Sandpile and anti-sandpile models

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SUBMITTED IN PARTIAL FULFILMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN MATHEMATICS INSTITUTE

Universiteit Leiden

Abstract

We introduce the mixed model of sandpile+anti-sandpile, which is called SA model. In the SA model, we are free to add or remove a particle from a chosen site. Because of the non-abelian property of the toppling operators and anti-toppling operators, the SA model becomes subtle, and the group property existing in the pure sandpile model and anti-sandpile model does not hold in the SA model.

Because of the non-local property of addition operators and anti-addition operators, the processes related to the sandpile, and anti-sandpile are not Feller. The traditional way of constructing the interacting particle processes in infinite volume, e.g., via Hille-Yoshida, does not work in these cases, other ways of construction are necessary. In the construction of sandpile+anti-sandpile process (SA process), we obtain the semigroup of the process for some special configurations and some special functions by series expansion and then using monotonicity of the process, we can extend it to the general case. The SA process shows a new transition phenomenon: it seems that the stationary measure is the result of a "competition" between the generators.

The sandpile model, anti-sandpile model are "self-organized" critical systems. In recent years, this is challenged because the special nature of the dynamics can be considered as an implicit fine tuning that makes sure the system can reach criticality, therefore the "self-organized" critical behavior of these systems can be thought of as a more conventional phase transition between "stabilizable" and "non-stabilizable". We discuss the conditions for a mixed system to reach a stable state both in finite volume and infinite volume.

Keywords: Self-organized Criticality(SOC); sandpile model; anti-sandpile model; SA-model; SA process; Stabilization

Ackwledgements

Firstly, I would like to express my appreciation to my supervisor, Dr. Frank Redig, for many insightful conversations during the development of the master project, and for helpful comments on the text. His great guidance makes sure the success of this master project. His suggestions on writing of a complete academic report are precious and will be helpful to me forever. He is not only my supervisor, but also a good friend.

I thank Dr. Florks Margaretha Spieksma who gave me the first class at Leiden University, for her help during my study as well as her advice on making slides and giving lectures.

I also would like to give my thanks to Dr. Erik van Zwet for his advice to the thesis and to Dr. M.F.E. de Jeu for reading my thesis.

My sincere thanks are given to Martin van der Schans, Erik van Werkhoven, Daniël Worm, Peter Bruin, Arjen Stolk, the people in room 205, the time we spent together is precious to me. Also I appreciate the help of all my friends and classmates at Leiden.

My earnest thanks to my family, who have been an inspiration throughout my life, I would thank all they have done for me.

Haiyan Liu
August 2006

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Chapter 1

Introduction

1.1 Motivation

The study of the sandpile model and the anti-sandpile model are motivated from the study of "self-organized" critical(SOC) phenomena. SOC is a term used in physics to describe a class of systems exhibiting criticality in a dynamically generated, i.e., "spontaneous" way. This means that the system has a dynamics leading to a stationary state which has features of equilibrium systems at the critical point, such as power law decay of correlations.

In standard critical phenomena, there are some control parameters, such as temperature, magnetic field, reproduction rate, etc. When the control parameter takes a special value, the so-called critical value, the behavior of the system changes radically. And when the parameter above or below the critical value, the system shows differently. The standard percolation model is an example of a system exhibiting a critical phenomenon.

In this model, there is a bond between two neighboring vertices of $\mathbb{Z}^2$, all of which are independently open with probability p and closed with probability $1-p$. The probability for a bond to be open is the control parameter of the percolation system. People are interested in the size of the open path containing a certain vertex for a given p , without loss if generality we can take this vertex to be the origin. The probability that the origin belongs to an infinite cluster is denoted:

$$\theta(p) := \mathcal{P}_p(O \longleftrightarrow \infty)$$

Where $\mathcal{P}_p$ is the product probability measure and $O \longleftrightarrow \infty$ means there is an infinite open path from O. $\theta(p) > 0$ implies the existence of infinite cluster containing "O". It is clear that $\theta(0) = 0$, $\theta(1) = 1$ and $\theta(p)$ is an increasing function. The *critical probability* is defined as:

$$p_c := \inf\{p : \theta(p) > 0\}$$

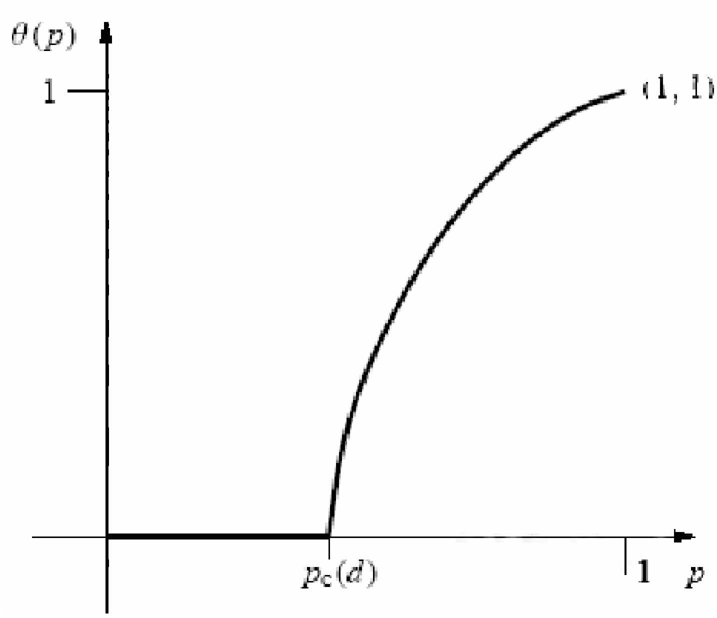

Figure 1.1: Sketch for percolation function

From the definition of p_c, it is not clear whether the percolation function is continuous at the point p_c. It has been shown that $\theta(p_c) = 0$ in dimension 2, see [19] and [10]. So the sketch in Figure 1.1 is known to be valid for the 2 dimensional case. If we let $S(n)$ be the sphere of radius n, and $\mathcal{C}(0)$ denote the cluster containing the origin and let $|\mathcal{C}(0)|$ denote its size. It is proved in [10] that when $d \geq 2$,

1. For $p < p_c$, $\mathcal{P}_p(O \to S(n)) \leq e^{-n\phi(p)}$ for some $\phi(p) > 0$, for all n, see Theorem 5.1.

2. For $p > p_c$, with probability one there is exactly one infinite open cluster, see, Theorem 6.1.

which tells us that when $p < p_c$, the cluster is almost surely of finite size, when $p > p_c$, with positive probability, the origin is in an infinite cluster and at the critical point p_c, we have power law for cluster size distribution, i.e., $\mathcal{P}_{p_c}(|\mathcal{C}(0)| \geq n) \sim \frac{1}{n^\delta}, d \geq 2$, and when $d = 2, \delta = \frac{91}{5}$, see[10]. For the p below and above p_c, the system behaves dramatically differently and on the critical point, the size of the open path containing the initial site decays along power law.

"Self-organized" critical phenomena is caused by driving the system to reach a critical state by their intrinsic dynamics, independently of tuning the value of parameters. In the book "How nature works?", Per Bak gives various kinds of natural phenomena where SOC is present, such as, forest fires, landslides, earthquakes, etc, of which the most canonical one is a sandpile since it is visual and can be studied via both simulations and experiment. A nice computer experiment can be found from in [21]

When playing with sand, we drop grains of sand to the flat ground individually. At the beginning, the pile is flat, each grain just stays at the place where it lands. We can understand the motion of each grain in terms of the physical properties, such as the place, the neighbor around it, the size of it. As the process continues, the slope of the sandpile become steeper and steeper. Eventually, some of the sand slides, which may even span all or most of pile. At the point that the sand slides, the system is far out of balance, and its behavior can no longer be understood in terms of the behavior of the individual grains. So the gradient of the slope is the control parameter of a sandpile and the value at which the sandpile becomes unstable is the critical value. When the gradient of the slope is lower the critical value, all the grains just keep at the place they are landed; while when the gradient of the slope is bigger than the critical value, the pile topples which is the intrinsic force.

1.2 Overview of the later chapters

Besides the sandpile model, the so called anti-sandpile model as well as the mixed model of sandpile and anti-sandpile(in chapter 3) are introduced and studied. The motivation of studying the mixed model comes from the generalization of the Abelian property that exists in the sandpile model and anti-sandpile model. In nature, there are systems that are driven by input of energy(grains), and other ones driven by output of energy(grains). While the models that driven by both input and output of energy(grains) are more general and show a richer spectrum of phenomena

The remainder of the thesis is organized as follows. Chapter 2 gives an overview of the previous research on the sandpile model. Here we will become familiar with the mathematical structure of the model in finite volume.

In Chapter 3, at the beginning a short introduction of the anti-sandpile model is given. One of the important thing is that the sandpile model and the anti-sandpile model are conjugate with each other which will be shown in section 2. The study of the anti-sandpile model becomes easier with the help of the sandpile model. At the last section of this chapter, the combined model of sandpile+anti-sandpile model on finite volume is also introduced and the structures of the dynamics of such model is studied.

Chapter 4 is devoted to infinite volume processes. The purpose of studying of infinite volume systems is to study the universality of behavior in large volumes. For instance, 1 liter of water boils at 100 ^oC, while when the volume increase, such as 2 liters, the water will boil at the same temperature which implies that the model shows universal behavior. In the sandpile model, the size and the diameter of avalanche clusters have a power law behavior in the limit of large volumes, we are interested in how the system behaves in the infinite volume. From the point view of interacting particle system, we should construct the process in infinite volume. The first difficulty contained in the construction of the process is that we can not rely on the standard tools such as Hille-Yoshida or graphical constructions. So in this chapter, I will give the construction of the sandpile+anti-sandpile process and study the ergodic properties of this process.

Chapter 5 will be about the stabilization of height configurations. The sandpile model, anti-sandpile model as well as sandpile+anti-sandpile model are self-organized critical systems while they can also be thought as "organized" critical systems which show the ordinary equilibrium behaviors. The condition for such systems to reach a stable state are discussed.

Chapter 2

Mathematical aspects of the sandpile model

This chapter contains the basic facts of the sandpile model which works as the mathematical background for the whole thesis. In this chapter, the dynamics of the model, recurrent configurations, allowed configurations, etc about the sandpile model are discussed. It will given you general impression of the model.

2.1 Definition of the sandpile model

For a finite subset $\Lambda \subset \mathbb{Z}^d$ with $| \Lambda |< \infty$, we associate to each site $x \in \Lambda$ an integer number $\eta_x \in \mathbb{Z}$ which denotes the heights or the number of grains at that site. $\eta = \{\eta_x\}_{x\in\Lambda}$ forms a height configuration on Λ. We assign to each site x two critical values: $\eta^{\dagger}_{xc}$ and η_{xc}. Site x is stable if $\eta^{\dagger}_{xc} \leq \eta_x < \eta_{xc}$, otherwise site x is unstable. A configuration η is called stable if all the sites are stable. There are two cases for a site x to be unstable, one is $\eta_x < \eta^{\dagger}_{xc}$, the other is $\eta_x \geq \eta_{xc}$. In the sandpile model, we only consider a configuration with $\eta_x \geq \eta^{\dagger}_{xc}$, therefore site x is unstable in the sandpile model implies that $\eta_x \geq \eta_{xc}$. When a site is unstable, the grains will be redistributed in the system. Generally, the redistribution rule contains two aspects, one is how many particles will be redistributed in an unstable site, second how to distribute those particles. Different kinds of redistribution ways decide different kinds of model. For example, in the Manna model, the redistribution rule is stochastic; while the BTW model, there is a deterministic principle to guide the whole system. In this thesis, I mainly focus on the BTW sandpile model.

In the BTW sandpile model, the critical values for different sites are the same, for simplicity, we take $\eta^{\dagger}_{xc} = 0$, $\eta_{xc} = 2d, \forall x \in \Lambda$. If $\eta_x \geq 2d$, site x is unstable in the BTW sandpile model, and 2d particles will be lost from it and each of its neighbors will receive one. For a site x is on the boundary of Λ, it has less than 2d neighbors, the extra particles are allowed to leave the system from the boundary. For example in the following configuration on next page, one grain is lost from the boundary as site of height "4" topples.

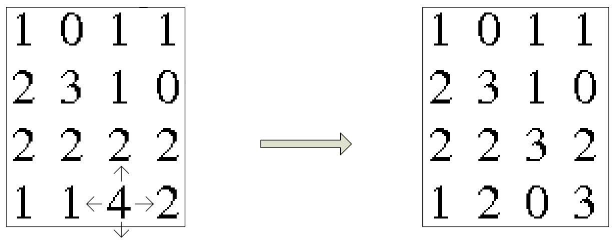

Figure 2.1: Grains leave the system from boundary

In the following, a mathematical definition of the models will be given. A height configuration $\eta = \{\eta_x\}_{x\in\Lambda}$ can be seen as a map from $\Lambda \to \mathbb{Z}$, i.e,

$$\eta : \Lambda \to \mathbb{Z}$$

We use some notations in the model,

- $\mathbb{X} = \mathbb{Z}^\Lambda$, it is the set of all configurations, no matter the heights are positive or not.
- $\mathcal{H} = \{0, 1, 2....\}^\Lambda$ denotes all the non-negative configurations.
- $\Omega_\Lambda = \{0, 1, 2..., 2d-1\}^\Lambda$ is the set of all stable configurations.

And a stable configuration η can be seen as a map from Λ to $\{0, 1, 2, ..., 2d-1\}$:

$$\eta : \Lambda \to \{0, 1, 2, ..., 2d-1\} \tag{2.1.1}$$

Of all the stable states, the "maximal" one is $\eta = \overline{2d-1}$, i.e, the configuration with $2d-1$ particles at every site. We give it a special name-"maximal state" and denote it by η^*. Any addition of a particle to any site of it will cause an avalanche in the system which will extend to all the sites in Λ.

2.2 Dynamics of the sandpile model

In a pure sandpile model, we only consider configurations in the set $\mathcal{H}$. The stabilization of a sandpile is a map which turns an unstable state into a stable one. Let $\mathcal{S}$ denote the stabilization operator, so:

$$\mathcal{S} : \mathcal{H} \to \Omega_\Lambda \tag{2.2.1}$$

In the process of the stabilization, toppling will happen at sites which are unstable, one site topples may cause other sites to become unstable. In order to introduce the toppling operators, we need the following matrix Δ, which describes the redistribution rule,

Definition 2.2.1. *For the sandpile model on $\Lambda \in \mathbb{Z}^d$ with $|\Lambda| < \infty$, the toppling matrix is defined as follows:*

$$\Delta_{xy} = \begin{cases} 2d & x = y \\ -1 & x \text{ and } y \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{2.2.2}$$

This is a symmetric matrix, on the diagonal the values are equal to $2d$, and $\sum_{y\in\Lambda} \triangle_{xy} \geq 0$ for all x, and $\sum_{y\in\Lambda} \triangle_{xy} > 0$ if $x \in \partial\Lambda$(the set of boundary sites of Λ), which means that the system is dissipative on the boundary sites. With the help of this matrix, we can define the toppling operators,

Definition 2.2.2. *The toppling operator at site x denoted by t_x is defined by the following expression: for any $\eta \in \mathcal{H}$:*

$$(t_x\eta)_y = \begin{cases} \eta_y - \triangle_{xy}, & \text{if the site } x \text{ is unstable.} \\ \eta_y & \text{otherwise.} \end{cases} \tag{2.2.3}$$

This definition tells that when a toppling happens at site x, $\eta_x \to \eta_x - 2d$, $\eta_y \to \eta_y + 1$ if x, y are nearest neighbors, and otherwise the heights remain the same. When site x is stable, t_x has no effect on η, in such a case, t_x is called *illegal.* When t_x operates on a configuration of which site x is unstable, it is called *legal.* After a legal toppling at site x, the total number of grains in the system will decrease $\sum_{y\in\Lambda} \triangle_{xy} \geq 0$, which means that there is no generation of sand during topplings in the system and particles can leave the system only if topplings happen at the boundary sites, it seems that there are some sink sites near the boundary collecting grains lost from the boundary.

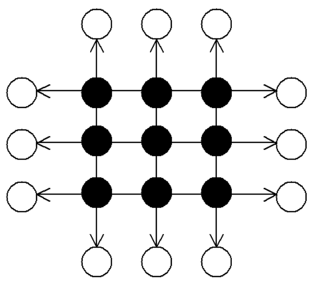

Figure 2.2: General graph with sink site indicated by ◯

Lemma 2.2.3. *The toppling operators $t_x (x \in \Lambda)$ commute.*

Proof. Without loss of generality, assume η is a configuration of which sites x and y are both unstable, i.e., both t_x and t_y operating on η are legal, then:

$$(t_x\, t_y\, \eta)_k = \eta_k - \triangle_{yk} - \triangle_{xk} = (t_y\, t_x\, \eta)_k \tag{2.2.4}$$

Since the equality holds for every $\eta \in \mathcal{H}$, we can conclude that $t_x t_y = t_y t_x$. $\square$

For a configuration $\eta \in \mathcal{H}$, if it is unstable, topplings will happen on it, the following proposition shows that an unstable configuration can always reach a stable via topplings.

Proposition 2.2.4. *If Λ is organized as a union of finite number of finite connected subsets of $\mathbb{Z}^d$, every configuration $\eta \in \mathcal{H}$ can reach a stable state after a finite number of legal topplings.*

Proof. If a site on the boundary topples infinitely many times, the system must loose infinite grains, which is impossible since our system is of finite volume and there are finite grains on each site.

Assume a site x near to the boundary topples infinite times, it must give infinite grains to its nearest neighbors on the boundary, which should topple infinitely many times in order to become stable; this is also impossible;

Continuing this argument, we can prove that any site connected to a boundary site could not topple infinite times. $\square$

From now on, we will always take Λ to be a connected subset of $\mathbb{Z}^d$, an unstable configuration can always reach a stable one via finite number of legal topplings, and since all the topplings commute, how you "organize" toppling is not important, so we will show them in sequel. For $\eta \in \mathcal{H}$, the stabilization operator $\mathcal{S}$ working on η can be expressed in the form a sequence of legal toppling operators:

$$\mathcal{S}(\eta) = t_{x_n} ... \, t_{x_1}\, (\eta) \in \Omega_\Lambda, x_k \in \Lambda(k = 1, ..., n). \tag{2.2.5}$$

In this expression, x_m is the place of m'th toppling , notice by Lemma 2.2.3, the order of a sequence of legal topplings has no effect on the final result, which only depends on the number of topplings at every site.

Definition 2.2.5. *If $t_{x_n} ... t_{x_1}$ is a sequence of legal topplings, the toppling number at site x is denoted by:*

$$n_x = \sum_{k=1}^{n} \mathcal{I}(x_k = x) \tag{2.2.6}$$

Use the Abelian property of the topplings, rearrange them in the expression (2.2.5), $\mathcal{S}(\eta) = t_{x_n} \dots t_{x_1}(\eta), x_k \in \Lambda, (k = 1, \dots n)$, we get:

$$\mathcal{S}(\eta) = \prod_{x \in \Lambda} t_x^{n_x} \eta = \eta - \Delta n, \text{ where } n = (n_x)_{x \in \Lambda} \tag{2.2.7}$$

The toppling numbers contained in the stabilization have a property:

Lemma 2.2.6. *If $\eta \in \mathcal{H}$, and $t_{x_n} \dots t_{x_1}$ is a sequence of legal topplings such that the resulting configuration is stable, then the toppling numbers n_x are "maximal", that is to say for every sequence of legal topplings $t_{y_m} \dots t_{y_1}$, and n' be the vector of toppling numbers, then:*

$$n'_x \leq n_x, \textit{ for every } x \in \Lambda$$

Proof. Assume ξ is the stable configuration resulted from the legal sequence of topplings $t_{x_n} \dots t_{x_1}$, then:

$$\xi = \eta - \Delta n \tag{2.2.8}$$

Suppose $t_{y_m} \dots t_{y_1}$ is another legal sequence of topplings with toppling numbers m such that $m_x \leq n_x$ for all $x \in \Lambda$(which is always possible since we can choose all the $m_x = 0$), and for a site $y \in \Lambda$, an extra legal toppling can be performed. Denote:

$$\zeta = \eta - \Delta m \tag{2.2.9}$$

Since an extra legal still be possible at site y of ζ, i.e, $\zeta_y \geq 2d$ and ξ is stable, $\xi_y < 2d$, we have that $\xi_y < \zeta_y$. From (2.2.8) and (2.2.9), we get $\xi_y = \eta_y - \sum_{x \in \Lambda} \Delta_{yx} n_x$ and $\zeta_y = \eta_y - \sum_{x \in \Lambda} \Delta_{yx} m_x$ respectively, then:

$$\eta_y - \textstyle\sum_{x \in \Lambda} \Delta_{yx} n_x < \eta_y - \sum_{x \in \Lambda} \Delta_{yx} m_x$$

Consequently,

$$(m_y - n_y)\Delta_{yy} < \sum_{x \neq y} (n_x - m_x)\Delta_{yx} \leq 0 \tag{2.2.10}$$

The last inequality holds because $\Delta_{xy} \leq 0$, for $x \neq y$ and $n_x \geq m_x$ by assumption. Then we get $m_y - n_y < 0$, therefore $m_y < n_y$. This means if m' denotes the toppling vector where we legally topples the site y once more, $m'_y \leq n_y$ still holds. Therefore, we know once the toppling sequence is legal, we must have the toppling numbers $n'_x \leq n_x$, for all $x \in \Lambda$. ☐

Proposition 2.2.7. *$\mathcal{S}$ is well-defined.*

Proof. We have to prove that for a configuration η the stabilization operator $\mathcal{S}$ can result in only one stable state.

Since $\mathcal{S}(\eta) = t_{x_n} ... t_{x_1}(\eta) = \eta - \triangle n \in \Omega_\Lambda$, n is the column of the toppling numbers, we should prove that for a given $\eta \in \mathcal{H}$, n is fixed in order to get a stable state. Assume there are two sequences of legal topplings characterized by the toppling numbers: $\prod_{x\in\Lambda} t_x^{n_x}$ and $\prod_{x\in\Lambda} t_x^{m_x}$ both of which can stabilize η. Then by the "maximal" property of the toppling numbers, for every $x \in \Lambda$,

$$n_x \leq m_x \text{ and } m_x \leq n_x \tag{2.2.11}$$

Therefore, we can conclude that

$$n = m$$

Then $\mathcal{S}(\eta)$ is uniquely determined for every $\eta \in \mathcal{H}$. □

An unstable state can stabilize itself without external influence. In the sandpile model, external influence comes from the addition of grains to sites.

Definition 2.2.8. *Let α_x be the operator that adds a particle to a nonnegative configuration on site x:*

$$(\alpha_x \eta)_y = \begin{cases} \eta_y + 1 & \text{if } y = x \\ \eta_y & \text{otherwise.} \end{cases} \tag{2.2.12}$$

It is obvious that the new configuration $\alpha_x\eta$ is still nonnegative. So the set $\mathcal{H}$ is closed under all such $\alpha_x (x \in \Lambda)$. When added a grain, a stable configuration may becomes unstable. For example, adding a grain to the maximal η^* on any site $x \in \Lambda$ will cause site x to become unstable, and afterwards topplings happen on site x first, and later may expand to every site of Λ. Now we give the definition of a new operator a_x, the result of a_x operating on η is the final result of adding a grain on site x and then stabilization:

Definition 2.2.9. *The addition operator $a_x : \Omega_\Lambda \to \Omega_\Lambda$:*

$$a_x \eta = \mathcal{S}(\alpha_x \eta) = \mathcal{S}(\eta + \delta_x), \forall \eta \in \Omega_\Lambda \tag{2.2.13}$$

Obviously, this operator is well-defined, since $\mathcal{S}$ is well-defined. Let n_η^x denote the column consisting of toppling numbers needed to stabilize $\alpha_x(\eta)$, then we have:

$$\eta + \delta_x - \Delta n_\eta^x = a_x \eta \tag{2.2.14}$$

Theorem 2.2.10. *All the operators a_x commute with each other.*

Proof. For any $\eta \in \Omega_\Lambda$ and any $x, y \in \Lambda$,

$$\begin{aligned} a_x\, a_y\, \eta &= \mathcal{S}(\mathcal{S}(\eta + \delta_y) + \delta_x) \\ &= \mathcal{S}(\eta + \delta_x + \delta_y) \\ &= \mathcal{S}(\mathcal{S}(\eta + \delta_x) + \delta_y) = a_y\, a_x\, \eta \end{aligned} \tag{2.2.15}$$

So, all the a_x commute. $\square$

The dynamics of a sandpile is guided by two forces, one is the relaxation through topplings(internal force), the other is addition of grains to the configuration(external force). Let $\mathcal{P} = (p_x)_{x\in\Lambda}$ be a positive probability distribution on Λ, with $p_x > 0, \forall x \in \Lambda$ and $\sum_{x\in\Lambda} p(x) = 1$. $X_1, X_2, ...$ are *i.i.d* with distribution $\mathcal{P}$. Starting from a chosen stable configuration $\eta_0 = \eta \in \Omega_\Lambda$, add a grain on X_1 and afterwards let the system stabilize itself if necessary, resulting in a stable configuration η_1. Then we add a grain on X_2 and afterwards the system stabilize itself again to reach a stable state η_2,...etc, we get a sequence of configurations, $\eta_0, \eta_1, ...,$, among which there is the following relation:

$$\eta_n = a_{X_n}\eta_{n-1} = \prod_{i=1}^{n} a_{X_i}\eta_0 \tag{2.2.16}$$

We know η_n is generated by adding a grain to η_{n-1} on site X_n and immediate relaxation, and it is also can be seen as a configuration generated by adding to each site of $\{X_1, X_2, ..., X_n\}$ a grain at the same time and then relaxation. $\{\eta_n : n \in \mathbb{N}\}$ is a Markov chain with the transition operator $\mathbf{P}$: $\forall f : \Omega_\Lambda \to \mathbb{R}$:

$$\mathbf{P}f(\eta) = \mathbf{E}[f(\eta_1) \mid \eta_0 = \eta] = \sum_{x\in\Lambda} p(x)\, f\,(a_x\eta), \tag{2.2.17}$$

In a Markov chain, a state ξ that can be reached infinitely many times is called "recurrent", otherwise it is called "transient". For ξ recurrent, then $P_\xi\{\xi_n = \xi, \text{ for infinitely many } n\} = 1$. It is a general property of a finite Markov Chain(a Markov Chain with finite possible states) to have at least one recurrent configuration. Define:

$$\mathcal{R} = \{\eta : P^\eta(\eta_n = \eta, \text{ for infinitely many n}) = 1\}$$

be the set of all recurrent configurations of sandpile model, it is non-empty since the size of the set of stable configurations: $|\Omega_\Lambda| = (2d)^{|\Lambda|} < \infty$.

