing $deg_p \circ <;>$:

$$< . >_p : D_V(X) \times Z_1^h \to \mathbb{Z}.$$

For $D \in D_V(X)$ and $E \in Z_1^h(X)$ we define

$$< D.E >_p = deg_p < D; E > .$$

Now we look at a few properties of this intersection product, the one similar to the projection formula and the proofs of all these propositions can be found in [Fu] or [QL]. Let $X$ and $Y$ be arithmetic varieties over $\mathbb{Z}_{(p)}$. Let $f : X \to Y$ be a proper morphism between $X$ and $Y$. We define the push forward $f_*$ of $k$-cycles on $X$ to $k$-cycles of $Y$ . For any subvariety $V$ of $X$, the image $W = f(V)$ is a closed subvariety of $Y$. Then there is an induced embedding of $R(W)$ (the field of rational functions on $W$) in $R(V)$ (the field of rational functions on $V$) which is a finite field extension if $W$ has same dimension as $V$. Now

$$deg(V/W) = \begin{cases} [R(V) : R(W)] \; if \;\; dim(W) = dim(V) \\ \quad 0 \quad if \;\; dim(V) < dim(W) \end{cases}$$

Now define

$$f_*[V] = deg(V/W)[W].$$

This extends to a homomorphism called the push forward of cycles from the $k$-cycles of $X$ to the $k$-cycles of $Y$:

$$f_* : Z_k(X) \to Z_k(Y).$$

Now we define the pull back of cycles. A morphism of varieties $f : X \to Y$ is said to be of relative dimension $n$ if for all subvarieties $V$ of $Y$ and all irreducible components $V'$ of $f^{-1}(V)$, $\;dim(V') = dim(V) + n$. Now let $f : X \to Y$ be a morphism of relative dimension $n$ .Then

$$f^* : Z_k Y \to Z_{n+k} X$$

is given by:

$$f^*[V] = [f^{-1}(V)]$$

where $[V]$ denotes the cycle corresponding to $V$.

We know that $f^*$ is defined on $D_V(Y)$.

$$f^* : D_V(Y) \to D_V X \tag{3.1}$$

We have a projection formula:

$$< D; f_*E >= f_* < f^*D; E >, \quad D \in D_V(Y), \ E \in Z_1^h(X). \tag{3.2}$$

Let us consider the arithmetic variety $X$ over $Spec(\mathbb{Z}_{(p)})$ with its structural morphism $\pi : X \to A_\circ$. Let us denote $\pi_*(\mathbb{Z}_{(p)})$ by $X_p$. We have

$$< X_p.E >_p = deg_p < X_p; E >= [k(E) : \mathbb{Q}] \ \ for\ all\ E \in \ Z_1^h(X).$$

For every $f : X \to Y$ finite, flat morphism of arithmetic varieties we have seen how the $f_*$ and $f^*$ are defined on the cycles of $X$ and $Y$ respectively. Let us now consider the action of $f_*$. We have a map $O_Y \to f_*O_{X)}$ which is called the direct image of $O_X$ in $O_Y$. Now we define the norm map for two arithmetic varieties as defined by Grothendieck in [EGA 2].

Let $X$, $Y$ be two arithmetic varieties defined over $\mathbb{Z}_{(p)}$. Let $f : X \to Y$ be a finite, flat morphism of rank $n \geq 1$. $f_*O_x$ is a locally free module over $O_Y$ of rank $n$. Let us denote $f_*O_X$ by $B$ and $f_*O_Y$ by $A$. Then $B$ is a locally free $A$ module of rank $n$. From 6.5 of [EGA 2] we have a map $N_{B/A}$:

$$N_{B/A} : B \to A$$

such that

$$N_{B/A}(I_B) = I_A$$

$$N_{B/A}(f.g) = N_{B/A}(f).N_{B/A}(g) \ \text{ where f, g are 2 sections of } B$$

$$N_{B/A}(s) = s^n \ \text{ s is a section of A}$$

$N_{B/A}$ acts on the invertible sheafs $\ell$ on B: .

