The LIBOR market model

Pietersz, R.

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Master's thesis

R. Pietersz

Leiden University[1]

and

ABN AMRO Bank N.V.
Market Risk Modeling and Product Analysis[2]

October 6, 2003

[1]Mathematical Institute, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands.
[2]Gustav Mahlerlaan 10, Postbus 283, 1000 EA Amsterdam, The Netherlands.

This Thesis has been submitted for the Master's Degree in Mathematics at the University of Leiden. Research for this Thesis took place during an internship at ABN AMRO Bank N.V., Market Risk Modeling and Product Analysis, from the 1st of January 2001 till the 1st of July 2001. The internship at the bank was under guidance of Drs. D. J. Boswinkel and Prof. Dr. A. C. F. Vorst. The supervisor of the Thesis was Prof. Dr. S. M. Verduyn Lunel, University of Leiden.

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Figure 1: A typical log-normal density function.

1 Introduction

A few years ago, a new model for valuing interest rate derivatives was introduced by Brace, Gątarek and Musiela (First as a working paper, 1995, School of Mathematics, University of New South Wales, later as [BGM97].), Jamshidian ([Jam97]) and Miltersen, Sandmann and Sondermann ([MSS97]). Almost surely, this model had been known and had been used in practice before these papers were published[1].

This model is generally named as "BGM", "BGM/J", referring to the above stated authors, or LIBOR market model, "LMM". In this thesis it will be referred to as LMM.

Here LIBOR stands for London Inter-Bank Offer Rate. Within the LMM the variables that are modeled are the LIBOR forward rates, which are directly observable from the market. This is in contrast with earlier models, which modeled unobservable variables (e.g. short rate models). Moreover, the LMM is engineered in such a way that forward rates are log-normally distributed, which is in line with current market practice for quoting cap prices using the Black formula. See also Figure 1.

This thesis presents the theory of the LMM as well as practical issues arising with a computer implementation. Also, a novel extension is made to incorporate the market observed so-called "volatility smile" into the LMM, utilizing the concept of forward rate dependent instantaneous volatility. The thesis ends with presenting results of some empirical tests to illustrate the performance of the LMM and smile-adjusted LMM.

2 General workings of the LIBOR market model

The LMM is a tool to price and hedge interest rate derivatives. The LMM does that by modeling the interest rate market, i.e., the LMM assumes certain market behavior, thereby creating a hypothetical LMM world. Within this hypothetical world, interest rate derivatives can be hedged and replicated exactly using the basic underlying securities, namely bonds. Also, the LMM has to be "fine-tuned" or so called "calibrated" to prevailing market conditions, i.e., the control

[1] See the discussion in [Reb98], Section 18.1.

Figure 2: Implementation structure of the LIBOR market model.

knobs of the LMM have to be set in such a way that the LMM internal model values match the actual market values as close as possible. This is called the *calibration* process. The calibration requires market data.

Let us look at the pricing of an interest rate derivative within a computer implementation of the LMM. Such an implementation basically consists of three parts, namely

(i) *Calibration.* The calibration part adjusts the parameters of the LMM as to minimize the difference between LMM internal model values and actual prevailing market values. The user has to specify to which values should be calibrated. The calibration part requires market data. When finished optimizing, it will pass its optimal parameters onto the pricer part. Several different calibrations are available.

    The calibration part is described in Section 4.

(ii) *Pricer.* The pricer part approximates the general formula (2) to compute prices of interest rate derivatives. It needs the time zero LIBOR forward rates, the parameters provided by the calibration part and it requires information from the derivative (The pricer part has to be able to obtain the payoff of the derivative for any market scenario the pricer part may wish to specify.). In the case of the LMM, the pricer part is either an analytic formula or a Monte Carlo (MC) simulation. In other areas (e.g. equity) numerical solvers for partial differential equations (PDEs) are used as well for the pricer part, but up to now, PDEs are unsuited for the LMM.

    The pricer part is described in Section 6.

(iii) *Derivative.* The derivative part returns the derivative-payoff given a certain market scenario specified by the pricer part.

    Some derivatives are described in Section 7.

See Figure 2. In general any pricer may be coupled to any calibration and any derivative, but there are some derivatives that require specific pricers with specific calibrations.

3 LIBOR market model theory

Within the LIBOR market model, all pricing is done using LIBOR forward rates only. For example, payoffs of interest rate derivatives are written in terms of forward rates and the forward rates themselves are modeled as geometric Brownian motions.

LIBOR forward rates are not traded in markets; one cannot go out and buy an amount of LIBOR forward rates. However, arbitrage derivatives pricing theory is based on hedging with tradable assets, e.g. bonds.

To solve for this, the LMM specifies equations of motion for the bond price processes. From these, equations are derived for the forward rates. Specifying the instantaneous volatility for the forward rates will then lead to conditions on the bond price equations. The resulting bond price dynamics will constitute the LMM pricing foundation.

Continuing, specific measures (the spot LIBOR measure and the terminal LIBOR measures) are calculated. The arbitrage pricing theory then tells us that prices of derivatives are given by the expected value under a particular LIBOR measure of the discounted payoff of the derivative. Lastly, the driving equations of the LIBOR forward rates under the various measures are calculated.

This whole exercise will then lead to the following situation: Roughly speaking, prices of derivatives are the expectation of the payoff. The payoff is written completely in terms of LIBOR forward rates. The equations governing the forward rates (under some LIBOR measure) are known as well. As a result the bond price processes can be completely forgotten about; the only rates that are dealt with are the forward rates. One always has to remember though that the pricing is based on a hedge with the underlying assets: bonds.

In Subsection 3.1 the general theory of pricing derivatives is reviewed. In Subsection 3.2 the LMM is introduced. Subsection 3.3 contains the no-arbitrage assumption for the LIBOR market model. In Subsection 3.4 useful measures and numeraires are defined. It also contains derivations of stochastic differential equations (SDEs) which the forward rates satisfy under the various measures. The final Subsection 3.5 gives a brief summary of the LMM.

3.1 Markets and general pricing of derivatives

[2] Consider a market $\mathcal{M}$ in which N assets are traded continuously from time 0 up to time T. There is uncertainty as to what the future prices of the assets will be. This uncertainty will be modeled through a d-dimensional Brownian motion W defined on its canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the filtration $\mathbb{F} = \{\mathcal{F}(t) : 0 \leq t \leq T\}$ to be the augmentation of the natural filtration generated by the Brownian motion, i.e., $\mathcal{F}(t)$ is the σ-field generated by $\sigma(W(s) : 0 \leq s \leq t)$ and the null-sets of $\mathcal{F}$. Asset i has price $B_i(t)$ at time t, $0 \leq t \leq T$, and the price process $B_i(\cdot)$ is assumed to be a positive Itô diffusion, i.e., $B_i(\cdot)$ is assumed to

[2] This Section has been based upon the first chapter of [Bjö96].

satisfy the stochastic differential equation

$$
\begin{aligned}
\frac{\mathrm{d}B_i(t)}{B_i(t)} &= \mu_i(t)\mathrm{d}t + \beta_i(t)\cdot \mathrm{d}W(t) \\
&= \mu_i(t)\mathrm{d}t + \sum_{j=1}^{d} \beta_{ij}(t)\mathrm{d}W_j(t), \quad 0 \le t \le T, \\
B_i(0) &= b_{0,i}, \qquad\qquad i = 1,\dots,N. \qquad (1)
\end{aligned}
$$

The processes $\mu_i : [0,T]\times\Omega \to \mathbb{R}$ and $\beta_i : [0,T]\times\Omega \to \mathbb{R}^d$ may be stochastic and are assumed to be locally bounded and previsible. $b_{0,i}$ is the time zero price of asset i. The above all for $i = 1,\dots,N$.

Several investors operate in this market and maintain portfolios of the assets.

Definition 1

(i) A portfolio process *π is any locally bounded $\mathbb{F}$-previsible process, $\pi : [0,T]\times\Omega \to \mathbb{R}^N$.*

(ii) The value process *of a portfolio π is the process $V^\pi : [0,T]\times\Omega \to \mathbb{R}$ defined by*

$$V^\pi(t) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \pi_i(t)B_i(t), \quad 0 \le t \le T.$$

(iii) A portfolio π is said to be self-financing *if its value process V^π satisfies*

$$\mathrm{d}V^\pi(t) = \sum_{i=1}^{N} \pi_i(t)\mathrm{d}B_i(t), \quad 0 \le t \le T.$$

(iv) A self-financing portfolio π is called admissible *in the market $\mathcal{M}$ if the corresponding value process V^π is lower bounded almost surely (abbreviated a.s.), i.e., if there exists a real number $K < \infty$ such that*

$$V^\pi(t) \ge -K \ \ \forall t, \ \ 0 \le t \le T, \ \ a.s. \qquad \square$$

A portfolio π holds an amount $\pi_i(t)$ of asset i at time t, $0 \le t \le T$, $i = 1,\dots,N$. An admissible self-financing portfolio π may be traded in the market $\mathcal{M}$ as well against the price $V^\pi(t)$ at time t, $0 \le t \le T$. Note that $\pi_i(t)$ is allowed to be negative, $i = 1,\dots,N$. This amounts to short-selling asset i. Condition (iv) of Definition 1 excludes portfolios with doubling-up strategies, which make almost sure profits starting with zero value, see [Øks00], Example 12.1.4.

Definition 2

(i) An arbitrage portfolio *π is a self-financing portfolio that has zero value at time 0 and that has a non-negative value at time T, almost surely, with positive probability of the value being strictly positive at time T.*

(ii) A market $\mathcal{M}$ is said to be arbitrage-free *if no admissible arbitrage portfolios exist in $\mathcal{M}$.*

(iii) An equivalent martingale probability measure $\mathbb{Q}$ of the market $\mathcal{M}$ is a probability measure on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$, and such that all assets are martingales under $\mathbb{Q}$. □

Theorem 3 (Absence of arbitrage) *If an equivalent martingale measure exists for the market $\mathcal{M}$ then $\mathcal{M}$ is arbitrage-free.*

Proof: Suppose π is an admissible arbitrage portfolio. Then V^π is a martingale under $\mathbb{Q}$. From the martingale property (see item (i) of Definition 30 in Appendix A),

$$\mathbb{E}^{\mathbb{Q}}\big[V^\pi(T)\big] = V^\pi(0) = 0,$$

so $\mathbb{E}^{\mathbb{Q}}[V^\pi(T)] = 0$ and $V^\pi(T) \geq 0$ a.s., hence $V^\pi(T) = 0$ a.s. which is in contradiction with $\mathbb{P}(V^\pi(T) > 0) > 0$ in item (i) of Definition 2. □

From here on, any portfolio is assumed to be self-financing and admissible.

The prices of the assets in the market $\mathcal{M}$ are denoted in a fixed pricing unit, say euros. However they may be expressed in terms of their relative value to any traded asset which has a positive value at all times, i.e., in terms of a so called *numeraire*.

Definition 4 *A* numeraire *B is the value process of a portfolio such that $B(t) > 0$ for all t, $0 \leq t \leq T$, almost surely.* □

An example of a numeraire is any of the assets B_i, $i = 1, \ldots, N$. If B is a numeraire, then the assets $B_1/B, \ldots, B_N/B$ together with the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ constitute a market as well, say $\tilde{\mathcal{M}}$, where prices are denoted in units of the numeraire asset B. The above-described transformation of markets is called *change of numeraire*.

Let $\mathbb{X}$ be a set of $\mathcal{F}(T)$-measurable random variables on the probability space $(\Omega, \mathcal{F})$. To each random variable $X \in \mathbb{X}$ a *contingent T-claim* will be associated (which will be denoted by X as well) which pays out the random amount X at time T.

Definition 5

(i) A portfolio π is said to hedge against the claim X *if*

$$V^\pi(T) = X \text{ a.s.}$$

If this is the case, then the claim X is said to be attainable *in the market $\mathcal{M}$.*

(ii) If all claims $X \in \mathbb{X}$ are attainable in the market $\mathcal{M}$ then $\mathcal{M}$ is said to be complete *with respect to $\mathbb{X}$.*

(iii) The price of a claim *X is the smallest value x at which there exists a portfolio π that hedges against X and that has initial value $V^\pi(0)$ equal to x.*

(iv) A portfolio that hedges against a claim X at minimal initial cost is called a hedging portfolio *of the claim X.* □

Note that if π is a hedging portfolio of a claim X at price x then $-\pi$ is a hedging portfolio of the claim $-X$ at price $-x$. This shows that the price of a hedge is equal to both a seller or a buyer of the claim, due to the ability of short-selling. If short-selling is prohibited or restricted, this symmetry breaks[3].

Define

$$\mathbb{X} \stackrel{\text{def}}{=} \big\{X \in L^1(\Omega, \mathcal{F}(T), \mathbb{P}) : \exists \mu > 1 : \mathbb{E}[X^\mu] < \infty\big\}.$$

The following result is taken from [KaS91], Theorem 5.8.12.

Theorem 6 (Completeness) *If there exists an equivalent martingale measure $\mathbb{Q}$ for the market $\mathcal{M}$ and if such a measure $\mathbb{Q}$ is unique, then every claim $X \in \mathbb{X}$ is attainable in the market $\mathcal{M}$.* □

The proof in [KaS91] actually does demonstrate the existence of a replicating portfolio for any claim $X \in \mathbb{X}$, using the Brownian-martingale integral representation theorem.

Propositon 7 *Suppose there exists an equivalent martingale measure $\mathbb{Q}$ for the market $\mathcal{M}$. Let X be a claim which is attainable in $\mathcal{M}$. Then the price of the claim X at time t, $0 \le t \le T$, is given by $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}(t)]$. In particular, if $\tilde{\mathbb{Q}}$ is an equivalent martingale measure for a market $\tilde{\mathcal{M}}$ that is obtained from $\mathcal{M}$ under a change of numeraire B, then the price of the claim X at time t, $0 \le t \le T$, is given by*

$$B(t)\mathbb{E}^{\tilde{\mathbb{Q}}}\big[\frac{X}{B(T)}\big|\mathcal{F}(t)\big]. \tag{2}$$

Proof: The statement follows from the fact that the time t value of the claim is equal to the time t value of a hedging portfolio π and from the fact that the value process of the hedging portfolio is a martingale under $\mathbb{Q}$. Using the martingale property (item (i) from Definition 30), the time t value of the claim is then

$$\begin{aligned} V^\pi(t) &= \mathbb{E}^{\mathbb{Q}}\big[V^\pi(T)\big|\mathcal{F}(t)\big] \\ &= \mathbb{E}^{\mathbb{Q}}\big[X\big|\mathcal{F}(t)\big], \quad 0 \le t \le T. \end{aligned}$$

The latter equality because π hedges against X, i.e., $V^\pi(T) = X$ a.s. □

3.2 LIBOR market model

[4] The LIBOR market $\mathcal{M}$ consists of $N+1$ assets, namely $N+1$ bonds. Regarding these bonds, a set of $N+1$ bond maturities $\{T_i\}_{i=1}^{N+1}$ is given, with

$$0 < T_1 < \cdots < T_{N+1} \tag{3}$$

and the maturity of the ith bond is T_i, $i = 1, \ldots, N+1$. Define $T_0 = 0$. The horizon time T of the LIBOR market is defined as the maturity date T_N of the Nth bond. The price process of the ith bond is denoted by $B_i(\cdot)$. Bond i is

[3] For further reading on this subject, see [CvK93] and [KaK96].
[4] Sections 3.2–3.4 have been based upon [Jam96].

$$
\begin{aligned}
\frac{\mathrm{d}B_i(t)}{B_i(t)} &= \mu_i(t)\mathrm{d}t + \beta_i(t)\cdot \mathrm{d}W(t) \\
&= \mu_i(t)\mathrm{d}t + \sum_{j=1}^{d} \beta_{ij}(t)\mathrm{d}W_j(t), \quad 0 \le t \le T_i, \\
B_i(0) &= b_{0,i}^{\mathrm{Market}}, \qquad\qquad i = 1,\dots,N+1, \qquad (4)
\end{aligned}
$$

where $b_{0,i}^{\mathrm{Market}}$ is the bond price observed in the market at time 0.