Theorem 2.2.11. $\mathcal{R}_1 = \{\eta \in \Omega_\Lambda : \exists (k_x)_{x\in\Lambda} > 0, \prod_{x\in\Lambda} a_x^{k_x}\eta = \eta\} = \mathcal{R}$.

Proof. For $\eta \in \mathcal{R}$, by the definition of recurrent configuration, we know there is a $\eta_n = \eta$. While in the Markov Chain starting from η, there must be a site sequence $X_1, ..., X_n$ such that $\eta_n = \prod_{j=1}^n a_{X_j}\eta$, define:

$$k_x = \textstyle\sum_{j=1}^n I(X_j = x), \forall x \in \Lambda$$

there for $\prod_{x\in\Lambda} a_x^{k_x}\eta = \eta$. Since we add particle to every site with positive probability, for n large enough, it makes sure that $k_x > 0, \forall x \in \Lambda$. This proves $\mathcal{R} \subseteq \mathcal{R}_1$.

Since $\mathcal{R}$ is non-empty and assume $\xi \in \mathcal{R}$, then for the maximal stable configuration $\eta^* = \overline{2d-1}$, we have the following expression:

$$\eta^* = \textstyle\prod_{x\in\Lambda} a_x^{2d-1-\xi_x}\xi$$

that is to say η^* can be reached from recurrent configuration ξ, therefore $\eta^* \in \mathcal{R}$. And for every $\eta \in \mathcal{R}_1$, $\exists (k_x)_{x\in\Lambda} > 0$, such that $\prod_{x\in\Lambda} a_x^{k_x}\eta = \eta$, then $\prod_{x\in\Lambda}(a_x^{k_x})^n\eta = \eta, \forall n \in \mathbb{N}$, we can choose n large enough that $nk_x \geq 2d, \forall x \in \Lambda$, for $\eta^* = \prod_{x\in\Lambda} a_x^{2d-1-\eta_x}\eta$, then

$$\begin{aligned} &\prod_{x\in\Lambda} a_x^{nk_x-(\eta_x^*-\eta_x)}\eta^* \\ =& \prod_{x\in\Lambda} a_x^{nk_x-(\eta_x^*-\eta_x)} \prod_{x\in\Lambda} a_x^{\eta_x^*-\eta_x}\eta \\ =& \prod_{x\in\Lambda} a_x^{nk_x}\eta = \eta \end{aligned} \tag{2.2.18}$$

which means that any $\eta \in \mathcal{R}_1$ can be reached from η^*, and hence $\eta^* \in \mathcal{R}$ implies $\eta \in \mathcal{R}$. It proves that $\mathcal{R}_1 \subseteq \mathcal{R}$.

By now we can conclude $\mathcal{R}_1 = \mathcal{R}$. □

2.3 Group structures in the dynamics of the sandpile model

Since [5], people have started to study the mathematical properties of the sandpile model. One of the most important ones is the abelian group contained in the model. This section contains three parts: one is two groups in the Abelian sandpile model; another is the invariant measure for the dynamics of the sandpile model; the other is about the toppling numbers. The reason that I put these three in the same section is that group structure is the key point to prove the uniform measure on the recurrent configurations is invariant, using the stationary property of the uniform measure, we can obtain a relation between topplings numbers and Green functions.

According to Theorem 2.2.11, for a fixed $\eta \in \mathcal{R}$, there exists $(k_x)_{x\in\Lambda} > 0$ such that $\prod_{x\in\Lambda} a_x^{k_x}\eta = \eta$. So $\prod_{x\in\Lambda} a_x^{k_x}$ operates on η as the identity operator. Through the following lemma, we will show that $\prod_{x\in\Lambda} a_x^{k_x}$ acts as the identity operator on the whole set of recurrent configurations.

Lemma 2.3.1. *Define:* $\mathcal{A} = \{\zeta \in \mathcal{R} : \prod_{x\in\Lambda} a_x^{k_x}\zeta = \zeta\}$
then $\mathcal{A} = \mathcal{R}$

Proof. $\mathcal{A} \subseteq \mathcal{R}$ holds by the definition of $\mathcal{A}$. On the other side, it remains to prove that: $\mathcal{R} \subseteq \mathcal{A}$:

For $\zeta \in \mathcal{A}$, $\prod_{x\in\Lambda} a_x^{k_x}(a_x\zeta) = a_x(\prod_{x\in\Lambda} a_x^{k_x}\zeta) = a_x\zeta$, then $a_x\zeta \in \mathcal{A}$, for every $x \in \Lambda$. $\eta^* = \prod_{x\in\Lambda} a_x^{2d-1-\zeta_x}\zeta$, which implies that $\eta^* \in \mathcal{A}$. Since the recurrent set $\mathcal{R}$ is the unique recurrent class that contains η^*, $\mathcal{R} \subseteq \mathcal{A}$. Hence, we get $\mathcal{R} = \mathcal{A}$.

□

Define:

$$\mathbf{G} = \{\textstyle\prod_{x\in\Lambda} a_x^{k_x}, k_x \in \mathbb{N}, \forall x \in \Lambda\}$$

By Lemma 2.3.1, we know there is an identity operator in $\mathbf{G}$ when working on $\mathcal{R}$, the following theorem tells us that $\mathbf{G}$ operating on the set $\mathcal{R}$ is an abelian group.

Theorem 2.3.2. $\mathbf{G}$ *operating on* $\mathcal{R}$ *defines an Abelian group.*

Proof. The proof has two points: Abelian property and the existence of the inverse operator.

The Abelian property is obvious. All the operators contained in $\mathbf{G}$ are generated by the set $\{a_x : x \in \Lambda\}$. Since all the a_x commute, we can conclude that all the operators in $\mathbf{G}$ keep the Abelian property.

Secondly, for a fixed $\eta \in \mathcal{R}$, there is an operator $\prod_{x\in\Lambda} a_x^{k_x}$ such that $\prod_{x\in\Lambda} a_x^{k_x}(\eta) = \eta$. By Lemma 2.3.1, we know $\prod_{x\in\Lambda} a_x^{k_x}$ is the identity operator of $\mathbf{G}$ acting on $\mathcal{R}$, denoted by e. Now define a new operator:

$$a_x^{-1} = a_x^{k_x-1} \prod_{y\in\Lambda, y\neq x} a_y^{k_y} \tag{2.3.1}$$

Then by Abelian property, we can get that $a_x^{-1}a_x = a_x a_x^{-1} = e$. Then a_x^{-1} is the inverse operator of a_x.

And for $g = \prod_{x\in\Lambda} a_x^{m_x} \in \mathbf{G}$, define

$$g^{-1} = \prod_{x\in\Lambda} (a_x^{-1})^{m_x} \tag{2.3.2}$$

By the definition of a_x^{-1}, we know all the a_x^{-1} and a_y also commute with each other.

$$g^{-1}g = \prod_{x\in\Lambda}(a_x^{-1})^{k_x} \prod_{x\in\Lambda} a_x^{k_x} = \prod_{x\in\Lambda}(a_x^{-1})^{k_x} \prod_{x\in\Lambda} a_x^{k_x} = e$$

Similarly, we can get $g^{-1}g = e$. Consequently, the inverse operator exists for every $g \in \mathbf{G}$. □

For a fixed $\eta \in \mathcal{R}$, define $O_\eta := \{g\eta : g \in \mathbf{G}\}$ as the orbit starting from η.

Lemma 2.3.3. *For $\eta \in \Omega_\Lambda$,*
1) $O_\eta = \mathcal{R}$
2) If $g\eta = g'\eta$, for some $g, g' \in \mathbf{G}$, $g\xi = g'\xi, \forall \xi \in \mathcal{R}$.

Proof. For $\eta \in \mathcal{R}$ and $g \in \mathbf{G}$, $g\eta$ is reached from η, then $g\eta \in \mathcal{R}$, and hence $O_\eta \subseteq \mathcal{R}$. We know $\eta^* = \prod_{x\in\Lambda} a_x^{2d-1-\eta_x}\eta$, then $\eta^* \in O_\eta$. From 2.2.8 we know, every recurrent configuration $\xi \in \mathcal{R}$ can be reached from η^*, then $\xi \in O_\eta$ since $\eta^* \in O_\eta$. Then $\mathcal{R} \subseteq O_\eta$, then we get $O_\eta = \mathcal{R}$.

Suppose for η, there exist $g, g' \in \mathbf{G}$ such that $g\eta = g'\eta$. Then define:

$$\mathrm{A} = \{\xi \in \mathcal{R} : g\xi = g'\xi\}$$

For $\xi \in \mathrm{A}$, $\mathbf{G}\xi = \{h\xi : h \in \mathbf{G}\} \subseteq \mathrm{A}$, which tells us that once the Markov chain enters A, it will never leave, i.e, A is an entrapped set of the Markov Chain. While we know $\eta^* = \prod_{x\in\Lambda} a_x^{2d-1-\eta_x}\eta$, then

$$\begin{aligned} g\eta^* &= g\prod_{x\in\Lambda} a_x^{2d-1-\eta_x}\eta = \prod_{x\in\Lambda} a_x^{2d-1-\eta_x} g\eta \\ &= \prod_{x\in\Lambda} a_x^{2d-1-\eta_x} g'\eta \\ &= g'\prod_{x\in\Lambda} a_x^{2d-1-\eta_x}\eta = g'\eta^* \end{aligned} \tag{2.3.3}$$

so $\eta^* \in \mathrm{A}$. For any $\zeta \in \mathcal{R}$, by (2.2.18), we know it can be reached from η^*, therefore $\zeta \in \mathrm{A}$, then we have $g\xi = g'\xi, \forall \xi \in \mathcal{R}$. □

Theorem 2.3.4. $\mid \mathcal{R} \mid = \mid \mathbf{G} \mid$

Proof. For all $\eta \in \mathcal{R}$,

$$\text{The map: } \Psi_\eta : \mathbf{G} \to \mathcal{R} : g \to g\eta$$

From Lemma 2.3.3, we know this map is a bijection from $\mathbf{G}$ to $\mathcal{R}$. Then the size of $\mathbf{G}$ and the size of $\mathcal{R}$ are equal. □

Corollary 2.3.5. *Take $\mathcal{R}_1$ be the set defined in Theorem 2.2.11. Define:*

$$\mathcal{R}_2 = \{\eta \in \Omega_\Lambda : \forall x \in \Lambda, \exists n_x \geq 1, a_x^{n_x}\eta = \eta\}$$

and

$$\mathcal{R}_3 = \{\eta \in \Omega_\Lambda : \exists x \in \Lambda, \exists n_x \geq 1, a_x^{n_x}\eta = \eta\}$$

then $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = \mathcal{R}$

Proof. From Theorem 2.2.11, $\mathcal{R} = \mathcal{R}_1$.

By the definition of $\mathcal{R}_1$, we know $\mathcal{R}_2 \subseteq \mathcal{R}_1$. It remains to prove $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

To prove $\mathcal{R}_1 \subseteq \mathcal{R}_2$, we know $\mid \mathbf{G} \mid = \mid \mathcal{R} \mid < \infty$, so every element in $\mathbf{G}$ is of finite order(or else there must be infinitely many element in $\mathbf{G}$). Then for every a_x, there must be a $n_x \in \mathbb{N}$ such that $a_x^{n_x} = e$. Then for any $\eta \in \mathcal{R}_1 = \mathcal{R}, a_x^{n_x}\eta = \eta$, which implies that $\mathcal{R}_1 \subseteq \mathcal{R}_2$. By now we proves $\mathcal{R}_1 = \mathcal{R}_2$

To prove that $\mathcal{R}_3 = \mathcal{R}_1$, firstly $\mathcal{R}_1 = \mathcal{R}_2 \subseteq \mathcal{R}_3$. It remains to prove that $\mathcal{R}_3 \subseteq \mathcal{R}_2$, so we have to show that for a configuration η, if there is a $x \in \Lambda$ and $n_x \in \mathbb{N}$ such that $a_x^{n_x}\eta = \eta$, then for each $y \in \Lambda$, there exists n_y such that $a_y^{n_y}\eta = \eta$.

Since the number of stable states is finite, for $\eta \in \Omega_\Lambda$, for each $y \in \Lambda$, there must be a n_y such that for a fixed p_x such that $a_y^{n_y}a_y^{p_y}\eta = a_y^{p_y}\eta$, otherwise, the set of production operator $\{a_y^n, n \in \mathbb{N}\}$ operating on $a_y^{p_y}\eta$ results in infinite number of stable configuration which is contradict with $\mid \Omega_\Lambda \mid < \infty$.

When we add $2d$ grains to site x, site x becomes unstable and topples one time immediately, the result is that height at site x keeps and each of its nearest neighbors receives one, which is equivalent with adding one grain to each of its neatest neighbors directly, i.e.,

$$a_x^{2d}\eta = \prod_{y \in \Lambda, |y-x|=1} a_y\eta, \forall \eta \in \Omega_\Lambda$$

Since Λ is a connected subset of $\mathbb{Z}^d$, for every y, there is a path connected it to x, so there must be a k such that

$$a_x^{kn_x}\eta = a_y^{p_y}(a_{x_1}...a_{x_n})\eta, \text{for a cerain } p_y > 0$$

For this p_y, we know there is a $n_y > 0 \in \mathbb{N}$ such that $a_y^{n_y} a_y^{p_y} \eta = a_y^{p_y} \eta$, Then

$$\begin{aligned} a_y^{n_y} \eta &= a_y^{n_y} a_x^{kn_x} \eta \\ &= a_y^{n_y} a_y^{p_y} (a_{x_1} ... a_{x_n}) \eta \\ &= (a_{x_1} ... a_{x_n}) a_y^{n_y} a_y^{p_y} \eta \\ &= (a_{x_1} ... a_{x_n}) a_y^{p_y} \eta = \eta \end{aligned} \tag{2.3.4}$$

□

Define:$\Psi : \mathbb{Z}^\Lambda \to \mathbf{G}$:

$$\Psi(k) = \prod_{x \in \Lambda} a_x^{k_x}, \forall k \in \mathbb{Z}^\Lambda \tag{2.3.5}$$

This map is a homomorphism. $Ker(\Psi) = \{k \in \mathbb{Z}^\Lambda : \prod_{x \in \Lambda} a_x^{k_x} = e\}$

Theorem 2.3.6. *Define: $\Delta\mathbb{Z}^\Lambda = \{\Delta n : n \in \mathbb{Z}^\Lambda\}$, then $Ker(\Psi) = \Delta\mathbb{Z}^\Lambda$ and hence* $\mathbf{G}$ *isomorphic with $\mathbb{Z}^\Lambda / \Delta\mathbb{Z}^\Lambda$*

Proof. For the toppling matrix:

$$\Delta_{xy} = \begin{cases} 2d & x = y \\ -1 & x \text{ and } y \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{2.3.6}$$

We know adding $2d$ particles to a site x, that site must topple and give each of its neighbors 1 particle, which has the same effect as adding 1 particles to each of its nearest neighbors. And hence we have the relation:

$$a_x^{\Delta_{xx}} = \prod_{y \in \Lambda, y \neq x} a_y^{-\Delta_{xy}} \tag{2.3.7}$$

Then multiply $\prod_{y \in \Lambda, y \neq x} a_y^{\Delta_{xy}}$ bo both sides,

$$\prod_{y \in \Lambda} a_y^{\Delta_{xy}} = e \tag{2.3.8}$$

So for any $(k_x)_{x \in \Lambda} \in \mathbb{Z}^\Lambda$, $\prod_{y \in \Lambda} a_y^{k_x \Delta_{xy}} = e$. Multiply the equation for all $x \in \Lambda$:

$$\prod_{x \in \Lambda} \prod_{y \in \Lambda} a_y^{k_x \Delta_{xy}} = e \tag{2.3.9}$$

Since $\Delta_{xy} = \Delta_{yx}$,

$$e = \prod_{x\in\Lambda}\prod_{y\in\Lambda} a_y^{k_x\Delta_{yx}} = \prod_{y\in\Lambda} a_y^{\sum_{x\in\Lambda}\Delta_{yx}k_x} = \prod_{y\in\Lambda} a_y^{(\Delta k)_y} \tag{2.3.10}$$

So, $\Delta\mathbb{Z}^\Lambda \subseteq \text{Ker}\Psi$

For any $m \in \text{Ker}(\Psi)$, then $\prod_{x\in\Lambda} a_x^{m_x} = e$, then there are two non-negative integer vectors m^+, m^- such that $m = m^+ - m^-$. Then, for all $\eta \in \mathcal{R}$, we have:

$$\prod_{x\in\Lambda} a_x^{m_x^+}\eta = \prod_{x\in\Lambda} a_x^{m_x^-}\eta$$

then there existence two non-negative vectors $k^+ = (k^+(x))_{x\in\Lambda} \geq 0, k^- = (k^-(x))_{x\in\Lambda} \geq 0$ such that

$$\eta + m^+ - \Delta k^+ = \zeta = \eta + m^- - \Delta k^-$$

Then we get that:

$$m = m^+ - m^- = \Delta(k^+ - k^-)$$

which implies that $m \in \Delta\mathbb{Z}^\Lambda$, it proves that $\text{Ker}(\Psi) \subseteq \Delta\mathbb{Z}^\Lambda$. Now we can conclude that

$$\text{Ker}(\Psi) = \Delta\mathbb{Z}^\Lambda \text{ and } \mathbf{G} \cong \mathbb{Z}^\Lambda / \Delta\mathbb{Z}^\Lambda \tag{2.3.11}$$

□

Corollary 2.3.7. $|\ \mathcal{R}\ |=|\ \mathbf{G}\ |=|\ \mathbb{Z}^\Lambda/\Delta\mathbb{Z}^\Lambda\ |= det(\Delta)$, $|\cdot|$ *denotes the number of elements in a set.*

Proof. In Theorem 2.3.4, it is proved that $|\ \mathcal{R}\ |=|\ \mathbf{G}\ |$. Here we still need to prove that: $|\ \mathbb{Z}^\Lambda/\Delta\mathbb{Z}^\Lambda\ |= \det(\Delta)$.

For a diagonal matrix A with entry value a_{xx}, then:

$$\mathbb{Z}^\Lambda/\mathbf{A}\mathbb{Z}^\Lambda = (\mathbb{Z}/a_{11}\mathbb{Z}) \oplus (\mathbb{Z}/a_{22}\mathbb{Z}) \oplus ... \oplus (\mathbb{Z}/a_{NN}\mathbb{Z}) \tag{2.3.12}$$

We know the size of the space on the right side is $\prod_{x\in\Lambda} a_{xx} = \det(A)$, and hence $\mathbb{Z}^\Lambda/\mathbf{A}\mathbb{Z}^\Lambda = \det(A)$.

For matrix Δ, we can turn it to a diagonal matrix A by some row and column operation, see[11]. And such operation preserves the determinant of the matrix, i.e., $\det(\Delta) = \det(A)$.

And we know these two quotient space $\mathbb{Z}^\Lambda \diagup \mathbf{A}\mathbb{Z}^\Lambda$ and $\mathbb{Z}^\Lambda \diagup \Delta\mathbb{Z}^\Lambda$ are isomorphic with each other. Then:

$$| \mathbb{Z}^\Lambda \diagup \Delta\mathbb{Z}^\Lambda | = | \mathbb{Z}^\Lambda \diagup \mathbf{A}\mathbb{Z}^\Lambda | = \det(A) = \det(\Delta) \tag{2.3.13}$$

□

Define the uniform measure on the recurrent set $\mathcal{R}$

$$\mu = \frac{1}{| \mathcal{R} |} \sum_{\eta \in \mathcal{R}} \delta_\eta \tag{2.3.14}$$

Since $a_x : \mathcal{R} \to \mathcal{R}$ is a bijection, the image measure

$$\mu \circ a_x = \frac{1}{| \mathcal{R} |} \sum_{\eta \in \mathcal{R}} \delta_{a_x \eta} = \frac{1}{| \mathcal{R} |} \sum_{\eta \in \mathcal{R}} \delta_\eta = \mu \tag{2.3.15}$$

that is to say, the uniform measure μ on $\mathcal{R}$ is stationary measure for the Markov Chain and all individual operators $a_x, x \in \Lambda$.

2.4 Expected Toppling numbers

The toppling matrix Δ in a BTW sandpile model coincides not only in the notation but also the analogous discrete property. The reason is that in a BTW sandpile model, the toppling matrix Δ there is to describe the diffusion of grains("energy") through out the lattice during the avalanche. The change of the number of grains(energy) at site y when it topples happens at site x is just give by Δ_{xy}. The continuous version of this evolution is described as:

$$\frac{\partial f}{\partial t} = D\Delta f \tag{2.4.1}$$

where Δ is now the Laplacian operator on $\mathbb{R}^d$. The Green function for the continuous evolution function satisfies $\Delta\mathbb{G}(x, y) = \delta_{x-y}$ and in the discrete case:

$$\Delta\mathbb{G} = \text{id}, \text{ then }, \mathbb{G}(x, y) = (\Delta^{-1})_{xy}. \tag{2.4.2}$$

To give a probability interpretation of $\mathbb{G}$, consider the a d-dimensional simple random walk on $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$, which will stop if it leaves the boundary of Λ, see [**?**] denote the transition function of the random walk by P,then:

$$(P - I)f = -\frac{1}{2d}\Delta f \tag{2.4.3}$$

Then $\mathbb{G}(I-P) = \frac{1}{2d}$, then $\mathbb{G} = \frac{1}{2d}\cdot(I-P)^{-1} = \frac{1}{2d}\sum_{n=0}^{\infty} P^n$. While $P^n_{x,y} = p_n(x,y)$ where $p_n(x,y)$ is the probability that taking n steps to reach y starting from x. Then $\mathbb{G}(x,y) = \frac{1}{2d}\sum_{n=0}^{\infty} p_n(x,y)$. And hence the probabilistic interpretation of $\mathbb{G}$ is $\mathbb{G}(x,y) = \frac{1}{2d}E_x$(number of visits at site y starting from x).

For any $\eta \in \mathcal{R}$, define $n(x,y,\eta)$ to be the toppling number at site y when adding a particle to η at site x, then:

$$\eta(y) + \delta_{x,y} - \sum_z \Delta_{yz}\, n\,(x,z,\eta) = (a_x\eta)(y) \tag{2.4.4}$$

Integrate the equation over the stationary measure μ:

$$\int \eta(y)\mu(d\eta) + \int \delta_{x,y}\mu(d\eta) - \sum_z \int \Delta_{yz}\, n\,(x,z,\eta)\mu(d\eta) = \int (a_x\eta)(y)\eta\mu(d\eta)$$

By the invariance of μ, we know $\int \eta(y)\mu(d\eta) = \int (a_x\eta)(y)\mu(d\eta)$. Then we get:

$$\sum_z (\Delta_{yz} \int n(x,z,\eta)\mu(d\eta)) = \delta_{xy} \tag{2.4.5}$$

i.e:

$$(\int n(x,y,\eta)\mu(d\eta)) = (\Delta^{-1})_{xy} = \mathbb{G}(x,y) \tag{2.4.6}$$

2.5 Allowed configurations

Generally given a configuration $\eta \in \Omega_\Lambda$, it is hard for us to decide whether it is recurrent or not using the definition. Here we will give another more convenient way to check whether a configuration is recurrent or not.