$$N_{B/A}(O_X) = O_Y,$$

$$N_{B/A}(\ell^1 \otimes_{O_X} \ell^2) = N_{B/A}(\ell^1) \otimes N_{B/A}(\ell^2) \ \text{ where } \ell^1,\ \ell^2 \text{ are invertible sheaves on } B,$$

$$N_{B/A}(\ell^{-1}) = N_{B/A}(\ell)^{-1} \ \ \ell \text{ is an invertible sheaf on } B,$$

$$N_{B/A}(\ell) = \ell^{\otimes n}.$$

$N_{B/A}$ extends to a map $N_{X/Y} : Pic(X) \to Pic(Y)$. Since there exists a one-one correspondence between isomorphism classes of invertible sheaves and Cartier divisors modulo linear equivalence, we get a map:

$$N_f : D_V(X) \to D_V(Y).$$

such that for any open set $U$ of $Y$ and all invertible sheafs $\ell$ on $O_X$ and for all meromorphic regular sections $s$ defined of $\ell$ defined on $f^{-1}(U)$ we have

$$div_U(N(s)) = N_f(div_{f^{-1}(U)}(s)). \tag{3.3}$$

From 21.5.6 of [EGA 4] we have that $f_* f^* E = nE$ for all $E \in D_V(Y)$. Hence from the properties of the norm map $N_f$ we can conclude that $N_f(f^* D) = nD$ for all divisors $D \in D_V(Y)$.

We now define the map $cyc_x$ on an arithmetic variety $X$ defined over $\mathbb{Z}_{(p)}$ of dimension $n$.

$$cyc : D_V(X) \to Z_{n-1}(X)$$
$$cyc(D) = \sum length(O_{Y(D),x}).\bar{x}$$

where $D$ is a Cartier divisor of $X$, and $Y(D)$ is the closed subscheme corresponding to the ideal $I_X(D) \subset O_X$. The sum runs over all $x \in X$ such that co-dimension of $\bar{x} = 1$ .

**2.3 Proposition.** Let $f : X \to Y$ be a finite flat morphism of arithmetic varieties over $\mathbb{Z}_{(p)}$. Then for all $D \in D_V(X)$

$$f_*(cyc(D)) = cyc(f_*(D)).$$

**Proof** Because of 21.10.17 from [EGA iv].

**2.4 Remark.** With the same hypothesis as in proposition 2.3 we have $cyc(f_*) : D_V(X) \to Z_0(Y)$ and $f_* cyc : D_V(X) \to Z_0(Y)$ and $cyc(f_*) = f_* cyc$.

We end this section with a proposition:

**2.5 Proposition.** For all $D \in D_V(X)$ and for all $E \in Z_1^h$, the following formula holds:

$$f_* < D; f^* E >=< N_f(D); E > \quad and \quad < D.f^* E >_p =< N_f(D).E >_p .$$

**Proof.** We know that elements of $Z_h^1$ are linear combinations of integral points. Hence it suffices to prove the proposition for one integral point $E$.
Consider $E$ an integral point. Consider the restriction of $f$ to $f^{-1}(E)$. Denote it by $f'$.

Let $(X_i)$ denote the irreducible components of $f^{-1}(E)$. For all $i$ let $\eta$ denote the generic point of $X_i$ and $m_i$ the geometric multiplicity of $X_i$. Then $f^*(E) = \sum_i m_i X_i$. Hence

$$< D; f^*E >= \sum_i m_i div(D_{|X_i}) = div(D_{|f^{-1}(E)}).$$

From eq 3.3, proposition 2.3 and remark 2.4 we have $f_* div(D_{|f^{-1}(E)}) = div N_{f'}(D_{|f^{-1}E})$
.
And $div N_{f'}(D_{|f^{-1}E}) = N_f(D)_{|E}$. Hence we have

$$N_f(D)_{|E} =< N_f(D); E >= f_*(D_{|f^{-1}(E)})$$

$$f_*(D_{|f^{-1}(E)}) = f_* < N_f(D); f^*E >$$

and hence $< D.f^*E >_p =< N_f(D).E >_p$ $\square$

## 3.3 Faltings Formula and the Moduli Space $A^p_{(g,n)}$

In this section we revisit the Faltings formula proved for abelian varieties.
Let $g$, $r \geq 0$ and $p$ be a prime number, $S$ an arithmetic variety over $\mathbb{Z}_{(p)}$. Let $A$ and $A^{'}$ be abelian schemes of dimension $g$. Let $f : A \to A^{'}$ be an $S$-isogeny of rank $p^r$.

**3.1 Remark.** Let $O_A$, $O_{A'}$ be the zero sections of $A$, $A^{'}$, respectively. $\omega_{A/S}$, $\omega_{A'/S}$ be the invertible sheaves $O_A^* \wedge^g \Omega_{A/S}$, $O_{A'}^* \wedge^g \Omega_{A'/S}$.
So we have an $O_S$-morphism $j : \omega_{A'/S} \to \omega_{A/S}$. $S$ being an arithmetic variety and $f$ being an isogeny we have that $f$ is etale on $A_{\mathbb{Q}}$, hence $j$ an isomorphism on the generic fiber $S_{\mathbb{Q}}$ .

