The FitzHugh-Nagumo equations on an unbounded domain

Houtman, G.

Gerard Houtman

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Thesis advisor: Dr. S.C. Hille

Mathematisch Instituut, Universiteit Leiden

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Chapter 1

Introduction

A.L. Hodgkin and A.F. Huxley wrote a series of five papers concerned with the flow of electric current through the surface membrane of a nerve fibre, based on their experiments with giant squid axons. In their final paper, written in 1952, *A Quantitative Description of Membrane Current and its Application to Conduction and Excitation in Nerve*, [11], they summarized all of the results and put them into mathematical models. In 1963 they won the Nobel Prize in Physiology or Medicine for their research.
In 1961 R. FitzHugh reduced the system from four dynamic variables to two by projection, while retaining the properties of physiological interest, in his paper *Impulses and Physiological States in Theoretical Models of Nerve Membrane* [7]. A similar result was achieved by J. Nagumo et al. in *An Active pulse transmission line simulating nerve axon*, [16], in 1962.

1.1 Hodgkin and Huxley

We will give a brief overview of the research by Hodgkin and Huxley, so that we have some understanding of the biological background. Between the inside and outside of a cell there is always a potential difference, due to the distribution of ions and the permeability of the cell-membrane. An inactive cell has, relative to outside the cell, a negative resting potential. The cell will undergo an action potential when it is depolarized. This is a temporary change in potential difference over the membrane. A.L. Hodgkin and A.F. Huxley did experiments to get a grip on this action potential. Basically, the electrical behavior of the membrane can be represented by the electrical circuit shown in figure 1.1, from [11].
It follows from their research that the total membrane current density (I) can be modeled as the sum of the capacity current density $C_M \frac{dV}{dt}$ and the ionic current I_i,

$$I = C_M \frac{dV}{dt} + I_i,$$

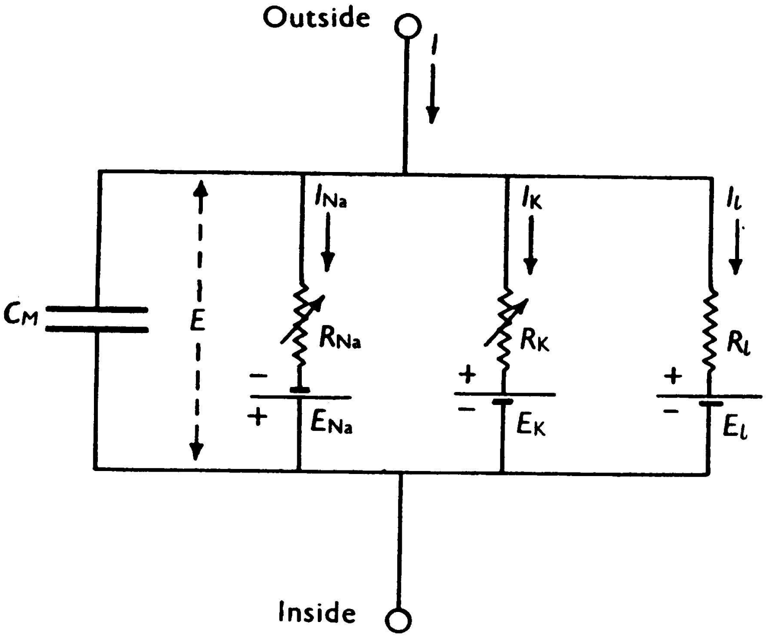

Figure 1.1: An electrical circuit that represents the electrical behavior of the membrane.

where V is the displacement of the membrane potential from its resting value and C_M the membrane capacity. We can split the ionic current into components carried by sodium ions (I_{Na}), potassium ions (I_K) and other ions (I_l), where l stands for leakage.

$$I_i = I_{Na} + I_K + I_l$$

They showed that the individual ionic currents can be expressed in terms of ionic conductances (g_{Na}, g_K and $\bar{g}_l$)

$$\begin{aligned} I_{Na} &= g_{Na}(V - V_{Na}), \\ I_K &= g_K(V - V_K), \\ I_l &= \bar{g}_l(V - V_l), \end{aligned}$$

where V_{Na}, V_K and V_l are the resting potentials for each of the ions. In Figure 1.1 the resistances come from these conductances: $R_{Na} = 1/g_{Na}; R_K = 1/g_K; R_l = 1/g_l$. Furthermore they make some assumptions on the sodium and potassium conductances, based on the results of their experiments.

$$\begin{aligned}
\frac{dn}{dt} &= \alpha_n(V)(1-n) - \beta_n(V)n, & g_K &= \bar{g}_K n^4, \\
\frac{dm}{dt} &= \alpha_m(V)(1-m) - \beta_m(V)m, & g_{Na} &= \bar{g}_{Na} m^3 h, \\
\frac{dh}{dt} &= \alpha_h(V)(1-h) - \beta_h(V)h.
\end{aligned}$$

Here $\bar{g}_K$ and $\bar{g}_{Na}$ are constants, n represents the potassium activation, m the sodium activation and h the sodium inactivation, $(0 \leq n, m, h \leq 1)$. The α's and β's are constants that vary with V.

$$\begin{aligned}
\alpha_n(V) &= 0.01(V+10)/(\exp\frac{V+10}{10} - 1), \\
\beta_n(V) &= 0.125\exp(V/80), \\
\alpha_m(V) &= 0.1(V+25)/(\exp\frac{V+25}{10} - 1), \\
\beta_m(V) &= 4\exp(V/18), \\
\alpha_h(V) &= 0.07\exp(V/20), \\
\beta_h(V) &= 1/(\exp\frac{V+30}{10} + 1).
\end{aligned}$$

Putting it all together we find the four-dimensional system of differential equations by Hodgkin and Huxley.

$$I = C_M \frac{dV}{dt} + \bar{g}_K n^4 (V - V_K) + \bar{g}_{Na} m^3 h (V - V_{Na}) + \bar{g}_l (V - V_l),$$

with

$$\begin{aligned}
\frac{dn}{dt} &= \alpha_n(V)(1-n) - \beta_n(V)n, \\
\frac{dm}{dt} &= \alpha_m(V)(1-m) - \beta_m(V)m, \\
\frac{dh}{dt} &= \alpha_h(V)(1-h) - \beta_h(V)h.
\end{aligned}$$

These equations are supplemented by suitable initial conditions.

1.2 FitzHugh and Nagumo

In FitzHugh's first article in 1960 on the Hodgkin-Huxley equations, [6], he used a quasi steady state approximation, setting

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} h = 0 \\ \frac{\partial}{\partial t} n = 0 \end{array} \right. \tag{1.1}$$

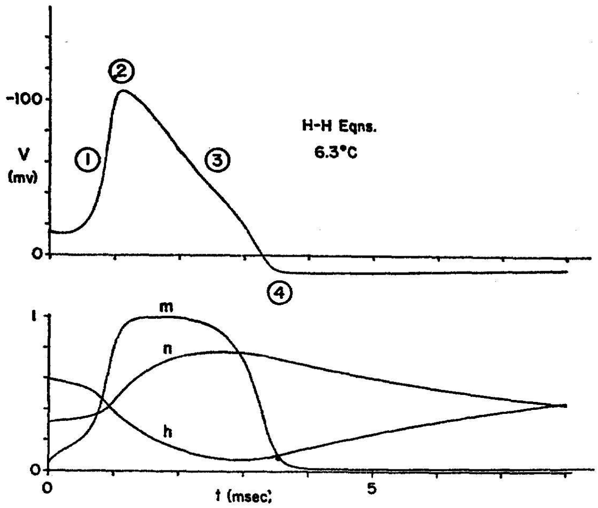

Figure 1.2: Numerical solutions of the Hodgkin-Huxley equations

From figure 1.2 from [6], we see that this is not a very good approximation. It was only used to get a better understanding of the complete sytem, since it is very difficult to consider all the variables at once.
In his second article on the Hodgkin-Huxley equations, [7], one year later he noted that the phase space (V, m, n, h) can be divided into two subsystems (V, m), the fast variables, and (n, h), the slow variables. Then he eliminated one dimension from each of the two planes by linear projection.That is, he noted that the curves of n and $-h$ had the same shapes, see Figure 1.2, so he replaced them by their average $w = 0.5(n - h)$, by projecting along lines of constant w onto the line $n + h = 0.85$. The (V, m)-plane was projected similarly along lines of constant u, with $u = V - 36m$. For a more detailed description, one should consult the paper by R. FitzHugh [7]. The resulting FitzHugh-Nagumo equations are the following:

$$\frac{\partial}{\partial t}u = I + u(u - \alpha)(1 - u) - w,$$
$$\frac{\partial}{\partial t}w = \varepsilon(u - \gamma w).$$

Here the model has been made dimensionless and α, ε and γ are constants with $0 < \alpha < 1$ and $\varepsilon \ll 1$. This system is a model for the homogeneous (with respect to the current) situation, i.e. the membrane potential is enforced to be space-homogeneous. For the general case, one may think of a chain of coupled homogeneous elements, but now the potential can "diffuse" through the membrane, to neighboring elements. Along a membrane ions and hence potential diffuse along the axon. Accordingly we add a diffusion term, $a\Delta u$, $a > 0$. Only to the first equation because only the u-variable contains the potential V. The current I is now a function of position.

$$\frac{\partial}{\partial t}u = a\Delta u + I + u(u-\alpha)(1-u) - w,$$
$$\frac{\partial}{\partial t}w = \varepsilon(u - \gamma w).$$

We will treat the case where $I = 0$. That is, we there is no external electric forcing on the axon.
We can consider these equations in a more general and abstract form.

$$\partial_t U_1(t) = a\Delta U_1(t) + p_3(U_1(t)) + c_1 U_2(t),$$
$$\partial_t U_2(t) = c_2 U_1(t) + c_3 U_2(t).$$

Here p_3 is a cubic polynomial and c_1, c_2 and c_3 are constants. Note that the partial differential equations are now considered as ordinary differential equations and differentiation with respect to spatial coordinates as an operation, i.e. $a\Delta$ is an operator.

1.3 Outline of this thesis

This thesis is a mix of general, abstract results and results that apply to the FitzHugh-Nagumo equations. We will study this two-dimensional system, considered as an explicit example of a nonlinear evolutionary equation,

$$\partial_t U(t) = AU(t) + F(t, U(t))$$

in a Banach space of (essentially) functions over a physical domain $\Omega \subset \mathbb{R}^n$. These FitzHugh-Nagumo equations, or more generally, reaction-diffusion equations,

$$\partial_t U(t) = a\Delta U(t) + F(t, U(t)),$$

have been studied thoroughly on bounded domains with suitable boundary conditions, e.g. Dirichlet, Neumann. The situation on unbounded domains however seems less well-documented, in particular the full space $\mathbb{R}^n$. Moreover, various techniques that work for bounded domains do not directly apply to the unbounded case. An obstruction that is often encountered is that the inclusions $L^p(\Omega) \subset L^{p'}(\Omega)$, if $p > p'$, that hold for bounded Ω, do not hold in unbounded domains.
Our first objective has been to collect various results from different sources that cover this case. As a starting point we used [18]. J. Rauch and J. Smoller studied the FitzHugh-Nagumo equations in their article. They prove local and global existence for these equations, however, at close inspection, it turns out that they only prove it for these equations in a Banach space of continuous bounded functions. Essentially they circumvent in this way the problem that the non-linearity F generally does not map the Banach space

in which one wants to situate the solutions, into itself. We are interested in L^p-spaces as the Banach spaces to find the solutions. Since the FitzHugh-Nagumo equations come from biology and many examples of evolutionary equations from physics or chemistry it is natural to consider these L^p-spaces. Solutions should be bounded, $L^\infty(\mathbb{R}^n)$, and volumes or densities have to be integrable, $L^1(\mathbb{R}^n)$, and so on. We use the theory presented in [12], where K. Ito and F. Kappel prove a theorem that ensures the existence and uniqueness of local (in time) so called mild solutions, using semigroups. We have reformulated their result and reproved it in what we feel is a more transparent way. These mild solutions are coupled to the variation of constants formula,

$$U(t) = T(t)\phi + \int_0^t T(t-s)F(s, U(s))ds.$$

Therefore our second objective has been to make use of this variation of constants formula as much as possible to obtain all of our results.
With this approach we try not to put to many regularity conditions on the solutions such as differentiability.
This thesis consists of three parts. In the first part, Chapter 2, the homogeneous differential equation,

$$\partial_t U(t) = AU(t),$$

is solved. What makes this section interesting is that we investigate in detail and exploit properties of the heat semigroup in the Fréchet space of Schwartz functions. Moreover, we pause to identify the domain of the infinitesimal generator of this semigroup, $A = a\Delta$, in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Although it is already known that, for $1 < p < \infty$, this domain is the Sobolev space $W^{2,p}(\mathbb{R}^n)$, the proof is not easy to find in literature. So in this chapter we will construct the proof and we will show that the heat-semigroup is the one being generated by $a\Delta$. The precise characterisation of the domain in $L^1(\mathbb{R}^n)$ seems to be unknown.
In Chapter 3 the nonlinearly perturbed system,

$$\partial_t U(t) = AU(t) + F(t, U(t)),$$

is considered. Here we will prove the local-existence-and-uniqueness-theorem from [12] using the Picard-Banach fixed point theorem. From this theorem we establish the existence of evolutionary operators $\hat{S}(t, s)$, such that for an initial condition ϕ we have

$$U(t; \phi) = \hat{S}(t, 0)\phi \text{ and } \hat{S}(t, s)\hat{S}(s, \sigma) = \hat{S}(t, \sigma) \text{ when } \sigma \leq s \leq t.$$

We then apply this result in Chapter 4 to the FitzHugh-Nagumo equations and find conditions on the L^p-spaces, such that there exists a local mild solution.

In the final part, Chapter 5, three methods for global solutions are presented. The first two are standard methods, based on the Lipschitz continuity of the nonlinearity. The third is a new method that can be used to prove global existence of positive solutions. In that case the state spaces are assumed to be Banach lattices, since one needs a partial ordering to define positivity and a relationship between ordering and norm, to obtain a-priori estimates. Unfortunately, time limitations prevented us from establishing global existence in the spaces identified in Chapter 4, using any of these methods.

1.4 Introduction of notation

Before we start, first a few conventions including the three types of topological vector spaces that we will use most throughout this thesis.

- $\mathbb{R}_+$: The nonnegative real numbers: $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\} = [0, \infty)$.
- $\mathbb{N}$ and $\mathbb{N}_0$: The positive integers: $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- ∂_i denotes partial derivatives with respect to the i-th variable.
- D^α: For partial derivatives we write $D^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index.
- $L^p(\Omega, X)$: The L^p-spaces consist of equivalence classes of measurable functions $\varphi : \Omega \to X$, where Ω is a measure space and X a Banach space, such that

$$\|\varphi\|_{L^p(\Omega,X)} = \left(\int_\Omega \|\varphi\|_X^p d\mu \right)^{1/p} < \infty.$$

  For $X = \mathbb{R}$, we write $L^p(\Omega)$ and

$$\|\varphi\|_{L^p(\Omega)} = \left(\int_\Omega |\varphi|^p d\mu \right)^{1/p}.$$

- $W^{k,p}(\mathbb{R}^n)$: For $1 \leq p \leq \infty$, and $k \in \mathbb{N}_0$, the *Sobolev space* $W^{k,p}(\mathbb{R}^n)$ is defined to be the subspace of $L^p(\mathbb{R}^n)$ consisting of all $\varphi \in L^p(\mathbb{R}^n)$ such that for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$, $D^\alpha \varphi \in L^p(\mathbb{R}^n)$ (in the sense of distributions). $W^{k,p}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|\varphi\|_{k,p} = \sum_{\alpha \in \mathbb{N}_0^n \ |\alpha| \leq k} \|D^\alpha \varphi\|_p. \tag{1.2}$$

- $\mathbb{S}(\mathbb{R}^n)$: The Schwartz space, see Section 2.1.

Chapter 2

Solutions to the heat equation

As a preparation for treating the FitzHugh-Nagumo equations we consider the following partial differential equation on $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} \frac{\partial}{\partial t} U(x,t) = a\Delta U(x,t), & t > 0, x \in \mathbb{R}^n, \\ U(x,0) = \phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

Here and in any similar equation a is positive. This equation is called the *heat equation*. We can also consider it as an abstract ordinary differential equation:

$$\begin{cases} \partial_t U(t) = a\Delta U(t), \\ U(0) = \phi, \end{cases} \tag{2.2}$$

where $U(t)$ is in a suitable Banach space, or, more general, a Fréchet space, of functions on $\mathbb{R}^n$. In section 2.1 we will solve the heat equation in the Fréchet space $\mathbb{S}(\mathbb{R}^n)$, which is not a Banach space. This abstract differential equation is a particular example of the following.

$$\begin{cases} \partial_t U(t) = AU(t) \\ U(0) = \phi \end{cases} \tag{2.3}$$

Here A is an operator on a Banach space X and the function U is supposed to take values in X. The first theorem solves the differential equation (2.3).

Theorem 2.1. *Let X a Banach space and A a linear operator on X. Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$. If $\phi \in D(A)$, the domain of A, then the differential equation* (2.3) *has a unique solution $U : \mathbb{R}_+ \to D(A) \in C^1(\mathbb{R}_+, X)$. The solution is given by $U(t) = T(t)\phi$ for all $t \in \mathbb{R}_+$.*

Proof. Since $\phi \in D(A)$, $U(\cdot) = T(\cdot)\phi \in C^1(\mathbb{R}_+, X)$ has values in $D(A)$,

$$\partial_t U(t) = \partial_t T(t)\phi = AT(t)\phi = AU(t)$$

and clearly $U(0) = \phi$. For the uniqueness, let V be a solution and define $W_t(s) = T(t-s)V(s)$ for $0 \le s \le t < \infty$, then

$$\partial_s W_t(s) = \partial_s[T(t-s)V(s)] = -T(t-s)AV(s) + T(t-s)AV(s) = 0.$$

Thus $W_t(s)$ is constant and $V(t) = W_t(t) = W_t(0) = T(t)V(0) = T(t)\phi$. $\square$

We will use this result by J.A. Goldstein, see [9], to solve (2.3) on $L^p(\mathbb{R}^n)$ in Section 2.3. First we solve (2.1) in $\mathbb{S}(\mathbb{R}^n)$. We will find a similar result in this Fréchet space as we do in the case where X is a Banach space.

2.1 The heat semigroup in $\mathbb{S}(\mathbb{R}^n)$

A Schwartz function is a rapidly decreasing smooth function, i.e. the *Schwartz space* $\mathbb{S}(\mathbb{R}^n)$ consists of functions $\varphi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\lim_{|x|\to\infty} P(x) D^\alpha \varphi(x) = 0$$

for each polynomial P and each multi-index $\alpha \in \mathbb{N}_0^n$. Note that this condition is equivalent to the condition

$$\sup_{x\in\mathbb{R}^n} |P(x)D^\alpha\varphi(x)| < \infty.$$

We can define a family of norms on $\mathbb{S}(\mathbb{R}^n)$

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| \qquad \alpha, \beta \in \mathbb{N}_0^n$$

Using these norms we can define a complete metric on $\mathbb{S}(\mathbb{R}^n)$

$$d_{\mathbb{S}}(\varphi,\psi) = \sum_{m=1}^{\infty} 2^{-m} \|\varphi-\psi\|_{(m)}/(1 + \|\varphi-\psi\|_{(m)})$$

where $\|\cdot\|_{(m)}$, $m = 1, 2, \ldots$ is an enumeration of $\{\|\cdot\|_{\alpha,\beta} | \alpha, \beta \in \mathbb{N}_0^n\}$. Hence, $(\mathbb{S}(\mathbb{R}^n), d_{\mathbb{S}})$ is a Fréchet space, i.e. a topological vector space with a complete metric, that is translation invariant, i.e. $d_{\mathbb{S}}(\varphi,\psi) = d_{\mathbb{S}(\varphi-\psi,0)}$ for all $\varphi, \psi \in \mathbb{S}(\mathbb{R}^n)$. This metric is not very convenient to work with, however it is clear that

$$\varphi_k \to \varphi \text{ in } (\mathbb{S}(\mathbb{R}^n), d_{\mathbb{S}}) \text{ if and only if } \|\varphi_k - \varphi\|_{\alpha,\beta} \to 0 \text{ for all } \alpha, \beta \in \mathbb{N}_0^n.$$

It is well known that the Fourier transformation $F : \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}(\mathbb{R}^n)$ is a linear homeomorphism, where F is given by:

$$F[\varphi](z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot z} \varphi(x) dx.$$

We define *convolution* by

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

A very convenient result follows if we combine this with Fourier transformation.

$$F[\varphi * \psi] = (2\pi)^{n/2} F[\varphi]F[\psi] \tag{2.4}$$

An other convenient property of the Schwartz space is that it is dense in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$. All of these results on the Schwartz space can be found in [9].
We have chosen the Schwartz-space because then Fourier transformation is a linear homeomorphism, which makes it easy to solve the differential equation. Define for $t > 0$ and $x \in \mathbb{R}^n$ the map

$$E_a(t) : x \mapsto E_a(t,x) = (4\pi a t)^{-n/2} e^{-|x|^2/(4at)}.$$

For $t = 0$, $E_a(0,x) = \delta(x)$, the delta-distribution. We can prove the following proposition, solving the heat equation in $\mathbb{S}(\mathbb{R}^n)$.

