Green functions on Riemann surfaces and an application to Arakelov theory

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Contents

Introduction

As the title indicates, this thesis consists of two parts. In the first part (§§ 1–3), we study Green functions for the Laplace operator on Riemann surfaces. The Green function on a Riemann surface is an integral kernel which solves the *Poisson equation*

$$2i\partial\bar\partial f = \phi.$$

More precisely, let X be a compact connected Riemann surface, and let μ be a smooth differential 2-form on X satisfying $\int_X \mu = 1$. Intuitively speaking, μ plays the role of a volume form on X, but at this point it need not be positive or even real. The *Green function* is a smooth function g_μ defined outside the diagonal on $X \times X$ such that for any smooth 2-form ϕ on X, the function

$$f(x) = \int_{y \in X \setminus \{x\}} g_\mu(x, y)\phi(y) \quad (x \in X)$$

satisfies $2i\partial\bar\partial f = \phi - \left(\int_X \phi\right)\mu$. In particular, f is a solution to the Poisson equation in the case where $\int_X \phi = 0$. The function g_μ is not unique; we normalise it by requiring that

$$\int_{y \in X \setminus \{x\}} g_\mu(x, y)\mu(y) = 0 \quad (x \in X).$$

In § 1 (inspired mainly by de Rham's book [17] on differentiable manifolds) we state some preliminaries on Riemannian manifolds, most importantly the existence and smoothness of a *Green form* which inverts the Laplace–de Rham operator on differential forms of arbitrary degree. We use these results in § 2 to deduce the existence and smoothness of the Green function for Riemann surfaces.

The main result of this thesis is proved in § 3. Suppose we have a compact connected Riemann surface X with a non-negative 2-form μ satisfying $\int_X \mu = 1$. Consider a finite atlas $\{(U^j, z^{(j)})\}_{j=1}^n$, and suppose that real numbers $c_1 > 0$, $M \geq 1$, $r \in (1/2, 1)$ are given such that the following hypotheses hold:

(1) Each $z^{(j)}U^j$ contains the open unit disc in $\mathbf{C}$.

(2) Write $\mu = iF^j\, dz^{(j)} \wedge d\bar z^{(j)}$ on U^j. Then $0 \leq F^j(x) < c_1$ for all $x \in U^j$ with $|z(x)| < 1$.

(3) The discs $\{x \in U^j \mid |z^{(j)}(x)| < r\}$ cover X.

(4) For all indices j and k, the function $|dz^{(j)}/dz^{(k)}|$ on the set of all $x \in U^j \cap U^k$ such that $|z^{(j)}(x)| < 1$ and $|z^{(j)}(x)| < 1$ is bounded by M.

Under these assumptions, we will prove that there exists a real number $C > 0$ such that for any compact connected Riemann surface satisfying the above hypotheses for certain values of c_1, r and M, the inequality

$$\sup_{X \times X \setminus \Delta} g_\mu \leq \frac{Cn}{(1-r)^{3/2}} \log\frac{1}{1-r} + \left(\frac{8}{3}\log 2 + \frac{1}{4}\right) nc_1 + \frac{n-1}{2\pi}\log M$$

holds. This result was first proved by F. Merkl, but without the dependence on r and without explicit constants. In the proof of this inequality, we mostly follow Merkl's approach, described in Edixhoven et al. [5].

In the second part (§§ 4–10), we describe how the estimate obtained in § 3 can be used in the setting of Arakelov's intersection theory for divisors on arithmetic surfaces. An *arithmetic surface* is a regular two-dimensional scheme X together with a projective flat morphism $p\colon X \to B$, where B is the spectrum of the ring of integers of some number field K. For simplicity, we will assume moreover that the generic fibre $X_K = X \times_B \operatorname{Spec} K$ of X is geometrically connected. To get a useful intersection theory, we take our divisors to be Weil divisors on X together with an $\mathbf{R}$-linear combination of the 'infinite fibres' of X, i.e. the curves over $\mathbf{R}$ or $\mathbf{C}$ obtained by changing the base to the completion of K with respect to an Archimedean valuation. As is the case for non-singular varieties over a field, there is a correspondence between divisors and line bundles.

In §§ 4–8, we describe Arakelov's intersection theory, as well as Faltings' analogue for arithmetic surfaces of the Riemann–Roch formula. The contribution 'at infinity' to the Arakelov intersection number of two horizontal divisors can be expressed using Green functions on the Riemann surfaces obtained from our arithmetic surfaces by changing the base to the complex numbers. This provides the connection with the first part of this thesis. The concept of height functions is briefly explained in § 4, where we also state the conventions on valuations of number fields that we will use. We collect some analytic preliminaries for Arakelov theory, such as the Arakelov–Green function and the concept of an admissible line bundle, in § 5. To state the Riemann–Roch–Faltings formula we also need an algebraic tool, called the determinant of cohomology, which is defined in § 7. The actual intersection theory is described in § 8.

In §§ 9–10 we study a particular application of Arakelov theory. If D is a horizontal divisor on an arithmetic surface, the *height* of D with respect to a fixed B-valued point P of X is the Arakelov intersection number $(D\,.\,P)$ of D and P. The problem we consider in § 9 is the estimation, on an arithmetic surface X of genus $g \geq 1$, the height of a divisor on the generic fibre of the form $D' - D$ with D and D' effective divisors of degree g such that some integer multiple of $D' - D$ is rationally equivalent to the zero divisor. We will prove a formula (Theorem 9.3) which relates this height to intersection properties of the horizontal divisors obtained by taking the closure in X of the divisors D and D'. After that, we bound one of the terms in this formula using a method which estimates the extrema of solutions of Poisson's equation on a graph; this method is described in Appendix A. Finally, in § 10, we consider the case where X is the semi-stable model over $\operatorname{Spec} \mathbf{Z}[\zeta_l]$ of the modular curve $X_1(l)$. In Theorem 10.7 we give an asymptotic bound for $l \to \infty$ of one of the other terms in the height formula of Theorem 9.3 by applying the methods of § 3, although an upper bound for the Arakelov $(1,1)$-form on $X_1(l)$ still needs to be inserted into our expression.

I should note that more sophisticated methods than those in this thesis (using the spectral theory of the Laplace operator) have already been used to compute (special values of) the Arakelov–Green function on the modular curves $X_0(N)$ for squarefree N; see Abbes and Ullmo [1]. In a related direction, estimates for the difference between the Arakelov–Green function and the 'hyperbolic Green function' (which is associated to the hyperbolic $(1,1)$-form $(dx\,dy)/y^2$ from the complex upper half-plane) have been obtained by Jorgenson and Kramer [10].

1. Manifolds and currents

In this section we give a brief overview of *currents* on differentiable manifolds. Currents are a kind of functionals on differential forms with compact support invented by de Rham [17]. They generalise the concept of *distributions* on Euclidean space, developed by Schwartz, to the setting of differentiable manifolds.

The theory in this section is more general than will be useful to us in the next section, but it does not seem to me that specialising to the case of Riemann surfaces leads to more clarity at this point. We do restrict ourselves to oriented manifolds in order to eliminate the need for distinguishing between so-called *even* and *odd* forms (see de Rham [17], § 5).

Definition. By a *(real) differentiable manifold* we mean a second-countable Hausdorff space with a smooth (i.e. C^∞) differentiable structure.

Let X be a differentiable manifold of dimension n. We write $\mathcal{E}^p_X$, or simply $\mathcal{E}^p$, for the sheaf of smooth p-forms on X. For each open subset $U \subseteq X$ we write $\mathcal{D}^p(U)$ for the $\mathbf{R}$-vector space of smooth p-forms with compact support in U. Note that the $\mathcal{D}^p(U)$ do not constitute a sheaf (not even a presheaf) unless $n = 0$; instead there are natural inclusions $\mathcal{D}^p(U) \subseteq \mathcal{D}^p(U')$ for $U \subseteq U'$ open subsets of X. Furthermore, we write

$$\mathcal{E}_X = \bigoplus_{p=0}^{n} \mathcal{E}^p_X \qquad \text{and} \qquad \mathcal{D}(U) = \bigoplus_{p=0}^{n} \mathcal{D}^p(U) \quad (U \subseteq X \text{ open}).$$

Then $(X, \mathcal{E}^0_X)$ is a locally ringed space and $\mathcal{E}_X$ is a sheaf of graded-commutative $\mathcal{E}^0_X$-algebras.

Let X be an oriented differentiable manifold of dimension n. In this section, we will use the convention that for any differential form α on X, the integral $\int_X \alpha$ is understood as the integral over X of the degree n component of α.

Besides the usual differential forms on manifolds, we will need *double forms* on a product of two manifolds X and Y. These play the role of integration kernels for linear operators $\mathcal{E}_X \to \mathcal{E}_Y$.

Definition. Let X and Y be differentiable manifolds. Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ denote the first and second projections, respectively; note that these are open mappings. We define $p_1^*\mathcal{E}_X$ to be the sub-$\mathcal{E}^0_{X\times Y}$-module of $\mathcal{E}_{X\times Y}$ generated by the pull-backs of differential forms on X via p_1, and we define $p_2^*\mathcal{E}_Y$ analogously. The sheaf of *double forms* on $X \times Y$ is the $\mathcal{E}^0_{X\times Y}$-module

$$\mathcal{E}_{X,Y} = p_1^*\mathcal{E}_X \otimes_{\mathcal{E}^0_{X\times Y}} p_2^*\mathcal{E}_Y.$$

This sheaf is in a natural way a direct sum

$$\mathcal{E}_{X,Y} = \bigoplus_{p,q\geq 0} p_1^*\mathcal{E}^p_X \otimes_{\mathcal{E}^0_{X\times Y}} p_2^*\mathcal{E}^q_X,$$

so that a double form can be decomposed in its components of degree (p, q) for $p, q \geq 0$.

Remark. Suppose X and Y are oriented differentiable manifolds of dimension m and n, respectively. If $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are coordinates on open subsets $U \subseteq X$ and $V \subseteq Y$, then a double form ϕ on $U \times V$ can be written uniquely as

$$\phi = \sum_{\substack{0\leq p\leq m\\ 0\leq q\leq n}} \sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}} \phi_{i_1,\ldots,i_p;j_1,\ldots,j_q}(dx_{i_1} \wedge \ldots \wedge dx_{i_p}) \otimes (dy_{j_1} \wedge \ldots \wedge dy_{j_q})$$

where the $\phi_{i_1,\ldots,i_p;j_1,\ldots,j_q}$ are smooth functions on $U \times V$. Notice that the wedge product induces an isomorphism

$$\begin{aligned} \mathcal{E}_{X,Y} &\longrightarrow \mathcal{E}_{X\times Y} \\ \alpha \otimes \beta &\longmapsto \alpha \wedge \beta \end{aligned};$$

however, this isomorphism changes by a sign $(-1)^{pq}$ on the component of degree (p, q) if X and Y are interchanged.

There is an obvious way to take the wedge product of double forms on $X \times Y$ with differential forms on X and on Y. We will only need the wedge product of a double form on $X \times Y$ with a form on Y. This is the unique $\mathcal{E}^0_{X,Y}$-bilinear map

$$\begin{aligned} \mathcal{E}_{X,Y} \times p_2^*\mathcal{E}_Y &\longrightarrow \mathcal{E}_{X,Y} \\ (\alpha, \beta) &\longmapsto \alpha \wedge_2 \beta \end{aligned}$$

which satisfies

$$(p_1^*\alpha_1 \otimes p_2^*\alpha_2) \wedge_2 p_2^*\beta = p_1^*\alpha_1 \otimes p_2^*(\alpha_2 \wedge \beta)$$

for all differential forms α_1 on X and α_2, β on Y. We will sometimes write $\alpha(x,y) \wedge \beta(y)$ for the value of the form $\alpha \wedge_2 p_2^*\beta$ at a point $(x,y) \in X \times Y$.

Double forms can be integrated over subsets of one (or both) of the manifolds X and Y. Let y be a point of Y, and write i_y for the map $X \to X \times Y$ defined by $x \mapsto (x,y)$. Then we can identify the pull-back $i_y^*\mathcal{E}_{X,Y}$ with $\mathcal{E}_X \otimes_{\mathbf{R}} \mathcal{E}_Y(y)$; here $\mathcal{E}_Y(y)$ is the fibre of $\mathcal{E}_Y$ at Y, which is a finite-dimensional $\mathbf{R}$-vector space. More precisely, there are natural isomorphisms

$$\begin{aligned} i_y^*\mathcal{E}_{X,Y} &\cong i_y^*p_1^*\mathcal{E}_X \otimes_{\mathcal{E}^0_X} i_y^*p_2^*\mathcal{E}_Y \\ &\cong (p_1 \circ i_y)^*\mathcal{E}_X \otimes_{\mathcal{E}^0_X} (p_2 \circ i_y)^*\mathcal{E}_Y \\ &\cong \mathcal{E}_X \otimes_{\mathcal{E}^0_X} (\mathcal{E}^0_X \otimes_{\mathcal{E}^0_{Y,y}} \mathcal{E}_{Y,y}) \\ &\cong \mathcal{E}_X \otimes_{\mathcal{E}^0_{Y,y}} \mathcal{E}_{Y,y} \\ &\cong \mathcal{E}_X \otimes_{\mathbf{R}} \mathcal{E}_Y(y). \end{aligned}$$

For any open subset $U \subseteq X$ and any double form ϕ defined on an open subset of $X \times Y$ containing $U \times \{y\}$, we can therefore write the element $i_y^*\phi$ of $\mathcal{E}_X(U) \otimes_{\mathbf{R}} \mathcal{E}_Y(y)$ as a finite sum

$$i_y^*\phi = \sum_i \alpha_i \otimes \beta_i$$

with $\alpha_i \in \mathcal{E}_X(U)$ and $\beta_i \in \mathcal{E}_Y(y)$, and we put

$$\int_{x \in U} \phi(x,y) = \sum_i \left(\int_U \alpha_i\right) \beta_i,$$

which is an element of $\mathcal{E}_Y(y)$ (provided the integrals $\int_U \alpha_i$ converge). It is clear that for fixed ϕ, the definition of $\int_{x\in U} \phi(x,y)$ does not depend on the choice of the α_i and β_i. We note that $\int_{x\in U} \phi(x,y)$ does not necessarily define a smooth differential form on an open subset of Y.

Definition. Let X be an oriented differentiable manifold of dimension n. For $0 \le p \le n$, a *(real-valued) current of degree* p on X is an $\mathbf{R}$-linear map

$$T\colon \mathcal{D}^{n-p}(X) \to \mathbf{R}$$

which is continuous in the following sense: if $U \subseteq X$ is a coordinate chart, $K \subseteq U$ is compact and $\{\phi_i\}_{i=0}^\infty$ is a sequence of smooth $(n-p)$-forms with support in K such that every partial derivative of every coefficient of the form ϕ_i (expressed in terms of the coordinates on U) converges uniformly to 0 as $i \to \infty$, then $T(\phi_i) \to 0$ as $i \to \infty$. The $\mathbf{R}$-vector space of currents of degree p on X is denoted by $(\mathcal{D}')^p(X)$. We write

$$\mathcal{D}'(X) = \bigoplus_{p=0}^n (\mathcal{D}')^p(X)$$

for the space of *(real-valued) currents* on X. If $T = \sum_{p=0}^n T^p$ is a current and $\phi = \sum_{p=0}^n \phi^p$ is a differential form with compact support (with $T^p \in (\mathcal{D}')^p(X)$ and $\phi^p \in \mathcal{D}^p(X)$), then we put

$$T(\phi) = \sum_{p=0}^n T^p(\phi^{n-p}).$$

Let X be an oriented differentiable manifold of dimension n. Important examples of currents on X include the currents $[\alpha]$ of degree p for $\alpha \in \mathcal{E}^p(X)$, which are defined by

$$[\alpha](\phi) = \int_X \alpha \wedge \phi \quad \text{for all } \phi \in \mathcal{D}^{n-p}(X).$$

We call $[\alpha]$ the current *represented* by α. More generally, we say that a current T of degree p is *represented on an open subset* $U \subseteq X$ by a p-form $\alpha \in \mathcal{E}^p(U)$ if

$$T(\phi) = \int_U \alpha \wedge \phi \quad \text{for all } \phi \in \mathcal{D}^{n-p}(U) \subseteq \mathcal{D}^{n-p}(X).$$

A current is called *smooth* on an open subset $U \subseteq X$ if it is represented by a smooth differential form on U.

Another important example is the current δ_a of degree n at a point $a \in X$ (widely known as the 'Dirac delta function'), defined by

$$\delta_a(f) = f(a).$$

Currents can be differentiated: the derivatives of a current T of degree p are defined by

$$dT(\phi) = (-1)^{p+1} T(d\phi).$$

If T is represented by a p-form α, this definition becomes the formula for integration by parts:

$$\int_X d\alpha \wedge \phi = (-1)^{p+1} \int_X \alpha \wedge d\phi.$$

Let X be an oriented n-dimensional Riemannian manifold, i.e. an oriented differentiable manifold of dimension n equipped with a Riemannian metric $\langle\ ,\ \rangle$. Let $\star$ be the Hodge star operator on X. This $\mathcal{E}^0_X$-linear map $\mathcal{E}_X \to \mathcal{E}_X$ sends forms of degree p to forms of degree $n-p$ and is defined in the following way. The metric gives rise to canonical isomorphisms

$$\flat\colon \mathcal{T}_X \xrightarrow{\sim} \mathcal{E}^1_X$$
$$\sharp\colon \mathcal{E}^1_X \xrightarrow{\sim} \mathcal{T}_X$$

between the tangent sheaf and the cotangent sheaf; these are inverses of each other. Taking exterior powers gives isomorphisms

$$\flat\colon \textstyle\bigwedge^p \mathcal{T}_X \xrightarrow{\sim} \mathcal{E}^p_X$$
$$\sharp\colon \mathcal{E}^p_X \xrightarrow{\sim} \textstyle\bigwedge^p \mathcal{T}_X$$

for $0 \le p \le n$. From this we get perfect pairings

$$\begin{aligned} \mathcal{E}^p_X \times \mathcal{E}^p_X &\longrightarrow \mathcal{E}^0_X \\ (\alpha, \beta) &\longmapsto \langle \alpha, \beta \rangle, \end{aligned}$$

where

$$\langle \alpha, \beta \rangle = \alpha(\sharp\beta) = \beta(\sharp\alpha).$$

Furthermore, we have a canonical isomorphism

$$\begin{aligned} \mathcal{E}^0 &\xrightarrow{\sim} \mathcal{E}^n \\ 1 &\mapsto \omega_X, \end{aligned}$$

where ω_X is the volume form of X, i.e. the unique n-form on X such that

$$(\omega_X(x))(e_1, \ldots, e_n) = 1$$

for every point $x \in X$ and every positively oriented orthonormal basis $(e_1, \ldots, e_n)$ of the tangent space of X at x. The star operator on forms of degree p is now defined as the unique isomorphism $\mathcal{E}^p \xrightarrow{\sim} \mathcal{E}^{n-p}$ such that

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega_X$$

for all open subsets $U \subseteq X$ and all $\alpha, \beta \in \mathcal{E}^p(U)$.

For any differential form α of degree p, the identity

$$\star\star\alpha = (-1)^{p(n-p)}\alpha$$

holds; this can easily be checked at each point $x \in X$ using an orthonormal basis of the tangent space at x. Thus the inverse of the star operator is given by

$$\star^{-1}\alpha = (-1)^{p(n-p)} \star\alpha \quad \text{for } \alpha \text{ of degree } p.$$

If α and β are two forms of the same degree p, we have

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega_X = \beta \wedge \star\alpha,$$

and hence, by the anticommutativity of the wedge product,

$$\begin{aligned} \star\alpha \wedge \beta &= (-1)^{p(n-p)}\alpha \wedge \star\beta \\ &= \alpha \wedge \star^{-1}\beta. \end{aligned}$$

This last formula implies that the current $[\star\alpha]$ representing a form $\star\alpha$ satisfies

$$[\star\alpha](\beta) = [\alpha](\star^{-1}\beta).$$

We extend the star operator to a linear map on the space of currents on X, in such a way that $(\mathcal{D}')^p(X)$ is sent to $(\mathcal{D}')^{n-p}(X)$ for $0 \le p \le n$, by putting

$$(\star T)\phi = T(\star^{-1}\phi).$$

A differential form $\alpha \in \mathcal{E}(X)$ is said to be *square-integrable* if the integral $\int_X \alpha \wedge \star\alpha$ exists (recall that the integral is defined as the integral of the degree n component of $\alpha \wedge \star\alpha$). This is the case if α has compact support, for example. We define an inner product $(\ ,\ )$ on the $\mathbf{R}$-vector space of square-integrable forms by putting

$$(\alpha, \beta) = \int_X \alpha \wedge \star\beta;$$

it is easy to check that this satisfies the axioms for an inner product (see de Rham [17], § 24).

We let δ be the *codifferential* on differential forms and currents on X, defined by

$$\delta\alpha = (-1)^p \star^{-1} d \star \alpha = (-1)^{n+p+1} \star d \star^{-1} \alpha \quad \text{for } \alpha \text{ of degree } p;$$

the second equality follows by an easy computation. This operator, which sends p-forms to $(p-1)$-forms, is the *metric dual* of d, in the sense that

$$(\alpha, d\beta) = (\delta\alpha, \beta)$$

for all square-integrable forms α and β such that the intersection of the supports of α and β is compact. An easy computation shows that the codifferential on currents satisfies

$$\delta T(\phi) = (-1)^{p+1} T(\delta\phi) \quad \text{for } T \text{ of degree } p.$$

We let

$$\Delta = d\delta + \delta d$$

be the *Laplace–de Rham operator* on forms and currents on X. The fact that δ is the metric dual of d implies that Δ is its own metric dual, i.e.

$$(\alpha, \Delta\beta) = (\Delta\alpha, \beta)$$

for all forms α and β such that the intersection of the supports of α and β is compact. Furthermore, it follows easily from the definitions of d and δ for currents that

$$\Delta T(\phi) = T(\Delta\phi)$$

for any current T and any differential form ϕ with compact support, and that

$$\star\Delta = \Delta\star.$$

A form (or current) α is called *harmonic* if $\Delta\alpha = 0$; the identity $\star\Delta = \Delta\star$ implies that $\star$ takes harmonic currents to harmonic currents.

Theorem 1.1 (Elliptic regularity). *Let X be an oriented Riemannian manifold, and let T be a current on X. If ΔT is smooth on an open subset $U \subseteq X$, and in particular if T is harmonic on U, then T is smooth on U.*

Proof. De Rham [17], § 29, corollaire 1.

Corollary 1.2. *Let X be a compact oriented Riemannian manifold. Then every harmonic current of degree 0 on X is represented by a locally constant function.*

Proof. Let T be a harmonic current of degree 0 on X. Then T is represented by a harmonic function f because of Theorem 1.1. Now

$$0 = (f, \Delta f) = (f, \delta df) = (df, df),$$

and the fact that $(\ ,\ )$ is positive definite implies $df = 0$, i.e. f is locally constant. □

Let X be an oriented Riemannian manifold. We are interested in the global solutions of the *Poisson equation*

$$\Delta S = T,$$

where T is a given current on X. For the equation to have a solution S, the current T will have to vanish on the harmonic forms ϕ with compact support on X, as the following computation shows:

$$T(\phi) = \Delta S(\phi) = S(\Delta\phi) = S(0) = 0.$$

Proposition 1.5 below shows that if X is compact, this condition is also sufficient. Note that $\mathcal{D}(X) = \mathcal{E}(X)$ if X is compact.

Definition. Let X be a Riemannian manifold, and let $\Delta \subseteq X \times X$ be the diagonal. Let $W \subseteq X \times X$ be an open subset containing Δ such that the *geodesic distance* $r(x, y)$ is defined for all $(x, y) \in W$ (see de Rham [17], § 27). Let k be a real number. A function $f: W \setminus \Delta \to \mathbf{R}$ is said to be $O(r^k)$ as $r \to 0$ if there exists an open subset $W' \subseteq W$, containing Δ, such that $r^{-k} f$ is bounded on $W' \setminus \Delta$. A differential form or double form α is said to be $O(r^k)$ if the components of α with respect to any chart of the form $(U \times U, \phi \times \phi)$, where (U, ϕ) is a chart on X, are $O(r^k)$.

Theorem 1.3. *Let X be a compact oriented Riemannian manifold of dimension n. There exist linear operators H and G on $\mathcal{D}'(X)$ which preserve degrees, take smooth forms to smooth forms and satisfy the relations*

$$\begin{array}{llll} dH = Hd = 0, & \delta H = H\delta = 0, & \star H = H\star, & H^2 = H, \\ dG = Gd, & \delta G = G\delta, & \star G = G\star, & GH = HG = 0, \end{array}$$
$$\Delta G = G\Delta = 1 - H.$$

These operators are self-dual in the sense that for all $T \in \mathcal{D}'(X)$ and $\phi \in \mathcal{E}(X)$,

$$GT(\phi) = T(G\phi) \quad \text{and} \quad HT(\phi) = T(H\phi).$$

Furthermore, there exist double forms

$$h \in \mathcal{E}_{X,X}(X \times X) \quad \text{and} \quad g \in \mathcal{E}_{X,X}(X \times X \setminus \Delta_X),$$

with Δ_X the diagonal in $X \times X$, such that the following holds. The form g is $O(r^{2-n})$, where r^{2-n} is to be interpreted as $\log r$ if $n = 2$. For all $\alpha \in \mathcal{E}(X)$ we have

$$H\alpha(x) = \int_{y \in X} h(x, y) \wedge \star\alpha(y) \quad \text{and} \quad G\alpha(x) = \int_{y \in X \setminus \{x\}} g(x, y) \wedge \star\alpha(y).$$

The fact that g is $O(r^{2-n})$ implies that the second integral converges for any $\alpha \in \mathcal{E}(X)$.

Proof. De Rham [17], § 31, théorème 23.

Corollary 1.4. *Let T be a current on a compact oriented Riemannian manifold. Then the equation $\Delta S = T$ has a solution S if and only if $HT = 0$.*

Proof. If $\Delta S = T$ for some current S, then $HT = H\Delta S = 0$ since $Hd = 0$ and $H\delta = 0$. Conversely, if $HT = 0$, the current $S = GT$ satisfies $\Delta S = T$. □

The operator H should be thought of as giving the 'harmonic component' of a current. It follows from $dH = 0$ and $\delta H = 0$ that $\Delta H = 0$, so HT is harmonic (and hence, by Theorem 1.1, smooth) for every current T. The operator G inverts the Laplace operator in the sense that if T is a current with $HT = 0$ (which, by the following lemma, means that T is 'orthogonal to harmonic forms'), then $S = GT$ is a solution to $\Delta S = T$.