A *LIBOR forward rate agreement* (FRA) for time T_i ($i = 1,\dots,N$) is an agreement to borrow (or lend) 1 euro from time T_i till time T_{i+1}. The *accrual period* δ_i of the ith forward is defined to be $\delta_i = T_{i+1} - T_i$, for $i = 1,\dots,N$. The lending/borrowing rate which is agreed upon is called the *LIBOR forward rate* for lending/borrowing from time T_i till time T_{i+1}. By convention, this rate is quoted as follows: The return at time T_{i+1} of 1 euro borrowed out at time T_i is equal to 1 plus the rate multiplied by the accrual period. To be precise, define the LIBOR forward rate $L_i : [0,T_i] \times \Omega \to \mathbb{R}$ by

$$
1 + \delta_i L_i(t) = \frac{B_i(t)}{B_{i+1}(t)}, \quad 0 \le t \le T_i, \quad i = 1,\dots,N. \qquad (5)
$$

See Figure 3 for an example of an LMM forward rate structure.

We want to be able to specify the instantaneous volatility of the LIBOR forward rates. If $\sigma_i : [0,T_i] \times \Omega \to \mathbb{R}^d$ are locally bounded previsible processes for $i = 1,\dots,N$, then the bond price processes need to be specified in such a way that the following holds

$$
\frac{\mathrm{d}L_i(t)}{L_i(t)} = \dots + \sigma_i(t)\cdot \mathrm{d}W(t), \quad 0 \le t \le T_i, \quad i = 1,\dots,N. \qquad (6)
$$

In the following, conditions on the β_i, $i = 1,\dots,N+1$, will be calculated such that equation (6) holds.

To this end define the process $s_i : [0,T_i] \times \Omega \to \mathbb{R}^d$ by $s_{ij}(t) = L_i(t)\sigma_{ij}(t)$, $0 \le t \le T_i$, $j = 1,\dots,d$, $i = 1,\dots,N$. Equation (6) then becomes

$$
\mathrm{d}L_i(t) = \dots + s_i(t)\cdot \mathrm{d}W(t), \quad 0 \le t \le T_i, \quad i = 1,\dots,N. \qquad (7)
$$

From equation (5) follows

$$
\begin{aligned}
\mathrm{d}L_i(t) &= \frac{1}{\delta_i}\mathrm{d}\Big(\frac{B_i(t)}{B_{i+1}(t)}\Big) \\
&\overset{(*)}{=} \frac{1}{\delta_i}\frac{B_i(t)}{B_{i+1}(t)}\Bigg(\Big(\mu_i(t) - \mu_{i+1}(t) - \big(\beta_i(t) - \beta_{i+1}(t)\big)\cdot \beta_{i+1}(t)\Big)\,\mathrm{d}t \\
&\qquad + \Big(\beta_i(t) - \beta_{i+1}(t)\Big)\cdot \mathrm{d}W(t)\Bigg), \\
&\qquad 0 \le t \le T_i, \quad i = 1,\dots,N. \qquad (8)
\end{aligned}
$$

Equality $(*)$ is achieved by applying Corollary 35 of Appendix A. Comparing equations (7) and (8), it may be concluded that the following condition needs to be satisfied by the β_is:

$$
\beta_i(t) - \beta_{i+1}(t) = \frac{\delta_i}{1 + \delta_i L_i(t)} s_i(t), \quad 0 \le t \le T_i, \quad i = 1,\dots,N. \qquad (9)
$$

Definition 8 *For $t \in [0,T]$, define $i(t)$ as the unique integer i which satisfies*

$$
T_{i-1} < t \le T_i.
$$

□

$i(t)$ denotes the index of the bond which is first to expire at time t. It follows

$$
\begin{aligned}
\beta_{i(t)}(t) - \beta_{i+1}(t) &= \sum_{j=i(t)}^{i} \big(\beta_j(t) - \beta_{j+1}(t)\big) \\
&= \sum_{j=i(t)}^{i} \frac{\delta_j}{1 + \delta_j L_j(t)} s_j(t), \quad i = i(t),\dots,N, \quad 0 \le t \le T. \qquad (10)
\end{aligned}
$$

Let $\beta : [0,T] \times \Omega \to \mathbb{R}^d$ be any locally bounded $\mathbb{F}$-previsible process, continuous on (T_i, T_{i+1}), $i = 1, \ldots, N+1$. It is seen that if the β_is satisfy

$$\beta_i(t) = \begin{cases} \beta(t) - \sum_{j=i(t)}^{i-1} \frac{\delta_j}{1+\delta_j L_j(t)} s_j(t), & 0 \le t \le T_{i-1}, \\ \beta(t), & T_{i-1} < t \le T_i, \end{cases} \tag{11}$$

then equation (7) is satisfied. This concludes our calculations of necessary and sufficient conditions on the β_is for (7) to hold.

Remark 9 The conditions on the β_is are to ensure that equation (6) holds. Now (6) is a condition on $B_i(t)/B_{i+1}(t)$, for $0 \le t \le T_i$, $i = 1, \ldots, N$. Thus (6) *does not* specify conditions on $B_i(t)$ for $T_{i-1} < t \le T_i$, $i = 1, \ldots, N+1$, i.e., the LIBOR forward rates do not care about a bond that is first to expire. This freedom is reflected in the formulas through the ability to fully specify the $\beta_{i(\cdot)}(\cdot)$ function through $\beta(\cdot)$. □

Specifying the bond price dynamics using (11) *ensures that the LIBOR forward rates satisfy equation (7) and thus also equation (6).* The bond price dynamics using (11) will subsequently be defined as

$$\frac{\mathrm{d}B_i(t)}{B_i(t)} = \mu_i(t)\mathrm{d}t + \beta_i(t) \cdot \mathrm{d}W(t) \tag{12}$$

$$= \begin{cases} \mu_i(t)\mathrm{d}t + \Big(\beta(t) - \big(\sum_{j=i(t)}^{i-1} \delta_j \frac{B_{j+1}(t)}{B_j(t)} s_j(t) \big) \Big) \cdot \mathrm{d}W(t), & 0 \le t \le T_{i-1}, \\ \mu_i(t)\mathrm{d}t + \beta(t) \cdot \mathrm{d}W(t), & T_{i-1} < t \le T_i. \end{cases}$$

3.3 No-arbitrage assumption

The no-arbitrage condition for the LIBOR market model on the drift terms μ is stated below.

Assumption 10 (No-arbitrage assumption for the LIBOR market model) *Assume that there exists a locally bounded $\mathbb{F}$-previsible process $\varphi^{\mathrm{MPR}} : [0,T] \times \Omega \to \mathbb{R}^d$ such that*

$$\mu_i(t) = \beta_i(t) \cdot \varphi^{\mathrm{MPR}}(t), \tag{13}$$

for t, $0 \le t \le T_i$, $i = 1, \ldots, N+1$. □

The process φ^{MPR} may be used to construct an equivalent martingale measure for the LIBOR market model. This will be done explicitly for the spot LIBOR measure and the terminal LIBOR measure, see Subsections 3.4.1 and 3.4.2, respectively, but we will omit the construction for the euro-denoted measure. Having constructed such an equivalent martingale measure then guarantees no-arbitrage, cf. Theorem 3. Hence the name "no-arbitrage assumption". If moreover the process φ^{MPR} is almost surely uniquely defined by (13) at all times, then the LIBOR market will be complete as well, see Theorem 6.

MPR stands for "market price of risk". Component j of $\varphi^{\mathrm{MPR}}(t)$ denotes the market price of risk for the source of uncertainty W_j at time $t \in [0,T]$, $j = 1, \ldots, d$. The market price of risk is the quotient of expected rate of return over the amount of uncertainty. Assumption 13 requires that the market price of risk per factor at a particular point in time is the same for all bonds i,

$i = 1, \ldots, N+1$.

Remark 11 For any portfolio price process $V : [0,T] \times \Omega \to \mathbb{R}$ write

$$\frac{\mathrm{d}V(t)}{V(t)} = \mu_V(t)\mathrm{d}t + \beta_V(t) \cdot \mathrm{d}W(t), \quad 0 \le t \le T,$$

for locally bounded previsible processes $\mu_V : [0,T] \times \Omega \to \mathbb{R}$ and $\beta_V : [0,T] \times \Omega \to \mathbb{R}^d$. Due to the self-financing property it follows that at each time $t \in [0,T]$, $\mu_V(t)$ and $\beta_V(t)$ are linear combinations of the $\mu_i(t)$ and $\beta_i(t)$, $i = 1, \ldots, N+1$. Therefore it follows that the no-arbitrage assumption will hold for any portfolio process V:

$$\mu_V(t) = \beta_V(t) \cdot \varphi^{\mathrm{MPR}}(t), \quad 0 \le t \le T. \tag{14}$$

□

3.4 Measures and numeraires

In this Section, several numeraires are introduced and their martingale measures are computed. The SDEs satisfied by the forward rates under the respective measures are computed as well.

3.4.1 Spot LIBOR measure

The spot LIBOR portfolio invests in the bonds using the following strategy

(i) At time 0, start with 1 euro, buy $(1)/B_1(0)$ T_1-bonds.

(ii) At time T_1, receive $\frac{1}{B_1(0)}$ euro, buy $(\frac{1}{B_1(0)})/B_2(T_1)$ T_2-bonds.

(iii) At time T_2, receive $\frac{1}{B_1(0)B_2(T_1)}$ euro, buy $(\frac{1}{B_1(0)B_2(T_1)})/B_3(T_2)$ T_3-bonds.

(.) Etc...

In general, between times T_i and T_{i+1}, the spot LIBOR portfolio holds an amount of $1/\prod_{j=1}^{i+1} B_j(T_{j-1})$ of T_{i+1}-bonds. Therefore the value $B(t)$ at time t, $0 \le t \le T$, of the spot LIBOR portfolio is

$$B(t) = \frac{B_{i+1}(t)}{\prod_{j=1}^{i+1} B_j(T_{j-1})}, \quad T_i \le t < T_{i+1}.$$

Note that the spot LIBOR portfolio is self-financing. The stochastic differential of the spot LIBOR price process is

$$\frac{\mathrm{d}B(t)}{B(t)} = \mu_{i(t)}(t)\mathrm{d}t + \beta_{i(t)}(t) \cdot \mathrm{d}W(t), \quad 0 \le t \le T.$$

Quotients of asset price processes over the spot LIBOR portfolio price process have to become martingales under the spot LIBOR measure. Therefore, the stochastic differential of a bond price over the numeraire price is calculated – this is done in the same way in which equation (8) was derived.

$$\begin{aligned}\frac{\mathrm{d}(B_i(t)/B(t))}{(B_i(t)/B(t))} &= \Big(\mu_i(t) - \mu_{i(t)}(t) - \big(\beta_i(t) - \beta_{i(t)}(t)\big) \cdot \beta_{i(t)}(t) \Big)\mathrm{d}t \\ &\quad + \big(\beta_i(t) - \beta_{i(t)}(t) \big) \cdot \mathrm{d}W(t), \quad 0 \le t \le T_i, \quad i = 1, \ldots, N+1.\end{aligned}$$

In the following the spot LIBOR measure will be constructed explicitly, given the existence of the process φ^{MPR} mentioned in Assumption 10.

Define the process $\varphi^{\mathrm{Spot}} : [0,T] \times \Omega \to \mathbb{R}^d$,

$$\varphi^{\mathrm{Spot}}(t) \stackrel{\mathrm{def}}{=} \varphi^{\mathrm{MPR}}(t) - \beta_{i(t)}(t), \quad 0 \le t \le T.$$

As φ^{MPR} satisfies (14) this will translate through elementary manipulations into φ^{Spot} satisfying

$$\mu_{V_1}(t) - \mu_{V_2}(t) - \big(\beta_{V_1}(t) - \beta_{V_2}(t)\big) \cdot \beta_{i(t)}(t) = \big(\beta_{V_1}(t) - \beta_{V_2}(t)\big) \cdot \varphi^{\mathrm{Spot}}(t), \tag{15}$$

for V_1, V_2 portfolio price processes and for t, $0 \le t \le T$. Define the local martingale $M : [0,T] \times \Omega \to \mathbb{R}$ by

$$M(t) \stackrel{\mathrm{def}}{=} \int_0^t \varphi^{\mathrm{Spot}}(s) \cdot \mathrm{d}W(s), \quad 0 \le t \le T,$$

and define the process $W^{\mathbb{Q}_{\mathrm{Spot}}} : [0,T] \times \Omega \to \mathbb{R}^d$ by

$$\begin{aligned} W^{\mathbb{Q}_{\mathrm{Spot}}}(t) &\stackrel{\mathrm{def}}{=} W(t) + \langle W, M\rangle(t) \\ &= W(t) + \int_0^t \varphi^{\mathrm{Spot}}(s)\mathrm{d}s, \quad 0 \le t \le T, \end{aligned} \tag{16}$$

where the second equality follows from Kunita-Watanabe. From Girsanov's theorem (Theorem (36), appendix A) it then follows that $W^{\mathbb{Q}_{\mathrm{Spot}}}$ is a local martingale under the measure $\mathbb{Q}_{\mathrm{Spot}}$ determined by its Radon-Nikodým derivative

$$\begin{aligned} \frac{\mathrm{d}\mathbb{Q}_{\mathrm{Spot}}}{\mathrm{d}\mathbb{P}}(t) &\stackrel{\mathrm{def}}{=} e^{M(t) - \frac{1}{2}\langle M\rangle(t)} \\ &= e^{\int_0^t \varphi^{\mathrm{Spot}}(s)\cdot \mathrm{d}W(s) - \frac{1}{2}\int_0^t \|\varphi^{\mathrm{Spot}}(s)\|^2 \mathrm{d}s}, \quad 0 \le t \le T. \end{aligned} \tag{17}$$

Since $\int_0^\cdot \varphi^{\mathrm{Spot}}(s)\mathrm{d}s$ is a finite variation process, $W^{\mathbb{Q}_{\mathrm{Spot}}}$ has the same quadratic variation structure as a Brownian motion. Moreover, $W^{\mathbb{Q}_{\mathrm{Spot}}}$ is a local martingale under $\mathbb{Q}_{\mathrm{Spot}}$. Lévy's characterization of Brownian motion (Theorem (37), appendix A) subsequently yields that $W^{\mathbb{Q}_{\mathrm{Spot}}}$ is a Brownian motion under $\mathbb{Q}_{\mathrm{Spot}}$.

The SDEs for the bond price processes over the spot LIBOR price process are expressed in terms of the $\mathbb{Q}_{\mathrm{Spot}}$-Brownian motion $W^{\mathbb{Q}_{\mathrm{Spot}}}$;

$$\begin{aligned} \frac{\mathrm{d}(B_i(t)/B(t))}{(B_i(t)/B(t))} &= \Big(\mu_i(t) - \mu_{i(t)}(t) - \big(\beta_i(t) - \beta_{i(t)}(t)\big) \cdot \beta_{i(t)}(t) \Big)\mathrm{d}t \\ &\quad + \big(\beta_i(t) - \beta_{i(t)}(t) \big) \cdot \big(\mathrm{d}W^{\mathbb{Q}_{\mathrm{Spot}}}(t) - \varphi^{\mathrm{Spot}}(t)\mathrm{d}t\big) \\ &= \big(\beta_i(t) - \beta_{i(t)}(t)\big) \cdot \mathrm{d}W^{\mathbb{Q}_{\mathrm{Spot}}}(t), \end{aligned}$$

for t, $0 \le t \le T_i$, $i = 1, \dots, N+1$, the latter equality in virtue of equation (15).