Proposition 2.2. *The partial differential equation* (2.1) *with initial condition* $\phi \in \mathbb{S}(\mathbb{R}^n)$ *has as its unique solution in* $\mathbb{S}(\mathbb{R}^n)$

$$U(x,t) = E_a(t) * \phi(x), \qquad x \in \mathbb{R}^n, t \geq 0.$$

Proof. Let $F[U(\cdot,t)](z)$ the Fourier transform of $U(\cdot,t)$ with respect to x. Since

$$F[\Delta\varphi](z) = -|z|^2 F[\varphi](z),$$

where $|z|^2 = z_1^2 + \cdots + z_n^2$, for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$, the differential equation transforms into

$$\begin{cases} \frac{\partial}{\partial t} F[U(\cdot,t)](z) = -a|z|^2 F[U(\cdot,t)](z) \\ F[U(\cdot,0)](z) = F[\phi](z) \end{cases}$$

This equation is easily solved

$$F[U(\cdot,t)](z) = F[\phi](z)e^{-a|z|^2 t}$$

Using equation (2.4) and the following result:

$$\text{If } \varphi(x) = e^{-\alpha|x|^2}, \text{ then } F[\varphi](z) = (2\alpha)^{-n/2} e^{-|z|^2/(4\alpha)},$$

we find

$$\begin{aligned} F[U(\cdot,t)](z) &= F[\phi](z)e^{-a|z|^2 t} = F[\phi](z)F[(2at)^{n/2}e^{-|\cdot|^2/(4at)}](z) \\ &= (2\pi)^{-n/2} F[\phi * (2\pi)^{n/2} E_a(t)](z) = F[E_a(t) * \phi](z). \end{aligned}$$

From this it follows, since Fourier transformation is a homeomorphism on $\mathbb{S}(\mathbb{R}^n)$, that $U(\cdot,t) = E_a(t) * \phi$. $\square$

The so-called *heat kernel* $E_a(t)$ that we have found defines a linear semigroup on $\mathbb{S}(\mathbb{R}^n)$, i.e. the family of linear operators $(T_a(t))_{t\geq 0}$ given by

$$T_a(t)\varphi = E_a(t) * \varphi \text{ for } t > 0 \text{ and} \tag{2.5}$$

$$T_a(0)\varphi = \varphi \tag{2.6}$$

is a semigroup on $\mathbb{S}(\mathbb{R}^n)$. It is also strongly continuous. Let us check the claims for semigroups:

1. $T(t)T(s)\varphi = T(t+s)\varphi$ for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$.
   $E_a(t) \in \mathbb{S}(\mathbb{R}^n)$ for all $a, t > 0$, so they have Fourier transforms:

   $$F[E_a(t)](z) = (2\pi)^{-n/2} e^{-a|z|^2 t} \tag{2.7}$$

   From (2.7) we derive that

   $$\begin{aligned} F[E_a(t) * E_a(s)](z) =&(2\pi)^{n/2} F[E_a(t)](z) F[E_a(s)](z) \\ =&(2\pi)^{n/2}(2\pi)^{-n/2} e^{-a|z|^2 t} \cdot (2\pi)^{-n/2} e^{-a|z|^2 s} \\ =&(2\pi)^{-n/2} e^{-a|z|^2 (t+s)} \\ =&F[E_a(t+s)](z) \end{aligned}$$

   So $E_a(t) * E_a(s) = E_a(t+s)$ and

   $$\begin{aligned} T(t)T(s)\varphi = E_a(t) * (E_a(s) * \varphi) &= (E_a(t) * E_a(s)) * \varphi \\ = E_a(t+s) * \varphi &= T(t+s)\varphi \end{aligned}$$

2. $T_a(t)$ is a continuous operator on $\mathbb{S}(\mathbb{R}^n)$.
   Since Fourier transformation is a homeomorphism $F : \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}(\mathbb{R}^n)$ and $F[E_a(t) * \varphi](z) = e^{-a|z|^2 t} F[\varphi](z)$, it suffices to show that $M_t : \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}(\mathbb{R}^n) : \varphi \mapsto \hat{E}_{a,t}\varphi$, where $\hat{E}_{a,t}(z) = e^{-a|z|^2 t}$, is continuous. So we have to show that $d_{\mathbb{S}}(M_t\varphi, M_t\psi) \to 0$ if $d_{\mathbb{S}}(\varphi, \psi) \to 0$, or equivalently

   $$\|M_t\varphi - M_t\psi\|_{\alpha,\beta} \to 0 \text{ if } d_{\mathbb{S}}(\varphi, \psi) \to 0 \text{ for all } \alpha, \beta \in \mathbb{N}_0^n$$

   If $\varphi, \psi \in \mathbb{S}(\mathbb{R}^n)$ then it is easily checked that

   $$\|\varphi\psi\|_{\alpha,\beta} \leq \sum_{\gamma+\delta=\beta} \|\varphi\|_{\alpha,\gamma} \|\psi\|_{0,\delta}$$

   In this case we get

   $$\|M_t\varphi - M_t\psi\|_{\alpha,\beta} = \|\hat{E}_{a,t}(\varphi - \psi)\|_{\alpha,\beta} \leq \sum_{\gamma+\delta=\beta} \|\hat{E}_{a,t}\|_{\alpha,\gamma} \|\varphi - \psi\|_{0,\delta}$$

   $\hat{E}_{a,t}$ is a Schwartz function, so $\|\hat{E}_{a,t}\|_{\alpha,\gamma} < \infty$. $\|\varphi - \psi\|_{0,\delta} \to 0$ since $d_{\mathbb{S}}(\varphi, \psi) \to 0$ so we conclude $\|M_t\varphi - M_t\psi\|_{\alpha,\beta} \to 0$.

So $(T_a(t))_{t\geq 0}$ is a semigroup on $\mathbb{S}(\mathbb{R}^n)$. Now we will show that it is strongly continuous.

Proposition 2.3. *$(T_a(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathbb{S}(\mathbb{R}^n)$.*

Proof. Let $\varphi \in \mathbb{S}(\mathbb{R}^n)$. We have to show that

$$\lim_{t\downarrow 0} d_{\mathbb{S}}(T_a(t)\varphi - \varphi) = 0$$

This is equivalent to

$$\|T_a(t)\varphi - \varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta (T_a(t)\varphi - \varphi)(x)| \to 0 \text{ as } t \downarrow 0,$$

for all α, β multi-indexes. Because Fourier transformation is a homeomorphism on $\mathbb{S}(\mathbb{R}^n)$ this is equivalent to

$$\sup_{z\in\mathbb{R}^n} |z^\alpha D^\beta F[T_a(t)\varphi - \varphi](z)| \to 0 \text{ as } t \downarrow 0.$$

We calculate the Fourier transform

$$F[T_a(t)\varphi - \varphi](z) = (e^{-a|z|^2 t} - 1)F[\varphi](z)$$

So we have to show that for all multi-indexes α and β

$$\sup_{z\in\mathbb{R}^n} |z^\alpha D^\beta ((e^{-a|z|^2 t} - 1)F[\varphi](z))| \to 0 \text{ as } t \downarrow 0,$$

i.e $z^\alpha D^\beta ((e^{-a|z|^2 t} - 1)F[\varphi](z))$ converges uniformly to 0 as $t \downarrow 0$. For the derivative we use the product-rule

$$D^\beta ((e^{-a|z|^2 t} - 1)F[\varphi](z)) = \sum_{\gamma+\delta=\beta} D^\gamma (e^{-a|z|^2 t} - 1) D^\delta F[\varphi](z) =$$
$$(e^{-a|z|^2 t} - 1) D^\beta F[\varphi](z)) + \sum_{\gamma+\delta=\beta, \gamma\neq 0} t P_\gamma(z,t) e^{-a|z|^2 t} D^\delta F[\varphi](z)$$

with $P_\gamma(z,t)$ a polynomial.
To show that $z^\alpha (e^{-a|z|^2 t} - 1) D^\beta F[\varphi](z)$ converges uniformly to 0 we show that $\frac{e^{-a|z|^2 t}-1}{1+|z|^2}$ converges uniformly to 0, since $z^\alpha(1+|z|^2)D^\beta F[\varphi](z)$ is uniformly bounded. The Mean Value Theorem yields

$$\left| \frac{e^{-a|z|^2 t} - 1}{t} \right| \leq \max_{0\leq\theta\leq t} \left| \frac{d}{d\tau} e^{-a|z|^2 \tau}|_{\tau=\theta} \right| = a|z|^2.$$

From this it follows that

$$\left| \frac{e^{-a|z|^2 t} - 1}{1+|z|^2} \right| \leq \frac{a|z|^2 t}{1+|z|^2} \leq at \text{ for all } z \in \mathbb{R}^n.$$

Hence $\frac{e^{-a|z|^2t}-1}{1+|z|^2} \to 0$ uniformly on $\mathbb{R}^n$ as $t \downarrow 0$. The function $z \to te^{-a|z|^2t}$ converges uniformly in $\mathbb{R}^n$ to 0 as $t \downarrow 0$, since $\sup_{z\in\mathbb{R}^n} |te^{-a|z|^2t}| = t$. The highest power of z_i in $P_\gamma(z,t)$ is γ_i, so $\left|\frac{P_\gamma(z,t)}{1+z^\gamma}\right|$ is uniformly bounded on $\mathbb{R}^n$ and so is $z^\alpha(1+z^\gamma)D^\delta F[\varphi](z)$. So the summation also converges uniformly. □

In a Banach space a strongly continuous linear semigroup $(T(t))_{t\geq 0}$ has a generator A_T. This generator is defined by

$$A_T\varphi = \lim_{t\downarrow 0} \frac{T_a(t)\varphi - \varphi}{t}$$

$\mathbb{S}(\mathbb{R}^n)$ is not a Banach space, but we can still calculate this limit. Usually the domain of the generator is not the entire space. However, for $(T_a(t))_{t\geq 0}$ the domain of the generator A_a is all of $\mathbb{S}(\mathbb{R}^n)$.

Lemma 2.4. *For all $\varphi \in \mathbb{S}(\mathbb{R}^n)$*

$$\lim_{t\downarrow 0} \frac{T_a(t)\varphi - \varphi}{t} = a\Delta\varphi.$$

Proof. Let $\varphi \in \mathbb{S}(\mathbb{R}^n)$ We have to show that

$$\lim_{t\downarrow 0} d_{\mathbb{S}}(\frac{T_a(t)\varphi - \varphi}{t}, a\Delta\varphi) = 0$$

So, as in Proposition 2.3, we will show that $z^\alpha D^\beta F[\frac{T_a(t)\varphi-\varphi}{t} - a\Delta\varphi](z)$ converges uniformly to 0. We calculate the Fourier transform

$$F[\frac{T_a(t)\varphi - \varphi}{t} - a\Delta\varphi] = \frac{e^{-a|z|^2t} - 1}{t} F[\varphi] + a|z|^2 F[\varphi]$$

We differentiate again with the product-rule

$$D^\beta((\frac{e^{-a|z|^2t} - 1}{t} + a|z|^2)F[\varphi](z)) =$$
$$(\frac{e^{-a|z|^2t} - 1}{t} + a|z|^2)D^\beta F[\varphi](z) - \sum_{i=1}^n 2az_i(e^{-a|z|^2t} - 1)D^{\beta-e_i}F[\varphi](z) +$$
$$\sum_{j=1}^n (4a^2z_j^2te^{-a|z|^2t} - 2a(e^{-a|z|^2t} - 1))D^{\beta-2e_j}F[\varphi](z) +$$
$$\sum_{\substack{0\leq k,l\leq n\\ k\neq l}} 4a^2z_kz_lte^{-a|z|^2t}D^{\beta-e_k-e_l}F[\varphi](z) + \sum_{\substack{\gamma+\delta=\beta\\ |\gamma|\geq 2}} +tP_\gamma(z,t)e^{-a|z|^2t}D^\delta F[\varphi](z).$$

With $P_\gamma(z,t)$ a polynomial with $\left|\frac{P_\gamma(z,t)}{1+z^\gamma}\right|$ uniformly bounded on $\mathbb{R}^n$. In Proposition 2.3 we have already seen that $(e^{-a|z|^2t}-1)\varphi(z)$ and $te^{-a|z|^2t}\varphi(z)$

converge to 0 in $\mathbb{S}(\mathbb{R}^n)$ for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$. So, what is left to prove is uniform convergence of $z^\alpha(\frac{e^{-a|z|^2t}-1}{t} + a|z|^2)D^\beta F[\varphi](z)$ as $t \downarrow 0$. Again we use the Mean Value Theorem.

$$\begin{aligned}\left|\frac{e^{-a|z|^2t} + a|z|^2t - 1}{t}\right| &\le \max_{0\le\theta\le t}\left|\frac{d}{d\tau}\{e^{-a|z|^2\tau} + a|z|^2\tau\}|_{\tau=\theta}\right| \\ &= |-a|z|^2(e^{-a|z|^2t} - 1)| = a|z|^2(1 - e^{-a|z|^2t}) \\ &= a|z|^2t(\frac{1 - e^{-a|z|^2t}}{t}) \le a|z|^2ta|z|^2 = a^2|z|^4t.\end{aligned}$$

So we conclude that $\frac{1}{a^2|z|^4}(\frac{e^{-a|z|^2t}-1}{t} + a|z|^2)$ converges uniformly to 0 and since $z^\alpha a^2|z|^4D^\beta F[\varphi](z)$ is uniformly bounded this concludes the proof. $\square$

In the following section we will show that the definition of the linear operator $T_a(t)$ by convolution, (2.6), extends to $L^p(\mathbb{R}^n)$ and $W^{k,p}(\mathbb{R}^n)$. It follows that $(T_a(t))_{t\ge0}$ is a semigroup on $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$ and on $W^{k,p}(\mathbb{R}^n)$ for $1 \le p \le \infty$ and $k \in \mathbb{N}$. It is strongly continuous on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and on $W^{2,p}(\mathbb{R}^n)$ for $1 < p < \infty$ and $k \in \mathbb{N}$.

2.2 The heat semigroup in $L^p(\mathbb{R}^n)$ and $W^{k,p}(\mathbb{R}^n)$

Proposition 2.6 shows that $(T_a(t))_{t\ge0}$ is also a semigroup on $L^p(\mathbb{R}^n)$ and $W^{k,p}(\mathbb{R}^n)$. For $W^{k,p}(\mathbb{R}^n)$ we need that $D^\alpha T_a(t)\varphi = T_a(t)D^\alpha\varphi$ for all $\varphi \in W^{k,p}(\mathbb{R}^n)$ and $|\alpha| \le k$.

Proposition 2.5. 1. *If $\varphi \in \mathbb{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, then $D^\alpha T_a(t)\varphi = T_a(t)D^\alpha\varphi$.* 2. *If $\varphi \in W^{k,p}(\mathbb{R}^n)$, and $|\alpha| \le k$, then $D^\alpha T_a(t)\varphi = T_a(t)D^\alpha\varphi$ in the sense of distributions.*

Proof. Let $\varphi \in \mathbb{S}(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$

$$\begin{aligned}D^\alpha T_a(t)(x)\varphi &= D^\alpha\int_{\mathbb{R}^n} E_a(t,y)\varphi(x-y)dy = \int_{\mathbb{R}^n} E_a(t,y)D_x^\alpha\varphi(x-y)dy \\ &= T_a(t)D^\alpha\varphi(x)\end{aligned}$$

Now take $\varphi \in W^{k,p}(\mathbb{R}^n)$, $|\alpha| \le k$ and $\psi \in \mathcal{D} \subset \mathbb{S}(\mathbb{R}^n)$. Since all functions are in $L^p(\mathbb{R}^n)$ or $L^q(\mathbb{R}^n)$, with $\frac{1}{p} + \frac{1}{q} = 1$ and sufficiently differentiable, we define $\langle\cdot,\cdot\rangle$, by $\langle\varphi,\psi\rangle = \langle\varphi,\psi\rangle_{p,q} = \langle T_\varphi,\psi\rangle_{\mathcal{D}',\mathcal{D}}$. Now we can use the following properties:

- $\langle D^\alpha\varphi,\psi\rangle = (-1)^{|\alpha|}\langle\varphi, D^\alpha\psi\rangle$
- $\langle T_a(t)\varphi,\psi\rangle = \langle\varphi, T_a(t)\psi\rangle$, since $T_a(t)\varphi = E_a(t) * \varphi$ and $E_a(t)$ is even.

The result follows:

$$
\begin{aligned}
\langle D^\alpha T_a(t)\varphi, \psi\rangle &= (-1)^{|\alpha|}\langle T_a(t)\varphi, D^\alpha\psi\rangle = (-1)^{|\alpha|}\langle \varphi, T_a(t)D^\alpha\psi\rangle \\
&= (-1)^{|\alpha|}\langle \varphi, D^\alpha T_a(t)\psi\rangle\rangle = \langle D^\alpha\varphi, T_a(t)\psi\rangle \\
&= \langle T_a(t)D^\alpha\varphi, \psi\rangle = \langle T_a(t)D^\alpha\varphi, \psi\rangle.
\end{aligned}
$$

□

Proposition 2.6. *$(T_a(t))_{t\geq 0}$ is a linear semigroup on $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and on $W^{k,p}(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and $k \in \mathbb{N}$.*

Proof. Proposition $C.2$ shows that $T_a(t)\varphi \in L^p(\mathbb{R}^n)$ for all $t \geq 0$ if $\varphi \in L^p(\mathbb{R}^n)$ and Proposition $C.1$ reveals that $\|E_a(t)\|_1 \leq 1$ for all $t > 0$. Young's Convolution Inequality, Lemma $A.1$, then shows

$$
\|T_a(t)\varphi\|_p = \|E_a(t) * \varphi\|_p \leq \|E_a(t)\|_1\|\varphi\|_p \leq \|\varphi\|_p.
$$

Then, for $W^{k,p}(\mathbb{R}^n)$

$$
\begin{aligned}
\|T_a(t)\varphi\|_{k,p} &= \|E_a(t) * \varphi\|_{k,p} = \sum_{|\alpha|\leq k} \|D^\alpha(E_a(t) * \varphi)\|_p \\
&= \sum_{|\alpha|\leq k} \|E_a(t) * (D^\alpha\varphi)\|_p \leq \|E_a(t)\|_1 \sum_{|\alpha|\leq k} \|D^\alpha\varphi\|_p \\
&= \|E_a(t)\|_1\|\varphi\|_{k,p} \leq \|\varphi\|_{k,p}
\end{aligned}
$$

□

To show that $(T_a(t))_{t\geq 0}$ is also a strongly continuous semigroup on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ we use that the embedding $\mathbb{S}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is continuous.