Definition 2.5.1. *Let $\eta \in \mathcal{H}$, for nonempty set $\mathcal{W} \subseteq \Lambda$, we call the pair $(\mathcal{W}, \eta_{\mathcal{W}})$ a forbidden sub-configuration(FSC) if for all $x \in \mathcal{W}$,*

$$\eta_x + 1 \leqslant \sum_{y \in \mathcal{W} \setminus \{x\}} (-\triangle_{xy}) \tag{2.5.1}$$

If for $\eta \in \mathcal{H}$ there exists a FSC $(\mathcal{W}, \eta_{\mathcal{W}})$, then we say that η contains a FSC. A configuration $\eta \in \Omega_\Lambda$ is called allowed if it does not contain any forbidden sub-configuration. The set of all stable allowed configurations is denoted by $\mathcal{R}'$.

Remark:

1) Sub-configurations of an allowed configuration are also allowed.

2) A configuration on Λ with only one site is always allowed[18].

There is some relationship between *allowed* configuration and the recurrent configuration.

Lemma 2.5.2. *$\mathcal{R}'$ is closed under the dynamics of sandpile, i.e, for all $g \in \mathbf{G}, \eta \in \mathcal{R}', g\eta \in \mathcal{R}'$.*

Proof. Since the element of g is a_x, while α_x and all the toppling operators t_y are the sub-elements of a_x. So it is suffices to prove that the set $\mathcal{R}'$ of all stable allowed configurations is closed under all the α_x and $t_x, x \in \Lambda$.

Clearly, if $\eta \in \mathcal{R}'$, $\eta + \delta_x \in \mathcal{R}'$ since all the heights do not decrease.

Suppose for a $\eta \in \mathcal{R}'$, assume $t_x\eta \notin \mathcal{R}'$, then there exists a FSC $(\mathcal{W}, (t_x\eta)_\mathcal{W})$. While when toppling happens at site x, only the height of site x decrease. If site $x \notin \mathcal{W}$, $(\mathcal{W}, (\eta)_\mathcal{W})$ is also a FSC, it is contradict with the fact that η is allowed. So site $x \in \mathcal{W}$. $\forall y \in \mathcal{W}$:

$$(t_x\eta)_y + 1 \leq \sum_{k \in \mathcal{W}\setminus\{y\}} (-\triangle_{yk}) \tag{2.5.2}$$

i.e:

$$\eta_y - \triangle_{xy} + 1 \leq \sum_{k \in \mathcal{W}\setminus\{y\}} (-\triangle_{yk}) \tag{2.5.3}$$

so:

$$\eta_y + 1 \leq \triangle_{xy} + \sum_{k \in \mathcal{W}\setminus\{y\}} (-\triangle_{yk}) = \sum_{k \in \mathcal{W}\setminus\{y,x\}} (-\triangle_{yk}) \tag{2.5.4}$$

Because there are at least two sites contained in $\mathcal{W}$, $\mathcal{W} \setminus \{x\}$ is non-empty. Then $(\mathcal{W} \setminus \{x\}, \eta_{\mathcal{W}\setminus\{x\}})$ is a FSC of η. It is a contradiction with η is allowed. □

A special allowed configuration is $\eta^* = \overline{2d-1}$, combine with $\mathcal{O}_{\eta^*} = \mathcal{R}$(Lemma 2.3.3), then : $\mathcal{R} \subseteq \mathcal{R}'$. Denote the boundary of Λ by $\partial\Lambda$.

Lemma 2.5.3. *For $x \in \Lambda$ denoted by $\vartheta_\Lambda(x)$ the number of neighbors of x in Λ. $\vartheta_\Lambda(x) \neq 2d$ if and only of x is a boundary, $x \in \partial\Lambda$. Then we have $\eta \in \mathcal{R}'$ if and only if:*

$$\prod_{x \in \partial\Lambda} a_x^{2d-\vartheta_\Lambda(x)}\eta = \eta \tag{2.5.5}$$

Investigate the meaning of equation 2.5.5, let n_x be the total toppling numbers at site x during the operation of $\prod_{x\in\partial\Lambda} a_x^{2d-\vartheta_\Lambda(x)}$, then:

$$\eta + \sum_{x \in \partial\Lambda} (2d - \vartheta_\Lambda(x)) - \Delta n = \eta \tag{2.5.6}$$

n should satisfy the following equation:

$$\Delta n = \sum_{x \in \partial\Lambda} (2d - \vartheta_\Lambda(x)) \tag{2.5.7}$$

Easily to check that $n = (1)_{x\in\Lambda}$ is a solution of it, and since $\det(\Delta) \neq 0$, it has unique solution $n = (1)_{x\in\Lambda}$. So, it remains to prove that after adding $2d - \vartheta_\Lambda(x)$ grains to each boundary site $x \in \partial\Lambda$, every site of Λ topples once. The specific proof can be referred to Lemma 3.23, [18].

Define: $\mathbf{H} = \{\prod_{x\in\partial\Lambda} a_x^{k_x}, k_x \in \mathbb{N}\}$

Lemma 2.5.4. $\mathbf{H}$ *operating on* $\mathcal{R}'$ *is a group.*

Proof. Since $\mathbf{H} \subset \mathbf{G}$, according to Lemma2.5.2, $\mathcal{R}'$ is closed under $\mathbf{H}$; by lemma 2.5.3, the inverse operator a_x^{-1} exists for all the $x \in \partial\Lambda$, and for any $h = \prod_{x\in\partial\Lambda} a_x^{k_x}$, it is easy to check the inverse operator of it is $h^{-1} = \prod_{x\in\partial\Lambda} (a_x^{-1})^{k_x}$. □

Theorem 2.5.5. *A stable configuration is recurrent if and only if it is allowed.*

Proof. Since $\eta^* \in \mathcal{R}'$ and $O_{\eta^*} = \mathcal{R}$(see, Lemma 2.3.3), $\mathcal{R} \subseteq \mathcal{R}'$ by lemma 2.5.2,. We only need to prove if η is allowed, it is also recurrent. Because $\mathbf{H}$ working on $\mathcal{R}'$ forms a group and $\eta^* \in \mathcal{R}'$, there is a $g \in \mathbf{H}$ such that $g\eta = \eta^*$, and hence $g\eta \in \mathcal{R}$. Let g^{-1} is the inverse operator of g in of $\mathbf{H}$. Then $\eta = g^{-1}(g\eta)$. Since $g\eta \in \mathcal{R}$ and $g^{-1} \in \mathbf{H} \subset \mathbf{G}$, then $\eta \in \mathcal{R}$, here also use the closed property of $\mathcal{R}$ under $\mathbf{G}$. This proves $\mathcal{R}' \subseteq \mathcal{R}$. □

When we check whether a given configuration is recurrent or not, we can use the "Burning algorithm" introduced by Dhar [5] Define a $\Lambda^* = \Lambda \cup \{*\}$, the $*$ is an artificial site, called the root, added to Λ, and $*$. To every site x on the boundary, $\vartheta_\Lambda(x)$ is the number of nearest neighbors to x, and $2d - \vartheta_\Lambda(x)$ edges go from x to the root. So the extended graph (Λ^*, E^*) is a graph that every site in Λ has exactly $2d$ outgoing edges.

The burning algorithm is used on a configuration $\eta \in \Omega_\Lambda$, and final result is a set $\Lambda' \in \Lambda$. The burning time of $*$ is zero, the initial result is $\Lambda_0 = \Lambda$, at time 1, we remove("burning") the sites x at which $\eta_x + 1$ is strictly bigger than the number of neighbors of x in Λ, which is called "burning one". Λ_1 is the set of the left sites after burning one; use the same algorithm on Λ_1, get Λ_2, and go on till no more sites can be burnt. The final set is $\mathcal{B}(\eta, \Lambda)$. For example,

There are nine sites in the initial configuration and after burning three times, all the sites are burnt out.

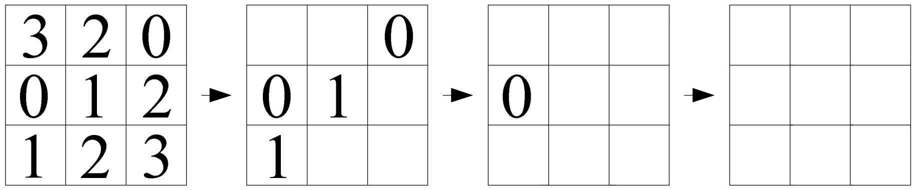

Figure 2.3: An Example for “burning algorithm”

$\forall \eta \in \mathcal{R}'$, it means that for any $\mathcal{W} \in \Lambda$, there exists a $x \in \mathcal{W}$ such that:

$$\eta_x + 1 > \sum_{y \in \mathcal{W} \diagup \{x\}} \Delta_{xy} \tag{2.5.8}$$

So, we start from $\mathcal{W}_1 = \Lambda$, then use the burning algorithm, we will burn out all the sites. So, if a configuration can be burnt all the sites using the “burning algorithm” if and only is recurrent.

Chapter 3

The anti-sandpile model

This chapter is about the so called anti-sandpile model. The anti-sandpile model and the sandpile model are conjugate with each other, which will be shown in section 2 of this chapter. The study of the anti-sandpile model becomes easier with the help of this conjugation. At the last section of this chapter, I combine the sandpile model and the anti-sandpile model together to form a mixed model which exhibits different property to the pure models.

3.1 Introduction

In the sandpile model, grains are added to a stable state which may cause the system to become unstable, afterwards, the system relax itself to a stable one via topplings. Differently from the sandpile model, in the anti-sandpile model, grains are randomly removed from the system and afterwards the system relaxes itself to a stable one by anti-topplings if necessary. The dynamics of the sandpile model and anti-sandpile model are similar but in opposite directions.

Some of the notations used in the sandpile model will also be used here, so I just rewrite them again. Λ is a subset of $\mathbb{Z}^d$ with finite sites. $\eta_x \in \mathbb{Z}$ denotes the height or the number of grains at site x. $\eta = (\eta_x)_{x\in\Lambda}$ forms a height configuration on Λ. $\mathbb{X} = \{\eta : \eta_x \in \mathbb{Z}, \forall x \in \Lambda\}$ is the set of all configurations with integer heights. In a system, two critical values are given, one is $\eta^\dagger_{xc} = 0$, the other $\eta_{xc} = 2d$, a site x is stable only if $0 \leq \eta_x < 2d$. The set of all stable states is denoted by $\Omega_\Lambda = \{\eta : 0 \leq \eta_x < 2d, \forall x \in \Lambda\}$. When $\eta_x < 0$, site x will receive $2d$ particles and each of its neighbors looses one. Particles are allowed to enter the system from the boundary when anti-topplings happen on the boundary sites. In the pure anti-sandpile model, our state space is $\mathcal{J} = \{..., -1, 0, 1, ..., 2d-1\}^\Lambda$. For $\eta \in \mathcal{J}$, if it is unstable, anti-topplings will happen till all the sites become stable.

3.2 Relation between sandpile model and anti-sandpile model

As $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$, the toppling matrix is the same as the one defined in chapter 2:

$$\Delta_{xy} = \begin{cases} 2d & x = y \\ -1 & x \text{ and } y \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{3.2.1}$$

The study of the dynamics of the anti-sandpile model starts from the study a single anti-toppling

Definition 3.2.1. *Let $t_x^\dagger$ denote the anti-toppling operator at site x, for $\eta \in \mathcal{J}$*

$$(t_x^\dagger \eta)_y = \begin{cases} \eta_y + \Delta_{xy} & \textit{if } \eta_x < 0 \\ \eta_y & \textit{otherwise.} \end{cases} \tag{3.2.2}$$

And $t_x^\dagger$ is "legal" if it operates on a site that is unstable,i.e., $\eta_x < 0$, otherwise, it is "illegal".

When anti-topplings happen on the boundary sites, grains enter the system. See the following picture.

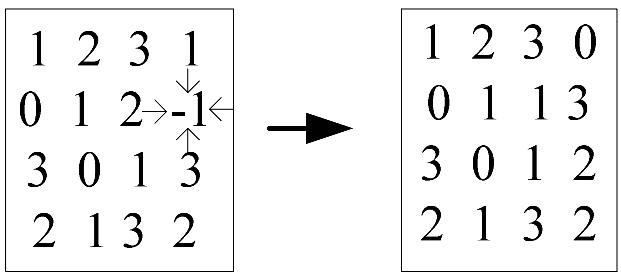

Figure 3.1: Grains enter when anti-topplings happen on the boundary.

Removing a grain from a site is equivalent with adding a hole to that site.

Definition 3.2.2. *Let $\alpha_x^\dagger$ denotes the operator of adding a hole at site x, for any configuration $\eta \in \mathcal{J}$:*

$$\left(\alpha_x^\dagger \eta\right)_y = \begin{cases} \eta_y - 1 & \textit{if } y = x \\ \eta_y & \textit{otherwise.} \end{cases} \tag{3.2.3}$$

Compare the anti-toppling operators with the toppling operators defined by (2.2.3), we found that these two kinds of operators are coupled by the following operator:

Definition 3.2.3. *Define "flip" operator* $\theta : \mathbb{X} \to \mathbb{X}$, $\forall \eta \in \mathbb{X}$*:*

$$\theta(\eta) = \overline{2d-1} - \eta \text{ } i.e \text{ } (\theta\eta)_x = (2d-1) - \eta_x \text{ } for \text{ } all \text{ } x \in \Lambda. \tag{3.2.4}$$

This is an operator that turns a height configuration to another one. It seems that there is a "mirror" at the place with every height $\frac{2d-1}{2}$, then $\theta\eta$ is the "mirror" image of θ and hence we know $\theta\theta\eta = \eta$, for any η, i.e., $\theta\theta = \text{id}$.

Before giving the relation of the sandpile model and anti-sandpile model model, we sea a simple example, e.g., take $\Lambda = \{1, 2, 3, -7\} \subset \mathbb{Z}$ and let

$$\eta = (0, -1, 1, -7) \in \mathcal{J}$$

then,

$$\theta\eta = (1, 2, 0, 8) \in \mathcal{H}$$

You may find some interesting phenomena, for x with $\eta_x < 0$, we have $(\theta\eta)_x > 2d$ and reverse part is also right, that is to say $t_x^\dagger$ is legal at η if and only if t_x is legal at $\theta\eta$. It seems that the stabilization of one is companied with the stabilization of the other. The following proposition describes the relation of sandpile model and anti-sandpile model.

Let $\mathcal{S}^\dagger : \mathcal{J} \to \Omega_\Lambda$ be the stabilization operator in the anti-sandpile model that transfers every configuration in $\mathcal{J}$ to a stable one through legal anti-topplings.

Proposition 3.2.4. *For* $\eta \in \mathcal{J}$*, and* $\mathcal{S}$ *is the stabilization operator in the sandpile model:* $\mathcal{S} : \mathcal{H} \to \Omega_\Lambda$*, then we have the following relations:*
a) $t_x^\dagger\eta = \theta t_x(\theta\eta), t_x^\dagger t_y^\dagger = t_y^\dagger t_x^\dagger$*;*
b) $\mathcal{S}^\dagger$ *is well defined and* $\mathcal{S}^\dagger\eta = \theta\mathcal{S}(\theta\eta)$*.*

Proof. $\Lambda \subset \mathbb{Z}^d$ and $\forall \eta \in \mathcal{J}$

Firstly, $\forall x \in \Lambda$, $\eta_x < 0 \Leftrightarrow (\theta\eta)_x = 2d - 1 - \eta_x > 2d - 1$. This equation means that $t_x^\dagger$ being legal on η is equivalent with t_x being legal on $\theta\eta$.

Secondly, Combining (2.2.3) and (3.2.2), we know :

$$(t_x^\dagger\eta)_y + (t_x\theta\eta)_y = \eta_y + (\theta\eta)_y = 2d - 1, \forall y \in \Lambda. \tag{3.2.5}$$

this tells us that after legal anti-toppling and legal toppling happen at the same site on η and $\theta\eta$ respectively, the resulting configurations are still conjugate with each other, i.e.,

$t_x^\dagger \eta = \theta t_x(\theta\eta)$, further more, sine $\theta\theta = \text{id}$, we have $t_x \theta\eta = \theta t_x^\dagger \eta$. For all the $\eta \in \mathcal{J}$, equation (3.2.5) holds, therefore:

$$t_x^\dagger = \theta\, t_x\, \theta, t_x = \theta\, t_x^\dagger\, \theta. \tag{3.2.6}$$

By the Abelian property of the toppling operators(Lemma 2.2.3) and $\theta^2 = \text{id}$, we get that all the anti-toppling operators $t_x^\dagger (x \in \Lambda)$ also commute.

Assume $... \circ t_{x_n}^\dagger \circ ... \circ t_{x_1}^\dagger$ is a sequence of legal anti-topplings on η, and from the proof of item a), we know $... \circ t_{x_n} \circ ... \circ t_{x_1}$ is a sequence of legal topplings on $\theta\eta$. From Proposition 2.2.4, we know there must be a $... \circ t_{x_n} \circ ... \circ t_{x_1}$ must be a finite sequence. Then $\mathcal{S}^\dagger$ is organized by finitely many legal anti-topplings that can stabilize η. From the proof of item a), we know $\theta \mathcal{S}^\dagger \theta$ can stabilize $\theta\eta \in \mathcal{H}$. $\mathcal{S}(\theta\eta) = \theta\mathcal{S}^\dagger\theta(\theta\eta) = \theta\mathcal{S}^\dagger\eta$. $\theta\theta = \text{id}$, we get $\mathcal{S}^\dagger\eta = \theta\mathcal{S}\theta\eta, \forall \eta \in \mathcal{J}$, and hence $\mathcal{S}^\dagger = \theta\mathcal{S}\theta$. Since $\mathcal{S}$ is well defined, and $\theta : \mathcal{J} \to \mathcal{H}$ is a bijection and $\theta\theta = \text{id}$, then $\mathcal{S}^\dagger$ is well-defined.

□

Let $a_x^\dagger$ be the operator that transfers a stable configuration to another via removing a grain from site x and the immediate relaxation of the system to a stable one by anti-topplings. Since removing a grain is equivalent with adding a hole, we call $a_x^\dagger$ the anti-addition operator associated to site x.

Definition 3.2.5. *Define* $a_x^\dagger : \Omega_\Lambda \to \Omega_\Lambda$, $\forall \eta \in \Omega_\Lambda$

$$a_x^\dagger \eta = \mathcal{S}^\dagger(\alpha^\dagger \eta) = \mathcal{S}^\dagger(\eta - \delta_x) \text{ for all } x \in \Lambda. \tag{3.2.7}$$

The anti-additions have the following properties.

Theorem 3.2.6. *Let* a_x *be the addition operator defined by (2.2.13), then*

$$a_x^\dagger = \theta\, a_x\, \theta, \text{ as well as } a_x = \theta\, a_x^\dagger\, \theta. \tag{3.2.8}$$

and hence $a_x^\dagger (x \in \Lambda)$ *commute.*

Proof. $\forall \eta \in \Omega$,

$$\begin{aligned} a_x^\dagger \eta &= \mathcal{S}^\dagger(\eta - \delta_x) \\ &= \theta \mathcal{S} \theta(\eta - \delta_x) = \theta \mathcal{S}(\overline{2d-1} - \eta + \delta_x) \\ &= \theta\, a_x\, (\overline{2d-1} - \eta) = \theta\, a_x\, \theta \eta \end{aligned} \tag{3.2.9}$$

Therefore we have $a_x^\dagger = \theta\, a_x\, \theta$.
similarly, we get $a_x = \theta\, a_x^\dagger\, \theta$.

$\forall x, y \in \Lambda$,

$$\begin{aligned} a_x^\dagger a_y^\dagger &= \theta\, a_x\, \theta\theta\, a_y\, \theta \\ &= \theta\, a_x\, a_y \theta = \theta\, a_y\, a_x \theta \\ &= \theta\, a_y\, \theta\theta\, a_x\, \theta = a_y^\dagger\, a_x^\dagger \end{aligned} \tag{3.2.10}$$

which proves the Abelian property. □

The above discussion gives a relation between the sandpile model and anti-sandpile model. When we deal with problems related to the anti-sandpile model, we can always turn them into the related form in the sandpile model, and after we have done with them, we can turn them back to the anti-sandpile form again.

3.3 Mathematical results about the anti-sandpile model

Similar to the sandpile model, the anti-sandpile model also shows many nice properties which are presented in this section, including the dynamics, recurrent configurations, invariant measure, etc.

Dynamics of the anti-sandpile model

Instead of adding grains as in the sandpile model, we remove grains from the system and afterwards let the system relax to a stable one by anti-topplings immediately, and repeat that independently again and again which forms the dynamics of the anti-sandpile model.

Let $\eta_0 = \eta \in \Omega$ be the initial configuration, we remove a grain from site X_1 and let the system relax to a stable one, denoted by η_1, then we independently remove one grain from X_2, let the system relax to $\eta_2 \in \Omega_\Lambda$, etc. Then the dynamics of the anti-sandpile is expressed in the form of the following Markov Chain:

$$\eta_n = a^\dagger_{\mathbf{X}_n}\eta_{n-1} \text{ with } \eta_0 = \eta \in \Omega$$

where $\mathbf{X}_n$ are i.i.d with distribution $(p_x)_{x\in\Lambda}, p_x > 0, \forall x \in \Lambda$, and $\sum_{x\in\Lambda} p(x) = 1$.

The transition operator P is defined as: $\forall f : \Omega_\Lambda \to \mathbf{R}$:

$$Pf(\eta) = \mathbf{E}[f(\eta_1) \mid \eta_0 = \eta] = \sum_{x\in\Lambda} p(x)f(a^\dagger_x\eta) \tag{3.3.1}$$

Let $\mathcal{R}^\dagger$ denote the set of all recurrent configurations of the anti-sandpile Markov Chain, it is non-empty since this is a finite Markov Chain.

Define: $\mathbf{G}^\dagger = \{\prod_{x\in\Lambda}(a^\dagger_x)^{k_x}, \forall k_x \in \mathbb{N}\}$ as the set of all production of anti-addition operators. The following theorem is to Theorem 2.2.4.

Theorem 3.3.1. *A configuration $\eta \in \Omega_\Lambda$ is recurrent in the anti-sandpile Markov Chain if and only if there is a $g^\dagger \in \mathbf{G}^\dagger$ such that $g^\dagger\eta = \eta$.*

Proof. The proof of this theorem is very similar to the proof of theorem 2.2.4. so I do not repeat it again here. □

For any set $\mathcal{A} \subseteq \mathbb{X}$, define $\theta\mathcal{A} := \{\theta\eta : \eta \in \mathcal{A}\}$.

Theorem 3.3.2. *1): $\mathcal{R}^\dagger = \theta\mathcal{R}$ and $\mid \mathcal{R}^\dagger \mid = \mid \mathcal{R} \mid = det(\Delta)$*
2): $\mathbf{G}^\dagger$ working on $\mathcal{R}^\dagger$ forms an Abelian group.
3): Define $\mu^\dagger = \frac{1}{|\mathcal{R}^\dagger|}\sum_{\eta\in\mathcal{R}^\dagger}\delta_\eta$ be the uniform measure on $\mathcal{R}^\dagger$, then $\mu^\dagger = \mu\theta$ and $\mu^\dagger$ is invariant under the individual operation of $a^\dagger_x, x \in \Lambda$.

Proof. For $\eta \in \mathcal{R}$, from Theorem 2.2.11, there is a $g = \prod_{x\in\Lambda} a_x^{k_x}$ such that $g\eta = \eta$, then $\theta(g(\eta)) = \theta\eta$, i.e

$$\theta g\theta(\theta\eta) = \theta\eta$$

Since $\theta g\theta = \theta\prod_{x\in\Lambda} a_x^{k_x}\theta = \prod_{x\in\Lambda}(\theta a_x\theta)^{k_x} = \prod_{x\in\Lambda}(a^\dagger_x)^{k_x} \in \mathbf{G}^\dagger$, according to Theorem 3.3.1, we know $\theta\eta \in \mathcal{R}^\dagger$. It proves that $\theta\mathcal{R} \subseteq \mathcal{R}^\dagger$; it can be proved that $\theta\mathcal{R}^\dagger \subseteq \mathcal{R}$ by the similar argument, then $\mathcal{R}^\dagger \subseteq \theta\mathcal{R}$. Therefore $\mathcal{R}^\dagger = \theta\mathcal{R}$. From Theorem 2.3.4, we know $\mid \mathcal{R} \mid = \det\Delta$, then $\mid \mathcal{R}^\dagger \mid = \mid \mathcal{R} \mid = \det\Delta$.

Define $\Phi : \mathbf{G} \to \mathbf{G}^\dagger$ such that $\Phi(g) = \theta g\theta, \forall g \in \mathbf{G}$, we know Φ is a bijection satisfying $\Phi(g_1g_2) = \theta g_1g_2\theta = \theta g_1\theta\theta g_2\theta = \Phi(g_1)\Phi(g_2)$, using $\theta\theta = \text{id}$ in the second equality.