It immediately follows that the above quotients are martingales under $\mathbb{Q}_{\mathrm{Spot}}$. So $\mathbb{Q}_{\mathrm{Spot}}$ is the measure that was looked for. $\mathbb{Q}_{\mathrm{Spot}}$ will be called the *spot LIBOR measure*.

Notation 12 The norm $\|\cdot\|$ used in equation (17) always denotes the L^2 norm $\|\cdot\|_2$, unless explicitly stated otherwise. □

An SDE is derived for the LIBOR forward rates expressed in terms of $W^{\mathbb{Q}_{\mathrm{Spot}}}$. Substituting (16) into equation (8) and using (15), gives

$$\begin{aligned}
\mathrm{d}L_i(t) &= \frac{1+\delta_i L_i(t)}{\delta_i}\Big(\big(\beta_i(t)-\beta_{i+1}(t)\big)\cdot\big(\beta_{i(t)}(t)-\beta_{i+1}(t)\big)\mathrm{d}t \\
&\qquad\qquad + \big(\beta_i(t)-\beta_{i+1}(t)\big)\cdot \mathrm{d}W^{\mathbb{Q}_{\mathrm{Spot}}}(t)\Big) \\
&= \sum_{j=i(t)}^{i} \frac{\delta_j s_j(t)\cdot s_i(t)}{1+\delta_j L_j(t)}\mathrm{d}t + s_i(t)\cdot \mathrm{d}W^{\mathbb{Q}_{\mathrm{Spot}}}(t),
\end{aligned}$$

for t, $0 \le t \le T_i$, $i = 1, \ldots, N$, where the latter equality uses equations (9) and (10). Note that the drift terms μ disappear in the above equation; the pricing of derivatives is independent of the real-world expected return of the underlying assets. Finally, recalling $\sigma_i(\cdot) \equiv L_i(\cdot)s_i(\cdot)$,

$$\frac{\mathrm{d}L_i(t)}{L_i(t)} = \sum_{j=i(t)}^{i} \frac{\delta_j L_j(t)\sigma_j(t)\cdot\sigma_i(t)}{1+\delta_j L_j(t)}\mathrm{d}t + \sigma_i(t)\cdot \mathrm{d}W^{\mathbb{Q}_{\mathrm{Spot}}}(t), \tag{18}$$

for t, $0 \le t \le T_i$, $i = 1, \ldots, N$.

3.4.2 Terminal LIBOR measure

Here the numeraire will be one of the bonds, say B_{n+1}, for some n, $n \in \{1, \ldots, N\}$. A portfolio that contains one bond is automatically self-financing.

Quotients of asset price processes over the bond price process have to become martingales under the terminal measure n. In particular, B_n/B_{n+1} will become a martingale. Thus the nth LIBOR forward rate, which is an affine transformation of B_n/B_{n+1}, will become a martingale under the terminal measure n. This will prove to be useful when computing the price of a caplet within the LIBOR market model (a caplet is some type of interest rate derivative and will be described in Section 4.1.1).

The stochastic differential of a bond price over the numeraire price is calculated – this is done in the same way in which equation (8) was derived.

$$\begin{aligned}
\frac{\mathrm{d}(B_i(t)/B_{n+1}(t))}{(B_i(t)/B_{n+1}(t))} &= \Big(\mu_i(t)-\mu_{n+1}(t)-\big(\beta_i(t)-\beta_{n+1}(t)\big)\cdot\beta_{n+1}(t)\Big)\mathrm{d}t \\
&\quad + \big(\beta_i(t)-\beta_{n+1}(t)\big)\cdot \mathrm{d}W(t), \\
&\qquad 0 \le t \le \min(T_i, T_{n+1}),\ \ i = 1, \ldots, N+1.
\end{aligned}$$

Exactly as in the case of the spot LIBOR measure, processes $\varphi^{T_{n+1}} : [0, T_{n+1}] \times \Omega \to \mathbb{R}^d$ and $W^{\mathbb{Q}_{T_{n+1}}} : [0, T_{n+1}] \times \Omega \to \mathbb{R}^d$ are defined together with a measure $\mathbb{Q}_{T_{n+1}}$ such that $W^{\mathbb{Q}_{T_{n+1}}}$ is a d-dimensional Brownian motion under $\mathbb{Q}_{T_{n+1}}$. To be precise, $\varphi^{T_{n+1}}$, $W^{\mathbb{Q}_{T_{n+1}}}$ and $\mathbb{Q}_{T_{n+1}}$ are defined by

$$\varphi^{T_{n+1}}(t) \stackrel{\mathrm{def}}{=} \varphi^{\mathrm{MPR}}(t) - \beta_{n+1}(t),\ \ 0 \le t \le T_{n+1},$$

$$W^{\mathbb{Q}_{T_{n+1}}}(t) \stackrel{\mathrm{def}}{=} W(t) + \int_0^t \varphi^{T_{n+1}}(s)\mathrm{d}s,\ \ 0 \le t \le T_{n+1}, \tag{19}$$

$$\frac{\mathrm{d}\mathbb{Q}_{T_{n+1}}}{\mathrm{d}\mathbb{P}}(t) \stackrel{\text{def}}{=} e^{\int_0^t \varphi^{T_{n+1}}(s)\cdot \mathrm{d}W(s) - \frac{1}{2}\int_0^t \|\varphi^{T_{n+1}}(s)\|^2 \mathrm{d}s}, \quad 0 \le t \le T_{n+1}.$$

Here $\varphi^{T_{n+1}}$ will satisfy

$$\mu_{V_1}(t) - \mu_{V_2}(t) - \big(\beta_{V_1}(t) - \beta_{V_2}(t)\big)\cdot \beta_{n+1}(t) = \big(\beta_{V_1}(t) - \beta_{V_2}(t)\big)\cdot \varphi^{T_{n+1}}(t), \tag{20}$$

for V_1, V_2 portfolio price processes and for t, $0 \le t \le T_{n+1}$.

The SDEs for the bond price processes over the $(n+1)$th bond price process are expressed in terms of the $\mathbb{Q}_{T_{n+1}}$-Brownian motion $W^{\mathbb{Q}_{T_{n+1}}}$;

$$\begin{aligned}
\frac{\mathrm{d}(B_i(t)/B_{n+1}(t))}{(B_i(t)/B_{n+1}(t))} &= \Big(\mu_i(t) - \mu_{n+1}(t) - \big(\beta_i(t) - \beta_{n+1}(t)\big)\cdot \beta_{n+1}(t)\Big)\mathrm{d}t \\
&\quad + \big(\beta_i(t) - \beta_{n+1}(t)\big)\cdot \big(\mathrm{d}W^{\mathbb{Q}_{T_{n+1}}}(t) - \varphi^{T_{n+1}}(t)\mathrm{d}t\big) \\
&= \big(\beta_i(t) - \beta_{n+1}(t)\big)\cdot \mathrm{d}W^{\mathbb{Q}_{T_{n+1}}}(t),
\end{aligned}$$

for t, $0 \le t \le \min(T_i, T_{n+1})$, $i = 1, \dots, N+1$, the latter equality in virtue of equation (20).

It immediately follows that the above quotients are martingales under $\mathbb{Q}_{T_{n+1}}$. So $\mathbb{Q}_{T_{n+1}}$ is the measure that was looked for. $\mathbb{Q}_{T_{n+1}}$ will be called the *nth terminal measure* or the *T_{n+1}-terminal measure*.

An SDE is derived for the LIBOR forward rates expressed in terms of $W^{\mathbb{Q}_{T_{n+1}}}$ where $n \in \{1, \dots, N\}$. Substituting (19) into equation (8) and using (20), gives

$$\begin{aligned}
\mathrm{d}L_i(t) &= \frac{1+\delta_i L_i(t)}{\delta_i}\Big(\big(\beta_i(t) - \beta_{i+1}(t)\big)\cdot\big(\beta_{n+1}(t) - \beta_{i+1}(t)\big)\mathrm{d}t \\
&\qquad + \big(\beta_i(t) - \beta_{i+1}(t)\big)\cdot \mathrm{d}W^{\mathbb{Q}_{T_{n+1}}}(t)\Big) \\
&= -\sum_{j=i+1}^{n} \frac{\delta_j s_j(t)\cdot s_i(t)}{1+\delta_j L_j(t)}\mathrm{d}t + s_i(t)\cdot \mathrm{d}W^{\mathbb{Q}_{T_{n+1}}}(t), \qquad (21)
\end{aligned}$$

for t, $0 \le t \le \min(T_i, T_{n+1})$, $i = 1, \dots, N$, where the latter equality uses equation (9). Here the summation convention is taken to be

$$\sum_{j=i}^{n} x_j \stackrel{\text{def}}{=} \begin{cases} \sum_{j=i}^{n} x_j, & i < n, \\ 0, & i = n, \\ -\sum_{j=n}^{i} x_j, & i > n, \end{cases}$$

for integers i and n and for summands $\{x_j\}_{j=i}^{n}$. Note that again the drift terms μ disappear in (21). Finally, recalling $\sigma_i(\cdot) \equiv L_i(\cdot)s_i(\cdot)$,

$$\frac{\mathrm{d}L_i(t)}{L_i(t)} = -\sum_{j=i+1}^{n} \frac{\delta_j L_j(t)\sigma_j(t)\cdot\sigma_i(t)}{1+\delta_j L_j(t)}\mathrm{d}t + \sigma_i(t)\cdot \mathrm{d}W^{\mathbb{Q}_{T_{n+1}}}(t), \tag{22}$$

for t, $0 \le t \le \min(T_i, T_{n+1})$, $i = 1, \dots, N$.

3.5 LIBOR market model summary

The LIBOR market model requires the following input:

(i) A set of bond maturities as in (3).

(ii) The time zero LIBOR forward rates $L_1(0), \ldots, L_N(0)$.

(iii) The instantaneous volatilities of the forward rates $\sigma_i(\cdot)$ for $i = 1, \ldots, N$.

$\sigma_i(\cdot)$, $i = 1, \ldots, N$ form the *parameters* of the LIBOR market model. In the process of calibration, these parameters are chosen in such a way that the LMM correctly prices certain securities that are traded actively in the markets. The calibration procedure is discussed in Section 4.

Prices of interest rate derivatives are given by the general pricing formula (2), i.e., prices are the expected value under a certain measure of the discounted payoff of the derivative. The payoff of the derivative is completely written in terms of the LIBOR forward rates. The SDEs that the LIBOR forward rates satisfy under the appropriate measures are known, cf. SDEs (18) and (22).

4 Calibration

The calibration is the computation of the parameters of the LIBOR market model, $\sigma_i(\cdot)$, $i = 1, \ldots, N$, so as to match as closely as possible model derived prices/values to market observed prices/values of actively traded securities. Typically, a calibration procedure in a computer implemented LMM can take a few seconds up to fifteen minutes.

In Subsection 4.1 the model implicit prices/values given the parameter functions are derived. Several ways in which to specify the instantaneous volatility are discussed in Subsection 4.2. Subsection 4.3 presents issues arising with a computer implementation of a calibration.

4.1 Calibration theory

For now, the instantaneous volatility is assumed to be a deterministic function $\sigma_i : [0, T_i] \to \mathbb{R}^d$ for $i = 1, \ldots, N$.

There are three security prices/market variables to which the LMM may be calibrated in reasonable time. These are:

(i) Caplet prices.

(ii) Forward rate correlations.

(iii) Swaption prices.

In the following Sections, the model derived values of these securities/market values are calculated and expressed in terms of the parameter functions $\sigma_i(\cdot)$ for $i = 1, \ldots, N$.

4.1.1 Caplets

A caplet is a call option on a LIBOR forward rate. (Caplets that are discussed here have European-style exercise features.) A caplet gives its owner the right, but not the obligation, to borrow money over the forward accrual period at the pre-negotiated strike rate of the caplet. The caplet payoff is paid out at the end of the forward accrual period. Consider a loan for the nth forward period, with a notional amount M. A caplet on this loan will be called a caplet on the nth forward rate. Suppose the strike rate is K. The price of such a caplet will be denoted by $C_n(T_n, K)$. The payoff of the caplet is then

$$M\delta_n\big(\ L_n(T_n) - K\ \big)_+,$$

paid out at time T_{n+1}. Here the function $(\cdot)_+ : \mathbb{R} \to \mathbb{R}$ is defined by $(x)_+ = \max(x, 0)$ for $x \in \mathbb{R}$. If the notional amount M is taken to be $10,000$ euro, then the payoff is said be quoted in *basispoints* (bps).

Using [Bla76] a closed form formula for the price of a caplet may be derived, assuming that the forward rates are log-normally distributed and have constant volatility. For caplet n, $n \in \{1, \ldots, N\}$, and volatility $\sigma > 0$ the formula reads

$$\begin{aligned} C_n^{\text{Black}}(\sigma) &= M\delta_n B_{n+1}(0)\big(\ L_n(0)N(d_1) - KN(d_2)\ \big), \\ d_1 &= \frac{\log(\frac{L_n(0)}{K}) + \frac{1}{2}\sigma^2 T_n}{\sigma\sqrt{T_n}}, \\ d_2 &= \frac{\log(\frac{L_n(0)}{K}) - \frac{1}{2}\sigma^2 T_n}{\sigma\sqrt{T_n}} = d_1 - \sigma\sqrt{T_n}, \end{aligned} \tag{23}$$

where $N : \mathbb{R} \to [0, 1]$ is the standard normal distribution function, $N(x) = \int_{-\infty}^{x}(1/\sqrt{2\pi})e^{-\frac{1}{2}y^2}\mathrm{d}y$, for $x \in \mathbb{R}$. The Black formula has since become so popular that in the financial markets, prices of caplets are actually quoted in terms of so called *Black implied volatilities*. The Black implied volatility of a caplet is the volatility with which the Black formula returns the market quoted price of the caplet.

In practice, caplets are not traded; they are always traded in the form of caps. A cap consists of multiple (different) caplets. Brokers quote prices of caps which again are expressed in terms of Black implied volatilities. The caplet volatilities may be obtained from the cap volatilities quoted in the markets using a boot-strapping algorithm.

The LIBOR market model was constructed in such a way that the LIBOR forward rates are log-normally distributed, cf. condition (6). As such, it may then be expected that the model-internal Black implied volatility for the nth caplet is some average of the instantaneous volatility $\sigma_n(\cdot)$. This is indeed the case, as will be shown next.