Lemma 2.7. *$\mathbb{S}(\mathbb{R}^n)$ embeds continuously into $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$.*

Proof. For $p = \infty$ there is nothing to prove, since $\|\cdot\|_\infty = \|\cdot\|_{0,0}$.
For $1 \leq p < \infty$. Let $\varphi, \{\varphi_k\}_{k=0}^\infty \subset \mathbb{S}(\mathbb{R}^n)$ such that $\varphi_k \to \varphi$ in $\mathbb{S}(\mathbb{R}^n)$ as $k \to \infty$. Now the result follows:

$$
\begin{aligned}
\|\varphi_k - \varphi\|_p^p &= \int_{\mathbb{R}^n} |\varphi_k(x) - \varphi(x)|^p dx \\
&\leq \int_{\mathbb{R}^n} |\frac{1}{1+|x|^2}|^p |(1+|x|^2)(\varphi_k(x) - \varphi(x))|^p dx \\
&\leq \|\frac{1}{1+|\cdot|^2}\|_p^p \|(1+|\cdot|^2)(\varphi_k - \varphi)\|_\infty^p < \infty.
\end{aligned}
$$

□

Now we can prove the strong continuity on $W^{k,p}(\mathbb{R}^n)$ and thus also for $L^p(\mathbb{R}^n)$, since $L^p(\mathbb{R}^n) = W^{0,p}(\mathbb{R}^n)$.

Proposition 2.8. *$(T_a(t))_{t\geq 0}$ is a C_0-semigroup on $W^{k,p}(\mathbb{R}^n)$ for $1 \leq p < \infty$ and $k \in \mathbb{N}_0$.*

Proof. What is left to prove is the strong continuity, i.e.

$$\lim_{t\to 0} \|T_a(t)\varphi - \varphi\|_{k,p} = 0 \text{ for all } \varphi \in W^{k,p}(\mathbb{R}^n)$$

So we take $\varphi \in W^{k,p}(\mathbb{R}^n)$ and $\varepsilon > 0$. $\mathbb{S}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$, so there is a $\psi \in \mathbb{S}(\mathbb{R}^n)$ such that $\|\psi - \varphi\|_{k,p} \leq \varepsilon/3$. Lemma 2.7 shows that $\mathbb{R}_+ \to L^p(\mathbb{R}^n) : t \mapsto T_a(t)\psi$ is continuous. $D^\alpha\psi \in \mathbb{S}(\mathbb{R}^n)$ and

$$\|T_a(t)\psi - \psi\|_{k,p} = \sum_{|\alpha|\leq k} \|D^\alpha(T_a(t)\psi - \psi)\|_p = \sum_{|\alpha|\leq k} \|T_a(t)D^\alpha\psi - D^\alpha\psi\|_p,$$

hence $\mathbb{R}_+ \to W^{k,p}(\mathbb{R}^n) : t \mapsto T_a(t)\psi$ is also continuous. So there exists a $\delta > 0$ such that $\|T_a(t)\psi - \psi\|_{k,p} \leq \varepsilon/3$ for all $0 \leq t \leq \delta$. The triangle-inequality gives us

$$\|T_a(t)\varphi - \varphi\|_{k,p} \leq \|T_a(t)\varphi - T_a(t)\psi\|_{k,p} + \|T_a(t)\psi - \psi\|_{k,p} + \|\psi - \varphi\|_{k,p}$$

$T_a(t)$ is a linear operator, so $T_a(t)\varphi - T_a(t)\psi = T_a(t)(\varphi - \psi)$ and using Proposition 2.5 and Young's convolution inequality, Lemma A.1, we find

$$\begin{aligned}
\|T_a(t)(\varphi - \psi)\|_{k,p} &\leq \sum_{|\alpha|\leq k} \|D^\alpha T_a(t)(\varphi - \psi)\|_p = \sum_{|\alpha|\leq k} \|T_a(t)D^\alpha(\varphi - \psi)\|_p \\
&= \sum_{|\alpha|\leq k} \|E_a(t) * D^\alpha(\varphi - \psi)\|_p \\
&\leq \sum_{|\alpha|\leq k} \|E_a(t)\|_1 \|(D^\alpha(\varphi - \psi)\|_p \\
&= \|E_a(t)\|_1 \|\varphi - \psi\|_{k,p} \leq \|\varphi - \psi\|_{k,p} \leq \varepsilon/3
\end{aligned}$$

So

$$\|T_a(t)\varphi - \varphi\|_{k,p} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

And this concludes the proof. □

2.3 The generator and its domain in $L^p(\mathbb{R}^n)$

We have found the C_0-semigroup on $L^p(\mathbb{R}^n)$. We now want to verify that its generator is indeed $a\Delta$ in $L^p(\mathbb{R}^n)$. If it is generated by $a\Delta$ on a suitable domain we can use Theorem 2.1 to solve the differential equation (2.3). In this section we will show that the generator of $(T_a(t))_{t\geq 0}$ is given by $A_a\varphi := A_{T_a}\varphi = a\Delta\varphi$ for all function $\varphi \in D(A_a)$. This domain is the Sobolev space $W^{2,p}(\mathbb{R}^n)$, for $1 < p < \infty$. Although this seems to be considered common knowledge, the proof is hard to find in literature. The value of this section is therefore to bring together the results and arguments needed to come to this conclusion. In [1] elliptic operators on $\mathbb{R}^n$, such as Δ, are considered. P. Cannarsa and V. Vespri prove some results that are closely related to our situation, such as: There exists a unique solution $U \in W^{1,p}(\mathbb{R}^n)$ of $(\lambda - a\Delta)U = f$ if $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and $\mathrm{Re}\lambda \geq C$ for some constant C. Unfortunately we can not use their approach to solve our problem. The proof that we will present consists of showing the following inclusions:

$$W^{2,p}(\mathbb{R}^n) \subset D(A_a) \text{ for } 1 \leq p < \infty, \tag{2.8}$$

$$D(A_a) \subset \{\varphi \in L^p(\mathbb{R}^n) | \Delta\varphi \in L^p(\mathbb{R}^n)\} \text{ for } 1 \leq p < \infty, \tag{2.9}$$

$$\{\varphi \in L^p(\mathbb{R}^n) | \Delta\varphi \in L^p(\mathbb{R}^n)\} \subset W^{2,p}(\mathbb{R}^n) \text{ for } 1 < p < \infty. \tag{2.10}$$

From these inclusions it follows that

$$D(A_a) = W^{2,p}(\mathbb{R}^n),$$

provided that $1 < p < \infty$. Inclusion 2.10 is not trivial at all. It follows from a delicate theorem on singular integrals from the book by E.M. Stein, [20], here stated in Theorem 2.16. In the following proposition the results are gathered.

Theorem 2.9. *The generator A_a of the semigroup $(T_a(t))_{t\geq 0}$ on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$ is given by $A_a\varphi = a\Delta\varphi$ (in the sense of distributions) and its domain $D(A_a)$ is the Sobolev space $W^{2,p}(\mathbb{R}^n)$.*

Inclusion (2.10) is only valid for $1 < p < \infty$. For the cases $p = 1$ and $p = \infty$ we have the following results from [20]:

1. When $n = 1$ inclusion (2.10) holds for $p = 1$ and for $p = \infty$.
2. When $n > 1$ the inclusion does not hold in the case $p = 1$ neither in the case $p = \infty$

At the end of this section we will have proven Theorem 2.9. We will first prove, in Lemma 2.10 that $\mathbb{S}(\mathbb{R}^n) \subset D(A_a)$. Next we show, in Lemma 2.11, since $\mathbb{S}(\mathbb{R}^n)$ is dense in $W^{2,p}(\mathbb{R}^n)$, that for every $\varphi \in W^{2,p}(\mathbb{R}^n)$ there exists a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset \mathbb{S}(\mathbb{R}^n)$ such that, in $L^p(\mathbb{R}^n)$, $\varphi_k \to \varphi$ and $\Delta\varphi_k \to \Delta\varphi$. Then we combine these two lemmas in Lemma 2.12 to prove the first inclusion, (2.8).

Lemma 2.10. $\mathbb{S}(\mathbb{R}^n) \subset D(A_a)$ *and $A_a\varphi = a\Delta\varphi$ for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$.*

Note that here $\mathbb{S}(\mathbb{R}^n)$ is considered as a subset of $L^p(\mathbb{R}^n)$.

Proof. Define the family of operators as in Lemma 2.7 by

$$G(t) = \frac{T_a(t) - I}{t} \qquad t > 0,$$
$$G(0) = a\Delta.$$

According to Lemma 2.4 $t \mapsto G(t)\varphi : [0, \varepsilon] \to \mathbb{S}(\mathbb{R}^n)$ is continuous for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$. Hence, Lemma 2.7, it is continuous from $[0, \varepsilon]$ to $L^p(\mathbb{R}^n)$. So the limit exists and the result follows. $\square$

Lemma 2.11. *For all $\varphi \in W^{2,p}(\mathbb{R}^n)$, $1 \le p < \infty$ there exists a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset \mathbb{S}(\mathbb{R}^n)$ such that $\varphi_k \to \varphi$ in $L^p(\mathbb{R}^n)$ and $\Delta\varphi_k \to \Delta\varphi$ in $L^p(\mathbb{R}^n)$.*

Proof. Let $\varphi \in W^{2,p}(\mathbb{R}^n)$. Since $\mathbb{S}(\mathbb{R}^n)$ is dense in $W^{2,p}(\mathbb{R}^n)$ there exists a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset \mathbb{S}(\mathbb{R}^n)$ such that $\varphi_k \to \varphi$ in $W^{2,p}(\mathbb{R}^n)$ and thus in $L^p(\mathbb{R}^n)$.
$\varphi \mapsto \Delta\varphi$ is a continuous linear mapping from $W^{2,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, since

$$\|\Delta\varphi\|_{L^p} \le \sum_{|\alpha| \le 2} \|D^\alpha \varphi\|_{L^p} = \|\varphi\|_{W^{2,p}}$$

So we also have $\Delta\varphi_k \to \Delta\varphi$ in $L^p(\mathbb{R}^n)$. $\square$

Lemma 2.12. *Let $1 \le p < \infty$, then $W^{2,p}(\mathbb{R}^n) \subset D(A_a)$ and $A_a\varphi = a\Delta\varphi$ for all $\varphi \in W^{2,p}(\mathbb{R}^n)$.*

Proof. Define the following graphs:

$$G_1 = \{(\varphi, a\Delta\varphi) | \varphi \in \mathbb{S}(\mathbb{R}^n)\}$$
$$G_2 = \{(\varphi, a\Delta\varphi) | \varphi \in W^{2,p}(\mathbb{R}^n)\}$$
$$G_{A_a} = \{(\varphi, A_a\varphi) | \varphi \in D(A_a)\}$$

Lemma 2.11 implies $G_2 \subset \overline{G_1}$, with $\overline{G_1}$ the closure in $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$. Lemma 2.10 implies $G_1 \subset G_{A_a}$. G_{A_a} is closed in $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ because the generator of a C_0-semigroup is a densely defined closed operator. So $\overline{G_1} \subset G_{A_a}$ and thus

$$G_2 \subset \overline{G_1} \subset G_{A_a}. \tag{2.11}$$

So $W^{2,p}(\mathbb{R}^n) \subset D(A_a)$ and $A_a\varphi = a\Delta\varphi$ for all $\varphi \in W^{2,p}(\mathbb{R}^n)$. $\square$

Since $\mathbb{S}(\mathbb{R}^n)$ is a subset of $D(A_a)$, dense in $L^p(\mathbb{R}^n)$ and invariant under the semigroup $(T_a(t))_{t \ge 0}$, Proposition $A.3$ implies that $\mathbb{S}(\mathbb{R}^n)$ is a core for $(A_a, D(A_a))$ in $L^p(\mathbb{R}^n)$. This means that the closure of $\mathbb{S}(\mathbb{R}^n)$ with respect to the graph norm is the domain $D(A_a)$. And this implies $\overline{G_1} = G_{A_a}$. As a corollary we can prove inclusion (2.9) in Lemma 2.14. To prove this result we need the following lemma. (For explanation of notation for distibutions, consult Appendix A.)

Lemma 2.13. *Let $\{\varphi_k\}_{k=1}^{\infty} \subset \mathbb{S}(\mathbb{R}^n)$ a sequence and assume there exist functions $\varphi, \psi \in L^p(\mathbb{R}^n)$ such that $\varphi_k \to \varphi$ in $L^p(\mathbb{R}^n)$ and $\Delta\varphi_k \to \psi$ in $L^p(\mathbb{R}^n)$. Then $\Delta\varphi = \psi$ in the sense of distributions.*

Proof. Let $\chi \in \mathcal{D}$, then

$$\begin{aligned}
\langle \Delta T_\varphi, \chi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T_\varphi, \Delta\chi\rangle_{\mathcal{D}',\mathcal{D}} = \langle \varphi, \Delta\chi\rangle_{p,q} &= \lim_{k\to\infty} \langle \varphi_k, \Delta\chi\rangle_{p,q} \\
&= \lim_{k\to\infty} \langle T_{\varphi_k}, \Delta\chi\rangle_{\mathcal{D}',\mathcal{D}} = \lim_{k\to\infty} \langle \Delta T_{\varphi_k}, \chi\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \lim_{k\to\infty} \langle T_{\Delta\varphi_k}, \chi\rangle_{\mathcal{D}',\mathcal{D}} = \lim_{k\to\infty} \langle \Delta\varphi_k, \chi\rangle_{p,q} = \langle \psi, \chi\rangle_{p,q} \\
&= \langle T_\psi, \chi\rangle_{\mathcal{D}',\mathcal{D}}.
\end{aligned}$$

□

Lemma 2.14. *For all $\varphi \in D(A_a)$ $A_a\varphi = a\Delta\varphi$ in the sense of distributions and consequently $D(A_a) \subset \{\varphi \in L^p(\mathbb{R}^n) | \Delta\varphi \in L^p(\mathbb{R}^n)\}$.*

Proof. For all $\varphi \in D(A_a)$ there exists a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset \mathbb{S}(\mathbb{R}^n)$, such that $\|\varphi - \varphi_k\|_{A_a} \to 0$, since $\mathbb{S}(\mathbb{R}^n)$ is a core of $(A_a, D(A_a))$. So $\varphi_k \to \varphi$ in $L^p(\mathbb{R}^n)$ and $a\Delta\varphi_k \to A_a\varphi$ in $L^p(\mathbb{R}^n)$. Because $\varphi_k \in \mathbb{S}(\mathbb{R}^n)$, Lemma 2.4 shows $A_a\varphi_k = a\Delta\varphi_k$. Thus according to Lemma 2.13 $A_a\varphi = a\Delta\varphi$.
A_a is a map from $D(A_a) \subset L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. So if $\varphi \in D(A_a)$ then $\varphi \in L^p(\mathbb{R}^n)$ and $A_a\varphi = a\Delta\varphi \in L^p(\mathbb{R}^n)$ □

Inclusion (2.10), as mentioned before, is the hardest to prove. The case $p = 2$ can easily be obtained through Fourier transformation, Lemma 2.15. The general result for $1 < p < \infty$ will be proven in Theorem 2.19.

Lemma 2.15. *If $\varphi \in L^2(\mathbb{R}^n)$ and $\Delta\varphi \in L^2(\mathbb{R}^n)$ then $\varphi \in W^{2,2}(\mathbb{R}^n)$.*

Proof. Assume $\varphi \in L^2(\mathbb{R}^n)$ and $\Delta\varphi \in L^2(\mathbb{R}^n)$. We have to prove that $D^\alpha\varphi \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le 2$. However, since $F[D^\alpha\varphi](z) = (-iz)^\alpha F[\varphi](z)$ and $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a linear homeomorphism it suffices to prove that

$$z^\alpha F[\varphi] \in L^2(\mathbb{R}^n) \text{ for all } |\alpha| \le 2$$

1. $|\alpha| = 1$
   Let $1 \le j \le n$. Use that $z_j^2 \le |z|^2 \le 1 + |z|^4$ for $z \in \mathbb{R}^n$, then

$$\begin{aligned}
\|z_j F[\varphi]\|_2^2 = \int_{\mathbb{R}^n} |z_j F[\varphi]|^2 dz &\le \int_{\mathbb{R}^n} (|F[\varphi]|^2 + |z|^4 |F[\varphi]|^2) dz \\
&\le \|F[\varphi]\|_2^2 + \|F[\Delta\varphi]\|_2^2 < \infty
\end{aligned}$$

2. $|\alpha| = 2$
   Let $1 \leq i, j \leq n$. Use that $z_i^4 \leq |z|^4$ and $z_i^2 z_j^2 \leq z_i^4 + z_j^4 \leq |z|^4$ for $i \neq j$, then

$$\begin{aligned}\|z_i z_j F[\varphi]\|_2^2 &= \int_{\mathbb{R}^n} |z_i z_j F[\varphi]|^2 dz = \int_{\mathbb{R}^n} z_i^2 z_j^2 |F[\varphi]|^2 dz \\ &\leq \int_{\mathbb{R}^n} |z|^4 |F[\varphi]|^2 dz \leq \|F[\Delta\varphi]\|_2^2 < \infty\end{aligned}$$

□

To understand the theorem presented in [20] that we will use we will first introduce the Bessel potential and the potential space. The *Bessel potential* $\mathcal{J}_\alpha$, $\alpha > 0$ is defined for $\varphi \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ by

$$\mathcal{J}_\alpha(\varphi) = G_\alpha * \varphi$$

with $F[G_\alpha](x) = (2\pi)^{-n/2}(1 + |x|^2)^{-\alpha/2}$. In [20] it is proven that such a G_α exists and that $G_\alpha \in L^1(\mathbb{R}^n)$. So the convolution is defined and $\mathcal{J}_\alpha$ maps $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. The definition of the potential space makes use of this Bessel potential. The *potential space* $\mathcal{L}_\alpha^p(\mathbb{R}^n)$, $\alpha > 0$, $1 \leq p \leq \infty$ is the subspace of $L^p(\mathbb{R}^n)$ consisting of all φ that can be written in the form $\varphi = \mathcal{J}_\alpha(\psi)$, $\psi \in L^p(\mathbb{R}^n)$. The $\mathcal{L}_\alpha^p$-norm of φ is defined to be the L^p-norm of ψ, i.e.

$$\|\varphi\|_{\mathcal{L}_\alpha^p} = \|\psi\|_p, \text{ if } \varphi = \mathcal{J}_\alpha(\psi). \tag{2.12}$$

The definition of the norm is consistent, because $\mathcal{J}_\alpha : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is injective, see [20]. The following theorem connects the potential spaces to the Sobolev spaces.

Theorem 2.16 (Stein,[20]). *If $1 < p < \infty$ and $k \in \mathbb{N}_0$ then $\mathcal{L}_k^p(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$ and the norms,* (2.12) *and* (1.2)*, are equivalent.*

The Bessel potential $\mathcal{J}_2$ is the inverse operator of $I - \Delta$ on Schwartz functions, i.e.

Lemma 2.17. *For all $\varphi \in \mathbb{S}(\mathbb{R}^n)$*

$$(I - \Delta)\mathcal{J}_2(\varphi) = \varphi = \mathcal{J}_2((I - \Delta)\varphi).$$

Proof. We use, as always for Schwartz functions, Fourier-transformation.

$$\begin{aligned}F[(I - \Delta)\mathcal{J}_2(\varphi)](x) =& F[(I - \Delta)(G_2 * \varphi)](x) \\ =& (1 + |x|^2)(2\pi)^{n/2}(2\pi)^{-n/2}(1 + |x|^2)^{-1}F[\varphi](x) \\ =& F[\varphi](x), \text{ and} \\ F[\varphi](x) =& (2\pi)^{n/2}(2\pi)^{-n/2}(1 + |x|^2)^{-1}(1 + |x|^2)F[\varphi](x) \\ =& F[G_2 * (I - \Delta)\varphi](x) \\ =& F[\mathcal{J}_2((I - \Delta)\varphi)](x).\end{aligned}$$

So, putting it all together, we find

$$F[(I-\Delta)\mathcal{J}_2(\varphi)] = F[\varphi] = F[\mathcal{J}_2((I-\Delta)\varphi)].$$

The result follows since $\varphi \in \mathbb{S}(\mathbb{R}^n)$. $\square$

With the use of this result on $\mathbb{S}(\mathbb{R}^n)$, we can prove a similar result for $L^p(\mathbb{R}^n)$.