Proposition 1.5. *Let X be a compact oriented Riemannian manifold, and let T be a current on X. Then $HT = 0$ if and only if $T(\phi) = 0$ for all harmonic forms ϕ. Consequently, the equation $\Delta S = T$ has a solution S if and only if T vanishes on harmonic forms.*

Proof. If $HT = 0$ and ϕ is a harmonic form, then

$$T(\phi) = \Delta GT(\phi) = GT(\Delta\phi) = 0.$$

Conversely, suppose T vanishes on harmonic forms. For every form ϕ, the form $H\phi$ is harmonic, so that

$$HT(\phi) = T(H\phi) = 0 \quad \text{for all } \phi \in \mathcal{E}(X).$$

We conclude that $HT = 0$. □

Corollary 1.6. *Let X be a compact connected oriented Riemannian manifold of dimension n, and let T be a current of degree n on X. Then $HT = 0$ is equivalent to $T(1) = 0$. Consequently, the equation $\Delta S = T$ has a solution S if and only if $T(1) = 0$.*

Proof. This follows from Proposition 1.5 since the only harmonic functions on X are the constant functions by Corollary 1.2. □

In the next section, it will be more natural to work with complex-valued differential forms instead of real-valued forms. We therefore introduce the notion of a *complex-valued* current, which is entirely analogous to that of a real-valued current except that complex-valued currents act on complex-valued differential forms and take complex values.

Definition. Let X be an oriented differentiable manifold. For $0 \le p \le n$, a *complex-valued current of degree* p on X is an $\mathbf{R}$-linear map

$$T\colon \mathcal{D}^{n-p}(X) \to \mathbf{C}$$

such that the real and imaginary components of T are real-valued currents on X. If $\phi = \Re\phi + i\Im\phi$ is a smooth complex-valued $(n-p)$-form on X with compact support, where $\Re\phi$ and $\Im\phi$ are real-valued $(n-p)$-forms, we put

$$T\phi = T(\Re\phi) + iT(\Im\phi),$$

and in this way we view T as a $\mathbf{C}$-linear map from the space of complex $(n-p)$-forms with compact support to $\mathbf{C}$. Like for real-valued currents, we define

$$dT(\phi) = (-1)^{p+1}T(d\phi)$$

for all complex-valued currents T of degree p and smooth complex-valued $(n-p)$-forms ϕ with compact support. If X is an oriented Riemannian manifold, the operators $\star$, δ and Δ are defined for complex-valued forms and currents by $\mathbf{C}$-linearity. If in addition X is compact, the operators H and G from Theorem 1.3 are also extended to complex-valued forms and currents by $\mathbf{C}$-linearity; it is then clear that the theorem remains true (with the same functions h and g) if real-valued forms and currents are replaced by complex-valued ones.

2. Green functions on Riemann surfaces

In this section, we will give the definition of the Green function of a compact connected Riemann surface X equipped with a smooth $(1,1)$-form μ which satisfies $\int_X \mu = 1$. We begin by recalling some facts about Riemann surfaces; for proofs, we refer to Forster's book [8].

Definition. By a *Riemann surface* we mean a second-countable Hausdorff space with a one-dimensional complex analytic structure.

For any Riemann surface X, we fix an orientation on X (as a two-dimensional real differentiable manifold) by requiring that for every $x \in X$ and for some (hence every) identification of the complex tangent space $\mathrm{T}_X(x)$ of X at x with $\mathbf{C}$, the $\mathbf{R}$-basis $(1,i)$ of $\mathrm{T}_X(x)$ is positively oriented.

On any Riemann surface X we have the sheaves of complex-valued[†] C^∞ differential forms and the differential d between them:

$$\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2.$$

The sheaf of 1-forms can be decomposed as

$$\mathcal{E}^1 = \mathcal{E}^{(1,0)} \oplus \mathcal{E}^{(0,1)},$$

where $\mathcal{E}^{(1,0)}$ and $\mathcal{E}^{(0,1)}$ consist of the functions that are locally of the form $f\,dz$ and $f\,d\bar z$, respectively, with z a holomorphic coordinate. Complex conjugation induces involutions on all the $\mathcal{E}^i$; they are given by

$$\begin{aligned} \bar f(x) &= \overline{f(x)},\\ \overline{f\,dz + g\,d\bar z} &= \bar g\,dz + \bar f\,d\bar z,\\ \overline{f\,dz\wedge d\bar z} &= -\bar f\,dz\wedge d\bar z \end{aligned}$$

on $\mathcal{E}^0$, $\mathcal{E}^1$ and $\mathcal{E}^2$, respectively. The *real* differential forms on X are those that are invariant under complex conjugation; they correspond to the usual real-valued differential forms on X viewed as a real differentiable manifold.

We define differential operators

$$\partial, \bar\partial \colon \mathcal{E}^0 \longrightarrow \mathcal{E}^1$$

by requiring that for all $U \subseteq X$ open and $f \in \mathcal{E}^0(U)$ we have

$$df = \partial f + \bar\partial f \quad \text{with } \partial f \in \mathcal{E}^{(1,0)}(U) \text{ and } \bar\partial f \in \mathcal{E}^{(0,1)}(U).$$

Furthermore, we define

$$\partial, \bar\partial \colon \mathcal{E}^1 \longrightarrow \mathcal{E}^2$$

by

$$\partial(f\,dz + g\,d\bar z) = \partial g \wedge d\bar z \quad\text{and}\quad \bar\partial(f\,dz + g\,d\bar z) = \bar\partial f \wedge dz.$$

We define an automorphism of $\mathcal{E}^1$, called the *(conformal) star operator*, by

$$*(\alpha+\beta) = -i\alpha + i\beta \quad \text{for } U \subseteq X \text{ open, } \alpha \in \mathcal{E}^{(1,0)}(U),\ \beta \in \mathcal{E}^{(0,1)}(U).$$

The star operator can be viewed as rotation by $\pi/2$ in the (complexified, suitably oriented) cotangent space: if $z = x + iy$ is a local coordinate with x and y real, then

$$*dx = dy \quad\text{and}\quad *dy = -dx.$$

The *(conformal) Laplace operator* is the differential operator

$$d * d = 2i\partial\bar\partial \colon \mathcal{E}^0 \longrightarrow \mathcal{E}^2.$$

The kernel of the Laplace operator is the sheaf of *harmonic functions*.

[†] Note that this differs from the notation in the previous section, where $\mathcal{E}$ was used for real differential forms.

Remark. We will see in the proof of Theorem 2.2 how the star operator $*$ is related to the Hodge operator $\star$ on oriented Riemannian manifolds. Notice that although the operators $\star$, δ and Δ are not defined for a Riemann surface without further structure, the conformal structure of a Riemann surface allows us to define the conformal Laplace operator, which takes 0-forms to 2-forms (in contrast to the Laplace–de Rham operator, which preserves degrees).

Let X be a Riemann surface. Without further notice, by a *current* we will mean a complex-valued current. In a similar way as for the differential d, we can also take the holomorphic and antiholomorphic differentials ∂ and $\bar{\partial}$ of currents T of degree p on X:

$$\left.\begin{aligned}\partial T(\phi) &= (-1)^{p+1}T(\partial\phi)\\ \bar{\partial}T(\phi) &= (-1)^{p+1}T(\bar{\partial}\phi)\end{aligned}\right\}\quad \text{for all } \phi \in \mathcal{E}^{1-p}(X).$$

Furthermore, we transfer the star operator to currents T of degree 1 by means of the formula

$$*T(\phi) = -T(*\phi) \quad \text{for all } \phi \in \mathcal{E}^1(X).$$

The Laplace operator is defined for currents of degree 0 in the same way as for smooth functions, namely as the operator

$$d * d = 2i\partial\bar{\partial}.$$

A simple calculation shows that for all currents T of degree 0 and all $f \in \mathcal{E}^0(X)$,

$$(d * dT)(f) = T(d * df).$$

Suppose T is a current of degree 2 on a compact connected Riemann surface X. Theorem 2.2 below gives a necessary and sufficient condition for the existence of a current S of degree 0 such that $d * dS = T$. To prove the theorem, we will apply Theorem 1.3 to X equipped with a Riemannian metric which is compatible with the complex structure in a sense which we now explain. The tangent sheaf of X has an automorphism $*$ which is uniquely defined by the requirement that $\alpha(*v) = -(*\alpha)v$ for all vector fields v and all 1-forms α. In a holomorphic coordinate $z = x + iy$, this automorphism is given by

$$*(v_x\partial_x + v_y\partial_y) = -v_y\partial_x + v_x\partial_y,$$

i.e. it rotates tangent vectors by $\pi/2$ with respect to the orientation and conformal structure defined by the complex structure on X.

Definition. A Riemannian metric $\langle\ ,\ \rangle$ on a Riemann surface X is called *compatible with the complex structure* if the operator $*$ on the tangent sheaf of X is an isometry.

Let $(U, x+iy)$ be any holomorphic chart. Then $\langle\ ,\ \rangle$ is compatible with the complex structure if and only if the matrix of $\langle\ ,\ \rangle$ with respect to the real coordinates (x, y) at every point of U is of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}\right)$ with $a > 0$. This follows directly by writing out the definition.

Lemma 2.1. *Let X be a Riemann surface. Then there exists a Riemannian metric $\langle\ ,\ \rangle$ on X which is compatible with the complex structure.*

Proof. We choose an atlas $\{(U_j, z_j)\}_{j\in I}$ on X and a smooth partition of unity $\{\phi_j\}_{j\in I}$ subordinate to the covering $\{U_j\}_{j\in I}$. For each $i \in I$, let x_j and y_j be the real coordinates on U_j such that $z_j = x_j + iy_j$, and write ∂_x^j and ∂_y^j for the partial derivatives with respect to x_j and y_j. For any C^∞ vector field v on U_j, we denote by v_x^j and v_y^j the real C^∞ functions on U_j such that $v = v_x^j\partial_x^j + v_y^j\partial_y^j$. We define a Riemannian metric $\langle\ ,\ \rangle$ on X by putting

$$\langle v, w\rangle = \sum_{j\in I}\phi_j \cdot (v_x^j w_x^j + v_y^j w_y^j).$$

Now the star operator is given in the real coordinates (x_j, y_j) by $*v = -v_y^j\partial_x^j + v_x^j\partial_y^j$. Therefore

$$\begin{aligned}\langle *v, *w\rangle &= \sum_{j\in I}\phi_j \cdot ((-v_y^j)(-w_y^j) + v_x^j w_x^j)\\ &= \langle v, w\rangle,\end{aligned}$$

and we conclude that the star operator is an isometry. □

Theorem 2.2. *Let X be a compact connected Riemann surface, and let μ be a smooth $(1,1)$-form on X such that $\int_X \mu = 1$. For any current T of degree 2 on X, there exists a unique current $G_\mu T$ of degree 0 such that*

$$d * dG_\mu T = T - T(1)\cdot[\mu] \quad \text{and} \quad G_\mu T(\mu) = 0.$$

If T is represented on an open subset $U \subseteq X$ by a 2-form $\phi \in \mathcal{E}^2(U)$, then $G_\mu T$ is represented on U by a function $f \in \mathcal{E}^0(U)$. Write Δ_X for the diagonal in $X \times X$. There is a unique C^∞ function

$$g_\mu\colon (X \times X) \setminus \Delta_X \longrightarrow \mathbf{R},$$

having a logarithmic (hence integrable) singularity along Δ_X, such that for all $\phi \in \mathcal{E}^2(X)$ the current $G_\mu[\phi]$ is represented by the function $G_\mu\phi \in \mathcal{E}^0(X)$ given by

$$G_\mu\phi(x) = \int_{y\in X\setminus\{x\}} g_\mu(x,y)\phi(y).$$

Proof. Using Lemma 2.1 we choose a Riemannian metric $\langle\ ,\ \rangle$ on X which is compatible with the complex structure. If $z = x+iy$ is a holomorphic coordinate on an open subset $U \subseteq X$, the Hodge operator $\star$ on 1-forms is given in the coordinates (x,y) by

$$\star dx = dy, \quad \star dy = -dx,$$

because of the definition of $\star$ (see § 1) and the compatibility of $\langle\ ,\ \rangle$ with the complex structure. Because $\mathcal{E}^1_U$ is generated over $\mathcal{E}^0_U$ by dx and dy, the $\mathcal{E}^0_U$-linear operators $\star$ and $*$ have the same effect on 1-forms. Now the codifferential is given by $\delta = -\star^{-1}d\star$ on forms and currents of degree 1. For any current S of degree 0 we therefore have

$$d * dS = -\star\delta dS = -\star\Delta S,$$

where Δ is the Laplace–de Rham operator. In particular, our two definitions of harmonic functions (as functions annihilated by the operators Δ and $d * d$, respectively) coincide.

Let M be the smooth function $G(\star\mu)$, and let G_μ be the linear operator on currents of degree 2 defined by

$$G_\mu T = -\star GT + [T(M)] + T(1)\left[M - \int_X M\mu\right].$$

Then

$$\begin{aligned}\Delta G_\mu T &= -\star\Delta GT + T(1)\Delta M\\ &= -\star T + \star HT + T(1)\cdot[\star\mu - \star H\mu]\\ &= -\star\big(T - T(1)\cdot[\mu] - H(T - T(1)\cdot[\mu])\big)\\ &= -\star(T - T(1)\cdot[\mu]);\end{aligned}$$

the last equality follows from Corollary 1.6 and the fact that

$$(T - T(1)\cdot[\mu])(1) = T(1) - T(1)\int_X \mu = 0.$$

Therefore,

$$d * dG_\mu T = T - T(1)\cdot[\mu].$$

Furthermore,

$$\begin{aligned}G_\mu T(\mu) &= -\star GT(\mu) + \int_X T(M)\mu + T(1)\int_X M\mu - T(1)\int_X M\mu\cdot\int_X \mu\\ &= -T(G(\star\mu)) + T(M)\\ &= 0,\end{aligned}$$

so we see that the current $G_\mu T$ has the required properties. Clearly $G_\mu T$ is unique up to a harmonic current S satisfying $S(\mu) = 0$. By Corollary 1.2, such a current is represented by a constant function h. The only such h satisfiying $\int_X h\mu = 0$ is $h = 0$, from which we conclude that $G_\mu T$ is unique. The equality

$$\Delta G_\mu T = -\star(T - T(1)\cdot[\mu])$$

implies that if T is smooth on an open subset $U \subseteq X$, then so is $G_\mu T$ because of Theorem 1.1.

Let $\phi \in \mathcal{E}^2(X)$ be a smooth 2-form. Then for all $x \in X$,

$$\begin{aligned}
G_\mu\phi(x) &= -G(\star\phi)(x) + \int_X M\phi + \int_X \phi\cdot\Big(M(x) - \int_X M\mu\Big)\\
&= -\int_{y\in X\setminus\{x\}} g(x,y)\wedge\star\star\phi(y) + \int_X M\phi + \int_X \phi\cdot\Big(M(x) - \int_X M\mu\Big)\\
&= \int_{y\in X\setminus\{x\}} \Big(-g^0(x,y) + M(y) + M(x) - \int_X M\mu\Big)\phi(y)\\
&= \int_{y\in X\setminus\{x\}} g_\mu(x,y)\phi(y),
\end{aligned}$$

where g^0 is the component of degree $(0,0)$ of the double form g and where

$$g_\mu(x,y) = -g^0(x,y) + M(x) + M(y) - \int_X M\mu.$$

This proves the existence of g_μ; the uniqueness is clear. □

Definition. Let X be a compact connected Riemann surface, and let μ be a smooth $(1,1)$-form on X such that $\int_X \mu = 1$. The function g_μ occurring in Theorem 2.2 is called the *Green function* associated to the Laplace operator on X.

Note that if μ is real (i.e. invariant under complex conjugation), then the Green function g_μ is real as well; this can be seen for example from the formula given for g_μ in the proof of Theorem 2.2. Furthermore, $g_\mu(x,y)$, viewed as a function of y for a fixed value of x, represents the current $G_\mu(\delta_x)$, as the following lemma shows.

Lemma 2.3. *Let X be a compact connected Riemann surface, and let μ be a smooth $(1,1)$-form on X such that $\int_X \mu = 1$. For all $x \in X$, let $g_{x,\mu}$ be the C^∞ function on $X \setminus \{x\}$ sending y to $g_\mu(x,y)$, and let $[g_{x,\mu}]$ be the current defined by*

$$[g_{x,\mu}](\phi) = \int_{X\setminus\{x\}} g_{x,\mu}\phi \quad \text{for all } \phi \in \mathcal{E}^2(X).$$

Then $g_{x,\mu}$ is the unique smooth function on $X \setminus \{x\}$ satisfying

$$d * d[g_{x,\mu}] = \delta_x - [\mu] \quad \text{and} \quad \int_{X\setminus\{x\}} g_{x,\mu}\mu = 0.$$

Proof. To prove that $d * d[g_{x,\mu}] = \delta_x - [\mu]$, we have to show that

$$\int_{X\setminus\{x\}} g_{x,\mu} d * df = f(x) - \int_X f\mu \quad \text{for all } x \in X \text{ and } f \in \mathcal{E}^0(X).$$

By Theorem 2.2, the function $\tilde f$ given by

$$\tilde f(x) = \int_{X\setminus\{x\}} g_{x,\mu} d * df$$

is the unique function in $\mathcal{E}^0(X)$ satisfying

$$d * d\tilde f = d * df \quad \text{and} \quad \int_X \tilde f\mu = 0.$$

But $f - \int_X f\mu$ also satisfies these equations, so we are done. To prove that $\int_{X\setminus\{x\}} g_{x,\mu}\mu = 0$, we note that Theorem 2.2 implies that the function m given by

$$m(x) = \int_{X\setminus\{x\}} g_{x,\mu}\mu$$

is the unique function in $\mathcal{E}^0(X)$ such that $d * dm = 0$ and $\int_X m\mu = 0$; but $m = 0$ satisfies these equalities, so we are done. The uniqueness of $g_{x,\mu}$ is clear from the fact that the difference of any two solutions is a harmonic (hence constant) function h with $\int_X h\mu = 0$. $\square$

Let (U, z) be a coordinate chart on X, let $x \in U$, and let μ and $g_{x,\mu}$ be as in Lemma 2.3. Let l_x be a C^∞ function on $X \setminus \{x\}$ which is of the form $y \mapsto \frac{1}{2\pi}\log|z(x) - z(y)|$ in some open neighbourhood $U' \subseteq U$ of x. A computation shows that in U' we have

$$d * d[l_x] = \delta_x$$

as currents of degree 2. Therefore $d * d[g_{x,\mu} - l_x]$ is represented by a smooth $(1,1)$-form on X, and now Theorem 2.2 implies that $g_{x,\mu} - l_x$ can be extended to a smooth function on X. It follows that g_μ can be decomposed as

$$g_\mu(x, y) = \frac{1}{2\pi}\log|z(x) - z(y)| + h(x, y)$$

outside the diagonal on $U' \times U'$, where h is a smooth function on $U' \times U'$ (depending on the choice of z).

Proposition 2.4. *Let X be a compact connected Riemann surface, and let μ be a smooth $(1,1)$-form on X such that $\int_X \mu = 1$. Then $g_\mu(x, y) = g_\mu(y, x)$ for all $x \neq y$.*

Proof. Arakelov [2], Proposition 1.1.

3. Estimation of Green functions

In this section we describe a method invented by F. Merkl (see Edixhoven et al. [5], § 18) of estimating the maxima of Green functions on Riemann surfaces. Theorem 3.1 below was proved by Merkl, but without explicit bounds on the various constants appearing in the estimate; most importantly, the parameter r_1 was assumed to be fixed once and for all, and the dependence of the estimate on r_1 is not apparent from the original formulation of Merkl's theorem (Edixhoven et al. [5], Theorem 18.1.1). Except in a few places, we follow Merkl's proof quite closely, while also doing the necessary computations to make the r_1-dependence of the constants explicit.

The situation is as follows. Let X be a compact connected Riemann surface, and let μ be a smooth real-valued $(1,1)$-form on X such that $\int_X \mu = 1$. We consider a finite atlas $\{(U^j, z^{(j)})\}_{j=1}^n$. For any index j with $1 \leq j \leq n$ and any $r > 0$, we define the open set

$$U_r^j = \bigl\{x \in U^j \bigm| |z^{(j)}(x)| < r\bigr\}.$$

We assume that there exist real numbers

$$1/2 < r_1 < 1, \quad c_1 > 0, \quad M \geq 1$$

such that our atlas satisfies the following hypotheses:

(1) Each $z^{(j)}U^j \subseteq \mathbf{C}$ contains the closed unit disc.

(2) Write $\mu = iF^j\, dz^{(j)} \wedge d\bar{z}^{(j)}$ on U^j. Then $0 \leq F(x) \leq c_1$ for all $x \in U_1^j$.

(3) The open sets $U_{r_1}^j$ with $1 \leq j \leq n$ cover X.

(4) For all j and k, the function $\bigl|dz^{(j)}/dz^{(k)}\bigr|$ on $U_1^j \cap U_1^k$ is bounded by M.

Our goal in this section is to provide an explicit upper bound for the Green function g_μ in terms of the parameters n, c_1, r_1 and M. We will prove the following result:

Theorem 3.1. *There exists a positive real number C such that the following holds. Let X be a compact Riemann surface, and let μ be a smooth $(1,1)$-form on X such that $\int_X \mu = 1$. Consider an atlas on X consisting of n charts and fulfilling the above hypotheses (1)–(4) for certain values of r_1, c_1 and M. Then the Green function g_μ on $X \times X \setminus \Delta$, where Δ is the diagonal, satisfies*

$$\sup_{X\times X\setminus\Delta} g_\mu \le \frac{Cn}{(1-r_1)^{3/2}} \log\frac{1}{1-r_1} + \left(\frac{8}{3}\log 2 + \frac{1}{4}\right) nc_1 + \frac{n-1}{2\pi}\log M.$$

Furthermore, for every index j and all $x \ne y \in U^j_{r_1}$ the inequality

$$\left|g_\mu(x,y) - \log|z^{(j)}(x) - z^{(j)}(y)|\right| \le \frac{Cn}{(1-r_1)^{3/2}} \log\frac{1}{1-r_1} + \left(\frac{8}{3}\log 2 + \frac{1}{4}\right) nc_1 + \frac{n-1}{2\pi}\log M$$

holds.

Remark. Theorem 3.1 remains true if hypothesis (1) is replaced by the weaker hypothesis that each $z^{(j)}U^j$ contains the *open* unit disc. Namely, if we replace $z^{(j)}$ by $(1+\epsilon)z^{(j)}$ with $\epsilon > 0$ so small that $(1+\epsilon)r_1 < 1$, we can apply the theorem with r_1 replaced by $(1+\epsilon)r_1$ and with the same values for n, c_1 and M; letting ϵ tend to zero gives the desired result.

Our approach to proving Theorem 3.1 is as follows. For all $a, b \in X$, we write $g_{a,b}$ for the unique C^∞ function on $X \setminus \{a,b\}$ representing the current $G_\mu(\delta_a - \delta_b)$, i.e. $g_{a,b}$ satisfies

$$d*d[g_{a,b}] = \delta_a - \delta_b \quad\text{and}\quad \int_{X\setminus\{a,b\}} g_{a,b}\mu = 0.$$

We choose a function l_a which looks like $\frac{1}{2\pi}\log|z - z(a)|$ in a neighbourhood of a, where z is a holomorphic coordinate, and we do the same for b. Then the function $g_{a,b} - l_a + l_b$ is bounded on X. We give an explicit bound for it in Lemma 3.5 for the case where a and b lie in the same open subset $U^j_{r_1}$ for one of the charts $(U^j, z^{(j)})$ of our atlas, and subsequently in Lemma 3.6 for general a and b. Then for all $a \in X$ we consider the function $g_{a,\mu}$ on $X \setminus \{a\}$ defined by

$$g_{a,\mu}(x) = \int_{b\in X\setminus\{x\}} g_{a,b}(x)\mu(b).$$

A straightforward computation using Fubini's theorem shows that this function satisfies

$$d*d[g_{a,\mu}] = \delta_a - [\mu] \quad\text{and}\quad \int_{X\setminus\{a\}} g_{a,\mu}\mu = 0.$$

Using Lemma 2.3, we see that $g_{a,\mu}(b) = g_\mu(a,b)$ with g_μ the Green function for the Laplace operator on X. Using the definition of $g_{a,\mu}$ as the integral of $g_{a,b}\mu(b)$, we will be able to derive an upper bound for g_μ from the bound for $|g_{a,b} - l_a + l_b|$ given in Lemma 3.6.

We begin by restricting our attention to one of the charts of our atlas, say (U, z). By assumption, z gives a homeomorphism between U_1 and the open unit disc in $\mathbf{C}$. Let r_2 be a real number such that

$$r_1 < r_2 < 1,$$

and write

$$r_3 = (1 + r_2)/2.$$

We choose a C^∞ function

$$\tilde\chi\colon \mathbf{R}_{\ge 0} \to [0,1]$$

such that $\tilde\chi(r) = 1$ for $r \le r_2$ and $\tilde\chi(r) = 0$ for $r \ge 1$ (see Figure 1). We also define a function $\chi \in \mathcal{E}^0(X)$ by putting

$$\chi(x) = \tilde\chi(|z(x)|) \quad \text{for } x \in U_1$$

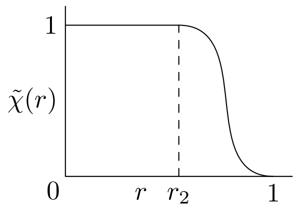

Figure 1: The function $\tilde\chi$.

and extending by 0 outside U_1. Furthermore, we put

$$\chi^{\mathrm{c}} = 1 - \chi.$$

We replace U by $U_{1/r_1} = \{x \in U \mid r_1|z(x)| < 1\}$; note that the hypotheses (1) to (4) remain fulfilled when we shrink the charts in this way. Then for all $a, b \in U_{r_1}$, the function

$$f_{a,b} = \frac{1}{2\pi}\log\left|\frac{(z - z(a))(\overline{z(a)}z - 1)}{(z - z(b))(\overline{z(b)}z - 1)}\right|$$

is defined on $U \setminus \{a, b\}$. Moreover, $f_{a,b}$ is harmonic on $U \setminus \{a, b\}$, since the logarithm of the modulus of a holomorphic function is harmonic. We extend $\chi^{\mathrm{c}} f_{a,b}$ to a smooth function on U by defining it to be zero in a and b.

Let A be the open annulus

$$A = U_1 \setminus \overline{U_{r_2}}.$$

Let (ρ, ϕ) be real polar coordinates on A such that $z = \rho\exp(i\phi)$. A straightforward calculation shows that in these coordinates the star operator is given by

$$*d\rho = \rho\, d\phi, \quad *d\phi = -\frac{d\rho}{\rho}.$$

We let $\langle\ ,\ \rangle_A$ be the inner product

$$\langle \alpha, \beta\rangle_A = \int_A \alpha \wedge *\beta.$$

on the $\mathbf{R}$-vector space of square-integrable real-valued 1-forms on A (see § 1). Furthermore, we write

$$\|\alpha\|_A^2 = \langle \alpha, \alpha\rangle_A.$$

We begin the proof of Theorem 3.1 with a general fact about harmonic functions on A.