From section 1.3 of [Ra] we have a unique vertical Cartier divisor $D_f$ on $S$ such that $\omega_{A'} \otimes O_S(D_f) \cong \omega_A$.
As $f$ is etale on $A_{\mathbb{Q}}$, the Cartier divisor $D_f$ commutes with base change. i.e. if $S^{'}$ is an arithmetic variety on $\mathbb{Z}_{(p)}$ and $\pi : S^{'} \to S$ is a morphism, then $D_{f_{S'}} = \pi^* D_f$.

**3.2 Remark.** Let $S = Spec(R)$ where $R \subset \bar{\mathbb{Q}}$ is a finite, flat $\mathbb{Z}_{(p)}$-algebra. Let $K$ be the field of fractions of $R$. Let $A_K$, $B_K$ be 2 abelian varieties defined over $K$. Let $f : A_K \to B_K$ be an isogeny between them of degree $d$. Let $A$ and $B$ be the Néron models of $A_K$ and $B_K$. Let $\omega_A$, $\omega_B$ be the invertible sheaf of relative differential forms of $A$, $B$. Let us put $\omega_A \otimes \omega_B^{-1} = \omega_U$. In fact $deg(\omega_U) = deg(V_U^{-1})$, where $V_U$ is a non zero ideal of $O_S$. From remark 3.1 we have $V_U^{-1} = O_S(D_f)$.

**3.3 Proposition.** If $R$ is the ring of integers of a number field $K$, then

$$deg(\omega_A) - deg(\omega_B) = deg(\omega_U) = deg(V_U^{-1}) = \log \#(O_S/V_U) - \frac{[K : \mathbb{Q}]}{2} \log d$$

holds true.

**Proof.** This formula is nothing but the reformulation of Faltings formula that we proved in chapter 1. We have seen that the infinite places contribute $\frac{[K:\mathbb{Q}]}{2}\log d$. All that is left is to compute the contribution at the finite places.
From the above discussion we have

$$\begin{aligned} deg(\omega_A) - deg(\omega_B) &= deg(\omega_U) = deg(V_U^{-1}) \\ deg(V_U^{-1}) &= \log \#(V_U^{-1}/R.1) - \frac{[K:\mathbb{Q}]}{2}\log d. \\ &= \log \#(V_U^{-1}.V_U/R.V_U) - \frac{[K:\mathbb{Q}]}{2}\log d \\ &= \log \#(R/R.V_U) - \frac{[K:\mathbb{Q}]}{2}\log d. \qquad \square \end{aligned}$$

With the same hypothesis as in remark 3.1 we proceed to give another expression for Faltings formula for the two abelian schemes $A$, $A^{'}$ in terms of the Cartier divisor $D_f$.

**3.4 Proposition.** With hypothesis as in the first paragraph of this section, let $A^{'}_{\bar{\mathbb{Q}}}$ and $A_{\bar{\mathbb{Q}}}$ denote the restriction of $A$, $A^{'}$ to $\bar{\mathbb{Q}}$, with $f$ the isogeny of rank $p^r$ between them, then we have

$$h_F(A^{'}_{\bar{\mathbb{Q}}}) - h_F(A_{\bar{\mathbb{Q}}}) = \left(\frac{r}{2} - \frac{< D_f.S >_p}{[K:\mathbb{Q}]}\right)\log p.$$

**Proof.** Let $L$ be a field $\subset \bar{\mathbb{Q}}$ containing $K$, such that the abelian varieties $A_L$ , $A^{'}_L$ (which are the fibers of $A$, $A^{'}$) are semi stable over $L$. Let $\mathbb{A}$ , $\mathbb{A}^{'}$ be the Néron models of the abelian varieties $A_L$ , $A^{'}_L$ over $O_L$ respectively. Let $R^{'}$ be the integral closure of $R$ in $L$. Let us put $T = Spec(R^{'})$ and $B = Spec(O_L)$. We have the inclusion $R \to R^{'}$, and the morphism $\pi : T \to S$. Hence the $T$-morphism $f_T$ extends to a $B$-morphism $F : \mathbb{A} \to \mathbb{A}^{'}$ .
As $F$ is etale on $\mathbb{A}_U$ where $U = Spec[1/p]$ the $O_B$-morphism of $J : \omega_{\mathbb{A}'} \to \omega_{\mathbb{A}}$ is an isomorphism on the open set $U$. From the remark 3.1 there exists a unique, effective, vertical, Cartier divisor $D_\circ$ on $B$ above $p$ such that