Lemma 2.18. *If $1 \leq p \leq \infty$ and $\varphi, \Delta\varphi \in L^p(\mathbb{R}^n)$, then $\mathcal{J}_2((I-\Delta)\varphi) = \varphi$ in the sense of distributions.*

Proof. Let $\psi \in \mathcal{D}$ a test function and $\varphi \in L^p(\mathbb{R}^n)$ such that $\Delta\varphi \in L^p(\mathbb{R}^n)$,then

$$\begin{aligned}
\langle T_{\mathcal{J}_2((I-\Delta)\varphi)}, \psi\rangle_{\mathcal{D}',\mathcal{D}} &= \langle \mathcal{J}_2((I-\Delta)\varphi), \psi\rangle_{p,q} = \langle G_2 * ((I-\Delta)\varphi), \psi\rangle_{p,q} \\
&= \langle (I-\Delta)\varphi, G_2 * \psi\rangle_{p,q} = \langle (I-\Delta)\varphi, \mathcal{J}_2(\psi)\rangle_{p,q} \\
&= \langle T_{(I-\Delta)\varphi}, \mathcal{J}_2(\psi)\rangle_{\mathcal{D}',\mathcal{D}} = \langle (I-\Delta)T_\varphi, \mathcal{J}_2(\psi)\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \langle T_\varphi, (I-\Delta)\mathcal{J}_2(\psi)\rangle_{\mathcal{D}',\mathcal{D}} = \langle T_\varphi, \psi\rangle_{\mathcal{D}',\mathcal{D}}
\end{aligned}$$

Here we used that G_2 is even, so $\langle G_2 * \varphi, \psi\rangle_{p,q} = \langle \varphi, G_2 * \psi\rangle_{p,q}$ for all $\varphi \in L^p(\mathbb{R}^n)$, $\psi \in L^q(\mathbb{R}^n)$ and that $\psi \in \mathcal{D} \subset \mathbb{S}(\mathbb{R}^n)$, so $(I-\Delta)\mathcal{J}_2(\psi) = \psi$, according to Lemma 2.3. $\square$

Now we have prepared ourselves to prove inclusion (2.10).

Theorem 2.19. *If $\varphi \in L^p(\mathbb{R}^n)$, $\Delta\varphi \in L^p(\mathbb{R}^n)$ and $1 < p < \infty$, then $\varphi \in W^{2,p}(\mathbb{R}^n)$.*

Proof. Assume $\varphi, \Delta\varphi \in L^p(\mathbb{R}^n)$, then, according to Lemma 2.18 $\mathcal{J}_2((I-\Delta)\varphi) = \varphi$, so $\varphi \in \mathcal{L}_2^p$. And thus, Theorem 2.16, $\varphi \in W^{2,p}(\mathbb{R}^n)$. $\square$

Theorem 2.19 was the third and final inclusion that we needed to prove. So it follows that $D(A_a) = W^{2,p}(\mathbb{R}^n)$, $1 < p < \infty$. In Lemma 2.14 it was shown that $a\Delta$ is the generator in $L^p(\mathbb{R}^n)$ of the semigroup $(T_a(t))_{t\geq 0}$. This concludes the proof of Theorem 2.9. Now we can combine this result with Theorem 2.1 and Corollary 2.20 follows, solving the homogeneous problem in $L^p(\mathbb{R}^n)$, provided that $1 < p < \infty$.

Corollary 2.20. *Let $1 < p < \infty$, then the differential equation* (2.3)*, with initial condition $U(0) = \phi \in W^{2,p}(\mathbb{R}^n)$ has a unique solution $U \in C^1(\mathbb{R}_+, L^p(\mathbb{R}^n))$. It is given by*

$$U(t) = T_a(t)\phi, \qquad t \geq 0.$$

Chapter 3

Well-posedness for abstract semilinear Cauchy problems

Theorem 2.1 showed that the homogeneous problem can be solved uniquely if we choose the initial condition in the domain of the operator A and assume that A is the generator of a strongly continuous semigroup. To solve the inhomogeneous problem we prove a theorem that requires seven more conditions. This theorem is of course applicable to the homogeneous problem, since the extra seven conditions are all satisfied by $F \equiv 0$. Let us state the inhomogeneous problem that we will consider in this chapter.

Problem 3.1. *Let X and Y be Banach spaces such that X embeds continuously and densely into Y. We identify X with its image. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on Y with generator $(A, D(A))$. We consider the following abstract semilinear Cauchy problem in Y:*

$$\partial_t U(t) = AU(t) + F(t, U(t)), \; t > 0, \tag{3.1}$$

$$U(0) = \phi. \tag{3.2}$$

In this equation $F(t, \cdot)$ is a map from X to Y, continuous for almost all t. We choose the initial condition $\phi \in X$.

In most of the results on local well-posedness A is the generator of an analytic semigroup. In this chapter we deal, apparently, with the more general case, where A is only assumed to generate a C_0-semigroup. However, it is not clear whether Conditions 3 and 4 below, that we use to solve the problem, imply that $(T(t))_{t\geq 0}$ is an analytic semigroup.

What also needs our attention is that in general $X \subsetneq Y$. For all $t \geq 0$ $F(t, X) \subset Y$, so we also find that in general $F(t, X) \not\subset X$. One way to treat this is to find a subspace X' of X that is also a Banach space, such that $F(t, X') \subset X'$. Here we do not try to find such a subspace. Instead, we will put conditions on the map $\Psi_{p,t}$, that is associated with the semigroup. In [18], the conditions are such that $F(t, X) \subset X$, since J.Rauch and J.

Smoller consider a smooth function F and X consists of bounded continuous functions. For L^p-spaces a nonlinear function typically maps a function out of $L^p(\mathbb{R}^n)$, so we can not simply use their results for our case. Here we present a theory following K. Ito and F. Kappel, [12], and S.C. Hille, [10], taking a semigroup perspective.
Following [12], we can define a solution of (3.1),(3.2) on $[0, \tau]$ in X in a classical sense and in a more general sense. The generalized solution is called a mild solution.

Definition 3.2.

1. *A classical solution of* (3.1),(3.2) *on* $[0, \tau]$ *in* X *is a function* $U(\cdot; \phi) \in C^1((0, \tau), X) \cap C([0, \tau], X)$*, with* $U(t; \phi) \in D(A)$ *for* $t > 0$ *such that* (3.1) *and* (3.2) *are satisfied.*

2. *A mild solution of* (3.1),(3.2) *on* $[0, \tau]$ *in* X *is a function* $U(\cdot; \phi) \in C([0, \tau], X)$ *that satisfies the Variation of Constants Formula*

$$U(t, \phi) = T(t)\phi + \int_0^t T(t-s)F(s, U(s))ds, \ 0 \leq t \leq \tau. \tag{3.3}$$

Under certain conditions, the mild solutions in Theorem 3.6 are even classical solutions, see [12]. The Variation of Constans Formula can be derived from the differential equation. Assume U is a classical solution, then

$$\begin{aligned}\partial_s[T(t-s)U(s)] &= -T(t-s)AU(s) + T(t-s)\partial_s U(s) \\ &= -T(t-s)AU(s) + T(t-s)AU(s) + T(t-s)F(s, U(s)) \\ &= T(t-s)F(s, U(s)).\end{aligned}$$

We integrate this from $s = 0$ to $s = t$,

$$\int_0^t \partial_s[T(t-s)U(s)]ds = \int_0^t T(t-s)F(s, U(s))ds.$$

From this the formula follows

$$T(0)U(t) - T(t)U(0) = \int_0^t T(t-s)F(s, U(s)ds.$$

$T(0)U(t) = U(t)$ and $U(0) = \phi$, so we get

$$U(t) = T(t)\phi + \int_0^t T(t-s)F(s, U(s))ds$$

So, if U is a classical solution of (3.1),(3.2), then it is also a mild solution. However, if U is a mild solution of (3.1),(3.2), it does not need to be a classical solution, since it does not need to be differentiable. So, a mild

solution is a generalization of a classical solution.
A solution of (3.1), (3.2) depends on the initial condition ϕ. However, most of the time there is no confusion about what this condition is, so we will write $U(\cdot)$ instead of $U(\cdot, \phi)$. In Theorem 3.6 the existence and uniqueness of mild solutions will be shown under certain conditions. Before we state these conditions we will first define a continuous linear map $\Psi_{p,t}$ from $L^p([0,t], Y)$ to Y, for $1 \leq p \leq \infty$ and $t > 0$:

$$\Psi_{p,t} : L^p([0,t], Y) \to Y : \varphi \mapsto \int_0^t T(t-s)\varphi(s)ds \tag{3.4}$$

Since $(T(t))_{t\geq 0}$ is strongly continuous on Y, the map $T : [0,t] \to \mathcal{L}(Y)$ is strongly measurable. According to Lemma $B.4$ it then follows that, because $\varphi \in L^p([0,t], Y)$ is measurable, the map $s \mapsto T(t-s)\varphi(s)$ is measurable from $[0,t]$ to Y. Because $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$, Proposition $A.4$, for some $M \geq 1$ and $\omega \in \mathbb{R}$, we see that $\int_0^t \|T(t-s)\varphi(s)\|_Y ds < \infty$ for all $\varphi \in L^p([0,t], Y)$. So according to Theorem $B.7$, $s \mapsto T(t-s)\varphi(s)$ is Bochner integrable and definition 3.4 makes sense. If there exists a $1 \leq p \leq \infty$ such that the map $s \mapsto F(s, U(s))$ is in $L^p([0,t], Y)$, then (3.3) can be written as

$$U(t) = T(t)\phi + \Psi_{p,t}\left[F(\cdot, U(\cdot))\right].$$

F defines a Nemytskii-mapping N_F from functions $U : \mathbb{R}_+ \to X$ to functions $N_F(U) : \mathbb{R}_+ \to Y$

$$N_F(U)(t) := F(t, U(t)).$$

Using this notation, (3.3) becomes

$$U(t) = T(t)\phi + \Psi_{p,t}\left[N_F(U)\right], \ 0 \leq t \leq \tau.$$

With this rewritten Variation of Constants Formula we also rewrite the definition of a mild solution. Note that this definition can only be used if there exists a $1 \leq p \leq \infty$ such that $N_F(U) \in L^p([0,t], Y)$ for all $U \in C([0,t], X)$. The conditions listed below ensure this.

Definition 3.3. *A mild solution of* (3.1),(3.2) *on* $[0,\tau]$ *in* X *is a function* $U(\cdot;\phi) \in C([0,\tau], X)$ *that satisfies*

$$U(t) = T(t)\phi + \Psi_{p,t}\left[N_F(U)\right], \ 0 \leq t \leq \tau. \tag{3.5}$$

We will now state the conditions under which local existence and uniqueness of mild solutions will be proven.

1. X, the domain of $F(t,\cdot)$, is independent of t and invariant under the semigroup $(T(t))_{t\geq 0}$, and

2. The restriction of $(T(t))_{t\geq 0}$ to X, $(T(t)|_X)_{t\geq 0}$, is a strongly continuous semigroup on X.

3. There exist $1 \leq p < \infty$ and $T > 0$ such that $\Psi_{p,t}(L^p([0,t], Y)) \subset X$, for all $t \in (0,T]$ and

4. There exists an $M > 0$ such that $\|\Psi_{p,t}\|_{Y,X} \leq M$ for all $t \in (0,T]$, where $\|\Psi_{p,t}\|_{Y,X} := \|\Psi_{p,t}\|_{\mathcal{L}(L^p([0,t],Y),X)}$. This norm makes sense in view of Lemma 3.4.

5. $F : [0,T] \times X \to Y$ is a generalized Carathéodory function.

6. N_F maps $L^\infty([0,T], X)$ into $L^\infty([0,T], Y)$, and

7. $N_F : L^\infty([0,T], X) \to L^\infty([0,T], Y)$ is locally Lipschitz continuous.

If $X = Y$, the local existence and uniqueness of a mild solution can be proven under weaker conditions, i.e. Conditions 1 through 4 are automatically satisfied. So we assume that X is a proper subset of Y. The map $\Psi_{p,t} : L^p([0,t], Y) \to Y$ is continuous. According to Condition 3 the image is contained in X. The following lemma shows that the map $\Psi_{p,t} : L^p([0,t], Y) \to X$ is continuous as well.

Lemma 3.4. *Let X, Y and Z be Banach spaces, such that X embeds continuously into Y. We identify X with its image. Let $L : Z \to Y$ be a linear and continuous map with $L(Z) \subset X$. Then the map $L : Z \to X$ is continuous*

Proof. Let $\{z_n\}_{n=1}^\infty \subset Z$ a sequence for which there exists a $z_0 \in Z$ and a $x_0 \in X$ such that $\|z_n - z\|_Z \to 0$ and $\|Lz_n - x_0\|_X \to 0$ as $n \to \infty$. Since X embeds continuously into Y there exists a constant $C > 0$ such that $\|\xi_1 - \xi_2\|_Y \leq C\|\xi_1 - \xi_2\|_X$ for all $\xi_1, \xi_2 \in X$. So it follows that $\|Lz_n - x_0\|_Y \leq C\|Lz_n - x_0\|_X \to 0$ as $n \to \infty$. L is a linear and continuous map from Z to Y, so in particular it is closed. Thus $Lz_0 = x_0$. Consequently $L : Z \to X$ is closed and linear and thus, according to the Closed Graph Theorem, continuous. □

It follows that, if $1 \leq p < \infty$, the mapping Ψ_p with $\Psi_p\varphi(t) := \Psi_{p,t}\varphi$ is a continuous map from $L^p([0,T], Y)$ into $C([0,T], X)$.

Lemma 3.5. *If Conditions* (3) *and* (4) *are satisfied, then Ψ_p is a continuous mapping from $L^p([0,T], Y)$ into $C([0,T], X)$.*

Before we prove this lemma we first derive the following property of $\Psi_{p,t}$. For $0 < h \leq t < t + h \leq T$ and $\varphi \in L^p([0,T], Y)$ with $1 \leq p < \infty$

$$\Psi_{p,t+h}\varphi = T(t)\Psi_{p,h}\varphi + \Psi_{p,t}\varphi(\cdot + h). \tag{3.6}$$

$$\begin{aligned}
\Psi_{p,t+h}\varphi &= \int_0^h T(t+h-s)\varphi(s)ds + \int_h^{t+h} T(t+s-h)\varphi(s)ds \\
&= T(t)\int_0^h T(h-s)\varphi(s)ds + \int_0^t T(t-s)\varphi(s+h)ds \\
&= T(t)\Psi_{p,h}\varphi + \Psi_{p,t}\varphi(\cdot + h).
\end{aligned}$$

Because it is a linear continuous operator and thus closed, it follows from Theorem B.8 that $T(t)$ can be put in front of the integral.

Proof. (Lemma 3.5) We will only prove right-hand continuity. Left-hand continuity can be proven analogously. Let $\varphi \in L^p([0,t],Y)$. Condition (3) implies that $\Psi_{p,t}\varphi \in X$ for all $0 \le t \le T$. We first prove continuity in $t = 0$. Let $0 < h \le T$, then it follows from Condition (4) that

$$\|\Psi_{p,h}\varphi\|_X = \|\Psi_{p,h}\varphi_h\|_X \le M\|\varphi_h\|_{L^p([0,T],Y)} = M\|\varphi\|_{L^p([0,h],Y)} \to 0 \text{ as } h \downarrow 0$$

where we have defined $\varphi_h(t) = \varphi(t)$ for $0 \le t \le h$ and $\varphi_h(t) = 0$ elsewhere. Now for $0 < t < T$. Take h such that $0 < h \le t < t+h \le T$. From equation (3.6) we obtain

$$\Psi_{p,t+h}\varphi - \Psi_{p,t}\varphi = T(t)\Psi_{p,h}\varphi + \Psi_{p,t}(\varphi(\cdot+h) - \varphi).$$

Using Condition (4) again we see that

$$\|\Psi_{p,t}(\varphi(\cdot+h) - \varphi)\|_X \le M\|\varphi(\cdot+h) - \varphi\|_{L^p([0,T],Y)} \to 0 \text{ as } h \downarrow 0.$$

Note that we use here the strong continuity of the right translation semigroup in $L^p(\mathbb{R}_+,Y)$ for $1 \le p < \infty$. This strong continuity does not hold for $p = \infty$. We have already shown that $\|\Psi_{p,h}\varphi\|_X \to 0$ as $h \downarrow 0$. This proves the right-hand continuity. $\square$

Now we are prepared to prove the main theorem of this chapter.

Theorem 3.6. *The semilinear Cauchy problem in Y, Problem 3.1, has a unique local mild solution in X if the conditions 1 through 7 are satisfied, i.e.*

1. *For any $\gamma_0 > 0$ there exist constants $\tau = \tau(\gamma_0)$, $0 < \tau \le T$, and $\gamma = \gamma(\gamma_0) \ge 0$ such that for all $\phi \in X$ with $\|\phi\|_X \le \gamma_0$ there exists a mild solution $U(\cdot;\phi) \in C([0,\tau],X)$ of (3.1),(3.2) on $[0,\tau]$ such that $\|U(t;\phi)\|_X \le \gamma$ for $0 \le t \le \tau$.*

2. *For all $\phi \in X$ and $\tau > 0$ there exists at most one mild solution $U \in C([0,\tau],X)$ such that $U(0) = \phi$.*

3. *For any $\gamma > 0$ and $\tau > 0$ there exists a constant $C = C(\gamma,\tau)$ such that*

$$\|U(\cdot,\phi_1) - U(\cdot,\phi_2)\|_{C([0,\tau],X)} \le C\|\phi_1 - \phi_2\|_X$$

for mild solutions $U(\cdot;\phi_j) \in C([0,\tau],X)$ with $\|U(t,\phi_j)\| \le \gamma$, $0 \le t \le \tau$, $(j = 1,2)$.

Theorem 11.2 in [12] deals with a slightly different and a more general differential equation. The conditions here are equivalent to the conditions they use. We will present a proof that makes use of the Picard-Banach Fixed Point Theorem.

Theorem 3.7 (Picard-Banach Fixed Point Theorem). [9] *Let M be a complete metric space with metric d. Let $0 < \theta < 1$ and let $S : M \to M$ satisfy*

$$d(S^n x, S^n y) \leq \theta d(x, y)$$

for some positive integer n and for all $x, y \in M$. Then S has a unique fixed point in M.