Lemma 3.2. *For every real harmonic function g on A such that $\|dg\|_A$ exists,*

$$\max_{|z|=r_3} g - \min_{|z|=r_3} g \le \frac{2\sqrt{\pi}}{1 - r_2}\|dg\|_A.$$

Proof. By the formula for the star operator in polar coordinates,

$$\begin{aligned} dg \wedge *dg &= (\partial_\rho g\, d\rho + \partial_\phi g\, d\phi) \wedge (\rho\partial_\rho g\, d\phi - \rho^{-1}\partial_\phi g\, d\rho) \\ &= \bigl((\partial_\rho g)^2 + (\rho^{-1}\partial_\phi g)^2\bigr)\rho\, d\rho\, d\phi. \end{aligned}$$

Using the mean value theorem, we can bound the left-hand side of the inequality we need to prove by

$$\begin{aligned} \max_{|z|=r_3} g - \min_{|z|=r_3} g &\le \pi \max_{|z|=r_3} |\partial_\phi g| \\ &= \pi|\partial_\phi g|(x) \quad \text{for some } x \text{ with } |z(x)| = r_3. \end{aligned}$$

Write $R = (1 - r_2)/2$, and let

$$D = \left\{ z \in U \;\middle|\; |z - z(x)| < R \right\} \subset A$$

be the open disc of radius R around x (recall that $r_3 = (1 + r_2)/2$). Choose polar coordinates (σ, ψ) on D such that $z - z(x) = \sigma \exp(i\psi)$. Because g is harmonic, so is $\partial_\phi g$, and Gauss' mean value theorem implies that

$$\partial_\phi g(x) = \frac{1}{\pi R^2} \int_D \partial_\phi g \, \sigma \, d\sigma \, d\psi.$$

On the space of real continuous functions on D, we have the inner product

$$(h_1, h_2) \mapsto \int_D h_1 h_2 \, \sigma \, d\sigma \, d\psi.$$

Applying the Cauchy–Schwarz inequality with $h_1 = \rho^{-1} \partial_\phi g$ and $h_2 = \rho$ gives

$$\begin{aligned}
\left| \int_D \partial_\phi g \, \sigma \, d\sigma \, d\psi \right| &\le \left[\int_D \left(\rho^{-1} \partial_\phi g\right)^2 \sigma \, d\sigma \, d\psi \right]^{1/2} \cdot \left[\int_D \rho^2 \sigma \, d\sigma \, d\psi \right]^{1/2} \\
&\le \left[\int_A (\rho^{-1} \partial_\phi g)^2 \rho \, d\rho \, d\phi \right]^{1/2} \cdot \left[\int_D \sigma \, d\sigma \, d\psi \right]^{1/2} \\
&\le \left[\int_A dg \wedge * dg \right]^{1/2} [\pi R^2]^{1/2} \\
&= \sqrt{\pi}\, R \|dg\|_A.
\end{aligned}$$

Combining the above results finishes the proof.[†] □

Lemma 3.3. *For all $a, b \in U_{r_1}$, there exists a function $\tilde{g}_{a,b} \in \mathcal{E}^0(X)$ such that*

$$d * d\tilde{g}_{a,b} = \begin{cases} d * d(\chi^{\mathrm{c}} f_{a,b}) & \text{on } U \\ 0 & \text{on } X \setminus \overline{U_1}. \end{cases}$$

It is unique up to an additive constant and fulfills

$$\|d\tilde{g}_{a,b}\|_A \le \|d(\chi^{\mathrm{c}} f_{a,b})\|_A.$$

Proof. First we note that the expression on the right-hand side of the equality defines a smooth 2-form on X, because $d * d(\chi^{\mathrm{c}} f_{a,b})$ vanishes for $|z| > 1$; this follows from the fact that there χ^{c} is constant and $f_{a,b}$ is harmonic. Since moreover $\chi^{\mathrm{c}} f_{a,b} = 0$ on U_{r_2}, we see that the support of this 2-form is contained in the closed annulus $\bar{A}$. By Stokes' theorem,

$$\int_{\bar{A}} d * d(\chi^{\mathrm{c}} f_{a,b}) = \int_{\partial \bar{A}} * d(\chi^{\mathrm{c}} f_{a,b}).$$

Notice that $f_{a,b}$ is invariant under the substitution $z \mapsto 1/\bar{z}$; this implies that $\partial_\rho f_{a,b}(z) = 0$ for $|z| = 1$. Furthermore, $\chi^{\mathrm{c}}(z) = 1$ and $d\chi^{\mathrm{c}}(z) = 0$ for $|z| = 1$, so we see that

$$d(\chi^{\mathrm{c}} f_{a,b})(z) = \chi^{\mathrm{c}}(z) df_{a,b}(z) = (\partial_\phi f_{a,b} \, d\phi)(z) \quad \text{if } |z| = 1.$$

Likewise, since $\chi^{\mathrm{c}} = 0$ and $d\chi^{\mathrm{c}}(z) = 0$ for $|z| = r_2$,

$$d(\chi^{\mathrm{c}} f_{a,b})(z) = \chi^{\mathrm{c}}(z) df_{a,b}(z) = 0 \quad \text{if } |z| = r_2.$$

This means that for z on the boundary of $\bar{A}$,

$$* d(\chi^{\mathrm{c}} f_{a,b})(z) = \begin{cases} -(\partial_\phi f_{a,b} \, d\rho)(z) & \text{if } |z| = 1 \\ 0 & \text{if } |z| = r_2. \end{cases}$$

[†] I wish to thank R. van der Hout for a helpful suggestion for the proof.

In particular, $*d(\chi^{\mathrm{c}} f_{a,b})$ vanishes when restricted to the submanifold $\partial\bar{A}$ of X. From this we conclude that

$$\int_{\bar{A}} d * d(\chi^{\mathrm{c}} f_{a,b}) = \int_{\partial\bar{A}} *d(\chi^{\mathrm{c}} f_{a,b}) = 0.$$

Applying Theorem 2.2 now shows that a function $\tilde{g}_{a,b}$ with the required property exists.

To prove the inequality $\|d\tilde{g}_{a,b}\|_A \leq \|d(\chi^{\mathrm{c}} f_{a,b})\|_A$, we note that

$$\begin{aligned}\|d(\chi^{\mathrm{c}} f_{a,b})\|_A^2 &= \|d\tilde{g}_{a,b} + d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\|_A^2 \\ &= \|d\tilde{g}_{a,b}\|_A^2 + 2\langle d\tilde{g}_{a,b}, d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\rangle_A + \|d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\|_A^2.\end{aligned}$$

The last term is clearly non-negative. Furthermore, integration by parts using Stokes' theorem gives

$$\begin{aligned}\langle d\tilde{g}_{a,b}, d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\rangle_A &= \int_A d\tilde{g}_{a,b} \wedge *d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b}) \\ &= \int_{\partial\bar{A}} \tilde{g}_{a,b} *d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b}) - \int_A \tilde{g}_{a,b}\, d * d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b}).\end{aligned}$$

The second term vanishes because $d * d\tilde{g}_{a,b} = d * d(\chi^{\mathrm{c}} f_{a,b})$ on A. From our earlier expression for $*d(\chi^{\mathrm{c}} f_{a,b})(z)$ on the boundary of A, we see that

$$\int_{\partial\bar{A}} \tilde{g}_{a,b} *d(\chi^{\mathrm{c}} f_{a,b}) = 0.$$

Finally, because $\partial\bar{A}$ is also the (negatively oriented) boundary of $X \setminus A$ and because $d * d\tilde{g}_{a,b} = 0$ on $X \setminus A$,

$$-\int_{\partial\bar{A}} \tilde{g}_{a,b} *d\tilde{g}_{a,b} = \int_{X\setminus A} d\tilde{g}_{a,b} \wedge *d\tilde{g}_{a,b} \geq 0.$$

Thus we have

$$\langle d\tilde{g}_{a,b}, d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\rangle_A \geq 0,$$

which proves the inequality. □

Lemma 3.4. *Write* $\lambda = \max_{r_2 \leq r \leq 1} |\tilde{\chi}'(r)|$. *Then*

$$\max_X \tilde{g}_{a,b} - \min_X \tilde{g}_{a,b} \leq c_3(r_1, r_2, \lambda),$$

where

$$\begin{aligned}c_3(r_1, r_2, \lambda) = 4\sqrt{\frac{1+r_2}{1-r_2}} &\left(\lambda \log \frac{1}{r_2 - r_1} + \lambda \log \frac{1}{1 - r_1} + \frac{1}{r_2 - r_1} + \frac{r_1}{1 - r_1}\right) \\ &+ \frac{4}{\pi}\left(\log \frac{1}{r_2 - r_1} + \log \frac{1}{1 - r_1}\right).\end{aligned}$$

Proof. First of all, we note that

$$\begin{aligned}\max_X \tilde{g}_{a,b} &= \max\left\{\sup_{U_{r_3}} \tilde{g}_{a,b},\ \sup_{X\setminus U_{r_3}} \tilde{g}_{a,b}\right\}, \\ \min_X \tilde{g}_{a,b} &= \min\left\{\inf_{U_{r_3}} \tilde{g}_{a,b},\ \inf_{X\setminus U_{r_3}} \tilde{g}_{a,b}\right\}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sup_{U_{r_3}} \tilde{g}_{a,b} &\leq \sup_{U_{r_3}}(\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) + \sup_{U_{r_3}} \chi^{\mathrm{c}} f_{a,b} \\ &= \max_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) + \max_{r_2 \leq |z| \leq r_3} \chi^{\mathrm{c}} f_{a,b}\end{aligned}$$

because of the maximum principle ($\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}$ is harmonic on U) and because $\chi^{\mathrm{c}}(z) = 0$ for $|z| < r_2$. In the same way, we find

$$\inf_{U_{r_3}} \tilde{g}_{a,b} \geq \min_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) + \min_{r_2 \leq |z| \leq r_3} \chi^{\mathrm{c}} f_{a,b}.$$

We extend $\chi f_{a,b}$ to a smooth function on $X \setminus \{a,b\}$ by putting $(\chi f_{a,b})(x) = 0$ for $x \notin U$. Then $\tilde{g}_{a,b} + \chi f_{a,b}$ is harmonic on $X \setminus \{a,b\}$, and the same method as above gives us

$$\begin{aligned} \sup_{X \setminus U_{r_3}} \tilde{g}_{a,b} &\leq \max_{|z|=r_3} (\tilde{g}_{a,b} + \chi f_{a,b}) - \min_{r_3 \leq |z| \leq 1} \chi f_{a,b} \\ &\leq \max_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) + \max_{|z|=r_3} f_{a,b} - \min_{r_3 \leq |z| \leq 1} \chi f_{a,b} \end{aligned}$$

and

$$\inf_{X \setminus U_{r_3}} \tilde{g}_{a,b} \geq \min_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) + \min_{|z|=r_3} f_{a,b} - \max_{r_3 \leq |z| \leq 1} \chi f_{a,b}.$$

These estimates imply that

$$\begin{aligned} \max_X \tilde{g}_{a,b} &\leq \max_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) + 2 \sup_A |f_{a,b}|, \\ \min_X \tilde{g}_{a,b} &\geq \min_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) - 2 \sup_A |f_{a,b}|, \end{aligned}$$

and hence

$$\max_X \tilde{g}_{a,b} - \min_X \tilde{g}_{a,b} \leq \max_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) - \min_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) + 4 \sup_A |f_{a,b}|.$$

By Lemma 3.2 and Lemma 3.3,

$$\begin{aligned} \max_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) - \min_{|z|=r_3} (\tilde{g}_{a,b} - \chi^{\mathrm{c}} f_{a,b}) &\leq \frac{2\sqrt{\pi}}{1 - r_2} \| d\tilde{g}_{a,b} - d(\chi^{\mathrm{c}} f_{a,b}) \|_A \\ &\leq \frac{2\sqrt{\pi}}{1 - r_2} (\| d\tilde{g}_{a,b} \|_A + \| d(\chi^{\mathrm{c}} f_{a,b}) \|_A) \\ &\leq \frac{4\sqrt{\pi}}{1 - r_2} \| d(\chi^{\mathrm{c}} f_{a,b}) \|_A. \end{aligned}$$

We have

$$\begin{aligned} \| d(\chi^{\mathrm{c}} f_{a,b}) \|_A &\leq \| d(\chi^{\mathrm{c}}) f_{a,b} \|_A + \| \chi^{\mathrm{c}} df_{a,b} \|_A \\ &\leq \| \tilde{\chi}'(\rho) f_{a,b} \, d\rho \|_A + \| df_{a,b} \|_A \\ &\leq \lambda \| d\rho \|_A \sup_A |f_{a,b}| + \| df_{a,b} \|_A. \end{aligned}$$

Now

$$\begin{aligned} \| d\rho \|_A^2 &= \int_A d\rho \wedge *d\rho \\ &= \int_A \rho \, d\rho \wedge d\phi \\ &= \pi(1 - r_2^2). \end{aligned}$$

Furthermore, for all $a, b \in U_{r_1}$ we have

$$|f_{a,b}(z)| = \frac{1}{2\pi} \left| \log|z - z(a)| + \log|\overline{z(a)} z - 1| - \log|z - z(b)| - \log|\overline{z(b)} z - 1| \right|.$$

The triangle inequality gives

$$r_2 - r_1 < |z - z(a)| < 1 + r_1 \quad \text{and} \quad 1 - r_1 < |\overline{z(a)} z - 1| < 1 + r_1$$

for all $a \in U_{r_1}$ and all $z \in A$. From this we see (keeping in mind that $1/2 < r_1 < r_2 < 1$) that

$$\left| \log|z - z(a)| \right| < \log \frac{1}{r_2 - r_1} \quad \text{and} \quad \left| \log|\overline{z(a)} z - 1| \right| < \log \frac{1}{1 - r_1}.$$

We conclude that for all $a, b \in U_{r_1}$,

$$\sup_A |f_{a,b}| \leq \frac{1}{\pi}\left(\log\frac{1}{r_2 - r_1} + \log\frac{1}{1 - r_1}\right).$$

Finally we estimate the quantity $\|df_{a,b}\|_A$. Because $f_{a,b}$ is a real function, we have

$$df_{a,b} = \partial_z f_{a,b}\, dz + \overline{\partial_z f_{a,b}}\, d\bar{z}.$$

Therefore,

$$\begin{aligned}
\|df_{a,b}\|_A^2 &= \int_A df_{a,b} \wedge * df_{a,b} \\
&= 2i \int_A |\partial_z f_{a,b}|^2\, dz \wedge d\bar{z} \\
&= 4 \int_0^{2\pi}\int_{r_2}^1 |\partial_z f_{a,b}|^2\, \rho\, d\rho\, d\phi \\
&\leq 4\pi(1 - r_2^2) \sup_A |\partial_z f_{a,b}|^2.
\end{aligned}$$

A straightforward computation gives

$$\partial_z f_{a,b} = \frac{1}{4\pi}\left(\frac{1}{z - z(a)} + \frac{\overline{z(a)}}{\overline{z(a)}z - 1} - \frac{1}{z - z(b)} - \frac{\overline{z(b)}}{\overline{z(b)}z - 1}\right).$$

Using our previous estimates for $|z - z(a)|$ and $|\overline{z(a)}z - 1|$, we see (again keeping in mind that $1/2 < r_1 < r_2 < 1$) that

$$\sup_A |\partial_z f_{a,b}| \leq \frac{1}{2\pi}\left(\frac{1}{r_2 - r_1} + \frac{r_1}{1 - r_1}\right).$$

From this we obtain

$$\|df_{a,b}\|_A \leq \sqrt{\frac{1 - r_2^2}{\pi}}\left(\frac{1}{r_2 - r_1} + \frac{r_1}{1 - r_1}\right).$$

Combining the estimates for $\sup_A |f_{a,b}|$ and $\|df_{a,b}\|_A$ yields the lemma. □

From now on we impose the normalisation condition

$$\int_X \tilde{g}_{a,b}\mu = 0$$

on $\tilde{g}_{a,b}$ for all $a, b \in U_{r_1}$; this can be attained by adding a suitable constant to $\tilde{g}_{a,b}$. Then the function $g_{a,b}$ defined earlier is equal to

$$g_{a,b} = \tilde{g}_{a,b} + \chi f_{a,b} - \int_X \chi f_{a,b}\mu$$

for all $a, b \in U_{r_1}$; note that the logarithmic singularities of $f_{a,b}$ are integrable. Indeed, it follows from the definition of $\tilde{g}_{a,b}$ that the equations

$$d * d[g_{a,b}] = \delta_a - \delta_b \quad \text{and} \quad \int_X g_{a,b}\mu = 0$$

that define $g_{a,b}$ uniquely also hold when $g_{a,b}$ is replaced by $\tilde{g}_{a,b} + \chi f_{a,b} - \int_X \chi f_{a,b}\mu$. Furthermore, we define

$$l_a = \begin{cases} \frac{\chi}{2\pi}\log|z - z(a)| & \text{on } U \\ 0 & \text{on } X \setminus \overline{U_1}; \end{cases}$$

note that the right-hand side defines a smooth function on $X \setminus \{a\}$ which is bounded from above by

$$\sup_{z \in U_1} \frac{\chi(z)}{2\pi}\log|z - z(a)| \leq \frac{1}{2\pi}\log(1 + r_1).$$

Lemma 3.5. *For all $a, b \in U_{r_1}$,*

$$\max_X |g_{a,b} - l_a + l_b| < c_3(r_1, r_2, \lambda) + \frac{1}{2\pi} \log \frac{1+r_1}{1-r_1} + \left(\frac{8}{3}\log 2 - \frac{1}{4}\right) c_1.$$

Proof. By the above equation for $g_{a,b}$ and the definitions of $f_{a,b}$ and l_a, we get

$$g_{a,b} - l_a + l_b = \tilde{g}_{a,b} - \int_X \chi f_{a,b}\mu + \frac{\chi}{2\pi} \log \left| \frac{\overline{z(a)}z - 1}{\overline{z(b)}z - 1} \right|,$$

where the last term is extended to zero outside U. We estimate each of the terms on the right-hand side. From $\int_X \tilde{g}_{a,b}\mu = 0$ and hypothesis (2) ($\mu = iF\,dz \wedge d\bar{z}$ with $0 \le F(z) \le c_1$) it follows that

$$\max_X \tilde{g}_{a,b} \ge 0 \ge \min_X \tilde{g}_{a,b};$$

together with the estimate for $\max_X \tilde{g}_{a,b} - \min_X \tilde{g}_{a,b}$ from Lemma 3.4, this implies

$$\max_X |\tilde{g}_{a,b}| \le c_3(r_1, r_2, \lambda).$$

Because the support of χ is contained in U_1, hypothesis (2) together with the definition of $f_{a,b}$ gives

$$\int_X \chi f_{a,b}\mu = \int_{U_1} \frac{\chi}{2\pi} \Big(\log|z - z(a)| + \log|\overline{z(a)}z - 1| - \log|z - z(b)| - \log|\overline{z(b)}z - 1| \Big)\, \mu.$$

Now

$$\int_{U_1} \frac{\chi}{2\pi} \log|z - z(a)|\, \mu \le \frac{c_1}{2\pi} \int_{\substack{|w|<1 \\ |w - z(a)|>1}} \log|w - z(a)|\, i\, dw \wedge d\bar{w}.$$

In order to bound this expression independently of r_1, we look at all a with $|z(a)| \le 1$. It is easy to see that the maximum of the expression is attained when $|z(a)| = 1$; by rotational symmetry we can take $a = 1$. In this case we have to integrate over the crescent-shaped domain

$$\big\{ w \in \mathbf{C} \bigm| |w| < 1 \text{ and } |w - 1| > 1 \big\},$$

which is contained in

$$\big\{ 1 + r\exp(i\phi) \bigm| 1 < r < 2, 2\pi/3 < \phi < 4\pi/3 \big\}.$$

Therefore, we get

$$\begin{aligned} \int_{U_1} \frac{\chi}{2\pi} \log|z - z(a)|\, \mu &< \frac{c_1}{\pi} \int_{\pi/3}^{2\pi/3} \int_1^2 \log(r)\, r\, dr\, d\phi \\ &= \left(\frac{4}{3}\log 2 - \frac{1}{2} \right) c_1. \end{aligned}$$

In a similar way, we find

$$\begin{aligned} \int_{U_1} \frac{\chi}{2\pi} \log|z - z(a)|\, \mu &\ge -c_1/2, \\ \int_{U_1} \frac{\chi}{2\pi} \log|\overline{z(a)}z - 1|\, \mu &< \left(\frac{4}{3}\log 2 - \frac{1}{2} \right) c_1, \\ \int_{U_1} \frac{\chi}{2\pi} \log|\overline{z(a)}z - 1|\, \mu &\ge -c_1/4. \end{aligned}$$

The same estimates hold for b; combining them, we get

$$\left| \int_X \chi f_{a,b}\mu \right| \le \left(\frac{8}{3}\log 2 - \frac{1}{4} \right) c_1.$$

Finally, we have

$$\begin{aligned} \max_X \frac{\chi}{2\pi} \log \left| \frac{\overline{z(a)}z - 1}{\overline{z(b)}z - 1} \right| &\le \frac{1}{2\pi} \sup_{U_1} \log \left| \frac{\overline{z(a)}z - 1}{\overline{z(b)}z - 1} \right| \\ &\le \frac{1}{2\pi} \log \frac{1 + r_1}{1 - r_1}, \end{aligned}$$

which finishes the proof. □

We are now going to apply Lemma 3.5 (which holds for any chart (U, z) satisfying the hypotheses (1) and (2)) to our atlas $\{(U^j, z^{(j)}) \mid 1 \le j \le n\}$. Besides including the index j in the notation for the coordinates, we denote by $l_a^{(j)}$ and $\chi^{(j)}$ the functions l_a and χ defined for the coordinate $(U^j, z^{(j)})$.

The following lemma is a generalisation of Lemma 3.5 to the situation where a and b are arbitrary points of X.

Lemma 3.6. *For all $a, b \in X$ and all j, k such that $a \in U^j_{r_1}$ and $b \in U^k_{r_1}$,*

$$\sup_X \bigl|g_{a,b} - l_a^{(j)} + l_b^{(k)}\bigr| \le c_5(r_1, r_2, \lambda, n, c_1, M),$$

where

$$c_5(r_1, r_2, \lambda, n, c_1, M) = n\left(c_3(r_1, r_2, \lambda) + \frac{1}{2\pi}\log\frac{1+r_1}{1-r_1} + \left(\frac{8}{3}\log 2 - \frac{1}{4}\right)c_1\right) \\ + \frac{n-1}{2\pi}\left(\log M + \log\frac{1}{r_2 - r_1} + \log(1+r_1)\right).$$

Proof. We first show that for any two coordinate indices j and k and for all $a \in U^k_{r_1} \cap U^j_{r_1}$,

$$\sup_X \bigl|l_a^{(k)} - l_a^{(j)}\bigr| \le \frac{1}{2\pi}\left(\log M + \log\frac{1}{r_2 - r_1} + \log(1+r_1)\right). \qquad (*)$$

To prove this, let $y \in X$. We distinguish three cases to prove that $l_a^{(k)}(y) - l_a^{(j)}(y)$ is bounded from above by the right-hand side of $(*)$; the inequality then follows by interchanging j and k.

Case 1: Suppose $y \in U^j_1$ with $\bigl|z^{(j)}(y) - z^{(j)}(a)\bigr| < (r_2 - r_1)/M$. In this case we have

$$\bigl|z^{(j)}(y)\bigr| < \bigl|z^{(j)}(a)\bigr| + \frac{r_2 - r_1}{M} < r_2,$$

hence $a, y \in U^j_{r_2}$. Let $[a, y]^j$ denote the line segment between a and y in the $z^{(j)}$-coordinate, i.e. the curve in $U^j_{r_2}$ whose $z^{(j)}$-coordinate is parametrised by

$$\hat{z}^{(j)}(t) = (1-t)z^{(j)}(a) + tz^{(j)}(y) \quad (0 \le t \le 1).$$

We claim that this line segment also lies inside $U^k_{r_2}$. Suppose this is not the case; then, because the 'starting point' $(z^{(j)})^{-1}\bigl(\hat{z}^{(j)}(0)\bigr) = a$ does lie in $U^k_{r_2}$, there exists a smallest $t \in (0, 1)$ for which the point

$$y' = (z^{(j)})^{-1}\bigl(\hat{z}^{(j)}(t)\bigr) \in U^j_{r_2}$$

lies on the boundary of $U^k_{r_2}$. It follows from the hypothesis (4) imposed on the coordinates that

$$\bigl|z^{(k)}(y') - z^{(k)}(a)\bigr| \le M\bigl|z^{(j)}(y') - z^{(j)}(a)\bigr|.$$

On the other hand,

$$\begin{aligned}\bigl|z^{(j)}(y') - z^{(j)}(a)\bigr| &= t\bigl|z^{(j)}(y) - z^j(a)\bigr| \\ &< (r_2 - r_1)/M,\end{aligned}$$

by assumption, and

$$\bigl|z^{(k)}(y') - z^{(k)}(a)\bigr| > r_2 - r_1$$

by the triangle inequality. This implies

$$\bigl|z^{(k)}(y') - z^{(k)}(a)\bigr| > M\bigl|z^{(j)}(y') - z^{(j)}(a)\bigr|,$$

a contradiction. Therefore, the line segment $[a, y]^j$ lies inside $U^j_{r_2} \cap U^k_{r_2}$. By hypothesis (4),

$$\bigl|z^{(k)}(y) - z^{(k)}(a)\bigr| \le M\bigl|z^{(j)}(y) - z^{(j)}(a)\bigr|.$$

Because $\chi^{(j)}(y) = \chi^{(k)}(y) = 1$, we find

$$\begin{aligned} l_a^{(k)}(y) - l_a^{(j)}(y) &= \frac{1}{2\pi} \log \left| \frac{z^{(k)}(y) - z^{(k)}(a)}{z^{(j)}(y) - z^{(j)}(a)} \right| \\ &\le \frac{1}{2\pi} \log M, \end{aligned}$$

which is bounded by the right-hand side of $(*)$.

Case 2: Suppose $y \notin U_1^j$. Then $l_a^{(j)}(y) = 0$, and thus

$$l_a^{(k)}(y) - l_a^{(j)}(y) = l_a^{(k)}(y) \le \frac{\log(1 + r_1)}{2\pi}.$$

Case 3: Suppose $y \in U_1^j$ and $\left|z^{(j)}(y) - z^{(j)}(a)\right| \ge (r_2 - r_1)/M$. Then

$$l_a^{(k)}(y) - l_a^{(j)}(y) \le \frac{\log(1 + r_1)}{2\pi} - \frac{\chi^{(j)}(y)}{2\pi} \log \frac{r_2 - r_1}{M},$$

which is also bounded by the right-hand side in $(*)$.