Then $\mathbf{G}$ and $\mathbf{G}^\dagger$ are isomorphic with each other. According to Theorem 2.3.2., we know that $\mathbf{G}$ acting on $\mathcal{R}$ forms an Abelian group, then $\mathbf{G}^\dagger$ acting on $\mathcal{R}^\dagger$ is also an Abelian group combined with the fact that $\mathcal{R}^\dagger = \theta\mathcal{R}$ and $\theta\theta = id$.

From 1) we know $\mathcal{R}^\dagger = \theta\mathcal{R}$ and $\mid \mathcal{R}^\dagger \mid=\mid \mathcal{R} \mid= \det(\Delta)$, and

$$\mu^\dagger = \frac{1}{|\mathcal{R}^\dagger|}\sum_{\eta\in\mathcal{R}^\dagger}\delta_\eta = \frac{1}{|\mathcal{R}|}\sum_{\eta\in\mathcal{R}}\delta_{\theta\eta} = \mu\theta$$

As to the invariance, since the uniform measure μ on $\mathcal{R}$ is invariant under individual $a_x, \forall x \in \Lambda$, and $a_x^\dagger = \theta a_x \theta$,

$$\mu^\dagger \circ a_x^\dagger = \mu\theta \circ a_x^\dagger = \mu\theta\theta \circ a_x\theta = \mu \circ a_x\theta = \mu\theta = \mu^\dagger$$

which proves that $\mu^\dagger$ is invariant under the operation of individual $a_x^\dagger$. □

Anti-toppling numbers:

For all $x, y \in \Lambda$, and all $\eta \in \Omega_\Lambda$, define $n(x, y, \eta)$ be the number of topplings happening at site y when a grain is added to site x, and $n^\dagger(x, y, \eta)$ as the anti-toppling number at site y when a grain is removed from site x, then with $n_x^\dagger(\eta) = (n^\dagger(x, y, \eta), y \in \Lambda)$

$$\eta - \delta_x + \Delta\, n_x^\dagger\,(\eta) = a_x^\dagger\eta \tag{3.3.2}$$

We have already shown that

$$\theta\eta + \delta_x - \Delta\, n_x\,(\theta\eta) = a_x\theta\eta, n_x(\theta\eta) = (n(x, y, \theta\eta), y \in \Lambda) \tag{3.3.3}$$

Combining these two equations together:

$$\eta + \theta\eta + \Delta(n_x^\dagger(\eta) - n_x(\theta\eta)) = a_x^\dagger\eta + a_x\theta\eta = \theta\; a_x\;\theta\eta + a_x\theta\eta \tag{3.3.4}$$

Take $a_x\theta\eta = \xi$:

$$\overline{2d-1} + \Delta(n_x^\dagger(\eta) - n_x(\theta\eta)) = \theta\xi + \xi = \overline{2d-1} \tag{3.3.5}$$

Since $\det\Delta \neq 0, n_x^\dagger(\eta) - n_x(\theta\eta) = 0, i.e., n^\dagger(x, y, \eta) = n(x, y, \theta\eta)$.
From equation (2.4.6):

$$\int_{\mathcal{R}^\dagger} n^\dagger(x, y, \eta)\mu^\dagger(d\eta)_x = \Delta_{xy}^{-1} = \mathbb{G}(x, y) \tag{3.3.6}$$

3.4 Sandpile+anti-sandpile model

In the sandpile model and the anti-sandpile model, we definitely add or take off a grain from a randomly chosen site. If at a chosen site, we are free to add or remove a grain with positive probability, in the case that we add a grain, the system relaxes to a stable one by topplings immediately after the adding; and in the case that we we choose to remove a grain, the system relaxes to a stable one by anti-topplings immediately. In this model, adding and removing grains are both possible, so the model is no longer the pure sandpile model or the pure anti-sandpile model, but a mixed model, we call it the "sandpile+anti-sandpile model", denoted by the $SA-$model. In the SA-model, the possible states are not purely in $\mathcal{H}$ or $\mathcal{J}$, but in $\mathbb{X}$-the set of all height configurations. Here, Ω_Λ still denotes the set of all stable configurations. $\eta_{xc} = 2d, \eta^\dagger_{xc} = 0$, so, if $\eta_x \geq 2d$, site x topples, and if $\eta_x < 0$, site x anti-topples. Then the dynamics of the $SA-$model is more complicated than the pure models.

Dynamics of sandpile+anti-sandpile model

$\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$. $\mathbf{P}$ and $\mathbf{Q}$ are two positive probability measures: $\sum_{x\in\Lambda} p(x) = 1$ and $p(x) > 0, \forall x \in \Lambda$; $q(x) \in (0,1), \forall x \in \Lambda$. Everytime, we randomly choose a site $x \in \Lambda$ according to the distribution $\mathbf{P}$ and we are free to choose to add a particle with probability $q(x)$ and then let the system relax itself by topplings or to add a hole with probability $1-q(x)$ and then let the system stabilize itself by anti-topplings. Start from $\eta_0 = \eta \in \Omega_\Lambda$, independently repeat the same steps again and again, we get the dynamics of the $SA-$ model,

$$\eta_n = b_{X_n}\eta_{n-1}, \eta_0 = \eta \tag{3.4.1}$$

With all the X_n are $i.i.d$ with distribution $\mathbf{P}$ and all the b_y are $i.i.d.$ random operators with prob $\mathcal{Q}$. Because of the independence among all the X_n and independence of all $b_y, y \in \Lambda$, we get $\{\eta_n : n \in \mathbb{N}\}$ is a Markov chain with the transition operator P:

$$\begin{aligned} Pf(\eta) &= E[f(\eta_{X_1}) \mid \eta_0 = \eta] \\ &= \sum_{x\in\Lambda} p(x)q(x)f(a_x\eta) + \sum_{x\in\Lambda} p(x)(1-q(x))f(a^\dagger_x\eta) \end{aligned} \tag{3.4.2}$$

Since $|\Omega_\Lambda| < \infty$, there are only finite number of possible states for the Markov Chain, and hence there is at least one recurrent configuration for the Markov Chain. From the 3.4.2, every stable configuration can be reached from all the other stable configurations, especially the recurrent ones, consequently, every stable configuration is recurrent in the mixed Markov Chain. Then Ω_Λ is also the set of all recurrent configurations of the $SA-$ model. For the $SA-$ model, we get following properties,

Lemma 3.4.1. *$t_x t_y^\dagger \neq t_y^\dagger t_x$ and $a_x a_y^\dagger \neq a_y^\dagger a_x$*

Proof. Consider $\Lambda = (1,2,3) \in \mathbb{Z}$, and $\eta = (-1,2,0)$. Then $t_2 t_1^\dagger \eta = (1,1,0) \neq t_1^\dagger t_2 \eta = (0,0,1)$. For $\xi = (0,1,0)$

$$a_1^\dagger a_2 \xi = (0,0,1)$$

but

$$a_2 a_1^\dagger \xi = (1,1,0)$$

Then $a_1^\dagger$ and a_2 do not commute. □

Remark: For a configuration $\eta \in \mathbb{X}$ with both kinds of unstable sites, the order of topplings and anti-topplings takes role on the stabilization of η, different order of legal topplings and legal anti-topplings may result in different final results. The stabilization of a configuration in the $SA-$ model will be discussed further in Chapter 5.

Define:

$$\widehat{\mathbf{G}} = \{\textstyle\prod_{m=1}^n b_{x_m}, \text{ with all } x_m \in \Lambda, b_{x_m} = a_{x_m} \text{ or } a_{x_m^\dagger}\, \forall n > 0\}$$

Different from the sandpile model and the anti-sandpile model, the mixed model looses the group property.

Theorem 3.4.2. *There does not exist $\widehat{\Omega} \subseteq \Omega$ such that $\widehat{\mathbf{G}}$ acting on $\widehat{\Omega}$ is a group.*

Proof. Assume there is a $\widehat{\Omega}$ such that $\widehat{\mathbf{G}}$ acting on $\widehat{\Omega}$ is a group.

Firstly, we will prove that $\widehat{\Omega}$ has to be equal to Ω.
In fact in order to form a group, $\widehat{\Omega}$ must be a closed set under the action of $\widehat{G}$. If $\widehat{\Omega} \neq \Omega$, there must be a $\eta \in \Omega$ but $\eta \notin \widehat{\Omega}$. While we know for any $\xi \in \widehat{\Omega}$, the Mixed Markov Chain starting from ξ can reach all the recurrent configurations of the Markov Chain; however any configuration in Ω is recurrent, therefore η can also be reached from ξ. Then we know η must be in $\widehat{\Omega}$, which is contradict with the assumption. Then we get that $\widehat{\Omega} = \Omega$.

Secondly, assume $\widehat{\mathbf{G}}$ acting on Ω is a group. Let $\widehat{B}$ be the set of all bijections from Ω to Ω, $\mid \widehat{B} \mid = \mid \Omega \mid !$. Since Ω is finite set, $\mid \Omega \mid !$ is finite. So, if $\widehat{G}$ operating on Ω could form a group, it must be a finite group. Then for any $a_x^\dagger$ and any $\eta \in \Omega$, there must be a $n < \infty$, such that $(a_x^\dagger)^n \eta = \eta$, from corollary 2.3.5, we know $\eta \in \mathcal{R}_1^\dagger = \mathcal{R}^\dagger$, which means that $\Omega \subseteq \mathcal{R}^\dagger$, i.e, all the stable configuration is recurrent in the Anti-sandpile Markov

Chain. However from the definition of "allowed" configuration, we know $\overline{0}$ is not allowed, and hence not recurrent in the Sandpile Markov Chain. From Theorem 3.3.2, we know $\mathcal{R}^\dagger = \theta\mathcal{R}$, then we know $\theta\overline{0} = \eta^* \notin \mathcal{R}^\dagger$ but $\eta^* \in \Omega$. $\square$

Chapter 4

The Infinite-Volume limit sandpile+anti-sandpile process

In Chapter 2 and 3, we studied the finite volume sandpile and anti-sandpile models, especially the invariant measure and recurrent configurations of the finite Markov Chains. We are interested in how the system behaves as the volume increases. In the infinite volume case, the systems become some processes. However, we are even not sure the existence of such processes because in the infinite volume case, both the addition and anti-addition operators are non-local which determines that the related process are not Feller, and hence the classical way used in the construction of a process in the interacting particle system does not work here even in one dimensional case. In this chapter, I mainly give the construction of a so-called sandpile+anti-sandpile process and study the invariant measure of it.

4.1 Feller processes

In the one-dimensional model, $\Omega = \{0, 1\}^{\mathbb{Z}}$ is the set of all stable states with the product topology, which organizes Ω as a compact metric space with metric,

$$d(\xi, \eta) = \sum_{x \in \mathbb{Z}} 2^{-|x|} \mid \eta(x) - \xi(x) \mid \tag{4.1.1}$$

This topology has the following properties,

1 **Convergence:** A series $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ and $\eta \in \mathbb{X}$, we say η_n converges to η, if $\forall \Lambda \subset \mathbb{Z}, \exists M_0 = n_0(\Lambda)$ such that $\forall n \geq M_0$, we have:

$$\eta_n(x) = \eta(x), \forall x \in \Lambda$$

2 **Continuity**: A function $f : \Omega \to \mathbb{R}$ is continuous at the point η if

$$\lim_{\Lambda\uparrow\mathbb{Z}} \sup_{\sigma,\xi} \mid f(\eta_\Lambda \sigma_{\Lambda^c}) - f(\eta_\Lambda \xi_{\Lambda^c}) \mid= 0$$

Where $(\eta_\Lambda\sigma_{\Lambda^c})(x) = \eta(x)$, for $x \in \Lambda$, and $\sigma(x)$, for $x \in \Lambda^c$.

3 **Local function**: A function $f : \Omega \to \mathbb{R}$ is *local* if it depends only on a fixed set $\mathbf{A}$ with finite sites. This means that $\forall \eta, \xi$ such that $\eta_{\mathbf{A}} = \xi_{\mathbf{A}}$ implies $f(\eta) = f(\xi)$.

Remarks: A local function is continuous, while a continuous function can be non-local, e.g., $f(\eta) = \sum_{x\in\mathbb{Z}} e^{-|x|}\eta(x)$, it is continuous but not local. But, continuous function can be approximated uniformly by local functions, this follows, e.g., from Stone-Weierstrass Theorem.

From the view of interacting particle systems, the first step of construction is getting the generator $\mathcal{L}$ of the process. Once we can show that there is a Markov semigroup $S(t)$ corresponding to the generator $\mathcal{L}$, a Markov process(with cadlag paths) starting from η can be defined via:

$$(S(t)f)(\eta) = \mathbf{E}^\eta f(\eta_t) \tag{4.1.2}$$

According to theorem 1.5 of [12], we know the uniqueness of Markov process defined by (4.1.2). So, the main work to get a process corresponding to a given generator $\mathcal{L}$ is to get the semigroup $S(t)$. As we know, if the process is *Feller, i.e.,* $f \in C(\Omega)$ *implies* $S(t)f \in C(\Omega)$, the work becomes easy thanks to Hille-Yoshida theorem[12]. It tell us that if we use $\mathcal{D}(\mathcal{L}) := \{f \in \mathbf{C}(\Omega) : \lim_{t\downarrow 0} \frac{S(t)f-f}{t}$ exists$\}$ to denote the domain of $\mathcal{L}$, the generator and the semigroup has the following relations:

(1): $\mathcal{L}f = \lim_{t\downarrow 0} \frac{S(t)f-f}{t}$, for $f \in \mathcal{D}(\mathcal{L})$

(2): $S(t)f = \lim_{n\to\infty}(I - \frac{t}{n}\mathcal{L})^{-n}f = e^{t\mathcal{L}}f$, for $f \in \mathbf{C}(\Omega)$

(3): For $f \in \mathcal{D}(\mathcal{L}), S(t)f \in \mathcal{D}(\mathcal{L})$, and $(d/dt)S(t)f = \mathcal{L}S(t)f = S(t)\mathcal{L}f$

In this way, the Feller semigroup $S(t)$ is given by (2), i.e,

$$S(t)f(\eta) = e^{t\mathcal{L}}f(\eta), \forall f \in C(\Omega), \eta \in \Omega$$

In the case that we are not sure a process is Feller, we could not get semigroup using Hille-Yoshida. Generally it is not easy to check whether a process is Feller or not by the definition. However a Feller process has the some good properties such as $S(t)f$ is right continuous as a function of t, for every $f \in C(\Omega)$; existence of invariant measures, etc. Therefore, one way to prove a certain process is not Feller is to the prove that one of such these properties does not hold.

4.2 Flip process

One kind of famous models in the interacting particle systems is the flip-process. In the one-dimensional infinite volume model, $\Omega = \{0,1\}^{\mathbb{Z}}$ is the set of stable states. Define θ_x to be "spin-flip" operator on Ω,

$$(\theta_x\eta)_y = \begin{cases} (\eta_x + 1)\text{mod } 2, & \text{if } y = x \\ \eta_y & \text{otherwise.} \end{cases} \tag{4.2.1}$$

The flip operator θ_x transfers the value of η_x from one to the other. Assign to each site a Poisson process N_t^x with rate 1, for $x \neq y$, N_t^x and N_t^y are independent. For configuration $\eta \in \Omega$, the flip rate from zero to one at site x is denoted by $c^0(x,\eta)$ and from one to zero is denote by $c^1(x,\eta)$.

An Example of Feller flip process

The simplest case of the flip process is the process with rate $c^0(x,\eta) = c^1(x,\eta) = 1, \forall x \in \mathbb{Z}$. In this case, there is no interaction among sites, each site evolves follows independently, therefore there is a Markov process $\eta_t(x)$ corresponding to every $x \in \mathbb{Z}$:

$$\eta_t(x) = (\theta_x^{N_t^x}\eta)(x) \tag{4.2.2}$$

The semigroup $S(t)$ for this process is defined as:

$$S(t)f(\eta) = E^\eta(f(\eta_t)), \forall f \in C(\Omega) \tag{4.2.3}$$

Since there is no interaction between sites, from (4.2.2) we know for a fixed value N_t^x, η_t is a continuous function of η. By the Dominated Convergence theorem(DOM), $S(t)f$ is also continuous which means that this process is Feller.

An Example of Non-Feller flip process

If the flip rate $c(x,\eta)$ loses the local property, the flip process can loose the Feller property. For instance let us consider the following flip process.

For $\eta \in \Omega = \{0,1\}^{\mathbb{Z}}$, if $\eta(x) = 0$, with rate 1 flip from zero to one and a site with height 1 will stay at 1 except if $\eta = \overline{1}$ in which case all sites flip to zero, for this process the generator is:

$$\begin{aligned}\mathcal{L}f(\eta) = &\sum_{x\in\mathbb{Z}} I(\eta(x)=0)[f(\theta_x\eta)-f(\eta)] \\ &+ \prod_{x\in\mathbb{Z}} I(\eta(x)=1)[f(\theta\eta)-f(\eta)]\end{aligned} \tag{4.2.4}$$

with θ the global flip-operator. Assume $S(t)$ is the semigroup corresponding to the generator (4.2.4),

Proposition 4.2.1. *The Markov process associated to generator (4.2.4) is not a Feller process.*

If $S(t)$ is the semigroup of a Feller process. For $\mu \in \mathcal{P}(\Omega)$(the probability measure on Ω), $T > 0$ $\mu_T = \frac{1}{T}\int_0^T \mu S(t)dt \in \mathcal{P}(\Omega)$. By the compactness of $\mathcal{P}(\Omega)$, we know there is a subsequence T_n such that $\mu_{T_n} \to \nu$. From Proposition 1.8, e) of [12], we know ν is invariant under $S(t)$. This means that for a Feller process, there is at leat one invariant measure. Now we start to give the proof.

Proof. Suppose measure μ is invariant and $\mu(I(\eta(0)=0)) = \delta > 0$, since $\frac{te^{-t}}{>}0$ for $t > 0$ that is to say the probability to flip at the origin is strictly positive. Then

$$\mathbb{P}_\mu(I(\eta_t(0)=0)) < \delta$$

so μ is not invariant, hence the only possible invariant measure are the ones such that $\mu(I(\eta(0)=0)) = 0$. Similar discussion on μ with $\mu(\eta(x)) = \delta > 0$, we can get the only possible invariant measure are the ones with $\mu(\eta(x)) = 0, \forall x \in \mathbb{Z}$. Then the only possible one is $\mu = \delta_1$(with $\delta_1(\{\bar{1}\}) = 1$). But δ_1 is not invariant since at $\bar{1}$ flips $\bar{0}$ definitely. So there is no invariant measure for this process. Then we can conclude that this flip process is non-Feller. ☐

4.3 Some mathematical tools

Generally, in the field of interacting particle systems, there are two methods that are used very often in the construction of processes, one is “coupling”, the other is “monotonicity”.

4.3.1 Coupling

Coupling is the most important and generally applicable technique used in the construction of the process. A coupling is simply a construction of two or more stochastic processes on a common probability space.

The convenience of coupling method can even be shown by the following simple example. For instance, we want to prove that two functions $f(\eta)$ and $g(\eta)$ are positively correlated when η is a real-valued random variable and f and g are two bounded increasing functions on the real lines. If we only use the variable η, it is hard to get the relation since the sign of $\mathbf{E}[f(\eta) - \mathbf{E}(f(\eta))][g(\eta) - \mathbf{E}(g(\eta))]$ is not decided. However, if one let η and ζ be two *i.i.d* variables, the proof is quite simple:

$$\begin{aligned} 0 &\leq \mathbf{E}[f(\eta) - g(\zeta)][g(\eta) - g(\zeta)] \\ &= \mathbf{E}f(\eta)g(\eta) + \mathbf{E}f(\zeta)g(\zeta) - \mathbf{E}f(\eta)g(\zeta) - \mathbf{E}f(\zeta)g(\eta) \\ &= 2\{\mathbf{E}f(\eta)g(\eta) - \mathbf{E}f(\eta)\mathbf{E}g(\eta)\} \\ &= 2\text{cov}\{f(\eta), g(\eta)\} \end{aligned} \tag{4.3.1}$$

The monotonicity of f and g makes "$\leq$" set up. In the third step, we use the independent and identical property of η and ζ.

4.3.2 Monotonicity

1. For $\eta, \zeta \in \Omega$, we say that $\eta \leq \zeta$ if $\eta(x) \leq \zeta(x), \forall x \in \mathbb{Z}$.
2. A function $f : \Omega \to \mathbb{R}$ is called "monotone" if for any $\eta, \zeta \in \Omega$ such that $\eta \leq \zeta$, we have $f(\eta) \leq f(\zeta)$. $\mathcal{M}$ denotes the set of all monotone bounded functions.
3. A Markov process $\{\eta_t, t \geq 0\}$ on Ω with Markov semigroup group $\{S(t), t \geq 0\}$ is monotone if $f \in \mathcal{M}$ implies $S(t)f \in \mathcal{M}$.
4. μ_1, μ_2 are two probability measure on Ω, we say $\mu_1 \leq \mu_2$, if:

$$\int f d\mu_1 \leq \int f d\mu_2 \text{ for all } f \in \mathcal{M} \tag{4.3.2}$$

In [12], there is a theorem gives us a way to prove that two measures have a monotone relation:

Theorem 4.3.1. *Suppose μ_1 and μ_2 are two probability measures on Ω. A necessary and sufficient condition for $\mu_1 \leq \mu_2$ is that there exists a probability measure ν on $\Omega \times \Omega$ which*

satisfies:
(a) $\nu\{(\eta,\zeta):\eta\in\mathbf{A}\}=\mu_1(\mathbf{A})$. *(b)* $\nu\{(\eta,\zeta):\zeta\in\mathbf{A}\}=\mu_2(\mathbf{A})$. *for all Borel sets in* Ω, *and* *(c)* $\nu\{(\eta,\zeta):\eta\leq\zeta\}=1$.

Proof. The proof is contained in [12], Theorem 2.4 in chapter 2. □

Remark: In order to prove a process is monotone, we only need to prove $S(t)f$ is monotone for any $f\in\mathcal{M}$, i.e, $(S(t)f)(\eta)\leq(S(t)f)(\zeta)$ for any $\eta\leq\zeta$, and hence we need:

$$(S(t)f)(\eta)=\mathbf{E}^{\eta}f(\eta_t)\leq(S(t)f)(\zeta)=\mathbf{E}^{\zeta}f(\zeta_t) \tag{4.3.3}$$

From theorem 4.3.1, we know it suffices to construct a coupling processes η_t and ζ_t and prove that:

$$\mathbf{P}^{\eta,\zeta}\{(\eta_t,\zeta_t):\eta_t\leq\zeta_t\}=1$$

where $\mathbf{P}^{\eta,\zeta}$ is the coupling measure of $\mathbf{P}^{\eta}$ and $\mathbf{P}^{\zeta}$, which satisfies
$\mathbf{P}^{\eta,\zeta}((\eta,\zeta):\eta\in A)=\mathbf{P}^{\eta}(A)$, $\mathbf{P}^{\eta,\zeta}((\eta,\zeta):\zeta\in A)=\mathbf{P}^{\zeta}(A)$

4.4 Infinite volume anti-sandpile process

In the one-dimensional anti-sandpile model, $\Omega=\{0,1\}^{\mathbb{Z}}$ is the set of all stable configurations. And the critical value $\eta^{\dagger}_{xc}=0$, i.e, when $\eta^{\dagger}_{x}<0$, site x is unstable and anti-topplings happen in the system.

4.4.1 Anti-addition operator

The dynamics of the anti-sandpile is guided by the addition of holes and relaxation of the system. Addition of a hole to a site may influence the whole configuration. The study of the anti-addition operators starts from simple configurations on which the anti-addition operators remain local.

Definition 4.4.1. *Let* $\Omega^{\dagger}_{f}$ *be the set of configurations with a finite number of critical sites, i.e.*

$$\Omega^{\dagger}_{f}=\{\eta\in\Omega:\mid\eta^{-1}(\{0\})\mid<\infty\} \tag{4.4.1}$$

where $\eta^{-1}(\{0\})=\{x\in\mathbb{Z}:\eta(x)=0\}$.

The set $\Omega^{\dagger}_{f}$ is dense in Ω. For every configuration in $\Omega^{\dagger}_{f}$, we can choose a finite interval that contains $\eta^{-1}(\{0\})$. For the convenience, we will introduce two notations:

$$k^{+}(x,\eta)^{\dagger}:=\inf\{y\geq0:\eta_{x+y}=1\} \tag{4.4.2}$$

and

$$k^-(x,\eta)^\dagger := \inf\{y>0 : \eta_{x-y}=1\} \tag{4.4.3}$$

where $\inf\emptyset := +\infty$. Then $[x+k^+(x,\eta), x-k^-(x,\eta)^\dagger]$ is the very set characterized by x with the boundary site of height 1 and the interval sites of height 0. So every anti-addition (addition of a hole) to the inside of $[x+k^+(x,\eta), x-k^-(x,\eta)^\dagger]$ has no influence on the outside of this interval.