$$\omega_{\mathbb{A}'} \otimes O_B(D_\circ) \cong \omega_{\mathbb{A}}.$$

And hence we have

$$D_{\circ T} = D_{f_T} = \pi^* D_f.$$

As the height function is invariant under the extension of ground field we have from proposition 3.3

$$h_F(A^{'}_{\bar{\mathbb{Q}}}) - h_F(A_{\bar{\mathbb{Q}}}) = \frac{1}{[L:\mathbb{Q}]}\Big(deg(\omega_{A^{'}_L}) - deg(\omega_{A_L})\Big);$$

$$\frac{1}{[L:\mathbb{Q}]}\Big(deg(\omega_{A'_L}) - deg(\omega_{A_L})\Big) = \frac{r}{2}\log p - \frac{1}{[L:\mathbb{Q}]}\log \#(O_L/O_L(D_{\circ T})).$$

$\#(O_L/O_L(D_{\circ T})$ is the norm of the ideal corresponding to the ideal $D_{\circ T}$, which is nothing but

$$p^{[k(D_{\circ T|T}):\mathbb{F}_p]} = p^{[k(\pi^* D_{f|T}):\mathbb{F}_p]} =< \pi^* D_f.T >_p .$$

Hence we have:

$$\#(O_L/O_L(D_{\circ T})) =< \pi^* D_f.T >_p .$$

By the projection formula we have

$$< \pi^* D_f.T >_p= [L:K] < D_f.S >_p .$$

Hence we have:

$$\begin{aligned} h_F(A'_{\bar{\mathbb{Q}}}) - h_F(A_{\bar{\mathbb{Q}}}) &= \frac{r}{2}\log p - \frac{1}{[L:\mathbb{Q}]}\log p^{[L:K]<D_f.S>_p}. \\ &= \left[\frac{r}{2} - \frac{[L:K] < D_f.S >_p}{[L:\mathbb{Q}]}\right]\log p \\ &= \left[\frac{r}{2} - \frac{< D_f.S >_p}{[K:\mathbb{Q}]}\right]\log p. \qquad \square \end{aligned}$$

We now look at the moduli space $A^p_{(g,n)}$ of $p$-ordinary principally polarised abelian schemes of type $(g,n)$ over the scheme $\mathbb{Z}_{(p)}$ where $g \geq 1$ and $n \geq 3$ and $p \not| n$. We need the basic properties of our moduli space $A^p_{(g,n)}$.
$A^p_{(g,n)}$ is a contravariant functor $A^p_{(g,n)} : Schemes \to Sets$ which assigns to every scheme $S$ the set $A^p_{(g,n)}(S)$ of isomorphism classes of $p$-ordinary abelian schemes of type $(g,n)$ defined over the scheme $S$.
The following theorem asserts that our $A^p_{(g,n)}$ is an arithmetic variety. We omit the proof which can be found in [Mu 2].

**3.5 Theorem** The fine moduli scheme $A^p_{(g,n)}$ exists when $n \geq 3$. $A^p_{(g,n)}$ is an arithmetic variety over $\mathbb{Z}_{(p)}$ and the geometric fibers of $A^p_{(g,n)}$ are irreducible.
**Proof** Proof in chapter 7 theorem 7.9 in [Mu 2].

Now let us consider $isog_g$ the fine moduli scheme of isogenies of rank $p^g$ of type $PN$ between principally polarised abelian schemes of type $(g,n)$ defined over $\mathbb{Z}_{(p)}$. $isog_g$ is an integral arithmetic variety on $\mathbb{Z}_{(p)}$.(chapter 7, section 3 of [Fa 3]) We have the morphisms $pr_1 : isog_g \to A^p_{(g,n)}$ which assigns to each isogeny $f : B_1 \to B_2 \in isog_g$ its source $B_1$ , and $pr_2 : isog_g \to A^p_{(g,n)}$ which assigns to each isogeny $f : B_1 \to B_2 \in isog_g$ its target $B_2$. These two morphisms define a map

$$pr = (pr_1, pr_2) : isog_g \to A^p_{(g,n)} \times A^p_{(g,n)}.$$

$A^p_{(g,n)}$ is a fine moduli scheme hence there exists a universal abelian scheme $A$. Now $pr_1^*A$ denoted by $A_1$ and $pr_2^*$ denoted by $A_2$ are the pull backs of the universal abelian scheme $A$ under $pr_1^*$ and $pr_2^*$ respectively. Hence we have a universal isogeny $f_{un} : A_1 \to A_2$. The following proposition guarantees us that our projection maps $pr_i$ are finite and flat.