According to the hypothesis (3) imposed on our atlas, the open sets $U_{r_1}^j$ cover X. Furthermore, X is connected. For arbitrary $a, b \in X$ and indices j and k such that $a \in U_{r_1}^j$ and $b \in U_{r_1}^k$, we can therefore choose a finite sequence of indices $j = j_1, j_2, \ldots, j_m = k$ with $m \le n$ and points $a = a_0$, $a_1, \ldots, a_m = b$ such that $a_i \in U_{r_1}^{j_i} \cap U_{r_1}^{j_{i+1}}$ for $1 \le i \le m - 1$. Using

$$g_{a,b} = \sum_{i=1}^{m} g_{a_{i-1},a_i}$$

we get

$$\begin{aligned} \sup_X \left|g_{a,b} - l_a^{(j)} + l_b^{(k)}\right| &= \sup_X \left| \sum_{i=1}^{m} \left(g_{a_{i-1},a_i} - l_{a_{i-1}}^{(j_i)} + l_{a_i}^{(j_i)} \right) + \sum_{i=1}^{m-1} \left(l_{a_i}^{(j_{i+1})} - l_{a_i}^{(j_i)} \right) \right| \\ &\le \sum_{i=1}^{m} \sup_X \left| g_{a_{i-1},a_i} - l_{a_{i-1}}^{(j_i)} + l_{a_i}^{(j_i)} \right| + \sum_{i=1}^{m-1} \sup_X \left| l_{a_i}^{(j_{i+1})} - l_{a_i}^{(j_i)} \right|. \end{aligned}$$

The lemma now follows from Lemma 3.5 and the inequality $(*)$. □

Using the preceding lemma, we can now prove Theorem 3.1. We choose a continuous partition of unity $\{\phi^j\}_{j=1}^n$ subordinate to the covering $\{U_{r_1}^j\}_{j=1}^n$. Let $a \in X$ and let j be an index such that $a \in U_{r_1}^j$. By the definition of $g_{a,\mu}$ we have

$$\begin{aligned} g_{a,\mu}(x) - l_a^{(j)}(x) &= \int_{b \in X} g_{a,b}(x)\mu(b) - l_a^{(j)}(x) \\ &= \sum_{k=1}^{n} \int_{b \in U_{r_1}^k} \phi^k(b)\big(g_{a,b}(x) - l_a^{(j)}(x)\big)\mu(b) \\ &= \sum_{k=1}^{n} \int_{b \in U_{r_1}^k} \phi^k(b)\big(g_{a,b}(x) - l_a^{(j)}(x) + l_b^{(k)}(x)\big)\mu(b) - \sum_{k=1}^{n} \int_{b \in U_{r_1}^k} \phi^k(b) l_b^{(k)}(x)\mu(b). \end{aligned}$$

For all j we have, by the same method as in the proof of Lemma 3.5,

$$\sup_{x \in X} \left| \int_{b \in U_{r_1}^k} \phi^k(b) l_b^{(k)}(x)\mu(b) \right| \le c_1/2.$$

Together with Lemma 3.6, this gives the inequality

$$\begin{aligned}
\sup_X \bigl| g_{a,\mu} - l_a^{(j)} \bigr| &\le c_5(r_1, r_2, \lambda, n, c_1, M) \sum_{j=1}^{n} \int_{b \in U_{r_1}^j} \phi^j(b)\mu(b) + \sum_{j=1}^{n} c_1/2 \\
&= c_5(r_1, r_2, \lambda, n, c_1, M) + nc_1/2 \\
&= n \left(c_3(r_1, r_2, \lambda) + \frac{1}{2\pi} \log \frac{1}{1-r_1} + \left(\frac{8}{3} \log 2 + 1/4 \right) c_1 \right) \\
&\quad + \frac{n-1}{2\pi} \left(\log M + \log \frac{1}{r_2 - r_1} + \log(1 + r_1) \right).
\end{aligned}$$

We also have

$$\begin{aligned}
\sup_X g_{a,\mu} &\le \sup_X \bigl(g_{a,\mu} - l_a^{(j)} \bigr) + \sup_X l_a^{(j)} \\
&\le \sup_X \bigl(g_{a,\mu} - l_a^{(j)} \bigr) + \frac{\log(1+r_1)}{2\pi}.
\end{aligned}$$

By varying the choice of $\tilde{\chi}$, we can choose λ as close to $1/(1-r_2)$ as we want. Substituting $\lambda = 1/(1-r_2)$ shows that

$$\begin{aligned}
c_3(r_1, r_2, 1/(1-r_2)) = \frac{4\sqrt{1+r_2}}{(1-r_2)^{3/2}} &\left(\log \frac{1}{r_2 - r_1} + \log \frac{1}{1-r_1} + \frac{1-r_2}{r_2-r_1} + r_1 \frac{1-r_2}{1-r_1} \right) \\
&+ \frac{4}{\pi} \left(\log \frac{1}{r_2 - r_1} + \log \frac{1}{1-r_1} \right).
\end{aligned}$$

We choose $r_2 = (1 + r_1)/2$; then we see that

$$c_3(r_1, r_2, 1/(1-r_2)) \le \frac{B}{(1-r_1)^{3/2}} \log \frac{1}{1-r_1}$$

for some constant $B > 0$. Finally,

$$\sup_X g_{a,\mu} \le \frac{Cn}{(1-r_1)^{3/2}} \log \frac{1}{1-r_1} + \left(\frac{8}{3} \log 2 + 1/4 \right) nc_1 + \frac{n-1}{2\pi} \log M$$

and, since $l_a^{(j)} = \log |z^{(j)} - z^{(j)}(a)|$ on $U_{r_1}^j$,

$$\begin{aligned}
\sup_{U_{r_1}^j} \bigl| g_{a,\mu} - \log |z^{(j)} - z^{(j)}(a)| \bigr| &\le \sup_X |g_{a,\mu} - l_a^{(j)}| \\
&\le \frac{Cn}{(1-r_1)^{3/2}} \log \frac{1}{1-r_1} + \left(\frac{8}{3} \log 2 + 1/4 \right) nc_1 + \frac{n-1}{2\pi} \log M
\end{aligned}$$

for some constant $C > 0$, which ends the proof of Theorem 3.1. $\square$

4. Valuations and height functions

Let K be a number field. A *valuation* on K is a function

$$\begin{aligned} K &\longrightarrow \mathbf{R}_{\geq 0} \\ x &\longmapsto |x| \end{aligned}$$

which for some positive constant c and all $x, y \in K$ satisfies

$$\begin{aligned} |x| = 0 &\iff x = 0, \\ |xy| &= |x||y|, \\ |x+y| &\leq c \max\{|x|, |y|\}. \end{aligned}$$

The properties of valuations on number fields are well known; for them we refer to texts on algebraic number theory, such as Neukirch [16]. We only state our conventions and the results that we will use.

For any valuation $|\ \ |$ on K we have $|0| = 0$ and $|1| = 1$. A valuation $|\ \ |$ is called *trivial* if $|x| = 1$ for all $x \in K^\times$, and *non-trivial* if it takes values other than 0 and 1. Any valuation $|\ \ |$ defines a Hausdorff topology on K, a basis of which consists of the sets $\bigl\{y \in K \bigm| |y - x| < r\bigr\}$ for $x \in X$ and $r > 0$.

By a *place* of K we mean an equivalence class of non-trivial valuations on K, two valuations being equivalent if they define the same topology on K. We will denote the completion of K with respect to a place v by K_v. For each place v of K the valuation $|\ \ |_v$ on K extends uniquely to a continuous valuation on K_v, which we also denote by $|\ \ |_v$. We will refer to the places corresponding to ultrametric and Archimedean valuations as *finite* and *infinite* places, respectively.

If v is a finite place of K, we will denote the valuation ring by $O_{K,v}$ and the residue class field (which is finite) by k_v. Each valuation $|\ \ |$ in the class corresponding to a finite place v is discrete, and for exactly one of them the function $-\log|\ \ |$ gives rise to a surjective group homomorphism

$$\operatorname{ord}_v\colon K^\times \to \mathbf{Z}.$$

The intersection of the $O_{K,v}$ is the ring of integers of K, denoted by O_K. If on the other hand v is an infinite place, K_v is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, and therefore equipped with a standard absolute value $|\ \ |$. For every place v of K, we fix a distinguished valuation $|\ \ |_v$ in the class v by

$$|x|_v = \begin{cases} (\#k_v)^{-\operatorname{ord}_v(x)} & \text{if } v \text{ is discrete;} \\ |x| & \text{if } K_v \cong \mathbf{R}; \\ |x|^2 & \text{if } K_v \cong \mathbf{C}. \end{cases}$$

Remark. For any place v of K, the additive group of K_v is commutative and locally compact as a topological group, so there exists a Haar measure μ_v on K_v; it is unique up to a constant scalar factor. Our choice of the valuations $|\ \ |_v$ is equivalent to requiring that multiplication by x changes volumes by a factor $|x|_v$, i.e.

$$\mu_v(xV) = |x|_v \mu_v(V)$$

for every measurable subset $V \subset K_v$ such that $\mu_v(V)$ is finite.

Proposition 4.1. *If $K \to K'$ is an extension of number fields, then for every place v of K we have the identity*

$$\prod_{v'|v} |x|_{v'} = |x|_v^{[K':K]} \quad \textit{for all } x \in K^\times,$$

where v' runs over the (finitely many) places of K' extending v. Furthermore, for any $x \in K^\times$ there are only finitely many places v of K such that $|x|_v \neq 1$, and we have the product formula

$$\prod_v |x|_v = 1 \quad \textit{for all } x \in K^\times,$$

where v runs over all places of K.

Proof. Neukirch [16], § III.1.

If a and b are two coprime integers (in particular, not both zero), the *height* of the point $(a:b) \in \mathbf{P}^1(\mathbf{Q})$ (or of the rational number a/b, if $b \neq 0$) is defined by

$$h_{\mathbf{Q}}((a:b)) = \log \max\{|a|, |b|\}.$$

This definition can be generalised to arbitrary number fields, but first we have to get rid of the special role that the Archimedean valuation on $\mathbf{Q}$ plays in this definition. Note that we have in fact

$$h_{\mathbf{Q}}((a:b)) = \sum_v \log \max\{|a|_v, |b|_v\},$$

where v runs over all places of $\mathbf{Q}$ (i.e., the finite and infinite places defined earlier). This last identity holds because $\max\{|a|_v, |b|_v\} = 1$ for all finite places v if a and b are coprime. If a and b are multiplied by an element of $\mathbf{Q}^\times$, the sum $\sum_v \log \max\{|a|_v, |b|_v\}$ does not change; this follows by a simple calculation using the product formula. Therefore, the second formula for $h_{\mathbf{Q}}((a:b))$ makes sense without the restriction that a and b are coprime integers.

We generalise the above idea to a height function on $\mathbf{P}^n(K)$, for any number field K, by putting

$$h_K((x_0 : \cdots : x_n)) = \sum_v \log \max\{|x_0|_v, \ldots, |x_n|_v\}$$

with v running over the places of K. Again, this is a well-defined quantity by the product formula. If $K \hookrightarrow K'$ is an extension of number fields, Proposition 4.1 implies that

$$h_{K'}(x) = [K':K] h_K(x) \quad \text{for all } x \in \mathbf{P}^n(K).$$

We can now define an *absolute height function* h on $\mathbf{P}^n(\bar{\mathbf{Q}})$ by setting

$$h(x) = \frac{1}{[K:\mathbf{Q}]} h_K(x) \quad \text{if } x \in \mathbf{P}^n(K);$$

the definition does not depend on the choice of the number field K containing x. We note that h is invariant with respect to the action of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on $\mathbf{P}^n(\bar{\mathbf{Q}})$.

Let K be a number field, and let X be a quasi-projective variety over K. For each embedding

$$i \colon X \hookrightarrow \mathbf{P}^n_K$$

and each point $P \in X(\bar{K})$, we define the height of P with respect to the embedding i as

$$h_i(P) = h(i(P)).$$

This definition obviously depends on the choice of the embedding. A more intrinsic description of heights can be given in terms of line bundles on arithmetic varieties; we will see this in more detail in § 9.

5. Analytic part of Arakelov theory

In the intersection theory on arithmetic surfaces that we are going to describe in § 8, the contributions to the intersection number coming from the infinite places are defined with the help of the so-called *Arakelov–Green function* for the Riemann surfaces associated to the arithmetic surface (see the introduction). Furthermore, the metrised line bundles on arithmetic surfaces that we will consider have to satisfy an analytic criterion called *admissibility*. In this section, we will define these concepts.

For any compact connected Riemann surface of positive genus, we can define a metric on $\mathcal{O}_{X\times X}(-\Delta)$, the sheaf of holomorphic functions vanishing on the diagonal in $X \times X$, by means of the Arakelov–Green function. Furthermore, denoting the sheaf of holomorphic differentials on X by Ω^1_X, we have a natural isomorphism $\Delta^*\mathcal{O}_{X\times X}(-\Delta) \xrightarrow{\sim} \Omega^1_X$ of sheaves of $\mathcal{O}_X$-modules, called the *adjunction isomorphism*. Via this isomorphism, we can therefore provide Ω^1_X with a natural metric; in the second part of this section we give the details of this construction.

Definition. Let X be a compact connected Riemann surface of genus $g \geq 1$. The g-dimensional complex vector space $\Omega^1(X)$ of holomorphic 1-forms on X is equipped with an inner product given by

$$\langle \alpha, \beta \rangle = \frac{i}{2}\int_X \alpha \wedge \bar{\beta}.$$

We choose an orthonormal basis $(\omega_1, \omega_2, \ldots, \omega_g)$ of $\Omega^1(X)$, and we define a volume form $\mu_X \in \mathcal{E}^2(X)$, called the *canonical* $(1,1)$*-form* or *Arakelov* $(1,1)$*-form*, by

$$\mu_X = \frac{i}{2g}\sum_{j=1}^{g} \omega_j \wedge \bar{\omega}_j.$$

It is easy to check that this definition does not depend on the choice of orthonormal basis and that $\int_X \mu_X = 1$.

Definition. The *Arakelov–Green function* of a compact connected Riemann surface X of positive genus is the real symmetric C^∞ function

$$g_{\mathrm{Ar}} = 2\pi g_{\mu_X}$$

outside the diagonal on $X \times X$.

Definition. Let X be a Riemann surface, and let $\mathcal{L}$ be a holomorphic line bundle on X equipped with a C^∞ Hermitian metric $\|\ \|$. The *curvature* of $(\mathcal{L}, \|\ \|)$ is the $(1,1)$-form defined locally by

$$\mathrm{curv}_{\|\ \|} = -\frac{1}{2\pi} d * d \log \|s\|,$$

where s is a local generating section of $\mathcal{L}$. This definition does not depend on the choice of s because the logarithm of the modulus of a holomorphic function is harmonic.

Definition. Let X be a compact connected Riemann surface, and let μ be a smooth $(1,1)$-form on X satisfying $\int_X \mu = 1$. A line bundle $\mathcal{L}$ on X equipped with a Hermitian metric $\|\ \|$ is called *admissible* with respect to μ if $\|\ \|$ is smooth and

$$\mathrm{curv}_{\|\ \|} = (\deg \mathcal{L})\mu.$$

Let X and μ be as in the above definition, and let $\mathcal{L}$ be a line bundle on X. Any two metrics on $\mathcal{L}$ which are admissible with respect to μ are related by multiplication by a strictly positive C^∞ function ϕ satisfying $d * d(\log \phi) = 0$. Because the only harmonic functions on X are the constant functions, we see that any two admissible metrics on $\mathcal{L}$ differ by multiplication by a positive constant. The following proposition claims that admissible metrics do exist.

Proposition 5.1. *Let X be a compact Riemann surface equipped with a smooth $(1,1)$-form μ such that $\int_X \mu = 1$, and let $\mathcal{L}$ be a line bundle on X. Then there exists a metric on $\mathcal{L}$ which is admissible with respect to μ.*

Proof. Because $\operatorname{Pic} X \cong \operatorname{Cl} X$, it is enough to prove the claim for line bundles of the form $\mathcal{O}_X(D)$ with $D = \sum_{P\in X} n_P P$ a divisor on X. Viewing 1 as a meromorphic section of $\mathcal{O}_X(D)$, we define $\|\ \|$ by

$$\log \|1\|(x) = 2\pi \sum_{P\in X} n_P g_\mu(P, x)$$

for x outside the support of D, i.e. in the open subset of X where 1 is a generating section of $\mathcal{O}_X(D)$. The fact that g_μ has singularities of the form $\frac{1}{2\pi}\log|z(x) - z(P)|$ for x in a coordinate neighbourhood (U, z) of P (see § 2) implies that $\|\ \|$ extends in a unique way to a C^∞ metric on $\mathcal{O}_X(D)$. Writing $g_{P,\mu}(x) = g_\mu(P, x)$ and applying Lemma 2.3, we see that

$$\begin{aligned}\operatorname{curv}_{\|\ \|} &= -d * d \sum_{P\in X} n_P g_{P,\mu} \\ &= (\deg D)\mu \\ &= (\deg \mathcal{O}_X(D))\mu\end{aligned}$$

outside the support of D, and by continuity this equality holds on the whole of X. □

Let X be a Riemann surface, and let Ω^1_X be the sheaf of holomorphic 1-forms on X. Let $\Delta\colon X \to X \times X$ be the diagonal embedding, and let $\mathcal{O}_{X\times X}(-\Delta)$ be the sheaf of holomorphic functions vanishing on the diagonal. We are going to construct an isomorphism

$$\alpha\colon \Delta^*\mathcal{O}_{X\times X}(-\Delta) \xrightarrow{\sim} \Omega^1_X$$

of line bundles on X, called the *adjunction isomorphism*. Since morphisms of sheaves can be defined locally, it is sufficient to consider a chart (U, z) on X. Let $p_1, p_2\colon U \times U \to U$ be the projections onto the first and second coordinates, respectively, and write $z_1 = z \circ p_1$ and $z_2 = z \circ p_2$. Then $\mathcal{O}_{U\times U}(-\Delta)$ is the free $\mathcal{O}_{U\times U}$-module of rank 1 generated by $(z_1 - z_2)$, and we have a morphism

$$\begin{aligned}\mathcal{O}_{U\times U}(-\Delta) &\longrightarrow \Delta_*\Omega^1_U \\ (z_1 - z_2)f &\longmapsto f|_\Delta dz\end{aligned}$$

of $\mathcal{O}_{U\times U}$-modules. It is easy to check that it does not depend on the choice of the coordinate z. From the fact that Δ^* and Δ_* are adjoint functors, we get a well-defined morphism

$$\begin{aligned}\alpha_U\colon \Delta^*\mathcal{O}_{U\times U}(-\Delta) &\longrightarrow \Omega^1_U \\ ((z_1 - z_2)f)|_\Delta &\longmapsto f|_\Delta dz\end{aligned}$$

of line bundles on U. It is easy to check that α_U is an isomorphism. Because the α_U are defined in a canonical way, we obtain the desired isomorphism α by glueing the α_U for U running through the chart domains of some atlas on X.

Suppose that X is a compact connected Riemann surface of positive genus. We put a metric $\|\ \|$ on the sheaf $\mathcal{O}_{X\times X}(-\Delta)$ using the Arakelov–Green function: outside the diagonal, we define $\|\ \|$ by

$$\log \|1\|(x, y) = -g_{\mathrm{Ar}}(x, y)$$

and we extend by continuity. In particular, for a chart (U, z) on X, the generator $z_1 - z_2$ of $\mathcal{O}_{U\times U}(-\Delta)$ (with z_1 and z_2 as above) satisfies

$$\log \|z_1 - z_2\| = \log|z_1 - z_2| - g_{\mathrm{Ar}}$$

on $U \times U$. By the description of g_μ in § 2, the function g_{Ar} can be written as $\log|z_1 - z_2| + h$ with h a smooth function on $U \times U$, so we see that $\log\|z_1 - z_2\|$ can be extended to a smooth function on $U \times U$. Via the isomorphism α, we now get a metric on Ω^1_X which on the chart domain U is given by

$$\log \|dz\|(x) = \lim_{y\to x}\bigl(\log|z(x) - z(y)| - g_{\mathrm{Ar}}(x, y)\bigr).$$

It can be shown that the metric on Ω^1_X is admissible with respect to the canonical $(1,1)$-form μ_X (see Arakelov [2], Theorem 4.1).

Suppose $\mathbf{K}$ is a topological field which is isomorphic to $\mathbf{C}$. Let n be a positive integer. *Analytic functions* from open subsets of $\mathbf{K}^n$ to $\mathbf{K}^n$ can be defined independently of the choice of an isomorphism $\mathbf{K} \to \mathbf{C}$, namely as functions represented locally by a vector of n convergent power series in n variables. Therefore we can define an *n-dimensional analytic variety over* $\mathbf{K}$ as a second-countable Hausdorff space which can be covered by charts $z: U \to U'$, where z is a homeomorphism between open subsets $U \subseteq X$ and $U' \subseteq \mathbf{K}^n$, such that the glueing functions are holomorphic. An analytic variety of dimension 1 over $\mathbf{K}$ is called a *Riemann surface over* $\mathbf{K}$.

Let X be a compact connected Riemann surface over $\mathbf{K}$. The definitions in this section apply to X as they do to Riemann surfaces over $\mathbf{C}$, except that we have to choose an element $i \in \mathbf{K}$ such that $i^2 = -1$. If we replace i by $-i$ (and simultaneously replace μ by $-\mu$ in Theorem 2.2, to comply with the requirement $\int_X \mu = 1$), it is clear from the definitions that the following things change sign: the orientation on X, the star operator $*$, the Laplace operator $d * d$, the current $[\alpha]$ representing a differential form α, the canonical $(1,1)$-form μ_X (if X has positive genus), and the curvature of a line bundle. The Arakelov–Green function g_{Ar} and the concept of admissibility are independent of the choice of i. This fact will be useful in the following sections, when we deal with Riemann surfaces over fields of the form $\bar{K}_v$ where K is a number field, K_v is its completion with respect to an Archimedean valuation v and $\bar{K}_v$ is an algebraic closure; these fields are isomorphic to $\mathbf{C}$, but not in a canonical way.

6. Arithmetic curves and surfaces

In this section we introduce arithmetic curves and surfaces, the main objects of interest in Arakelov theory. We will assume knowledge of the basic concepts from algebraic geometry treated in the books by Hartshorne [9] and Liu [12].

For the whole of this section, we fix a number field K. Let O_K be its ring of integers, and write

$$B = \operatorname{Spec} O_K.$$

Definition. An *arithmetic variety* over B is an integral regular projective flat B-scheme. *Arithmetic curves* and *arithmetic surfaces* are arithmetic varieties of Krull dimension 1 and 2, respectively.

Remark. If X is an arithmetic variety over B, then X is Noetherian because B is Noetherian and X is of finite type over B.

Remark. In this context, flatness is equivalent to surjectivity of the underlying continuous map of topological spaces (see Liu [12], Proposition 4.3.9).

Examples of arithmetic curves over B include the curves $\operatorname{Spec} O_L$ with L a finite extension of the number field K. The following proposition implies that these are in fact the only ones.

Proposition 6.1. *Let $f: C \to B$ be a proper flat morphism with C a regular integral scheme of Krull dimension 1. Let L be the function field of C. Then L is a finite extension of K, and C is isomorphic to $\operatorname{Spec} O_L$, where O_L is the ring of integers of L.*

Proof. Let $C_K = C \times_B \operatorname{Spec} K$ be the generic fibre of C. It is non-empty because f is flat; we can therefore choose a non-empty affine open subset $U = \operatorname{Spec} A$ of C_K. Since C_K is of finite type over K and integral, A is a finitely generated integral K-algebra; the field of fractions of A equals L. The transcendence degree of L over K is equal to the Krull dimension of A (see Eisenbud [6], § 13.1, Theorem A), which is 0 because f is flat and the dimensions of B and C are equal (see Liu [12], Theorem 4.3.12). This implies that L is a finite extension of K.

By the regularity of C, each ring $\mathcal{O}_C(U)$ with U a non-empty open subset of C is a domain which is integrally closed in its field of fractions L. In particular, every such $\mathcal{O}_C(U)$ contains O_L, which is the integral closure of $\mathbf{Z}$ in L. This means that there is a natural morphism $i: C \to \operatorname{Spec} O_L$. The composition $C \xrightarrow{i} \operatorname{Spec} O_L \to B$ equals f, and since f is proper and $\operatorname{Spec} O_L \to B$ is separated, i is proper as well (see Hartshorne [9], Proposition II.4.8(e)). Since C is regular of Krull dimension 1, for each point $x \in C$ the stalk $\mathcal{O}_{C,x}$ is a valuation ring of L. Conversely, let R be a valuation ring of L; because O_L is the intersection of all valuation rings of L, there exists a canonical morphism

$\operatorname{Spec} R \to \operatorname{Spec} O_L$. The valuative criterion of properness (Hartshorne [9], Theorem II.4.7) implies that there is a unique morphism $\operatorname{Spec} R \to C$ making the diagram

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r] \arrow[d] & C \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & \operatorname{Spec} O_L
\end{tikzcd}$$

commutative; here the morphism $\operatorname{Spec} L \to C$ is given by the inclusion $\{\eta\} \to C$, where η is the generic point of C. The morphism $\operatorname{Spec} R \to C$ induces a canonical isomorphism $R \cong \mathcal{O}_{C,x}$, where x is the image of the closed point of $\operatorname{Spec} R$ under the morphism $\operatorname{Spec} R \to C$ (see Hartshorne [9], proof of Theorem II.4.7). This means that $x \mapsto \mathcal{O}_{C,x}$ gives a bijection between the points of C and the valuation rings of L when we view each $\mathcal{O}_{C,x}$ naturally as a subring of L. The valuation rings of L correspond in turn bijectively to the points of $\operatorname{Spec} O_L$, so the continuous map $\operatorname{sp}(i)$ of topological spaces that underlies i is bijective. On any one-dimensional Noetherian scheme, the closed sets are the whole space and finite sets of closed points, so $\operatorname{sp}(i)$ is a homeomorphism. Finally, for each $x \in C$ the map on stalks $i_x^\#\colon O_{L,i(x)} \to \mathcal{O}_{C,x}$ is a local homomorphism between valuation rings of L, hence it is an isomorphism (see Hartshorne [9], Theorem I.6.1A). We conclude that i is an isomorphism. □

Let X be an arithmetic variety over B. It follows from the definition of regularity that the generic fibre $X_K = X \times_B \operatorname{Spec} K$ is regular (and therefore smooth over K, since K is perfect). For simplicity we will from now on require that moreover X_K is geometrically connected. This is always the case for the arithmetic surfaces we are interested in (which are models of geometrically connected curves). In general we can make sure this condition is fulfilled by replacing K by its algebraic closure inside the function field of X (see Liu [12], Proposition 8.3.8).

Our goal in this section and the next two is to describe an intersection theory for arithmetic surfaces, resembling classical intersection theory on surfaces over a field (see for example Hartshorne [9], § V.1), in particular on fibred surfaces (i.e. surfaces which are proper and flat over a projective curve). When trying to do this, one runs into trouble if the 'infinite fibres' of the projection $X \to B$ are not taken into account. We write S_{fin} and S_{inf} for the sets of finite and infinite places of K, respectively (see § 4). The fibres of X over finite places of K are defined by

$$X_v = X \times_B \operatorname{Spec} k_v \quad \text{for every } v \in S_{\text{fin}},$$

where k_v is the residue class field of v and the map $\operatorname{Spec} k_v \to B$ is induced by the canonical ring homomorphism $O_K \to k_v$. We define fibres over the infinite places of K (the Archimedean valuations) in a similar way by putting

$$X_v = X \times_B \operatorname{Spec} K_v \quad \text{for every } v \in S_{\text{inf}}.$$

Furthermore, for each infinite place v we fix an algebraic closure $\bar{K}_v$ of K_v; note that $\bar{K}_v$ is a topological field isomorphic to $\mathbf{C}$. We let $\mathfrak{X}_v$ be the projective analytic variety over $\bar{K}_v$ associated to $X \times_B \operatorname{Spec} \bar{K}_v$ (see Serre [18], § 2). If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, we write $\mathcal{F}_v$ for the analytic sheaf on $\mathfrak{X}_v$ obtained by pulling back $\mathcal{F}$ by the natural morphism $\mathfrak{X}_v \to X$.