The anti-addition operator at site x is denoted by $a_x^\dagger$. For a finite interval $\Lambda \subset \mathbb{Z}$, we use $(a_x^\dagger)_\Lambda$ denotes the anti-addition operator that be strict on Λ, i.e, $(a_x^\dagger)_\Lambda \eta = ((a_x^\dagger)_\Lambda \eta_\Lambda)\eta_{\Lambda^c}$

Let $\overline{e_x}$ is a configuration such that $\overline{e_x}(x)=1$ and $\overline{e_x}(y)=0, \forall y\neq x$. For all $\eta \in \Omega_f^\dagger$, when $\eta(x)=1$ $a_x^\dagger\eta = \eta - \overline{e_x}$. When $\eta_x = 0$, $k^+(x,\eta)^\dagger < \infty, k^-(x,\eta)^\dagger < \infty$, imagine there is a "mirror" in the middle of the interval $[x+k^+(x,\eta), x-k^-(x,\eta)^\dagger]$, then after the operation of $a_x^\dagger$, the heights of the boundary of this set becomes 0, the height of the "mirror image" of site x becomes 1 and all the others keep the same. E.g., for a configuration

$$\cdots 1100\ \underbrace{10\dot{0}0 \mid 0001}\ 1111\cdots$$

Here denote the place of $\dot{0}$ is x, $k^+(x,\eta)^\dagger = 5$, $k^-(x,\eta)^\dagger = 2$, "|" is the mirror. Then the configuration $a_x^\dagger\eta$ is

$$\cdots 1100\ \underbrace{00\dot{0}0 \mid 0100}\ 1111\cdots$$

In the case that at least one of $k^+(x,\eta)^\dagger$ is not finite, we can choose a sequence of configurations $\eta_n = \eta^{\Lambda n}$ such that, $\eta^{\Lambda n}(x) = \eta(x)$, for $x\in\Lambda_n$, and and $\eta^{\Lambda_n}(x)=1$ for $x\in\Lambda_n^c$, then $\eta^{\Lambda n}\downarrow\eta$ with the $k^+(x,\eta_n)^\dagger < \infty, k^-(x,\eta_n)^\dagger<\infty$ for all n. Then we take $a_x^\dagger\eta = \lim_{n\to\infty} a_x^\dagger\eta_n$. For example:

$$\cdots 000101\ \underbrace{000\dot{0}0}\ 0000\cdots$$

Take off a particle from the site of "$\dot{0}$", the final state is:

$$\cdots 000100\ \underbrace{000\dot{0}0}\ 0000\cdots$$

Summarizing the discussion above, we get the following proposition.

Proposition 4.4.2. *The final state of adding a hole at any site x are expressed in five cases:*

1. $k^+(x,\eta)^\dagger = 0, i.e., \eta_x = 1$, *then:*

$$a_x^\dagger\eta = \eta - \bar{e}_x \tag{4.4.4}$$

2. $k^+(x,\eta)^\dagger > 0$, $k^+(x,\eta)^\dagger \bigvee k^-(x,\eta)^\dagger < \infty$, *then*

$$a_x^\dagger \eta = \eta - \overline{e}_{x+k^+(x,\eta)^\dagger} - \overline{e}_{x-k^-(x,\eta)^\dagger} + \overline{e}_{x+k^+(x,\eta)-k^-(x,\eta)} \tag{4.4.5}$$

3. $k^+(x,\eta)^\dagger = \infty, k^-(x,\eta)^\dagger < \infty$, *then*

$$a_x^\dagger \eta = \eta - \overline{e}_{x-k^-(x,\eta)^\dagger} \tag{4.4.6}$$

4. $k^+(x,\eta)^\dagger < \infty, k^-(x,\eta)^\dagger = \infty$, *then:*

$$a_{x^\dagger} \eta = \eta - \overline{e}_{x+k^+(x,\eta)^\dagger} \tag{4.4.7}$$

5. $k^+(x,\eta)^\dagger = k^-(x,\eta)^\dagger = \infty$, *then:*

$$a_x^\dagger \eta = \eta \tag{4.4.8}$$

Combine with the definition of addition and anti-addition operators in finite volume, we know $a_x^\dagger = \theta a_x \theta$ also holds in the infinite volume case. And hence $a_x^\dagger a_y^\dagger = a_y^\dagger a_x^\dagger$ by the abelian property of addition operators, see (3.17) of [13].

In the infinite volume case, if a configuration η is recurrent, we know all the sub-configurations of η must be recurrent and the reverse part is also right. So the definition of the recurrent is given,

Definition 4.4.3. *A configuration $\eta \in \Omega$ is recurrent if and only if any sub-configuration $\eta_\Lambda, \Lambda \in \mathbb{Z}$ is recurrent. Where η_Λ is the restriction of η to Λ. $\mathcal{R}^\dagger$ be the set of all such recurrent configurations.*

4.4.2 Anti-sandpile process

Let $\{N_t^x\}_{x\in\mathbb{Z}}$ be a collection of independent rate 1 Poisson processes. $a_x^\dagger$ acting on $\eta \in \Omega$ follows the process of N_t^x. We want to define a process informally described by:

$$\eta_t = \prod_{x\in\mathbb{Z}} (a_x^\dagger)^{N_t^x} \eta \tag{4.4.9}$$

The formal generator for this process is: for a local function $f : \Omega \to \mathbb{R}$

$$\mathcal{L}^\dagger f(\eta) = \sum_{x\in\mathbb{Z}} (f(a_x^\dagger \eta) - f(\eta)) = \sum_{x\in\mathbb{Z}} (a_x^\dagger - I) f(\eta) \tag{4.4.10}$$

Firstly, we can not be sure whether such a process exists or not, since in the infinite volume case, the addition operator and anti-addition operators loose the local property, so we could not get the generator but a formal one. However by the non-local property of $a_x^\dagger$, the formal generator (4.4.10) is not continuous on Ω, and the process is not a Feller process. Therefore the classical techniques, such as Hille-Yoshida, break down here. In [13], the infinite volume one-dimensional sandpile model is constructed, the resulting process is non-Feller and the only stationary measure is δ_1(The dirac measure concentrating on $\eta^* = \overline{1}$, i.e, of all heights are 1). As to the infinite volume one-dimensional anti-sandpile model, we can use the similar method as in [13] to construct the process, however, I would not plan to give the specific steps, but use the conjugate relation of sandpile model and anti-sandpile model to give the semigroup and stationary measure for the anti-sandpile process directly.

The Markov process $\eta_t = \prod_{x\in\mathbb{Z}} a_x^{N_t^x}\eta$ is a process with formal generator, for all bounded local function f on Ω,

$$\mathcal{L}f(\eta) = \textstyle\sum_{x\in\mathbb{Z}}[f(a_x\eta) - f(\eta)]$$

the corresponding semigroup is denoted by $S(t)$, see [13]. If we $S^\dagger(t)$ to denote the semigroup of the process with generator (4.4.10.)

Since the two generators have the following relations,

$$\mathcal{L}^\dagger = \theta\mathcal{L}\theta$$

and

$$S^\dagger(t)f(\eta) = E^\eta f(\eta_t) = E^\eta\theta f(\theta\eta_t) = \theta S(t)f(\theta\eta) = \theta S(t)\theta f(\eta)$$

Then

$$S^\dagger(t) = \theta S(t)\theta \tag{4.4.11}$$

The anti-sandpile process is not a Feller process, since the sandpile process with generator $\mathcal{L}$ is not Feller[13], and the stationary measure for this process is $\theta\delta_1 = \delta_0$.

4.4.3 Properties of the anti-sandpile process

The infinite volume one-dimensional anti-sandpile process is not a Feller process. So some of the properties of Feller process do not hold in general, however, for some "special" configuration η and "special" function f, the generator $\mathcal{L}^\dagger$ and semigroup $S^\dagger(t)$ still have a good relation. In order to give the properties, we need the "$N-$local" function and “decent’ configurations which are introduced in [13]. I just put them here. We use Ω_1 as the set of configurations with infinite “0” at either side of the origin.

$$\Omega_1 = \{\eta \in \Omega : \sum_{x<0}(1-\eta(x)) = \sum_{x>0}(1-\eta(x)) = \infty\}$$

For $\eta \in \Omega_1$, we order all the sites of height “0” in a sequence as:

$$X_0(\eta) = \min\{x \geq 0, \eta(x) = 0\}, X_1(\eta) = \min\{x > X_0(\eta), \eta(x) = 0\}, \text{ etc.}$$

And define

$$X_{-1}(\eta) = \max\{x < 0, \eta(x) = 0\}, X_{-2}(\eta) = \max\{x < X_{-1}(\eta), \eta(x) = 0\}, \text{ etc.}$$

Then we define the intervals:

$$\begin{aligned} \mathbf{I}_0 &= (X_{-1}, X_0] \cap \mathbb{Z} \\ \mathbf{I}_1 &= (X_0, X_1] \cap \mathbb{Z} \\ \mathbf{I}_{-1} &= (X_{-2}, X_{-1}] \cap \mathbb{Z}, etc \end{aligned} \tag{4.4.12}$$

Then a configuration is called *decent* if $\eta \in \Omega_1$ and

$$a(\eta) = \limsup_{n\to\infty} \frac{|I_{-n}(\eta)| + \cdots + |I_n(\eta)|}{2n} = a(\eta) < \infty$$

We use Ω_{dec} denote the set of all decent configurations. Now, we define:

- $\Omega_1^\dagger = \theta\Omega_1$
- $X_k^\dagger(\eta) = X_k(\theta\eta), I_k^\dagger = I_k(\theta\eta), \forall k \in \mathbb{Z}, \forall \eta \in \Omega_1^\dagger$
- $\Omega_{dec}^\dagger = \theta\Omega_{dec}$

A function $f : \Omega_1 \to \mathbb{R}$ is called “N-local”, if it only depends on the heights η_x, $x \in \cup_{k=-N}^N I_k(\eta)$[14]. For N-local function f, define $\theta \circ f : \Omega_1^\dagger \to \mathbb{R}$:

$$\theta \circ f(\eta) := f(\theta\eta) \tag{4.4.13}$$

It is a function that only depends on the heights of $\cup_{k=-N}^N I_k^\dagger(\eta)$. Immediately, we have:

$$f(a_x\theta\eta) - f(\theta\eta) = 0, \forall x \in \mathbb{Z}\backslash \cup_{k=-N-1}^{N+1} \mathbf{I}_k^\dagger \tag{4.4.14}$$

So $\mathcal{L}f(\theta\eta) = \sum_{x\in\cup_{k=-N-1}^{N+1} I_k^\dagger(\eta)} [f(a_x\theta\eta) - f(\theta\eta)]$, and hence:

$$\mathcal{L}^\dagger f(\eta) = \theta \circ \mathcal{L} \circ \theta f(\eta) = \sum_{x\in\cup_{k=-N-1}^{N+1} I_k^\dagger(\eta)} [f(a_x^\dagger \eta) - f(\eta)] \tag{4.4.15}$$

Lemma 4.4.4. $\{a_n : n \geq 0\}$ *is a sequence of positive real numbers such that:*

$$\limsup_{n\to\infty} \tfrac{a_n}{n} = a < \infty$$

Then the series $\sum_{n=1}^{\infty} t^n a_n{}^n \diagup n!$ *converges for* $| t |< \frac{1}{a.e}$

Take $a(n) = \frac{1}{2}\sum_{k=-n}^{n} | \mathbf{I}_k^\dagger(\eta) |$, then for $\eta \in \Omega_{dec}^\dagger$, we have:

$$\sum_{n=1}^{\infty} t^n (\sum_{k=-n}^{n} | \mathbf{I}_k^\dagger(\eta) |)^n \diagup 2^n n! \text{ converges, for } | t |< \frac{1}{a^\dagger(\eta)\cdot e}. \tag{4.4.16}$$

$\forall \eta \in \Omega_f^\dagger$, $| \eta^{-1}(0) |=| \{x \in \mathbb{Z} : \eta(x) = 0\} |< \infty$. Then for $|x|$ large enough, $\eta(x) = 1$, then for n large enough, $| \mathbf{I}_n^\dagger(\eta) |= 1$ and $| \mathbf{I}_{-n}^\dagger(\eta) |= 1$, $\frac{1}{2n}\sum_{k=-n}^{n} | \mathbf{I}_k^\dagger(\eta) | \to 1$, and hence $\eta \in \Omega_{dec}^\dagger$. Then for the three sets that we use often in this section have the relation:

$$\Omega_f^\dagger \subset \Omega_{dec}^\dagger \subset \Omega_1^\dagger \tag{4.4.17}$$

The following proposition shows that for $\eta \in \Omega_{dec}$ and $N-$local, the generator and the semigroup have the following relation,

Proposition 4.4.5. *Let* $\eta \in \Omega_{dec}^\dagger$, f *be bounded and* $\mathbf{N}$*-local, when* $t < \frac{1}{4a^\dagger(\eta)\cdot e}$, *the series* $\sum_{n=0}^{k} \frac{t^n}{n!}((\mathcal{L}^\dagger)^n f)(\eta)$ *converges absolutely to* $\sum_{n=0}^{\infty} \frac{t^n}{n!}(\mathcal{L}^\dagger)^n f(\eta)$ *which equals* $S^\dagger(t)f(\eta)$. *The semigroup is right continuous:*

$$\lim_{t\to 0} \frac{(S^\dagger(t)f)(\eta) - f(\eta)}{t} = \mathcal{L}^\dagger f(\eta) \tag{4.4.18}$$

Proof. We know $S^\dagger(t) = \theta S(t)\theta$ and $\mathcal{L}^\dagger = \theta\mathcal{L}\theta$. And for $\eta \in \Omega_{dec}^\dagger, \theta\eta \in \Omega_{dec}$, and from Theorem 4.1 of [13], we know for $t < \frac{1}{4e\cdot a(\theta\eta)}$, $\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{L}^n f(\theta\eta)$ converges absolutely and equals $S(t)f(\theta\eta)$; $\lim_{t\downarrow 0}\frac{S(t)f(\theta\eta)-f(\theta\eta)}{t}=\mathcal{L}f(\theta\eta)$. Then when $t<\frac{1}{4e\cdot a(\theta\eta)}=\frac{1}{4e\cdot a^{\dagger}(\eta)}$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}(\mathcal{L}^{\dagger})^n f(\eta) &= \sum_{n=0}^{\infty}\frac{t^n}{n!}(\theta\mathcal{L}\theta)^n f(\eta)\\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\theta\mathcal{L}^n\theta f(\eta)=\theta\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathcal{L}^n f(\theta\eta)\\ &= \theta S(t)\theta f(\eta)=S^{\dagger}(t)f(\eta)\end{aligned} \tag{4.4.19}$$

and we also have that:

$$\lim_{t\downarrow 0}\frac{\theta S(t)f(\theta\eta)-\theta f(\theta\eta)}{t}=\theta\mathcal{L}f(\theta\eta)=\mathcal{L}^{\dagger}f(\eta) \tag{4.4.20}$$

i.e.,

$$\lim_{t\downarrow 0}\frac{S^{\dagger}(t)f(\eta)-f(\eta)}{t}=\mathcal{L}^{\dagger}f(\eta) \tag{4.4.21}$$

□

4.5 Sandpile+anti-sandpile process

Now we combine both a_x and $a_x^{\dagger}$ in the same process. Associate to each site x two independent Poisson Processes $N_t^{x,\alpha}$ with rate α and $N_t^{x,\beta}$ with rate β. All such Poisson processes are independent with each other. On the event times of $N_t^{x,\alpha}$, addition operator a_x acts on η and on the event times of $N_t^{x,\beta}$, $a_x^{\dagger}$ operates on η. Then we get a mixed type process with the formal generator:

$$\mathcal{L}_{\alpha\beta}=\alpha\mathcal{L}+\beta\mathcal{L}^{\dagger}, f-\text{local} \tag{4.5.1}$$

where $\mathcal{L}=\sum_{x\in\mathbb{Z}}(a_x-\mathbf{I})$, $\mathcal{L}^{\dagger}=\sum_{x\in\mathbb{Z}}(a_x^{\dagger}-\mathbf{I})$. We call this process SA(sandpile+anti-sandpile)-process. In the later part of the thesis, when I mention $SA-$process, I always mean the process with formal generator (4.5.1).

The most typical method to construct a process related to the sandpile model is the method used in [13], i.e, start from simple configuration in Ω_f, construct a "monotone" process and then extend it to the general configuration. And using the conjugate property of the sandpile model and the anti-sandpile model, we know the construction of the pure anti-sandpile process should start from $\Omega_f^{\dagger}=\theta\Omega_f$, it is also a monotone process. From experience, when starting to construct the $SA-$process, we should start from a configuration in $\Omega_f\cap\Omega_f^{\dagger}$, however this method fails, because $\Omega_f\cap\Omega_f^{\dagger}=\emptyset$. Another way is necessary.

4.5.1 Construction of the SA-process

In both the sandpile process and anti-sandpile process, we know for the decent configurations and $N-$local functions, the semigroup and generator has a good relation that for $\eta \in \Omega_{dec}$, and $t < \frac{1}{4ea(\eta)}$, the series: $\sum_{n=0}^{\infty} \frac{t^n \mathcal{L}^n f(\eta)}{n!}$ converges absolutely and, for $\eta \in \Omega_{dec}^{\dagger}$, $t < \frac{1}{4e \cdot a^{\dagger}(\eta)}$, $\sum_{n=0}^{\infty} \frac{t^n (\mathcal{L}^{\dagger})^n f(\eta)}{n!} = \theta \sum_{n=0}^{\infty} \frac{t^n \mathcal{L}^n f(\theta\eta)}{n!}$ also converges absolutely, which gives us an idea to construct the SA-process.

The construction of the $SA-$process is divided two steps:

1. At first, for configuration η such that $\eta \in \Omega_{dec} \cap \Omega_{dec}^{\dagger}$ and $N-$ local function f, we get the semigroup $S(t)$ by series expansion;

2. Using the monotonicity of the process, we extend the definition of semigroup to every configuration and every continuous function.

For $\eta \in \Omega_{dec} \cap \Omega_{dec}^{\dagger}$, and define: $B_n(\eta) = \cup_{k=-n}^{n} I_k(\eta)$ and $B_n^{\dagger}(\eta) = \cup_{k=-n}^{n} I_k^{\dagger}(\eta)$.

Lemma 4.5.1. *Take $\eta \in \Omega_{dec} \cap \Omega_{dec}^{\dagger}$, for all $x \in B_n(\eta)$,*

(1) $B_n(a_x\eta) \subseteq B_{n+1}(\eta)$

(2) $B_n^{\dagger}(a_x\eta) \subseteq B_n^{\dagger}(\eta)$

and for all $x \in B_n^{\dagger}(\eta)$, we have:

(3) $B_n(a_x^{\dagger}\eta) \subseteq B_n(\eta)$

(4) $B_n^{\dagger}(a_x^{\dagger}\eta) \subseteq B_{n+1}^{\dagger}(\eta)$

Proof. For $x \in B_n(\eta)$, when a_x operates on η, the number of sites of height "1" in $B_{n+1}(\eta)$ will increase 1, consequently, $B_n(a_x\eta) \subseteq B_{n+1}(\eta)$; the number of sites of height "1" increases means that one of the intervals $I_k^{\dagger}$ is split into two, so $B_n^{\dagger}(a_x\eta) \subseteq B_n^{\dagger}(\eta)$.

By the conjugate property, we get: $B_n^{\dagger}(a_x^{\dagger}\eta) = B_n(\theta a_x^{\dagger}\eta) = B_n(a_x\theta\eta)$ and $B_n(a_x^{\dagger}\eta) = B_n^{\dagger}(\theta a_x^{\dagger}\eta) = B_n^{\dagger}(a_x\theta\eta)$. From the relation of (1) and (2), we know: $B_n(a_x\theta\eta) \subseteq B_{n+1}(\theta\eta) = B_{n+1}^{\dagger}(\eta)$ and $B_n^{\dagger}(a_x\theta\eta) \subseteq B_n^{\dagger}(\theta\eta) = B_n(\eta)$. And hence we have: $B_n^{\dagger}(a_x^{\dagger}\eta) \subseteq B_{n+1}^{\dagger}(\eta)$ and $B_n(a_x^{\dagger}\eta) \subseteq B_n(\eta)$, which are the statement of (3), (4). □

Then for a function f, if it depends on the heights of $B_N(\eta) \cup B_N^{\dagger}(\eta)$, then $f(a_x\eta) = f(\eta)$ and $f(a_x^{\dagger}\eta) = f(\eta)$ for $x \notin B_{N+1}(\eta) \cup B_{N+1}^{\dagger}(\eta)$, therefore,

$$\mathcal{L}_{\alpha\beta} f(\eta) = \alpha \sum_{x \in B_{N+1}(\eta) \cup B_{N+1}^{\dagger}(\eta)} [f(a_x\eta) - f(\eta)] + \beta \sum_{x \in B_{N+1}(\eta) \cup B_{N+1}^{\dagger}(\eta)} [f(a_x^{\dagger}\eta) - f(\eta)]$$

Then we get the following proposition,

Theorem 4.5.2. *For a N-local function f, and any $\eta \in \Omega_{dec} \cap \Omega^{\dagger}_{dec}$, we have:*

$$| \mathcal{L}^n_{\alpha\beta} f(\eta) | \leq (\alpha+\beta)^n \mid B_{N+n}(\eta) \cup B^{\dagger}_{N+n}(\eta) \mid^n 2^n \|f\|_\infty$$

Proof. For n=1,
$\mathcal{L}_{\alpha\beta} f(\eta) = \alpha \sum_{x \in B_{N+1} \cup B^{\dagger}_{N+1}} [f(a_x\eta) - f(\eta)] + \beta \sum_{x \in B_{N+1} \cup B^{\dagger}_{N+1}} [f(a^{\dagger}_x\eta) - f(\eta)]$. So,

$$| \mathcal{L}_{\alpha\beta} f(\eta) | \leq (\alpha+\beta) \mid B_{N+1}(\eta) \cup B^{\dagger}_{N+1}(\eta) \mid 2\|f\|_\infty$$

Assume for $n=k$, the claim is right, then we try to get the expression for $n=k+1$, Denoting $\mathcal{L}^k_{\alpha\beta} f(\eta) = g^k\eta$, since f is $N-$local, then $\mathcal{L}_{\alpha\beta} f(\eta)$ only depends on the heights of $B_{N+1}(\eta) \cup B^{\dagger}_{N+1}(\eta)$, then by the induction, we know $g^k(\eta)$ is a function that depends on the heights of $B_{N+k}(\eta) \cup B^{\dagger}_{N+k}(\eta)$, then we have:

$$\begin{aligned} \mathcal{L}_{\alpha\beta} g^k(\eta) = \alpha & \sum_{x \in B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta)} [g^k(a_x\eta) - g^k(\eta)] \\ & + \beta \sum_{x \in B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta)} [g^k(a^{\dagger}_x\eta) - g^k(\eta)] \end{aligned} \quad (4.5.2)$$

Then:

$$\begin{aligned} | \mathcal{L}^{k+1}_{\alpha\beta} f(\eta) | \leq & (\alpha+\beta) \mid B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta) \mid |g^k(\eta)| \\ & + \alpha \sum_{x \in B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta)} | g^k(a_x\eta) | \\ & + \beta \sum_{x \in B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta)} | g^k(a^{\dagger}_x\eta) | \end{aligned} \quad (4.5.3)$$

While: $| g^k(a_x\eta) | \leq (\alpha+\beta)^k \mid B_{N+k}(a_x\eta) \cup B^{\dagger}_{N+k}(a_x\eta) \mid^k 2^k \|f\|_\infty$
from Lemma 4.5.1, we know: $B_{N+k}(a_x\eta) \subseteq B_{N+k+1}(\eta), B^{\dagger}_{N+k}(a_x\eta) \subseteq B^{\dagger}_{N+k+1}(\eta)$, then:

$$\alpha \sum_{x \in B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta)} | g^k(a_x\eta) | \leq \alpha(\alpha+\beta)^k \mid B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta) \mid^{k+1} 2^k \|f\|_\infty \quad (4.5.4)$$

Similarly, we get:

$$\beta \sum_{x \in B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta)} | g^k(a^{\dagger}_x\eta) | \leq \beta(\alpha+\beta)^k \mid B_{N+k+1}(\eta) \cup B^{\dagger}_{N+k+1}(\eta) \mid^{k+1} 2^k \|f\|_\infty \quad (4.5.5)$$

Then sum up the above inequalities:

$$| \mathcal{L}_{\alpha\beta}^{k+1} f(\eta) | \leq (\alpha+\beta)^{k+1} | B_{N+k+1}(\eta) \cup B_{N+k+1}^{\dagger}(\eta) |^{k+1} 2^{k+1} \|f\|_{\infty} \tag{4.5.6}$$

□

The following lemma will be used in the proof of proposition 4.5.4, I just include it here.