Proof. [Theorem 3.6] Conditions 1 and 2 together with Proposition A.4 show that there exist $\omega \in \mathbb{R}$ and $M_1 \geq 1$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq M_1 e^{\omega t}$ for all $t \geq 0$. We define $M_0 = \max_{0 \leq t \leq T} M_1 e^{\omega t}$, where T is defined by Conditions 3 and 4. Choose $\gamma = 2M_0\gamma_0$ and let $\tau \leq T$. Let ϕ be such that $\|\phi\|_X \leq \gamma_0$. We define a metric space

$$M_{\tau,\gamma,\phi} = \{U \in C([0, \tau], X) | U(0) = \phi, \|U\|_\infty \leq \gamma\}.$$

The metric is defined by the supremum-norm $\|U\|_\infty = \sup_{t \in [0,\tau]} \|U(t)\|_X$: $d(\varphi, \psi) = \|\varphi - \psi\|_\infty$. Furthermore we define a solution operator

$$S_{\tau,\gamma,\phi} : M_{\tau,\gamma,\phi} \to C([0, \tau], X) : U \mapsto U_0 + \Psi_p \left[N_F(U)\right],$$

where $U_0(t) = T(t)\phi$. $M_{\tau,\gamma,\phi}$ is complete because it is a closed subset of $C([0, t], X)$, hence it is a complete metric space. We will show that $S_{\tau,\gamma,\phi}$ maps into $M_{\tau,\gamma,\phi}$ and that, if τ is chosen appropriately small, it is a contraction , i.e. there exists a $0 < \theta < 1$ such that $\|S_{\tau,\gamma,\phi}U - S_{\tau,\gamma,\phi}V\|_\infty \leq \theta\|U - V\|_\infty$ for all $U, V \in M_{\tau,\gamma,\phi}$. It then follows from the Picard-Banach Fixed Point Theorem that there exists a fixed point of $S_{\tau,\gamma,\phi}$, i.e. a mild solution.
To prove that $S_{\tau,\gamma,\phi}$ maps into $M_{\tau,\gamma,\phi}$, we show the following

1. $U \in C([0, \tau], X)$ implies $S_{\tau,\gamma,\phi}U \in C([0, \tau], X)$.
   Since $(T(t)|_X)_{t \geq 0}$ is a strongly continuous semigroup on X (Condition 2), U_0 is continuous from $[0, T]$ into X. If $U \in C([0, \tau], X)$, then it is measurable, according to Lemma B.3. Lemma 2.1 in [10] then implies that $N_F(U) : [0, \tau] \to Y$ is measurable. $U \in C([0, \tau], X) \subset L^\infty([0, \tau], X)$, so Condition 6 implies $N_F(U) \in L^\infty([0, \tau], Y)$. Since $[0, \tau]$ is bounded we also have $N_F(U) \in L^p([0, \tau], Y)$, for all $1 \leq p < \infty$. Now Lemma 3.5 implies that $\Psi_p N_F(U) \in C([0, \tau], X)$, which shows that $S_{\tau,\gamma,\phi}U \in C([0, \tau], X)$.

2. $U(0) = \phi$ implies $S_{\tau,\gamma,\phi}U(0) = \phi$.
   Clearly $S_{\tau,\gamma,\phi}U(0) = U_0(0) = \phi$.

3. $\|U\|_\infty \leq \gamma$ implies $\|S_{\tau,\gamma,\phi}U\|_\infty \leq \gamma$.
   The triangle inequality implies

$$\|S_{\tau,\gamma,\phi}U\|_\infty \leq \|U_0\|_\infty + \|\Psi_p N_F(U)\|_\infty. \tag{3.7}$$

Since $U_0(t) = T(t)\phi$ we see that

$$\|U_0(t)\|_X \leq \|T(t)\|_{\mathcal{L}(X)} \|\phi\|_X \leq M_0\gamma_0 = \gamma/2$$

for all $0 \leq t \leq T$ and thus

$$\|U_0\|_\infty \leq \gamma/2. \tag{3.8}$$

For $\|\Psi_p N_F(U)\|_\infty$ we first bound $\|\Psi_{p,t} N_F(U)\|_X$ for $t \in [0, \tau]$.

$$\|\Psi_{p,t} N_F(U)\|_X \leq \|\Psi_{p,t}\|_{Y,X} \|N_F(U)\|_{L^p([0,t],Y)}$$

Condition 7 tells us that for all φ, $\psi \in L^\infty([0,T], X)$ such that $\|\varphi\|_\infty \leq \gamma$ and $\|\psi\|_\infty \leq \gamma$ there exists a constant $L_\gamma > 0$ such that

$$\|N_F(\varphi) - N_F(\psi)\|_\infty \leq L_\gamma \|\varphi - \psi\|_\infty.$$

This implies that

$$\begin{aligned}\|N_F(\varphi)\|_\infty &\leq \|N_F(\varphi) - N_F(0)\|_\infty + \|N_F(0)\|_\infty \\ &\leq L_\gamma \|\varphi\|_\infty + \|N_F(0)\|_\infty.\end{aligned}$$

Now we can make the following estimation.

$$\begin{aligned}\|N_F(U)\|^p_{L^p([0,t],Y)} &= \int_0^t \|N_F(U)(s)\|^p_Y ds \\ &\leq \int_0^t (L_\gamma \|U\|_\infty + \|N_F(0)\|_\infty)^p ds = L^p t\end{aligned}$$

Here we used that $\|U\|_\infty \leq \gamma$ and that there exists a constant K such that $\|N_F(0)\|_\infty \leq K$ and defined $L := \gamma + K$. Consequently, using Condition 4, we see that

$$\|\Psi_{p,t} N_F(U)\|_X \leq MLt^{1/p}.$$

Hence

$$\|\Psi_p N_F(U)\|_\infty \leq ML\tau^{1/p}. \tag{3.9}$$

Now, putting equations (3.7), (3.8) and (3.9) together we find

$$\|S_{\tau,\gamma,\phi} U\|_\infty \leq \gamma/2 + ML\tau^{1/p}.$$

So if necessary we adapt τ such that $M\tau^{1/p} \leq \gamma/2$. Then we see that

$$\|S_{\tau,\gamma,\phi} U\|_\infty \leq \gamma.$$

To show that $S_{\tau,\gamma,\phi}$ is a contraction we do the following.

$$\begin{aligned}\|S_{\tau,\gamma,\phi}U(t) - S_{\tau,\gamma,\phi}V(t)\|_X &= \|\Psi_{p,t}\{N_F(U) - N_F(V)\}\|_X \\ &\leq \|\Psi_{p,t}\|_{Y,X}\|N_F(U) - N_F(V)\|_{L^p([0,t],Y)}\end{aligned}$$

And because of Condition 7 we have

$$\begin{aligned}\int_0^t \|N_F(U)(s) - N_F(V)(s)\|_Y^p ds &\leq \int_0^t L_\gamma^p \|U(s) - V(s)\|_X^p ds \\ &\leq \int_0^t L_\gamma^p \|U - V\|_\infty^p ds \\ &\leq L_\gamma^p t \|U - V\|_\infty^p\end{aligned}$$

This is true for all $0 \leq t \leq \tau$, so we also see that

$$\|S_{\tau,\gamma,\phi}U - S_{\tau,\gamma,\phi}V\|_\infty \leq ML_\gamma \tau^{1/p} \|U - V\|_\infty.$$

If necessary we adapt τ again such that $ML_\gamma \tau^{1/p} < 1$. We define $\theta = ML_\gamma \tau^{1/p}$ and find that $\theta < 1$. This concludes the proof of the first statement in the theorem.

For the second statement, let U and V mild solutions in $C([0,\tau],X)$ of (3.1),(3.2), such that $U(0) = V(0) = \phi$, with $\|U(t)\|_X \leq \gamma$ and $\|V(t)\|_X \leq \gamma$ for all $0 \leq t \leq \tau$. Then it follows from the variation of constants formula and Condition (4), that for all $0 \leq t \leq \tau$

$$\begin{aligned}\|U(t) - V(t)\|_X^p &\leq M^p \|N_F(U) - N_F(V)\|_{L^p([0,t],Y)}^p \\ &\leq M^p L_\gamma^p \int_0^t \|U(s) - V(s)\|_X^p ds.\end{aligned}$$

Using Gronwall's Lemma, we find that $\|U(t) - V(t)\|_X \leq 0$ for all $0 \leq t \leq \tau$, i.e. $U \equiv V$.

For the third statement let $U_1(\cdot, \phi_1)$ and $U_2(\cdot, \phi_2)$ mild solutions in $C([0,\tau],X)$ of (3.1),(3.2), such that $\|U_1(t)\|_X \leq \gamma$ and $\|U_2(t)\|_X \leq \gamma$ for all $0 \leq t \leq \tau$. Then

$$\begin{aligned}\|U_1(t) - U_2(t)\|_X &\leq \|T(t)(\phi_1 - \phi_2)\|_X + M\|N_F(U_1) - N_F(U_2)\|_{L^p([0,t],Y)} \\ &\leq M_0 \|(\phi_1 - \phi_2)\|_X + ML_\gamma \left(\int_0^t \|U_1(s) - U_2(s)\|_X^p ds \right)^{\frac{1}{p}}.\end{aligned}$$

Now we use the inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$ and $p \geq 1$.

$$\|U_1(t) - U_2(t)\|_X^p \leq 2^{p-1} M_0^p \|(\phi_1 - \phi_2)\|_X^p + 2^{p-1} M^p L_\gamma^p \int_0^t \|U_1(s) - U_2(s)\|_X^p ds.$$

It then follows from Gronwall that

$$\|U_1(t) - U_2(t)\|_X^p \leq 2^{p-1} M_0^p e^{2^{p-1} M^p L_\gamma^p t} \|\phi_1 - \phi_2\|_X^p.$$

So if we define $C^p := 2^{p-1}M_0^p e^{2^{p-1}M^p L_\gamma^p \tau}$, then we have for all $0 \leq t \leq \tau$, that

$$\|U_1(t) - U_2(t)\|_X \leq C\|\phi_1 - \phi_2\|_X.$$

Hence $\|U_1 - U_2\|_{C([0,\tau],X)} \leq C\|\phi_1 - \phi_2\|_X$. $\square$

$$U(t+s,\phi) = U(s, U(t,\phi)).$$

Hence we can define continuous *evolutionary operators* $\hat{S}(t,s) : X \to X$ for $t, s \in [0, \tau_\phi)$ and $s \leq t$, such that $U(t;\phi) = \hat{S}(t,0)\phi$. Furthermore

$$\hat{S}(t,t) = I \text{ for all } t \in [0,\tau_\phi),$$
$$\hat{S}(t,s)\hat{S}(s,\sigma) = \hat{S}(t,\sigma) \text{ for } \sigma \leq s \leq t.$$

If F does not depend on $t \in \mathbb{R}_+$ then we can define a strongly continuous semigroup S, such that $U(t;\phi) = S(t)\phi$, by $\hat{S}(t,s) = S(t-s)$. Then

$$S(0) = \hat{S}(t,t) = I \text{ and}$$
$$S(t)S(s) = \hat{S}(t+s,t)\hat{S}(t,0) = \hat{S}(t+s,0) = S(t+s)$$

In [12] it is shown that if $\phi \in D(A)$ and $A\phi + F(0,\phi) \in X$ the mild solution satisfies a Lipschitz condition on each compact subinterval of $[0,\tau)$. This Lipschitz condition implies that $U(\cdot,\phi)$ is strongly absolutely continuous. In [14] J. Komura shows that such a function is differentiable almost everywhere. If not X but Y is reflexive, then one can also prove differentiability of the unique mild solution. In this case the solution is a classical solution, if $\phi \in D(A)$ and $A\phi + F(0,\phi) \in X$, see [12].

Chapter 4

The FitzHugh-Nagumo equations in an abstract setting

In this Chapter we will use the theory of the previous section to find Banach spaces X and Y such that the FitzHugh-Nagumo equations have a unique local mild solution in X. To see that we can apply Theorem 3.6 to these equations we write them as follows.

$$\begin{aligned}\partial_t U_1(t) &= a\Delta U_1(t) + F_1(U_1(t), U_2(t)),\\ \partial_t U_2(t) &= F_2(U_1(t), U_2(t)),\end{aligned}$$

with initial conditions $U_1(0) = \phi_1 \in X_1$ and $U_2(0) = \phi_2 \in X_2$ and mappings $F_1(x, y) = p_3(x) + c_1 y$, with p_3 a cubic polynomial and $F_2(x, y) = c_2 x + c_3 y$. If we write $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ $A = a\Delta \oplus 0$ and $T = T_a \oplus I$, we see that we are dealing with a particular case of Problem 3.1. The mappings $F_i : X \to Y_i$, $i = 1, 2$, are in this case independent of $\mathbb{R}_+$. So, if we can find Banach spaces X_1, X_2, Y_1 and Y_2 such that the conditions for Theorem 3.6 are satisfied, we can prove local existence and uniqueness for the FitzHugh-Nagumo equations. Now we will go through the seven conditions to find possible Banach spaces of Theorem 3.6.

- Conditions 1 and 2.
  We want to study solutions in $L^p(\mathbb{R}^n)$ and in Proposition 2.8 we proved that $(T_a(t))_{t\geq 0}$ is a strongly continuous semigroup on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. So Condition 1, Condition 2 and the assumption on the semigroup on Y in Problem 3.1, i.e. strong continuity, imply that we may choose L^p-spaces, with $1 \leq p < \infty$, for X_1 and Y_1 or an intersection thereof.

- Conditions 3 and 4.

We can use the following proposition to satisfy Conditions 3 and 4 for X_1 and Y_1.

Proposition 4.1. *Let $t > 0$, $1 \le q < \infty$ and $1 \le p, r \le \infty$ such that*

$$q \le r,\ \frac{1}{r} > \frac{1}{q} - \frac{2}{n} \text{ and } p > [1 + \frac{n}{2}(\frac{1}{r} - \frac{1}{q})]^{-1},$$

then

$$\Psi_{p,t}(L^p([0,t], L^q(\mathbb{R}^n))) \subset L^r(\mathbb{R}^n)$$

and there exists a constant $C = C(n, a, q, r, p)$, such that for $t > 0$

$$\|\Psi_{p,t}\|_{L^q,L^r} \le Ct^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})+1-\frac{1}{p}}$$

The proof can be found in Appendix C. Thus for Condition 3 we have to find p, q and r that satisfy the conditions of Proposition 4.1. Then Condition 4 is also satisfied, since $\|\Psi_{p,t}\|_{L^q,L^r} \le CT^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})+1-\frac{1}{p}} =: M$ for $t \in (0, T]$. Note that actually $\Psi_{p,t}(L^p([0,t], L^q(\mathbb{R}^n))) \subset L^r \cap L^q(\mathbb{R}^n)$, so if we choose $X_1 = L^r \cap L^q(\mathbb{R}^n)$ and $Y_1 \subset L^q(\mathbb{R}^n)$ Conditions 3 and 4 are satisfied and X_1 embeds continuously into Y_1, because of the following lemma.

Lemma 4.2. *Let $1 \le p \le q \le \infty$. If $\varphi \in L^p \cap L^q(\mathbb{R}^n)$, then $\varphi \in L^s(\mathbb{R}^n)$ for all $p \le s \le q$ and $\|\varphi\|_s \le 2^{1/s}(\|\varphi\|_p + \|\varphi\|_q)$.*

A proof can be found in [10].
The semigroup acting on X_2 and Y_2 is I, the identity. Therefore we have to take $X_2 = Y_2$, since $t \mapsto I$ is a group and Lemma 4.3 in [10] then implies that $\Psi_{p,t}$ is surjective. Now Conditions 3 and 4 are easily satisfied for the second equation with $\|\Psi_{p,t}\|_{Y_2,X_2} \le T^{1/p}$ for all $t \in (0, T]$.

- Conditions 5, 6 and 7.
  For these conditions the only difficulty is the cubic polynomial in F_1. To deal with it we use the following theorem, which is proven in [5] by D.G. de Figuiredo. Recall that $N_F(\varphi)(t) := F(\varphi(t))$ for $\varphi : \Omega \to \mathbb{R}$.

Theorem 4.3. *Let Ω be an open set of $\mathbb{R}^m$ and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function. Suppose that there exist a constant $C > 0$, a function $B \in L^q(\Omega)$, $1 \le q \le \infty$ and $r > 0$ such that*

$$|F(t, x)| \le C|x|^r + B(t)$$

for all $t \in \Omega$ and $x \in \mathbb{R}$, then N_F maps $L^{qr}(\Omega)$ into $L^q(\Omega)$ and N_F is continuous and bounded.

The other mappings are all of the form $x \mapsto cx$, so these are continuous and bounded as well. $F : \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and if we choose X and Y such that F maps X into Y, then $N_F : L^\infty([0,T], X) \to L^\infty([0,T], Y)$ is Lipschitz continuous.

Theorem 4.3 and the estimate $|p_3(s)| \leq c_1|s| + c_2|s|^3$ for all $s \in \mathbb{R}$ yield that

$$f \in L^p(\mathbb{R}^n) \text{ implies } p_3(f) \in L^{p/3}(\mathbb{R}^n).$$

We want p_3 to map into Y_1 and we need that X_1 embeds into Y_1. So we let

$$X_1 = L^{p_1} \cap L^{p_1'}(\mathbb{R}^n),$$

with $1 \leq p_1 \leq p_1' < \infty$. Then $p_1' \geq p_1/3$, $X_1 \subset L^{p_1} \cap L^{p_1/3}(\mathbb{R}^n)$ and

$$p_3(X_1) \subset \bigcap_{\substack{p_1 \leq 3q \leq p_1' \\ q \geq 1}} L^q(\mathbb{R}^n).$$

We now choose

$$Y_1 = L^{s_1} \cap L^{s_1'}(\mathbb{R}^n),$$

with $p_1 \leq s_1 \leq s_1' \leq p_1'$ and $p_1 \leq 3s_1 \leq 3s_1' \leq p_1'$. Such a choice for s_1 and s_1' is possible, provided that $p_1' \geq 3p_1$. The first condition ensures that X_1 embeds continuously into Y_1, Lemma 4.2, and the second that F_1^1 maps X_1 into Y_1.

We also need to satisfy Conditions 3 and 4. Proposition 4.1 shows that

$$\Psi_{p,t}(L^p([0,t], L^{s_1}(\mathbb{R}^n))) \subset L^{s_1} \cap L^r(\mathbb{R}^n)$$

for all $r \geq s_1$ that satisfy

$$s_1 \leq r, \ \frac{1}{r} > \frac{1}{s_1} - \frac{2}{n} \text{ and } p > [1 + \frac{n}{2}(\frac{1}{r} - \frac{1}{s_1})]^{-1},$$

We want

$$\Psi_{p,t}(L^p([0,t], L^{s_1}(\mathbb{R}^n))) \subset L^{p_1}(\mathbb{R}^n),$$

so it follows that $s_1 \leq p_1$. And since we already had $p_1 \leq s_1$, we find $s_1 = p_1$.
We also want

$$\Psi_{p,t}(L^p([0,t], L^{s_1'}(\mathbb{R}^n))) \subset L^{p_1'}(\mathbb{R}^n),$$

so we find an extra condition from Proposition 4.1

$$\frac{1}{p_1'} > \frac{1}{s_1'} - \frac{2}{n}.$$

What is left to choose are the spaces X_2 and Y_2. We have to satisfy the following conditions: $X_2 = Y_2$, X_2 embeds continuously into Y_1 and X_1 embeds continuously into Y_2. We define

$$X_2 = Y_2 = L^{p_2} \cap L^{p_2'}(\mathbb{R}^n).$$

If we let $p_2 \leq p_1 \leq s_1' \leq p_2'$ then X_2 embeds continuously into Y_1 and if we let $p_1 \leq p_2 \leq p_2' \leq p_1'$ then X_1 embeds continuously into Y_2. From this it follows that $p_2 = p_1$ and $s_1' \leq p_2' \leq p_1'$.
Let us now summarize the results. We have chosen the following Banach spaces

$$X_1 = L^{p_1} \cap L^{p_1'}(\mathbb{R}^n),\ Y_1 = L^{p_1} \cap L^{s_1'}(\mathbb{R}^n),\ X_2 = Y_2 = L^{p_1} \cap L^{p_2'}(\mathbb{R}^n). \quad (4.1)$$

The conditions on these spaces are

$$1 \leq p_1 \leq s_1' \leq p_1'/3 < \infty,\ \frac{1}{p_1'} > \frac{1}{s_1'} - \frac{2}{n},\ s_1' \leq p_2' \leq p_1'. \quad (4.2)$$

Before we state the results in a theorem we first need to check that there exist such p_1, p_1', s_1', p_2' that satisfy the above conditions. Using the following steps one can find all possible solutions.