Like in the case of varieties over a field, we can define divisors and line bundles on an arithmetic variety X, and (at least for curves and surfaces) they are in a sense equivalent. We could use the customary definition of these objects if it weren't for the fact that this does not lead to a satisfactory intersection theory; more precisely, it is impossible to define the degree of divisors on curves in such a way that a principal divisor has degree zero. We therefore 'enhance' our varieties by viewing the infinite places of K as extra points of B and by placing the infinite fibres of X defined above on a more or less equal footing with the fibres above closed points of B. Because of the occurrence of the infinite places, the divisors and line bundles that are relevant to us have to be defined somewhat differently than usual.

Definition. A *metrised line bundle* on an arithmetic variety X is a pair $(\mathcal{L}, \|\ \|)$, where $\mathcal{L}$ is a line bundle on X and $\|\ \|$ is a family $\{\|\ \|_v \mid v \in S_{\text{inf}}\}$, where each $\|\ \|_v$ is the norm associated to some continuous Hermitian metric on the analytic line bundle $\mathcal{L}_v$ on $\mathfrak{X}_v$. An *isometry* between metrised line bundles $(\mathcal{L}, \|\ \|_{\mathcal{L}})$ and $(\mathcal{M}, \|\ \|_{\mathcal{M}})$ is an isomorphism of line bundles $\mathcal{L} \xrightarrow{\sim} \mathcal{M}$ such that for all $v \in S_{\text{inf}}$ the induced isomorphism $\mathcal{L}_v \xrightarrow{\sim} \mathcal{M}_v$ of analytic line bundles is an isometry with respect to $\|\ \|_{\mathcal{L}}$ and $\|\ \|_{\mathcal{M}}$.

If there is no risk of confusion, the metric will often be omitted from the notation for a metrised line bundle, i.e. a metrised line bundle $(\mathcal{L}, \| \enspace \|)$ is also denoted simply by $\mathcal{L}$.

The structure sheaf $\mathcal{O}_X$ is metrised in a natural way: if v is an infinite place, $(\mathcal{O}_X)_v$ is the structure sheaf of the analytic variety $\mathfrak{X}_v$, and its metric is defined by

$$\|f\|_v(x) = |f(x)|_v^{1/[K_v:\mathbf{R}]},$$

where $| \enspace |_v$ is the unique extension of the valuation on K_v (defined in § 4) to $\bar{K}_v$. (Notice that $\|f\|_v(x)$ is just the usual absolute value of the complex number $f(x)$ when $\bar{K}_v$ is identified with $\mathbf{C}$.) The tensor product of two metrised line bundles $(\mathcal{L}, \| \enspace \|_\mathcal{L})$ and $(\mathcal{M}, \| \enspace \|_\mathcal{M})$ is obtained by taking the usual tensor product of line bundles and defining the metric on the $(\mathcal{L}\otimes\mathcal{M})_v$ using the formula

$$\|s \otimes t\|_{\mathcal{L}\otimes\mathcal{M},v} = \|s\|_{\mathcal{L},v} \cdot \|t\|_{\mathcal{M},v},$$

where s and t are local generating sections of $\mathcal{L}$ and $\mathcal{M}$, respectively. For any metrised line bundle $(\mathcal{L}, \| \enspace \|)$, the dual $\mathcal{L}^\vee$ of $\mathcal{L}$ is made into a metrised line bundle by requiring the natural isomorphism

$$\mathcal{L} \otimes \mathcal{L}^\vee \xrightarrow{\sim} \mathcal{O}_X$$

to be an isometry.

Let $f\colon X \to Y$ be a morphism of arithmetic varieties over B. Then metrised line bundles on Y can be pulled back to X via f in the following way. For any metrised line bundle $(\mathcal{L}, \| \enspace \|)$ on Y, we write $f^*\mathcal{L}$ for the usual inverse image of the line bundle $\mathcal{L}$ (see Hartshorne [9], § II.5). Let v be an infinite place of K, let $\mathfrak{Y}_v$ be the projective analytic variety associated to $Y \times_B \operatorname{Spec} \bar{K}_v$, and let $f_v\colon \mathfrak{X}_v \to \mathfrak{Y}_v$ be the analytic map associated to $f \times \mathrm{id}\colon X \times_B \operatorname{Spec} \bar{K}_v \to Y \times_B \operatorname{Spec} \bar{K}_v$. Then we have $(f^*\mathcal{L})_v \cong f_v^*\mathcal{L}_v$, and we define the metric on $(f^*\mathcal{L})_v$ as the pull-back of the metric on $\mathcal{L}_v$.

Definition. The group of *Arakelov divisors* on an arithmetic variety X is the Abelian group

$$\operatorname{Div} X = \bigoplus_Y \mathbf{Z} \oplus \bigoplus_{v \in S_{\mathrm{inf}}} \mathbf{R}$$

with Y ranging over the integral closed subschemes of codimension one in X and with S_{inf} the set of infinite places of K, as before. In other words, an Arakelov divisor is a finite formal sum

$$D = \sum_Y n_Y Y + \sum_v a_v X_v$$

with the $n_Y \in \mathbf{Z}$ and the $a_v \in \mathbf{R}$. (Here Y and X_v are just formal symbols.) We call

$$D_{\mathrm{fin}} = \sum_Y n_Y Y \quad \text{and} \quad D_{\mathrm{inf}} = \sum_v a_v X_v,$$

the *finite* and *infinite* components of D.

After this general description of arithmetic varieties, we will now consider the simplest kind of arithmetic variety, namely the spectrum of the ring of integers of a number field. Since K is an arbitrary number field and since we have assumed our arithmetic varieties to be geometrically connected, it is enough to consider the case $X = B = \operatorname{Spec} O_K$. We take a look at metrised line bundles, the Picard group, divisors, and the class group in this setting.

A metrised line bundle $(\mathcal{L}, \| \enspace \|)$ on the base scheme B consists of an invertible O_K-module L and Hermitian metrics on the one-dimensional $\bar{K}_v$-vector spaces $L \otimes_{O_K} \bar{K}_v$. Such a module is called a *metrised O_K-module.*

Definition. The *Picard group* of B, denoted by $\operatorname{Pic} B$, is the group of isometry classes of metrised line bundles on B under the tensor product.

Like in the case of curves over a field, we can define the *degree* of a line bundle. This is done by taking a non-zero rational section

$$s \in \mathcal{L}_\eta \cong K \otimes_{O_K} L,$$

where η is the generic point of B; this gives an isomorphism

$$\begin{aligned} i\colon \mathcal{L}_\eta &\xrightarrow{\sim} K \\ s &\longmapsto 1. \end{aligned}$$

For each finite place v of K, the stalk

$$\mathcal{L}_v \cong O_{K,v} \otimes_{O_K} L$$

is a free $O_{K,v}$-module of rank one. When mapped into K via the $O_{K,v}$-linear map

$$\mathcal{L}_v \hookrightarrow \mathcal{L}_\eta \xrightarrow{i} K,$$

it coincides with some power of the maximal ideal $\mathfrak{m}_v$ of $O_{K,v}$, say $\mathfrak{m}_v^{n_v}$. The *degree* of $(\mathcal{L}, \|\ \|)$ is now defined as

$$\deg(\mathcal{L}, \|\ \|) = -\sum_{v\in S_{\mathrm{fin}}} n_v \log \#k_v - \sum_{v\in S_{\mathrm{inf}}} [K_v : \mathbf{R}] \log \|s\|_v,$$

where s is viewed as an element of each $L \otimes_{O_K} \bar{K}_v$ via the natural map

$$\mathcal{L}_\eta \longrightarrow \mathcal{L}_\eta \otimes_K \bar{K}_v \cong L \otimes_{O_K} \bar{K}_v.$$

Suppose we start with another rational section, say xs with $x \in K^\times$. Then the map $\mathcal{L}_v \to K$ changes by multiplication by x^{-1}, so each n_v changes by

$$-\operatorname{ord}_v(x) = \log(|x|_v)/\log(\#k_v).$$

Furthermore, for every infinite place v our normalisation of the valuation $|\ \ |_v$ on K implies that

$$\|xs\|_v = |x|_v^{1/[K_v:\mathbf{R}]} \|s\|_v.$$

The expression defining the degree therefore changes by

$$-\sum_{v\in S_{\mathrm{fin}}} \log|x|_v - \sum_{v\in S_{\mathrm{inf}}} \log|x|_v = 0$$

because of the product formula. It follows that our definition of the degree is independent of the choice of s.

The Arakelov divisors on B are easy to describe: a divisor is a finite formal sum

$$D = \sum_{v\in S_{\mathrm{fin}}} n_v v + \sum_{v\in S_{\mathrm{inf}}} a_v v$$

with $n_v \in \mathbf{Z}$ and $a_v \in \mathbf{R}$. The *degree* of such a divisor is defined as

$$\deg D = \sum_{v\in S_{\mathrm{fin}}} n_v \log \#k_v + \sum_{v\in S_{\mathrm{inf}}} a_v [K_v : \mathbf{R}];$$

thus we obtain a surjective group homomorphism $\deg\colon \operatorname{Div} B \to \mathbf{R}$. For $x \in K^\times$, the divisor of x is defined as

$$\operatorname{div}(x) = \sum_{v\in S_{\mathrm{fin}}} \operatorname{ord}_v(x)\, v - \sum_{v\in S_{\mathrm{inf}}} \frac{1}{[K_v : \mathbf{R}]} \log(|x|_v)\, v.$$

Divisors of the form $\operatorname{div}(x)$ are called *principal divisors*.

Definition. The *(divisor) class group* of B, denoted by $\operatorname{Cl} B$, is the Abelian group

$$\operatorname{Cl} B = \operatorname{Div} B/(\text{principal divisors}).$$

For every Arakelov divisor

$$D = \sum_{v \in S_{\text{fin}}} n_v v + \sum_{v \in S_{\text{inf}}} a_v v$$

we define a metrised line bundle $(\mathcal{O}_B(D), \|\ \|_D)$ as follows: $\mathcal{O}_B(D)$ is the usual line bundle $\mathcal{O}_B(D_{\text{fin}})$ defined by

$$\mathcal{O}_B(D)(U) = \{x \in K^\times \mid \operatorname{ord}_v(x) + n_v \geq 0 \text{ for all } v \in U\} \cup \{0\} \quad (U \subseteq B \text{ open}),$$

where $v \in U$ means that the prime ideal of O_K associated to v lies in U; for every $v \in S_{\text{fin}}$ the metric on the one-dimensional $\bar{K}_v$-vector space $(\mathcal{O}_B(D))_v$ is defined by

$$\|x\|_{D,v} = \exp(-a_v)|x|_v^{1/[K_v:\mathbf{R}]}.$$

Proposition 6.2. *The association $D \mapsto \mathcal{O}_B(D)$ induces a group isomorphism*

$$\phi\colon \operatorname{Cl} B \xrightarrow{\sim} \operatorname{Pic} B.$$

Proof. For all Arakelov divisors D and D' on B, there is an isomorphism $\mathcal{O}_B(D+D') \cong \mathcal{O}_B(D) \otimes \mathcal{O}_B(D')$ as ordinary line bundles, and it follows from the definition of the metric on the tensor product that this isomorphism is an isometry. Therefore ϕ is a group homomorphism; to show that it is an isomorphism, we will construct an inverse. Let $(\mathcal{L}, \|\ \|)$ be a metrised line bundle on B. Let s be a non-zero rational section of $\mathcal{L}$, i.e. an element of the stalk $\mathcal{L}_\eta$ of $\mathcal{L}$ at the generic point of B; then we get an isomorphism

$$\begin{aligned} \mathcal{L}_\eta &\xrightarrow{\sim} K \\ s &\longmapsto 1. \end{aligned}$$

For every $v \in S_{\text{fin}}$, the image of the stalk $\mathcal{L}_v$ under the composed map

$$\mathcal{L}_v \to \mathcal{L}_\eta \to K$$

is a sub-$\mathcal{O}_{B,v}$-module of K which is free of rank 1, so it coincides with some power of the maximal ideal $\mathfrak{m}_v$ of $\mathcal{O}_{B,v}$, say $\mathfrak{m}_v^{-n_v}$. Then $\mathcal{L} \cong \mathcal{O}_B\bigl(\sum_v n_v v\bigr)$ as line bundles, and there is a unique Arakelov divisor D with finite part $\sum_v n_v v$ such that $(\mathcal{L}, \|\ \|) \cong (\mathcal{O}_B(D), \|\ \|_D)$ as metrised line bundles. We let $\phi^{-1}(\mathcal{L})$ be the class of D in $\operatorname{Cl} B$. This class is independent of the choice of s, because changing s comes down to multiplication by an element of $K^\times$, and hence to changing D by a principal divisor. It is now clear that $\phi \circ \phi^{-1}$ is the identity on $\operatorname{Pic} B$, and it is also easy to check that $\phi^{-1} \circ \phi$ is the identity on $\operatorname{Cl} B$. We conclude that ϕ is a group isomorphism. □

The product formula implies that the degree of a principal divisor is zero, so the degree map on divisors induces a map $\deg\colon \operatorname{Cl} B \to \mathbf{R}$. It is a matter of straightforward computation to check that the degree maps on both sides of the isomorphism $\operatorname{Cl} B \xrightarrow{\sim} \operatorname{Pic} B$ are compatible, i.e. $\deg D = \deg \mathcal{O}_B(D)$ for all Arakelov divisors D on B.

Remark. Let $\operatorname{Div}^0 B$ and $\operatorname{Cl}^0 B$ denote the kernels of the degree maps $\operatorname{Div} B \to \mathbf{R}$ and $\operatorname{Cl} B \to \mathbf{R}$, respectively. Then $\operatorname{Cl}^0 B$ can be viewed as an extension of Cl_K, the ideal class group of K, in the following way. Let V be the hyperplane in $\bigoplus_{v \in S_{\text{inf}}} \mathbf{R}$ defined by

$$V = \left\{ (a_v)_{v \in S_{\text{inf}}} \in \bigoplus_{v \in S_{\text{inf}}} \mathbf{R} \;\middle|\; \sum_{v \in S_{\text{inf}}} [K_v : \mathbf{R}] a_v = 0 \right\};$$

this is in a natural way a subgroup of $\operatorname{Div}^0 B$. Write I_K for the group of fractional ideals of O_K. There is a canonical surjective group homomorphism

$$\begin{aligned} \operatorname{Div}^0 B &\longrightarrow I_K \\ \sum_{v \in S_{\text{fin}}} n_v v + D_{\text{inf}} &\longmapsto \prod_{v \in S_{\text{fin}}} \mathfrak{p}_v^{n_v} \end{aligned}$$

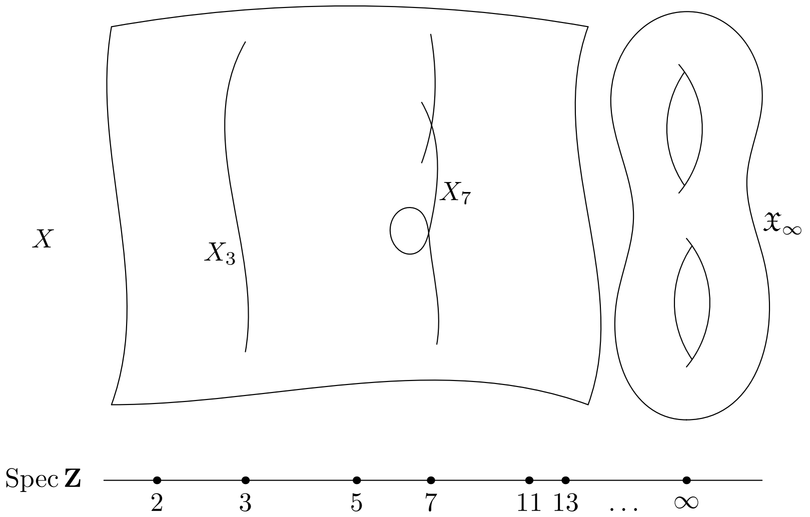

Figure 2: Artist's impression of an arithmetic surface over $\operatorname{Spec}\mathbf{Z}$.

with kernel V, where $\mathfrak{p}_v$ is the prime ideal of O_K associated to the finite place v. We therefore have the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & O_K^\times \arrow[r] \arrow[d] & K^\times \arrow[r] \arrow[d, "\mathrm{div}"] & K^\times/O_K^\times \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & V \arrow[r] & \operatorname{Div}^0 B \arrow[r] & I_K \arrow[r] & 0.
\end{tikzcd}$$

Here the map $O_K^\times \to V$ sends an element $x \in O_K^\times$ to the vector $(-[K_v : \mathbf{R}]^{-1} \log |x|_v)_{v \in S_{\mathrm{inf}}}$, and the map $K^\times/O_K^\times \to I_K$ is induced by the map that associates to an element $a \in K^\times$ the fractional O_K-ideal aO_K. Since $K^\times/O_K^\times \to I_K$ is injective, the snake lemma gives us an exact sequence of cokernels

$$0 \longrightarrow V/\Lambda \longrightarrow \operatorname{Cl}^0 B \longrightarrow \operatorname{Cl}_K \longrightarrow 0,$$

where Λ is the image of $O_K^\times$ in V. Via this exact sequence we can express at once two important results from algebraic number theory, namely the finiteness of the ideal class group and Dirichlet's unit theorem, by saying that $\operatorname{Cl}^0 B$ is a compact topological group.

We now turn to the next simplest species of arithmetic variety, namely arithmetic surfaces. They can be viewed as *models* of regular projective curves over K. Arakelov's intersection theory on arithmetic surfaces, which we will describe in § 8, can be used to compute heights of points on these curves.

Figure 2 depicts an arithmetic surface over $\mathbf{Z}$, the generic fibre of which has genus 2. This surface has a smooth fibre at 3 and a singular fibre at 7.

For every regular projective curve X_K over K, the following theorem asserts that there exists a model of X_K over B, i.e. an arithmetic surface over B of which X_K is the generic fibre.

Theorem 6.3. *Let X_K be a geometrically connected regular projective curve over K. Then there exists an arithmetic surface X over B such that*

$$X \times_B \operatorname{Spec} K \cong X_K.$$

Proof. Liu [12], Proposition 10.1.8.

Let X be an arithmetic surface whose generic fibre X_K has positve genus. Just as for curves, we have divisors and line bundles, and they are in a certain sense equivalent.

The integral closed subschemes of codimension 1 of X come in two kinds. If Y is such a subscheme, the composed map

$$Y \longrightarrow X \xrightarrow{p} B$$

is either constant or surjective on the underlying topological spaces. In the first case Y is an irreducible component of a finite fibre of p. In the second case Y is flat and finite over B and equal to the closure in X of a closed point of X_K (see Liu [12], Proposition 8.3.4). A divisor which is a linear combination of irreducible components of fibres (finite or infinite) is called *vertical*; one whose components are flat over B is called *horizontal*.

If $f \in K(X)$ is a non-zero rational function, we define the *divisor of f* as

$$\operatorname{div}(f) = \sum_Y \operatorname{ord}_Y(f)\, Y + \sum_{v \in S_{\mathrm{inf}}} \operatorname{ord}_v(f)\, X_v,$$

where ord_Y is the normalised discrete valuation associated to Y, and $\operatorname{ord}_v(f)$ (the 'order of vanishing of f at X_v') is defined as

$$\operatorname{ord}_v(f) = -\int_{\mathfrak{X}_v} \log|f|_v\, \mu_v.$$

Here μ_v is the Arakelov $(1,1)$-form on $\mathfrak{X}_v$. The Arakelov divisors of the form $\operatorname{div}(f)$ are called *principal divisors*.

Definition. The *(divisor) class group* of X, denoted by $\operatorname{Cl} X$, is the Abelian group

$$\operatorname{Cl} X = \operatorname{Div} X/(\text{principal divisors}).$$

Definition. Recall from § 5 that a line bundle $\mathcal{L}$ on a compact connected Riemann surface $\mathfrak{X}$ equipped with a metric $\|\ \|$ is admissible with respect to a smooth $(1,1)$-form μ with $\int_{\mathfrak{X}} \mu = 1$ if $\|\ \|$ is smooth and

$$\operatorname{curv}_{\|\ \|} = (\deg \mathcal{L})\mu.$$

Let X be an arithmetic surface whose generic fibre has positive genus, and let $(\mathcal{L}, \|\ \|)$ be a metrised line bundle on X. Then $(\mathcal{L}, \|\ \|)$ is called *admissible* if for every infinite place v of K, the metrised line bundle $(\mathcal{L}_v, \|\ \|_v)$ on $\mathfrak{X}_v$ is admissible with respect to the Arakelov $(1,1)$-form μ_v.

According to Proposition 5.1, any line bundle $\mathcal{L}$ on X can be made into an admissible line bundle by equipping each analytic line bundle $\mathcal{L}_v$ with an admissible metric. Each of these metrics is unique up to scaling by a positive constant. The structure sheaf $\mathcal{O}_X$ with its natural Hermitian metric is an admissible line bundle of curvature 0. If $(\mathcal{L}, \|\ \|_{\mathcal{L}})$ and $(\mathcal{M}, \|\ \|_{\mathcal{M}})$ are two admissible line bundles on X, the definition of the metric on the tensor product $\mathcal{L} \otimes \mathcal{M}$ implies that $\mathcal{L} \otimes \mathcal{M}$ is again an admissible line bundle, and that the dual of an admissible line bundle is admissible.

Definition. The *Picard group* of X, denoted by $\operatorname{Pic} X$, is the group of isometry classes of admissible line bundles on X under the tensor product.

To every Arakelov divisor

$$D = D_{\mathrm{fin}} + \sum_{v \in S_{\mathrm{inf}}} a_v X_v$$

we associate an admissible line bundle $(\mathcal{O}_X(D), \|\ \|_D)$ in the following way. Disregarding the metric, $\mathcal{O}_X(D)$ is the usual line bundle $\mathcal{O}_X(D_{\mathrm{fin}})$ (see Hartshorne [9], § II.6). For each infinite place v, let D_v denote the divisor D_{fin} pulled back to $\mathfrak{X}_v$, and write

$$D_v = \sum_{P \in \mathfrak{X}_v} n_P P \quad \text{with } n_P \in \mathbf{Z}.$$

Then we define $\|\ \|_{D,v}$ to be the unique admissible metric on $\mathcal{O}_{\mathfrak{X}_v}(D_v)$ satisfying

$$\int_{\mathfrak{X}_v} \log\|1\|_{D,v} \mu_v = -a_v;$$

note that the logarithmic singularities of $\log\|1\|_{D,v}$ are integrable. More explicitly, this metric is given by the formula

$$\log\|1\|_{D,v} = \sum_{P \in \mathfrak{X}_v} n_P g_{\mathrm{Ar}}(P, x) - a_v,$$

which follows by applying Lemma 2.3 in a similar way as in the proof of Lemma 5.1.

Proposition 6.4. *The association $D \mapsto \mathcal{O}_X(D)$ induces an isomorphism*

$$\operatorname{Cl} X \xrightarrow{\sim} \operatorname{Pic} X.$$

Proof. This is done in the same way as for curves (Proposition 6.2).

7. The determinant of cohomology

Let A be a field or a Dedekind ring, let $Z = \operatorname{Spec} A$, and let $f\colon Y \to Z$ be a projective morphism of schemes. We will introduce a concept of 'relative cohomology' of Y over Z. The case where $A = O_K$ and Y is an arithmetic variety over $\operatorname{Spec} O_K$ will be needed in § 8 to formulate one of the key results of Faltings' paper [7], namely the Riemann–Roch formula for arithmetic surfaces.

The cohomology functors are the right derived functors of the global sections functor $\Gamma(Y, \ )$. This functor can be decomposed as $\Gamma(Z, \ ) \circ f_*$, where f_* is the direct image functor. We are going to concern ourselves with the right derived functors of the direct image functor f_*. If Z is affine (as it is in the case of arithmetic surfaces), this is actually equivalent to studying the cohomology of $\mathcal{O}_Y$-modules, at least for coherent $\mathcal{O}_Y$-modules. Our approach extends more easily to the case where the base scheme is any Dedekind scheme (i.e. a normal integral Noetherian scheme of dimension 0 or 1), however.

Definition. Let A be a commutative ring, and let M be a projective A-module of finite rank r. Then M is locally free of rank r, so the maximal antisymmetric tensor power

$$\det M = \bigwedge^r M$$

is locally free of rank 1. It is therefore an invertible A-module, called the *determinant* of M.

Lemma 7.1. *Let A be a commutative ring, and let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of finitely generated projective A-modules. Suppose M' and M'' are of rank r and s, respectively. Then there is a canonical isomorphism of A-modules

$$\begin{aligned} \phi\colon \det M' \otimes_A \det M'' &\xrightarrow{\sim} \det M \\ (x_1 \wedge \ldots \wedge x_r) \otimes (y_1 \wedge \ldots \wedge y_s) &\longmapsto \alpha x_1 \wedge \ldots \wedge \alpha x_r \wedge \tilde{y}_1 \wedge \ldots \wedge \tilde{y}_s, \end{aligned}$$

where $\tilde{y}_i$ denotes an arbitrary element of M such that $\beta\tilde{y}_i = y_i$.

Proof. It follows from the alternating property of the wedge product that the image of an element $(x_1 \wedge \ldots \wedge x_r) \otimes (y_1 \wedge \ldots \wedge y_s) \in \det M$ does not depend on the choice of the $\tilde{y}_i$, so ϕ is a well-defined map which is clearly A-linear. To prove that it is an isomorphism, it suffices to check that this is locally the case, so we may assume A is a local ring. Since projective A-modules are free, we can choose bases $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_s\}$ of M' and M'', respectively; then $\{\alpha x_1, \ldots, \alpha x_r, \tilde{y}_1, \ldots, \tilde{y}_s\}$ is a basis of M, where each $\tilde{y}_i \in \beta^{-1}\{y_i\}$ is arbitrary. Now $\{x = x_1 \wedge \ldots \wedge x_r\}$ and $\{y = y_1 \wedge \ldots \wedge y_s\}$ are bases of $\det M'$ and $\det M''$, respectively, so that $\{x \otimes y\}$ is a basis of $\det M' \otimes_A \det M''$. This basis is sent to the basis $\{\alpha x_1 \wedge \ldots \wedge \alpha x_r \wedge \tilde{y}_1 \wedge \ldots \wedge \tilde{y}_s\}$ of $\det M$ by the map ϕ, which is therefore an isomorphism. □

If A is a Dedekind ring, not all finitely generated A-modules are projective; a finitely generated A-module is projective if and only if it is torsion-free. We still need to define a determinant for these modules, however. The following lemma shows how to do this.

Lemma 7.2. *Let A be a Dedekind ring, and let M be a finitely generated A-module. Then there exists a short exact sequence of A-modules*

$$0 \longrightarrow E \longrightarrow F \longrightarrow M \longrightarrow 0$$

with E and F finitely generated projective A-modules. The invertible A-module

$$\det M = \det F \otimes (\det E)^\vee$$

is, up to canonical isomorphism, independent of the choice of E and F and of the morphisms $E \to F \to M$.