**3.6 Proposition.** The projections $pr_i$ are finite, flat and of rank $e_{(g,p)} = \#Sp_{2g}(\mathbb{Z})/(\Gamma_\circ(p))$, where $\Gamma_\circ(p)$ is the group of matrices $\begin{bmatrix} M_1 M_2 \\ M_3 M_4 \end{bmatrix} \in Sp_{2g}(\mathbb{Z})$ such that $M_3$ is congruent to 0 mod $p$.
**Proof** The proof can be found in [Fa 3], chapter vii, proposition 4.1 .

In fact $e_{(g,p)}$ is the number of isotropic subgroups of order $p^g$ of the $p$-torsion group of a principally polarised abelian variety of dimension $g$ over an algebraically closed field of characteristic zero.

**3.7 Lemma.**

$$e_{(g,p)} = \prod_{i=1}^{g} (p^i + 1).$$

**Proof.** From the above paragraph we know that $e_{(g,p)} = \#Sp_{2g}(\mathbb{Z})/(\Gamma_\circ(p))$. Now consider the $\mathbb{F}_p$-vector space $F = \mathbb{F}_p^{2g}$ with the standard symplectic pairing. $e_{(g,p)}$ is the nothing but the number of isotropic subspaces of $F$. So hence $e_{(g,p)} = \dfrac{T}{\#Gl_g(\mathbb{F}_p)}$, where $T$ is the number of free orthogonal sets of $F$ with $g$ vectors.
Now we calculate $T$. We have to count the number of sets of cardinallity $g$ of the $2g$-dimensional $\mathbb{F}_p$ vector space $F$. We can choose our first vector $v_1$ of $F$ in $p^{2g} - 1$ ways. Let us denote the span of $v_1$ by $V$. Then with respect to the standard bilinear form on $F$ we have $F = V \oplus V'$ where $V'$ denotes the orthogonal space of $V$ of dimension $(2g - 1)$. Now the number of ways in which we can calculate the second vector $v_2$ orthogonal to $v_1$ is $p^{2g-1} - p$ (as we have to discard the multiples of $v_1$ from $V'$ in choosing $v_2$). Similarly we compute $v_3$ by discarding the multiples of both $v_1, v_2$ from $W'$, where $W'$ is the orthogonal set of the span of $v_1, v_2$ in $F$ which is equal to $(p^{2g-2} - p^2)$. Continuing this way we have

$$T = \prod_{i=0}^{g-1} (p^{2g-i} - p^i).$$

Now we want to compute the number of elements of $Gl_g(\mathbb{F}_p)$. We can choose our first column for a matrix $\in Gl_g(\mathbb{F}_p)$ (which can be seen as a vector in the vector space $F' = \mathbb{F}_p^g$) in $p^g - 1$ ways. We can choose our second vector in $(p^g - p)$ ways as we have to make sure that the second vector is not a scalar multiple of the first vector. Similarly we can choose our second vector in $(p^g - p^2)$ ways. In continuing this way we have

$$\#Gl_g(\mathbb{F}_p) = \prod_{i=0}^{g-1}(p^g - p^i)$$

Hence

$$e_{(g,p)} = \frac{\prod_{i=0}^{g-1}(p^{2g-i} - p^i)}{\prod_{i=0}^{g-1}(p^g - p^i)}.$$

$$\frac{\prod_{i=0}^{g-1}(p^{2g-i} - p^i)}{\prod_{i=0}^{g-1}(p^g - p^i)} = \frac{\prod_{i=0}^{g-1}(p^{2g-2i} - 1)}{\prod_{i=0}^{g-1}(p^{g-i} - 1)}$$

$$\frac{\prod_{i=0}^{g-1}(p^{2g-2i} - 1)}{\prod_{i=0}^{g-1}(p^{g-i} - 1)} = \prod_{i=0}^{g-1}(p^i + 1). \qquad \square$$

## 3.4 Average Height Formula

In this section we prove the average height formula for principally polarised abelian varieties over $\bar{\mathbb{Q}}$ of dimension $g$ with good reduction at $p$. We first look at the moduli space $A_{g,n}^p$ of all principally polarised abelian schemes defined over the scheme $\mathbb{Z}_{(p)}$. The points of these space are the isomorphism classes of $p$-ordinary principally polarised abelian schemes of type $(g, n)$ over $\mathbb{Z}_{(p)}$. We see how the objects of $A_{g,n}^p$ are related to our principally polarised abelian varieties over $\bar{\mathbb{Q}}$ of dimension $g$ with good reduction at $p$. We now see how we can define the height of a principally polarised abelian variety when it is seen as a point on the moduli space.