Lemma 4.5.3. *For* $a, b \in \mathbb{R}^{+} \cup \{0\}, (a+b)^n \leq 2^n(a^n + b^n)$

Proof. For $a, b \in \mathbb{R}^{+} \cup \{0\}$, $(a+b)^n \leq 2^n(\max\{a,b\})^n \leq 2^n(a^n + b^n)$ □

Proposition 4.5.4. *Let* $\eta \in \Omega_{dec} \cap \Omega_{dec}^{\dagger}$*,* f *be a bounded and* N*-local function, then for* $t < \min\{\frac{1}{8e(\cdot\alpha+\beta)\cdot a(\eta)}, \frac{1}{8e\cdot(\alpha+\beta)\cdot a^{\dagger}(\eta)}\}$*, the series* $\sum_{n=0}^{\infty} \frac{t^n}{n!}(\mathcal{L}_{\alpha\beta}^n f)(\eta)$ *converges absolutely. Then define:*

$$S(t)f(\eta) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\mathcal{L}_{\alpha\beta}^n f)(\eta) \tag{4.5.7}$$

Proof. Use Lemma 4.5.3 on inequality 4.5.6:

$$\mathcal{L}_{\alpha\beta}^n f(\eta) \leq 2^n(\alpha+\beta)^n(| B_{N+n}(\eta) |^n + | B_{N+n} |^n)2^n \|f\|_{\infty} \tag{4.5.8}$$

For $\eta \in \Omega_{dec} \cap \Omega_{dec}^{\dagger}$, we have:

$$\limsup_{n\to\infty} \frac{|B_{N+n}(\eta)|}{2n} = \limsup_{n\to\infty} \frac{|B_{N+n}(\eta)|}{2(N+n)} = a(\eta) < \infty$$

and:

$$\limsup_{n\to\infty} \frac{|B_{N+n}^{\dagger}(\eta)|}{2n} = \limsup_{n\to\infty} \frac{|B_{N+n}^{\dagger}(\eta)|}{2(N+n)} = a^{\dagger}(\eta) < \infty$$

By lemma 4.4.4, we know:

1) For $t < \frac{1}{8e(\alpha+\beta)a(\eta)}$, $\sum_{n=0}^{\infty} \frac{4^n(\alpha+\beta)^n t^n |B_{N+n}(\eta)|^n}{n!}$ converges.

2) For $t < \frac{1}{8e(\alpha+\beta)a^{\dagger}(\eta)}$, $\sum_{n=0}^{\infty} \frac{4^n(\alpha+\beta)^n t^n |B_{N+n}^{\dagger}(\eta)|^n}{n!}$ converges.

Then for $t < \min\{\frac{1}{8e(\alpha+\beta)a(\eta)}, \frac{1}{8e(\alpha+\beta)a^{\dagger}(\eta)}\}$, $\sum_{n=0}^{\infty} \frac{t^n}{n!}(\mathcal{L}_{\alpha\beta}^n f)(\eta)$ converges absolutely. □

4.5.2 Extending the definition of semigroup according to initial measure

Lemma 4.5.5. *Assume μ be an positive translation invariant mixing measure on Ω, $\rho(\eta) = \int I(\eta(0) = 0)d\mu$ denote the density of "0" in the whole configuration, then if $\rho(\eta) \in (0,1)$, $\eta \in \Omega_{dec} \cap \Omega^{\dagger}_{dec}$.*

Proof. For any $\eta \in \Omega_{dec} \cap \Omega^{\dagger}_{dec}$, we know:

$$a(\eta) = \limsup_{n\to\infty} \tfrac{1}{2n} \textstyle\sum_{k=-n}^{n} \mid \mathbf{I_k}(\eta) \mid < \infty$$

$$a^{\dagger}(\eta) = \limsup_{n\to\infty} \tfrac{1}{2n} \textstyle\sum_{k=-n}^{n} \mid \mathbf{I^{\dagger}_k}(\eta) \mid < \infty$$

Then for a η such that $\rho(\eta) > 0$, we know the average size of all such interval $I_k(\eta)$ equals to $\frac{1}{\rho(\eta)}$, i.e.,

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} \mid \mathbf{I_k}(\eta) \mid = \frac{1}{\rho(\eta)} = a(\eta) < \infty \tag{4.5.9}$$

This tells us that for η such that $\rho(\eta) > 0$, $\eta \in \Omega_{dec}$. Use the same discussion, for η such that $1 - \rho(\eta) > 0$, $a^{\dagger}(\eta) = \frac{1}{1-\rho(\eta)} < \infty$, i.e, $\eta \in \Omega^{\dagger}_{dec}$. In a word, $\rho(\eta) \in (0,1)$ implies $\eta \in \Omega_{dec} \cap \Omega^{\dagger}_{dec}$. □

Proposition 4.5.6. *Let μ be the initial measure on Ω, which is translation invariant and mixing measure concentrating on $\Omega_{dec} \cap \Omega^{\dagger}_{dec}$ with:*

$$\mu(\eta(0) = 0) = \rho \in (0,1)$$

Then the process can be constructed in the following time intervals,

(1) For $\beta < \alpha$, $0 \le t < \frac{\rho}{\alpha-\beta}$

(2) For $\alpha = \beta$, $t \in [0,\infty)$

(3) For $\beta > \alpha$, $0 \le t < \frac{1-\rho}{\beta-\alpha}$

Proof. μ is the initial state measure. And according to Proposition 4.5.4 and lemma 4.5.5, we know the semigroup $S(t)$ is defined in the form of (4.5.7) for $0 \le t < t^*_1(\mu) = \min\{\frac{\rho}{8e(\cdot\alpha+\beta)}, \frac{1-\rho}{8e\cdot(\alpha+\beta)}\}$.

Under the condition that $0 \le t < t^*_1(\mu)$, $\rho_t = \int I(\eta(0) = 0)dS(t)\mu(\mathrm{t})$ is the probability of "0"'s at time t with initial measure μ,

$$\rho_t = \textstyle\int I(\eta(0) = 0)dS(t)\mu = \int S(t)(I(\eta(0) = 0))d\mu$$

Differentiate this equation:

$$\frac{d\rho_t}{dt} = \int \frac{dS(t)I(\eta(0) = 0)}{dt} d\mu \tag{4.5.10}$$

Then for $t \in [0, t_1^*(\mu))$, substitute $S(t)$ by (4.5.7)

$$\frac{dS(t)I(\eta(0) = 0)}{dt} = \mathcal{L}_{\alpha\beta} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\mathcal{L}_{\alpha\beta}^n f)(\eta) = \mathcal{L}_{\alpha\beta} S(t) f(\eta) \tag{4.5.11}$$

Then substitute (4.5.11) to (4.5.10):

$$\frac{d\rho_t}{dt} = \int \mathcal{L}_{\alpha\beta} S(t) I(\eta(0) = 0) d\mu = \int \mathcal{L}_{\alpha\beta} I(\eta(0) = 0) dS(t)\mu \tag{4.5.12}$$

we know for any μ translation invariant, $\int \mathcal{L} I(\eta(0) = 0) d\mu = -1$ and $\int \mathcal{L}^\dagger I(\eta(0) = 0) d\mu = 1$[13], then get:

$$\frac{d\rho_t}{dt} = \beta - \alpha, t \in [0, t_1^*(\mu)) \tag{4.5.13}$$

Get the solution for this equation:

$$\rho_t = \rho + (\beta - \alpha)t, \text{ with } \rho = \int I(\eta(0) = 0) d\mu > 0, t \in [0, t_1^*(\mu)) \tag{4.5.14}$$

We can check that for any small $\varepsilon > 0$,

$$0 < \rho_{t_1^*(\mu)-\varepsilon} = \rho + (\beta - \alpha)(t_1^*(\mu) - \varepsilon) < 1 \tag{4.5.15}$$

Let $\mu_{t_1^*(\mu)-\varepsilon} := S(t_1^*(\mu) - \varepsilon)\mu$, which denotes the measure at time $t_1^*(\mu) - \varepsilon$. If we take $\eta_{t_1^*(\mu)-\varepsilon}$ according to $\mu_{t_1^*(\mu)-\varepsilon}$, from (4.5.15)

$$\mu_{t_1^*(\mu)-\varepsilon}(\eta_{t_1^*(\mu)} - \varepsilon)(I(\eta(0) = 0)) \in (0, 1)$$

According to lemma 4.5.5, $\eta_{t_1^*(\mu)-\varepsilon} \in \Omega_{dec} \cap \Omega_{dec}^\dagger$. Then using the result in proposition 4.5.4 to $\eta_{t_1^*(\mu)-\varepsilon}$ extracting according $\mu_{t_1^*(\mu)-\varepsilon}$, the result of 4.5.4 can be extend up to $t_1^*(\mu_1)$ with $\mu_1 = \mu_{t_1^*(\mu)-\varepsilon}$, denote $\mu_2 = \mu_{t_1^*(\mu_1)-\varepsilon}$, etc. Once we can prove that the configuration distributed according to $\mu_{t_1^*(\mu_k)-\varepsilon}$ has density between 0 and 1, we can extend the process to a further time $t_1^*(\mu_{t_t^*(\mu_k)-\varepsilon})$. So the extending will stop at the first time s such that $\mu_s(I(\eta_s(0) = 1)) \neq (0, 1)$ for η_s with distribution μ_s. And before that time, the measure for the process exists and is translation invariant, so the density before that time still has the form of (4.5.13)

Let $0 < \rho_t = \rho + (\beta - \alpha)t < 1$, the solution for this inequality is:

- For $\beta < \alpha$, $0 \leq t < \frac{\rho}{\alpha-\beta}$.
- For $\alpha = \beta$, $t \in [0, \infty)$.
- For $\beta > \alpha$, $0 \leq t < \frac{1-\rho}{\beta-\alpha}$

Then we can give the definition of semigroup in time interval satisfying the latter conditions. □

The semigroup defined above is for initial configurations $\eta \in \Omega_{dec} \cap \Omega^{\dagger}_{dec}$ and $N-$local function f. The following theorem tells us that this process is monotone, then we can extend the definition to a larger set.

Theorem 4.5.7. *The sandpile+anti-sandpile is monotone.*

Proof. We use η_t and ξ_t to denote the process with initial state η and ξ respectively. In order to prove that a process is monotone, we only need to find a coupled process (η_t, ξ_t) with $\eta \leq \xi$, and show that there is a coupled measure $P^{\eta,\xi}$ such that $P^{\eta,\xi}\{(\eta_t, \xi_t) : \eta_t \leq \xi_t\} = 1$.

The strategy of the coupling is referred to [13]. Shortly, for $\eta \leq \xi$, some addition on ξ may make a site x of height "1" becomes 0. In η either $\eta(x) = 0$, then we do nothing on η or $\eta(x) = 1$, then there correspondence with a unique site $y(x, \eta, \xi)$, $a_{y(x,\eta,\xi)}$ operating on η generate "0" at site x.

Some anti-addition on ξ may cause some site x of height 0 to become 1. In ξ, if $\xi(x) = 1$, then we do nothing on ξ and if $\xi(x) = 0$, there is a unique site $y(x, \xi, \eta)$ such that $a^{\dagger}_{y(x,\xi,\eta)}$ operating on η make site x turn from 0 to 1.

□

We have defined $S(t)f(\eta)$ for $\eta \in \Omega_{dec} \cap \Omega^{\dagger}_{dec}$, which is characterized by the initial distribution μ such that $\mu(I(\eta(0))) \in (0, 1)$ and for $N-$local function f. Next we want to use the "monotonicity" of the process to extend the definition to all configurations $\eta \in \Omega$ and all continuous function $f \in C(\Omega)$.

Step 1 : For every $\eta \in \Omega$, we can define new configuration η_n such that $\eta_n(x) = \eta(x)$ for $x \in [-n, n]$, otherwise 0. So we know $\eta_n \uparrow \eta$ as $n \uparrow \infty$.

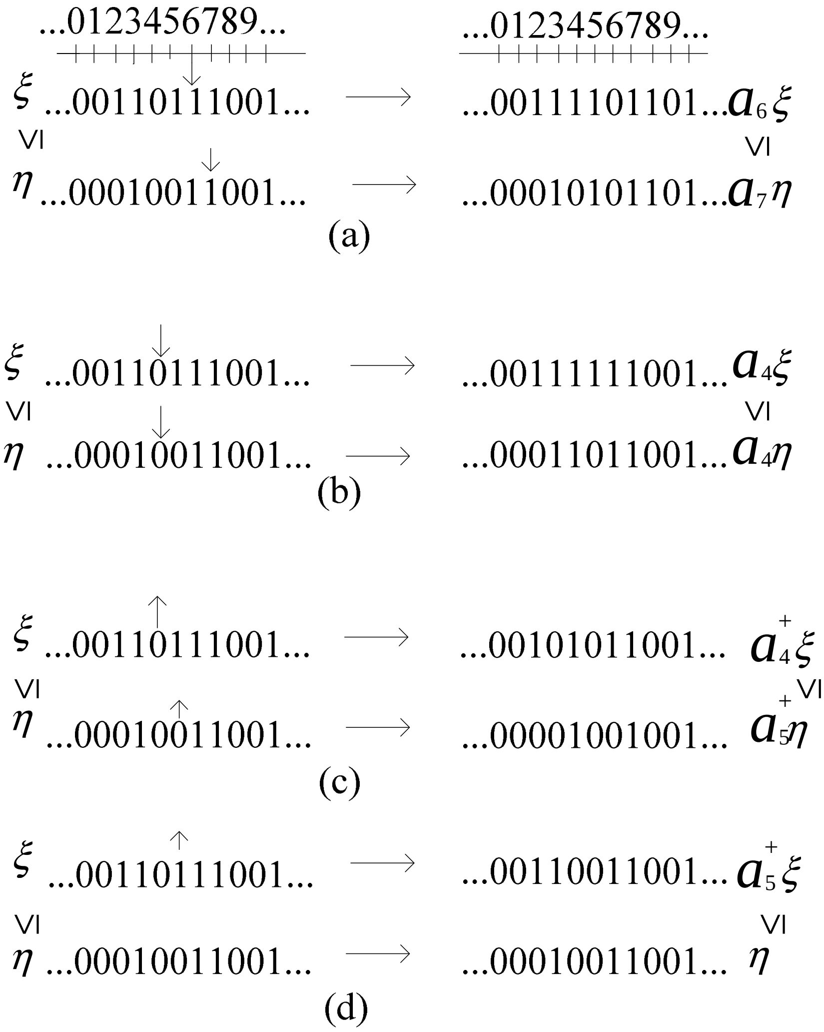

Figure 4.1: An Example to show coupling

Step 2 : For a fixed n, define a new configuration η_n^{dec} such that

$$\begin{aligned}
&\eta_n^{dec}(x) = \eta(x), \forall x \in [-n, n] \\
&\eta_n^{dec}(n + 2l - 1) = 1, \eta_n^{dec}(n + 2l) = 0, \text{ for all positive integer } l \\
&\eta_n^{dec}(-n - 2l + 1) = 1, \eta_n^{dec}(-n - 2l) = 0, \text{ for all positive integer } l
\end{aligned} \tag{4.5.16}$$

Obviously $\eta_n^{dec} \in \Omega_{dec} \cap \Omega_{dec}^\dagger$ with $a(\eta) = 1$ and $a^\dagger(\eta) = 1$.

Step 3 : For a fixed n, for $m \in \mathbb{N}$, define η_n^m as:

$$\eta_n^m(x) = \begin{cases} 0 & \text{for } x \in [-n-m, -n) \cap (n, n+m] \\ \eta_n^{dec}(x) & \text{otherwise.} \end{cases} \tag{4.5.17}$$

Then for a fixed n, $\eta_n^m \in \Omega_{dec} \cap \Omega_{dec}^{\dagger}$, $\eta_n^m \downarrow \eta_n$ as $m \uparrow \infty$. Then

$$S(t)f(\eta_n) := \lim_{m\uparrow\infty} S(t)f(\eta_n^m) \text{ and } S(t)f(\eta) := \lim_{n\uparrow\infty} S(t)f(\eta_n) \tag{4.5.18}$$

Step 5 : A local function f , there must be a N such that f is also N-local function. Then $\{f : f \text{ is local }\} \subseteq \{f : \text{ f is bouned and N-local }\} \subseteq C(\Omega)$, since for any $f \in C(\Omega)$, it can be approximated by local function, certainly it also can be approximated by the function in $\{f : \text{ f is bouned and N-local }\}$, then the definition extending to any $f \in C(\Omega)$.

With the former 5 steps, the semigroup can be extend to every stable configuration and every continuous function, which means that we get the process starting from any initial measure.

4.5.3 The invariant measures for the SA-process

Theorem 4.5.8. *Let $\mathcal{I}$ be the set of invariant measures for the process with generator 4.5.1. Then we have:*

(a) For $\alpha > \beta$, $\mathcal{I} = \{\delta_1\}$.

(b) For $\alpha < \beta$, $\mathcal{I} = \{\delta_0\}$.

(c) For $\alpha = \beta$, $\mathcal{I} \supset \{\lambda\delta_0 + (1-\lambda)\delta_1, \lambda \in [0,1]\}$

Proof. For ν is translation invariant and mixing with $\nu(I(\eta(0) = 0)) = \rho \in (0,1)$, we put $\rho(t) = \int I(\eta(0) = 0) d\nu S(t)$, which is the density of "0"'s in the configuration at time t, for t with the condition in theorem 4.5.6, we know the density at time t is,

$$\rho_t = \rho + (\beta - \alpha)t, \text{ with } \rho = \int I(\eta(0) = 0) d\nu > 0$$

Let $\overline{0}$ be the configuration with all the sites of height 0, and $\overline{1}$ be the configuration with all the sites of height 1. Let the measure δ_0 and δ_1 be defined as $\delta_0(\{\overline{0}\}) = 1$ and $\delta_1(\{\overline{1}\}) = 1$.

Our discussion will also be divided into three cases:

Case 1: For $\alpha > \beta$, when $0 \le t < \frac{\rho}{(\alpha-\beta)}$, ρ_t is a decreasing function of time t, and for $t \ge \frac{\rho}{(\alpha-\beta)}$, $\rho_t = 0$. Then for translation invariant measure μ_ρ with $\mu_\rho(I(\eta(0) =$

$0)) = \rho \in (0,1)$, $\mu_\rho S(t) = \delta_1$, for $t \geq \frac{\rho}{\alpha-\beta}$. And by the monotonicity of the process, $\delta_1 S(t) = \lim_{t\downarrow 0} \mu_\rho S(t)$, for all $t \geq 0$.

Now we begin to prove that δ_1 is invariant, i.e, $\delta_1 S(t) = \delta_1, \forall t > 0$. For a fixed $t > 0$, there must be a α^t such that $t > \frac{\alpha^t}{\alpha-\beta}$. Then for all $\rho \leq \alpha^t$, we have $t > \frac{\rho}{\alpha-\beta}$. Therefore for $0 < \rho < \alpha^t$, $\mu_\rho S(t) = \delta_1$. Then $\delta_1 S(t) = \lim_{\rho\downarrow 0} \mu_\rho S(t) = \delta_1$. Then$\delta_1$ is an invariant measure.

For δ_0, we know $\lim_{\rho\uparrow 1} \mu_\rho \to \delta_0$, with μ_ρ translation invariant and $\mu_\rho(I(\eta(0) = 0)) = \rho \in (0,1)$. For all $\rho \in (0,1)$, $\mu_\rho S(t) = \delta_1$, for $t \geq \frac{1}{\alpha-\beta}$. Then

$$\delta_0 S(t) = \lim_{\rho\uparrow 1} \mu_\rho S(t) = \delta_1, \forall t \geq \frac{1}{\alpha - \beta} \tag{4.5.19}$$

For all positive probability measure measure μ on Ω and all bounded monotone function f, we have:

$$\int f d\delta_0 \leq \int f d\mu \leq \int f d\delta_1 \tag{4.5.20}$$

By the monotonicity of the process, we know $\delta_0 S(t) \leq \mu S(t) \leq \delta_1 S(t)$, combined with $\delta_0 S(t) = \delta_1, \forall t \geq \frac{1}{\alpha-\beta}$ and δ_1 is invariant, we get $\mu S(t) = \delta_1, \forall t \geq \frac{1}{\alpha-\beta}$. And hence δ_1 is unique invariant measure.

Case 2: For $\alpha < \beta$, use the similar discussion as in Case 1, we can get: δ_0 is invariant and for every probability measure μ, $\mu S(t) = \delta_0, \forall t \geq \frac{1}{\alpha-\beta}$. Then $\mathcal{I} = \{\delta_0\}$

Case 3: For $\alpha = \beta$, $\rho(t) = \rho \in (0,1)$. For every $\rho \in (0,1)$, μ_ρ is the translation invariant measure with $\mu_\rho[I(\eta(0)) = 0] = \rho$. By the monotonicity of the process, we know $\delta_1 S(t) = \lim_{\rho\downarrow 0} \mu_\rho S(t)$, all $t > 0$. Then for fixed $t > 0$,

$$\delta_1 S(t)[I(\eta(0)) = 0] = \lim_{\rho\downarrow 0} \mu_\rho S(t)[I(\eta(0)) = 0] = \lim_{\rho\downarrow 0} \rho = 0 \tag{4.5.21}$$

The second equality holds because the process keeps density when $\alpha = \beta$. Then $\delta_1 S(t) = \delta_1$, and hence δ_1 is translation invariant. Similarly we can prove that δ_0 is also invariant. Easily to check that the linear combination $\lambda\delta_0 + (1-\lambda)\delta_1$ are also invariant. Then $I \supset \{\lambda\delta_0 + (1-\lambda)\delta_1, \lambda \in [0,1]\}$. □

Both δ_0 and δ_1 are translation invariant, and we also know that they are ergodic measure by Proposition 4.11 of [12]. For the case $\alpha = \beta$, we have the following conjecture.

Conjecture 4.5.9. $I_e = \{\delta_0, \delta_1\}$

For any product measure μ_ρ with $\mu_\rho(\eta(0) = 0) = \rho$, we know that $\mu_\rho S(t)$ is not translation invariant. So we conjecture that the only possible translation invariant measure with the density of "0" being ρ is $\rho\delta_0 + (1-\rho)\delta_1$.

4.5.4 Discussion of the result

Let $S_1(t)$ be the semigroup of corresponding to the formal generator

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}} [f(a_x\eta) - f(\eta)] \tag{4.5.22}$$

and $S_2(t)$ be the semigroup corresponding to the formal generator

$$\mathcal{L}^\dagger f(\eta) = \sum_{x\in\mathbb{Z}} [f(a_x^\dagger\eta) - f(\eta)] \tag{4.5.23}$$

And $S(t)$ be the semigroup for the process with generator (4.5.1). From [13], we know δ_1 is the only invariant measure for $S_1(t)$ and from section 4.4.2, we know δ_0 is the unique invariant measure for $S_2(t)$.

In a Feller process, we have

$$\mu S(t) = \mu \text{ iff } \int \mathcal{L}f d\mu = 0, f \in \mathcal{D}(\mathcal{L})$$

Therefore if a generator is of the form $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, i.e., the summation of two Feller generators, and μ is invariant for $\mathcal{L}_1$ but not for $\mathcal{L}_2$, then μ could not be invariant for $\mathcal{L}$. This is precisely what happens in SA process.

In SA process, the generator, $\mathcal{L}_{\alpha,\beta} = \alpha\mathcal{L} + \beta\mathcal{L}^\dagger$, we know that:

1. For $\alpha > \beta$, δ_1 is the invariant measure for $\mathcal{L}_{\alpha,\beta}$, $\mathcal{L}$ but not for $\mathcal{L}^\dagger$.

2. For $\alpha < \beta$, δ_0 is the invariant measure for $\mathcal{L}^\dagger$ and $\mathcal{L}_{\alpha,\beta}$ but not for $\mathcal{L}$.

The reason that item 1 and item 2 hold is that $\mathcal{L}$ and $\mathcal{L}^\dagger$ are not generators for Feller processes. In a word, the non-locality of addition and anti-addition operator is responsible for this kind of "competition" between two generators for the stationary measure.

Chapter 5

Stabilization of a configuration

The sandpile model, anti-sandpile model and sandpile+anti-sandpile are "Self-organized" critical(SOC) systems, which can achieve criticality without tuning the parameters of the system. The reason that these systems can achieve criticality is that the rate of driving(adding or removing grains) is very slow compared to the rate of relaxation, which is an implicit tuning of parameters. From this point, these three models can also be considered as examples of ordinary critical systems. So there is a transition point for a system to be stabilizable and non-stabilizable.

5.1 Stabilization of infinite-volume anti-sandpile configuration

5.1.1 Introduction

Let $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$, we know that for any configuration $\eta \in \mathcal{H} = \{\eta : \eta(x) \geq 0, \forall x \in \Lambda\}$, it can be stabilized in a finite number of legal topplings, and the order of the topplings has no effect on the final configuration, see Proposition 2.2.4 and 2.2.7. For the anti-sandpile configuration $\xi \in \mathcal{J} = \{\eta : \eta(x) \leq 2d-1, \forall x \in \Lambda\}$, it can always be stabilized by finitely many legal anti-topplings and such stabilization is also well-defined, see Proposition 3.2.4. The reason for such phenomena is that grains of sand are allowed to enter or leave the system from the boundary when topplings or anti-topplings happen on the boundary sites.