To compute the price $C_n^{\text{Model}}(T_n, K)$ of the nth caplet within the LMM, $n \in \{1, \ldots, N\}$, the nth terminal measure $\mathbb{Q}_{T_{n+1}}$ will be used. Under this measure, the nth LIBOR forward rate becomes a martingale, since from SDE (22),

$$\frac{\mathrm{d}L_n(t)}{L_n(t)} = \sigma_n(t) \cdot \mathrm{d}W^{\mathbb{Q}_{T_{n+1}}}(t),$$

for t, $0 \le t \le T_n$. This SDE has solution

$$L_n(t) = L_n(0)e^{\int_0^t \sigma_n(s)\cdot \mathrm{d}W^{\mathbb{Q}_{T_{n+1}}}(s) - \frac{1}{2}\int_0^t \|\sigma_n(s)\|^2 \mathrm{d}s}, \quad 0 \le t \le T_n,$$

which can be verified using Itô's formula. Therefore, $L_n(T_n) = L_n(0)e^Z$ where Z is an $\mathcal{F}(T_n)$-measurable random variable which is normally distributed *under* $\mathbb{Q}_{T_{n+1}}$, $Z \sim \mathcal{N}(-\frac{1}{2}\tau^2, \tau^2)$, where

$$\tau^2 \stackrel{\text{def}}{=} \int_0^{T_n} \|\sigma_n(s)\|^2 \mathrm{d}s.$$

The LMM price $C_n^{\text{Model}}(T_n, K)$ is now given by formula (2), i.e., the LMM price of the nth caplet is

$$\begin{aligned} C_n^{\text{Model}}(T_n, K) &= M\delta_n B_{n+1}(0)\mathbb{E}^{\mathbb{Q}_{T_{n+1}}}\left[\frac{\big(L_n(T_n)-K\big)_+}{B_{n+1}(T_{n+1})}\right] \\ &= M\delta_n B_{n+1}(0)\mathbb{E}^{\mathbb{Q}_{T_{n+1}}}\left[\big(L_n(T_n)-K\big)_+\right] \end{aligned} \tag{24}$$

since $B_{n+1}(T_{n+1}) = 1$. This expectation can be calculated using basic manipulations of integration calculus. The actual calculation may be found in appendix B; the result is given here

$$C_n^{\text{Model}}(T_n, K) = M\delta_n B_{n+1}(0)\big(\ L_n(0)N(d_1) - KN(d_2)\ \big), \tag{25}$$

where

$$\begin{aligned} d_1 &= \frac{\log(\frac{L_n(0)}{K}) + \frac{1}{2}\tau^2}{\tau}, \\ d_2 &= \frac{\log(\frac{L_n(0)}{K}) - \frac{1}{2}\tau^2}{\tau} = d_1 - \tau. \end{aligned}$$

The LMM price of a caplet may also be quoted in terms of its Black implied volatility. Denote by $\sigma_n^{\text{Black,Model}}$ the Black implied volatility within the LMM for the nth caplet. Comparing formula (25) with the Black formula (23) the following Corollary is obtained.

Corollary 13 *The Black implied volatility of caplet n, $n \in \{1, \ldots, N\}$, within the LIBOR market model is given by*

$$\sigma_n^{\text{Black,Model}} = \sqrt{\frac{1}{T_n}\int_0^{T_n} \|\sigma_n(s)\|^2 \mathrm{d}s}. \tag{26}$$

□

4.1.2 Forward rate correlations

Definition 14 *The* instantaneous correlation $\rho_{ij}^{\text{Model}} : [0, \min(T_i, T_j)] \times \Omega \to [-1, 1]$ between two forward rates i and j, $i, j \in \{1, \ldots, N\}$, *is defined as the instantaneous cross-variation of the two rates divided by the square root of the instantaneous quadratic variation of both rates. The* instantaneous quadratic variation (cross-variation) at time t *is the derivative with respect to time of the total quadratic variation (cross-variation) process at time t. In the form of a formula;*

$$\rho_{ij}^{\text{Model}}(t) \stackrel{\text{def}}{=} \frac{\frac{\mathrm{d}}{\mathrm{d}t}\langle L_i, L_j\rangle(t)}{\sqrt{\big(\frac{\mathrm{d}}{\mathrm{d}t}\langle L_i\rangle(t)\big)\big(\frac{\mathrm{d}}{\mathrm{d}t}\langle L_j\rangle(t)\big)}}, \quad 0 \le t \le \min(T_i, T_j).$$

□

Propositon 15 *The time t correlation $\rho_{ij}^{\text{Model}}(t)$, $i, j \in \{1, \ldots, N\}$, within the LIBOR market model is given by*

$$\rho_{ij}^{\text{Model}}(t) = \frac{\sigma_i(t) \cdot \sigma_j(t)}{\|\sigma_i(t)\|\|\sigma_j(t)\|}, \quad 0 \le t \le \min(T_i, T_j). \tag{27}$$

Proof: An application of Corollary 32 of Appendix A together with equation (6). □

Remark 16 Firstly, note that the LMM internal forward rate correlations are deterministic, because the instantaneous volatility $\sigma.(\cdot)$ is deterministic.

Secondly, note that the above formula gives the LMM-internal forward rate correlations *at all times* t, $0 \leq t \leq T$. If a trader has a view on some future forward rate correlation, he could choose to calibrate the LMM to his particular anticipated future correlation. In practice however, the LMM is only calibrated to *time zero* forward rate correlations (if at all), where the market time zero forward correlation $\rho_{ij}^{\text{Market}}(0)$ is taken to be the observed historic correlation. □

4.1.3 Swaptions

A *swap agreement* is an agreement between two parties to swap fixed for floating interest rate payments on some notional loan amount. The floating interest may for example be the LIBOR rate. A swap agreement consists of a number of swaplets. Each swaplet prescribes the swap of fixed for floating interest rate over a certain accrual time. The floating rate is determined (*set*) at the beginning of the accrual period, the actual payment is made at the end of the accrual period. The rate of the fixed leg at which the swap agreement has zero value is called the *swap rate*.

Consider a swap agreement consisting of a number of swaplets, the first swaplet being set at time T_i and paying out at time T_{i+1}, the last swaplet being set at time T_{j-1} and paying out at time T_j, for some $i < j$, $i, j \in \{1, \ldots, N\}$. The swap thus consists of $j - i$ swaplets. The pre-negotiated rate of the fixed leg at which the swap has zero value, i.e., the swap rate, will be denoted by $S_{i:j}$. To be precise, it may be shown (for example [Reb98], equation (1.25′)) that the swap rate $S_{i:j} : [0, T_i] \times \Omega \to \mathbb{R}$ is equal to

$$S_{i:j}(t) = \frac{B_i(t) - B_j(t)}{\sum_{k=i}^{j-1} \delta_k B_{k+1}(t)}, \tag{28}$$
$$0 \leq t \leq T_i, \quad j = i+1, \ldots, N+1, \quad i = 1, \ldots, N,$$

and it is thus defined that way.

A *swaption* could be called an option on the swap rate. (Swaptions that are discussed here all have European-style exercise features.) A swaption gives its owner the right, but not the obligation, to enter into a certain swap agreement at the pre-negotiated strike rate. The swaption provides a cash flow at the end of each swaplet period. Consider a swap as described above with notional amount M. Suppose the strike rate is K and the swaption expiry time is T_i. The cash flow emanating from the swaption at time T_k, $k = i+1, \ldots, j$, is then

$$M\delta_k\big(\, S_{i:j}(T_i) - K \,\big)_+.$$

Using [Bla76], as in the case for options on forward rates, again a closed form formula may be derived for swaption prices, given the assumption that swap rates are log-normally distributed and have constant volatility. For the above described swaption with instantaneous volatility $\sigma(t)$ at time t, $0 \leq t \leq T_i$,

$\sigma : [0, T_i] \to [0, \infty)$, the Black price of a swaption is

$$
\begin{aligned}
& & MA\big(\ S_{i:j}(0)N(d_1) - KN(d_2)\ \big), \\
d_1 &=& \frac{\log(\frac{S_{i:j}(0)}{K}) + \frac{1}{2}\int_0^{T_i}\sigma^2(s)\mathrm{d}s}{\sqrt{\int_0^{T_i}\sigma^2(s)\mathrm{d}s}}, \\
d_2 &=& \frac{\log(\frac{S_{i:j}(0)}{K}) - \frac{1}{2}\int_0^{T_i}\sigma^2(s)\mathrm{d}s}{\sqrt{\int_0^{T_i}\sigma^2(s)\mathrm{d}s}} = d_1 - \sqrt{\int_0^{T_i}\sigma^2(s)\mathrm{d}s}, \\
A &=& \sum_{k=i+1}^{j}\delta_k B_k(0). \qquad (29)
\end{aligned}
$$

A is called the *present value of a basis point (PVBP)*. The Black formula for swaptions has since become so popular as well, that in the financial markets, prices of swaptions are actually quoted in terms of Black implied volatilities, alike the case for caplets.

So it is standard market practice to assume that both forward rates and swap rates are log-normally distributed[5]. To examine this simultaneous assumption more closely, the swap rate defined in equation (28) is written in terms of forward rates (divide through by $B_i(t)$),

$$S_{i:j}(t) = \frac{1 - \prod_{k=i}^{j-1}\frac{1}{1+\delta_k L_k(t)}}{\sum_{k=i}^{j-1}\delta_k \prod_{m=i}^{k}\frac{1}{1+\delta_m L_m(t)}},$$

$0 \le t \le T_i$, $j = i+1, \ldots, N+1$, $i = 1, \ldots, N$. From the above equation it may be seen that the simultaneous assumption of log-normal distributed forward rates *and* log-normal distributed swap rates is not consistent. Conclusively, within a LIBOR market model, swaptions cannot be priced using Black's model.

But as it turns out, swap rates are actually very close to being log-normally distributed within the LIBOR market model (as within any model assuming log-normality of forward rates). Namely, a good approximation of the volatility of the logarithm of the swap rate will follow from the following procedure:

(i) *Determining the instantaneous volatility of the logarithm of the swap rate.* This instantaneous volatility will in general be stochastic since swap rates are not log-normally distributed. It will be expressed in terms of swap rates, bond prices and forward rates.

(ii) *Approximate the instantaneous volatility of the swap rate by evaluating any stochastic terms at time zero.* As a result a deterministic instantaneous volatility of the swap rate is obtained. The model-internal approximate swaption Black implied volatility will then be some average of that deterministic instantaneous volatility.

Next the formula for the instantaneous volatility of a swap rate within the LMM is stated. A proof may be found in Section III of [HuW00].

[5] A discussion on this topic may be found in [Reb99b].

Propositon 17 (Swap rate instantaneous volatility expressed in terms of forward rate instantaneous volatility) *Within the LMM, the swap rate* $S_{i:j}$*, for* $i < j$*,* $i, j \in \{1, \ldots, N\}$*, satisfies the following SDE, where the terms that only contribute to the finite variation part of* $S_{i:j}$ *have been left out;*

$$\frac{\mathrm{d}S_{i:j}(t)}{S_{i:j}(t)} = \cdots + \sigma_{i:j}(t) \cdot \mathrm{d}W(t), \quad 0 \leq t \leq T_i, \tag{30}$$

where $\sigma_{i:j} : [0, T_i] \times \Omega \to \mathbb{R}^d$ *is defined by*

$$\sigma_{i:j}(t) \stackrel{\text{def}}{=} \sum_{k=i}^{j-1} \frac{\delta_k L_k(t) \gamma_k^{i:j}(t)}{1 + \delta_k L_k(t)} \sigma_k(t) \tag{31}$$

with $\gamma_k^{i:j} : [0, T_i] \times \Omega \to \mathbb{R}$*,*

$$\gamma_k^{i:j}(t) \stackrel{\text{def}}{=} \frac{\prod_{l=i}^{j-1} \big(1 + \delta_l L_l(t)\big)}{\Big(\prod_{l=i}^{j-1} \big(1 + \delta_l L_l(t)\big) \Big) - 1} - \frac{\sum_{l=i}^{k-1} \delta_l \prod_{m=l+1}^{j-1} \big(1 + \delta_m L_m(t)\big)}{\sum_{l=i}^{j-1} \delta_l \prod_{m=l+1}^{j-1} \big(1 + \delta_m L_m(t)\big)},$$

for t*,* $0 \leq t \leq T_i$*,* $k = i, \ldots, j-1$*.* □

Denote by $\sigma_{i:j}^{\text{Black,Model,Approx}}$ the approximate Black implied volatility within the LMM for the above described swaption. The approximation is achieved by evaluating the stochastic terms occurring in the instantaneous volatility of the swap rate at time zero.

Corollary 18 *The approximate Black implied volatility of the above described swaption within the LIBOR market model is given by*

$$\begin{aligned}\left(\sigma_{i:j}^{\text{Black,Model,Approx}}\right)^2 &= \frac{1}{T_i}\int_0^{T_i} \Big\| \sum_{k=i}^{j-1} \frac{\delta_k L_k(0)\gamma_k^{i:j}(0)}{1+\delta_k L_k(0)}\sigma_k(s)\Big\|^2 \mathrm{d}s \\ &= \frac{1}{T_i}\sum_{k=i}^{j-1}\sum_{l=i}^{j-1} \frac{\delta_k L_k(0)\gamma_k^{i:j}(0)}{1+\delta_k L_k(0)} \frac{\delta_l L_l(0)\gamma_l^{i:j}(0)}{1+\delta_l L_l(0)} \\ &\qquad \int_0^{T_i} \rho_{kl}^{\text{Model}}(s)\|\sigma_k(s)\|\|\sigma_l(s)\|\mathrm{d}s. \qquad (32)\end{aligned}$$

Proof: Evaluating the stochastic terms in the instantaneous volatility $\sigma_{i:j}(\cdot)$ at time zero leaves us with a deterministic instantaneous volatility. Classic Black theory then gives us the above formula. Equation (27) has been used to write the vector product of the $\sigma_\cdot(\cdot)$ in terms of the correlation ρ^{Model}. □

Remark 19 Note that a swaption is a security whose price is sensitive to the future forward rate correlations, cf. the appearance of $\rho^{\text{Model}}(\cdot)$ in the above approximating formula. In a sense, it may be said that the market expresses its views on the future forward rate correlations through the prices of swaptions. Calibrating the LMM to a range of swaption volatilities could thus be viewed as obtaining the implied future forward rate correlation matrix. □

4.2 Ways in which to specify the instantaneous volatility

Several ways in which to specify the instantaneous volatility will be discussed.

The various forms of instantaneous volatility serve two purposes.

(i) *Separating the influence of the parameters on the model values.*

A way of specifying the instantaneous volatility will lead to certain model values being only dependent on certain parameters, thereby enabling more robust minimization or even enabling a minimization over two sets of parameters, each set having an independent influence on the model values (see Section 4.2.1).

(ii) *Parameter control.*

Sometimes the most general form of instantaneous volatility will yield too many LMM parameters when calibrating only to a small number of calibration objects. Specifying the instantaneous volatility in a less general form will then lead to a reduction of the number of parameters and ultimately to a more stable calibration procedure (see Sections 4.2.2 and 4.2.3).

For an overview of the various ways in which to specify the instantaneous volatility, see Figure 4.

4.2.1 Spherical coordinates vs. Euclidean coordinates

Spherical coordinates may be used to denote the instantaneous volatility. Using spherical coordinates will make some minimization schemes more robust, see the remark at the end of this Section.

Figure 4: Overview of ways in which to specify the instantaneous volatility.

Definition 20 *Given a vector of angles* $\theta = (\theta_1, \ldots, \theta_{d-1}) \in \mathbb{R}^{d-1}$ *(which will be called* spherical coordinates*), we associate to it Euclidean coordinates* $f(\theta)$ *of a point on the* $(d-1)$*-dimensional unit sphere* $S^{d-1} = \{x \in \mathbb{R}^d : \; \|x\| = 1\}$ *where the mapping* $f : \mathbb{R}^{d-1} \to S^{d-1}$ *is given by*

$$f_j(\theta) \stackrel{\text{def}}{=} \begin{cases} \cos(\theta_j) \prod_{k=1}^{j-1} \sin(\theta_k), & \text{if } \; j = 1, \ldots, d-1, \\ \prod_{k=1}^{d-1} \sin(\theta_k), & \text{if } \; j = d. \end{cases}$$

□

Definition 21 *Let* $\Sigma_i : [0, T_i] \to [0, \infty)$ *and* $\theta_i : [0, T_i] \to \mathbb{R}^{d-1}$ *be functions. The instantaneous volatility structure* $\sigma_i : [0, T_i] \to \mathbb{R}^d$*,* $i = 1, \ldots, N$*, is said to be written in terms of* spherical coordinates *if*

$$\sigma_{ij}(t) = \Sigma_i(t) f_j(\theta_i(t)), \;\; j = 1, \ldots, d, \;\; 0 \le t \le T_i, \;\; i = 1, \ldots, N. \tag{33}$$

Σ_i *is named the total instantaneous volatility of the ith forward.* □

Note that $\Sigma_i^2(t) = \sum_{j=1}^d \sigma_{ij}^2(t)$, $f_j(\theta_i(t)) = \sigma_{ij}(t)/\Sigma_i(t)$, $j = 1, \ldots, d$, $0 \le t \le T_i$, $i = 1, \ldots, N$. See also Figure 5.