The following 3 paragraphs are very technical and we state them only vaguely without any proof, just to give an idea of how the Faltings height function is defined on equivalence classes of abelian varieties in the moduli space.

**4.1 Remark.** Faltings has described a way to compactify moduli space of principally polarised abelian varieties $A_g$ in chapter iv of [CL]. We here summarize the main consequences of the toroidal compactification of the moduli space.
1. There exists a proper scheme $\tilde{A}_g$ over $Spec(\mathbb{Z})$ containing $A_g$ as a dense open subscheme.
2. There exists a semi-abelian scheme $\chi_g \underset{\epsilon}{\overset{\pi}{\rightleftarrows}} \tilde{A}_g$ whose each fiber is an extension of an abelian variety by a torus and which extends the universal scheme over $A_g$.

**4.2 Remark.** Let $\omega = \epsilon^*(\wedge^g \Omega_{\chi_g/\tilde{A}_g})$. Global sections of $\omega^{\otimes k}$ define a morphism from $f : \tilde{A}_g \to \mathbb{P}^n$, with image $\bar{A}_g$ is $Proj\left(\bigoplus_{k\in\mathbb{N}} \Gamma(\tilde{A}_g, \omega^{\otimes k})\right)$. And from some non trivial results of Satake compactification we have that some power $\omega^{\otimes n}$ of $\omega$ descends to an invertible sheaf $\ell_n$ on $\bar{A}_g$; $\ell_n$ is ample on $\bar{A}_g$.

Now consider $A$ a principally polarised abelian variety over a number field $K$ with semi-stable reduction over $O_K$. $A_K$ defines a $K$-point $[A_K]$ of $A_g$. Since $\tilde{A}_g$ is proper, $[A_K]$ extends to an $O_K$-valued point $< A_K >$: $Spec(O_K) \to \tilde{A}_g$. The pull back of $\chi_g$ by the map $< A_K >$ is a semi-abelian scheme extending $A_K$. From chapter [CL] we have that the pullback of $\chi_g$ by $< A_K >$ is equal to the Néron model $N(A_K)^\circ$. And we have $\omega_{A_K} =< A_K >^* \omega$. Then we put the moduli theoretic height $h_{geom} = 1/n(h_{\ell_n}([X_K]))$. Since $X_K$ has semi-stable reduction over $O_K$, $h_{geom}$ is nothing but the Faltings height $h_F(X_K)$.

**4.3 Remark.** If $A$ is a principally polarised abelian variety over $\bar{\mathbb{Q}}$ of dimension $g$ with good reduction at $p$, then there exists a $p$-ordinary abelian scheme $A^{'}$ defined over a integral, finite and flat $\mathbb{Z}_{(p)}$ algebra $R \subset \bar{\mathbb{Q}}$ such that $A^{'}_{\bar{\mathbb{Q}}} = A$ from definition 1.6 of this chapter. In fact for each $n \geq 3$ prime to $p$ ,we an find an $A^{'}$ of type $(g,n)$ such that $A^{'}_{\bar{\mathbb{Q}}} = A$.

**4.4 Remark.** 1. Let $(A;\lambda)$ be a $\bar{\mathbb{Q}}$ principally polarised abelian variety of dimension $g$ with good reduction at $p$. Let $G_1$,...,$G_e$ ($e = e_{(g,p)}$) be the isotropic subgroups of $A[p]$ of rank $p^g$ and $A_i = A/G_i$. Then the isogenies $f_i : A \to A_i$ which are of rank $p^g$ are of type $P$ (by choosing a suitable polarisation) by theorem 1.13 of this chapter for all $1 \leq i \leq e$.
2. From remark 1.11 for each of these abelian varieties $A_i$ there exists an abelian scheme $A^{'}_i$ such that $A^{'}_{i\bar{\mathbb{Q}}}$=$A_i$, $A^{'}_{i\bar{\mathbb{Q}}}$ are of type $(g,n)$ and the isogenies $f_i$ are of type $PN$.

**4.5 Remark.** By virtue of the above remark 4.4 all our isogenies $f_i$'s can now be viewed as isogenies between $A$ and $A_i$ and hence as an element of $isog_g$.
Now we are all set to prove the average height formula.

