Daniell integrals and their induced measures

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1 Introduction

In the world of mathematics, measures play an integral role in many constructions. For example, probability laws can be described by measures, and most of integration theory rests on the pillar of measure theory. Constructing new measures is typically done by constructing set functions on suitable spaces and then applying Carathéodory's Extension Theorem to yield a measure on a larger space. However, this direct approach can often be tedious and undesirable. In this thesis, we consider an alternative approach to constructing measures that is based on Daniell integrals.

Introduced in 1918 by Percy Daniell in his paper 'A general form of Integral' [Dan18], Daniell integrals on certain sets of functions are defined axiomatically. The integrals then induce a measure on suitable spaces related to those sets of functions. The goal of this thesis is to fully outline and give an overview of the process in which Daniell integrals are defined, how measures arise from such integrals, and to introduce a mechanism for constructing new measures together with some examples of this mechanism in action.

The thesis is split into two main chapters. The first chapter concerns the Daniell integrals themselves: how they are defined and extended to larger spaces, with two examples of Daniell integrals given in Section 2.3. This chapter also provides in-depth coverage of how measures are induced by Daniell integrals (Section 2.4). Some parts of the theory are given more attention than is typically done in the literature. For example, in the construction of the concept of a Daniell integral, two often-used extensions are considered and we highlight their equivalence. Furthermore, the theory of the induced measures contains two separate approaches (one using Carathéodory's Extension Theorem and one using the theory of the extension of Daniell integrals) to give a more complete overview of the underlying ideas.

The second chapter primarily contains examples of measures that were induced by Daniell integrals in the manner described by the first chapter. We consider the Wiener measure, a measure on the space of trajectories of Wiener processes or Brownian motions, introduced by Norbert Wiener in his work *Differential Space* [Wie23] from 1923. Wiener used a specific type of Daniell integral in his original approach, which we follow here. Inspired by this integral, we introduce a generalization which makes use of so-called projective systems (Section 3.2). This generalization then forms the basis for a measure-constructing approach that we will apply in Section 3.2.3 in order to construct (probability) measures on an arbitrary infinite dimensional separable real Hilbert space. We also revisit some examples of Daniell-induced-measures by considering them as special cases of this generalization. The second chapter also introduces Gaussian measures, though their introduction serves mostly as a convenient example of a probability measure to be used in the final section.

2 Daniell integrals

2.1 A brief history of integration

Integration as a mathematical operation has quite a long history, dating all the way back to the Greek mathematician Eudoxus in about 408 B.C. ([Bur11]. p. 117). Early signs of integration can be found in the theory of quadratures: the process of finding the area of a geometric object by approximating it with squares, which have trivial area. This theory was only limited to sufficiently nice and symmetric shapes, and every subsequent historic step in integration would keep expanding the list of integrable objects.

The first major step beyond the Greeks' quadratures was taken in the 17th century when both Isaac Newton and Wilhelm Leibniz developed a theory of calculus where integration was seen as an opposite to differentiation: the integral was the *antiderivative* ([Sti89], p. 157). Although very practical for solving real world problems using differential equations and the like, this integral calculus had a shaky foundation.

The first major formalization of the integral, the Riemann integral, came from Bernhard Riemann in the 19th century ([Kat09], p. 785). In order the determine the area under the graph of a function, Riemann's integral partitioned the domain of the graph into pieces ever decreasing in size. Each piece's endpoints would then be lifted to the function value and the sum of the corresponding rectangles approximated the area under the graph. What made this more rigorous than other definitions was the novel and precise definition of a limit, like the epsilon-delta formulation of Augustin Cauchy, which made the process of partition pieces 'ever decreasing in size' rigorous.

Still, the Riemann integral had limitations in what functions could be integrated. It was known that the main condition for integration was continuity: if a function were to be sufficiently discontinuous, then the Riemann integral definition would fall apart. Furthermore, the complete classification of Riemann integrable functions was an awkward task, leading to yet another limitation of the concept. For a non-trivial example of a non-Riemann integrable function, take the Dirichlet function: a function which equals one on the rationals and zero otherwise

$$\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

The Dirichlet function is nowhere continuous, and cannot be Riemann integrated.

At the start of the 20th century, Henri Lebesgue managed to extend the integral even further with the use of measure theory ([Sti89], p. 531-533). He generalized the Riemann integral so that functions like the Dirichlet function could be integrated (and has integral 0), and any function which could be Riemann integrated would have a Lebesgue integral with the same value as the Riemann integral. The integral of a function would now be approximated in a limiting fashion by integrals of *simple functions*: functions that are linear combinations of indicator functions. The integrals of these indicator functions were then defined to simply be the measure of the set corresponding to the indicator function.

However, the Lebesgue integral was by no means the only extension of the Riemann integral in the 20th century. A different perspective was given in 1918 by Percy Daniell, a Chilean-born English mathematician, in his paper 'A general form of Integral' [Dan18]. The crucial difference between the methods of Daniell and Lebesgue was the point of axiomatization. Where Lebesgue used the measure as the driving force for defining the integral, Daniell instead defined the integrals themselves axiomatically and forewent any measure theory in the construction.

There used to be a time where the Daniell integral was regarded as the superior approach of the two, before the Lebesgue integral became the more popular choice once more ([Bog06b], p. 445). This was because the Daniell integral was more convenient when working on locally compact spaces, as one could then avoid having to work with measures that are not σ-finite. Furthermore, the Daniell integral was deemed easier to introduce to students since it avoided all the additional measure theoretic constructions of the Lebesgue integral.

Almost immediately after publication of Daniell's papers from 1918 onwards, Norbert Wiener started applying the concept of the Daniell integral to probabilistic concepts like Brownian motion ([Wie23], p. 132-174). Although the Daniell integral does not deal a priori with measure theory, there is a strong link between the two: under mild spatial assumptions, every Daniell integral on a vector lattice of functions on a space induces a unique measure on suitable σ-algebra of subsets of that space (see Section 2.4). It was this link that Wiener used to construct the now well known Wiener measure: a probability law on the space of continuous functions on $[0, 1]$, vanishing at zero (see Section 3.1.2).

Eventually, Lebesgue's definition gained in popularity once again, as it became clear that defining either the measure or the integral first can both yield equivalent integrals. Furthermore, measure theory's growth in the last century indicates the importance of its study, rendering the pedagogical argument unconvincing. Still, the Daniell integral can be used to construct measures in for example functional analysis (as is done in this thesis), and it can still be more convenient to use in certain frameworks, like the aforementioned locally compact spaces.

2.2 Definition and elementary properties

The following section follows the main results of Chapter 16 of the book 'Real Analysis' by H. L. Royden [Roy88], which itself follows Daniell's original approach found in [Dan18]. Several details have been added and different notation is introduced whenever needed. Recall the process in which the Lebesgue integral is defined:

Step 1: Given some measure μ on a measurable space, the Lebesgue integral of indicator functions $\mathbb{1}_A$ on measurable sets A is defined by $\int \mathbb{1}_A d\mu = \mu(A)$.

Step 2: Linear combinations of indicator functions form simple functions and the Lebesgue integral of a simple function is defined by $\int \sum_{i=1}^n c_i \mathbb{1}_{A_i} d\mu = \sum_{i=1}^n c_i \mu(A_i)$ for $c_i \in \mathbb{R}$ and disjoint measurable sets A_i.

Step 3: The Lebesgue integral of a non-negative measurable function f is defined by

$$\int f d\mu = \sup \left\{ \int g d\mu : g \text{ is a simple function with } 0 \leq g \leq f \right\}.$$

This integral can be shown to be additive and positively homogeneous with respect to scaling for all measurable f, as pointwise limits of simple functions g.

Step 4: The Lebesgue integral is extended to any integrable function f by splitting $f = f^+ - f^-$ into positive and negative parts $f^+ := \sup(f, 0)$ and $f^- := -\inf(f, 0)$, respectively. The integral of f is then defined by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

Here one is to assume that either $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$.

2.2.1 Daniell integration on the space L_0 of elementary functions

For some set X, let L_0 denote a set of bounded, real-valued functions over X: a starting point which we shall call the set of *elementary functions*. The specific choice of L_0 here depends on the setting and we will see several examples throughout this thesis. In essence, the Daniell approach considers the simple functions of step 2 above as merely one element of a larger class of possible sets of elementary functions and starts by defining an elementary integral on L_0: a Daniell integral. These elementary functions for which an integral can be conveniently expressed can then be seen as the 'squares' of a larger class of functions, analogous to the approximating squares used by the ancient Greeks.