1. Choose p_1 such that $1 \leq p_1 < \infty$.
2. Choose p_1' such that $p_1' \geq 3p_1$, $p_1' > n$ and $p_1' < \infty$.
3. Choose p_2' such that $p_2' \leq p_1'$, $p_2' \geq p_1$ and $p_2' > \frac{p_1' n}{n+2p_1'}$.
4. Choose s_1' such that $s_1' \geq p_1$, $s_1' \leq p_1'/3$, $s_1' > \frac{p_1' n}{n+2p_1'}$ and $s_1' \leq p_2'$.

Since $s_1' > \frac{p_1' n}{n+2p_1'}$ is equivalent with $\frac{1}{p_1'} > \frac{1}{s_1'} - \frac{2}{n}$, the above steps imply the conditions. Furthermore the conditions in the steps all follow from the original conditions. This may not be clear at once for $p_1' > n$ and $p_2' > \frac{p_1' n}{n+2p_1'}$, so we will show how they can be deduced.

- $s_1' \leq p_1'/3$ together with $\frac{1}{p_1'} > \frac{1}{s_1'} - \frac{2}{n}$ imply that $\frac{1}{p_1'} > \frac{3}{p_1'} - \frac{2}{n}$. From this it follows that $-\frac{2}{p_1'} > -\frac{2}{n}$, or $p_1' > n$.
- $\frac{1}{p_1'} > \frac{1}{s_1'} - \frac{2}{n}$ is equivalent with $s_1' > \frac{p_1' n}{n+2p_1'}$. Combining this with $s_1' \leq p_2'$ we find $p_2' \geq s_1' > \frac{p_1' n}{n+2p_1'}$

The above steps can only be followed if all conditions are compatible. For most of the conditions this is clear. We will only show that $p_1'/3 > \frac{p_1' n}{n+2p_1'}$, using that $p_1' > n$. This allows one to choose p_2' as in step 3 in view of step 2.

$$\frac{p_1' n}{n+2p_1'} < \frac{p_1' n}{n+2n} = \frac{p_1' n}{3n} = p_1'/3$$

Now the functional-analytic set-up is such that Conditions $1-7$ of Theorem 3.6 hold. Then we have the following theorem:

Theorem 4.4. *Let $X = X_1 \oplus X_2$ as in* (4.1) *and* (4.2). *Then the FitzHugh Nagumo equations as defined at the beginning of this section have unique mild solutions in X locally in time.*

Chapter 5

Global existence

Now that we have found a local solution the next step is a global (in time) solution. In [17], J. Rauch proves the existence of global solutions to the FitzHugh-Nagumo equations. These results, however, only apply to bounded domains. It is not straightforward to adjust the methods used in his article so that it can be applied to unbounded domains, due to embeddings and particular inequalities, that do not hold there. So we have to try to find other methods. Remember that we defined a maximal solution $U(\cdot, \phi)$ on the maximal interval of existence $[0, \tau_\phi)$. We will use the following theorem, whose proof follows [19].

Theorem 5.1. *Let $U(\cdot, \phi)$ be the maximal solution of Problem* 3.1 *and let $[0, \tau_\phi)$ be the maximal interval of existence. If $\tau_\phi < \infty$ then*

$$\lim_{t \uparrow \tau_\phi} \|U(t, \phi)\|_X = \infty.$$

Proof. Suppose $U(\cdot, \phi)$ is the maximal solution and $[0, \tau_\phi)$ the maximal interval of existence and $\lim_{t \uparrow \tau_\phi} \|U(t, \phi)\|_X \neq \infty$. Then we have that

$$\liminf_{t \uparrow \tau_\phi} \|U(t, \phi)\|_X = M' < \infty.$$

So there exists a sequence $\{t_n\}_{n=0}^{\infty}$ such that $t_n < \tau_\phi$ for all $n \geq 0$, $t_n \uparrow \tau_\phi$ as $n \to \infty$ and $\|U(t_n, \phi)\|_X \leq M := M' + 1$ for all n. We define for all $n \geq 0$

$$\phi_n := U(t_n, \phi).$$

Since $\|\phi_n\|_X \leq M$ we can use Theorem 3.6 to see that there exists a $\tau = \tau(M) > 0$, independent of n, such that a unique solution exists for $0 \leq t \leq \tau$ with initial condition ϕ_n. Then, because of uniqueness,

$$U(t, \phi_n) = U(t_n + t, \phi) \text{ for all } 0 \leq t \leq \tau.$$

From this it follows that, because of maximality, $t_n + \tau \leq \tau_\phi$ for all $n \geq 0$, but this is a contradiction, since $t_n \uparrow \tau_\phi$ as $n \to \infty$. $\square$

So to prove that there exists a global solution we have to find a locally bounded function $T \mapsto M_T$ on $[0, \infty)$, the so-called a-priori estimate, such that

$$\|U(t, \phi)\|_X \leq M_T, \tag{5.1}$$

for all $0 \leq t \leq T$ and $T < \tau_\phi$. This implies that

$$\lim_{t \uparrow a} \|U(t, \phi)\|_X \leq \sup_{t \in [0,a]} M_T < \infty,$$

for all finite a. Hence, no blow-up in finite time can occur.
Classically the energy method is used, see [17]. An other way is with Lyapunov functions, see [19]. Yet an other method is the use of invariant regions, see [18]. All of these, however, show global existence on a bounded domain and our goal is to show global existence on an unbounded domain. And the results are not easily adapted to our case, since for instance the Poincaré inequality does not hold on an unbounded domain.
In the following sections we will present three different methods to find a bound as in (5.1). In Section 5.1 we assume that N_F is globally Lipschitz continuous. In Section 5.2 we only use local Lipschitz continuity, but then we also need to assume that $U(t) \in L^\infty(\mathbb{R}^n)$ for all $t \geq 0$. And finally in Section 5.3 we show how to find a bound if $U(t) \geq 0$ for all $t \geq 0$. Due to time limitations we did not succeed in applying any of the latter two methods to the FitzHugh-Nagumo system.

5.1 Global Lipschitz continuity

We assume that $F : \mathbb{R}_+ \times X \to Y$ is globally Lipschitz continuous as a function of $x \in X$, uniformly for t in compact intervals, i.e. for each $T > 0$ there exists a constant L_T such that $\|F(t, x) - F(t, x')\|_Y \leq L_T \|x - x'\|_X$ for all $t \in [0, T]$ and $x, x' \in X$. Now we can easily prove global existence.

Theorem 5.2. *Assume that the seven conditions for Theorem* 3.6 *are satisfied, with $T = \infty$ and that F is globally Lipschitz continuous as a function of $x \in X$, uniformly for t in compact intervals. Then the semilinear Cauchy problem, Problem* 3.1*, has a unique global mild solution in X.*

Proof. We write $U(\cdot)$ for the mild solution. The mild solution is the unique fixedpoint of the Variation of Constants formula, i.e.

$$U(t) = T(t)\phi + \Psi_{p,t}\left[N_F(U)\right] \text{ for all } t \geq 0.$$

We will show that $\|U(t)\|_X$ is bounded for all $t \geq 0$. We can estimate this norm as before:

$$\|U(t)\|_X \leq \|T(t)\|_{\mathcal{L}(X)} \|\phi\|_X + \|\Psi_{p,t}\|_{Y,X} \|N_F(U)\|_{L^p([0,t],Y)}. \tag{5.2}$$

Suppose that the maximal interval of existence $[0, \tau_\phi)$ of $U(\cdot, \phi)$ is finite, i.e. $\tau_\phi < \infty$. Choose $T \geq \tau_\phi$ and let L_T the Lipschitz constant as above. Then for $0 \leq s < \tau_\phi$:

$$\|N_F(U)(s)\|_Y = \|F(s, U(s))\|_Y \leq L_T \|U(s)\|_X + \|F(s, 0)\|_Y.$$

Now we can estimate $\|N_F(U)\|_{L^p([0,t],Y)}$.

$$\begin{aligned}
\|N_F(U)\|_{L^p([0,t],Y)} &= \left(\int_0^t \|N_F(U)(s)\|_Y^p ds\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^t (L_T \|U(s)\|_X + \|F(s,0)\|_Y)^p ds\right)^{\frac{1}{p}} \\
&\leq L_T \|U(\cdot)\|_{L^p([0,t],X)} + t^{\frac{1}{p}} \|N_F(0)\|_{L^p([0,t],Y)}.
\end{aligned}$$

Now we substitute this into inequality (5.2).

$$\begin{aligned}
\|U(t)\|_X \leq &\|T(t)\|_{\mathcal{L}(X)} \|\phi\|_X + \|\Psi_{p,t}\|_{Y,X} \|N_F(U)\|_{L^p([0,t],Y)} \\
\leq &M_X e^{\omega_x t} \|\phi\|_X + \|\Psi_{p,t}\|_{Y,X} \|N_F(0)\|_{L^p([0,t],Y)} \\
&+ L \|\Psi_{p,t}\|_{Y,X} \|U(\cdot)\|_{L^p([0,t],X)}
\end{aligned}$$

From this inequality we can also see that, using the inequality $(x+y)^p \leq 2^{p-1}(x^p + y^p)$,

$$\begin{aligned}
\|U(t)\|_X^p \leq &2^{p-1} M_X^p e^{\omega_x pt} \|\phi\|_X^p + 2^{p-1} \|\Psi_{p,t}\|_{Y,X}^p \|N_F(0)\|_{L^p([0,t],Y)}^p \\
&+ 2^{p-1} L^p \|\Psi_{p,t}\|_{Y,X}^p \int_0^t \|U(s)\|_X^p ds
\end{aligned}$$

To bound $\|U(t)\|_X^p$ we will use Gronwall's inequality, Lemma B.9. However, since we do not have any measurability results for the map $t \mapsto \|\Psi_{p,t}\|_{Y,X}$, we first use that $\|\Psi_{p,t}\|_{Y,X} \leq M$ for $t \in (0, T]$. Thus we find:

$$\begin{aligned}
\|U(t)\|_X^p \leq &2^{p-1} M_X^p e^{\omega_x pt} \|\phi\|_X^p + 2^{p-1} M^p \|N_F(0)\|_{L^p([0,t],Y)}^p \\
&+ 2^{p-1} L^p M^p \int_0^t \|U(s)\|_X^p ds.
\end{aligned}$$

Now the bound follows from Gronwall's inequality:

$$\begin{aligned}
\|U(t)\|_X^p \leq &2^{p-1} M^p \|N_F(0)\|_{L^p([0,t],Y)}^p + 2^{p-1} M_X^p \|\phi\|_X^p e^{\omega_x pt} e^{2^{p-1} L^p M^p t} \\
&+ 2^{2p-2} L^p M^{2p} \int_0^t e^{2^{p-1} L^p M^p (t-s)} \|N_F(0)\|_{L^p([0,s],Y)}^p ds,
\end{aligned}$$

for $0 \leq t \leq \tau_\phi$. Hence $\lim_{t \uparrow \tau_\phi} \|U(t)\|_X < \infty$, contradicting the assumption that $\tau_\phi < \infty$. $\square$

For the FitzHugh-Nagumo equations we can not use this theorem since the cubic polynomial is not globally Lipschitz continuous. It is locally Lipschitz continuous, so maybe we can prove global existence if we use the theorem presented in the following section. We included this case to stress the relevance of global Lipschitz continuity of F for global existence of solutions. Note that if $F : \mathbb{R}_+ \times X \to Y$ is globally Lipschitz continuous, uniformly in $t \in \mathbb{R}_+$ in compact intervals, it may happen that there is "blow-up at infinity", i.e. $\lim_{t\to\infty} \|U(t)\|_X = \infty$. If F is globally Lipschitz continuous, globally on $\mathbb{R}_+$, i.e. the Lipschitz constantL_T is independent of T, then this cannot happen.

5.2 Local Lipschitz continuity

If $F : \mathbb{R}_+ \times X \to Y$ is not globally Lipschitz continuous we can still prove global existence, but in a less general setting. However, it is general enough for our case. In Problem 3.1 we defined the abstract function $F : \mathbb{R}_+ \times X \to Y$. We restrict ourselves now to the case where X and Y are finite intersections of L^p-spaces over $\mathbb{R}^n$. We write for $t \geq 0$, $\varphi \in X$ and $x \in \mathbb{R}^n$

$$F(t, \varphi)(x) = f(t, \varphi(x)) \tag{5.3}$$

with $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. A condition that allows us to prove global existence is $U(t) \in L^\infty(\mathbb{R}^n)$ for almost all $t \in [0, \tau_\phi]$ and we have the a-priori estimate:

$$\|U(t)\|_\infty \leq M(t), \tag{5.4}$$

for some positive continuous function M on $\mathbb{R}_+$ for almost all $t \in [0, \tau_\phi]$.

Proposition 5.3. *Assume that the seven conditions for Theorem* 3.6 *are satisfied, with $T = \infty$. Let X and Y be finite intersections of L^p-spaces over $\mathbb{R}^n$, $1 \leq p < \infty$, such that X is continuously embedded into Y. Let F be defined as above and let f be locally Lipschitz continuous in $\mathbb{R}$ essentially uniform on compact sets in $\mathbb{R}_+$, with $f(t, 0) = 0$ for almost all $t \geq 0$. Assume further that $U(t) \in L^\infty(\mathbb{R}^n)$ for almost all $t \in [0, \tau_\phi]$ and satisfies* (5.4) *for some continuous function M on $[0, \tau_\phi]$. Then the semi linear Cauchy problem in Y, Problem* 3.1, *has a unique global mild solution in X for each initial value $\phi \in X$.*

Proof. The proof is essentially the same as the proof of Theorem 5.2. So it comes down to bound $\|U(t)\|_X$ and we saw that the only difficulty was $\|N_F(U)\|_{L^p([0,t],Y)}$. Suppose that the mild solution $U(t; \phi)$ for initial condition ϕ has finite time interval $[0, \tau_\phi)$ of maximal existence. Let $M' = \max_{t\in[0,\tau_\phi]} M(t)$. Since $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous there exists a constant L depending on τ_ϕ and M' such that

$$|f(t, x) - f(t, x')| \leq L|x - x'|$$

for all $x, x' \in \mathbb{R}^n$ such that $|x| \leq M'$ and $|x'| \leq M'$ and for almost all $t \in [0, \tau_\phi]$. From this it follows that for all these bounded $x \in \mathbb{R}^n$ and for almost all $t \in [0, \tau_\phi]$

$$|f(t,x)| \leq |f(t,0)| + L|x| = L|x|.$$

Consequently, using (5.3) and (5.4),

$$|F(t,U(t))(x)| = |f(t,U(t)(x))| \leq L|U(t)(x)|,$$

for almost all $t \in [o, \tau_\phi$ and for all $x \in \mathbb{R}^n$, because we assumed that $U(t) \in L^\infty(\mathbb{R}^n)$ and $\|U(t)\|_\infty \leq M'$ for almost all $t \in [0, \tau_\phi]$. Because Y is the finite intersection of L^p-spaces it now also follows that

$$\|F(t,U(t))\|_Y \leq L\|U(t)\|_Y,$$

for almost all $t \in [0, \tau_\phi]$. Since X is a continuously embedded subspace of Y there exists a constant $C > 0$ such that for all $x \in X$

$$\|x\|_Y \leq C\|x\|_X.$$

So it follows that

$$\|F(t,U(t))\|_Y \leq LC\|U(t)\|_X.$$

Now we can put a bound on $\|N_F(U)\|_{L^p([0,t],Y)}$, for all $0 \leq t < \tau_\phi$:

$$\|N_F(U)\|_{L^p([0,t],Y)} = \left(\int_0^t \|F(s,U(s))\|_Y^p ds\right)^{\frac{1}{p}} \leq LC\left(\int_0^t \|U(s)\|_X^p ds\right)^{\frac{1}{p}}$$

If we put this into inequality (5.2), using that $\|\Psi_{p,t}\|_{Y,X} \leq M$, we find

$$\|U(t)\|_X \leq M_x e^{\omega_x t}\|\phi\|_X + LCM\|U(\cdot)\|_{L^p([0,t],X)}.$$

Then we use Gronwall's inequality for $\|U(t)\|_X^p$, $0 \leq t < \tau_\phi$:

$$\|U(t)\|_X^p \leq \frac{\omega_x p 2^{p-1} M_x^p \|\phi\|_X^p}{\omega_x p - 2^{p-1}L^p C^p M^p} e^{\omega_x p t} - \frac{2^{2p-2} M_x^p \|\phi\|_X^p L^p C^p M^p}{\omega_x p - 2^{p-1}L^p C^p M^p} e^{2^{p-1}L^p C^p M^p t}.$$

Hence no blow-up at $t = \tau_\phi$ can occur, contradicting that $\tau_\phi < \infty$. □

The approach found in [17] is to establish that a solution $U(t) \in L^\infty(\Omega)$ for $t > 0$ and derive an a-priori estimate of the form (5.4) and the apply the result that we formulated as Theorem 5.1. Unfortunately this cannot easily be extended.

5.3 Global existence of positive solutions

In this final section, we use positivity of solutions to prove global existence. We will need that X and Y are Banach lattices such that the partial ordering is compatible with the embedding, i.e. $j : X \hookrightarrow Y$ is a Riesz homomorphism, see Appendix D. We also have to put a restriction on the nonlinearity $F : X \to Y$. The semigroup is assumed to be positive, i.e. $y \geq 0$ implies $T(t)y \geq 0$ for all $t \geq 0$. Before we prove the main theorem of this section we first show a useful relation between integration, the supremum and the partial ordering in a Banach lattice.

Lemma 5.4. *Let X a Banach lattice and let $\varphi : [0,t] \to X$ be Bochner integrable, then $s \mapsto \varphi(s)^+$ is Bochner integrable for $s \in [0,t]$ and*

$$\left[\int_0^t \varphi(s)ds\right]^+ \leq \int_0^t \varphi(s)^+ ds.$$

Proof. Since $\varphi : [0,t] \to X$ is Bochner integrable, it follows from Theorem $B.7$ that

$$\int_0^t \|\varphi(s)\|_X ds < \infty.$$

For all $s \in [0,t]$, we have that

$$|\varphi(s)^+| \leq |\varphi(s)^+| + |\varphi(s)^-| = |\varphi(s)|.$$

Hence, since X is a Banach lattice,

$$\|\varphi(s)^+\|_X \leq \|\varphi(s)\|_X.$$

Consequently

$$\int_0^t \|\varphi(s)^+\|_X ds \leq \int_0^t \|\varphi(s)\|_X ds < \infty$$

and $\varphi(s)^+$ is Bochner integrable. Then,

$$\int_0^t \varphi(s)ds = \int_0^t \varphi(s)^+ ds - \int_0^t \varphi(s)^- ds. \tag{5.5}$$

The positive cone X^+ is closed in X. Hence both integrals in (5.5) are in X^+ and we conclude

$$\int_0^t \varphi(s)ds \leq \int_0^t \varphi(s)^+ ds.$$

Consequently

$$\left[\int_0^t \varphi(s)ds\right]^+ \leq \left[\int_0^t \varphi(s)^+ ds\right]^+ = \int_0^t \varphi(s)^+ ds.$$

□

We will use this result in the following theorem. By a positive solution $U(t)$ we mean a solution such that $U(t) \geq 0$ on its maximal interval of existence.

Theorem 5.5. *Assume that the seven conditions for Theorem* 3.6 *are satisfied, with $T = \infty$ and $(X, \leq)$ and $(Y, \preceq)$ Banach lattices. Let the dense embedding $j : X \hookrightarrow Y$ be a Riesz homomorphism and identify X with $j(X) \subset Y$. Let $F : X \to Y$ such that there exist $a, b > 0$ such that $\|F^+(x)\|_Y \leq a + b\|x\|_X$ for all $x \in X^+$ and assume that the semigroup $(T(t))_{t\geq 0}$ generated by A is linear, strongly continuous in X and positive. Then any positive mild solution $U(t)$ in X to the Cauchy problem*

$$\begin{aligned} \partial_t U(t) &= AU(t) + F(U(t)),\ t \geq 0, \\ U(0) &= \phi, \end{aligned}$$

exists globally.