In the infinite volume case, the system keeps density, the stabilization of a system must depends on the density of the system. For a sandpile configuration η, when its density is small, it is easy to be stabilized, when the density is large, it is harder. In [8], Dickman takes the expectation of the density of the sandpile model $\rho_c = E_{\text{sandpile model}}(\eta(0))$ as the critical density. When $\rho < \rho_c$, when the system is unstable(active), topplings will happen till every site is stable(inactive). In the absence of activity, there is addition. In the infinite-size limit, it is conjectured for $\rho < \rho_c$, the activity density is 0, which means that the configuration can always be stabilized by finitely many legal topplings at each site; for $\rho > \rho_c$, the avalanche size is non-local, this means that infinite number of topplings happen during the stabilization.

In [15], Ronald Meester extends the conjecture of Dickman, it is conjectured that for all stationary and ergodic measures ν from which the expected density of the sandpile configuration is ρ, and there is a"critical density" ρ_c such that,

- for $\rho < \rho_c$, every site changes finitely times to get a stable configuration almost surely.
- for $\rho_c < \rho \leq 2d - 1$, there is at least a ν such that $\nu(\eta(0)) = \rho$, all the sites topples infinitely times almost surely.
- for $\rho > 2d-1$ and any ν with $\nu(\eta(0)) = \rho$, all the sites topple infinitely times almost surely.

Take μ_Λ as the uniform measure on $\mathcal{R}_\Lambda$(the recurrent set on Λ), μ is invariant. Let μ is the volume limit of the stationary measure μ_Λ and $\rho_{\mathbb{Z}^d}$ be the expected density of the Abelian sandpile model under μ. In [9], A. Fey-den Boer and F. Redig get a complete proof for this conjecture and conclude that for all $d \geq 1, \rho_c = d$. In $d = 1$, $\rho_c = 1 = \rho_{\mathbb{Z}}$, while for $d \geq 2, \rho_c < \rho_{\mathbb{Z}^d}$.

In the following part of this section, I will extend the results in [9] to the infinite volume anti-sandpile model.

5.1.2 Stabilization of the anti-sandpile configurations in $\mathbb{Z}^d$

Let $\mathcal{H} = \{0, 1, 2, ...\}^{\mathbb{Z}^d}$ and $\mathcal{J} = \{... - 1, 0, 1, 2, ...2d - 1\}^{\mathbb{Z}^d}$. For $\eta \in \mathcal{J}$, if $\eta(x) < 0$, site x is unstable and the anti-toppling happens. For for $d \in \mathbb{N}$, we recall the flip operator $\theta : \mathcal{J} \to \mathcal{H}$:

$$\theta\eta = \overline{2d - 1} - \eta, \forall \eta \in \mathcal{J} \tag{5.1.1}$$

then $\theta\mathcal{J} = \mathcal{H}$.

The definition of the stabilization of a sandpile configuration, i.e., $\eta \in \mathcal{H}$ is introduced in [9]. If we take η_Λ as the restriction of η on Λ. Then a configuration $\eta \in \mathcal{H}$ is called stabilizable if and only if for every sequence of volume $\Lambda_n \uparrow \mathbb{Z}^d$, there exists $m_{\Lambda_n} \in \mathbb{N}^{\Lambda_n}$ such that $\eta_{\Lambda_n} - \Delta_{\Lambda_n} m_{\Lambda_n} = \xi_{\Lambda_n} \in \Omega_{\Lambda_n}$ with $m_{\Lambda_n} = (m_{\Lambda_n})_{x \in \Lambda_n}$, and $m_{\Lambda_n}(x) \to m(x) < \infty$.

$\mathbb{S} = \{\eta \in \mathcal{H} : \eta \text{ is stabilizable. }\}$ be the set of all such stabilizable sandpile configurations. A probability measure ν on $\mathcal{H}$ is called stabilizable if $\nu(\mathbb{S}) = 1$.

In the following, the corresponding definitions related to anti-sandpile are given. For $\Lambda \subset \mathbb{Z}^d$, $\Omega_\Lambda = \{0, 1, 2, ..., 2d-1\}^\Lambda$ be the set of all stable configurations on Λ. For $\eta \in \mathcal{J}, \eta_\Lambda$ is the restriction of η on Λ.

Definition 5.1.1. *For $\eta \in \mathcal{J}$ is called stabilizable if and only if for every volume sequence $\Lambda_n \uparrow \mathbb{Z}^d$, there exists $m^\dagger_{\Lambda_n} \in \mathbb{N}^{\Lambda_n}$ such that*

$$\eta_{\Lambda_n} + \Delta_{\Lambda_n} m^\dagger_{\Lambda_n} = \zeta_{\Lambda_n} \in \Omega_{\Lambda_n} \tag{5.1.2}$$

and for every $x \in \mathbb{Z}^d, m^\dagger_{\Lambda_n}(x) \to m(x) < \infty$ as $n \to \infty$.

We use $\mathbb{S}^\dagger$ to denote the set of all such stabilizable configurations.

$m^\dagger_\Lambda(x)$ is non-decreasing in Λ. Therefore a configuration $\eta \in \mathcal{J}$ is not stabilizable if and only if there is a site $x \in \mathbb{Z}^d$ such that $m^\dagger_\Lambda(x) \uparrow \infty$.

By the conjugacy between sandpile and anti-sandpile, we get the following relation,

Lemma 5.1.2. $\mathbb{S}^\dagger = \theta\mathbb{S}$

Definition 5.1.3. *Let ν is a probability measure on $\mathcal{J}$, it is called stabilizable by anti-topplings if $\nu(\mathbb{S}^\dagger) = 1$.*

By the relation between stabilization by topplings and that by anti-topplings, we get that ν is stabilizable by topplings if and only of $\nu\theta$ is stabilizable by anti-topplings.

Theorem 5.1.4. *For every invariant and ergodic measure ν on $\mathcal{J}$, we have $\nu(\mathbb{S}^\dagger) = 0$, or 1.*

Proof. First we notice that for all $\Lambda \in \mathbb{Z}^d$ and $\eta \in \mathbb{Z}^d$, $(\tau_k\eta)(x) = \eta(x-k)$. Then for every sequence of volume $\Lambda_n \uparrow \infty$, $\tau_{-k}\Lambda_n \uparrow \infty$. And the heights in $(\tau_k\eta)_{\Lambda_n}$ are correspondence with the heights of $\eta_{\tau_{-k}\Lambda_n}$. Then if $m^\dagger_{\Lambda_n} \in \mathbb{N}^{\Lambda_n}$ satisfies:

$$(\tau_k\eta)_{\Lambda_n} + \Delta_{\Lambda_n} m^\dagger_{\Lambda_n} = \xi_{\Lambda_n} \in \Omega_{\Lambda_n}$$

Denote $\widetilde{m}^\dagger_{\tau_{-k}\Lambda_n}(x) = m^\dagger_{\Lambda_n}(x+k)$ for $x \in \tau_{-k}\Lambda_n$, then,

$$\eta_{\tau_{-k}\Lambda_n} + \Delta_{\tau_{-k}\Lambda_n} \widetilde{m}^\dagger_{\tau_{-k}\Lambda_n} = \widetilde{\xi}_{\tau_{-k}\Lambda_n} \in \Omega_{\tau_{-k}\Lambda_n}$$

with $\xi_{\tau_{-k}\Lambda_n}(x) = \xi_{\Lambda_n}(x+k), \forall x \in \tau_{-k}\Lambda_n$.

If $\tau_k\eta$ is not stabilizable, then there is a sequence $\Lambda_n \uparrow \mathbb{Z}^d$, and there is a site $y \in \Lambda$ such that $m^\dagger_{\Lambda_n}(y) \uparrow \infty$. Then for $x = y - k \in \tau_{-k}\Lambda_n$ $m^\dagger_{\tau_{-k}\Lambda_n}(x) = m^\dagger_{\Lambda_n}(x+k) = m^\dagger_{\Lambda_n}(y) \to \infty$.

It contradicts $\eta \in \mathbb{S}^\dagger$. Then $\tau_k \mathbb{S}^\dagger \subseteq \mathbb{S}^\dagger$. Let τ_{-k} operates on two side, we get $\tau_{-k}\tau_k \mathbb{S}^\dagger = \mathbb{S}^\dagger \subseteq \tau_{-k}\mathbb{S}^\dagger$. Then for every $k \in \mathbb{Z}^d$, $\tau_k \mathbb{S}^\dagger = \mathbb{S}^\dagger$, which proves that $\mathbb{S}^\dagger$ is translation invariant.

Take $f(\eta) = I_{\mathbb{S}^\dagger}$, $\tau_k f = f$, then by the ergodicity of ν, we know $f(\eta) = I_{\mathbb{S}^\dagger}$ is a constant $\nu - a.s.$, then $I_{\mathbb{S}^\dagger} = 1, \nu - a.s.$ or $I_{\mathbb{S}^\dagger} = 0, \nu - a.s.$. Then $\nu(\mathbb{S}^\dagger) = \nu(I_{\mathbb{S}^\dagger}) = 1$, or 0. □

As to the stabilization, we have the following properties:

Proposition 5.1.5. *1. $\mathbb{S}^\dagger$ is translation invariant measurable set.*

2. It $\eta \in \mathbb{S}^\dagger$ and $\eta \leq \xi$, then $\xi \in \mathbb{S}^\dagger$.

3. If ν is a stabilizable probability measure, and $\nu \leq \mu$, then μ is stabilizable measure on $\mathcal{J}$.

Proof. Since $m^\dagger_{\Lambda_n}$ is measurable since it is increasing. $\mathbb{S}^\dagger = \{\eta : \limsup_{n\uparrow\infty} m^\dagger_{\Lambda_n} < \infty, \forall x\}$, then it is also Borel measurable set. The invariance comes from Theorem the proof of Theorem 5.1.4.

Take $\eta \leq \xi \in \mathcal{J}$, and $\eta \in \mathbb{S}^\dagger$, and suppose $\xi \notin \mathbb{S}^\dagger$, there must be a site $x \in \mathbb{Z}^d$ such that $m_{\Lambda_n}(x) \uparrow \infty$. For the same sequence Λ_n, $\eta_{\Lambda_n} \leq \xi_{\Lambda_n}$, we know in order to stabilize η_{Λ_n}, by the abelian property of the reverse topplings, we can stabilize ξ_{Λ_n} first, and then take off $\xi_x - \eta_x$ grains of sand from the site x. Then we know $m^\xi_{\Lambda_n}(y) \leq m^\eta_{\Lambda_n}(y)$, for every $y \in \Lambda_n$ and every n. So if ξ is not stabilizable, neither is η. This contradicts the fact that η is stabilizable.

If ν is stabilizable, $\nu(\mathbb{S}^\dagger) = 1$. For any $\nu \leq \mu$, we claim that $\nu\theta \geq \mu\theta$: for any monotone function $f : \mathcal{H} \to \mathbb{R}$, $-f \circ \theta$, defined as $-f \circ \theta(\eta) := -f(\theta\eta)$, is also a monotone function on $\mathcal{J}$. Then we have:

$$\int_{\mathcal{H}} f(\eta) d\nu\theta = \int_{\mathcal{J}} f(\theta\eta) d\nu$$

$$\int_{\mathcal{H}} f(\eta) d\mu\theta = \int_{\mathcal{J}} f(\theta\eta) d\mu$$

Since $\nu \leq \mu$, and $-f \circ \theta$ is monotone, we know:

$$\int_{\mathcal{J}} f(\theta\eta) d\nu \geq \int_{\mathcal{J}} f(\theta\eta) d\mu$$

Then we get that:

$$\int_{\mathcal{H}} f(\eta) d\nu\theta \geq \int_{\mathcal{H}} f(\eta) d\mu\theta \tag{5.1.3}$$

which implies that $\nu\theta \geq \mu\theta$. If ν is stabilizable under anti-topplings, $\nu\theta$ is stabilizable under topplings, then by Proposition 2.1(Monotonicity) in [9], we know $\mu\theta$ is stabilizable by topplings, then μ is stabilizable by anti-topplings. □

This proposition gives us some idea that if we can find a measure that is stabilizable, then every measure that is dominated by that measure is also stabilizable. Then we define the following "critical densities":

Lemma 5.1.6. *Let $\mathcal{P}(J)$ be the set of all translation invariant and ergodic probability measures on $\mathcal{J}$. Define:*
$\rho_c^{\dagger,-} = \sup\{\rho \leq 2d-1 : \exists \nu \in \mathcal{P}(\mathcal{J}) :$ *with* $\nu(\eta(0)) = \rho,$ *and* $\nu(\mathbb{S}^{\dagger}) \neq 1\}$.
$\rho_c^{\dagger,+} = \inf\{\rho \leq 2d-1 : \forall \nu \in \mathcal{P}(\mathcal{J}) :$ *with* $\nu(\eta(0)) = \rho,$ *and* $\nu(\mathbb{S}^{\dagger}) = 1\}$.
Then $\rho_c^{\dagger,-} = \rho_c^{\dagger,+}$

Proof. Define:
$S = \{\rho \leq 2d-1 :$ such that $\forall \nu \in \mathcal{P}(J)$ with $\nu(\eta(0)) = \rho, \nu$ is stabilizable$\}$.

If we can prove S is an interval, then we prove the lemma. Suppose $\rho \in S$ and $\rho' > \rho$. For any measure $\nu' \in \mathcal{P}(J)$ with $\nu'(\eta(0)) = \rho'$. And there is a measure $\nu \in \mathcal{P}(J)$ such that $\nu(\eta(0)) = \rho$ and $\nu < \nu'$. Since ν is stabilizable, according to Proposition 5.1.5 c), we can get that ν' is stabilizable. □

Now we define the "critical density" for the infinite volume anti-sandpile model as $\rho_c^{\dagger} := \rho_c^{\dagger,-} = \rho_c^{\dagger,+}$.

We know that the Abelian sandpile in finite volume Λ has a unique recurrent class $\mathcal{R}_\Lambda$, and the uniform measure μ_Λ is the invariant measure, and it is proved that in [1] that $\mu = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_\Lambda$ is a measure on infinite volume height configurations, and it is translation invariant. Let $\mathcal{R}_\Lambda^{\dagger}$ be the recurrent class of the anti-sandpile model on finite subset Λ and $\mu_\Lambda^{\dagger}$ is the uniform measure on $\mathcal{R}_\Lambda^{\dagger}$. Since $\mathcal{R}_\Lambda^{\dagger} = \theta \mathcal{R}_\Lambda$, then $\mu_\Lambda^{\dagger} = \mu_\Lambda \theta$. If we take $\mu^{\dagger}$ as the infinite volume limit of $\mu_\Lambda^{\dagger}$, then $\mu^{\dagger} = \mu\theta$, which is also translation invariant. From now on, we will always use the notations μ and $\mu^{\dagger}$ as the limit of uniform measure μ_Λ, $\mu_\Lambda^{\dagger}$. Define $\rho_{\mathbb{Z}^d} := \mu(\eta(0))$, $\rho_{\mathbb{Z}^d}^{\dagger} := \mu^{\dagger}(\eta(0))$, then $\rho_{\mathbb{Z}^d}^{\dagger} = 2d - 1 - \rho_{\mathbb{Z}^d}$.

In the following we proves a lower bound for the critical value.

Theorem 5.1.7. *For a $\eta \in \mathcal{J}$, suppose that η has a distribution ν and ν is translation invariant and ergodic measure such that $\nu(\eta(0)) = \rho < 0$. Then ν is almost surely not stabilizable.*

Proof. Take $\eta \in \mathcal{J}$ such that $\nu(\eta(0)) = \rho < 0$, suppose η is stabilizable, then for any finite subset Λ of $\mathbb{Z}^d$, there exists $m_\Lambda(x), \forall x \in \Lambda$ such that:

$$\eta_\Lambda + \Delta m_\Lambda = \xi_\Lambda \in \Omega_\Lambda$$

with $m_\Lambda(x) \uparrow m(x) < \infty$, then taking limit $\Lambda \uparrow \mathbb{Z}^d$

$$\eta + \Delta m = \xi, \text{ for some } \xi \in \Omega \tag{5.1.4}$$

this gives that;

$$(-\Delta)m = \eta - \xi \tag{5.1.5}$$

Let $\{X_n, n \in \mathbb{N}\}$ be the simple random walk(SRW) starting at the origin, and assume P is the transition operator, i.e., $Pf(x) = \frac{1}{2d}\sum_{e:|e|=1} f(x+e)$ then for any function $f : \mathbb{Z}^d \to \mathbb{R}$,

$$M_n = f(X_n) - f(X_0) - \sum_{i=1}^{n-1}(P - I)f(X_i) \tag{5.1.6}$$

is a mean zero martingale. According to (2.4.3), we know $(P - I)f = -\frac{1}{2d}\Delta f$, substitute it to (5.1.8),

$$M_n = f(X_n) - f(X_0) - \frac{1}{2d}\sum_{i=1}^{n-1}(-\Delta)f(X_i) \tag{5.1.7}$$

Take $f(x) := m(x)$, then

$$M_n = m(X_n) - m(X_0) - \frac{1}{2d}\sum_{i=1}^{n-1}(-\Delta)m(X_i) \tag{5.1.8}$$

using (5.1.7), we get

$$M_n = m(X_n) - m(X_0) - \frac{1}{2d}\sum_{i=1}^{n-1}(\eta(X_i) - \xi(X_i)) \tag{5.1.9}$$

is a mean zero martingale w.r.t $\mathcal{F}_n = \sigma\{X_0, ..., X_n\}$. Taking expectation over the random walk,

$$\frac{1}{n}(E_0(m(X_n) - m(X_0))) = \frac{1}{n}\frac{1}{2d}E_0(\sum_{i=1}^{n-1}(\eta(X_i) - \xi(X_i))) \tag{5.1.10}$$

Using the fact that $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1}\eta(X_i) = \rho$, $\xi(x) \geq 0$ and $m(0) < \infty$, take lim sup on (5.1.12),

$$0 \leq \limsup_{n\to\infty} \frac{1}{n}E_0 m(X_n) \leq \frac{\rho}{2d} < 0 \tag{5.1.11}$$

which gives a contradiction. □

Item 1 of the following lemma is contained in [6] as the item 2 is the follows form item

1. These two items are also used in the stabilization of sandpile configurations, see [18], so I just put them here.

Lemma 5.1.8. *1. Suppose that $\xi \in \mathcal{R}_\Lambda$ is minimally recurrent, i.e., diminishing the height by one at any site $x \in \Lambda$ creates a forbidden subconfiguration. Then the total number of grains $\sum_{x\in\Lambda} \xi(x)$ equals the number of edges in Λ.*
2. Suppose that ν is a translation invariant probability measure concentrating on minimally recurrent configurations, i.e., such that $\nu-a.s.$ every restriction η_{Λ_0} is minimally recurrent in Λ_0. Then $\int \eta(0)d\nu = d$.

We know the ν in item 2 is stabilizable and the corresponding density $\nu(\eta(0)) = d$, by the conjugacy, we know there is a $\nu^\dagger = \nu\theta$ is stabilizable in the anti-sandpile and the expected density of the anti-sandpile is $d-1$. This implies that $\rho_c^\dagger \leq d-1$. In the following, we will show that for any $\delta > 0$, there exist a non stabilizable measure with density $d-1-\delta$. Here we use the example in Appendix, which tells us that for any $\delta > 0$, there is a non stabilizable measure ν for the sandpile configuration such that the expected density for the sandpile $\rho = d+\delta, \forall\delta > 0$. Then the $\nu\theta$ is the non stabilizable measure with the expected density $d-1-\delta$. Then $\rho_c^\dagger > d-1-\delta, \forall\delta > 0$, then $\rho_c^\dagger \geq d-1$.

By now we can conclude that $\rho_c^\dagger = d-1$. For $d = 1$, $\rho_{\mathbb{Z}}^\dagger = 0 = d - 1\rho_c^\dagger$.

Summarizing what we have done, we conclude that for the anti-sandpile model, $\nu \in \mathcal{P}(\mathcal{J})$ with the expected density of the anti-sandpile is ρ, i.e., $\nu(\eta(0)) = \rho, \eta \in \mathcal{J}$, for all $d \geq 1$,

a) For $\rho < 0$, ν is not stabilizable.

b) For $d-1 < \rho \leq 2d-1$, ν is stabilizable.

c) For $0 \leq \rho < d-1$, there is a ν with $\nu(\eta(0)) = \rho$ which is not stabilizable.

d) For $d = 1$, item a), b), c) become $0 < \nu(\eta(0)) < 1$ is stabilizable, and for ν such that $\nu(\eta(0)) < 0$, ν is not stabilizable.

5.2 Stabilization of a finite volume Symmetric SA configuration

5.2.1 Introduction

In the finite volume case, the stabilization of pure sandpile configuration and pure anti-sandpile configuration is well defined. While during the stabilization of SA configuration, both topplings and anti-topplings are possible to happen, and since the topplings and anti-topplings do not commute with each other, the order of the operators in the stabilization takes a role on the final results. For example, $\Lambda = \{1, 2, 3, 4\} \subset \mathbb{Z}$, and $\eta = (0, -1, 3, 1)$, next table shows all the possible ways as well as the corresponding results of stabilization.

Order of operators	Final configuration
$2^\dagger$, $1^\dagger$, 3, 4	(1, 1, 1, 0)
$2^\dagger$, 3, $1^\dagger$, 4	(1, 1, 1, 0)
$2^\dagger$, 3, 2, 4, 3	(0, 1, 0, 1)
$2^\dagger$, 3, 4, $1^\dagger$	(1, 1, 1, 0)
$2^\dagger$, 3, 4, 2, 3	(0, 1, 0, 1)
3, 4, 3	(0, 1, 0, 1)

In this table $2^\dagger$ means one anti-toppling at site 2 and "2" one means toppling at site 2. From this table, we know, for a fixed configuration, there are many ways to stabilize it, which may be of different size; and the final results are not unique. When the volume is very large, there may be many ways, so we are interested in whether every order of legal topplings and anti-topplings are of finite order, i.e, whether all the possible orders of legal operators can eventually stabilize a configuration.

At the beginning I want discuss the stabilization of a configuration in the traditional sandpile+anti-sandpile model. However, we found that in the SA model, an anti-toppling may make a site with negative mass become positive, and such positive mass may generate extra topplings later. While in the so-called "symmetric sandpile+anti-sandpile model", this problem disappears. The basic reason is that in the symmetric model, a site with negative mass would never become positive via anti-topplings. In the following part of this chapter, I will concentrate on the "symmetric sandpile+anti-sandpile" model.

5.2.2 Stabilization of a Symmetric sandpile+anti-sandpile(S-SA model) configuration

Definition of the Symmetric sandpile+anti-sandpile model

For $\Lambda \subset \mathbb{Z}^d$, $\eta(x) \in \mathbb{Z}$ denotes the height of site x. Then $\mathbb{X} = \{\eta : \eta(x) \in \mathbb{Z}, \forall x \in \Lambda\}$ be the set of all height configurations on Λ.

Site $x \in \Lambda$ is called stable if $-2d + 1 \leq \eta(x) \leq 2d - 1$, otherwise site x is called unstable. We use $\Omega_\Lambda^s = \{-2d + 1, ..., -1, 0, 1, 2, ...2d - 1\}^\Lambda$ to denote the set of all stable configurations in the S-SA model. When $\eta(x) \leq -2d$, $\eta(y) \to \eta(y)+\Delta(x, y)$, which is called anti-toppling at site x; and when $\eta(x) \geq 2d$, $\eta(y) \to \eta(y)-\Delta(x, y)$, which is called toppling at site x, with Δ the same as defined before. And use $\mathcal{K} = \{-2d+1, ..., -1, 0, 1, 2, ...\}^{\Lambda}$ and $\mathcal{J} = \{, ..., -2, -1, 0, 1, 2, ..., 2d-1\}^{\Lambda}$ as before. Then only topplings happens in the stabilization of $\eta \in \mathcal{K}$ and only anti-topplings happen in the stabilization of $\eta \in \mathcal{J}$.

We still use the t_x and $t_x^{\dagger}$ to denote toppling and anti-toppling operators at site x, then we get that in the symmetric model,

Lemma 5.2.1. *1) All the $t_x, x \in \Lambda$ commute;*
2) All the $t_x^{\dagger}, x \in \Lambda$ commute;
3) $t_x^{\dagger} t_y \neq t_y t_x^{\dagger}$

The proof of item 1) and 2) are the same as in chapter 1 and chapter 2; and for the item 3), we can see the following example, for $\Lambda = \{1, 2, 3, 4\} \subset \mathbb{Z}$, $\eta = (1, -2, 2, 0)$, then if $t_2^{\dagger} t_3 \eta = (1, -1, 0, 1)$ but $t_3 t_2^{\dagger} \eta = (0, 0, 1, 0)$.

For the S-SA model, we get that,

Theorem 5.2.2. *In the symmetric models, every legal sequence of topplings on $\eta \in \mathcal{K}$ is of finite size ; and every legal sequence of anti-topplings on $\eta \in \mathcal{J}$ is also of finite size.*

Proof. For $\eta \in \mathcal{K}$, define a new configuration ξ such that $\xi(x) = \eta \bigvee 0$, so $\xi \in \mathcal{H}$. And $\eta \leq \xi$, then if T is legal sequence of topplings on η, T is also legal on ξ. From proposition 2.2.4 and Lemma 2.2.6, we know $|T| < \infty$.