We assume that L_0 is a vector lattice. First, this means that L_0 is a vector space, i.e., if $f, g \in L_0$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in L_0$. Second, there exists a partial ordering '$\leq$' that is compatible with the linear structure of L_0: if $f \leq g$ then for all $\alpha \in \mathbb{R}$ and $h \in L_0$ we have

$$\alpha f \leq \alpha g,$$

and

$$f + h \leq g + h.$$

Finally, this partial ordering is such that for all $f, g \in L_0$ the supremum and infimum of the two functions exist in L_0: we denote these by $f \vee g, f \wedge g \in L_0$, respectively. Furthermore, we define all these operations pointwise, which will become important in for example the third condition of the definition of a Daniell integral. Note that $f \vee g = (f - g) \vee 0 + g$ and $f \wedge g = f + g - (f \vee g)$. Therefore, for a space S to be a vector lattice it suffices that S is a vector space with a partial ordering such that $f^+ := f \vee 0 \in S$ for all $f \in S$.

It turns out that if one wants to extend this elementary Daniell integral to a 'useful' integral, i.e., an integral with all the properties of the Lebesgue integral, then the property of being a vector lattice of bounded functions on X is sufficient (see for example [Roy88]).

We now define a Daniell integral I_0 on L_0 as a real functional on L_0 with several natural properties.

Definition 2.1. *A real functional $I_0 : L_0 \to \mathbb{R}$ on some vector lattice L_0 is called a Daniell integral if for all $f, g \in L_0$ and $\alpha, \beta \in \mathbb{R}$ the following conditions are met:*

(1) $I_0(\alpha f + \beta g) = \alpha I_0(f) + \beta I_0(g)$ (linearity)

(2) If $f \geq 0$, then $I_0(f) \geq 0$ (non-negativity)

(3) If $f_n \downarrow 0$[1], then $I_0(f_n) \to 0$ (continuity with respect to monotone convergence)

We start with some useful properties of Daniell integrals that will be used later.

Lemma 2.2. *Let $f, g \in L_0$ with $f \leq g$ for all X. Then $I_0(f) \leq I_0(g)$.*

Proof. Since $f \leq g$, we have $g - f \geq 0$ and so by non-negativity of Daniell integrals we get $I_0(g - f) \geq 0$. We now conclude by linearity that $I_0(g) - I_0(f) \geq 0$, and hence that $I_0(f) \leq I_0(g)$. □

The third property, continuity with respect to monotone convergence, is equivalent to another similar but useful property.

Lemma 2.3. *For a Daniell integral I_0 on some vector lattice L_0 of real functions over a set X, the following are equivalent:*

(a) If $f_n \downarrow 0$ and $f_n \in L_0$, then $I_0(f_n) \to 0$.

(b) If $(f_n)_{n\in\mathbb{N}}$ is an increasing[2] sequence of functions in L_0, and if $\varphi \in L_0$ is a function such that $\varphi \leq \lim_{n\to\infty} f_n$, then $I_0(\varphi) \leq \lim_{n\to\infty} I_0(f_n)$.

Proof. We show both implications separately.

(a) ⇒ (b): Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of functions in L_0, and $\varphi \in L_0$ a function such that $\varphi \leq \lim_{n\to\infty} f_n$. Then $\varphi \vee f_n \in L_0$ and $\lim_{n\to\infty} \varphi \vee f_n = \lim_{n\to\infty} f_n$. Hence, we can define a non-negative sequence $(g_n)_{n\in\mathbb{N}}$ in L_0 by

$$g_n := (\varphi \vee f_n) - f_n \geq 0.$$

Notice that this sequence is also monotonically decreasing to 0. Namely, let $x \in X$. Since $(f_n)_{n\in\mathbb{N}}$ is increasing and $\varphi \leq \lim_{n\to\infty} f_n$, there exists some $m \in \mathbb{N}$ such that m is the first index for which $f_m(x) \geq \varphi(x)$ (m is possibly infinite if $\varphi = \lim_{n\to\infty} f_n$, in which case $g_n = \lim_{n\to\infty} f_n - f_n \downarrow 0$ is clear). For $n < m$, we have $(\varphi \vee f_n)(x) = \varphi(x)$ and so $g_n(x) = (\varphi - f_n)(x) \geq 0$, which decreases monotonically. For $n \geq m$, we have $(\varphi \vee f_n)(x) = f_n(x)$ and so $g_n(x) = 0$, so that $g_n(x) \downarrow 0$ indeed holds. Property (a) now implies that

$$\lim_{n\to\infty} I_0(g_n) = \lim_{n\to\infty} I_0((\varphi \vee f_n) - f_n) = 0,$$
$$\lim_{n\to\infty} I_0(\varphi \vee f_n) = \lim_{n\to\infty} I_0(f_n).$$

Since $\varphi \leq \varphi \vee f_n$, it follows by Lemma 2.2 that

[1] Note that $f_n \downarrow 0$ denotes a pointwise limit, i.e., $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in X$. All limits of sequences of functions in this thesis are taken as pointwise limits.

[2] By 'increasing' we in fact mean nondecreasing.

$$I_0(\varphi) \leq \lim_{n\to\infty} I_0(\varphi \vee f_n) = \lim_{n\to\infty} I_0(f_n).$$

This proves the first implication.

(b) ⇒ (a): Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence of functions in L_0 with $\lim_{n\to\infty} f_n = 0$. Then $-f_n \uparrow 0$ and so $\varphi \equiv 0$ is a function in L_0 such that $\varphi \leq \lim_{n\to\infty} -f_n = 0$. It then follows from property (b) that

$$I_0(\varphi) \leq \lim_{n\to\infty} I_0(-f_n) = -\lim_{n\to\infty} I_0(f_n),$$

or equivalently by the linearity of I_0

$$\lim_{n\to\infty} I_0(f_n) \leq I_0(\varphi) = 0.$$

Since $f_n \geq 0$ for all n, it follows from non-negativity of I_0 that $\lim_{n\to\infty} I_0(f_n) \geq 0$, and so we conclude that

$$\lim_{n\to\infty} I_0(f_n) = 0,$$

which proves the second implication. □

2.2.2 Extension to the spaces $L_0^{\uparrow}$ and $L_0^{\downarrow}$

Next in Daniell's construction, step 3 in the extension process of the Lebesgue integral is mimicked, extending our Daniell integral on L_0 by considering limits of increasing sequences of functions in L_0. Denote by $L_0^{\uparrow}$ the set of extended real-valued functions f on X which can be approximated pointwise by an increasing sequence $(f_n)_{n\in\mathbb{N}} \subset L_0$. In other words, these are all the functions f for which there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset L_0$ such that $f_n \uparrow f$ converges pointwise. Note that this pointwise convergence implies that $L_0^{\uparrow} \subset \{f : X \to \mathbb{R} \cup \{+\infty\}\}$, and so in general $f \in L_0^{\uparrow}$ is not necessarily real-valued anymore. Also note that we have that $L_0 \subset L_0^{\uparrow}$, since for any $f \in L_0$ the sequence defined by $f_n := f$ for all n shows us that $f \in L_0^{\uparrow}$.

Following the extension of L_0 to $L_0^{\uparrow}$, we would also like to extend our Daniell integral I_0 on L_0 to a linear functional $I^{\uparrow}$ on $L_0^{\uparrow}$. For any $f \in L_0^{\uparrow}$, the canonical choice for $I^{\uparrow}(f)$ is $I^{\uparrow}(f) = \lim_{n\to\infty} I_0(f_n)$, since the integrals $I_0(f_n)$ are all well-defined. Existence of this limit (which may be infinite) is guaranteed, since if $n \geq m$, then $f_n - f_m \geq 0$ and so by Lemma 2.2 we have

$$I_0(f_n) \geq I_0(f_m),$$

so that $\{I_0(f_n)\}$ is an increasing sequence of real numbers. This also shows by monotonicity that $I^{\uparrow}(f) = \lim_{n\to\infty} I_0(f_n) = \sup_n I_0(f_n)$ exists, as an element of $\mathbb{R} \cup \{+\infty\}$. It remains to show that the limit is well-defined, that is to say the integral $I^{\uparrow}(f) = \lim_{n\to\infty} I_0(f_n)$ is independent of the choice of sequence f_n with $\lim_{n\to\infty} f_n = f$. This result follows from the following lemma.

Lemma 2.4. *Let $(f_n)_{n\in\mathbb{N}}$ and $(g_m)_{m\in\mathbb{N}}$ be two increasing sequences in L_0. If $\lim_{n\to\infty} f_n \leq \lim_{m\to\infty} g_m$, then*

$$\lim_{n\to\infty} I_0(f_n) \leq \lim_{m\to\infty} I_0(g_m).$$

Proof. Fix $N \in \mathbb{N}$ arbitrarily. Then we have

$$f_N \leq \lim_{n\to\infty} f_n \leq \lim_{m\to\infty} g_m.$$

Then in particular by Lemma 2.3 property (b) we have

$$I_0(f_N) \leq \lim_{n\to\infty} I_0(g_m).$$

Since N was chosen arbitrarily, it follows that

$$\lim_{n\to\infty} I_0(f_n) \leq \lim_{n\to\infty} I_0(g_m),$$

which completes the proof. □

Let us now take two increasing sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_m)_{m\in\mathbb{N}}$ in L_0 with $f = \lim_{n\to\infty} f_n = \lim_{m\to\infty} g_m$, so that

$$\lim_{n\to\infty} f_n \leq \lim_{m\to\infty} g_m,$$

and

$$\lim_{m\to\infty} g_m \leq \lim_{n\to\infty} f_n.$$

Applying Lemma 2.4 twice yields

$$\lim_{n\to\infty} I_0(f_n) = \lim_{m\to\infty} I_0(g_m),$$

and so the limit $I^{\uparrow}(f) = \lim_{n\to\infty} I_0(f_n)$ is indeed well-defined.