Proof. We write $U(t) = U(t)^+ - U(t)^-$. We assume that $U(t) \geq 0$ for all $t \geq 0$, so $U(t)^+ = U(t)$ and $U(t)^- = 0$. A mild solution satisfies the variation of constants formula,

$$U(t) = T(t)\phi + \int_0^t T(t-s)F(U(s))ds.$$

Since $[x + x']^+ \leq x^+ + x'^+$ for all $x, x' \in X$, and $U(t)^+ = U(t)$ it follows that

$$U(t) \leq [T(t)\phi]^+ + \left[\int_0^t T(t-s)F(U(s))ds\right]^+.$$

$T(t)$ is a positive and linear operator, so we find

$$[T(t)\phi]^+ = \left[T(t)[\phi^+ - \phi^-]\right]^+ = [T(t)\phi^+ - T(t)\phi^-]^+ \leq T(t)\phi^+.$$

The inequality follows because $T(t)\phi^+$ and $T(t)\phi^-$ are both in X^+ and we have for $x, x' \in X^+$ that

$$[x - x']^+ \leq x^+ + (-x')^+ = x^+.$$

$\int_0^t T(t-s)F(U(s))ds$ is a Bochner-integral in Y, with values in $j(X)$. So if we do not identify X with its image $j(X)$, in X we have to consider $j^{-1}\int_0^t T(t-s)F(U(s))ds$. If we use Proposition $D.1.1$ we find that

$$\begin{aligned} j\left(\left[j^{-1}\int_0^t T(t-s)F(U(s))ds\right]^+\right) &= \left[jj^{-1}\int_0^t T(t-s)F(U(s))ds\right]^\dagger \\ &= \left[\int_0^t T(t-s)F(U(s))ds\right]^\dagger. \end{aligned}$$

Now we can apply Lemma 5.4 to see that

$$\left[\int_0^t T(t-s)F(U(s))ds\right]^\dagger \preceq \int_0^t \left[T(t-s)F(U(s))\right]^\dagger ds.$$

Consequently,

$$j\left(\left[j^{-1}\int_0^t T(t-s)F(U(s))ds\right]^+\right) \preceq \int_0^t \left[T(t-s)F(U(s))\right]^\dagger ds$$
$$= jj^{-1}\int_0^t \left[T(t-s)F(U(s))\right]^\dagger ds.$$

j is a Riesz homomorphism, so from Proposition $D.1$ we get

$$\left[j^{-1}\int_0^t T(t-s)F(U(s))ds\right]^+ \leq j^{-1}\int_0^t \left[T(t-s)F(U(s))\right]^\dagger ds.$$

Now we identify X again with its image in Y. Thus we have

$$\left[\int_0^t T(t-s)F(U(s))ds\right]^+ \leq \int_0^t \left[T(t-s)F(U(s))\right]^\dagger ds.$$

Because $(T(t))_{t\geq 0}$ is a positive semigroup we see that

$$\left[\int_0^t T(t-s)F(U(s))ds\right]^+ \leq \int_0^t T(t-s)[F(U(s))]^\dagger ds.$$

Now substituting this into the variation of constans formula, we find

$$U(t) \leq T(t)\phi^+ + \int_0^t T(t-s)[F(U(s))]^\dagger ds.$$

X is a Banach lattice and all terms are in X^+, so it follows that

$$\|U(t)\|_X \leq \|T(t)\|_{\mathcal{L}(X)}\|\phi^+\|_X + \|\Psi_{p,t}\|_{Y,X}\|[F(U(s))]^\dagger\|_{L^p([0,t],Y)}.$$

Because $\|F^+(x)\|_Y \leq a + b\|x\|_X$ for all $x \in X^+$, it follows that

$$\|U(t)\|_X \leq \|T(t)\|_{\mathcal{L}(X)}\|\phi^+\|_X + \|\Psi_{p,t}\|_{Y,X}\left(at + b^p\|U\|^p_{L^p([0,t],X)}\right)^{1/p}.$$

Finally Gronwall's Inequality bounds $\|U(t)\|^p_X$ for all $t \geq 0$, using that $\|\Psi_{p,t}\|_{Y,X} \leq M$, and the inequality : $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. $\square$

If we apply this result to the FitzHugh-Nagumo equations we get the following result.

Corollary 5.6. *Let X and Y be defined as in Theorem* 4.4. *Then any mild solution in X to the FitzHugh-Nagumo equations that remains positive exists globally.*

Proof. We will check the conditions

1. Every L^p-space is a Banach lattice: Let $\varphi, \psi \in L^p(\mathbb{R}^n)$ and assume $|\varphi| \leq |\psi|$, i.e. $|\varphi(x)| \leq |\psi(x)|$ for almost all $x \in \mathbb{R}^n$ then

$$\|\varphi\|_p^p = \int_{\mathbb{R}^n} |\varphi(x)|^p dx \leq \int_{\mathbb{R}^n} |\psi(x)|^p dx = \|\psi\|_p^p.$$

   It follows that also every finite intersection of L^p-spaces is a Banach lattice. So X_1, X_2, Y_1 and Y_2 are Banach lattices. Now $X = X_1 \bigoplus X_2$ becomes a Banach lattice if we define $X^+ = X_1^+ \bigoplus X_2^+$ and for $x, x' \in X$ $x \leq x'$ if and only if $x_1 \leq x_1'$ and $x_2 \leq x_2'$, with $x = x_1 \oplus x_2$ and $x' = x_1' \oplus x_2'$. And $\|x\|_X = \|x_1\|_{X_1} + \|x_2\|_{X_2}$.

2. The map $F : X \to Y$ is defined by

$$F(x_1 \oplus x_2) = (f(x_1) + c_1 x_2, c_2 x_1 + c_3 x_2),$$

   with $f(x) = a_0 x(x - a_1)(x - a_2)$. The cubic polynomial is of course the only part that needs our attention. It is not hard to see that there exists a constant $C_f > 0$ such that for all $x \geq 0$, $f^+(x) \leq C_f x$. Obviously $k(x) = C_f x$ is the tangentline in $a_1 \leq x \leq a_2$ such that $k(0) = 0$. So we have to solve the equation $f'(x)x = f(x)$. We find $x = \frac{x_1+x_2}{2}$. Now we put $C_f = f'(\frac{x_1+x_2}{2})$. It follows that there exist $a, b > 0$ such that $\|F^+(x)\|_Y \leq a + b\|x\|_Y$ for all $x \in X^+$.

3. We have to check that the operator $T(t) = T_a(t) \oplus I$ is positive on X and Y. So we show it for $L^p(\mathbb{R}^n)$. Clearly I is a positive operator, so it suffices to show that $T_a(t)$ is a positive operator. $E_a(t, x) > 0$ for all $t > 0$ and for all $x \in \mathbb{R}^n$. Let $\varphi \in L^p(\mathbb{R}^n)^+$, i.e. $\varphi(x) \geq 0$ for almost all $x \in \mathbb{R}^n$, then for all $y \in \mathbb{R}^n$ we have that $E_a(t, x - y)\varphi(x) \geq 0$ for almost all $x \in \mathbb{R}^n$. It then follows that

$$(T_a(t)\varphi)(x) = \int_{\mathbb{R}^n} E_a(t, x - y)\varphi(y) dy \geq 0,$$

   for almost all $x \in \mathbb{R}^n$. So also $T_a(t)\varphi \in L^p(\mathbb{R}^n)^+$.

4. Finally, the natural embedding of these L^p-spaces is clearly a Riesz homomorphism. This concludes the proof.

$\square$

Chapter 6

Final thoughts

The last year of my study in mathematics I spent working on this thesis. Finally it is finished. Although, it is sort of open ended, since we were not able to prove global existence. However, in mathematics you should never expect to complete a research. There is always more to do, more to examine, more to discover. For me, though, it is the final step in completing my master in mathematics. So, for further research, maybe someone else can continue where I stopped. To conclude, we list four suggestions for further research.

- Global existence of solutions of the FitzHugh-Nagumo equations on an unbounded domain has not been proven, yet. The method that we have described, using the positivity of a solution, might be used as a starting point. Maybe it is possible to use a similar approach for non-positive solutions.
- An other unanswered question is whether the presented theory also holds for Fréchet spaces. In Chapter 2 we have found the heat semigroup on $\mathbb{S}(\mathbb{R}^n)$. Furthermore, we can define a generator, in view of Lemma 2.4. Is it possible to extend the results in Chapter 3 to this space?
- The domain of the infinitesimal generator Δ of the heat semigroup in $L^1(\mathbb{R}^n)$ is unknown. This is probably not an easy problem to solve. However, since $L^1(\mathbb{R}^n)$ is one of those spaces that we are naturally interested in, it might be worth solving this problem.
- Finally, the projection that FitzHugh used to simplify the Hodgkin-Huxley equations. How does it work from a functional analytic point of view and what is the relation between the solutions of the FitzHugh-Nagumo equations and those of the Hodgkin-Huxley equations?

Appendix A

Preliminaries for Chapter 2

Semigroups are used throughout this thesis. So in this appendix we will give a definition of and state some results on semigroups. Some prior knowledge on distributions is also needed to fully understand Chapter 2. Furthermore, Young's convolution inequality is often used in this chapter.

Lemma A.1 (Young's convolution inequality). *Let $\varphi \in L^p(\mathbb{R}^n)$ and $\psi \in L^q(\mathbb{R}^n)$ then*

$$\|\varphi * \psi\|_r \leq \|\varphi\|_p \|\psi\|_q$$

for $p, q, r \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. where

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)dy$$

A.1 Semigroups

All statements in this section are proven in [4]. Let's start with the definition of a semigroup.

Definition A.2. *A semigroup $(T(t))_{t\geq 0}$ on a metric space (X, d) is a family of operators on X that satisfy the following properties*

1. *$T(t) : X \to X$ is continuous with respect to d for all $t \geq 0$.*
2. *$T(0) = I$.*
3. *$T(t)T(s) = T(t+s)$ for all $t, s \geq 0$.*

It is called a *strongly continuous semigroup*, or C_0-semigroup if for all $x \in X$ $t \mapsto T(t)x$ is continuous for $t \geq 0$. Note that, using Property 3, continuity in $t = 0$ is sufficient for strong continuity.
A Banach space X is a complete metric space with respect to the metric $d(x, y) = \|x - y\|_X$. If $(T(t))_{t\geq 0}$ is a linear semigroup on a Banach space X,

i.e. the operators $T(t)$ are linear, then continuity, Property 1, is equivalent to boundedness. To a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X one may associate an operator A_T, the *generator* of the semigroup, defined by the following limit:

$$A_T\varphi = \lim_{t\to 0} \frac{T(t)\varphi - \varphi}{t}$$

This generator is a densely defined closed operator on X and its domain, $D(A_T)$, consits of all functions $\varphi \in X$ such that the above limit exists. So, loosely speaking, the generator is the derivative of the semigroup in $t = 0$. A similar result holds for $t > 0$,

$$\frac{d}{d\tau} T(\tau)|_{\tau=t}\varphi = T(t)A_T\varphi \text{ for all } \varphi \in D(A_T) \text{ and } t \geq 0.$$

Furthermore, the domain $D(A_T)$ is invariant under the semigroup and the generator and the semigroup commute:

$$A_T T(t)\varphi = T(t)A_T\varphi \text{ for all } \varphi \in D(A_T) \text{ and } t \geq 0.$$

A subspace D of the domain $D(A)$ of a linear operator $A : D(A) \subset X \to X$ is called a *core* for A if D is dense in $D(A)$ with respect to the graph norm: $\|x\|_A = \|x\|_X + \|Ax\|_X$.
Two more results on semigroups that are used in this thesis are the following propositions.

Proposition A.3. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X. A subspace D of $D(A)$ that is dense in X and invariant under the semigroup is a core for A.*

Proposition A.4. *For every strongly continuous semigroup $(T(t))_{t\geq 0}$, there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$$

for all $t \geq 0$.

A.2 Distributions

This section is based on [8].
For an open subset Ω of $\mathbb{R}^n$, we define

$$C_c^\infty(\Omega) = \bigcup_{K\subset\Omega \text{ compact}} C_c^\infty(K)$$

For a compact subset K of $\mathbb{R}^n$, $C_c^\infty(K)$ is a Fréchet space. The topology is defined by the norms:

$$\|\varphi\|_\alpha = \sup_{x\in K} |D^\alpha\varphi(x)|,$$

for $\alpha \in \mathbb{N}_0^n$. Hence a sequence $\{\varphi_k\}_{k=0}^\infty$ converges in $C_c^\infty(K)$ to φ if and only if $\{D^\alpha \varphi_k\}_{k=0}^\infty$ converges uniformly to $D^\alpha \varphi$. We say that $\{\varphi_k\}_{k=0}^\infty$ converges in $C_c^\infty(\Omega)$ to φ if there exists a compact $K \subset \Omega$ such that $\{\varphi_k\}_{k=0}^\infty \subset K$ and the sequence converges to φ in $C_c^\infty(K)$. A linear functional $T : C_c^\infty(\Omega) \to \mathbb{R}$ is called continuous if for each compact $K \subset \Omega$, $T|_{C_c^\infty(K)}$ is continuous, i.e. if $\varphi_k \to \varphi$ in $C_c^\infty(K)$, then $T\varphi_k \to T\varphi$ in $\mathbb{R}$. Now we can define distributions:

Definition A.5. *Let $\Omega \subset \mathbb{R}^n$ open. A distribution on Ω is a continuous linear functional on $C_c^\infty(\Omega)$.*

Schwartz introduced the notation: $\mathcal{D}(\Omega)$ for $C_c^\infty(\Omega)$, $\mathcal{D}'(\Omega)$ for the distributions on Ω, $\mathcal{D}$ for $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{D}'$ for the distributions on $\mathbb{R}^n$. If $F \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, then we write $\langle F, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$ for the value of F in φ. Every $\varphi \in L^p(\Omega)$, $1 \le p \le \infty$ defines a distribution on Ω, namely, the functional T_φ. For every $\psi \in \mathcal{D}(\Omega)$ it is defined by

$$\langle T_\varphi, \psi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} := \langle \varphi, \psi \rangle_{p,q,\Omega},$$

where $\langle \cdot, \cdot \rangle_{p,q,\Omega}$ is defined for $\varphi \in L^p(\mathbb{R}^n)$ and $\psi \in L^q(\mathbb{R}^n)$ with $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ by

$$\langle \varphi, \psi \rangle_{p,q,\Omega} := \int_\Omega \varphi(x)\psi(x)dx.$$

We write $\langle \cdot, \cdot \rangle_{p,q}$ for $\langle \cdot, \cdot \rangle_{p,q,\mathbb{R}^n}$. We call two functions $\varphi, \psi \in L^p(\Omega)$ the same in the sense of distributions if

$$\langle T_\varphi, \chi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle T_\psi, \chi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \text{ for all } \chi \in \mathcal{D}(\Omega).$$

The power of distribution theory is the possibility to define derivatives of a function $\varphi \in L^p(\Omega)$, even when it is not differentiable in the classical sense. If $\varphi \in C^{|\alpha|}(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$ integration by parts yields the following:

$$\int_\Omega D^\alpha \varphi(x)\psi(x)dx = (-1)^{|\alpha|} \int_\Omega \varphi(x) D^\alpha \psi(x)dx$$

So for a distribution $F \in \mathcal{D}'(\Omega)$ we define the derivative $D^\alpha F$ as

$$\langle D^\alpha F, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = (-1)^{|\alpha|} \langle F, D^\alpha \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$$

for all $\varphi \in \mathcal{D}(\Omega)$. Using the last two definitions we come to the following. Let $1 \le p \le \infty$. If $\varphi \in L^p(\Omega)$, then $D^\alpha \varphi \in L^p(\Omega)$ (in the sense of distributions) means that there exists a $\psi \in L^p(\Omega)$ such that

$$\langle T_\psi, \chi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle T_\varphi, D^\alpha \chi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \text{ for all } \chi \in \mathcal{D}(\Omega)$$

If the derivative exists then (in the sense of distributions) $D^\alpha \varphi = \psi$ and the distributions $D^\alpha T_\varphi, T_{D^\alpha \varphi}$ and T_ψ are the same.

Appendix B

Preliminaries for Chapter 3

In Chapter 3 we integrate vector valued functions. In this appendix we will give a precise definition and prove some results that we use in Chapter 3. For Theorem 3.6 we need generalized Carathéodory functions and Nemitsii mappings, so we will introduce those. Since we use Gronwall's lemma in this theorem we will prove it here.

B.1 Bochner integration of vector valued functions

This section is based on [2]. The following definitions are needed for measure theory:

Definition B.1. *Let (Ω, Σ, μ) a finite measure space and X a Banach space.*

1. *A function $\varphi : \Omega \to X$ is called simple if there exist $x_1, x_2, \ldots, x_n \in X$ and $E_1, E_2, \ldots, E_n \in \Sigma$ such that $\varphi = \sum_{i=1}^{n} x_i \chi_{E_i}$.*

2. *A function $\varphi : \Omega \to X$ is called μ-measurable if there exist a sequence of simple functions $\{\varphi_n\}_{n=1}^{\infty}$ such that*
$$\lim_{n\to\infty} \|\varphi_n(\omega) - \varphi(\omega)\|_X = 0,$$
for μ-almost every $\omega \in \Omega$.

3. *A function $\varphi : \Omega \to X$ is called weakly μ-measurable if for each $x^* \in X^*$ the function $x^*\varphi : \Omega \to \mathbb{R}$ is μ-measurable.*

It is easily verified that the finite sum of measurable functions is measurable and that the pointwise (almost everywhere) limit of measurable functions is measurable. Pettis' Measurability Theorem provides a different criterium for measurability relating it to separably valued functions. A function $\varphi : \Omega \to X$ is called *μ-essentially separably valued* if there exists $E \in \Sigma$ such that $\mu(E) = 0$ and $\varphi(\Omega \backslash E)$ is a norm-separable subset of X.

Theorem B.2 (Pettis' Measurability Theorem). *A function $\varphi : \Omega \to X$ is μ-measurable if and only if*

1. *φ is μ-essentially separably valued and*

2. *φ is weakly μ-measurable*

A proof can be found in [2]. The following lemma shows that a continuous map from a σ-compact space with a Borel measure to a Banach space is measurable. We use Pettis' Measurability Theorem to prove it.

Lemma B.3. *Let Ω be a σ-compact space with Borel measure μ and let X a Banach space. If $\varphi : \Omega \to X$ is continuous, then it is μ-measurable.*

Proof. First we show that φ is μ-essentially separably valued.
Ω is σ-compact, so $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, with Ω_i compact $\forall i$. Thus

$$\varphi(\Omega) = \varphi(\bigcup_{i=1}^{\infty} \Omega_i) = \bigcup_{i=1}^{\infty} \varphi(\Omega_i)$$

Since φ is a continuous function, $\varphi(\Omega_i)$ is compact for all i, so for every open cover

$$\varphi(\Omega_i) \subset \bigcup_{x_i \in \varphi(\Omega_i)} B(x_i, \varepsilon), \text{ with } \varepsilon \geq 0$$

there exists a finite number n_ε, such that

$$\varphi(\Omega_i) \subset \bigcup_{j=1}^{n_\varepsilon} B(x_{i,\varepsilon,j}, \varepsilon).$$

Now let $\varepsilon = \frac{1}{k}$, with $k \in \mathbb{N}$ and define for all i

$$V_i = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{n_{\frac{1}{k}}} x_{i,\frac{1}{k},j}$$

then

$$\varphi(\Omega_i) \subset \overline{V_i}$$

So if we define

$$V = \bigcup_{i=1}^{\infty} V_i$$

then

$$\varphi(\Omega) \subset \overline{V},$$

i.e. φ is μ-essentially separably valued.
What is left to prove is that φ is weakly μ-measurable. $\varphi : \Omega \to X$ is continuous, so $x^*\varphi : \Omega \to \mathbb{R}$ is continuous $\forall x^* \in X^*$. This implies that $x^*\varphi : \Omega \to \mathbb{R}$ is Borel-measurable for all $x^* \in X^*$, and thus $\varphi : \Omega \to X$ is weakly μ-measurable. $\square$

In Chapter 3 we integrate the vector valued function $s \to T(t-s)\varphi(s)$. It follows from Lemma $B.4$ that this function is indeed measurable, since $(T(t))_{t\geq 0}$ is a strongly continuous semigroup and thus *strongly measurable*, i.e. a map T from a measure space, (Ω, μ), to $\mathcal{L}(X)$, such that for all $x \in X$ the map $\omega \mapsto T(\omega)x$ is μ-measurable from Ω to X. Note that $Y = \mathcal{L}(X)$ is a Banach space with respect to the operator norm. μ-measurability of $T : \Omega \to Y$ is usually called uniform μ-measurability. In the following lemma we will only need strong measurability of T.