Remark 22 After specifying the instantaneous volatility in terms of spherical coordinates, the LMM parameters will be $\Sigma.(\cdot)$ and $\theta.(\cdot)$. Looking at formula (26) reveals that the model caplet prices are only dependent on $\Sigma.(\cdot)$. Likewise, looking at formula (27) shows that the model forward rate correlations are only dependent on $\theta.(\cdot)$. This presents the opportunity for a separated minimization whenever the LMM is calibrated to caplet prices and forward rate correlations only. □

Figure 5: Spherical coordinates in three dimensions. Here the point P denotes the point $f(\theta_1, \theta_2)$. s denotes Σ.

4.2.2 Time homogeneity

Definition 23 *A LIBOR market model is said to be* time homogeneous *if there exists a function* $\lambda : [0, T] \to \mathbb{R}^d$ *such that*

$$\sigma_i(t) = \lambda(T_i - t), \quad 0 \leq t \leq T_i, \quad i = 1, \ldots, N. \qquad \square$$

4.2.3 The Bell form

Definition 24 *Let* a, b, c *and* d *be real numbers. The instantaneous volatility structure* $\sigma_i : [0, T_i] \to \mathbb{R}^d$, $i = 1, \ldots, N$, *is said to be of the* Bell form[6] *if it is written in terms of spherical coordinates and if*

$$\Sigma_i(t) = a(T_i - t) + be^{-c(T_i - t)} + d, \quad 0 \leq t \leq T_i, \quad i = 1, \ldots, N,$$

where Σ_i *is the total volatility of* σ_i *as given in Definition 21.*

4.3 Calibration in practice

For practical purposes, the instantaneous volatility $\sigma_i : [0, T_i] \to \mathbb{R}^d$ is taken to be constant on the intervals $[T_{j-1}, T_j)$, $j = 1, \ldots, i$, for $i = 1, \ldots, N$ (except for the Bell form). This will reduce the parameter set of the LMM to matrices and vectors, depending on the form that has been chosen for the instantaneous volatility. In any case the parameters of the LMM are reduced to a finite number of variables. Abstractly, denote the parameters of the LMM by σ.

[6] The Bell form for the instantaneous volatility is due to Rebonato, [Reb98].
Yet another way of specifying the instantaneous volatility in order to control the number of parameters is the use of constant volatility or mean-reversion, see [JDP01].

Before starting a calibration, a list of "calibration objects" should be given. A calibration object can be either a caplet price, a forward rate correlation or a swaption price; see Section 4.1. Each of the entries in the list requires a description of the object itself (e.g. for a caplet; which forward period the caplet is associated with and what the expiry date is) together with the *observed market value* of the security price/correlation number. Note that caplet and swaption prices are quoted here in implied volatilities. Say a calibration has M calibration objects, with market values x_k^{Market}, $k = 1, \ldots, M$. Given a set of parameters σ, it is possible to compute the model values of the M calibration objects, cf. formulas (26), (27) and (32). This will yield M model values $x_k^{\text{Model}}(\sigma)$, $k = 1, \ldots, M$. Suppose for each calibration object k, a sense of error $\text{Err}_k(\cdot; x_k^{\text{Market}})$ is given for the model value x_k^{Model}, i.e., $\text{Err}_k(x_k^{\text{Model}}; x_k^{\text{Market}})$ is a measure for how far away the model value is from the market value for the kth calibration object. See Section 4.3.1 for specific Err functions. Note that for different calibration objects, different error functions may be used. Adding the errors for all the calibration objects, a sense of error is obtained as to how far the LIBOR market model with parameter set σ is away from the market. The calibration procedure is the minimization of this error over the parameter set σ so as to get the LMM to resemble the market as close as possible. *The whole calibration procedure will thus be the following minimization*

$$\min_{\sigma} \sum_{k=1}^{M} \text{Err}_k(x_k^{\text{Model}}(\sigma); x_k^{\text{Market}}).$$

4.3.1 Minimization of metric vs. principle components analysis

Three senses of error are discussed in this Section.

(i) *(Metric type of error)*

If x^{Market} and x^{Model} are the market respectively model values of a calibration object, then an error sense is readily given by

$$\text{Err}(x^{\text{Model}}; x^{\text{Market}}) = w|x^{\text{Model}} - x^{\text{Market}}|^{\gamma},$$

where $w > 0$ is a weight and $\gamma > 0$ is an appropriately chosen constant.

(ii) *(Zero error if within spread)*

If a calibration object is a caplet price or a swaption price, the market will quote bid and offer prices. Any price that is in between the bid and offer prices is said to be *within the spread*. A possible natural way then to assign a sense of error is to assign zero if the model price is within the spread and to assign some positive number if the model price is outside of the spread. Concretely, for a market spread $x^{\text{Market,Bid}}$, $x^{\text{Market,Offer}}$, define an error function by

$$\text{Err}(x^{\text{Model}}; x^{\text{Market,Bid}}, x^{\text{Market,Offer}}) =$$

$$\begin{cases} 0, & \text{if } x^{\text{Market,Bid}} \leq x^{\text{Model}} \leq x^{\text{Market,Offer}}, \\ x^{\text{Market,Bid}} - x^{\text{Model}}, & \text{if } x^{\text{Model}} \leq x^{\text{Market,Bid}}, \\ x^{\text{Model}} - x^{\text{Market,Offer}}, & \text{if } x^{\text{Model}} \geq x^{\text{Market,Offer}}. \end{cases}$$

(iii) *(Error minimization using principle components analysis)*[7]

Suppose the LMM is calibrated to the caplet prices and the complete time zero forward rate correlation matrix only. In this method, the calibration to the caplets and to the forward correlation can be done separately, so suppose the total instantaneous volatility $\Sigma_i(\cdot)$, for $i = 1, \ldots, N$, has already been calculated. A principle components analysis on the correlation matrix will yield a model in which approximately

$$\Delta L_i = \sum_{j=1}^{N} \alpha_{ij} x_j \Delta t.$$

Here $\Delta t > 0$ is a small time step. x_j is a random variable with mean 0 and variance s_j^2. x_j is called the *jth factor*. The factors themselves are independent of each other. The variances of the factors are assumed to be in descending order, so $s_1^2 \geq \cdots \geq s_N^2$. α_{ij} measures the influence of the jth factor on the ith forward rate. The influence on the forward rates of one factor is independent of another factor ($\sum_{i=1}^{N} \alpha_{ij}\alpha_{ik} = 0$, $j, k \in \{1, \ldots, N\}$, $j \neq k$) and the relative influence on the forward rates of a factor is 1 ($\sum_{i=1}^{N} \alpha_{ij}^2 = 1$, $j \in \{1, \ldots, N\}$). The above all for $i, j \in \{1, \ldots, N\}$.

Actually, if $\{\mathrm{Cov}_{ij}\}_{i,j=1}^{N}$ is the market observed historic forward rate covariation matrix, then the α matrix and s vector are determined by an eigenvalue decomposition $\mathrm{Cov} = \alpha S \alpha^\top$. Here α is the orthogonal matrix $\{\alpha_{ij}\}_{i,j=1}^{N}$ containing eigenvectors of Cov. S is a diagonal matrix with the eigenvalues of Cov on its diagonal, in descending order. s_j^2 is then equal to the jth entry on the diagonal of S, $j = 1, \ldots, N$.

It now seems natural to set the instantaneous volatility $\sigma_{ij}(t)$ proportional to $s_j\alpha_{ij}$, $i = 1, \ldots, N$, $j = 1, \ldots, d$, restricted by the total instantaneous volatility $\sum_{j=1}^{d} s_j\alpha_{ij}$ equaling $\Sigma_i(t)$. Therefore, set $\sigma_{ij}(t)$ equal to $\Sigma_i(t)s_j\alpha_{ij}/\sqrt{\sum_{j=1}^{d} s_j^2\alpha_{ij}^2}$, $i = 1, \ldots, N$, $j = 1, \ldots, d$. Note that $\sigma_{ij}(t)$ can be set in a time homogeneous way or in a way that keeps the correlation structure constant over time.

Having done this, the market correlation matrix is not entirely replicated within the model, unless the number of factors d is equal to the number of forwards N. The error that has been minimized is unclear from the construction; there is only an intuitive idea that the first d factors account for the most variance, so that the resulting model correlation matrix will largely resemble the market correlation matrix.

4.3.2 A useful result when calibrating to caplet volatilities and forward rate correlations only

[8] Suppose the LMM is calibrated to caplet prices and forward rate correlations only and that spherical coordinates are used. Assume for simplicity that the instantaneous volatility is constant, say $\sigma_{ik}(\cdot) = \Sigma_i b_{ik}$, where the total instantaneous volatility Σ_i equals $\sigma_i^{\mathrm{Black,Market}}$, for the LMM to price the caplets

[7] This method is due to Hull and White, [HuW00].

[8] This result is due to Rebonato, [Reb99a].

correctly, and $b_{ik} = f_k(\theta_i)$, $\theta_i \in \mathbb{R}^{d-1}$, $k = 1, \ldots, d$, $i = 1, \ldots, N$. Using formula (27) for the model forward rate correlation, it follows that $\rho_{ij}^{\text{Model}}(t) = b_i \cdot b_j$, $0 \leq t \leq \min(T_i, T_j)$, $i, j \in \{1, \ldots, N\}$. Denote by $\rho^{\text{Market}} = \{\rho_{ij}^{\text{Market}}\}_{i,j=1}^N$ the $(N \times N)$ market observed forward rate correlation matrix, denote by B the $(N \times d)$ matrix $\{b_{ik}\}_{i=1}^N{}_{k=1}^d$. From the above statement, $\rho^{\text{Model}} = BB^\top$. Note that B may be regarded as dependent on the θs, so write $B(\theta)$ for this dependence. Calibrate the LMM to the forward rate correlation matrix by minimizing

$$\min_{\theta_i \in \mathbb{R}^{d-1}, i=1,\ldots,N} \|\rho^{\text{Market}} - B(\theta)B^\top(\theta)\|,$$

where $\|\cdot\|$ is some norm, possibly fitted out with weights.

Note that the resulting optimal $(N \times d)$ matrix B has the following properties:

(i)
$$\sum_{k=1}^d b_{ik}^2 = 1, \quad i = 1, \ldots, N.$$

(ii) The discrepancies between $BB^\top$ and ρ^{Market} have been minimized in the above described way.

Rebonato provides an algorithm that calculates an $(N \times d)$ matrix A which has *the above stated properties* plus an additional property:

(iii)
$$\sum_{i=1}^N a_{ik} a_{ik'} = 0, \quad k, k' \in \{1, \ldots, d\}, \quad k \neq k'.$$

The matrix B may thus be replaced by the matrix A to obtain the extra feature that the influence of the Brownian motions on the forward rates are independent of each other.

The matrix A is constructed as in the following Lemma from elementary linear algebra. A proof may be found in for example [Reb99a], Theorem 2.

Lemma 25 *Let B be an $(N \times d)$ real matrix of full rank, $N > d$. Then, since the matrix $BB^\top$ is real and symmetric, it may be diagonalized, so*

$$BB^\top = PSP^\top.$$

Here P is an orthogonal $(N \times N)$ matrix, i.e., $PP^\top$ is the $(N \times N)$ identity matrix. S is a diagonal $(N \times N)$ matrix that has the eigenvalues of $BB^\top$ on its diagonal. Note that $\operatorname{rank}(BB^\top) = \operatorname{rank}(B) = d$*, therefore $BB^\top$ has d non-zero eigenvalues. The entries of the diagonal of S may be assumed to be in descending order, obtaining*

$$S = \begin{pmatrix} s_1^2 & & & & & \\ & \ddots & & & \varnothing & \\ & & s_d^2 & & & \\ & & & 0 & & \\ & \varnothing & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Let C_i be the ith column vector of P, $i = 1, \ldots, N$, and write $P = (C_1 \ \ldots \ C_N)$. Define the $(N \times d)$ matrix A by

$$A \stackrel{\text{def}}{=} \begin{pmatrix} s_1 C_1 & \ldots & s_d C_d \end{pmatrix}.$$

Then

$$BB^\top = AA^\top.$$

□

Having constructed A as in the above Lemma, it is checked here that A indeed satisfies the above stated conditions (i), (ii) and (iii). The orthogonality condition (iii) is satisfied by A due to the orthogonality of the column vectors $C_1, \ldots, C_N$. Conditions (i) and (ii) are satisfied by A due to the conclusion of the Lemma, $BB^\top = AA^\top$. This is readily seen for condition (ii). Regarding condition (i), note that the diagonal of $BB^\top$ consists of ones, i.e., $(BB^\top)_{ii} = 1$ for $i = 1, \ldots, N$, which ensures that A satisfies the unitary norm condition (i).

5 Extending the LIBOR market model to calibrate to the volatility smile

Within the plain vanilla LMM, all cap prices have the same Black implied volatility regardless of the strike rate K of the cap. This follows from equation (26).

In interest rate markets, caps are quoted at different volatilities for different strikes. The volatility shows a smile-like dependency with respect to the strike rate. This phenomenon is called the volatility smile.

As mentioned above, the plain vanilla LMM cannot account for the volatility smile. This Section presents an extension of the LMM which will enable it to reproduce the volatility smile. The extended LMM will be called forward rate dependent local volatility (FRDLV) LMM. Local volatility here means the same as instantaneous volatility.

The extension is based on assuming that the instantaneous volatility of the forward rates is not only a function of time, but also of the forward rate itself. Such an instantaneous volatility structure will be referred to as *forward rate dependent local volatility.* FRDLV is a novel application of the so-called "spot dependent local volatility" developed for equity derivatives by Dupire, see [Dup93b].

5.1 Forward rate dependent local volatility LMM

Suppose continuous functions $\sigma_i : [0, T_i] \times [0, \infty) \to \mathbb{R}^d$ are given, for $i = 1, \ldots, N$. We want to be able to specify the instantaneous volatility of the forward rates as (compare with equation (6))

$$\frac{\mathrm{d}L_i(t)}{L_i(t)} = \cdots + \sigma_i(t, L_i(t)) \cdot \mathrm{d}W(t), \quad 0 \le t \le T_i, \quad i = 1, \ldots, N.$$

Therefore we continue from equation (6) and repeat the whole theory developed there exactly. All the results obtained in Section 3 will remain valid when $\sigma_i(t)$ is replaced by $\sigma_i(t, L_i(t))$, for $i = 1, \ldots, N$.

The FRDLV-equivalent of equation (12) should be given some care though. The bond price dynamics should be written completely in terms of the bond prices themselves. Therefore, the FRDLV equivalent equation for (12) becomes

$$\frac{\mathrm{d}B_i(t)}{B_i(t)} = \mu_i(t)\mathrm{d}t + \beta_i(t)\cdot \mathrm{d}W(t)$$

$$= \begin{cases} \mu_i(t)\mathrm{d}t \\ +\Bigg(\beta(t) - \Big(\sum_{j=i(t)}^{i-1} (1 - \frac{B_{j+1}(t)}{B_j(t)})\sigma_j\big(t, \frac{1}{\delta_j}(\frac{B_j(t)}{B_{j+1}(t)} - 1)\big)\Big) \Bigg)\cdot \mathrm{d}W(t), \\ \qquad\qquad 0 \le t \le T_{i-1}, \\ \mu_i(t)\mathrm{d}t + \beta(t)\cdot \mathrm{d}W(t), \;\; T_{i-1} < t \le T_i. \end{cases}$$

This concludes the theoretical justification of the FRDLV LIBOR market model.

5.2 Determining the instantaneous volatility to incorporate smile

In this Section the instantaneous volatility function $\sigma_i(\cdot,\cdot)$, $i = 1,\ldots,N$, will be determined such that the FRDLV LMM prices caplets correctly for all strikes.