One problem that arises is that $L_0^{\uparrow}$ is not a vector space: if $0 \neq f \in L_0^{\uparrow}$ so that $f_n \uparrow f$ for some sequence $(f_n)_{n\in\mathbb{N}}$ in L_0, then $-f$ is not necessarily the limit of some increasing sequence. However, $L_0^{\uparrow}$ is a lattice, since if $g_n \uparrow g$ is another sequence in L_0, then

$$(f_n \vee g_n) \uparrow (f \vee g),$$
$$(f_n \wedge g_n) \uparrow (f \wedge g).$$

If we restrict linearity only to multiplication by non-negative constants, then since the limit operation is linear, we have extended our Daniell integral on L_0 to a real-valued positively-linear functional on $L_0^{\uparrow}$ given by

$$I^{\uparrow}(f) := \lim_{n\to\infty} I_0(f_n),$$

where $f_n \uparrow f$ and $f_n \in L_0$ for all n.

Completely analogously to $L_0^{\uparrow}$, we can also define the set $L_0^{\downarrow}$ as the set of functions f on X which can be approximated pointwise by a *decreasing* sequence $(f_n)_{n\in\mathbb{N}}$ in L_0. In a similar fashion as before, we note that $L_0^{\downarrow} \subset \{f : X \to \mathbb{R} \cup \{-\infty\}\}$. The set $L_0^{\downarrow}$ will play an important role in Section 2.2.4. For any $f \in L_0^{\uparrow}$ with corresponding increasing sequence $(f_n)_{n\in\mathbb{N}}$ in L_0 we have that $-(f_n)_{n\in\mathbb{N}}$ is decreasing with limit $-f \in L_0^{\downarrow}$, and vice versa. Hence, it follows that $L_0^{\downarrow} = -L_0^{\uparrow} = \{-f : f \in L_0^{\uparrow}\}$ and that we can also extend our Daniell integral on L_0 to a positively-linear functional on $L_0^{\downarrow}$: for $f_n \downarrow f$ and $f_n \in L_0$ for all n we have

$$I^{\downarrow}(f) := \lim_{n\to\infty} I_0(f_n) = \lim_{n\to\infty} -I_0(-f_n) = -I^{\uparrow}(-f).$$

For the sake of brevity, we will sometimes write $f \in L_0^{\updownarrow}$ to mean $f \in L_0^{\uparrow}$ or $f \in L_0^{\downarrow}$ and $I^{\updownarrow}$ to mean either $I^{\uparrow}$ or $I^{\downarrow}$. Furthermore, we have the following properties which will prove useful later.

Lemma 2.5. *If $f, g \in L_0^{\updownarrow}$ such that $f \leq g$, then $I^{\updownarrow}(f) \leq I^{\updownarrow}(g)$.*

Proof. Since $I^{\updownarrow}$ is linear, the proof is analogous to the proof of Lemma 2.2. □

Lemma 2.6. *For all $f \in L_0^{\updownarrow}$, $g \in L_0$ we have $f \pm g \in L_0^{\updownarrow}$.*

Proof. Let $f \in L_0^{\uparrow}$ so that there exists some sequence $(f_n)_{n\in\mathbb{N}} \subset L_0$ such that $f_n \uparrow g$. Define a new sequence $\{h_n\}$ with

$$h_n := f_n \pm g \in L_0,$$

for all n since L_0 is a vector space. Then $h_n \uparrow (f \pm g)$ and we conclude that $f \pm g \in L_0^{\uparrow}$. The proof for $f \in L_0^{\downarrow}$ is analogous to the case for $g \in L_0^{\uparrow}$. □

Lemma 2.7. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions on X. If $f_n \uparrow f$ and $f_n \in L_0^{\uparrow}$ for all n, then $f \in L_0^{\uparrow}$. Furthermore, $\lim_{n\to\infty} I^{\uparrow}(f_n) = I^{\uparrow}(f)$.*

Proof. Suppose that $f_n \uparrow f$. Since every $f_n \in L_0^\uparrow$, for every n we have a sequence $(f_{nm})_{m\in\mathbb{N}}$ of functions in L_0 such that $f_{nm} \uparrow f_n$. Define

$$g_n := f_{1n} \vee f_{2n} \vee \cdots \vee f_{nn},$$

that is, g_n is the supremum of the n-th functions for the first n sequences $(f_{nm})_{m\in\mathbb{N}}$. Since L_0 is a lattice, all $f_{in} \in L_0$, and $(f_{nm})_{m\in\mathbb{N}}$ is increasing, it follows that $(g_n)_{n\in\mathbb{N}}$ is an increasing sequence in L_0.

It remains to show that $g_n \uparrow f$, which would prove that $f \in L_0^\uparrow$. Since $f_{nm} \uparrow f_n$ for all n, we have $f_{nm} \leq f_n$ for all n and m and hence

$$g_n = f_{1n} \vee f_{2n} \vee \cdots \vee f_{nn} \leq f_1 \vee f_2 \vee \cdots \vee f_n = f_n.$$

This implies that $\lim_{n\to\infty} g_n \leq \lim_{n\to\infty} f_n$. Now fix some $k \in \mathbb{N}$ arbitrarily. Then for all $n \geq k$ we have

$$g_n = f_{1n} \vee f_{2n} \vee \cdots \vee f_{kn} \vee \cdots \vee f_{nn} \geq f_{kn}.$$

This implies that $\lim_{n\to\infty} g_n \geq \lim_{n\to\infty} f_{kn} = f_k$ for all k, and so that

$$\lim_{n\to\infty} g_n \geq \lim_{k\to\infty} f_k.$$

Combining these results, we get

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} f_n = f,$$

so that $g_n \uparrow f$ indeed holds. It follows that $f \in L_0^\uparrow$. By the arguments above, we have

$$f_{kn} \leq g_n \leq f_n,$$

for all $n \geq k$, and $g_n \leq g_{n+1}$. Using Lemmas 2.2 and 2.5, we can translate these inequalities to the integrals and get

$$I_0(f_{kn}) \leq I_0(g_n) \leq I^\uparrow(f_n),$$

for all $n \geq k$. By definition of $I^\uparrow$, it now follows that

$$\lim_{n\to\infty} I^\uparrow(f_n) = \lim_{n\to\infty} \lim_{k\to\infty} I_0(f_{kn}) \leq \lim_{n\to\infty} I_0(g_n)$$

and

$$\lim_{n\to\infty} I^\uparrow(f_n) \geq \lim_{n\to\infty} I_0(g_n),$$

and so we have

$$\lim_{n\to\infty} I^\uparrow(f_n) = \lim_{n\to\infty} I_0(g_n) = I^\uparrow(\lim_{n\to\infty} g_n) = I^\uparrow(f).$$

We conclude that $f \in L_0^\uparrow$ and $\lim_{n\to\infty} I^\uparrow(f_n) = I^\uparrow(f)$. □

Lemma 2.7 above has an analogue for functions in $L_0^\downarrow$ as well: the proof of that result is constructed using the infimum to define the sequence g_n, instead of the supremum.

2.2.3 Extension to the vector lattice L

For the final step, we finish the extension process of L_0 by introducing the set L of *integrable* functions on X. It will be a vector lattice containing L_0, and it comes equipped with a Daniell integral I that extends I_0 on L_0. To this end, we define the upper and lower integrals.

Definition 2.8. *For an arbitrary function $f : X \to \mathbb{R}$, the upper and lower integrals $\bar{I}(f)$ and $\underline{I}(f)$ are defined by*

$$\bar{I}(f) := \inf_{g \geq f, g \in L_0^{\uparrow}} I^{\uparrow}(g),$$

and

$$\underline{I}(f) := \sup_{h \leq f, h \in L_0^{\downarrow}} I^{\downarrow}(h) = -\bar{I}(-f), \tag{1}$$

respectively.

Lemma 2.9. *Let f, g be arbitrary real-valued functions on X. The upper integral $\bar{I}$ has the following properties:*

(1) $\bar{I}(cf) = c\bar{I}(f)$ for all $c \geq 0$ (positive homogeneity)

(2) If $f \leq g$, then $\bar{I}(f) \leq \bar{I}(g)$ (monotonicity)

(3) $\bar{I}(f+g) \leq \bar{I}(f) + \bar{I}(g)$ (subadditivity)

(4) If in particular, $f \in L_0^{\uparrow}$, then $\bar{I}(f) = I^{\uparrow}(f)$ (extension of $I^{\uparrow}$)

Proof. (1) and (2) follow directly from the properties of $I^{\uparrow}$.