**4.6 Theorem.** Let $(A;\lambda)$ be a $\bar{\mathbb{Q}}$ principally polarised abelian variety of dimension $g$ with good reduction at $p$. Let $G_1$,...,$G_e$ ($e = e_{(g,p)}$) be the isotropic subgroups of $A[p]$ of rank $p^g$ and $A_i = A/G_i$. Then there exists an integer $m = m(g,p)$ such that the following formula

$$\sum_{i=1}^{e} (h_F(A_i) - h_F(A)) = \left(\frac{eg}{2} - m\right) \log p$$

holds. Here $h_F(A)$ denotes the Faltings height defined in chapter 1.
**Proof.** Let us choose an $n \geq 3$ prime to $n$, then by theorem 3.5 $A^p_{g,n}$ exists and also $isog_p$ exists. Let us denote $A^p_{g,n}$ by $X$. We have already seen in section 3 that there exists a universal isogeny $f_{un} : A_1 \to A_2$ over $isog_g$ of rank $p^g$ and of type $PN$ between $A_1 = pr_1^* A$ and $A_2 = pr_2^* A$ where $A$ is the universal abelian scheme over $A^p_{g,n}$.
From remark 3.1 we have an effective, vertical Cartier divisor $D_{f_{un}}$ on $isog_g$ such that

$$\omega_{A_2} \otimes O(D_{f_{un}}) \cong \omega_{A_1}.$$

We have $pr_1 : isog_g \to X$. Since $isog_g$ and $X$ are both integral arithmetic varieties over $\mathbb{Z}_{(p)}$, we have the norm map $N_{pr_1} : D_V(isog_g) \to D_V(X)$.
Now consider $N_{pr_1}(D_{fun}) \in D_V(X)$, which is vertical and effective on $X$. From the properties of the norm function $N_{pr_1}$ stated in section 2 we have $N_{pr_1}(D_{fun})$ contained in the special fiber $X_p$ of $X$. From theorem 3.5 we can conclude that $X_p$ is irreducible. Hence we have $N_{pr_1}(D_{fun}) = mX_p$ for some positive integer $m$.

Let us now consider our principally polarised abelian variety $(A, \lambda)$ over $\bar{\mathbb{Q}}$ of dimension $g$ with good reduction at $p$. From remark 4.3 we have an $(A^{'},\lambda^{'})$ defined over a integral, finite and flat $\mathbb{Z}_{(p)}$-algebra $R \subset \bar{\mathbb{Q}}$ such that $A^{'}_{\bar{\mathbb{Q}}} = A$ and is of type $(g, n)$. Hence there exists a point $x^{'}$ corresponding to $(A^{'},\lambda^{'})$ in $X(R)$. Let us denote by $x$ the image of $x^{'}$ in $X$, which is an integer point being integral and closed.

Now consider the isogeny $f_i : A \to A_i$. From remark 4.5 each $f_i$ corresponds to a point $E_i$ in $isog_g(\bar{\mathbb{Q}})$. Let $F_i$ be the Zariski closure of this point $E_i$ in $isog_g$.

Let $K$ be a number field containing all the fields $k(F_i)$. Let us denote $[K : k(F_i)]$ by $a_i$, $[K : k(x)]$ by $a$ and $[k(x) : \mathbb{Q}]$ by $d$. The $F_i$'s are closed and integral points and hence integer points of $isog_g$. Now consider the map
$pr_1^* : Z_1^h(X) \to Z_1^h(isog_g)$. We have

$$pr_1^*(ax) = \sum_{i=1}^{e} a_i F_i. \tag{3.4}$$

We use proposition 3.4 and the universal Cartier divisor $D_{fun}$ to get the average height formula.
From proposition 3.4 we have

$$h_F(A_i) = h_F(A) + \left[\frac{g}{2} - \frac{< D_{fun}.F_i >}{[k(F_i) : \mathbb{Q}]}\right] \log p.$$

So we have:

$$h_F(A_i) = h_F(A) + \left[\frac{g}{2} - a_i\frac{< D_{fun}.F_i >}{[K : \mathbb{Q}]}\right] \log p.$$

From equation 1.4, we have:

$$\sum_{i=1}^{e} h_F(A_i) = eh_F(A) + \left(\frac{eg}{2} - \frac{1}{d} < D_{fun}.pr_1^* x >_p\right) \log p.$$

From proposition 2.5 of this chapter we have:

$$< D_{fun}.pr_1^* x >_p = < N_{pr_1}(D_{fun}).x >_p .$$

We have just seen that:

$$N_{pr_1}(D_{fun}) = mX_p.$$

Hence

$$\begin{aligned} < D_{fun}.pr_1^* x >_p &= m < X_p.x >_p = m[k(x) : \mathbb{Q}] \quad \text{(from arguments in section 2)} \\ &= md. \end{aligned}$$