To this end, caplets are considered for each forward $i = 1,\ldots,N$, with *fixing time* S, $0 \le S \le T_i$, *and payment time* T_{i+1}. To say that a caplet has fixing time S means that the rate at which the payoff is calculated is taken to be the rate observed at time S. The time-T_{i+1} payoff of such a caplet with strike rate K will be

$$M\delta_i\big(\,L_i(S) - K\,\big)_+.$$

Denote the FRDLV LMM price of such a caplet by $C_i^{\mathrm{Model}}(S,K)$. Similar to result (24),

$$C_i^{\mathrm{Model}}(S,K) = M\delta_i B_{i+1}(0)\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\big[\big(L_i(S) - K\big)_+\big], \tag{34}$$

for $0 \le S \le T_i$, $K \ge 0$, $i = 1,\ldots,N$. Also

$$\frac{\mathrm{d}L_i(t)}{L_i(t)} = \sigma_i(t, L_i(t))\cdot \mathrm{d}W^{\mathbb{Q}_{T_{i+1}}}(t), \;\; 0 \le t \le T_i, \;\; i = 1,\ldots,N. \tag{35}$$

The following theorem shows that the model caplet prices satisfy a partial differential equation (PDE) together with appropriate boundary conditions. Later on, it will be shown that this PDE with initial value specification determines the model caplet prices uniquely.

Theorem 26 *Assume that the instantaneous volatility functions* $\sigma_i : [0,T_i] \times [0,\infty) \to \mathbb{R}^d$ *are such that the* $\mathbb{Q}_{T_{i+1}}$*-transition measure of* $L_i(\cdot)$*, induced by equation (35), has a transition density* $p_i : D_i \to [0,\infty)$ *on domain* $D_i = \{(s,y;t,x) \in [0,\infty)^4 :\; 0 \le s < t \le T_i\}$*, i.e.,*

$$\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\big[f(L_i(t))\big|L_i(s) = y\big] = \int_0^\infty f(x)p_i(s,y;t,x)\mathrm{d}x,$$

for continuous functions $f : [0,\infty) \to \mathbb{R}$ *and* $0 \le s < t \le T_i$, $y \in [0,\infty)$*, such that the function which maps* (S,K) *onto* $p_i(0,L_i(0);S,K)$ *is of class* $C^{1,2}((0,T_i] \times [0,\infty))$*, for* $i = 1,\ldots,N$*. Then the model caplet prices* $C_i : [0,T_i] \times [0,\infty) \to [0,\infty)$ *given by equation (34) are continuous and satisfy* *(i)* C_i^{Model} *is of class* $C^{1,2}((0,T_i)\times(0,\infty))$ *and satisfies the PDE*

$$\frac{\partial C_i^{\text{Model}}}{\partial S}(S,K) = \frac{1}{2}\|\sigma_i(S,K)\|^2 K^2 \frac{\partial^2 C_i^{\text{Model}}}{\partial K^2}(S,K), \tag{36}$$

for $0 < S < T_i$*,* $K > 0$*,*

(ii) with boundary conditions

$$C_i^{\text{Model}}(0,K) = M\delta_i B_{i+1}(0)\big(\, L_i(0) - K\,\big)_+, \quad K \geq 0,$$

(iii) and

$$C_i^{\text{Model}}(S,0) = M\delta_i B_{i+1}(0) L_i(0), \quad 0 \leq S \leq T_i,$$

(iv) furthermore, C_i^{Model} *is bounded on* $[0,T_i]\times[0,\infty)$*,*

for $i = 1,\ldots,N$*.*

Proof: Part (i)[9]:

The statements made on the continuity and differentiability of the model caplet prices follow from the assumption of existence and sufficient differentiability of the transition density function. Since L_i has a transition density under $\mathbb{Q}_{T_{i+1}}$, then in particular,

$$\begin{aligned} C_i^{\text{Model}}(S,K) &= M\delta_i B_{i+1}(0)\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[(L_i(S)-K)_+\right] \\ &= M\delta_i B_{i+1}(0)\int_K^\infty (x-K)_+ p_i(0,L_i(0);S,x)\mathrm{d}x, \end{aligned}$$

for $0 < S \leq T_i$, $K \geq 0$, $i = 1,\ldots,N$. Differentiating the above stated equation twice with respect to the strike rate K yields

$$\begin{aligned} \frac{\partial^2 C_i^{\text{Model}}}{\partial K^2}(S,K) &= M\delta_i B_{i+1}(0) p_i(0,L_i(0);S,K), \\ & \quad 0 < S \leq T_i, \;\; K \geq 0, \;\; i = 1,\ldots,N. \end{aligned} \tag{37}$$

The function which maps (S,K) onto $p_i(0,L_i(0);S,K)$ is of class $C^{1,2}((0,T_i]\times[0,\infty))$, for all $i = 1,\ldots,N$, by assumption. Thus according to the forward Kolmogorov equation 39 in appendix C, p_i satisfies the PDE

$$\frac{\partial p_i}{\partial S}(S,K) = \frac{1}{2}\frac{\partial^2\big(\|\sigma_i\|^2 K^2 p_i\big)}{\partial K^2}(S,K),$$

for $0 < S \leq T_i$, $K \geq 0$, $i = 1,\ldots,N$. Integrating the latter equation twice with respect to K, then using (37) and multiplying through by $M\delta_i B_{i+1}(0)$ produces

$$\frac{\partial C_i^{\text{Model}}}{\partial S}(S,K) = \frac{1}{2}\|\sigma_i(S,K)\|^2 K^2 \frac{\partial^2 C_i^{\text{Model}}}{\partial K^2}(S,K) + A(S) + B(S)K,$$

for $0 \leq S \leq T_i$, $K \geq 0$, $i = 1,\ldots,N$. Here $A : [0,\infty) \to \mathbb{R}$ and $B : [0,\infty) \to \mathbb{R}$ are arbitrary functions. As $K \to \infty$, $C_i^{\text{Model}}(S,K) \to 0$ uniformly in S (follows

[9] The derivation for Part (i) followed here is a modification of the calculations made in Section 2 of [ABR97].

from equation (34)). Therefore the functions A and B are identically zero. Resultingly, the model caplet prices will satisfy PDE (36).

Part (ii) is a special case of equation (34).

Part (iii): Consider

$$\begin{aligned} C_i^{\mathrm{Model}}(S,0) &= M\delta_i B_{i+1}(0)\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[L_i(S)\right] \\ &= M\delta_i B_{i+1}(0)L_i(0), \end{aligned}$$

the latter equality since L_i is a martingale under $\mathbb{Q}_{T_{i+1}}$, $i = 1, \dots, N$.

Part (iv): For $0 \le K_1 \le K_2$, $(x - K_1)_+ \ge (x - K_2)_+$ for all real $x \ge 0$. Using the previous statement with K_1 equal to 0 and K_2 equal to K yields

$$\begin{aligned} C_i^{\mathrm{Model}}(S,K) &= M\delta_i B_{i+1}(0)\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[(L_i(S) - K)_+\right] \\ &\le M\delta_i B_{i+1}(0)\mathbb{E}^{\mathbb{Q}_{T_{i+1}}}\left[L_i(S)\right] \\ &= M\delta_i B_{i+1}(0)L_i(0), \end{aligned}$$

for $0 \le S \le T_i$, $K \ge 0$, $i = 1, \dots, N$.

Conclusively, C_i^{Model} is bounded by $M\delta_i B_{i+1}(0)L_i(0)$, $i = 1, \dots, N$. $\square$

Reversing the reasoning, suppose that we are equipped with market caplet prices $C_i^{\mathrm{Market}}(S,K)$, for $0 \le S \le T_i$, $K \ge 0$, $i = 1, \dots, N$. The instantaneous volatility $\sigma_i(\cdot,\cdot)$ needs to be specified in such a way that the model caplet prices $C_i^{\mathrm{Model}}(\cdot,\cdot)$ equate the market caplet prices $C_i^{\mathrm{Market}}(\cdot,\cdot)$, for $i = 1, \dots, N$. Proceed by defining the instantaneous volatility in such a way that (baring in mind (36))

$$\|\sigma_i(t,x)\|^2 \stackrel{\mathrm{def}}{=} 2\frac{\frac{\partial C_i^{\mathrm{Market}}}{\partial S}(t,x)}{x^2\frac{\partial^2 C_i^{\mathrm{Market}}}{\partial K^2}(t,x)}, \quad 0 \le t \le T_i, \quad x \ge 0, \quad i = 1, \dots, N. \tag{38}$$

By construction, the market caplet prices C_i^{Market}, $i = 1, \dots, N$, will then satisfy the PDE and boundary conditions of Theorem 26. Equality of model and market caplet prices will follow if *uniqueness* is proven for the PDE/initial value problem. Because of linearity of PDE (36) it will be sufficient to prove uniqueness for homogeneous boundary conditions. This is done in the following theorem, which is a standard result from PDE theory. A proof may be found in [ReR93], Theorem 4.25. The proof uses the maximum principle.

Theorem 27 (Uniqueness of PDE (36)) *Let the continuous function $\sigma : [0,T] \times [0,\infty) \to \mathbb{R}$ be given. Consider the following conditions on a continuous function $C : [0,T] \times [0,\infty) \to \mathbb{R}$.*

(i) C is of class $C^{1,2}((0,T) \times (0,\infty))$ and satisfies the PDE

$$\frac{\partial C}{\partial S}(S,K) = \frac{1}{2}\sigma(S,K)^2 K^2 \frac{\partial^2 C}{\partial K^2}(S,K), \tag{39}$$

for $0 < S < T$, $K > 0$,

(ii) with homogeneous boundary conditions

$$C(0,K) = 0, \quad K \ge 0, \quad C(S,0) = 0, \quad 0 \le S \le T,$$

(iii) furthermore, C is bounded on $[0,T] \times [0,\infty)$.

Then C is identically zero on $[0,T] \times [0,\infty)$. □

Summarizing, the FRDLV LMM will price caplets at different strikes and expiry times correctly if the total instantaneous volatility satisfies equation (38).

5.3 Writing the instantaneous volatility in terms of implied volatilities

In this Subsection, it is assumed that the market caplet price functions C_i^{Market} are given in terms of their Black implied volatilities, i.e.,

$$C_i^{\text{Market}}(S,K) = C_i^{\text{Black}}(\sigma_i^{\text{Black,Market}}(S,K); S,K),$$

where C_i^{Black} is the Black formula (23) and $\sigma_i^{\text{Black,Market}}(S,K)$ is the market observed Black implied volatility for caplet i with strike K and fixing time S, $i = 1,\dots,N$. From there on, the total instantaneous volatility defined by equation (38) is written completely in terms of the market observed Black implied volatility. This will have three advantages.

(i) Computing the total instantaneous volatility will become numerically more stable.

(ii) In practice, caplet prices are boot-strapped from cap implied volatilities and are thus already expressed in terms of implied volatilities.

(iii) The caplets occurring within caps all expire at the starting time of the corresponding forward accrual period. So the only information on the price of an ith caplet available in the market is on those caplets expiring at time T_i, for $i = 1,\dots,N$. As a result an educated guess has to be made for the caplet prices $C_i^{\text{Market}}(S,\cdot)$ for $0 \le S < T_i$, $i = 1,\dots,N$. Having fitted a curve $\sigma_i^{\text{Black,Market}}(T_i,\cdot)$, it will then be natural to set $\sigma_i^{\text{Black,Market}}(S,\cdot)$ equal to $\sigma_i^{\text{Black,Market}}(T_i,\cdot)$ for $0 \le S < T_i$, $i = 1,\dots,N$.

The instantaneous volatility expressed in terms of the implied volatility is stated in the following proposition, which is without proof. It may be found as formula (22.7) in [Wil98] and as formula (16) in [ABR97].

Propositon 28 *The total instantaneous volatility $\|\sigma_i\| : [0,T_i] \times [0,\infty) \to [0,\infty)$, $i = 1,\dots,N$, expressed in terms of the implied volatility $\sigma_i^{\text{Black,Market}} : [0,T_i] \times [0,\infty) \to [0,\infty)$ is, writing $\hat\sigma$ for $\sigma_i^{\text{Black,Market}}(t,x)$,*

$$\|\sigma_i(t,x)\| = \sqrt{\frac{\hat\sigma^2 + 2t\hat\sigma\frac{\partial\hat\sigma}{\partial T}}{(1 + x d_1(t,x)\sqrt{t}\frac{\partial\hat\sigma}{\partial K})^2 + x^2 t\hat\sigma(\frac{\partial^2\hat\sigma}{\partial K^2} - d_1(t,x)\sqrt{t}(\frac{\partial\hat\sigma}{\partial K})^2)}} \tag{40}$$

for $(t,x) \in [0,T_i] \times [0,\infty)$. Here $d_1 : [0,T_i] \times [0,\infty) \to \mathbb{R}$ is defined by

$$d_1(t,x) \stackrel{\text{def}}{=} \frac{\log(\frac{L_i(0)}{x}) + \frac{1}{2}\hat\sigma^2 t}{\hat\sigma\sqrt{t}}, \quad (t,x) \in [0,T_i] \times [0,\infty).$$ □

5.4 Calibration issues

The FRDLV LMM is only studied here for a Monte Carlo pricer. The instantaneous volatility may be expressed in terms of spherical coordinates. For the FRDLV LMM, the instantaneous volatility thus becomes

$$\sigma_{ij}(t) = \Sigma_i(t, L_i(t)) f_j(\theta_i(t)), \quad 0 \le t \le T_i, \quad j = 1, \dots, d, \quad i = 1, \dots, N, \tag{41}$$

compare with equation (33). $\Sigma_i : [0, T_i] \times [0, \infty) \to [0, \infty)$, $i = 1, \dots, N$, is determined by expression (40). This leaves us with the freedom of specifying the θ-functions as (i) constant (ii) time homogeneous or (iii) constant in between forward start and end dates.

The FRDLV LMM may also be calibrated to the three calibration object types described in Subsection 4.1. Each of these types is discussed below.

(i) *(Caplet prices)*

The FRDLV LMM automatically prices the caplets correctly for the whole range of strikes, due to its forward rate dependency feature. This was what the FRDLV LMM was engineered for.

(ii) *(Forward rate correlations)*

Equation (27) still holds, may it be that $\sigma_i(t)$ has to be replaced by $\sigma_i(t, L_i(t))$. Using this in conjunction with specification (41) gives

$$\rho_{ij}^{\text{Model}}(t) = f(\theta_i(t)) \cdot f(\theta_j(t)), \quad 0 \le t \le \min(T_i, T_j), \quad i, j \in \{1, \dots, N\}.$$

Note that the correlation structure is again deterministic whereas the instantaneous volatility itself is stochastic. This is due to the fact that only the total instantaneous volatility is stochastic/forward rate dependent. In particular, it follows that Remark 16 also applies to the FRDLV LMM.

(iii) *(Swaption prices)*

The instantaneous volatility of a swap rate within the FRDLV LMM, see equation (31), is no longer only stochastic through its dependence on the λ-functions, but now also through the forward rate dependency of the instantaneous volatility of the forward rates. So for approximation purposes the forward rate volatility appearing in the swap rate instantaneous volatility will be evaluated at time zero as well, yielding

$$\begin{aligned}(\sigma_{i:j}^{\text{Black,Model,Approx}})^2 = \\ \frac{1}{T_i} \sum_{k=i}^{j-1} \sum_{l=i}^{j-1} \frac{\delta_k L_k(0) \gamma_k^{i:j}(0)}{1 + \delta_k L_k(0)} \frac{\delta_l L_l(0) \gamma_l^{i:j}(0)}{1 + \delta_l L_l(0)} \\ \|\sigma_k(0, L_k(0))\| \|\sigma_l(0, L_l(0))\| \int_0^{T_i} \rho_{kl}^{\text{Model}}(s) \mathrm{d}s.\end{aligned} \tag{42}$$

for $i, j, k, l \in \{1, \dots, N\}$, $i < j$ and $k < l$.