(3) We have

$$\bar{I}(f+g) = \inf_{h \geq f+g, h \in L_0^{\uparrow}} I^{\uparrow}(h) \leq \inf_{h_1 \geq f, h_1 \in L_0^{\uparrow}} I^{\uparrow}(h_1) + \inf_{h_2 \geq g, h_2 \in L_0^{\uparrow}} I^{\uparrow}(h_2),$$

since if $h_1 \geq f$ and $h_2 \geq g$ with $h_1, h_2 \in L_0^{\uparrow}$, then $h := h_1 + h_2 \geq f + g$ and $h \in L_0^{\uparrow}$.

(4) Let $f \in L_0^{\uparrow}$. Then since $f \geq f$, we have

$$\bar{I}(f) = \inf_{g \geq f, g \in L_0^{\uparrow}} I^{\uparrow}(g) \leq I^{\uparrow}(f).$$

Meanwhile, since $g \geq f$ implies $I^{\uparrow}(g) \geq I^{\uparrow}(f)$, we have that $I^{\uparrow}(f) \leq \bar{I}(f)$ and hence that $\bar{I}(f) = I^{\uparrow}(f)$. □

An analogue to Lemma 2.9 above also holds for $\underline{I}$, although instead of subadditivity we have superadditivity:

$$\underline{I}(f+g) \geq \underline{I}(f) + \underline{I}(g).$$

This analogue can be easily obtained by using the results from Lemma 2.9 and applying equation (1).

Next, we define the extended set L of integrable functions on X to be the functions for which the upper and lower integrals coincide.

Definition 2.10. *Let L_0 be a vector lattice of elementary functions on X together with Daniell integral I_0. The set L of functions $f : X \to \mathbb{R}$ with*

$$-\infty < \bar{I}(f) = \underline{I}(f) < \infty,$$

is called the set of integrable functions. For any integrable f, we write

$$I(f) := \bar{I}(f) = \underline{I}(f).$$

To justify the words 'upper' and 'lower' in the definitions of $\bar{I}(f)$ and $\underline{I}(f)$, respectively: for any $f : X \to \mathbb{R}$, we have that

$$0 = I(0) = I(f-f) = \bar{I}(f-f) = \inf_{g \geq f-f, g \in L_0^\uparrow} I^\uparrow(g) \leq \bar{I}(f) + \bar{I}(-f).$$

Hence $\underline{I}(f) = -\bar{I}(-f) \leq \bar{I}(f)$ as one would naturally expect.

Now we would like to say that L is an extension of $L_0^\updownarrow$, and this is almost the case. The problem is that for functions $f \in L_0^\updownarrow$ the integral $I^\updownarrow(f)$ as well as some values of f might be infinite. Such functions are excluded from L by definition, and to circumvent this issue we introduce the restriction

$$L_0^{\updownarrow,\text{fin}} := \{f \in L_0^\updownarrow : f \text{ is real-valued and } |I^\updownarrow(f)| < \infty\},$$

so that $L_0^{\updownarrow,\text{fin}} \subset L$ holds. The validity of this last inclusion is one of the results of the following proposition.

Proposition 2.11. *The set of integrable functions L is a vector lattice, and the functional I is linear and extends the functional $I^\updownarrow$ on $L_0^{\updownarrow,fin}$ to L.*

Proof.

1. First, we show that L has a vector space structure and that I is linear. Let $f \in L$. Then for any scalar $c \geq 0$ we have by Lemma 2.9 that

$$\bar{I}(cf) = c\bar{I}(f) = c\underline{I}(f) = \underline{I}(cf) < \infty.$$

Since $\underline{I}(f) := -\bar{I}(-f)$, it follows that for any $c \leq 0$ we have that

$$\bar{I}(cf) = \bar{I}(-|c|f) = -\underline{I}(|c|f) = c\underline{I}(f) = c\bar{I}(f) = \underline{I}(cf) < \infty.$$

Now let $f, g \in L$. Then Lemma 2.9 yields

$$\bar{I}(f+g) \leq \bar{I}(f) + \bar{I}(g) = I(f) + I(g),$$

and likewise

$$-\underline{I}(f+g) = \bar{I}(-f-g) \leq \bar{I}(-f) + \bar{I}(-g) = -I(f) - I(g),$$

or

$$\underline{I}(f+g) \geq I(f) + I(g).$$

Since $\underline{I}(f+g) \leq \bar{I}(f+g)$, it follows that

$$\underline{I}(f+g) = \bar{I}(f+g) = I(f+g) = I(f) + I(g) < \infty.$$

This shows that $f+g \in L$ and that I is linear. We conclude that L is a linear space.

2. Second, we show that L is a lattice. As noted before, it suffices to show that for any $f \in L$, we have that $f^+ = f \vee 0 \in L$. Let $f \in L$ and let $\varepsilon > 0$. Consider functions $g, h \in L_0^\uparrow$ with $-h \leq f \leq g$ that are 'ε-close' to f in the sense that

$$I^\uparrow(g) < I(f) + \varepsilon < \infty,$$

and

$$I^\uparrow(h) \leq -I(f) + \varepsilon < \infty.$$

Note that such g and h exist precisely because $I(f)$ is finite. Since $L_0^\uparrow$ is a lattice, we have that $g \vee 0 \in L_0^\uparrow$. Furthermore, since $g = g \vee 0 + g \wedge 0$, and $g \wedge 0 \leq g$, which implies that $I^\uparrow(g \wedge 0) \leq I^\uparrow(g) < \infty$, we have

$$I^\uparrow(g \vee 0) \leq I^\uparrow(g) - I^\uparrow(g \wedge 0) < \infty.$$

Likewise, $h \wedge 0 \in L_0^\uparrow$ with $I^\uparrow(h \wedge 0) > -\infty$. It follows that $-(h \wedge 0) \leq f^+ \leq g \vee 0$, and hence that

$$-\infty < -I^{\uparrow}(h \wedge 0) \le \underline{I}(f^+) \le \bar{I}(f^+) \le I^{\uparrow}(g \vee 0) < \infty.$$

Now note that since $g \ge -h$, we have that

$$g \vee 0 + h \wedge 0 = g \vee 0 + h - h \vee 0 = g \vee 0 + (-h) \wedge 0 + h \le g \vee 0 + g \wedge 0 + h = g + h.$$

Hence, it follows that

$$I^{\uparrow}(g \vee 0) + I^{\uparrow}(h \wedge 0) \le I^{\uparrow}(g) + I^{\uparrow}(h) \le I(f) + \varepsilon - I(f) + \varepsilon = 2\varepsilon.$$

Combining this with $-I^{\uparrow}(h \wedge 0) \le \underline{I}(f^+) \le \bar{I}(f^+) \le I^{\uparrow}(g \vee 0)$, it follows that

$$\bar{I}(f^+) - \underline{I}(f^+) \le I^{\uparrow}(g \vee 0) + I^{\uparrow}(h \wedge 0) < 2\varepsilon.$$

Since ε was chosen arbitrarily, this implies that $\bar{I}(f^+) = \underline{I}(f^+)$ and hence that $f^+ \in L$. We conclude that L is a vector lattice.

3. Third, we show that I extends $I^{\updownarrow}$ on $L_0^{\updownarrow,\text{fin}}$ to L. Let $f \in L_0^{\uparrow,\text{fin}}$. By Lemma 2.9 (4), we have $\bar{I}(f) = I^{\uparrow}(f)$. By definition, there exists an increasing sequence $f_n \uparrow f$ with $f_n \in L_0$ for all n. For all n, we have $-f_n \in L_0 \subset L_0^{\uparrow}$, and so by $\bar{I}(f) = I^{\uparrow}(f)$ we have that

$$\underline{I}(f_n) = -\bar{I}(-f_n) = -I_0(-f_n) = I_0(f_n).$$

Furthermore, since $f \ge f_n$, we have $\underline{I}(f) \ge \underline{I}(f_n) = I_0(f_n)$. It follows that

$$\underline{I}(f) \ge \lim_{n\to\infty} \underline{I}(f_n) = \lim_{n\to\infty} I_0(f_n) = I^{\uparrow}(f) > -\infty.$$

Since $\underline{I}(f) \le \bar{I}(f)$, we have $I(f) = \underline{I}(f) = \bar{I}(f) = I^{\uparrow}(f)$ and so $f \in L$. We conclude that I restricted to $L_0^{\uparrow,\text{fin}}$ equals $I^{\uparrow}$. In other words, I extends $I^{\uparrow}$ to L. Since $L_0^{\downarrow} = -L_0^{\uparrow}$, and L is linear, it immediately also follows that $L_0^{\downarrow,\text{fin}} \subset L$ and that I extends $I^{\downarrow}$ to L.

□

Because the linear functional I is indeed an extension of $I^{\updownarrow}$ to L, for the remainder of this section it is no longer necessary to distinguish the integrals I_0, $I^{\uparrow}$, $I^{\downarrow}$ and I, and so we will choose not to do so. The following lemma is used in the proof of Proposition 2.14.