Lemma B.4. *Let (Ω, μ) a measure space and X a Banach space. If $\varphi : \Omega \to X$ is μ-measurable and $T : \Omega \to \mathcal{L}(X)$ is strongly μ-measurable, then*

$$\Omega \to X : \ \omega \mapsto T(\omega)\varphi(\omega)$$

is μ-measurable.

To prove this lemma we first show that

Lemma B.5. *If $\varphi : \Omega \to X$ is μ-measurable and $\psi : \Omega \to \mathbb{R}$ is measurable, then $\varphi\psi : \Omega \to X$ is μ-measurable.*

Proof. There are simple functions φ_n, ψ_n $n \in \mathbb{N}$ and subsets $A, A' \in \Omega$ with $\mu(A) = \mu(A') = 0$ such that

$$\begin{aligned} \lim_{n\to\infty} \varphi_n(\omega) = \varphi(\omega) \qquad & \forall \omega \in \Omega \setminus A \text{ and} \\ \lim_{n\to\infty} \psi_n(\omega) = \psi(\omega) \qquad & \forall \omega \in \Omega \setminus A'. \end{aligned}$$

Let $\omega \in \Omega \setminus (A \cup A')$, then

$$\|\varphi_n\psi_n(\omega) - \varphi\psi(\omega)\|_X \leq \|\varphi_n(\omega)\|_X |\psi_n(\omega) - \psi(\omega)| + \|\varphi_n(\omega) - \varphi(\omega)\|_X |\psi(\omega)| \to 0$$

The functions $\varphi_n\psi_n, (n \in \mathbb{N})$ are simple functions, $\mu(A \cup A') = 0$ and we have seen that $\lim_{n\to\infty}(\varphi_n\psi_n)(\omega) = (\varphi\psi)(\omega)$ for all $\omega \in \Omega \setminus (A \cup A')$, so $\varphi\psi$ is μ-measurable. □

Now we can prove Lemma $B.4$

Proof. Since φ is μ-measurable there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of simple functions such that $\lim_{n\to\infty} \|\varphi_n(\omega) - \varphi(\omega)\|_X = 0$, μ almost everywhere. So

$$T(\omega)\varphi(\omega) = \lim_{n\to\infty} T(\omega)\varphi_n(\omega)$$

μ almost everywhere. If we write

$$\varphi_n(\omega) = \sum_{j=1}^{N_n} x_{n,j}\chi_{E_{n,j}}(\omega)$$

then

$$T(\omega)\varphi_n(\omega) = \sum_{j=1}^{N_n} (T(\omega)x_{n,j})\chi_{E_{n,j}}(\omega)$$

The map $\Omega \to X : \omega \mapsto T(\omega)x_{n,j}$ is μ-measurable and the map $\Omega \to \mathbb{R} : \omega \mapsto \chi_{E_{n,j}}(\omega)$ is measurable, so according to Lemma $B.5$

$$\Omega \to X : \omega \mapsto (T(\omega)x_{n,j})\chi_{E_{n,j}}(\omega).$$

is measurable. The finite sum of measurable functions is measurable and the pointwise limit of measurable functions is measurable, so $\Omega \to X : \omega \mapsto T(\omega)\varphi(\omega)$ is μ-measurable. □

We can integrate measurable vector valued functions.

Definition B.6. *Let (Ω, Σ, μ) a finite measure space and X a Banach space. A μ-measurable function $f : \Omega \to X$ is called Bochner integrable if there exists a sequence of simple functions $\{f_k\}_{k=0}^{\infty}$ such that*

$$\lim_{k\to\infty} \int_\Omega \|f_k - f\|_X d\mu = 0.$$

Then for $E \in \Sigma$ the integral is defined by

$$\int_E f d\mu = \lim_{k\to\infty} \int_E f_k d\mu.$$

From [2] we have the following result

Theorem B.7. *A μ-measurable function $f : \Omega \to X$ is Bochner integrable if and only if*

$$\int_\Omega \|f\|_X d\mu < \infty.$$

A final result on vector valued functions that we need is

Theorem B.8. *Let T be a closed linear operator with domain in X and having values in a Banach space Y. If φ and $T\varphi$ are Bochner integrable with respect to μ, then for all $E \in \Sigma$,*

$$T\left(\int_E \varphi d\mu\right) = \int_E T\varphi d\mu.$$

For a proof we refer to [2].

B.2 Gronwall

The following lemma is a general form of Gronwall's Lemma.

Lemma B.9 (Gronwall). *Let A, B, C and D be real valued continuous functions on $[a, b]$, such that $D(x) \geq 0$ for all $x \in [a, b]$. If*

$$A(x) \leq B(x) + C(x) \int_a^x D(y)A(y)dy \text{ for all } x \in [a, b], \tag{B.1}$$

then also

$$A(x) \leq B(x) + C(x) \int_a^x e^{\int_y^x C(z)D(z)dz} B(y)D(y)dy \text{ for all } x \in [a, b].$$

The proof uses the following lemma

Lemma B.10. *Let A and B be real valued continuous functions, then the differential equation*

$$\begin{cases} \frac{dY}{dx}(x) = A(x)Y(x) + B(x), \\ Y(x_0) = Y_0 \end{cases} \tag{B.2}$$

has a unique solution

$$Y(x) = Y_0 e^{\int_{x_0}^x A(y)dy} + \int_{x_0}^x e^{\int_y^x A(z)dz} B(y)dy.$$

A proof can be found in every elementary book on differential equations.

Proof. (Gronwall) Define for $a \leq x \leq b$,

$$E(x) = \int_a^x D(y)A(y)dy$$

and

$$F(x) = A(x) - C(x)E(x).$$

It follows from $B.1$ that $F(x) \leq B(x)$ for all $x \in [a, b]$. Since A and D are continuous, E is continuously differentiable and we can deduce the following differential equation for E:

$$\begin{cases} \frac{dE}{dx}(x) = D(x)A(x) = C(x)D(x)E(x) + D(x)F(x), \\ E(a) = 0 \end{cases} \tag{B.3}$$

Lemma $B.10$ now shows that we can write

$$E(x) = \int_a^x e^{\int_y^x C(z)D(z)dz} D(y)F(y)dy.$$

The result follows if we use $F(x) \le B(x)$ and $D(x) \ge 0$.

$$\begin{aligned}
A(x) &\le B(x) + C(x)E(x) \\
&= B(x) + C(x) \int_a^x e^{\int_y^x C(z)D(z)dz} D(y)F(y)dy \\
&\le B(x) + C(x) \int_a^x e^{\int_y^x C(z)D(z)dz} B(y)D(y)dy.
\end{aligned}$$

□

B.3 Generalized Carathéodory functions and Nemytskii mappings

Let (Ω, μ) a measure space and X and Y Banach spaces. Follwing [10], we define a *generalized Carathéodory function* as a function $F : \Omega \times X \to Y$ such that

1. For each fixed $x \in X$ the function $F(\cdot, x) : \Omega \to Y$ is μ-measurable, and
2. For almost all fixed $\omega \in \Omega$ the function $F(\omega, \cdot) : X \to Y$ is continuous.

Denote by $M(\Omega, X)$ the equivalence classes of μ-measurable functions from Ω to X. With a function $\varphi \in M(\Omega, X)$ we can define an outer superposition mapping. The mapping $N_F : M(\Omega, X) \to M(\Omega, Y)$, with

$$N_F(\varphi)(\omega) := F(\omega, \varphi(\omega))$$

is called a *Nemytskii mapping*.
A Nemytskii mapping maps measurable functions to measurable functions.

Lemma B.11. *If $F : \Omega \times X \to Y$ is a generalized Carathéodory function and $\varphi : \Omega \to X$ is μ-measurable, then $\omega \mapsto N_F(\varphi)(\omega) : \Omega \to Y$ is μ-measurable*

A proof can be found in [10] and for Carathéodory functions, i.e. Ω is an open subset of $\mathbb{R}^n$ and $X = Y = \mathbb{R}$, in [5].

Appendix C

Estimates on $(T_a(t))_{t\geq 0}$

This appendix is based on [10]. In this appendix we will show which L^p-spaces we can choose such that Conditions 3 and 4 are satisfied for the FitzHugh-Nagumo equations. In Proposition $C.2$ we will prove that $T_a(t), t > 0$ is a bounded operator from $L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ if $1 \leq q \leq r \leq \infty$. We will use this result in Proposition $C.3$ to find conditions on p, q and r such that $\Psi_{p,t}(L^p([0,t], L^q(\mathbb{R}^n))) \subset L^r(\mathbb{R}^n)$ and $\|\Psi_{p,t}\|_{\mathcal{L}(L^q,L^r)}$ is bounded for $t \in (0,T]$. Recall that for $t > 0$

$$E_a(t) : x \mapsto E_a(t,x) = (4\pi a t)^{-n/2} e^{-|x|^2/(4at)}. \tag{C.1}$$

The following proposition gives a bound for $\|E_a^{(n)}(t)\|_s$, that we of course will use in Proposition $C.2$. To avoid confusion we write $E_a^{(n)}(t)$ if $x \in \mathbb{R}^n$.

Proposition C.1. *Let $1 \leq s \leq \infty$, then there exists a constant $C = C(n,a,s)$, such that*

$$\|E_a^{(n)}(t)\|_s \leq C t^{\frac{n}{2}(\frac{1}{s}-1)}.$$

Proof. First, we prove the proposition for $L^s(\mathbb{R})$, $1 \leq s < \infty$.

$$\|E_a^{(1)}(t)\|_s^s = \int_{\mathbb{R}} (4\pi a t)^{-s/2} e^{-sx^2/(4at)} dx = (4\pi a t)^{(1-s)/2} s^{-1/2}$$

The result follows with $C(1,a,s) = (4\pi a)^{\frac{1}{2}(\frac{1}{s}-1)} s^{-\frac{1}{2s}}$. By using the case $n = 1$ we obtain

$$\|E_a^{(n)}(t)\|_s \leq \prod_{i=1}^{n} \|E_a^{(1)}(t)\|_s = C(n,a,s) t^{\frac{n}{2}(\frac{1}{s}-1)},$$

where $C(n,a,s) = C(1,a,s)^n = (4\pi a)^{\frac{n}{2}(\frac{1}{s}-1)} s^{-\frac{n}{2s}}$. For $L^\infty(\mathbb{R}^n)$ the proof is trivial and $C(n,a,\infty) = (4\pi a)^{-\frac{n}{2}}$. $\square$

Proposition C.2. *Let $1 \leq q \leq r \leq \infty$ and $t > 0$, then $T_a(t)$ maps $L^q(\mathbb{R}^n)$ continuously into $L^r(\mathbb{R}^n)$ and there exists a $C = C(n, a, q, r)$ such that*

$$\|T_a(t)\|_{\mathcal{L}(L^q,L^r)} \leq Ct^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}.$$

Proof. If $\varphi \in L^q(\mathbb{R}^n), 1 \leq q \leq \infty$ and $r \geq q$ then there exists a $1 \leq s \leq \infty$ such that $\frac{1}{s} + \frac{1}{q} = 1 + \frac{1}{r}$. So by Young's Convolution Inequality (Lemma A.1), and Proposition C.1

$$\|T_a(t)\varphi\|_r = \|E_a(t) * \varphi\|_r \leq \|E_a(t)\|_s\|\varphi\|_q \leq C(n, a, s)t^{\frac{n}{2}(\frac{1}{s}-1)}\|\varphi\|_q.$$

It follows that $T_a(t)\varphi \in L^r(\mathbb{R}^n)$ and since $\frac{1}{s} + \frac{1}{q} = 1 + \frac{1}{r}$ we find

$$\|T_a(t)\|_{L^q,L^r} \leq C(n, a, q, r)t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}$$

where $C(n, a, q, r) = (4\pi a)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}(1 + \frac{1}{r} - \frac{1}{q})^{\frac{n}{2}(1+\frac{1}{r}-\frac{1}{q})}$. The map is linear and bounded, hence continuous. □

Proposition C.3 (Proposition 4.1). *Let $t > 0$, $1 \leq q < \infty$ and $1 \leq p, r \leq \infty$ such that*

$$q \leq r,\ \frac{1}{r} > \frac{1}{q} - \frac{2}{n} \text{ and } p > [1 + \frac{n}{2}(\frac{1}{r} - \frac{1}{q})]^{-1},$$

then

$$\Psi_{p,t}(L^p([0, t], L^q(\mathbb{R}^n))) \subset L^r(\mathbb{R}^n)$$

and there exists a constant $C = C(n, a, q, r)$, such that

$$\|\Psi_{p,t}\|_{\mathcal{L}(L^q,L^r)} \leq Ct^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})+1-\frac{1}{p}}$$

We will use Hölder's inequality to prove this result.

Lemma C.4 (Hölder's Inequality). *If $p, q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi \in L^p(\mathbb{R}^n)$ and $\psi \in L^q(\mathbb{R}^n)$, then $\varphi\psi \in L^1(\mathbb{R}^n)$ and*

$$\|\varphi\psi\|_1 \leq \|\varphi\|_p\|\psi\|_q.$$

Proof. [Proposition C.3] Let $\varphi \in L^p([0, t], L^q(\mathbb{R}^n))$, then, by Proposition C.2,

$$\|\Psi_{p,t}\varphi\|_r \leq \int_0^t \|T_a(t - s)\varphi(s)\|_r ds \leq \int_0^t C(n, a, q, r)(t - s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|\varphi(s)\|_q ds$$

Since $\varphi \in L^p([0, t], L^q(\mathbb{R}^n))$, we can use Hölder's inequality if $C(n, a, q, r)(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \in L^{p'}([0, t])$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Note that $p' = \frac{p}{p-1}$. Therefore we need

$$\frac{n}{2}(\frac{1}{r} - \frac{1}{q})(\frac{p}{p - 1}) > -1.$$

This inequality implies

$$1+\frac{n}{2}(\frac{1}{r}-\frac{1}{q})>1-(1-\frac{1}{p})=\frac{1}{p}\geq 0, \tag{C.2}$$

so the conditions under which we can use Hölder's inequality are

$$\begin{aligned}&\frac{1}{r}>\frac{1}{q}-\frac{2}{n} \text{ and}\\ &p>[1+\frac{n}{2}(\frac{1}{r}-\frac{1}{q})]^{-1}.\end{aligned}$$

Now we use Hölders inequality and find

$$\begin{aligned}\|\Psi_{p,t}\varphi\|_r &\leq \|C(n,a,q,r)(t-s)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|_{L^{(p/(p-1))}([0,t])}\|\varphi\|_{L^p([0,t],L^q(\mathbb{R}^n))}\\ &= C(n,a,q,r)t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})+1-\frac{1}{p}}\|\varphi\|_{L^p([0,t],L^q(\mathbb{R}^n))}.\end{aligned}$$

Note that $C=C(n,a,q,r)$ does not depend on p. This concludes the proof. □

Appendix D

Preliminaries for Chapter 5: Riesz spaces

In Section 5.3 the Banach spaces X and Y are assumed to be Banach lattices. In Banach lattices there exists a partial ordering. So we can define a positive solution, i.e. $U(t) \geq 0$ for all $t \geq 0$. Then we can prove global existence. In this appendix we will introduce Riesz spaces and Banach lattices and prove a useful result for Riesz homomorphisms. Therefore we first need to define a lattice and an ordered vectorspace.
A *lattice* L is a partially ordered set in which any two elements have a supremum and an infimum. We write $\varphi \vee \psi$ for the supremum of φ and ψ and $\varphi \wedge \psi$ for their infimum. A real linear vectors pace L is called an *ordered vectors pace* if L is partially ordered such that

- if $\varphi, \psi \in L$ then $\varphi \leq \psi$ implies $\varphi + h \leq \psi + h$ for all $h \in L$, and
- $\varphi \geq 0$ implies $af \geq 0$ for all $a \in \mathbb{R}_+$.

A real linear vector space is called a *Riesz space* if it is a lattice and an ordered vector space.
In a Riesz space we define $\varphi^+ = \varphi \vee 0$ and $\varphi^- = \varphi \wedge 0$. It follows that $\varphi = \varphi^+ - \varphi^-$ and we define $|\varphi| = \varphi^+ + \varphi^-$. In a Riesz space we can define positivity using the positive cone. The *positive cone* of a Riesz space L is defined by $L^+ = \{l \in L | l \geq 0\}$. An operator T on L is called a *positive operator* if it maps L^+ into itself, i.e. $T(L^+) \subset L^+$. Similarly a map $R : L \to M$, where M is a Riesz space, is called positive, if $R(L^+) \subset M^+$.
A Riesz space can be mapped into an other Riesz space by a Riesz homomorphism: Let $(L, \leq)$ and $(M, \preceq)$ be Riesz spaces. A linear mapping $\pi : L \to M$ is a *Riesz homomorphism* if $\varphi, \psi \in L$ such that $\varphi \wedge \psi = 0$ implies $\pi(\varphi) \wedge \pi(\psi) = 0$. A Riesz homomorphism is always a positive map and the partial ordering is invariant under this map. To be precise:

Proposition D.1. *Let $(L, \leq)$ and $(M, \preceq)$ be Riesz spaces, $\pi : L \to M$ a Riesz homomorphism and let $\varphi, \psi \in L$, then*

1. $\pi(\varphi^+) = \pi(\varphi)^+$,

2. $\pi(L^+) = \pi(L) \cap M^+$ *and*

3. If π *is injective, then* $\varphi \leq \psi$ *if and only if* $\pi(\varphi) \preceq \pi(\psi)$.

Proof. The first part can be found in [15]. Then the second part. Every Riesz homomorphism is positive, see [15], so $\pi(L^+) \subset M^+$. Thus $\pi(L^+) \subset \pi(L) \cap M^+$. Let $m \in \pi(L) \cap M^+$, then there exists a $l \in L$ such that $\pi(l) = m$. Then $\pi(l^+) = \pi(l)^+ = m^+ = m$, since $m \in M^+$. So there exists a $l' = l^+ \in L^+$ such that $\pi(l') = m$. Hence $m \in \pi(L^+)$ and $\pi(L^+) = \pi(L) \cap M^+$. For the third part assume that $\pi(\varphi) \preceq \pi(\psi)$, then $\pi(\psi - \varphi) = \pi(\psi) - \pi(\varphi) \in M^+ \cap \pi(L) = \pi(L^+)$. So $\psi - \varphi \in L^+ + \ker(\pi)$. We assumed that π is injective, so $\psi - \varphi \in L^+$ and thus $\varphi \leq \psi$. The other implication is immediate, see [15]. $\square$

We can define a seminorm on a Riesz space. A *Riesz seminorm* is a map $\rho : L \to \mathbb{R}_+$, such that

- $\rho(0) = 0$, $\rho(\varphi + \psi) \leq \rho(\varphi) + \rho(\psi)$ and $\rho(a\varphi) = |a|\rho(\varphi)$ for all $\varphi, \psi \in L$ and $a \in \mathbb{R}$, and
- If $\varphi, \psi \in L$ then $|\varphi| \leq |\psi|$ implies $\rho(\varphi) \leq \rho(\psi)$.

A norm on L is called a *Riesz norm* if it is a norm and a Riesz seminorm. A *normed Riesz space* is a Riesz space with a Riesz norm. A *Banach lattice* is a norm complete Riesz space. Note that in a Banach lattice the positive cone is closed.

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