6 Pricer methods

For the computational pricer part of the LIBOR market model (see the block marked with “2” in Figure 2) several methods exist. *All these methods approximate formula (2) numerically.* For the LIBOR market model, Monte Carlo simulation is commonly used.

Remark 29 Another valuation method would be to solve a certain partial differential equation (PDE) which is associated to (2). This method is frequently used for e.g. equity derivatives. Due to the high-dimensionality of the LIBOR market model, such a PDE method is (currently) not practically executable. However, if in the future such a method would become practical, then such an algorithm could simply be plugged into block “2” of Figure 2, and the LMM would be ready to price derivatives using the new pricer method.

This remark shows how flexible the LIBOR market model general workings are. □

6.1 Monte Carlo

In a Monte Carlo simulation, the processes $\log(L_i(\cdot))$, $i = 1, \ldots, N$, are simulated instead of the forward rates themselves; this is to ensure positivity of the simulated forward rates. The SDE for the logarithm of the LIBOR forward rates is, using Itô’s formula,

$$\mathrm{d}\log(L_i(t)) = \frac{\mathrm{d}L_i(t)}{L_i(t)} - \frac{1}{2}\frac{\mathrm{d}\langle L_i\rangle(t)}{L_i^2(t)}, \quad 0 \leq t \leq T_i, \quad i = 1, \ldots, N.$$

This equation may be written in terms of e.g. $W^{\mathbb{Q}_{\mathrm{Spot}}}$. Using $\mathrm{d}\langle L_i\rangle(t)/L_i^2(t) = \|\sigma_i(t)\|^2\mathrm{d}t$, $0 \leq t \leq T_i$, $i = 1, \ldots, N$, together with SDE (18), gives

$$\mathrm{d}\log(L_i(t)) = \Big(\big(\sum_{j=i(t)}^{i} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\sigma_j(t)\big) - \frac{1}{2}\sigma_i(t) \Big) \cdot \sigma_i(t)\mathrm{d}t + \sigma_i(t) \cdot \mathrm{d}W^{\mathbb{Q}_{\mathrm{Spot}}}(t),$$

for t, $0 \leq t \leq T_i$, $i = 1, \ldots, N$. A discretized version of the above SDE with time step $\Delta t > 0$ will read

$$\Delta\log(L_i(t)) = \Big(\big(\sum_{j=i(t)}^{i} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\sigma_j(t)\big) - \frac{1}{2}\sigma_i(t) \Big) \cdot \sigma_i(t)\Delta t + \sigma_i(t) \cdot \varepsilon\sqrt{\Delta t},$$

for t, $0 \leq t \leq T_i$, $i = 1, \ldots, N$. Here ε is a d-dimensional standard normally distributed random variable, i.e., $\varepsilon \sim \mathcal{N}(0, I)$, where 0 is the d-dimensional zero vector and I is the $(d \times d)$-identity matrix. A discretized equation for the logarithm of the forward rates may be obtained in a similar fashion for any terminal measure as well.

The discretized SDE may be used to sample forward rate paths in a computer implementation.

7 Interest rate derivatives

This Section describes various exotic interest rate derivatives which can be valued with the LIBOR market model. Caps and swaptions have already been described in Sections 4.1.1 and 4.1.3, respectively.

7.1 Spread options

The spread between two rates is simply the difference between the rates. Here the two rates may be any LIBOR or swap rate or even the same rate observed at different times. A spread option is an option on the spread. If R_1 and R_2 are two rates, a spread call option with strike K and fixing times T_1 and T_2 pays out an amount proportional to

$$\left(R_1(T_1) - R_2(T_2) - K \right)_+.$$

Lower correlation between rates will lead to higher spreads. So intuitively, a spread option will be more expensive if the correlation between the rates is lower.

7.2 Discrete barrier caps and floors

A barrier option is an option of which the payoff is contingent on some price or market value hitting some level, called the barrier. An example is a down-and-out barrier, which pays out normally until some process has come below the barrier. Other examples are down-and-in, up-and-out and up-and-in barriers. If the barrier is hit, the barrier option is said to be knocked in (in case of a down-and-in or up-and-in barrier) or knocked out (in case of a down-and-out or up-and-out barrier).

For a discrete barrier option, only a predesignated set of fixing times matters. The discrete barrier may knock in or out only if at one of the fixing times, the underlying market value has surpassed the barrier.

A discrete barrier cap is then a cap with a discrete barrier feature.

Consider a skewed interest rate market, i.e., higher volatilities for lower strike. Assume that this skew is caused by higher volatility if the LIBOR forward rate goes down and a lower volatility if the LIBOR forward rate goes up. (In equity markets, this concept of deviation from the Black-Scholes world is referred to as "vol-by-money"ness and is modeled by for example spot dependent local volatility. FRDLV is the fixed income market equivalent to spot dependent local volatility.) Consider the price of a down-and-out barrier cap in the presence of such a skewed interest rate market compared to a flat Black-Scholes world. If LIBOR goes up, the volatility will go down, and hence the cap is more likely to end up lower in-the-money. If LIBOR goes down, the volatility will go up, so the option will be knocked out by the barrier more likely. Thus intuitively, it is expected that such a barrier option is cheaper in the presence of a skew.

8 Results

As an illustration of the capabilities of the LMM and FRDLV LMM, several interest rate products are priced for various scenarios using different calibration techniques. Results and performances are compared.

Horizon	5.5 Years
Accrual Period	6 Months
Number of Forwards	10
Number of Factors	2
Yield Curve/Volatility Curve	EURO, Taken from 01/09/00
Number of Monte Carlo runs	100,000
Steps per Forward Period in MC Simulation	4

Table 1: Specification of Deal 1.

Expiry (Years)	Tenor (Years)	Approximate Price of Swaption (Bp)	MC Price of Swaption (Bp)	MC Standard Error (Bp)
1	1	60.73	59.16	0.20
1	2	123.68	123.58	0.41
1	3	181.63	182.95	0.56
1	4	241.28	240.99	0.68
2	1	75.64	74.92	0.28
2	2	146.51	147.29	0.54
2	3	212.88	213.40	0.73
3	1	82.66	82.73	0.33
3	2	160.75	161.14	0.61
4	1	88.21	87.84	0.35

Table 2: Results for Deal 1.

8.1 Deal 1: Accuracy of swaption approximation

The purpose of this Deal is to test the accuracy of the swaption price approximation formula (32). To this end, swaption prices were calculated for a LIBOR market model both with the approximating formula (32) and through Monte Carlo simulation. The results were compared.

The specification of Deal 1 is summarized in Table 1. The instantaneous volatility had been written in terms of spherical coordinates. Both the total and per-factor instantaneous volatility were taken to be time homogeneous, see Section 4.2.2. The LMM was calibrated to the caplet volatilities and time zero forward rate correlations only. The total instantaneous volatilities may then be obtained inductively from the market caplet volatilities. The market time zero forward rate correlation structure was taken to be

$$\rho_{ij}^{\text{Market}}(0) = e^{-\beta(T_i - T_j)}, \quad i, j = 1, \ldots, N, \tag{43}$$

where β equaled 0.85. The per-factor instantaneous volatilities were subsequently computed using the Hull and White method outlined in Subsection 4.3.1, item (iii).

The results may be found in Table 2. This was for a scenario in which rates and volatilities were approximately 5% and 15%, respectively. Next, the

	Volatility (%)		
Rate (%)	15	30	60
1	0.92	1.35	1.93
5	0.96	1.38	1.75
10	0.05	0.27	0.92

Table 3: Maximum relative error (in %) for all swaptions (except the 1 into 1 year swaption) under different scenarios.

	Volatility (%)								
	15			30			60		
Rate (%)	Appr. Price	MC Price	Error	Appr. Price	MC Price	Error	Appr. Price	MC Price	Error
1	13.95	13.62	0.33	20.12	19.39	0.73	33.15	31.76	1.39
5	60.73	59.16	1.57	89.75	86.39	3.36	150.19	143.77	6.42
10	495.70	495.59	0.11	496.33	496.52	0.19	518.99	516.65	2.34

Table 4: Comparing approximating and Monte Carlo swaption prices (Bp) for the 1 into 1 year swaption.

whole yield and volatility curves where proportionally increased/decreased to give rate scenarios of about 1%, 5% and 10% and volatility scenarios of about 15%, 30% and 60%. All the above swaptions were Monte Carlo-priced (the 1% rates scenario at strike rate 0.01, the other scenarios at strike rate 0.05) and approximated. The maximum relative error for each scenario of all the swaptions except for the 1 into 1 year swaption is reported in Table 3. This was done except for the 1 into 1 year swaption as this particular swaption proved to present the largest relative errors. The results for the 1 into 1 year swaption are stated in Table 4.

The above stated results show that (for the examined scenarios) the swaption price approximation formula works really well and that it may be used to calibrate the LMM toward the swap market. The results in [HuW00] show the same.

8.2 Deal 2: Factor dependency of spread options

The purpose of this Deal is to examine the influence of the number of factors used in the LMM on the price of spread options. To this end, a spread option was priced using Monte Carlo simulation while varying the number of factors.

The specification of Deal 2 is summarized in Table 5. The instantaneous volatility had been written in terms of spherical coordinates. Both the total and per-factor instantaneous volatility were taken to be constant over time. The LMM was calibrated to

(i) all caplet volatilities,

(ii) swaption volatilities of swap rates 1 and 2.

The total instantaneous volatilities were set equal to the market observed caplet volatilities (in order to price all caplets correctly). The per-factor volatilities

Horizon	13 Years
Accrual Period	6 Months
Number of Forwards	25
Number of Factors	Variable
Yield Curve/Volatility Curve/Swaption Surface	EURO, Taken from 14/02/01
Number of Monte Carlo runs	100,000
Steps per Forward Period in MC Simulation	4
Spread Option	
Rate 1	1 Year into 10 Year Swap
Rate 2	1 Year into 1 Year Swap
Strike Rate	0.01
Expiry Date	14/02/03
Number of Fixings	1
Number of Payments	10
Market Volatility Swap Rate 1	0.1036
Market Volatility Swap Rate 2	0.1541

Table 5: Specification of Deal 2.

Number of Factors	MC Price of Spread Option (Bp)	MC Standard Error (Bp)
1	34.69	0.27
2	51.32	0.34
3	50.29	0.34
4	50.84	0.34
5	51.74	0.36
7	50.22	0.33
10	52.14	0.36

Table 6: Results for Deal 2.

were adjusted to calibrate to the two swaption volatilities and their correlation. Whenever the number of factors was larger than one, these fits were very close. In the case of only one factor, there will be a model instantaneous correlation of 1, at all times, which means that the model fits will be nothing like the market values.

The results may be found in Table 6. As expected, see Subsection 7.1, the one-factor spread option price makes no sense. Ignoring the one-factor case, the spread option price proves to be unaffected by the number of factors. This is contrary to results presented by Sidenius ([Sid00]), who reports of a serious increase in the price of a spread option when moving from 3 to 10 factors. He points out that this increase is due to differences in the calibration when varying the number of factors. Our results however show that if the LMM is calibrated to the relevant calibration objects, the influence of the number of factors is negligible. The relevant calibration objects for a spread option are the volatilities of the two rates.

Horizon	3 Years
Accrual Period	6 Months
Number of Forwards	5
Number of Factors	3
Yield Curve	EURO, Taken from 14/02/01
Number of Monte Carlo runs	100,000
Steps per Forward Period in MC Simulation	2,000

Table 7: Specification of Deal 3.

On the other hand, for risk management purposes one would like to value all interest rate derivatives in some portfolio using only one particular calibration. In such a case, the factor dependency in the prices of exotic derivatives will be inevitable, as Sidenius' results show.

This also illustrates that one should pay due attention as to how the LMM is calibrated when pricing interest rate derivatives.

8.3 Deal 3: Calibration of FRDLV LMM

Whereas spot dependent local volatility has been successfully applied in practice for equity derivatives, forward rate dependent local volatility for LMM will unlikely be applied in practice, according to our empirical results. The reason for the failure of the FRDLV LMM is twofold:

(i) The smile and skew in fixed income markets is not as pronounced as within equity markets.

(ii) The spreads in equity markets are relatively much higher than in fixed income markets. Much more precision is needed for interest rate derivatives.

As introducing forward rate dependency of local volatility is a very subtle change to the whole LMM, very small time steps have to be taken with a Monte Carlo simulation in order for the influence of forward rate dependency to kick in and become noticeable. Since the smile is more flawed than in equity markets (item (i)) and because higher precision is needed (item (ii)), for simulated prices to be sufficiently in range with their fitted prices, a severely small time step has to be taken leading to exorbitant amounts of computational time. This renders the FRDLV LMM useless for any practical applications.

For the purpose of showing that FRDLV LMM works at least in theory, market conditions were specified with smile and skew phenomena alike those observed in equity markets. A range of twelve strikes was chosen together with five forwards, yielding in total an amount of sixty distinct caplets to which the FRDLV LMM was fitted to. All sixty caplets were then priced using Monte Carlo. The outcomes were compared to Monte Carlo prices of the caplets obtained by an ordinary LMM that had only been calibrated to the caplets for one particular strike. Performances were compared.

The specification of Deal 3 is summarized in Table 7. The instantaneous

Discrete Barrier Cap	
Type	Down and Out
Number of Fixings	5
Barrier Rate	0.4
Strike Rate	0.5

Table 8: Specification of discrete barrier cap.

	Price of Discrete Barrier Cap (Bp)	MC Standard Error (Bp)
With FRDLV	115.12	0.63
Without FRDLV	119.67	0.67

Table 9: Results for Deal 4.

volatility had been written in terms of spherical coordinates. The total instantaneous volatility was determined by formula (40). The per-factor volatility was taken to be constant over time. The FRDLV LMM was calibrated to the caplet volatilities (including strike dependency) and the time zero forward rate correlations only. The market forward rate correlation structure was taken to be the same as in Deal 1, see equation (43). The per-factor volatilities were calibrated toward this correlation structure. The local volatility surfaces were pre-sampled and an interpolator was subsequently used whenever a local volatility was needed during the actual Monte Carlo simulation.

The results may be found in Figures 6 and 7.

The results show that the FRDLV LMM correctly prices caplets at different strikes, as opposed to the plain vanilla LMM, for which it is impossible to do so.

8.4 Deal 4: Skew dependency of barrier caps

The purpose of this Deal is to empirically illustrate the intuitive argument made in Subsection 7.2 about discrete down-and-out barrier caps being cheaper in the presence of a skewed interest rate market. To this end, a skewed interest rate market was considered and a discrete down-and-out barrier cap was priced with and without forward rate dependency (without forward rate dependency here means that for any evaluation of the instantaneous volatility, the forward rate was simply taken to be the time zero forward rate). The prices of the barrier caps with and without forward rate dependency are compared.

The specification of Deal 4 is the same as for Deal 3. The specification of the discrete barrier cap may be found in Table 8. All time zero LIBOR forward rates were in between the barrier and strike rate.

The result may be found in Table 9. The expected difference in prices for with or without FRDLV is small; this may be due to the fact that only a short maturity barrier cap was looked at.

Figure 6: Results for Deal 3 with FRDLV. For caplets with various strike rates and expiry times, the absolute error is plotted between market observed and MC simulated prices for the LMM *with* FRDLV.

Figure 7: Results for Deal 3 without FRDLV. For caplets with various strike rates and expiry times, the absolute error is plotted between market observed and MC simulated prices for the LMM *without* FRDLV.

9 Conclusion

The LIBOR market model is a state of the art tool for pricing and hedging interest rate derivatives. It is flexible; the LMM is able to price many different kinds of interest rate derivatives.

Care should be taken though when calibrating the LMM as an inappropriate calibration may affect prices of interest rate derivatives considerably.