Lemma 2.12. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative functions and let $f := \sum_{n=1}^{\infty} f_n$. Then $\bar{I}(f) \le \sum_{n=1}^{\infty} \bar{I}(f_n)$.*

Proof. We may assume that $\bar{I}(f_n) < \infty$ for all n, because if this is not the case, then the inequality holds trivially. Now, for any $\varepsilon > 0$ there exists a sequence of 'ε-close' functions $g_n \in L_0^{\uparrow}$ such that for all n we have $f_n \le g_n$ and

$$I(g_n) \le \bar{I}(f_n) + \varepsilon \cdot 2^{-n}.$$

Since $g_n \in L_0^{\uparrow}$, for every fixed n there exists a sequence $\varphi_{n,m} \uparrow g_n$ such that $\varphi_{n,m} \in L_0$ for all $m \in \mathbb{N}$. Using this, we define a new sequence $\psi_{n,1} := \varphi_{n,1} \ge 0$, $\psi_{n,m} := \varphi_{n,m} - \varphi_{n,m-1} \ge 0$ for $m > 1$. We can assume that $\{\psi_{n,m}\}_{m\in\mathbb{N}}$ is non-negative by replacing $\varphi_{n,m}$ by $\varphi_{n,m} \vee 0$, which does not change the limit nor the fact that the sequence is in L_0. It follows that we have a telescoping series

$$g_n = \sum_{m=1}^{\infty} \psi_{n,m},$$

as every term of $\varphi_{n,m}$ gets cancelled out except for the limit $\lim_{m\to\infty} \varphi_{n,m} = g_n$ by definition. As a result, we have that

$$g := \sum_{n=1}^{\infty} g_n = \sum_{n,m\geq 1} \psi_{n,m}.$$

This shows that $g \in L_0^{\uparrow}$: we have an increasing sequence of partial sums $h_k \uparrow g$ for $h_k \in L_0$ described by

$$h_k := \sum_{n=1}^{k} \sum_{m=1}^{k} \psi_{n,m}.$$

The fact that the sequence is increasing follows from the fact that all $\psi_{n,m}$ are non-negative. We now have

$$I(g) = \sum_{n,m\geq 1} I(\psi_{n,m}) = \sum_{n=1}^{\infty} I(g_n) \leq \sum_{n=1}^{\infty} \bar{I}(f_n) + \varepsilon < \infty.$$

Since $f_n \leq g_n$, it naturally follows that $f \leq g$, and hence by Lemma 2.9 we have

$$\bar{I}(f) \leq I(g) \leq \sum_{n=1}^{\infty} \bar{I}(f_n) + \varepsilon.$$

Since ε was chosen as an arbitrary positive number, we conclude that

$$\bar{I}(f) \leq \sum_{n=1}^{\infty} \bar{I}(f_n),$$

which completes the proof. □

While the following proposition is used to prove Proposition 2.14, it also gives us a condition for which increasing sequences of functions in L are closed under taking pointwise limits. In Section 2.4.1 we will see that this result, which is in fact a version of the monotone convergence theorem for the Daniell integral I, is quite important.

Proposition 2.13. *Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of functions in L and let $f = \lim_{n\to\infty} f_n$ be a real-valued function. Then $f \in L$ if and only if $\lim_{n\to\infty} I(f_n) < \infty$. In this case, we have $I(f) = \lim_{n\to\infty} I(f_n)$.*

Proof. By definition, we have $f \geq f_n$ and hence $\bar{I}(f) \geq I(f_n)$. This means that if $\lim_{n\to\infty} I(f_n) = \infty$, then $\bar{I}(f) = \infty$ and $f \notin L$. This proves the first implication.

For the second implication, suppose that $\lim_{n\to\infty} I(f_n) < \infty$. Let $f_n \uparrow f$ be an increasing sequence in L. Define $g := f - f_1$. Then since f_n is increasing, $g \geq 0$ and

$$g = \sum_{n=1}^{\infty} (f_{n+1} - f_n),$$

since all the functions other than f and f_1 are cancelled out in the infinite sum. Applying Lemma 2.12, we get

$$\bar{I}(g) \leq \sum_{n=1}^{\infty} I(f_{n+1} - f_n) = \sum_{n=1}^{\infty} I(f_{n+1}) - I(f_n) = \lim_{n\to\infty} I(f_n) - I(f_1).$$

As a result, Lemma 2.9 implies that

$$\bar{I}(f) = \bar{I}(f_1 + g) \leq I(f_1) + \bar{I}(g) \leq \lim_{n\to\infty} I(f_n).$$

Since $f \geq f_n$, we have $\underline{I}(f) \geq f_n$ and hence

$$\underline{I}(f) \geq \lim_{n\to\infty} I(f_n).$$

From this we deduce that $\bar{I}(f) = \underline{I}(f) = \lim_{n\to\infty} I(f_n) < \infty$, and so $f \in L$. We conclude that the stated equivalence holds. □

An analogous result holds for decreasing sequences of functions in L, if we require that $\lim_{n\to\infty} I(f_n) > -\infty$ instead. Furthermore, a consequence of Proposition 2.13 above is that the extension process from L_0 to L is complete in the following sense: if E is an operator on sets that extends a set of elementary functions L_0 to a set of integrable functions L using the process outlined in this section, then $E(L_b) = L$, where L_b is the set of all integrable functions that are bounded.

We now combine the results above to show that the extension process of L_0 to L produces a Daniell integral on a vector lattice, as expected.

Proposition 2.14. *The functional I is a Daniell integral on L.*

Proof. By Proposition 2.11 part 1, I is a linear functional and by part 2, L is a vector lattice. Let $f \in L$ with $f \geq 0$. Then

$$I(f) = \bar{I}(f) = \inf_{g \geq f, g \in L_0^{\uparrow}} I(g) \geq 0,$$

since $g \geq f \geq 0$, which implies $I(g) \geq 0$ for all g by the non-negativity of I on $L_0^{\uparrow}$. Hence I is non-negative on L.

It remains to show continuity with respect to monotone convergence. Consider a sequence $f_n \downarrow 0$ of functions in L. Since the zero function $0 \in L$ with $I(0) = 0$, Proposition 2.13 tells us that $\lim_{n\to\infty} I(-f_n) = I(0) = 0$, as $-f_n \uparrow 0$. Therefore, $I(f_n) \to 0$ as $n \to \infty$ and we see that I is continuous with respect to monotone convergence. We conclude that I is indeed a Daniell integral on L. ☐

2.2.4 Summable functions: an alternative extension

Sometimes in the literature the set L_0 is extended in a different manner than Daniell's method of the previous section. In this section, we consider an approach similar to the approach found in Chapter 2 of [GS78]. Recall the definition

$$L_0^{\uparrow,\text{fin}} := \{f \in L_0^{\uparrow} : f \text{ is real-valued and } I(f) < \infty\}.$$

Instead of considering the set of functions $f : X \to \mathbb{R}$ for which $-\infty < \bar{I}(f) = \underline{I}(f) < \infty$, one considers the set $\hat{L}$ of functions f that can be written as a difference $f = f_1 - f_2$ for $f_1, f_2 \in L_0^{\uparrow,\text{fin}}$. This is similar to the fourth and final step of the Lebesgue approach. Restricting to $L_0^{\uparrow,\text{fin}}$ guarantees that $I^{\uparrow}(f_1) - I^{\uparrow}(f_2)$ is well-defined, which leads to the following definition.

Definition 2.15. *The set $\hat{L} := L_0^{\uparrow,fin} - L_0^{\uparrow,fin}$ of functions $f : X \to \mathbb{R}$ such that $f = f_1 - f_2$ for some $f_1, f_2 \in L_0^{\uparrow,fin}$ is called the set of summable functions. For any summable function $f = f_1 - f_2$, we write*

$$\hat{I}(f) := I^{\uparrow}(f_1) - I^{\uparrow}(f_2).$$

The facts that $\hat{L}$ is a vector lattice and $\hat{I}$ is a Daniell integral follow from the properties of $L_0^{\uparrow}$ and $I^{\uparrow}$ together with the definitions. Since the zero function is in $L_0^{\uparrow,\text{fin}}$, it immediately follows that $L_0 \subset L_0^{\uparrow,\text{fin}} \subset \hat{L}$, so that $\hat{I}(f) = I_0(f)$ for all $f \in L_0$. Furthermore, the definition for $\hat{I}$ is well-defined: let $f = f_1 - f_2 = g_1 - g_2$ for $f_1, f_2, g_1, g_2 \in L_0^{\uparrow,\text{fin}}$. Since $f_1 - f_2$ and $g_1 - g_2$ are not necessarily elements of $L_0^{\uparrow,\text{fin}}$ (else this extension would be trivial), we rewrite the equation as

$$\begin{aligned} f_1 + g_2 &= g_1 + f_2 \\ I^{\uparrow}(f_1 + g_2) &= I^{\uparrow}(g_1 + f_2) \\ I^{\uparrow}(f_1) + I^{\uparrow}(g_2) &= I^{\uparrow}(g_1) + I^{\uparrow}(f_2) \\ I^{\uparrow}(f_1) - I^{\uparrow}(f_2) &= I^{\uparrow}(g_1) - I^{\uparrow}(g_2) = \hat{I}(f), \end{aligned}$$

from which it follows that Definition 2.15 is well-defined.