So,

$$\sum_{i=1}^{e} h_F(A_i) = eh_F(A) + \left(\frac{eg}{2} - \frac{md}{d}\right) \log p.$$

Hence we have

$$\sum_{i=1}^{e} (h_F(A_i) - h_F(A)) = \left(\frac{eg}{2} - m\right) \log p. \qquad \square$$

# 4 Conclusion

In the last chapter we have seen a proof of the average height formula for principally polarised abelian varieties with good reduction at a prime $p$. In chapter two we have seen that there exist at least three approaches to prove the average height formula for semi-stable elliptic curves defined over a number field $K$. As we saw in the last chapter, Autissier's approach of looking at the moduli spaces of abelian varieties is the only existing method of proving the average height formula for abelian varieties. We conclude our thesis with a few remarks on why the other two methods are tough to be generalised to prove the average height formula for abelian varieties, and why we need so many conditions to prove the average height formula for abelian varieties.

We have seen in chapter two that Robin de Jong's approach of proving the height formula involves Arakelov intersection theory on regular minimal models of elliptic curves. We have see that the three lemmas 3.4, 3.5 and 3.6 give us our average height formula. Lemma 3.4 interprets the Faltings height formula in terms of Arakelov intersection product $(O, \omega)$ of the zero section $O$ and the dualizing sheaf $\omega$ and the Arakelov invariants $A(X_v)$ of the Riemann surfaces $X_v$ . In other words from proposition 2.18 we can say that 3.4 relates the Faltings height formula of the elliptic curve $E$ and the self-intersection product $(O, O)$.

We then saw in lemma 3.5 that the average sum $\sum_C \left((O, \omega) - (O', \omega')\right) = 0$. Then we saw lemma 3.6 and proposition 1.9 give us the average height formula. So the Green function which gives our Arakelov intersections at infinity plays a very important role in this method. In the case of abelian varieties though we have an intersection theory as we have seen in chapter three, we still do not have something analogous to the Green function which gives the intersection pairings at infinity.

Another method that we have seen in chapter two involves the basic properties of the cusp form $\Delta$ and makes use of proposition 2.18 and lemma 3.4. This approach is definitely not possible to be generalised as the modular discriminant $\Delta$ does not have any analogue in the case of abelian varieties.

In fact the method involving the modular discriminant $\Delta$, is essentially the same as the approach of Robin de Jong. In [Ro] Robin de Jong proved the following proposition which links both these methods:

**1.1 Proposition.** Let $X$ and $X^{'}$ be 1-dimensional complex tori related by an isogeny $f : X \to X^{'}$. Then we have

$$\prod_{P \in ker(f), P \neq 0} G(0, P) = \frac{\sqrt{N}. \left\|\Delta\right\|^{1/12}(X^{'})}{\left\|\Delta\right\|^{1/12}(X)}$$

**Proof.** Proposition 4.7 of [Ro].

We have seen a brief sketch of the approach of Autissier. He uses the theory of $L^1_2$-singular hermitian line bundles which generalizes the Arakelov intersection product for smooth hermitian line bundles. He computes the height of $h_{\hat{M}}(T_{N*}(Y))$ using this theory, and the average height formula follows immediately once we see that

$$\frac{1}{12}\frac{1}{[K:\mathbb{Q}]}h_{\hat{M}}(Y) = h_F(E).$$

This approach was generalised by Autissier himself to the case of abelian varieties. But the only moduli space that he could use is $A^p_{g,n}$. Hence he had to consider only principally polarised abelian varieties with good reduction at $p$, and only isotropic subgroups of order $p^g$ while taking the quotients of the abelian variety under consideration.

But it is natural to expect the average height formula even when we take quotients with respect to all subgroups of order $p^g$. For this we need either a more developed intersection theory or a more general moduli space retaining the useful properties.

Another important question that is not yet answered is the behaviour of the constant $m(g, p)$, which comes in the average height formula for abelian varieties. The average height formula for a semi-stable elliptic curve defined over a number field $K$, when we take quotients with respect to cyclic subgroups of order $p$ is

$$\frac{1}{e_N}\sum_C \left(h_F(E^{'}) - h_F(E)\right) = \frac{1}{2}\log p - \frac{1}{(p+1)}\log p.$$

Here we see that $m(g, p) = e_N \frac{1}{p+1} = 1$ for all primes $p$. But for higher dimensions we do not yet know how $m(g, p)$ behaves as $(g, p)$ varies. So it would be interesting to know more about this constant $m(g, p)$.

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