Extending the LMM with forward rate dependent local volatility to enable the LMM to replicate the volatility smile observed in markets is theoretically sound but impractical due to the high amount of computational time needed. Future research on incorporating the volatility smile into the LMM will need to be on CEV[10], stochastic volatility[11] or jumps[12].

10 Summary

The LIBOR market model (LMM) is a state of the art tool for pricing and hedging interest rate derivatives. This thesis presents the theory of the LMM as well as practical issues arising with a computer implementation. Also, a novel extension is made to incorporate the market observed so-called “volatility smile” into the LMM, utilizing the concept of forward rate dependent instantaneous volatility, a concept that has already been successfully applied for equity derivatives. The smile-adjusted LMM proves to be theoretically sound but practically not useful due to the high amount of computational time needed. The thesis ends with presenting results of some empirical tests to illustrate the performance of the LMM and smile-adjusted LMM.

[10]The Constant Elasticity of Variance (CEV) model was proposed by [CoR76].

[11]See [Dup93a].

[12]For an overview of modeling the smile, see Chapter 17 in [Hull00].

A Tools from stochastic calculus

Throughout this paper, several theorems from stochastic calculus have been used. Those theorems are stated in this appendix. Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ on which a d-dimensional Brownian motion $W : [0,T] \times \Omega \to \mathbb{R}^d$ is given. The filtration $\mathbb{F}$ is the augmentation of the natural filtration generated by the Brownian motion.

Definition 30

(i) A martingale *X is an $\mathbb{F}$-adapted process, $X : [0,T] \times \Omega \to \mathbb{R}$, such that $X(t)$ is integrable for all $t \in [0,T]$ and such that*

$$\mathbb{E}\big[X(t)\big|\mathcal{F}(s)\big] = X(s) \text{ a.s.}, \;\; 0 \leq s \leq t \leq T.$$

(ii) A local martingale *X is a process, $X : [0,T] \times \Omega \to \mathbb{R}$, such that there exists a sequence $\{T_i\}_{i=1}^{\infty}$, $T_i \nearrow \infty$ as $i \to \infty$ a.s., such that X^{T_i} is a martingale for all i.*

(iii) Let X and Y be continuous local martingales. Then there is a unique (up to indistinguishability) adapted, continuous process of bounded variation $\langle X, Y\rangle : [0,T] \times \Omega \to \mathbb{R}$ starting from zero such that $XY - \langle X, Y\rangle$ is a continuous local martingale. $\langle X, Y\rangle$ is called the cross-variation *of X and Y.*

(iv) Write $\langle X\rangle = \langle X, X\rangle$. $\langle X\rangle$ is called the quadratic variation *of X and will be non-decreasing.* □

A proof of the following result may be found in for example [ReY91], Proposition 2.7.

Theorem 31 (Kunita-Watanabe identity) *Let $M : [0,T] \times \Omega \to \mathbb{R}$, $N : [0,T] \times \Omega \to \mathbb{R}$ be continuous local martingales and let $H : [0,T] \times \Omega \to \mathbb{R}$ be a locally bounded previsible process. Then*

$$\langle \int_0^{\cdot} H(s) \mathrm{d}M(s), N\rangle(t) = \int_0^t H(s) \mathrm{d}\langle M, N\rangle(s), \;\; 0 \leq t \leq T. \qquad \square$$

Corollary 32 (The cross-variation between two processes) *Let $X : [0,T] \times \Omega \to \mathbb{R}$ and $Y : [0,T] \times \Omega \to \mathbb{R}$ be $\mathbb{F}$-adapted continuous semi-martingales satisfying*

$$\frac{\mathrm{d}X(t)}{X(t)} = \mu_X(t)\mathrm{d}t + \beta_X(t) \cdot \mathrm{d}W(t), \tag{44}$$

$$\frac{\mathrm{d}Y(t)}{Y(t)} = \mu_Y(t)\mathrm{d}t + \beta_Y(t) \cdot \mathrm{d}W(t), \tag{45}$$

$0 \leq t \leq T$, for $\mathbb{F}$-previsible locally bounded processes $\mu_X, \mu_Y : [0,T] \times \Omega \to \mathbb{R}$, $\beta_X, \beta_Y : [0,T] \times \Omega \to \mathbb{R}^d$. Then

$$\mathrm{d}\langle X, Y\rangle(t) = X(t)Y(t)\beta_X(t) \cdot \beta_Y(t)\mathrm{d}t, \;\; 0 \leq t \leq T.$$

Proof: Apply Kunita-Watanabe twice. □

A proof of the following result may be found in for example [KaS91], Theorem 3.3.3.

Theorem 33 (Itô's formula) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^2(\mathbb{R}^n)$ and let $X : [0,T] \times \Omega \to \mathbb{R}^n$ be an $\mathbb{F}$-adapted continuous n-dimensional semi-martingale. Then*

$$\mathrm{d}f(X(t)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(t))\mathrm{d}X_i(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(X(t))\mathrm{d}\langle X_i, X_j\rangle(t),$$

for t, $0 \le t \le T$. □

Corollary 34 (The product rule for stochastic differentiation) *Suppose $X, Y : [0,T] \times \Omega \to \mathbb{R}$ are $\mathbb{F}$-adapted continuous semi-martingales. Then*

$$\mathrm{d}(XY)(t) = X(t)\mathrm{d}Y(t) + Y(t)\mathrm{d}X(t) + \mathrm{d}\langle X, Y\rangle(t), \quad 0 \le t \le T.$$

Proof: Apply Itô's formula to the function $u : \mathbb{R}^2 \to \mathbb{R}$, $u(x,y) = xy$ for $(x,y) \in \mathbb{R}^2$. □

Corollary 35 (The stochastic differential of the quotient of two processes) *Let X and Y be as in Corollary 32. Then*

$$\begin{aligned} \frac{\mathrm{d}(X/Y)(t)}{(X/Y)(t)} &= \Big(\mu_X(t) - \mu_Y(t) - \big(\beta_X(t) - \beta_Y(t)\big)\cdot\beta_Y(t) \Big)\mathrm{d}t \\ &\qquad \big(\beta_X(t) - \beta_Y(t)\big)\cdot \mathrm{d}W(t), \quad 0 \le t \le T. \end{aligned} \tag{46}$$

Proof: Firstly apply the product rule for stochastic differentiation to get

$$\frac{\mathrm{d}(X/Y)(t)}{(X/Y)(t)} = \frac{1}{X(t)}\mathrm{d}X(t) + Y(t)\mathrm{d}(\frac{1}{Y})(t) + \frac{Y(t)}{X(t)}\mathrm{d}\langle X, \frac{1}{Y}\rangle(t), \tag{47}$$

$0 \le t \le T$. Secondly, apply Itô's formula to the function $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$ for $x \in \mathbb{R}\setminus\{0\}$, to get

$$\begin{aligned} \frac{\mathrm{d}(1/Y)(t)}{(1/Y)(t)} &= -\frac{1}{Y(t)}\mathrm{d}Y(t) + \frac{1}{Y(t)}\mathrm{d}\langle Y\rangle(t) \\ &= \big(-\mu_Y(t) + \|\beta_Y(t)\|^2\big)\mathrm{d}t - \beta_Y(t)\cdot\mathrm{d}W(t), \end{aligned} \tag{48}$$

$0 \le t \le T$. Thirdly apply Kunita-Watanabe to get

$$\frac{Y(t)}{X(t)}\mathrm{d}\langle X, \frac{1}{Y}\rangle(t) = -\beta_X(t)\cdot\beta_Y(t)\mathrm{d}t, \quad 0 \le t \le T. \tag{49}$$

Fourthly and lastly substitute equations (44), (45), (48) and (49) into (47) to obtain (46). □

For a proof of the following theorem see [ReY91], Proposition 1.4.

Theorem 36 (Girsanov's theorem) *Let $M : [0,T] \times \Omega \to \mathbb{R}$ be a continuous local martingale such that the process $Z : [0,T] \times \Omega \to [0,\infty)$ defined by*

$$Z(t) \stackrel{\text{def}}{=} e^{M(t)-\frac{1}{2}\langle M\rangle(t)}, \quad 0 \le t \le T,$$

is a uniformly integrable martingale and let $X : [0,T] \times \Omega \to \mathbb{R}^d$ be a continuous local martingale, i.e., each of the components of X is a continuous local martingale. Define the process $\tilde{X} : [0,T] \times \Omega \to \mathbb{R}^d$ by

$$\tilde{X}_j(t) \stackrel{\text{def}}{=} X_j(t) - \langle X_j, M\rangle(t), \quad 0 \le t \le T, \quad j = 1, \ldots, d.$$

Then the process $\tilde{X}$ is a d-dimensional continuous local martingale on $(\Omega, \mathcal{F}, \mathbb{Q})$ where the probability measure $\mathbb{Q}$ is defined by

$$\mathbb{Q}(A) \stackrel{\text{def}}{=} \mathbb{E}\big[Z(T)1_A\big] \quad \text{for} \quad A \in \mathcal{F}(T).$$

$\mathbb{Q}$ is mutually absolutely continuous with respect to $\mathbb{P}$. □

The process Z is called the *Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$* and is denoted by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$.

For a proof of the following theorem, see [ReY91], Theorem 3.6, or also [GiS79], Theorem 3 in Chapter 1, §3.

Theorem 37 (Lévy's characterization of Brownian motion) *Let $X : [0,T] \times \Omega \to \mathbb{R}^d$ be a continuous local martingale starting from the origin, which satisfies*

$$\langle X_j, X_k\rangle(t) = \delta_{jk}t, \quad 0 \le t \le T, \quad 1 \le j,k \le d,$$

then X is a d-dimensional Brownian motion. □

Here δ_{jk} equals 1 if $j = k$ and 0 otherwise.

B Calculation of a caplet price within the LMM

In this appendix the derivation of (25) from (24) will be presented. To this end we compute $C = M\delta_n B_{n+1}(0)\mathbb{E}[(L_n(T_n) - K)_+]$ with $L_n(T_n) = L_n(0)e^Z$ where Z is normally distributed, $Z \sim \mathcal{N}(-\frac{1}{2}\tau^2, \tau^2)$. A Lemma will be used[13].

Lemma 38 *Let $(\Omega, \mathbb{P})$ be a probability space and suppose $Z : \Omega \to \mathbb{R}$ is a normally distributed random variable, $Z \sim \mathcal{N}(\beta, \tau^2)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Then*

$$\mathbb{E}\big[e^Z f(Z)\big] = e^{\beta+\frac{1}{2}\tau^2}\mathbb{E}\big[f(Z+\tau^2)\big].$$

Proof: Substitute $y = x - \tau^2$ when calculating the integral

$$\mathbb{E}\big[e^Z f(Z)\big] = \int_{-\infty}^{\infty} e^x f(x)\frac{1}{\sqrt{2\pi\tau^2}}e^{-\frac{(x-\beta)^2}{2\tau^2}}\mathrm{d}x$$

to get the desired result. □

[13]Lemma 38 has been taken from [Ken99]. For another interesting application of this Lemma, see [KAP00].

Using the above Lemma on the function f defined by $f(x) = 1_{\{L_n(0)e^x \geq K\}}$ for $x \in \mathbb{R}$ yields,

$$\begin{aligned} C &= M\delta_n B_{n+1}(0)\Big(\, L_n(0)\mathbb{E}\big[e^Z 1_{\{L_n(0)e^Z \geq K\}}\big] - K\mathbb{E}\big[1_{\{L_n(0)e^Z\geq K\}}\big]\,\Big) \\ &= M\delta_n B_{n+1}(0)\Big(\, L_n(0)\mathbb{E}\big[1_{\{e^{\tau^2}L_n(0)e^Z \geq K\}}\big] - K\mathbb{E}\big[1_{\{L_n(0)e^Z\geq K\}}\big]\,\Big). \end{aligned} \tag{50}$$

Note that

$$\{\, e^{\tau^2}L_n(0)e^Z \geq K \,\} = \Big\{\frac{-Z-\frac{1}{2}\tau^2}{\tau} \leq \frac{\log(\frac{L_n(0)}{K}) + \frac{1}{2}\tau^2}{\tau}\Big\}$$

and $(-Z - \frac{1}{2}\tau^2)/\tau$ is standard normally distributed, $\sim \mathcal{N}(0,1)$. Subsequently, $\mathbb{E}[1_{\{e^{\tau^2}L_n(0)e^Z\geq K\}}] = N(d_1)$, where $d_1 = (\log(L_n(0)/K) + \frac{1}{2}\tau^2)/\tau$. Likewise, $\mathbb{E}[1_{\{L_n(0)e^Z\geq K\}}] = N(d_2)$, where $d_2 = (\log(L_n(0)/K) - \frac{1}{2}\tau^2)/\tau$. Substituting this into equation (50) gives the desired result (25).

C Forward Kolmogorov equation

A theorem from diffusion theory is stated. In this Theorem, a statement is made about the generator of an Itô diffusion. For a proof of this statement see [Øks00], Theorem 7.3.3. For a proof of the forward Kolmogorov equation, see Exercise 8.3 in the same book.

Theorem 39 (Forward Kolmogorov equation or Fokker-Planck equation) *Let $X : [0,\infty)\times\Omega \to \mathbb{R}$ be an Itô diffusion in $\mathbb{R}$, i.e., suppose X satisfies*

$$\mathrm{d}X(t) = \mu(t, X(t))\mathrm{d}t + \sigma(t, X(t))\cdot \mathrm{d}W(t), \quad t \geq 0,$$

where $\mu : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ and $\sigma : [0,\infty)\times\mathbb{R}\to\mathbb{R}^d$ are continuous functions. If $f : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is in the domain of the generator A of X and if f is of class $C^{1,2}([0,\infty)\times\mathbb{R})$, then

$$(Af)(t,x) = \frac{\partial f}{\partial t}(t,x) + \mu(t,x)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\|\sigma(t,x)\|^2\frac{\partial^2 f}{\partial x^2}(t,x),$$

for $(t,x)\in[0,\infty)\times\mathbb{R}^N$. Assume that the transition measure of X has a transition density $p : D \to [0,\infty)$ on domain $D = \{(s,y;t,x)\in\mathbb{R}^4 :\ 0 \leq s < t\}$ such that

$$\mathbb{E}\big[f(X(t))\big|X(s) = y\big] = \int_0^\infty f(x)p(s,y;t,x)\mathrm{d}x,$$

for continuous functions $f : \mathbb{R}\to\mathbb{R}$, $0 \leq s < t$, $y\in\mathbb{R}$, and assume that the function which maps (t,x) onto $p(s,y;t,x)$ for $(t,x)\in(s,\infty)\times\mathbb{R}$ is of class $C^{1,2}((s,\infty)\times\mathbb{R})$ for all $(s,y)\in[0,\infty)\times\mathbb{R}$. Then p satisfies the forward Kolmogorov equation

$$\frac{\partial p}{\partial t}(s,y;t,x) = (A^*p)(s,y;t,x), \quad 0\leq s<t, \quad x,y\in\mathbb{R},$$

where A^* is the adjoint operator to A, defined by $\langle Af,\phi\rangle = \langle f, A^*\phi\rangle$ for f in the domain of A, ϕ of class $C^{1,2}([0,\infty)\times\mathbb{R})$. Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2([0,\infty)\times\mathbb{R},\mathcal{B},\mathcal{L})$ ($\mathcal{B}$ is the Borel σ-field – $\mathcal{L}$ is Lebesgue measure). Finally,

$$(A^*\phi)(t,x) = \frac{\partial^2(\|\sigma\|^2\phi)}{\partial x^2}(t,x) - \frac{\partial(\mu\phi)}{\partial x}(t,x),$$

for $(t,x)\in[0,\infty)\times\mathbb{R}$, ϕ of class $C^{1,2}([0,\infty)\times\mathbb{R})$. □

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