Naturally, we would like to compare the extensions $\hat{L}$ and L. It is clear that $\hat{L} \subset L$, for L is a vector lattice and hence the difference between two elements $f_1, f_2 \in L_0^{\uparrow,\text{fin}}$ is once again an element of L. The other inclusion does not hold in general. It turns out that we need to slightly alter our definitions of L and $\hat{L}$ if we want to say more about the connection between these two vector lattices.

Let us start by extending $\hat{L}$. In order to define this extension, we first need a notion of sets of measure zero without actually having defined a measure. In Section 2.4 it will become clear that this condition is indeed equivalent to a set of measure zero with respect to an actual measure.

Definition 2.16. *A $Z \subset X$ is called a set of measure zero if, given any $\varepsilon > 0$, there exists an increasing sequence of non-negative functions $f_n \in L_0$ such that $\lim_{n\to\infty} I_0(f_n) < \varepsilon$ and*

$$\lim_{n\to\infty} f_n(x) \geq 1,$$

for all $x \in Z$. We will define the empty set to be a set of measure zero as well.

As is standard in measure theory, we say that an increasing sequence of functions f_n converges to f 'almost everywhere' (or a.e. for short) if $f_n(x) \uparrow f(x)$ for all x on a set of *full measure*, i.e., except for $x \in Z$ where Z is a set of measure zero.

Next, we consider a set quite similar to $L_0^{\uparrow,\text{fin}}$. Let us write $L_0^{\uparrow,\text{a.e.}}$ for all the extended real-valued functions f on X such that there exists an increasing sequence of functions in L_0 that converge to f almost everywhere and such that $\lim_{n\to\infty} I_0(f_n) < \infty$. Just like before, we can define a real functional $I^\uparrow$ (which we will denote by the same symbol as the functional on $L_0^{\uparrow,\text{fin}}$) defined by

$$I^\uparrow(f) := \lim_{n\to\infty} I_0(f_n),$$

for all $f \in L_0^{\uparrow,\text{a.e.}}$. The fact that this definition is well-defined (among all the other results for $I^\uparrow$) follows from the results in Section 2.2.2. Functions in $L_0^{\uparrow,\text{a.e.}}$ have the following nice property.

Lemma 2.17. *Every function $f \in L_0^{\uparrow,a.e.}$ is finite almost everywhere.*

Proof. Let $Z \subset X$ be the set of all x for which $f(x) = +\infty$. Let $f_n \uparrow f$ be a sequence in L_0. Without loss of generality we may assume that $f_n(x) \geq 0$ for all $x \in Z$, simply by considering $\tilde{f}_n := f_n - f_1 \in L_0$. Now let $\varepsilon > 0$ be arbitrary and consider some upper bound $C \in \mathbb{R}$ such that $\lim_{n\to\infty} I_0(f_n) < C < \infty$. Then for all $x \in Z$ we have

$$\lim_{n\to\infty} \frac{\varepsilon f_n(x)}{C} \geq 1,$$

since $\lim_{n\to\infty} f_n(x) = +\infty$, and

$$\lim_{n\to\infty} I_0\left(\frac{\varepsilon f_n(x)}{C}\right) = \frac{\varepsilon}{C} \lim_{n\to\infty} I_0(f_n) < \varepsilon.$$

This proves that Z is a set of measure zero by Definition 2.16, and hence every function $f \in L_0^{\uparrow,\text{a.e.}}$ is finite almost everywhere. □

Mirroring our construction of $\hat{L}$, we consider the set $\hat{L}_\infty$ of functions f that can be written as a difference $f = f_1 - f_2$ a.e. for $f_1, f_2 \in L_0^{\uparrow,\text{a.e.}}$. Since functions $L_0^{\uparrow,\text{a.e.}}$ also have bounded integrals, $\hat{I}(f_1 - f_2) = I^\uparrow(f_1) - I^\uparrow(f_2)$ is again well-defined. Furthermore, all $f \in L_0^{\uparrow,\text{a.e.}}$ are finite a.e. by Lemma 2.17, and so we do not run into any problems of subtracting infinity from infinity, except on a set of measure zero.

This does mean, however, that $\hat{L}_\infty$ contains functions with extended real values and hence it is strictly speaking not a vector lattice. We will instead call $\hat{L}_\infty$ a *vector lattice of extended real-valued functions*, which we define just like a vector lattice, but for $f + g$ we require that all possible functions $h = f + g$ belong to $\hat{L}_\infty$ again. For the $x \in X$ where $f(x) + g(x) = \infty - \infty$ is ambiguous, we let $h(x)$ range over all possible choices. This definition is taken from Section 16.1 of [Roy88], and can be seen as a bit unorthodox, given that we have defined our Daniell integral on L_0 on a vector lattice of real-valued functions. Several books (cf. [GS78], [Roy88], [Loo11], [RN12]) on Daniell integration seem to shy away from discussing the relationship between

L and $\hat{L}$, as well as what happens to the vector space structure once functions with extended real values are allowed. We will not solve this issue here, but we will shine a light on the connection between L and $\hat{L}$ in what follows.

Similar to $\hat{L}_\infty$, we introduce the vector lattice of extended real-valued functions L_∞ of functions in L that are allowed to have values on the extended real number line, with the same integral I. We can show that $\hat{L}_\infty \subset L_\infty$ once again. Let $f \in L_0^{\uparrow,\text{a.e.}}$. Then there exists an increasing sequence $(f_n)_{n\in\mathbb{N}}$ in L_0 such that $f_n \uparrow f$ almost everywhere. Define $g := \lim_{n\to\infty} f_n$ as a limit 'everywhere'. Then clearly $g \in L_0^\uparrow$, and $I^\uparrow(g) = \lim_{n\to\infty} I_0(f_n) = I^\uparrow(f)$. It now follows from an argument analogous to Lemma 2.9 (4) that

$$\bar{I}(f) = \inf_{g\geq f, g\in L_0^\uparrow} I^\uparrow(g) = \inf_{g\geq f, g\in L_0^{\uparrow,\text{a.e.}}} I^\uparrow(g) = I^\uparrow(f) = \underline{I}(f),$$

and we find that $L_0^{\uparrow,\text{a.e.}} \subset L_\infty$. We can now show that $\hat{L}_\infty \subset L_\infty$, since by Lemma 2.9 we have the subadditivity and superadditivity of $\bar{I}$ and $\underline{I}$ respectively, which together with the fact that all $f_1 - f_2 \in \hat{L}_\infty$ are finite a.e. show that

$$\hat{I}(f_1 - f_2) = \underline{I}(f_1) - \bar{I}(f_2) \leq \underline{I}(f_1 - f_2) \leq \bar{I}(f_1 - f_2) \leq \bar{I}(f_1) - \underline{I}(f_2) = \hat{I}(f_1 - f_2).$$

Now that we have extended both L and $\hat{L}$ to vector lattices of extended real numbers L_∞ and $\hat{L}_\infty$ respectively, we show the equivalence of the functionals I and $\hat{I}$. The precise statement can be found in Proposition 2.18 below.

Proposition 2.18. *For every $f \in L_\infty$, there exist $g \in \hat{L}_\infty$ and $h_f \in L_\infty$ with $I(|h_f|) = 0$, such that we have the decomposition*

$$f = g + h_f.$$

Proof. Let $f \in L_\infty$. It is a fact that $\{f \in L_0^\uparrow : I^\uparrow(f) < \infty\} \subset L_0^{\uparrow,\text{a.e.}} \subset \hat{L}_\infty$, since convergence everywhere is also convergence almost everywhere. This means that we have

$$I(f) = \inf_{g\geq f, g\in L_0^\uparrow} I(g) = \inf_{g\geq f, g\in L_0^\uparrow, I(g)<\infty} I(g) \geq \inf_{g\geq f, g\in L_0^{\uparrow,\text{a.e.}}} I(g) \geq \inf_{g\geq f, g\in \hat{L}_\infty} I(g).$$