Proof. Choose a finitely generated free A-module F and a surjective A-linear map $F \to M$; this is possible because M is finitely generated. Then $E = \ker F$ is torsion-free, hence projective; this shows the existence of the desired short exact sequence. If

$$0 \longrightarrow E \longrightarrow F \xrightarrow{\beta} M \longrightarrow 0,$$

$$0 \longrightarrow E' \longrightarrow F' \xrightarrow{\beta'} M \longrightarrow 0$$

are two such sequences, we consider the short exact sequence

$$0 \longrightarrow E'' \longrightarrow F'' \longrightarrow M \longrightarrow 0,$$

where $E'' = E \times E'$ and where F'' is the torsion-free (hence projective) A-module

$$F'' = \{(f, f') \in F \times F' \mid \beta(f) = \beta'(f')\}.$$

We have a commutative diagram

$$\begin{tikzcd}
 & \ker p \arrow[r, "\sim"] \arrow[d] & \ker q \arrow[d] & & \\
0 \arrow[r] & E'' \arrow[r] \arrow[d, "p"] & F'' \arrow[r] \arrow[d, "q"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r] & F \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where $p\colon E'' \to E$ and $q\colon F'' \to F$ are the natural surjections. According to Lemma 7.1, there are canonical isomorphisms

$$\begin{aligned}
\det F'' \otimes (\det E'')^\vee &\cong \det(\ker q) \otimes \det F \otimes \bigl(\det(\ker p) \otimes \det E\bigr)^\vee \\
&\cong \det F \otimes (\det E)^\vee,
\end{aligned}$$

and similarly with E and F replaced by E' and F'. This gives a canonical isomorphism

$$\det F \otimes (\det E)^\vee \cong \det F' \otimes (\det E')^\vee,$$

and we conclude that $\det M$ is unique up to canonical isomorphism. □

Remark. With this definition, any short exact sequence

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

of finitely generated modules over a Dedekind ring A gives rise to a canonical isomorphism

$$\det M' \otimes_A \det M'' \xrightarrow{\sim} \det M$$

like in Lemma 7.1.

Definition. Let A be a Dedekind ring, and let $\mathcal{F}$ be a coherent $\mathcal{O}_{\operatorname{Spec} A}$-module, so that $\mathcal{F} = \tilde{M}$ with M a finitely generated A-module. The *determinant* of $\mathcal{F}$ is the invertible sheaf $\det \mathcal{F}$ on $\operatorname{Spec} A$ defined by

$$\det \mathcal{F} = (\det M)^\sim;$$

by the previous lemma, it is unique up to canonical isomorphism.

Definition. Let $f\colon Y \to Z$ be a continuous map of topological spaces. The direct image functor f_* is a left exact functor from the category of sheaves of Abelian groups on Y to the category of sheaves of Abelian groups on Z. Its right derived functors are called the *higher direct image functors* and denoted by $\mathrm{R}^i f_*$ for $i \geq 0$.

Let A be a Dedekind ring, let $Z = \operatorname{Spec} A$, and let $f\colon Y \to Z$ be a projective morphism of schemes. Suppose $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules. Then the higher direct image sheaves look as follows (see Hartshorne [9], § III.8): for every affine open subset $U \subseteq Z$ and all $i \geq 0$,

$$(\mathrm{R}^i f_* \mathcal{F})(U) \cong \mathrm{H}^i(f^{-1}U, \mathcal{F}|_{f^{-1}U}).$$

The $\mathrm{R}^i f_* \mathcal{F}$ are quasi-coherent sheaves of $\mathcal{O}_Z$-modules. In particular, because in our case $Z = \operatorname{Spec} A$ is itself affine,

$$\mathrm{R}^i f_* \mathcal{F} \cong \mathrm{H}^i(Y, \mathcal{F})^\sim.$$

If Y is of dimension n, then $\mathrm{R}^i f_* \mathcal{F} = 0$ for all $i > n$ by Grothendieck's vanishing theorem (Hartshorne [9], Theorem III.2.7). Moreover, if $\mathcal{F}$ is coherent, all the $\mathrm{R}^i f_* \mathcal{F}$ are finitely generated $\mathcal{O}_Z$-modules (hence coherent) as a consequence of Serre's finiteness theorem (Hartshorne [9], Theorem III.5.2).

Definition. Let A be a Dedekind ring, let $f\colon Y \to \operatorname{Spec} A$ be a projective morphism, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_Y$-modules. The *determinant of cohomology* of $\mathcal{F}$ is the line bundle on $\operatorname{Spec} A$ defined by

$$\det \mathrm{R}f_*\mathcal{F} = \bigotimes_{i=0}^{n} (\det \mathrm{R}^i f_*\mathcal{F})^{\otimes(-1)^i},$$

where n is the dimension of Y. Similarly, if Y is an n-dimensional projective scheme over a field k, and $\mathcal{G}$ is a coherent sheaf of $\mathcal{O}_Y$-modules, the determinant of cohomology of $\mathcal{G}$ is the one-dimensional k-vector space

$$\det \mathrm{H}(Y, \mathcal{G}) = \bigotimes_{i=0}^{n} \bigl(\det \mathrm{H}^i(Y, \mathcal{G})\bigr)^{\otimes(-1)^i}.$$

8. Arakelov intersection theory

In this section we describe the intersection pairing constructed by Arakelov [2] on arithmetic surfaces. We also state Faltings' arithmetic analogue of the Riemann–Roch theorem from classical intersection theory on surfaces.

Classical two-dimensional intersection theory (see for example Hartshorne [9], § 5.1) is concerned with intersection numbers of curves on a non-singular projective surface X over a field. In the case where the ground field is algebraically closed, the intersection number of two integral curves meeting transversally is just the number of intersection points. In general, care is to be taken of the degrees of the intersection points and of the intersection multiplicities at points of X; furthermore, one also wants to define the self-intersection of a curve. This leads to the definition of an intersection number for any two integral curves on X. Once this is done, there is a unique way to extend this intersection number to a symmetric bilinear pairing

$$\begin{aligned} \operatorname{Div} X \times \operatorname{Div} X &\longrightarrow \mathbf{Z} \\ (D, E) &\longmapsto (D \,.\, E); \end{aligned}$$

here $\operatorname{Div} X$ is the group of Weil divisors on X. The assumption of projectivity implies that the intersection number of any divisor with a principal divisor is zero. The intersection pairing therefore induces a bilinear map $\operatorname{Cl} X \times \operatorname{Cl} X \to \mathbf{Z}$, where $\operatorname{Cl} X$ is the divisor class group of X.

Now let K be a number field with ring of integers O_K, and let X be an arithmetic surface over $B = \operatorname{Spec} O_K$ whose generic fibre has positive genus. If C and D are two distinct integral curves on X, and P is a closed point of X, it is still possible to define the intersection multiplicity of C and D in P. However, we can no longer define global intersection numbers in such a way that the intersection number of a principal divisor with any other divisor is zero. The problem originates from the fact that there is no definition of the degree of a divisor on the base scheme B such that the degree of every principal divisor vanishes. The solution is to add 'points at infinity' to B by considering the infinite places of K as well as the finite places, which correspond to the closed points of B. It was shown by Arakelov that a useful intersection theory can be constructed in the context of Arakelov divisors and admissible line bundles on X.

Let $\operatorname{Div} X$ be the group of Arakelov divisors on X. We are going to define an intersection pairing

$$\begin{aligned} \operatorname{Div} X \times \operatorname{Div} X &\longrightarrow \mathbf{R} \\ (D, E) &\longmapsto (D \,.\, E). \end{aligned}$$

We require this map to be bilinear, so we immediately reduce to the case where D is an infinite fibre X_v, an integral component of a finite fibre X_v, or an integral horizontal curve on X. Let i_D denote the composed map $\tilde{D} \to D \to X$, where $\tilde{D} \to D$ is the normalisation of D and $D \to X$ is the canonical morphism. We define

$$(D \,.\, E) = \begin{cases} \deg(i_D^*\mathcal{O}_X(E)) \cdot \log \# k_v & \text{if } D \text{ is an irreducible component of } X_v \text{ with } v \in S_{\mathrm{fin}}; \\ \deg(i_D^*\mathcal{O}_X(E)) \cdot [K_v : \mathbf{R}] & \text{if } D = X_v \text{ with } v \in S_{\mathrm{inf}}. \\ \deg(i_D^*\mathcal{O}_X(E)) & \text{if } D \text{ is horizontal.} \end{cases}$$

Here deg denotes the usual degree over k_v in the first case, the usual degree over K_v in the second case, and the Arakelov degree on $\tilde{D}$ in the third case; note that $\tilde{D}$ is the spectrum of the ring of integers of a number field by Proposition 6.1. It follows immediately that $(D \,.\, E) = 0$ if E is a principal divisor. Furthermore, it can be shown that this intersection pairing is symmetric (see Arakelov [2], § 1). Because the class group of X is isomorphic to $\operatorname{Pic} X$, the Picard group of X, we conclude that there is a symmetric bilinear intersection pairing

$$\begin{aligned} \operatorname{Pic} X \times \operatorname{Pic} X &\longrightarrow \mathbf{R} \\ (\mathcal{L}, \mathcal{M}) &\longmapsto (\mathcal{L} \,.\, \mathcal{M}). \end{aligned}$$

The following lemma gives some first properties of this intersection pairing in the case where one of the line bundles is the pull-back of a line bundle on the base curve.

Lemma 8.1. *Let X be an arithmetic surface over B, , and let $(\mathcal{L}, \|\ \|)$ be a metrised line bundle on B.*

(a) For every Arakelov divisor D on X which is the image of some section of p, we have

$$(\mathcal{O}_X(D) \,.\, p^*\mathcal{L}) = \deg \mathcal{L}.$$

(b) For every vertical Arakelov divisor E on X, we have

$$(\mathcal{O}_X(E) \,.\, p^*\mathcal{L}) = 0.$$

(c) For every admissible line bundle $\mathcal{M}$ on X such that the intersection of $\mathcal{M}$ with every fibre has degree 0 (by which we mean that $(\mathcal{O}_X(X_v) \,.\, \mathcal{M}) = 0$ for all $v \in S_{\mathrm{fin}} \cup S_{\mathrm{inf}}$), we have

$$(\mathcal{M} \,.\, p^*\mathcal{L}) = 0.$$

Proof. If D is the image of a section S, then

$$\begin{aligned} (\mathcal{O}_X(D) \,.\, p^*\mathcal{L}) &= \deg S^* p^* \mathcal{L} \\ &= \deg \mathcal{L}, \end{aligned}$$

which proves (a). For (b), we write i_K for the canonical morphism $X_K \to X$, and i_K^* for the functor which associates to any metrised line bundle $(\mathcal{M}, \|\ \|)$ on X the restriction of the ordinary line bundle $\mathcal{M}$ to X_K. Because of the commutative diagram

$$\begin{tikzcd} X_K \arrow[r] \arrow[d, "i_K"] & \operatorname{Spec} K \arrow[d] \\ X \arrow[r, "p"] & B \end{tikzcd}$$

and the fact that every line bundle on $\operatorname{Spec} K$ is trivial, the line bundle $i_K^* p^*(\mathcal{L}, \|\ \|)$ is trivial. In particular, the restriction of $\mathcal{L}$ to X_v is trivial for every infinite place v of K. A similar argument using the commutative diagram

$$\begin{tikzcd} \tilde{C} \arrow[r] \arrow[d, "i_C"] & \operatorname{Spec} k_v \arrow[d] \\ X \arrow[r, "p"] & B \end{tikzcd}$$

shows that for any irreducible component C of a finite fibre X_v, the line bundle $i_C^* p^* \mathcal{L}$ is trivial, where i_C is the natural map $\tilde{C} \to C \to X$ ($\tilde{C} \to C$ again being the normalisation of C). For any vertical Arakelov divisor E on X the definition of the intersection product gives us

$$(\mathcal{O}_X(E) \,.\, p^*\mathcal{L}) = 0.$$

To prove (c), we choose an Arakelov divisor

$$D = \sum_{v \in S_{\mathrm{fin}}} n_v v + \sum_{v \in S_{\mathrm{inf}}} a_v v$$

on B such that $\mathcal{L} = \mathcal{O}_B(D)$. Then $p^*\mathcal{L}$ is the admissible line bundle $\mathcal{O}_X(p^{-1}D)$, where

$$p^{-1}D = \sum_{v \in S_{\mathrm{fin}}} n_v X_v + \sum_{v \in S_{\mathrm{inf}}} a_v X_v;$$

for all $v \in S_{\mathrm{fin}}$, we view X_v as a divisor on X by identifying it with the sum of its integral components (counted with multiplicities). This implies that for every admissible line bundle $\mathcal{M}$ having degree 0 on each fibre,

$$(\mathcal{M} \,.\, p^*\mathcal{L}) = \sum_{v \in S_{\mathrm{fin}}} n_v(\mathcal{M} \,.\, X_v) + \sum_{v \in S_{\mathrm{inf}}} a_v(\mathcal{M} \,.\, X_v).$$

Each term vanishes by the definition of the intersection pairing. □

One of the fundamental theorems of Arakelov intersection theory that we will use, namely Faltings' equivalent of the Riemann–Roch formula for arithmetic surfaces, makes use of a metrisation of the line bundles $\det \mathrm{R}p_*\mathcal{L}$ on B for admissible line bundles $\mathcal{L}$ on X.

Theorem 8.2. *Let $\mathfrak{X}$ be a Riemann surface of genus $g \geq 1$. There is a unique way to assign metrics to the one-dimensional complex vector spaces $\det \mathrm{H}(\mathfrak{X}, \mathcal{L})$, for every admissible line bundle $\mathcal{L}$ on $\mathfrak{X}$, such that the following axioms hold:*

(1) Any isometry $f\colon \mathcal{L} \xrightarrow{\sim} \mathcal{M}$ of admissible line bundles on $\mathfrak{X}$ induces an isometry

$$\det f\colon \det \mathrm{H}(\mathfrak{X}, \mathcal{L}) \xrightarrow{\sim} \det \mathrm{H}(\mathfrak{X}, \mathcal{M})$$

of metrised one-dimensional $\mathbf{C}$-vector spaces.

(2) If the metric on $\mathcal{L}$ is changed by a factor $\alpha > 0$, the metric on $\det \mathrm{H}(\mathfrak{X}, \mathcal{L})$ changes by a factor $\alpha^{\chi(\mathcal{L})}$, where

$$\chi(\mathcal{L}) = \dim \mathrm{H}^0(\mathfrak{X}, \mathcal{L}) - \dim \mathrm{H}^1(\mathfrak{X}, \mathcal{L})$$

is the Euler characteristic of $\mathcal{L}$.

(3) For every admissible line bundle $\mathcal{L}$ on $\mathfrak{X}$ and every point $P \in \mathfrak{X}$, the canonical exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(P) \longrightarrow P_*P^*\mathcal{L}(P) \longrightarrow 0$$

induces an isometry

$$\det \mathrm{H}(\mathfrak{X}, \mathcal{L}(P)) \xrightarrow{\sim} \det \mathrm{H}(\mathfrak{X}, \mathcal{L}) \otimes P^*\mathcal{L}(P),$$

where $\mathcal{L}(P)$ is metrised such that the canonical isomorphism $\mathcal{L}(P) \xrightarrow{\sim} \mathcal{L} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{X}(P)$ is an isometry.

(4) The metric on $\det \mathrm{H}(\mathfrak{X}, \Omega^1_\mathfrak{X})$ (which is canonically isomorphic to $\bigwedge^g \mathrm{H}^0(\mathfrak{X}, \Omega^1_\mathfrak{X})$ by Serre duality) comes from the inner product $(\alpha, \beta) \mapsto \frac{i}{2}\int_\mathfrak{X} \alpha \wedge \bar\beta$ on $\mathrm{H}^0(\mathfrak{X}, \Omega^1_\mathfrak{X})$.

Proof. Faltings [7], Theorem 1.

In this way, we obtain a metrised line bundle $\det \mathrm{R}p_*\mathcal{L}$ on B for any admissible line bundle $\mathcal{L}$ on the arithmetic surface X.

Proposition 8.3 (Projection formula). *Let X be an arithmetic surface over B whose generic fibre has positive genus. For all admissible line bundles $\mathcal{L}$ on X and $\mathcal{E}$ on B, we have*

$$\deg \det \mathrm{R}p_*(\mathcal{L} \otimes p^*\mathcal{E}) = \deg \det \mathrm{R}p_*\mathcal{L} + \chi(\mathcal{L}) \deg \mathcal{E},$$

where

$$\chi(\mathcal{L}) = \mathrm{rank}_{O_K} \mathrm{H}^0(X, \mathcal{L}) - \mathrm{rank}_{O_K} \mathrm{H}^1(X, \mathcal{L})$$

is the Euler characteristic of $\mathcal{L}$ along the fibres of p.

Proof. There are canonical homomorphisms of coherent $\mathcal{O}_B$-modules

$$(\mathrm{R}^i p_*\mathcal{L}) \otimes_{\mathcal{O}_B} \mathcal{E} \longrightarrow \mathrm{R}^i p_*(\mathcal{L} \otimes_{\mathcal{O}_X} p^*\mathcal{E}) \quad (i = 0, 1),$$

which are isomorphisms because $\mathcal{E}$ is flat over B (see Liu [12], Proposition 5.2.32). This implies that there is a canonical isomorphism

$$\det(\mathrm{R}p_*\mathcal{L}) \otimes_{\mathcal{O}_B} \mathcal{E}^{\otimes\chi(\mathcal{L})} \xrightarrow{\sim} \det \mathrm{R}p_*(\mathcal{L} \otimes_{\mathcal{O}_X} p^*\mathcal{E}). \qquad (*)$$

For all $v \in S_{\mathrm{inf}}$, pulling back these sheaves to $\mathfrak{X}_v$ gives a canonical isomorphism

$$\det \mathrm{H}(\mathfrak{X}_v, \mathcal{L}_v) \otimes_{\bar K_v} \mathcal{E}_v^{\otimes\chi(\mathcal{L})} \xrightarrow{\sim} \det \mathrm{H}(\mathfrak{X}_v, \mathcal{L}_v \otimes_{\bar K_v} \mathcal{E}_v).$$

This isomorphism is an isometry because $\mathcal{E}_v$ is isomorphic (albeit not canonically) to $\bar K_v$ as a metrised $\bar K_v$-vector space. Taking the degree on both sides of $(*)$ gives the projection formula. □

Let $\omega_{X/B}$ be the relative dualising sheaf of X over B (see Liu [12], § 6.4). For every point $b \in B$ such that the fibre X_b is regular, the restriction of $\omega_{X/B}$ to X_b is canonically isomorphic to the sheaf of differentials $\Omega_{X_b/k(b)}$, where $k(b)$ is the residue field of B at b. The case where b is the generic point of B implies that for every infinite place v of K, the analytic line bundle $(\omega_{X/B})_v$ on $\mathfrak{X}_v$ is canonically isomorphic to $\Omega^1_{\mathfrak{X}_v}$, which we have provided with a metric in § 5. Via this isomorphism each $(\omega_{X/B})_v$ acquires an admissible metric, and in this way we make $\omega_{X/B}$ into an admissible line bundle on X.

Theorem 8.4 (Adjunction formula). *Let C be a section of the projection map $X \to B$, and denote its image also by C. Then*

$$(\mathcal{O}_X(C) \,.\, \omega_{X/B} \otimes \mathcal{O}_X(C)) = 0$$

or, equivalently,

$$(C \,.\, C) = -\deg C^*\omega_{X/B}.$$

Proof. Because X is regular, each fibre X_v with $v \in S_{\mathrm{fin}}$ intersects C at a single point, and X_v is regular at that point (see Liu [12], Corollary 9.1.32). The theorem now follows from Theorem 4.1 in Arakelov's paper [2], which does not assume that X is regular but requires that C does not pass through singular points of the fibres. □

Definition. Let C be a reduced curve over a field, and let $\pi\colon \tilde{C} \to C$ be the normalisation of C. Consider the sheaf $\mathcal{S} = \pi_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C$ on C. For every closed point x of C, the stalk $\mathcal{S}_x$ is of finite length over $\mathcal{O}_{C,x}$, and vanishes if x is a regular point of C (see Liu [12], § 7.5). We define

$$\delta_x = \mathrm{length}_{\mathcal{O}_{C,x}} \mathcal{S}_x.$$

A closed point x of C is called an *ordinary double point* if $\pi^{-1}\{x\}$, which is a finite scheme over the residue field $k(x)$ of C at x, has exactly two points over an algebraic closure of $k(x)$ and if $\delta_x = 1$.

Definition. An arithmetic surface X over B is said to be *semi-stable* if for every finite place v of B, the fibre X_v is geometrically reduced and the only singular points of X_v are ordinary double points.

If X is a semi-stable arithmetic surface over B, the relative dualising sheaf $\omega_{X/B}$ 'behaves nicely' (see Liu [12], Lemma 10.3.12). Moreover, if X_K is a smooth geometrically connected projective curve over K, the semi-stable reduction theorem (see Liu [12], Remark 10.4.2 and Theorem 10.4.3) shows that there exists a semi-stable model of X_K over the spectrum of the ring of integers of some finite extension of K. In many situations where $\omega_{X/B}$ plays a role, we therefore restrict ourselves to the case where X is semi-stable. An example of this is Faltings' version of the Riemann–Roch formula:

Theorem 8.5 (Riemann–Roch–Faltings). *Let X be a semi-stable arithmetic surface over B whose generic fibre has positive genus. Then for any admissible line bundle $\mathcal{L}$ on X,*

$$\deg\det \mathrm{R}p_*\mathcal{L} = \frac{1}{2}(\mathcal{L} \,.\, \mathcal{L} \otimes \omega^\vee_{X/B}) + \deg\det p_*\omega_{X/B}.$$

Proof. Faltings [7], Theorem 3; see also Moret-Bailly [14], théorème 6.13, for the formulation used here.

9. The height of a torsion line bundle

Let K be a number field with ring of integers O_K, write $B = \operatorname{Spec} O_K$, and let $p\colon X \to B$ be an arithmetic variety. The K-valued points of the generic fibre X_K correspond bijectively to the O_K-valued points of X, i.e. to the sections of p; this follows from the valuative criterion of properness. Let $S\colon B \to X$ be such a section. For any metrised line bundle $\mathcal{L}$ on X, we define the *height* of S with respect to $\mathcal{L}$ as

$$h_{\mathcal{L}}(S) = \frac{1}{[k(S):\mathbf{Q}]} \deg S^*\mathcal{L},$$

where $k(S)$ is the function field of S and $\deg S^*\mathcal{L}$ denotes the Arakelov degree of the metrised line bundle $S^*\mathcal{L}$ on B. It can be shown that if $i\colon X \to \mathbf{P}^n_B$ is a closed immersion, the height function $h_{\mathcal{L}}$ is equal to the height h_i defined in § 4 if we take $\mathcal{L}$ to be the line bundle $i^*\mathcal{O}_{\mathbf{P}^n_B}(1)$ equipped with a suitable metric (see Szpiro [19], § 3). We extend this idea of heights to arbitrary divisors on X_K by linearity.

Now suppose X is an arithmetic variety whose generic fibre has positive genus. From the definition of the Arakelov intersection product, we see that $S^*\mathcal{L} = (\mathcal{O}_X(S)\,.\,\mathcal{L})$; this gives a very useful interpretation of heights as intersection numbers. In this section we will estimate the height of a torsion line bundle on X_K, or more accurately speaking of a divisor on X possessing an integer multiple which is rationally equivalent to zero.

Height estimates of torsion line bundles play a role in determining the asymptotic running time of a recent algorithm by Edixhoven et al. [5] to compute Galois representations associated to a modular form. The main problem is computing the field of definition of certain torsion points of the Jacobian variety J of the modular curve $X_1(l)$ for a prime number l. This is done numerically, and to estimate the necessary precision for the computations, bounds on the heights of these torsion points are needed. There is a canonical height function on J, the Néron–Tate height, but this is not useful since the Néron–Tate height of a torsion point is zero. Still, height functions in the above sense, i.e. the functions $h_{\mathcal{L}}$ associated to a fixed model of J over B and a metrised line bundle $\mathcal{L}$, can be expected to assume 'small' values at the torsion points, since the difference between any two height functions on a variety over K is bounded (see Szpiro [19], lemmes 3.1–3.4).

The estimation of heights on J is reduced in [5] to the estimation of heights on a semi-stable model of $X_1(l)$ over the ring of integers $\mathbf{Z}[\zeta_l]$ of the cyclotomic field $\mathbf{Q}(\zeta_l)$. In this section and the next, we will describe some aspects of this estimation. We will start by considering a more general situation; we specialise to the case of modular curves in § 10.

Let X be a semi-stable arithmetic surface over B whose generic fibre X_K has genus $g \geq 1$. Let $\mathcal{T}_K$ be a line bundle on X_K whose class in $\operatorname{Pic} X_K$ is a torsion element; in particular, $\deg \mathcal{T}_K = 0$. Suppose we are given a divisor D of degree g on X_K such that $\dim_K \mathrm{H}^0(X_K, \mathcal{T}_K(D)) = 1$ (and hence, by Riemann–Roch, $\mathrm{H}^1(X_K, \mathcal{T}_K(D)) = 0$). Let D' be the divisor of any non-zero global section of $\mathcal{T}_K(D)$; since such a section exists and is unique up to multiplication by an element of $K^\times$, we see that D' is the unique effective divisor on X_K such that $\mathcal{T}_K \cong \mathcal{O}_{X_K}(D' - D)$.

We denote the closure of $\{D\}$ in X by D as well, and similarly for D'. Furthermore, let P be a section of p, i.e. a B-valued point of X. We view the intersection number $(P\,.\,D)$ as the height of D with respect to the line bundle $\mathcal{O}_X(P)$. The problem we are going to attack is the following: we know an estimate for the height of D and we want to derive from this an estimate for the height of D'. In other words, we want to bound the real number

$$(P\,.\,D') - (P\,.\,D) = \deg P^*\mathcal{O}_X(D' - D).$$

Lemma 9.1. *Let $(\mathcal{L}, \|\ \|)$ be an admissible line bundle on X, and let $\mathcal{L}_K$ denote the restriction of $\mathcal{L}$ to X_K. Suppose the class of $\mathcal{L}_K^{\otimes n}$ in the Picard group of X_K is trivial for some integer $n \geq 1$. Then there exists a vertical divisor E on X such that $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(E)$. Furthermore, for any metrised line bundle $\mathcal{M}$ on B, we have*

$$(\mathcal{L}\,.\,p^*\mathcal{M}) = 0.$$

Proof. Let s be a rational section of $\mathcal{L}^{\otimes n}$ such that $s|_{X_K}$ is a trivialising section of $\mathcal{L}_K^{\otimes n}$. Writing E for the divisor of s, we get an isomorphism

$$\begin{aligned}\mathcal{O}_X(E) &\xrightarrow{\sim} \mathcal{L}^{\otimes n}\\ 1 &\longmapsto s.\end{aligned}$$

of admissible line bundles on X. Restricting both sides to the generic fibre X_K, we get isomorphisms

$$\begin{array}{ccccc}\mathcal{O}_X(E)|_{X_K} & \xrightarrow{\sim} & \mathcal{L}_K^{\otimes n} & \xrightarrow{\sim} & \mathcal{O}_{X_K}\\ 1 & \longmapsto & s|_{X_K} & \longmapsto & 1\end{array}$$

of line bundles on X_K, so the support of E does not contain horizontal prime divisors. In other words, E is a vertical divisor. Moreover, the equality

$$(\mathcal{L} \,.\, \mathcal{L}') = \frac{1}{n}(\mathcal{O}_X(E) \,.\, \mathcal{L}')$$

holds for all admissible line bundles $\mathcal{L}'$ on X. In particular, for every metrised line bundle $\mathcal{M}$ on B we have

$$(\mathcal{L} \,.\, p^*\mathcal{M}) = \frac{1}{n}(\mathcal{O}_X(E) \,.\, p^*\mathcal{M}),$$

which vanishes by Lemma 8.1(b). □

We define a metrised line bundle

$$\mathcal{T} = \mathcal{O}_X(D' - D) \otimes p^*P^*\mathcal{O}_X(D' - D)^\vee$$

on X. We will see later (in the proof of Proposition 9.4) that a tensor power of $\mathcal{T}$ is isomorphic to the line bundle $\mathcal{O}_X(\Phi)$, where Φ is a divisor consisting of irreducible components of finite fibres of X which do not intersect P. If X is smooth over B, or more generally if the fibres of X are irreducible, there are no such components; we therefore think of $\mathcal{T}$ as 'almost' a torsion line bundle (note that the restriction of $\mathcal{T}$ to X_K is isomorphic to $\mathcal{T}_K$) and of Φ as a 'correction' for the non-smoothness of X. The following lemma summarises some useful properties of the line bundle $\mathcal{T}$.