For $\xi \in \mathcal{J}$, every legal anti-toppling in the S-SA model is also legal in the anti-sandpile model introduced in chapter 3. So for legal sequence of anti-topplings $\mathcal{A}$ on ξ in the sense of S-SA model, $\mathcal{A}$ is also legal on ξ in the sense of anti-sandpile model. According to proposition 3.2.4, we know $|\mathcal{A}| < \infty$. $\square$

The stabilization of a configuration $\eta \in \mathbb{X}$ is to turn η to a configuration in Ω_{Λ}^{s} via legal topplings and anti-topplings, obviously the stabilization is not well defined since topplings and anti-topplings do not commute.

Stabilization of finite volume S-SA configuration

For $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$, $\mathbb{Z}^{\Lambda}$ denotes the set of all height configurations. In order to stabilize a configuration $\eta \in \mathbb{Z}^{\Lambda}$, topplings and anti-topplings are both possible to happen. Since toppling operators and anti-topplings operators do not commute, the order of the topplings and reversed topplings will influence the final result.

Definition 5.2.3. *For $\eta \in \mathbb{Z}^{\Lambda}$, T is a sequence of topplings and anti-topplings, $T = t_{x_n}^{\alpha_n} \circ ... \circ t_{x_1}^{\alpha_1}$ with $\alpha_n \in \{\cdot, \dagger\}$ is said to be "legal sequence of operators"(LSO) in η if $t_{x_i}^{\alpha_i}$ is legal in $t_{x_{i-1}}^{\alpha_{i-1}} \circ ... \circ t_{x_1}^{\alpha_1}(\eta), \forall i \in \{1, ..., n\}$. And $|T| := n$ is the size of $\mathcal{T}$.*

For $\eta \in \mathbb{X}$, T^η is a legal sequence of operations in η and $T^\eta(\eta) = ...t_{x_n}^{\alpha_n} \circ ... \circ t_{x_1}^{\alpha_1}(\eta)$ with $\alpha_n \in \{\cdot, \dagger\}$.

Definition 5.2.4. *A configuration $\eta \in \mathbb{X} = \mathbb{Z}^\Lambda$ can be stabilized by LSO T^η if $T^\eta(\eta) \in \Omega_\Lambda^s$ and $|T^\eta| < \infty$.*

For a LSO T^η in $\eta \in \mathbb{Z}^\Lambda$, define $n_\Lambda(T^\eta)$ as the total number of topplings contained in T^η and $n_\Lambda^\dagger(T^\eta)$ as the total number of anti-topplings contained in T^η.

For a configuration $\eta \in \mathbb{Z}^\Lambda$, define the "mass" function $||\eta|| := \sum_{x\in\Lambda} |\eta(x)|$, then for every $\eta \in \mathbb{X} = \mathbb{Z}^d$, $||\eta|| < \infty$, this is because $|\eta(x)| < \infty$ and $|\Lambda| < \infty$. As to the "mass" function, we get the following result,

Lemma 5.2.5. *For $\eta \in \mathbb{Z}^\Lambda$ with $|\Lambda| < \infty$, and t_x and $t_y^\dagger$ are legal on η.*

$$||t_x\eta|| \leq ||\eta|| \text{ and } ||t_y^\dagger\eta|| \leq ||\eta||, \forall x, y \in \Lambda \tag{5.2.1}$$

especially $||t_x\eta|| \leq ||\eta|| - 1$ and $||t_y^\dagger\eta|| \leq ||\eta|| - 1$, $\forall x, y \in \partial\Lambda$.

Proof. Without generality we assume t_x is legal on η, then

$$t_x\eta(y) = \begin{cases} \eta(y) - 2d & x = y \\ \eta(y) + 1 & |y - x| = 1 \\ \eta(y) & \text{otherwise} \end{cases}$$

we know $t_x\eta(x) \geq 0$, then $|t_x\eta(x)| = |\eta(x)| - 2d$; $|t_x\eta(y)| \leq |\eta(y)| + 1$ for $|y - x| = 1$, otherwise $|t_x\eta(y)| = |\eta(y)|$. Then $||t_x\eta|| \leq ||\eta||$. When $x \in \partial\Lambda$, and $\vartheta_\Lambda(x)$ be the number of "lacking neighbors", $\vartheta_\Lambda(x) \geq 1$,

$$\begin{aligned} ||t_x\eta|| &= \sum_{y\in\Lambda} |t_x\eta(y)| \\ &\leq \sum_{y\in\Lambda} |\eta(y)| - 2d + (2d - \vartheta_\Lambda(x)) \\ &\leq ||\eta|| - 1 \end{aligned} \tag{5.2.2}$$

When $t_y^\dagger$ s legal on η, t_y is legal on $-\eta$. Then we have $||t_y^\dagger\eta|| = ||t_y(-\eta)|| \leq ||-\eta|| = ||\eta||$, when $x \in \partial\Lambda$, using (5.2.2), we get $||t_y^\dagger\eta|| = ||t_y(-\eta)|| \leq || - \eta|| - 1 = ||\eta|| - 1$ □

With the help of this lemma, then we can get the following theorem,

Theorem 5.2.6. *For $\eta \in \mathbb{Z}^\Lambda$, every legal sequence of operators(LSO) T^η is of finite size, i.e, $|T^\eta| < \infty$.*

Proof. Suppose T^η is a legal sequence of operators of infinite volume. Then $n_\Lambda(T^\eta) = \infty$ or $n^\dagger_\Lambda(T^\eta) = \infty$ or both hold. Since $|\Lambda| < \infty$, then there is a site x which topples or anti-topples infinitely many times. Since $|\Lambda|$ is connected and every $\eta(y) < \infty, y \in \Lambda$, then it is easy to see that there is a site $z \in \partial\Lambda$ that topples or anti-topples infinitely many times. According to Lemma 5.2.5, we get for every $n \in \mathbb{N}$, $0 \leq ||T^\eta \eta|| \leq ||\eta|| - n$, which is a contradiction since for a fixed $\eta \in \mathbb{Z}^\Lambda$, $||\eta|| < \infty$. □

By now we get that in the "symmetric" sandpile+anti-sandpile model, every way of stabilization of a finite volume configuration can eventually stops.

5.3 Stabilization of infinite volume configuration in S-SA model

For a configuration $\eta \in \mathbb{X}$, assign to each site $x \in \mathbb{Z}^d$ two independent rate 1 Poisson process $N^x_t, N^{x,\dagger}_t$, which are independent for different sites. On the event times of N^x_t, t_x operates on the configuration and on the event time of $N^{x,\dagger}_t$, $t^\dagger_x$ operates on η. Then a configuration $\eta \in \mathbb{Z}^{\mathbb{Z}^d}$ evolves towards η_t according to a Markov process with generator:

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}^d} I_{(\eta(x)>2d-1)}[f(\eta - \triangle_{x,\cdot}) - f(\eta)] + \sum_{x\in\mathbb{Z}^d} I_{(\eta(x)<-2d+1)}[f(\eta + \triangle_{x,\cdot}) - f(\eta)] \tag{5.3.1}$$

Let η_t be the configuration at time t and $n(x,t,\eta)$ and $n^\dagger(x,t,\eta)$ denotes the number of legal topplings and legal anti-topplings before time t, we know they are increasing as t increases. Define:$n(x,\infty,\eta) := \lim_{t\uparrow\infty} n(x,t,\eta)$ and $n^\dagger(x,\infty,\eta) := \lim_{t\uparrow\infty} n^\dagger(x,t,\eta)$.

Definition 5.3.1. *A configuration $\eta \in \mathbb{X}$ is stabilizable by the process with the generator (5.3.1) if for any $x \in \mathbb{Z}^d$, $n(x,\infty,\eta) < \infty$ and $n^\dagger(x,\infty,\eta) < \infty$.*

Therefore for η that is stabilizable, for all $t \geq 0$

$$n(x,t,\eta) < \infty \text{ and } n^\dagger(x,t,\eta) < \infty$$

under this case, with $n(t,\eta) = (n(x,t,\eta))_{x\in\mathbb{Z}^d}$ and $n^\dagger(t,\eta) = (n^\dagger(x,t,\eta))_{x\in\mathbb{Z}^d}$

$$\eta_t = \eta - \Delta(n(t,\eta) - n^\dagger(t,\eta)) \tag{5.3.2}$$

$\mathcal{H} = \{\eta : \eta(x) \geq 0, \forall x \in \mathbb{Z}^d\}$ is a special subset of $\mathbb{Z}^{\mathbb{Z}^d}$, the stabilization of $\eta \in \mathcal{H}$ under the process with generator is the same as under the process with generator

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}^d} I_{(\eta(x)>2d-1)}[f(\eta - \triangle_{x,\cdot}) - f(\eta)]$$

The condition for a configuration $\eta \in \mathcal{H}$ to be stabilized by the process with the upper generator has been studied in [9].

Now we give the main theorem of this section,

Theorem 5.3.2. *For $\eta \in \mathbb{Z}^{\mathbb{Z}^d}$, if $\xi \le \eta \le \zeta$ with $-\xi \in \mathcal{H}$, $\zeta \in \mathcal{H}$ and both $-\xi$ and ζ are stabilizable by the process with generator (5.3.1), η is also stabilizable by that process.*

Before giving the proof of this theorem, we give the following lemma,

Lemma 5.3.3. *For $\eta \in \mathbb{Z}^{\mathbb{Z}^d}$, if $\xi \le \eta \le \zeta$ with $-\xi \in \mathcal{H}$, $\zeta \in \mathcal{H}$, for t_x legal on η and $t_y^\dagger$ legal on η,*

$$\xi \le t_x\eta \le t_x\zeta \text{ and } t_y^\dagger\xi \le t_y^\dagger\eta \le \zeta$$

and $t_x\zeta \in \mathcal{H}$ and $-(t_y^\dagger\xi) \in \mathcal{H}$.

Proof. If t_x is legal on η, t_x is also legal on ζ, then $(t_x\eta)(y) = \eta(y) - \Delta(x,y) \le \zeta(y) - \Delta(x,y) = (t_x\zeta)(y), \forall y \in \mathbb{Z}^d$, which implies that $t_x\eta \le t_x\zeta$ and $t_x\zeta \in \mathcal{H}$.

Since $-\xi \in \mathcal{H}$, $\xi(x) \le 0, \forall x \in \mathbb{Z}^d$, and we know $(t_x\eta)(x) = \eta(x) - 2d \ge 0$ since t_x is legal implies that $\eta(x) \ge 2d$; and $(t_x\eta)(y) \ge \eta(y) \ge \xi(y), \forall y \ne x$, hence $\xi \le t_x\eta$.

Similarly, we can prove that for $t_y^\dagger$ is legal on η, $t_y^\dagger\xi \le t_y^\dagger\eta \le \zeta$ with $-(t_y^\dagger\xi) \in \mathcal{H}$. □

Now we can give the to proof of Theorem 5.3.2,

Proof. Assume η is not stabilizable under the process with generator (5.3.1), then there must be sites, x, y such that $n(x,\infty,\eta) = \infty$ or $n^\dagger(y,\infty,\eta) = \infty$, or both happen.

Without generality assume $n(x,\infty,\eta) = \infty$, if we let (ξ_t, η_t, ζ_t) be coupled by simultaneously topple or anti-topple if necessary. Then according to the lemma 5.3.3, during the stabilization process,

$$\xi_t \le \eta_t \le \zeta_t, \forall t \ge 0 \tag{5.3.3}$$

with $-\xi_t \in \mathcal{H}$ and $\zeta_t \in \mathcal{H}$. Then if an anti-toppling is legal on η_t, it is also legal on ξ_t; a toppling is legal on η_t, it is also legal on ζ_t which makes sure that $n(x,t,\eta) \le n(x,t,\zeta), \forall t \ge 0$. Take limit as $t \to \infty$, we get:

$$\infty = n(x,\infty,\eta) \le n(x,\infty,\zeta) \tag{5.3.4}$$

this contradicts the fact that ζ is stabilizable under the process with generator (5.3.1).

Then η is stabilizable under the process with generator (5.3.1). □

For any $\eta \in \mathbb{X}$, define two new configurations: η^+ and η^- such that,

$$\eta^+(x) := \eta(x) \bigvee 0, \text{ and } \eta^-(x) := \eta(x) \bigwedge 0$$

So $\eta^+ \in \mathcal{H}$ and $-\eta^- \in \mathcal{H}$. Then we get,

Corollary 5.3.4. *For $\eta \in \mathbb{Z}^{\mathbb{Z}^d}$, if both η^+ and $-\eta^-$ are stabilizable under the process with generator (5.3.1), then η is also stabilizable under the process with generator (5.3.1).*

The reverse part of Theorem 5.3.2 does not hold. For instance:

$$\eta = ...2, -2, 2, -2, 2, -2, ...$$

It is the configuration that -2 and 2 appear alternatively. Each site jumps at most 1 time. So this configuration is stabilizable by the process with generator (5.3.1), while, for $\eta^+ = ...0, 2, 0, 2, 0, 2, ... \in \mathcal{H}$ where 0 and 2 appears alternatively, we know this configuration is not stabilizable. It shows that the "negative" mass(holes) helps the stabilization.

5.4 Open question

For a fixed configuration η, what is the relation between the stabilization with the background of traditional sandpile+anti-sandpile model and the stabilization with the background symmetric sandpile+anti-sandpile model? The answer is that: a class of special ways of stabilization is to turn a configuration to Ω^s firstly, and then from Ω^s to Ω.

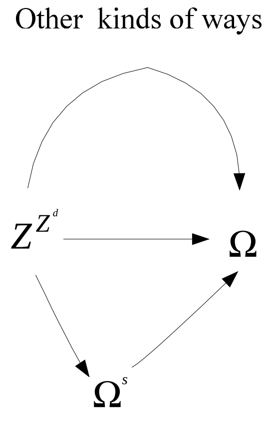

Figure 5.1: Relation of a stabilizaition in two different models

And for $\eta \in \mathbb{Z}^d$, and ν is translation invariant and ergodic measure, with $\nu(\eta(0)) = \rho$. If we take $\nu[(\eta \bigvee 0)(0)] = \rho^+$ and $\nu[(\eta \bigwedge 0)(0)] = \rho^-$. Then these tree "densities" have the following relation,

$$\rho = \rho^+ + \rho^- \tag{5.4.1}$$

As to the stabilization, if $0 < \rho^+ < d$, we know $\eta \bigvee 0$ is stabilizable. $-(\eta \bigwedge 0) \in \mathcal{H}$ with density $-\rho^-$, then if $-\eta^- < d$, i.e., $\eta^- > -d$, $-(\eta \bigwedge 0)$ is stabilizable, and hence $\eta \bigwedge 0$ is also stabilizable. And when $\rho^- < -2d + 1 - \rho^+$, $\rho < -2d + 1$, in this case, η can never be stabilized by the process with generator 5.3.1. It seems that for a fixed ρ^+ there is a "critical point" $\rho_c^-(\rho^+)$ on which the system shows transition between stabilizable and non-stabilizable. Similarly for $\rho^- > -d$. Then we have the following conjecture,

Conjecture 5.4.1. *ν is a translation invariant measure with the expected densities: $\nu(\eta(0)) = \rho$, $\nu[(\eta \bigvee 0)(0)] = \rho^+$ and $\nu[(\eta \bigwedge 0)(0)] = \rho^-$.For fixed $\eta^+ < d$, i.e, $\eta \bigvee 0$ is stabilizable, there is a critical value $\rho_c^-(\rho^+)$ such that,*

- *For $\rho^- > \rho_c^-(\rho^+)$, η is always stabilizable.*
- *For $\rho^- < -2d + 1 - \rho^+$, η is almost surely not stabilizable.*
- *For $-2d + 1 - \rho^+ < \rho^- < \rho_c^-(\rho^+)$, there is a measure μ with $\nu(\eta(0) \bigwedge 0) = \rho^-$ such that for η distributed according to ν, η is almost surely not stabilizable.*

For ρ^- fixed, we have the similar form of the conjecture.

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Ergodic theorems

In the following are some definitions and theorems that are used several times in Chapter 4 and Chapter 5 about the translation invariant measures and ergodic measures.

Define the shift transform $\tau_k : \mathbb{X} \to \mathbb{X}$:

$$(\tau_k \eta)(x) =: \eta(x-k), \forall \eta \in \mathbb{X} \tag{A .2}$$

And for each $f : \mathbb{X} \to \mathbb{R}$:

$$\tau_k f(\eta) = f(\tau_k \eta), \forall \eta \in \mathbb{X} \tag{A .3}$$

It easy to check $\mathcal{H}, \mathcal{J}, \Omega$ are invariant under the shift translation.

Definition A .2. *Let $\mathcal{A} \subseteq \mathbb{X}$ be invariant under the shift translation, μ be a non-negative measure on $\mathcal{A}$. Then μ is called "invariant" under the shift transform if for every $f \in C(\mathcal{A})$,*

$$\int f d(\tau_k \mu) = \int \tau_k f d\mu = \int f d\mu, \forall k \in \mathbb{Z}^d$$

i.e, $\tau_k \mu = \mu, \forall k \in \mathbb{Z}^d$.

Let $\mathcal{T}_{\mathcal{A}}$ be the set of all non-negative translation invariant measures on $\mathcal{A}$.

Definition A .3. *A $\mu \in \mathcal{T}_{\mathcal{A}}$ is said to be "ergodic" if whenever $\tau_k f = f$ for all $k \in \mathbb{Z}^d$ and f is measurable function on $\mathcal{A}$, it follows that f is constant a.s. relative to μ.*

The following two propositions describe some important properties of the ergodic measure, I just extract them from [12]

Theorem A .4 (Ergodic theorem). *If $\mu \in \mathcal{T}_{\mathcal{A}}$ and f is a bounded measurable function on $\mathcal{A}$, then*

$$\lim_{k \to \infty} \frac{\sum_{0 \le x \le k} \tau_x f}{\mid \{k \in \mathbb{Z}^d : 0 \le x \le k\} \mid} \tag{A .4}$$

exists a.s., and in L_1 relative to μ. For $d = 1$, it suffices that $f \in L_1(\mu)$

Corollary A .5. *Assume h is the limit of (A.4) and $\mu \in \mathcal{T}_{\mathcal{A}}$, then $\tau_m h = h$ $\mu-a.s$, $\forall m \in \mathbb{Z}^d$.*

Proof. Assume h is the limit of (A.4), then

$$\int | \lim_{k\to\infty} \frac{\sum_{0\le x\le k} \tau_x f}{|\{k\in\mathbb{Z}^d : 0\le x\le k\}|} - h \,|d\mu \to 0, \forall \mu \in \mathcal{T}_{\mathcal{A}}$$

then for all $m \in \mathbb{Z}^d$,

$$\begin{aligned}&\int | \lim_{k\to\infty} \frac{\sum_{0\le x\le k} \tau_m \tau_x f}{| \{x \in \mathbb{Z}^d : 0 \le x \le k\} |} - \tau_m h \,|d\mu \\ &= \int \tau_m \,| \lim_{k\to\infty} \frac{\sum_{0\le x\le k} \tau_x f}{| \{x \in \mathbb{Z}^d : 0 \le x \le k\} |} - h \,|d\mu \\ &\to 0, \forall \mu \in \mathcal{T}_{\mathcal{A}} \end{aligned} \tag{A .5}$$

while we know,

$$\lim_{k\to\infty} \frac{\sum_{0\le x\le k} \tau_m \tau_x f}{|\{x\in\mathbb{Z}^d : 0\le x\le k\}|} = \lim_{k\to\infty} \frac{\sum_{0\le x\le k} \tau_k f}{|\{x\in\mathbb{Z}^d : 0\le x\le k\}|}$$

Then we get that:

$$\tau_m h = h, \mu - a.s., \forall m \in \mathbb{Z}^d \tag{A .6}$$

□

Generally it is hard to check whether a measure is ergodic form the definition. The following proposition will give us a useful way to decide the ergodicity of a measure in $\mathcal{T}_{\mathcal{A}}$.

Proposition A .6. *A measure $\mu \in \mathcal{T}_{\mathcal{A}}$ is ergodic if and only if for every f and $g \in C(\mathcal{A})$,*

$$\lim_{k\to\infty} \frac{\sum_{0\le x\le k} \int (\tau_x f) g d\mu}{| \{x \in \mathbb{Z}^d : 0 \le x \le k\} |} = \int f d\mu \int g d\mu \tag{A .7}$$

Proof. For μ is ergodic, by Corollary A.5, the limit of (A.4) is translation invariant, and hence constant a.s(μ). So the left of (A.7) equals $\int fgd\mu = \int fd\mu \int gd\mu$.

For the converse, suppose (A.7) holds for every $f, g \in C(\mathcal{A})$. Since every bounded measurable function can be approximated in $L_1(\mu)$ by a continuous function, it follows that (A.7) holds for bounded measurable function as well. Now assume f is bounded measurable function and $\tau_k f = f, \forall k \in \mathbb{Z}^d$. Then (A.7) gives $\int fgd\mu = \int fd\mu \int gd\mu$. Take $f = g$, then f is constant $\mu-$a.s. By the definition of ergodic, we know μ is ergodic.

□

Proposition A .7. *Suppose μ_1 and $\mu_2 \in \mathcal{T}_{\mathcal{A}}$ are both ergodic. Then either $\mu_1 = \mu_2$ or μ_1 and μ_2 are mutually singular.*

Proof. Suppose that $\mu_1 \neq \mu_2$. Then there is an $f \in C(\mathcal{A})$ such that $\int f d\mu_1 \neq \int f d\mu_2$. Since both μ_1 and μ_2 are ergodic, then take $g = 1$ in (A.7), then we get:

$$\lim_{k \to \infty} \frac{\sum_{0 \leq x \leq k} \int \tau_x f d\mu_i}{\mid \{x \in \mathbb{Z}^d : 0 \leq x \leq k\} \mid} = \int f d\mu_i, i = 1, 2 \tag{A .8}$$

For μ is ergodic, combine with (A.6), we know $\int f d\mu_i$ is constant $\mu - a.s.$. Assume $\phi_1 = \int f d\mu_1$ and $\phi_2 = \int f d\mu_2$, then $\mu_1\{\phi_1 = \int f d\mu_1\} = 1$ and $\mu_2\{\phi_2 = \int f d\mu_2\} = 1$, while $\mu_1\{\phi_2 = \int f d\mu_2\} = 0$, or 1 a.s., and $\mu_2\{\phi_1 = \int f d\mu_1\} = 0$, or 1, a,s. Therefore μ_1 and μ_2 are mutually singularly.

□

An constructive example

In this appendix, I include the example given in [9]. In this example, the addition leads to infinitely many topplings at the origin as $\lim_{\Lambda \uparrow \mathbb{Z}^d}$.

Considering the Abelian sandpile model on a finite set $\Lambda \subset \mathbb{Z}^d$. Let $\vartheta_\Lambda(x)$ denotes the number of "lacking neighbors" of site x in the graph Λ. For η is recurrent, for each boundary site $x \in \partial\Lambda$, we add $\vartheta_\Lambda(x)$ grains to each x, then according to 2.5.5, 2.5.6, 2.5.7, we know each site $x \in \Lambda$ will topples exactly once and after topplings, the configuration will remain unaltered, i.e., still be η. If Λ is a rectangle of d-dimension, we add d grains to the corner sites and $d-1$ to other boundary sites. And from the definition of allowed configuration, we know for η is recurrent, and $W \subset \Lambda$, the restriction η_W is also allowed, and hence is also recurrent.

Now we give the example in $d = 2$, and the generalization to $d \neq 2$ is easy. Let ω, ω' be independent and distributed according to a Bernoulli measure P_p on $\{0,1\}^{\mathbb{Z}}$, with $P_p(\omega(x) = 1) = p$. Considering two dimensional random field $\zeta(x,y) = \omega(x) + \omega'(y)$. If $w(x) = 1$, we add one grain to each lattice site of the vertical line $\{(x,y), y \in \mathbb{Z}\}$ and if $\omega'(y) = 1$, then we add one grain to each lattice site of the horizontal line $\{(x,y), x \in \mathbb{Z}\}$. If we add according ζ, then there are almost surely infinitely many rectangles $R_1, ...R_n, ...$ surrounding the origin with corner sites where we add two grains and other boundary sites where we add at one grain.

If we add such a configuration ζ to a recurrent configuration η, we have that the number of topplings at the origin in the finite volume Λ is at least the number of rectangles R_i that are inside Λ.

Let M be set of all "minimal recurrent configurations", i.e, taking off one grain from $\eta \in M$ generates a non-recurrent configuration. ν is a translation invariant probability measure on M. From Lemma 5.1 9, we know $\nu(\eta(0)) = d$. Therefore the distribution ν_p of $\eta + \zeta$ with η drawn from ν is not stabilizable. Since we can choose p arbitrary close to zero, and density of $\rho \in (d, d+2)$ can be attained by ν_p.