Likewise,

$$\sup_{g\leq f, g\in \hat{L}_\infty} I(g) \geq I(f).$$

Since

$$\inf_{g\geq f, g\in \hat{L}_\infty} I(g) \geq \sup_{g\leq f, g\in \hat{L}_\infty} I(g),$$

we have

$$I(f) = \inf_{g\geq f, g\in \hat{L}_\infty} I(g).$$

By definition of the infimum this implies that there exists a (not necessarily decreasing) sequence $(g_n)_{n\in\mathbb{N}} \in \hat{L}_\infty$ such that $I(g_n) \downarrow \bar{I}(f)$ and $g_n \geq f$ for all n. To mold this sequence into a decreasing sequence, we use the following trick: define

$$\bar{g}_n := g_1 \wedge g_2 \wedge \cdots \wedge g_n.$$

Then $\bar{g}_n \geq f$ for all n, $\bar{g}_n \downarrow \bar{g}$ where $\bar{g} := \lim_{n\to\infty} g_n$, so $\bar{g} \geq f$. Moreover, we have $\bar{g}_n \leq g_n$ and consequently $\hat{I}(\bar{g}_n) \leq \hat{I}(g_n)$, from which it follows that $\hat{I}(\bar{g}_n) \downarrow \bar{I}(f)$. Analogous to Lemma 2.13, there exists a monotone convergence theorem for $\hat{I}$ on $\hat{L}_\infty$, which we will simply cite here as Corollary 2.6.1. of [GS78]. From this result and the fact that $\lim_{n\to\infty} \hat{I}(\bar{g}_n) = I(f) > -\infty$, it follows that $\bar{g} \in \hat{L}_\infty$ and

$$\hat{I}(\bar{g}) = \lim_{n\to\infty} \hat{I}(\bar{g}_n) = I(f).$$

Defining $h_f := f - \bar{g} \in L_\infty$ (which is finite a.e.) with $I(|h_f|) = I(|f - \bar{g}|) = I(\bar{g}) - I(f) = 0$, we conclude that there exists some $g \in \hat{L}_\infty$ and $h_f \in L_\infty$ with $I(|h_f|) = 0$ such that we have the required decomposition

$$f = g + h_f.$$

□

Looking closely, this proof shows us the need of extending $\hat{L}$ to $\hat{L}_\infty$. Namely, the extended real-valued functions $L_0^\uparrow$ with finite integrals are only a subset of $\hat{L}_\infty$, but not $\hat{L}$.

Let us denote

$$N := \{h \in L_\infty : I(|h|) = 0\}.$$

From Proposition 2.18 we find that $L_\infty \subset \hat{L}_\infty + N$. Since $\hat{L}_\infty + N \subset L_\infty$ by the extended vector space properties of L, we have that

$$L_\infty = \hat{L}_\infty + N,$$

which also implies that

$$L_\infty/N \cong \hat{L}_\infty/\hat{N},$$

where $\hat{N} := \hat{L}_\infty \cap N$. Since all functions $h \in N$ have integral zero, this shows in particular that the functionals I and $\hat{I}$ are equivalent: they coincide on all of $\hat{L}_\infty$. Therefore, when integrating it does not matter which of the extensions of $L_0^\updownarrow$ we consider moving forward.

Before we move on, we briefly showcase another definition for L that highlights the connection between L and $\hat{L}$. We introduce the smaller vector lattice $L' \subset L$ as the set of real-valued functions for which the new upper and lower integrals

$$\bar{I}'(f) := \inf_{g \geq f, g \in L_0^{\uparrow,\text{fin}}} I^\uparrow(g),$$

and

$$\underline{I}'(f) := \sup_{h \leq f, h \in L_0^{\downarrow,\text{fin}}} I^\downarrow(h) = -\bar{I}'(-f),$$

coincide and are finite. The integral I' on L' can then also be shown to be a Daniell integral, and we have the inclusion $\hat{L} \subset L'$. It can now be shown directly that an analogue of 2.18 holds for the vector lattices $\hat{L}$ and L', since $L_0^{\uparrow,\text{fin}} \subset \hat{L}$.

All in all, we have generalized a Daniell integral I_0 on a vector lattice L_0 of bounded real functions on X to a Daniell integral on a larger vector lattice L or $\hat{L}$. These two extensions are related by Proposition 2.18 above, and they both define equivalent Daniell integrals. We repeat that L_∞ and $\hat{L}_\infty$ are not *true* vector lattices, and so we cannot define Daniell integrals on these spaces. Furthermore, the vector lattice L' is a smaller extension of L_0 than the vector lattice L, which means that this extension is slightly less general. For these reasons, we will only consider the original extension vector lattices L and $\hat{L}$ together with their integrals I and $\hat{I}$ moving forward.

2.3 Examples of Daniell integrals

Now that we have explored the extension theorems of the Daniell integral, it is time to look at some examples.

2.3.1 Simple functions

One natural example of a Daniell integral is the Lebesgue integral. Let (X, Σ) be a measurable space. For L_0 we take the set of simple functions on X: functions f of the form $f(x) = \sum_{i=1}^n c_i \mathbb{1}_{A_i}(x)$, where each $c_i \in \mathbb{R}$ and the $A_i \in \Sigma$ are disjoint measurable sets.

Lemma 2.19. *The set L_0 of simple functions on X is a vector lattice.*

Proof. For $f, g \in L_0$ and $\alpha, \beta \in \mathbb{R}$, we write

$$f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}(x),$$

$$g(x) = \sum_{i=1}^{n} d_i \mathbb{1}_{B_i}(x).$$

We may assume without loss of generality that $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i = X$. If this is not the case, then we can add $A_{n+1} = X \setminus \bigcup_{i=1}^{n} A_i$ (resp. $B_{n+1} = X \setminus \bigcup_{i=1}^{n} B_i$) which is disjoint to all other A_i (resp. B_i) and f and g are unchanged if we simply let $c_{n+1} = d_{n+1} = 0$. We can now partition each A_i (resp. B_i) into sets $A_i \cap B_j$ (resp. $B_i \cap A_j$) for $1 \leq j \leq n$ such that

$$A_i = \bigcup_{j=1}^{n} (A_i \cap B_j),$$

and

$$B_i = \bigcup_{j=1}^{n} (B_i \cap A_j).$$

Since the A_i and B_i are both pairwise disjoints collections of sets we have

$$\mathbb{1}_{A_i}(x) = \sum_{j=1}^{n} \mathbb{1}_{A_i \cap B_j}(x),$$

and

$$\mathbb{1}_{B_i}(x) = \sum_{j=1}^{n} \mathbb{1}_{B_i \cap A_j}(x),$$

and hence it follows that

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \sum_{i=1}^{n} \alpha c_i \mathbb{1}_{A_i}(x) + \sum_{j=1}^{n} \beta d_j \mathbb{1}_{B_j}(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{i,j} \mathbb{1}_{A_i \cap B_j},$$

where $C_{i,j} = (\alpha c_i + \beta d_j)$ and the order of summation can be interchanged since we are dealing with finite sums. Hence, $\alpha f + \beta g$ is again a simple function.

Next, notice that $f \wedge g = -(-f \vee -g)$ and $f \vee g = \frac{1}{2}(|f - g| + f + g)$, so that if we show that $|f| \in L_0$ for $f \in L_0$, then L_0 must be a lattice. But for any $f \in L_0$, since the A_i are disjoint we have

$$|f| = |\sum_{i=1}^{n} c_i \mathbb{1}_{A_i}(x)| = \sum_{i=1}^{n} |c_i| \mathbb{1}_{A_i}(x),$$

which is again a simple function. □

Now let us define a Daniell integral I_0 on L_0. To this end, let μ be a finite (positive) measure on (X, Σ).

Lemma 2.20. *Let (X, Σ, μ) be a finite measure space, and consider functions f on X of the form*

$$f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}(x),$$

where the $A_i \in \Sigma$ are pairwise disjoint. Then the functional I_0 given by

$$I_0(f) := \sum_{i=1}^{n} c_i \mu(A_i),$$

defines a Daniell integral on L_0.

Proof. We check the conditions of Definition 2.1 to see that this is indeed a Daniell integral:

(1) Linearity: for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in L_0$ we have by the proof of Lemma 2.19 that

$$I_0(\alpha f + \beta g) = I_0(\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha c_i + \beta d_j)\mathbb{1}_{A_i \cap B_j}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha c_i + \beta d_j)\mu(A_i \cap B_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha c_i \mu(A_i \cap B_j) + \sum_{i=1}^{n}\sum_{j=1}^{n}\beta d_j \mu(A_i \cap B_j) = \alpha \sum_{i=1}^{n} c_i \mu(A_i) + \beta \sum_{j=1}^{n} d_j \mu(B_j) = \alpha I_0(f) + \beta I_0(g).$$

(2) Non-negativity: let $f \in L_0$ such that $f(x) \geq 0$ for all x. Then

$$I_0(f) = \sum_{i=1}^{n} c_i \mu(A_i) \geq 0,$$

since $\mu(A_i) \geq 0$ for all A_i and $c_i \geq 0$ for all i and because of the fact that the A_i are pairwise disjoint.

(3) Continuity with respect to monotone convergence: let $f_n \downarrow 0$ for $(f_n)_{n\in\mathbb{N}}$ a sequence in L_0 and write $f_n(x) = \sum_{i=1}^{N_n} c_i^{(n)} \mathbb{1}_{A_i^{(n)}}(x)$ with $c_i^{(n)} \geq 0$ and $A_i^{(n)} \in \Sigma$ pairwise disjoint. Then by the monotone convergence theorem for the Lebesgue integral (see for example Theorem 2.8.2. of [Bog06a]), it follows immediately that

$$\lim_{n\to\infty} I_0(f_n) = \lim_{n\to\infty} \sum_{i=1}^{N_n} c_i^{(n)} \mathbb{1}_{A_i^{(n)}}(x) = \lim_{n\to\infty} \int_X f_n d\mu = 0.$$

□

Note that we have assumed that μ is a finite measure, so that we avoid any possible subtraction of infinities. This assumption is also needed since any Daniell integral is by definition finite-valued. From here onwards we will see that the extensions of this Daniell integral correspond to the Lebesgue integral on larger and larger sets of functions.