Lemma 9.2. *There is a canonical isomorphism*

$$P^*\mathcal{T} \cong \mathcal{O}_B.$$

Furthermore, the identity

$$(\mathcal{T} \,.\, \mathcal{O}_X(E)) = (D' - D \,.\, E)$$

holds for any vertical Arakelov divisor E on X, and we have

$$(\mathcal{T} \,.\, \mathcal{T}) = (\mathcal{T} \,.\, \mathcal{O}_X(D' - D)).$$

Proof. The fact that $p \circ P$ is the identity on B implies that

$$P^*\mathcal{T} = P^*\mathcal{O}_X(D' - D) \otimes P^*\mathcal{O}_X(D' - D)^\vee,$$

which is naturally isomorphic to $\mathcal{O}_B$; this proves the first claim. For the second claim, note that the restriction of $\mathcal{T}$ to X_K is isomorphic (in a non-canonical way) to the torsion line bundle $\mathcal{O}_{X_K}(D' - D)$, since the restriction of $p^*P^*\mathcal{O}_X(D' - D)$ to X_K is trivial. Therefore, Lemma 8.1(b) implies that if E is a vertical divisor, then

$$\begin{aligned}(D' - D \,.\, E) - (\mathcal{T} \,.\, \mathcal{O}_X(E)) &= (p^*P^*\mathcal{O}_X(D' - D) \,.\, E)\\ &= 0.\end{aligned}$$

Finally, let $n \geq 1$ be such that the restriction to $\mathcal{T}^{\otimes n}|_{X_K}$ is trivial, and let E be a vertical divisor with $\mathcal{T}^{\otimes n} \cong \mathcal{O}_X(E)$ as in Lemma 9.1. Then we have

$$\begin{aligned}(\mathcal{T} \,.\, \mathcal{T}) &= \frac{1}{n}(\mathcal{O}_X(E) \,.\, \mathcal{T})\\ &= \frac{1}{n}(E \,.\, D' - D)\\ &= (\mathcal{T} \,.\, \mathcal{O}_X(D' - D)),\end{aligned}$$

which proves the last formula. □

Theorem 9.3. *Let $\omega_{X/B}$ be the relative dualising sheaf of X over B, made into an admissible line bundle as in § 8. Then*

$$2(P\,.\,D'-D) = (\mathcal{O}_X(D')\,.\,\mathcal{O}_X(D')\otimes\omega_{X/B}^\vee) - (\mathcal{O}_X(D)\,.\,\mathcal{O}_X(D)\otimes\omega_{X/B}^\vee) \\ - (\mathcal{T}\,.\,\mathcal{O}_X(D'+D)\otimes\omega_{X/B}^\vee).$$

Proof. To compute the number $(P.\,D'-D) = \deg P^*\mathcal{O}_X(D'-D)$, we apply the projection formula (Proposition 8.3) with $\mathcal{E} = P^*\mathcal{O}_X(D'-D)^\vee$ and $\mathcal{L} = \mathcal{O}_X(D')$. This gives

$$\begin{aligned}\deg\det\mathrm{R}p_*(\mathcal{T}\otimes\mathcal{O}_X(D)) &= \deg\det\mathrm{R}p_*(\mathcal{O}_X(D')\otimes p^*P^*\mathcal{O}_X(D'-D)^\vee) \\ &= \deg\det\mathrm{R}p_*\mathcal{O}_X(D') + \chi(\mathcal{O}_X(D'))\deg P^*\mathcal{O}_X(D'-D)^\vee \\ &= \deg\det\mathrm{R}p_*\mathcal{O}_X(D') - \deg P^*\mathcal{O}_X(D'-D).\end{aligned}$$

On the other hand, the Riemann–Roch–Faltings theorem applied to $\mathcal{T}\otimes\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ gives

$$\begin{aligned}\deg\det\mathrm{R}p_*(\mathcal{T}\otimes\mathcal{O}_X(D)) &= \frac{1}{2}(\mathcal{T}\otimes\mathcal{O}_X(D)\,.\,\mathcal{T}\otimes\mathcal{O}_X(D)\otimes\omega_{X/B}^\vee) + \deg\det p_*\omega_{X/B}, \\ \deg\det\mathrm{R}p_*\mathcal{O}_X(D') &= \frac{1}{2}(\mathcal{O}_X(D')\,.\,\mathcal{O}_X(D')\otimes\omega_{X/B}^\vee) + \deg\det p_*\omega_{X/B}.\end{aligned}$$

Thus we find

$$\begin{aligned}\deg P^*\mathcal{O}_X(D'-D) &= \frac{1}{2}(\mathcal{O}_X(D')\,.\,\mathcal{O}_X(D')\omega_{X/B}^\vee) - \frac{1}{2}(\mathcal{T}\otimes\mathcal{O}_X(D)\,.\,\mathcal{T}\otimes\mathcal{O}_X(D)\otimes\omega_{X/B}^\vee) \\ &= \frac{1}{2}(\mathcal{O}_X(D')\,.\,\mathcal{O}_X(D')\otimes\omega_{X/B}^\vee) - \frac{1}{2}(\mathcal{O}_X(D)\,.\,\mathcal{O}_X(D)\otimes\omega_{X/B}^\vee) \\ &\quad - \frac{1}{2}(\mathcal{T}\,.\,\mathcal{T}\otimes\mathcal{O}_X(2D)\otimes\omega_{X/B}^\vee).\end{aligned}$$

The theorem now follows by applying the formula $(\mathcal{T}\,.\,\mathcal{T}) = (\mathcal{T}\,.\,\mathcal{O}_X(D'-D))$ from Lemma 9.2. □

Remark. The equation of Theorem 9.3 can be proved more easily by noting that it can be rewritten as

$$(\mathcal{O}_X(D'+D-2P)\otimes\omega_{X/B}^\vee\,.\,p^*P^*\mathcal{O}_X(D'-D)) = 0,$$

where we have used that

$$\begin{aligned}(P\,.\,D'-D) &= \deg P^*\mathcal{O}_X(D'-D) \\ &= \deg P^*p^*P^*\mathcal{O}_X(D'-D) \\ &= (\mathcal{O}_X(P)\,.\,p^*P^*\mathcal{O}_X(D'-D)).\end{aligned}$$

The equation now follows from Lemma 8.1(c) since $\mathcal{O}_X(D'+D-2P)\otimes\omega_{X/B}^\vee$ is of degree 0 on each fibre (cf. Liu [12]; $\mathcal{O}_X(D'+D-2P)$ is of degree $2g-2$ on each fibre by Proposition 9.1.30 and $\omega_{X/B}$ has the same property by Proposition 9.1.35). Although the derivation just given is more elementary, we have also retained the first proof, since it illustrates the way in which the Riemann–Roch theorem relates properties of the push-forward of sheaves via the map $X \to B$ to intersection theory on X. This principle has been taken to great height by Grothendieck (see Hartshorne [9], Appendix A).

Each of the terms on the right-hand side in the formula of Theorem 9.3 is estimated from above by Edixhoven and de Jong in §§ 15–18 of the paper [5]. The term $(\mathcal{O}_X(D')\,.\,\mathcal{O}_X(D')\otimes\omega_{X/B}^\vee)$ is rewritten using the Riemann–Roch–Faltings formula:

$$\frac{1}{2}(\mathcal{O}_X(D')\,.\,\mathcal{O}_X(D')\otimes\omega_{X/B}^\vee) = \deg\det\mathrm{R}p_*\mathcal{O}_X(D') - \deg\det p_*\omega_{X/B}.$$

Next, Faltings' characterisation of the determinant of cohomology on a compact Riemann surface $\mathfrak{X}$ as a fibre of the metrised line bundle $\mathcal{O}(-\Theta)$ on the variety $J^{g-1}(\mathfrak{X})$ parametrising divisor classes of degree $g-1$ on $\mathfrak{X}$ (with the metric changed by a factor $\exp(-\delta(\mathfrak{X})/8)$, where $\delta(\mathfrak{X})$ is Faltings'

δ-invariant) is applied, as well as the arithmetic Noether formula (see Moret-Bailly [15]). This leads to the inequality

$$\begin{aligned}
\frac{1}{2}(\mathcal{O}_X(D') \,.\, \mathcal{O}_X(D') \otimes \omega_{X/B}^\vee) = &\sum_{v\in S_{\rm inf}} [K_v : \mathbf{R}] \int_{Q\in\mathfrak{X}_v} \log\|\theta\|_v([D'_v - Q])\mu_v(Q) \\
&+ \frac{1}{2}\deg\det p_*\omega_{X/B} - \frac{1}{8}(\omega_{X/B} \,.\, \omega_{X/B}) \\
&- \frac{1}{8}\sum_{v\in S_{\rm fin}} \delta(X_v)\log\#k_v + \frac{g}{2}[K:\mathbf{Q}]\log 2\pi \\
&- \log\#R^1p_*\mathcal{O}_X(D') \\
\le &\sum_{v\in S_{\rm inf}} [K_v : \mathbf{R}] \sup_{J^{g-1}(\mathfrak{X}_v)} \log\|\theta\|_v + \frac{1}{2}\deg\det p_*\omega_{X/B} \\
&+ \frac{g}{2}[K:\mathbf{Q}]\log 2\pi,
\end{aligned}$$

where we have used that $(\omega_{X/B}, \omega_{X/B}) \ge 0$ (see Faltings [7], Theorem 5). In the above inequality, $\delta(X_v) = \sum_{x\in X_v}[k(x) : k_v]\delta_x$ is the number of singular points of X_v (counted with degrees), μ_v is the canonical $(1,1)$-form on $\mathfrak{X}_v$, and $\|\theta\|_v$ is a real-valued function on $J^{g-1}(\mathfrak{X}_v)$ derived from the θ-function.

In [5], the right-hand side of the above inequality is estimated from above in the case where X is the semi-stable model of a modular curve of the form $X_1(pl)$, with p and l distinct prime numbers. The result is that

$$(\mathcal{O}_X(D') \,.\, \mathcal{O}_X(D') \otimes \omega_{X/B}^\vee) = O((pl)^6) \quad \text{as } pl \to \infty;$$

we do not go into the details, since the methods used are beyond the scope of this thesis.

As for the other terms in the equation of Theorem 9.3, we we will give an estimation of the term $-(\mathcal{O}_X(D) \,.\, \mathcal{O}_X(D) \otimes \omega_{X/B}^\vee)$ in the next section for the case where X is the semi-stable model over $\operatorname{Spec}\mathbf{Z}[\zeta_l]$ of a modular curve $X_1(l)$, with l a prime number such that $X_1(l)$ has genus $g \ge 1$, and where D is an effective divisor of degree g with support in the cusps of the modular curve.

Furthermore, we are now going describe the estimation of the term $-(\mathcal{T} \,.\, \mathcal{O}_X(D'+D) \otimes \omega_{X/B}^\vee)$. There is a curious resemblance to the subject of the first three sections: namely, pursuing a line of thought in [5], we will use bounds on the solutions of the Poisson equation on a graph. We will apply the following proposition.

Proposition 9.4. *Let D^+ and D^- be effective horizontal divisors of degree $d \ge 0$ such that the class of $\mathcal{O}_X(D^+ - D^-)|_{X_K}$ in the Picard group is a torsion element. Let P be a section of $X \to B$, and put*

$$\mathcal{V} = \mathcal{O}_X(D^+ - D^-) \otimes p^*P^*\mathcal{O}_X(D^+ - D^-)^\vee.$$

Then for any admissible line bundle $\mathcal{L}$ on X, we have

$$|(\mathcal{V} \,.\, \mathcal{L})| \le d \sum_{v\in S_{\rm fin}} (\#W_v - 1) \sum_{C\in W_v} |(\mathcal{O}_X(C) \,.\, \mathcal{L})|\,,$$

where W_v is the set of integral components of the fibre X_v for every finite place v of K.

Proof. Choose a positive integer n such that $\mathcal{O}_X(nD^+ - nD^-)|_{X_K}$ is trivial, and let s be a rational section of $\mathcal{O}_X(nD^+ - nD^-)$ such that $s|_{X_K}$ is a trivialising section of $\mathcal{O}_X(nD^+ - nD^-)|_{X_K}$. Furthermore, let t be the rational section $s \otimes p^*P^*s$ of the line bundle

$$\mathcal{V}^{\otimes n} = \mathcal{O}_X(nD^+ - nD^-) \otimes p^*P^*\mathcal{O}_X(nD^+ - nD^-)^\vee;$$

since s is determined up to multiplication by an element of $K^\times$, the section t is independent of the choice of s. Let Φ be the divisor of t. Then we get a canonical isomorphism

$$\begin{aligned}
\mathcal{O}_X(\Phi) &\xrightarrow{\sim} \mathcal{V}^{\otimes n} \\
1 &\mapsto t
\end{aligned}$$

of admissible line bundles on X. Restricting both sides to X_K, from the fact that $t|_{X_K}$ is a canonical trivialising secton of $\mathcal{V}^{\otimes n}|_{X_K}$ we get canonical isomorphisms

$$\begin{array}{ccccc} \mathcal{O}_X(\Phi)|_{X_K} & \xrightarrow{\sim} & \mathcal{V}^{\otimes n}|_{X_K} & \xrightarrow{\sim} & \mathcal{O}_{X_K} \\ 1 & \longmapsto & t|_{X_K} & \longmapsto & 1 \end{array}$$

of line bundles on X_K. Therefore, the support of Φ does not contain horizontal prime divisors. Furthermore, we have a canonical isomorphism

$$\begin{aligned} P^*\mathcal{V}^{\otimes n} &\cong P^*\mathcal{O}_X(nD^+ - nD^-) \otimes P^*\mathcal{O}_X(nD^+ - nD^-)^\vee \\ &\cong \mathcal{O}_B \end{aligned}$$

sending P^*t to 1, from which we conclude that the support of Φ consists of irreducible components of finite fibres of X which do not intersect P.

We write

$$\Phi = \sum_C n\phi(C)C \quad \text{with } \phi(C) \in \frac{1}{n}\mathbf{Z},$$

where C runs over the irreducible components of finite fibres of X. Then we get

$$\begin{aligned} (\mathcal{V} \,.\, \mathcal{L}) &= \frac{1}{n}(\mathcal{O}_X(\Phi) \,.\, \mathcal{L}) \\ &= \sum_C \phi(C)(\mathcal{O}_X(C) \,.\, \mathcal{L}). \end{aligned}$$

In order to estimate $|(\mathcal{V} \,.\, \mathcal{L})|$, we need a bound on the numbers $|\phi(C)|$. We have seen before that $\phi(C) = 0$ for all C which intersect P. Now let v be a closed point of B, and let C be an integral component of X_v. By Lemma 8.1,

$$\begin{aligned} (\mathcal{O}_X(C) \,.\, \mathcal{O}_X(D^+ - D^-)) - (\mathcal{O}_X(C) \,.\, \mathcal{V}) &= (\mathcal{O}_X(C) \,.\, \mathcal{O}_X(D^+ - D^-) \otimes \mathcal{V}^\vee) \\ &= (\mathcal{O}_X(C) \,.\, p^*P^*\mathcal{O}_X(D^+ - D^-)) \\ &= 0. \end{aligned}$$

This implies

$$\begin{aligned} \sum_{C' \in W_v} (C \,.\, C')\phi(C') &= \frac{1}{n}(C \,.\, \Phi) \\ &= (\mathcal{O}_X(C) \,.\, \mathcal{V}) \\ &= (C \,.\, D^+ - D^-). \end{aligned}$$

To bound the numbers $\phi(C)$, we interpret the system of equations

$$\begin{cases} \sum_{C' \in W_v} (C \,.\, C')\phi(C') = (C \,.\, D^+ - D^-) & \text{for all } C \in W_v \\ \phi(C) = 0 & \text{if } P \text{ intersects } C \end{cases} \tag{$*$}$$

as the Poisson equation on a graph in order to apply the estimates from Appendix A. Let Γ be the intersection graph of X_v, i.e. the graph whose vertices are the irreducible components of X_v and where two vertices C and C' are connected by an edge if and only if C and C' are distinct intersecting components of X_v. By the connectedness principle (Hartshorne [9], Corollary III.11.3), this graph is connected. We arbitrarily assign a direction to each edge to make Γ into a directed graph. Furthermore, we fix a metric on Γ (in the sense of Appendix A) by defining $\sigma(a)$ to be the positive integer

$$\sigma(a) = (C \,.\, C')/\log \#k_v$$

for any arrow a connecting two vertices C and C'. Then the function τ defined in Proposition A.1 is given by

$$\tau(C, C') = \begin{cases} (C \,.\, C')/\log \#k_v & \text{if } C \neq C', \\ 0 & \text{if } C = C'. \end{cases}$$

According to Proposition A.1,

$$\begin{aligned}
d^*d\phi(C) &= \sum_{C'\in W_v} \tau(C,C')(\phi(C)-\phi(C')) \\
&= \sum_{C'\neq C} \frac{(C\,.\,C')}{\log\#k_v}(\phi(C)-\phi(C')) \\
&= -\sum_{C'\in W_v} \frac{(C\,.\,C')}{\log\#k_v}\phi(C'),
\end{aligned}$$

where we have used that $\sum_{C'\in W_v}(C\,.\,C') = 0$ for all $C\in W_v$ (see Liu [12], Proposition 9.1.21). Substituting this into our system of equations $(*)$, we see that the function ϕ is a solution of the equation

$$d^*d\phi = \rho,$$

where

$$\rho(C) = -(C\,.\,D^+ - D^-)/\log\#k_v \quad \text{for all } C\in W_v,$$

under the normalising condition that $\phi(C)=0$ for the unique component C which intersects P. In order to estimate the $\phi(C)$, we apply Proposition A.2. Since D^+ and D^- are effective divisors of degree d, the number ρ^+ is bounded by d (with equality if and only if no irreducible component C intersects both D^+ and D^-). Furthermore, the fact that $\tau(C,C')\geq 1$ for all intersecting components C and C' implies that $R(C,C')\leq \#W_v - 1$ for all $C, C'\in W_v$. Therefore,

$$\max_{C\in W_v} |\phi(C)| \leq d(\#W_v - 1),$$

and finally

$$\begin{aligned}
|(\mathcal{V}\,.\,\mathcal{L})| &\leq \sum_{v\in S_{\mathrm{fin}}}\sum_{C\in W_v} |\phi(C)|\,|(\mathcal{O}_X(C)\,.\,\mathcal{L})| \\
&\leq \sum_{v\in S_{\mathrm{fin}}} d(\#W_v - 1)\sum_{C\in W_v} |(\mathcal{O}_X(C)\,.\,\mathcal{L})|,
\end{aligned}$$

which is the inequality we had to prove. $\square$

We can now bound the intersection number $(\mathcal{T}\,.\,\mathcal{O}_X(D'+D)\otimes\omega_{X/B}^\vee)$ occurring in Theorem 9.3. Because $(C\,.\,D'+D)\geq 0$ (since $D'+D$ and C are effective divisors without common components) and $(C\,.\,\omega_{X/B})\geq 0$ (since $X\to B$ is minimal and C is effective; see Liu [12], Corollary 9.3.26),

$$\begin{aligned}
\sum_{C\in W_v} \left|(\mathcal{O}_X(C)\,.\,\mathcal{O}_X(D'+D)\otimes\omega_{X/B}^\vee)\right| &\leq \sum_{C\in W_v}(C\,.\,D'+D) + \sum_{C\in W_v}(C\,.\,\omega_{X/B}) \\
&= (X_v\,.\,D'+D) + (X_v\,.\,\omega_{X/B}).
\end{aligned}$$

The first term equals $\deg_K(D'+D)\log\#k_v = 2g\log\#k_v$ (see Liu [12], Proposition 9.1.30), while the second term equals $(2g-2)\log\#k_v$ ([12], Proposition 9.1.35). Applying Proposition 9.4, we find

$$\left|(\mathcal{T}\,.\,\mathcal{O}_X(D'+D)\otimes\omega_{X/B}^\vee)\right| \leq (4g-2)g\sum_{v\in S_{\mathrm{fin}}}(\#W_v - 1)\log\#k_v,$$

which finishes the estimation of the last term in Theorem 9.3.

10. The case of a modular curve

In this section we study the asymptotic behaviour (as $N \to \infty$) of the intersection number $-(\mathcal{O}_{\mathcal{X}}(D) \,.\, \mathcal{O}_{\mathcal{X}}(D) \otimes \omega^{\vee}_{\mathcal{X}/B})$ appearing in Theorem 9.3, in the case where $\mathcal{X}$ is the semi-stable model over $\mathbf{Z}[\zeta_N]$ of the modular curve $X_1(N)$, where $N \geq 3$ is an integer such that $X_1(N)$ has genus $g \geq 1$, and where D is a divisor of the form $\sum_{i=1}^{g} P_i$ with each $P_i\colon B \to \mathcal{X}$ a cusp of $\mathcal{X}$. At the end we specialise (for simplicity) to the case where N is a prime number. The material in this section serves as an illustration of the method of § 3 and is somewhat more sketchy than the rest of this thesis. Our approach is based on § 18 of Edixhoven's article [5], but we use a different atlas on the modular curves $X(N)$.

We start with the complex analytic description of modular curves; we refer to Miyake's book [13] on modular forms for details.

Let $\mathbf{H} = \{\tau \in \mathbf{C} \mid \Im\tau > 0\}$ be the complex upper half-plane, and define

$$\mathbf{H}^* = \mathbf{H} \sqcup \mathbf{P}^1(\mathbf{Q}) = \mathbf{H} \sqcup \mathbf{Q} \sqcup \{\infty\},$$

made into a Hausdorff space by taking the open subsets of $\mathbf{H}$ together with all sets of the form

$$\begin{aligned} D_\infty(R) &= \{\tau \in \mathbf{H} \mid \Im\tau > R\} \cup \{\infty\} && (R > 0),\\ D_x(r) &= \{\tau \in \mathbf{H} \mid (\Re\tau - x)^2 + (\Im\tau - r)^2 < r^2\} \cup \{x\} && (x \in \mathbf{Q}, r > 0) \end{aligned}$$

as a basis for the topology. Notice that this is a finer topology than the subspace topology from $\mathbf{P}^1(\mathbf{C})$, and that $\mathbf{P}^1(\mathbf{Q})$ has the discrete topology as a subspace of $\mathbf{H}^*$.

The group $\mathrm{SL}_2(\mathbf{Z})$ acts on $\mathbf{H}$ via Möbius transformations:

$$\gamma\tau = \frac{a\tau + b}{c\tau + d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \text{ and } \tau \in \mathbf{H}.$$

This action can be extended in a unique way to a continuous action on $\mathbf{H}^*$, and $\mathbf{P}^1(\mathbf{Q})$ is permuted transitively under this action.

For every positive integer N, the modular curve $X_0(N)$ is defined as the quotient $\Gamma_0(N)\backslash\mathbf{H}^*$, where $\Gamma_0(N)$ is the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

Similary, we define $X_1(N) = \Gamma_1(N)\backslash\mathbf{H}^*$ and $X(N) = \Gamma(N)\backslash\mathbf{H}^*$, where

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; a \equiv d \equiv \pm 1 \pmod{N}, c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; a \equiv d \equiv \pm 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}.$$

The image of $\mathbf{P}^1(\mathbf{Q})$ in a modular curve is a finite set, called the set of *cusps* of the curve. The cusp which is the image of ∞ is also denoted by ∞.

Each of the modular curves $X_0(N)$, $X_1(N)$, $X(N)$ is in a natural way a compact Riemann surface (see Miyake [13], § 1.8). The inclusions

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbf{Z})$$

induce natural ramified coverings

$$X(N) \longrightarrow X_1(N) \longrightarrow X_0(N) \longrightarrow \mathbf{P}^1(\mathbf{C}),$$

where we have identified $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}^*$ with $\mathbf{P}^1(\mathbf{C})$ using the modular j-function

$$\begin{aligned} j\colon \mathbf{H}^* &\longrightarrow \mathbf{P}^1(\mathbf{C})\\ \tau &\longmapsto q^{-1} + 744 + 196884q + \cdots, \end{aligned}$$

where $q = \exp(2\pi i\tau)$. For any modular curve X, the function j induces a function $X \to \mathbf{P}^1(\mathbf{C})$, which is also denoted by j.

We will describe an atlas for $X(N)$ explicitly using discs around the cusps. Although we will ultimatly need to estimate the Arakelov–Green function on $X_1(N)$, computations are easier on $X(N)$ because the ramified covering $j\colon X(N) \to \mathbf{P}^1(\mathbf{C})$ is normal. Let us start with the cusp at infinity. We define a continuous map $z_\infty\colon D_\infty(1/N) \to \mathbf{C}$ by

$$z_\infty(\tau) = \begin{cases} \exp(2\pi i\tau/N + 2\pi/N^2) & \text{if } \tau \neq \infty, \\ 0 & \text{if } \tau = \infty. \end{cases}$$

The image of z_∞ equals the open unit disc. Suppose two points τ and τ' of $D_\infty(1/N)$ are identified by the action of $\Gamma(N)$. Then there is an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$ such that $\gamma\tau = \tau'$. If $c = 0$, then $\tau' = \tau \pm b$; otherwise, c is a non-zero multiple of N and

$$\begin{aligned} \Im\tau' &= \Im\frac{a\tau + b}{c\tau + d} = \frac{\Im\tau}{|c\tau + d|^2} \\ &\leq \frac{\Im\tau}{(c\,\Im\tau)^2} \leq \frac{1}{N^2\,\Im\tau}, \end{aligned}$$

a contradiction since both $\Im\tau$ and $\Im\tau'$ are greater than $1/N$ by assumption. Therefore, two points $\tau, \tau' \in D_\infty(1/N)$ are identified by the quotient map $\mathbf{H}^* \to X(N)$ if and only if $\tau' = \tau + Nk$ for some $k \in \mathbf{Z}$. The same holds for the map z_∞, and by the uniqueness of the quotient space we see that z_∞ induces a homeomorphism from an open subset $U_\infty \subset X(N)$ to the open unit disc in $\mathbf{C}$.

The action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{H}^*$ induces an action on $X(N)$ with kernel $\Gamma(N)$; the action is transitive on the cusps. For every cusp κ we choose an element $\gamma_\kappa \in \mathrm{SL}_2(\mathbf{Z})$ such that $\gamma_\kappa\infty = \kappa$ (with the convention $\gamma_\infty = 1$), and we put

$$U_\kappa = \gamma_\kappa U_\infty, \quad z_\kappa = z_\infty \circ \gamma_\kappa^{-1}.$$

A different choice for γ_κ comes down to multiplying γ_κ on the right by an element of the stabiliser of the cusp ∞, i.e. a matrix of the form $\pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b \in \mathbf{Z}$, and multiplying z_κ by $\exp(\mp 2\pi i b/N)$. In particular, the set U_κ does not depend on the choice of γ_κ.

From now on we suppose that $N > 1$. Fix a real number R with $1/2 < R < \sqrt{3}/2$, let U'_∞ be the image of $D_\infty(R)$ in $X(N)$, and write $U'_\kappa = \gamma_\kappa U'_\infty$ for every cusp κ. Note that the assumption $N > 1$ implies $U'_\kappa \subset U_\kappa$.

Lemma 10.1. *The open sets U'_κ, with κ running over the cusps of $X(N)$, cover $X(N)$. For every cusp κ, the image of U'_κ under the map z_κ is the disc*

$$\bigl\{w \in \mathbf{C} \bigm| |w| < r_1(N)\bigr\},$$

where

$$r_1(N) = \exp(-2\pi R/N + 2\pi/N^2).$$

Proof. To prove the first assertion it is enough to show that the open sets $\gamma D_\infty(R)$ with $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ cover $\mathbf{H}^*$. This follows directly from the fact that the standard fundamental domain

$$\{\tau \in \mathbf{H} \mid -1/2 \leq \Re\tau \leq 0 \text{ and } |\tau| \geq 1\} \cup \{\tau \in \mathbf{H} \mid 0 < \Re\tau < 1/2 \text{ and } |\tau| > 1\} \cup \{\infty\}$$

for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{H}^*$ is contained in $D_\infty(R)$ since $R < \sqrt{3}/2$. The second claim follows from the definition of z_κ. ☐

Lemma 10.2. *For every integer $N \geq 3$, the number of cusps of $X(N)$ equals $\frac{1}{2}N^2\prod_{p|N}(1 - p^{-2})$, where p runs over the prime divisors of N.*

Proof. Miyake [13], Theorem 4.2.10.

Lemma 10.3. *For any two cusps κ and λ, we have*

$$\left|\frac{dz_\kappa}{dz_\lambda}\right| \le N^2 \exp\bigl(2\pi(1-1/N^2)\bigr) \quad \textit{on } U_\kappa \cap U_\lambda.$$

Proof. This is obvious in the case $\kappa = \lambda$. Otherwise, we first apply γ_λ^{-1} to reduce to the case $\lambda = \infty$. Then for all $\tau \in D_\infty(1/N) \cap \gamma_\kappa \cdot D_\infty(1/N)$, we have

$$\begin{aligned} dz_\infty(\tau) &= \frac{2\pi i}{N} \exp\left(\frac{2\pi i}{N}\tau + \frac{2\pi}{N^2}\right) d\tau, \\ dz_\kappa(\tau) &= \frac{2\pi i}{N} \exp\left(\frac{2\pi i}{N}\gamma_\kappa^{-1}\tau + \frac{2\pi}{N^2}\right) d(\gamma_\kappa^{-1}\tau), \end{aligned}$$

so that

$$\frac{dz_\kappa}{dz_\infty}(\tau) = \exp\left(\frac{2\pi i}{N}(\gamma_\kappa^{-1}\tau - \tau)\right) \frac{d(\gamma_\kappa^{-1}\tau)}{d\tau}.$$

From this it follows by a simple calculation that for $\gamma_\kappa^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \ne 0$, we have

$$\left|\frac{dz_\kappa}{dz_\infty}\right|(\tau) = \frac{1}{|c\tau + d|^2} \exp\left(\frac{2\pi}{N}\bigl(\Im\tau - \Im(\gamma_\kappa^{-1}\tau)\bigr)\right).$$

Now both τ and $\gamma_\kappa^{-1}\tau$ are in $D_\infty(1/N)$, so we have $\Im\tau > 1/N$ and $\Im(\gamma_\kappa^{-1}\tau) > 1/N$; furthermore,

$$\begin{aligned} \Im(\gamma_\kappa^{-1}\tau) &= \Im\frac{a\tau + b}{c\tau + d} = \frac{\Im\tau}{|c\tau + d|^2} \\ &\le \frac{\Im\tau}{(\Im\tau)^2} < N, \end{aligned}$$

and similarly we find $\Im\tau < N$. From this we see that

$$\left|\frac{dz_\kappa}{dz_\infty}\right|(\tau) < N^2 \exp\left(\frac{2\pi}{N}(N - 1/N)\right),$$

which proves the claim. $\square$

Proposition 10.4. *Let S be an infinite set of integers greater than 2. For all $N \in S$, let μ_N be a smooth real-valued $(1,1)$-form on $X(N)$ satisfying $\int_{X(N)} \mu_N = 1$. Suppose that for each $N \in S$ a real number $c(N) > 0$ is given such that for any cusp κ of $X(N)$, the function f_κ on the disc U_κ around the cusp κ such that*

$$\mu = i f_\kappa\, dz_\kappa \wedge d\bar z_\kappa$$

satisfies

$$0 \le f_\kappa(x) < c(N) \quad \textit{for all } x \in U_\kappa.$$

Then we have

$$\sup_{X(N)\times X(N)\setminus\Delta} g_{\mu_N} = O\bigl(N^{7/2}\log N + N^2 c(N)\bigr) \quad \textit{as } N \to \infty,$$

and for all cusps κ and all $x \ne y \in U'_\kappa$ we have

$$\bigl|g_{\mu_N}(x,y) - \log|z_\kappa(x) - z_\kappa(y)|\bigr| = O\bigl(N^{7/2}\log N + N^2 c(N)\bigr) \quad \textit{as } N \to \infty.$$

Proof. As before, we fix a real number R with $1/2 < R < \sqrt{3}/2$, and for all $N \in S$ and every cusp κ of $X(N)$ we write U'_κ for the image of the set $\gamma_\kappa D_\infty(R)$ under the natural map $\mathbf{H}^* \to X(N)$, where γ_κ is any element of $\mathrm{SL}_2(\mathbf{Z})$ with $\gamma_\kappa\infty = \kappa$. For every $N \in S$ we apply Theorem 3.1 with the following parameters, provided by Lemmata 10.1, 10.2 and 10.3:

$$\begin{aligned} n &= \frac{1}{2}N^2 \prod_{p|N}(1 - p^{-2}), \\ r_1 &= \exp(-2\pi R/N + 2\pi/N^2), \\ M &= N^2 \exp(2\pi(1 - 1/N^2)), \\ c_1 &= c(N). \end{aligned}$$

It is clear that
$$n = O(N^2) \quad \text{and} \quad M = O(N^2) \qquad \text{as } N \to \infty,$$
and a simple calculation gives
$$\frac{1}{1-r_1} = O(N) \quad \text{as } N \to \infty.$$
Therefore, Theorem 3.1 gives the bounds
$$\begin{aligned}\sup_{X(N)\times X(N)\setminus\Delta} g_{\mu_N} &\le \frac{Cn}{(1-r_1)^{3/2}} \log\frac{1}{1-r_1} + \left(\frac{8}{3}\log 2 + \frac{1}{4}\right) nc_1 + \frac{n-1}{2\pi}\log M \\ &= O\bigl(N^2 \cdot N^{3/2}\log N + N^2 c(N) + N^2\log(N^2)\bigr) \\ &= O\bigl(N^{7/2}\log N + N^2 c(N)\bigr) \quad \text{as } N \to \infty\end{aligned}$$
and
$$\bigl|g_{\mu_N}(x,y) - \log|z_\kappa(x) - z_\kappa(y)|\bigr| = O\bigl(N^{7/2}\log N + N^2 c(N)\bigr) \quad \text{as } N \to \infty$$
for all cusps κ and all $x \ne y \in U'_\kappa$. □

Remark. To improve the above bound, it could be advantageous to take smaller discs U_κ (but the same U'_κ) when more is known about the asymptotic behaviour of $c(N)$. Although this has the effect that the parameter r_1 gets closer to 1, thereby increasing the exponent 7/2, this does not have to be a problem if $c(N)$ can be estimated more sharply on smaller discs.

Corollary 10.5. *Let S be the set of integers $N \ge 3$ such that $X_1(N)$ has positive genus. For each $N \in S$, define*
$$\mu_N = N^{-1} h^* \mu_{\mathrm{Ar}},$$
where μ_{Ar} is the canonical $(1,1)$-form on $X_1(N)$ and where h is the natural map $X(N) \to X_1(N)$. Then μ_N satisfies $\int_{X(N)} \mu_N = 1$. Let $c(N)$ be as in Proposition 10.4. Then
$$\sup_{X_1(N)\times X_1(N)\setminus\Delta} g_{\mu_{\mathrm{Ar}}} = O\bigl(N^{9/2}\log N + N^3 c(N)\bigr) \quad \text{as } N \to \infty$$
and
$$\bigl|\log\|dq\|(\infty)\bigr| = O\bigl(N^{9/2}\log N + N^3 c(N)\bigr) \quad \text{as } N \to \infty,$$
where the first ∞ denotes the cusp at infinity of $X_1(N)$.

Proof. A straightforward check using Lemma 2.3 shows that for all $a \in X_1(N)$ and all $x \in X(N)$,
$$g_{a,\mu_{\mathrm{Ar}}}(h(x)) = \sum_{\substack{b\in X(N)\\ h(b)=a}} e(b) g_{b,\mu_N}(x),$$
where $e(b)$ is the ramification index of h at b. The first claim now follows from Proposition 10.4 since $\sum_{h(b)=a} e(b) = N$ for all a. Furthermore, since h is totally ramified at ∞,
$$g_{\mathrm{Ar}}(\infty, h(x)) = g_{\infty,\mu_{\mathrm{Ar}}}(h(x)) = N g_{\infty,\mu_N}(x) = N g_{\mu_N}(\infty, x).$$
From the description of the metric on the sheaf of holomorphic differentials in § 5, we get
$$\log\|dq\|(\infty) = \lim_{x\to\infty}(\log|q(x)| - g_{\mathrm{Ar}}(\infty, x)).$$
Now $q = z_\infty^N \exp(-2\pi/N)$; letting x tend to the cusp ∞ in $X(N)$ this implies
$$\begin{aligned}\log\|dq\|(\infty) &= \lim_{h(x)\to\infty}\bigl(\log|q(h(x))| - g_{\mathrm{Ar}}(\infty, h(x))\bigr) \\ &= \lim_{x\to\infty}\bigl(N\log|z_\infty(x)| - 2\pi/N - N g_{\mu_N}(\infty, x)\bigr),\end{aligned}$$
and the second claim now follow from Proposition 10.4. □

Remark. In order to find a more explicit bound for the asymptotic behaviour of the Arakelov–Green function on $X(N)$ as $N \to \infty$, we should also estimate $c(N)$. Unfortunately, this does not appear to be easy. In the case where $N = pl$, where p and l are distinct odd prime numbers, we can slightly adapt the proof of Lemma 18.2.8 in Edixhoven's paper [5] to prove that $c(N) = O(N^8)$. We do not give the details, since that would expand this thesis too much; therefore, we leave an unknown expression $c(N)$ in our estimates.

Let $N \geq 3$ be a squarefree integer such that the modular curve $X_1(N)$ has positive genus. Consider the number field $K = \mathbf{Q}(\zeta_N)$ with ζ_N a primitive N-th root of unity; it is well known that the ring of integers of K is $\mathbf{Z}[\zeta_N]$. Suppose we have a semi-stable model $\mathcal{X}$ of $X_1(N)$ over $B = \operatorname{Spec} \mathbf{Z}[\zeta_N]$. If N is a prime number, such a model is provided by the paper [3] by Deligne and Rapoport.† We want to estimate the term $-(D \,.\, D - \omega_{\mathcal{X}/B})$ from Theorem 9.3 in the case where D is an effective divisor of degree g with support in the cusps of $X_1(N)$, using $\mathcal{X} \to B$ as our arithmetic surface. We write

$$D = \sum_{i=1}^{g} P_i,$$

where the P_i are (not necessarily distinct) cusps of $X_1(N)$.

Lemma 10.6. *For all i, j we have*

$$-(\mathcal{O}_{\mathcal{X}}(D) \,.\, \mathcal{O}_{\mathcal{X}}(D) \otimes \omega_{\mathcal{X}/B}^{\vee}) = (g+1) \sum_{i=1}^{g} (\mathcal{O}_{\mathcal{X}}(P_i) \,.\, \omega_{\mathcal{X}/B}) + \sum_{i<j} (\mathcal{T}_{i,j} \,.\, \mathcal{O}_{\mathcal{X}}(P_i - P_j)).$$

where $\mathcal{T}_{i,j}$ is the admissible line bundle

$$\mathcal{T}_{i,j} = \mathcal{O}_{\mathcal{X}}(P_i - P_j) \otimes p^* P^* \mathcal{O}_{\mathcal{X}}(P_i - P_j)^{\vee}$$

on $\mathcal{X}$.

Proof. Since $\mathcal{O}_{\mathcal{X}}(P_i - P_j)$ has degree 0 on each fibre (see Liu [12], Proposition 9.1.30), Lemma 8.1 implies that

$$\begin{aligned}(\mathcal{O}_{\mathcal{X}}(P_i - P_j) \otimes \mathcal{T}_{i,j}^{\vee} \,.\, \mathcal{O}_{\mathcal{X}}(P_i - P_j)) &= (p^* P^* \mathcal{O}_{\mathcal{X}}(P_i - P_j) \,.\, \mathcal{O}_{\mathcal{X}}(P_i - P_j)) \\ &= 0.\end{aligned}$$

We rewrite this as

$$(P_i \,.\, P_j) = \frac{1}{2}(P_i \,.\, P_i) + \frac{1}{2}(P_j \,.\, P_j) - \frac{1}{2}(\mathcal{T}_{i,j} \,.\, \mathcal{O}_{\mathcal{X}}(P_i - P_j)).$$

The desired formula now follows by writing out $-(\mathcal{O}_{\mathcal{X}}(D) . \mathcal{O}_{\mathcal{X}}(D) \otimes \omega_{\mathcal{X}/B}^{\vee})$ and applying the above formula for $(P_i \,.\, P_j)$ and the adjunction formula. □

For simplicity, we restrict ourselves in the following theorem to the case where N is a prime number, so that we can use the results on the semi-stable model $\mathcal{X}$ of $X_1(N)$ in the article [3] of Deligne and Rapoport (we use the scheme which is denoted by $\mathfrak{M}_{\Gamma'_{00}(p)}$ in [3], with $p = N$).

Theorem 10.7. *For N running through the odd prime numbers such that $X_1(N)$ has positive genus, we have*

$$-(\mathcal{O}_{\mathcal{X}}(D) \,.\, \mathcal{O}_{\mathcal{X}}(D) \otimes \omega_{\mathcal{X}/B}^{\vee}) = O\bigl(N^{15/2} \log N + N^6 c(N)\bigr) \quad \text{as } N \to \infty.$$

Proof. We claim that for any cusp P of $\mathcal{X}$,

$$(\mathcal{O}_{\mathcal{X}}(P) \,.\, \omega_{\mathcal{X}/B}) = O\bigl(N^{13/2} \log N + N^5 c(N)\bigr) \quad \text{as } N \to \infty.$$

After applying a modular automorphism of $\mathcal{X}$, we may assume that P is the cusp at infinity. The line bundle $P^* \omega_{\mathcal{X}/B}$ on B is free and generated by $P^* d(1/j)$, where $j \colon \mathcal{X} \to \mathbf{P}^1_B$ is the modular j-function (see Deligne and Rapoport [3]). By the definition of the Arakelov intersection pairing and the fact that $\frac{d(1/j)}{dq}(\infty) = 1$ for every $v \in S_{\mathrm{inf}}$, we get

$$\begin{aligned}(\mathcal{O}_{\mathcal{X}}(P) \,.\, \omega_{\mathcal{X}/B}) &= \deg P^* \omega_{\mathcal{X}/B} \\ &= -\sum_{v \in S_{\mathrm{inf}}} [K_v : \mathbf{R}] \log \|P^* d(1/j)\|_v \\ &= -\sum_{v \in S_{\mathrm{inf}}} [K_v : \mathbf{R}] \log \|P^* dq\|_v \\ &= -\sum_{v \in S_{\mathrm{inf}}} [K_v : \mathbf{R}] \log \|dq\|_v(\infty).\end{aligned}$$

† Katz and Mazur [11] give similar results for general N, but I haven't studied these.

Using Corollary 10.5 and the fact that $\sum_{v\in S_{\mathrm{inf}}}[K_v : \mathbf{R}] = [\mathbf{Q}(\zeta_N) : \mathbf{Q}] < N$, we obtain

$$(\mathcal{O}_{\mathcal{X}}(P) \,.\, \omega_{\mathcal{X}/B}) = O\bigl(N^{11/2}\log N + N^4 c(N)\bigr) \quad \text{as } N \to \infty.$$

The genus of $X_1(N)$ is $O(N^2)$ (see Miyake [13], Theorems 4.2.5 and 4.2.11), so we find

$$(g+1)\sum_{i=1}^{g}(P_i \,.\, \omega_{\mathcal{X}/B}) = O\bigl(N^{15/2}\log N + N^6 c(N)\bigr) \quad \text{as } N \to \infty.$$

Finally, we estimate the term $\sum_{i<j}(\mathcal{T}_{i,j} \,.\, \mathcal{O}_{\mathcal{X}}(P_i - P_j))$. By the Manin–Drinfeld theorem (Drinfeld [4], Theorem 1), the restriction of $\mathcal{O}_{\mathcal{X}}(P_i - P_j)$ to $\mathcal{X}_K$ is a torsion line bundle. Using Proposition 9.4, we find

$$\begin{aligned}|(\mathcal{T}_{i,j} \,.\, \mathcal{O}_{\mathcal{X}}(P_i - P_j))| &\le \sum_{v\in S_{\mathrm{fin}}}(\#W_v - 1)\sum_{C\in W_v}|(C \,.\, P_i - P_j)|\\ &\le 2\sum_{v\in S_{\mathrm{fin}}}(\#W_v - 1)\log \#k_v.\end{aligned}$$

According to Deligne and Rapoport [3], théorème V.2.12, the arithmetic surface $\mathcal{X}$ is smooth over $\mathbf{Z}[\zeta_N, 1/N]$, and the fibre above the point $(N) \in \operatorname{Spec}\mathbf{Z}[\zeta_N]$ has two irreducible components intersecting transversally. Therefore,

$$\sum_{i<j}(\mathcal{T}_{i,j} \,.\, P_i - P_j) = O(g^2\log N) = O(N^4\log N) \quad \text{as } N \to \infty,$$

and by Lemma 10.6 we get

$$-(\mathcal{O}_{\mathcal{X}}(D) \,.\, \mathcal{O}_{\mathcal{X}}(D)\otimes\omega_{\mathcal{X}/B}^{\vee}) = O\bigl(N^{15/2}\log N + N^6 c(N)\bigr) \quad \text{as } N \to \infty,$$

which ends the proof of Theorem 10.7. □

Appendix A. The Poisson equation on a graph

Let Γ be a finite connected directed graph given by a set V of vertices, a set A of arrows, and two maps $s, t\colon A \to V$, the *source* and *target* maps (i.e. each arrow $a \in A$ goes from the vertex $s(a)$ to the vertex $t(a)$). We define a linear map $d\colon \mathbf{R}^V \to \mathbf{R}^A$ by

$$d\phi = \phi \circ t - \phi \circ s,$$

so that for every function $\phi \in \mathbf{R}^V$, the change in ϕ across an arrow $a \in A$ equals $d\phi(a)$.

Suppose σ is a *metric* on Γ, i.e. a function $\sigma\colon A \to \mathbf{R}$ taking strictly positive values.† We equip $\mathbf{R}^V$ with the standard inner product $\langle\ ,\ \rangle$ and $\mathbf{R}^A$ with the inner product $\langle\ ,\ \rangle$ defined by

$$\langle f, g\rangle = \sum_{a\in A} \sigma(a) f(a) g(a).$$

Let d^* be the adjoint of d with respect to these inner products, i.e. the unique linear map $\mathbf{R}^A \to \mathbf{R}^V$ such that

$$\langle d^* f, \phi\rangle = \langle f, d\phi\rangle \quad \text{for all } f \in \mathbf{R}^A,\ \phi \in \mathbf{R}^V.$$

We consider the Laplace operator on Γ; this is the self-adjoint operator

$$d^* d\colon \mathbf{R}^V \to \mathbf{R}^V.$$

The kernel of d^*d contains the kernel of d; conversely, if $\phi \in \mathbf{R}^V$ is annihilated by d^*d, then

$$0 = \langle \phi, d^* d\phi\rangle = \langle d\phi, d\phi\rangle,$$

so $d\phi = 0$ since $\langle\ ,\ \rangle$ is positive definite. Because Γ is connected, the kernel of d (and of d^*d) consists of the constant functions. From the fact that d^*d is self-adjoint and the finiteness of V it follows that for any function $\rho \in \mathbf{R}^V$, the Poisson equation $d^*d\phi = \rho$ has a solution ϕ and only if $\langle \rho, 1\rangle = 0$. The function ϕ, if it exists, is unique up to addition of a constant function.

Proposition A.1. *For all v, $w \in V$, let $\tau(v, w)$ be the real number*

$$\tau(v, w) = \sum_a \sigma(a),$$

where a runs over the arrows connecting v and w (in either direction). Then the Laplace operator is given by

$$d^* d\phi(v) = \sum_{w\in V} \tau(v, w)(\phi(v) - \phi(w)).$$

Proof. A straightforward computation shows that the operator d^* is given by

$$d^* f(v) = \sum_{\substack{a\in A\\ t(a)=v}} \sigma(a) f(a) - \sum_{\substack{a\in A\\ s(a)=v}} \sigma(a) f(a).$$

Using this, another short computation gives

$$d^* d\phi(v) = \sum_{\substack{a\in A\\ t(a)=v}} \sigma(a)\bigl(\phi(v) - \phi(s(a))\bigr) + \sum_{\substack{a\in A\\ s(a)=v}} \sigma(a)\bigl(\phi(v) - \phi(t(a))\bigr),$$

from which the stated formula follows. □

† We do not use σ as a distance. Intuitively, we can view Γ as an electric circuit consisting of resistors, where resistor a has resistance $1/\sigma(a)$. The functions denoted by ϕ correspond to potentials, so that $d\phi(a)$ is the potential difference over resistor a.

For every vertex v, we define a function $\delta_v \in \mathbf{R}^V$ by $\delta_v(w) = 1$ if $v = w$ and $\delta_v(w) = 0$ if $v \neq w$. Fix a vertex $v_0 \in V$. For any two vertices $v, w \in V$, let $\phi_{v,w}$ be the unique solution of $d^*d\phi_{v,w} = \delta_v - \delta_w$ satisfying $\phi_{v,w}(v_0) = 0$. First suppose v and w are connected by at least one arrow. Then

$$\begin{aligned}\phi_{v,w}(v) - \phi_{v,w}(w) &= \langle \phi_{v,w}, \delta_v - \delta_w \rangle = \langle \phi_{v,w}, d^*d\phi_{v,w} \rangle \\ &= \langle d\phi_{v,w}, d\phi_{v,w} \rangle = \sum_{b \in A} \sigma(b)\, d\phi_{v,w}(b)^2 \\ &\leq \tau(v,w)(\phi_{v,w}(v) - \phi_{v,w}(w))^2,\end{aligned}$$

from which it follows that

$$0 \leq \phi_{v,w}(v) - \phi_{v,w}(w) \leq 1/\tau(v,w).$$

From the expression for d^*d from Proposition A.1, we see that if x is a vertex distinct from v and w, so that $d^*d\phi_{v,w}(x) = 0$, then $\phi_{v,w}(x)$ is a weighted average of the values of ϕ in the points connected to x. This implies that $\phi_{v,w}$ attains its extrema at v and w, and therefore

$$|\phi_{v,w}(x) - \phi_{v,w}(y)| \leq 1/\tau(v,w) \quad \text{for all } x, y \in V.$$

Definition. A *chain* of length n in Γ is a sequence $(v_0, \ldots, v_n)$ of vertices such that v_{i-1} and v_i are connected by an arrow for $i = 1, 2, \ldots, n$. For any two vertices v and w, we define

$$R(v,w) = \min_{(v_0,\ldots,v_n)} \sum_{i=1}^{n} 1/\tau(v_{i-1}, v_i).$$

where the minimum is taken over all chains $(v_0, \ldots, v_n)$ with $n \geq 0$, $v_0 = v$ and $v_n = w$.

Now let v and w be arbitrary vertices, and choose a chain $(v = v_0, v_1, \ldots, v_n = w)$ for which $\sum_{i=1}^{n} 1/\tau(v_{i-1}, v_i)$ equals $R(v,w)$. Then we have

$$\phi_{v,w} = \sum_{i=1}^{n} \phi_{v_{i-1},v_i},$$

and for any pair of vertices $x, y \in V$ this implies that

$$\begin{aligned}\bigl|\phi_{v,w}(x) - \phi_{v,w}(y)\bigr| &\leq \sum_{i=1}^{n} \bigl|\phi_{v_{i-1},v_i}(x) - \phi_{v_{i-1},v_i}(y)\bigr| \\ &\leq \sum_{i=1}^{n} 1/\tau(v_{i-1}, v_i) \\ &= R(v,w).\end{aligned}$$

Proposition A.2. *Let Γ be a finite connected directed graph with set of vertices V, equipped with a metric σ. Let $\rho \in \mathbf{R}^V$ be a function with $\sum_{v \in V} \rho(v) = 0$. We define*

$$\rho^+ = \sum_{\substack{v \in V \\ \rho(v) > 0}} \rho(v) \quad \Biggl(= - \sum_{\substack{v \in V \\ \rho(v) < 0}} \rho(v)\Biggr).$$

Then any solution ϕ to the Poisson equation $d^*d\phi = \rho$ satisfies

$$\phi(v) - \phi(w) \leq \rho^+ R(v,w) \quad \text{for all } v, w \in V.$$

In particular, if ϕ is normalised in such a way that it assumes both non-positive and non-negative values, then

$$\max_{v \in V} |\phi(v)| \leq \rho^+ \max_{v,w \in V} R(v,w).$$

Proof. For any two vertices v and w, we have

$$\begin{aligned}\phi(v) - \phi(w) &= \langle \phi, \delta_v - \delta_w \rangle = \langle \phi, d^*d\phi_{v,w} \rangle \\ &= \langle d^*d\phi, \phi_{v,w} \rangle = \langle \rho, \phi_{v,w} \rangle.\end{aligned}$$

Because $\phi_{v,w}$ attains its maximum and minimum in v and w, respectively, we can bound this as

$$\begin{aligned}\langle \rho, \phi_{v,w} \rangle &\leq \rho^+(\phi_{v,w}(v) - \phi_{v,w}(w)) \\ &\leq \rho^+ R(v,w),\end{aligned}$$

which proves the first inequality. The second inequality follows immediately from the first. $\square$

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