First consider the set $L_0^{\uparrow,\text{fin}}$, which consists of all finite functions f with finite integral that are the pointwise limit of an increasing sequence of functions f_n in L_0. It is a consequence of for example Theorem 2.1.5.(v) (together with Remark 2.1.6.) from [Bog06a] that for any sequence of Σ-measurable functions f_n with limit $f := \lim_{n\to\infty} f_n$, the function f is once again Σ-measurable. Since all functions in L_0 are clearly measurable, this shows that

$$L_0^{\uparrow} \subset \{f : X \to \mathbb{R} \cup \{\infty\} : f \text{ is measurable and } \inf_{x\in X} f(x) > -\infty\}.$$

Here the condition $\inf_{x\in X} f(x) > -\infty$ is crucial, since if f is not bounded from below, then any increasing sequence of functions that approximates f from below would also need to be unbounded from below. But this cannot happen if L_0 consists of all the finite simple functions.

To prove the other inclusion, we need the following Lemma, which connects the set $L_0^{\uparrow}$ to the non-negative measurable functions.

Lemma 2.21. *(Theorem 8.8, [Sch05], p. 62) Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a measurable function such that $f(x) \geq 0$ for all $x \in X$. Then there exists an increasing sequence of simple functions $(f_n)_{n\in\mathbb{N}} \subset L_0$ such that $f_n \uparrow f$.*

Proof. For every $n \in \mathbb{N}$, we define the level sets

$$A_k^n := \begin{cases} \{k2^{-n} \leq f < (k+1)2^{-n}\}, & k = 0, 1, 2, \ldots, n2^n - 1 \\ \{f \geq n\}, & k = n2^n. \end{cases}$$

These A_k^n are clearly disjoint and measurable, their union is all of X and they slice up the range of f into horizontal pieces, at least until a height of n. For each n we essentially split up the range $[0, n)$ of f into $n2^n$ intervals of length 2^{-n}, and so we can define approximating simple functions f_n by

$$f_n(x) := \sum_{k=0}^{n2^n} k2^{-n} \mathbb{1}_{A_k^n}.$$

From this construction we note the following:

- For all n, $f_n(x) \leq f(x)$. This is because for $x \in A_k^n$ we have that $k2^{-n} \leq f(x)$ and the A_k^n are disjoint, so that the f_n are step functions.
- If $f(x) < n$, then $|f_n(x) - f(x)| < 2^{-n}$.
- For all n, $f_n(x) \leq f_{n+1}(x)$. This is because for larger n, f_n approximates f better and better while at the same time $f_n \leq f$ holds for all n.

Now let $x \in X$. If $f(x) = +\infty$, then $f_n(x) = n$ by definition (since the approximation cuts off at $n < \infty$). Clearly, then, we have $f_n(x) \uparrow f(x)$, since $\lim_{n\to\infty} n = +\infty$.

Assume now that $f(x) < \infty$. Then there exists some $m \in \mathbb{N}$ such that $f(x) < m$. As a result, for all $n \geq m$ we have

$$|f_n(x) - f(x)| < 2^{-n}.$$

From this and the other results we conclude that $f_n \uparrow f$, which finishes the proof. □

Now take any measurable $f : X \to \mathbb{R} \cup \{\infty\}$ such that $a := \inf_{x\in X} f(x) > -\infty$. Such a function can be translated to a non-negative measurable function by considering $f + |a| \geq 0$. By Lemma 2.21, there then exists an increasing sequence of simple functions $f_n \uparrow (f + |a|)$, and since L_0 is a vector space with $\mathbb{1} \in L_0$, the functions $g_n := f_n - |a|$ are simple functions for all n. It follows that $g_n \uparrow f$, and hence $f \in L_0^\uparrow$. Combining this with the other inclusion from earlier, we find

$$L_0^\uparrow = \{f : X \to \mathbb{R} \cup \{\infty\} : f \text{ is measurable and } \inf_{x\in X} f(x) > -\infty\}.$$

The Daniell integral I_0 defined above now extends to $I^\uparrow$ on $L_0^\uparrow$ as usual, and in particular we find the Lebesgue integral for non-negative measurable functions, since these functions form a subset of $L_0^\uparrow$.

Next we consider the extension from $L_0^{\uparrow,\text{fin}}$ to $\hat{L}$. After taking differences of functions in $L_0^{\uparrow,\text{fin}}$, the condition that $\inf_{x\in X} f(x) > -\infty$ vanishes and we are left with

$$\hat{L} = \{f : X \to \mathbb{R} : f \text{ is measurable}, |I(f)| < \infty.\}$$

The extended integral I on $\hat{L}$ is now the Lebesgue integral by definition, and $\hat{L}$ consists of all real-valued Lebesgue integrable functions, by the condition that $|I(f)| < \infty$.

This shows that, at least for real-valued functions on some measurable space (X, Σ), the Lebesgue integral is a special case of a Daniell integral constructed from the simple functions playing the role of the elementary functions.

2.3.2 Functions vanishing outside a finite set

Consider a set X and L_0 the space of real functions f on X such that there exists some finite set $N_f \subset X$ with the property that $f(x) \neq 0$ for all $x \in N_f$ and $f(x) = 0$ otherwise (the functions f are zero for all but finitely many x). Let $f, g \in L_0$ with N_f and N_g the corresponding supporting sets. Then the set $N := N_f \cup N_g$ is a finite set such that for all $x \in N^c$ we have $f(x) + g(x) = f(x) \vee g(x) = f(x) \wedge g(x) = 0$. Hence, L_0 is a vector lattice. We can define a Daniell integral I_0 on L_0 as follows.

Lemma 2.22. *The functional*

$$I_0(f) := \sum_{x\in X} f(x),$$

defines a Daniell integral on L_0.

Proof. We check the conditions of Definition 2.1 to see that this is indeed a Daniell integral:

(1) Linearity: for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in L_0$ we have

$$I_0(\alpha f + \beta g) = \sum_{x\in X}(\alpha f + \beta g)(x) = \sum_{x\in X} \alpha f(x) + \beta g(x) = \alpha \sum_{x\in X} f(x) + \beta \sum_{x\in X} g(x) = \alpha I_0(f) + \beta I_0(g).$$

(2) Non-negativity: let $f \in L_0$ such that $f(x) \geq 0$ for all x. Then

$$I_0(f) = \sum_{x\in X} f(x) \geq 0.$$

(3) Continuity with respect to monotone convergence: let $f_n \downarrow 0$ for $(f_n)_{n\in\mathbb{N}}$ a sequence in L_0. Then we must have a monotone sequence of corresponding supporting sets $N_{f_1} \supseteq N_{f_2} \supseteq \cdots \supseteq \emptyset$, implying that $|N_{f_1}| \geq |N_{f_n}|$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and $x \in X$. Then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $0 \leq f_n(x) \leq \frac{\varepsilon}{|N_{f_1}|}$. For such $n \geq N$, we then also have

$$I_0(f_n) = \sum_{x\in N_{f_n}} f_n(x) \leq \frac{\varepsilon}{|N_{f_1}|} \sum_{x\in N_{f_n}} 1 = \varepsilon \frac{|N_{f_n}|}{|N_{f_1}|} < \varepsilon.$$

Since non-negativity of I_0 proves that $I_0(f_n) \geq 0$ for all $n \in \mathbb{N}$, we conclude that $\lim_{n\to\infty} I_0(f_n) = 0$.

□

We proceed by analysing the properties of $L_0^\uparrow$. Recall that $L_0^\uparrow$ is the set of all extended real-valued functions on X such that $f_n \uparrow f$ for some sequence $(f_n)_{n\in\mathbb{N}}$. If we take $X = \mathbb{N}$, we can take for example the sequence

$$f_n(m) := \begin{cases} 1 & \text{if } m \leq n \\ 0 & \text{else} \end{cases}$$

Then every $f_n \in L_0$, and $f_n \uparrow f$ where $f \equiv 1$, which is clearly not in L_0.

Note that no $f \in L_0^\uparrow$ can have the property that $f < 0$ for infinitely many x. To see this, let $f_n \uparrow f$ be the increasing sequence corresponding to f. Since f_1 is non-zero only for finitely many x, it is also strictly negative for finitely many x. Since $f_n \geq f_1$ for all $n \geq 1$, it follows that $f_n(x) < 0$ can only occur on the finite set N_{f_1}. As a result, we have that $L_0^\uparrow$ consists of all extended real-valued functions on X such that $f(x) \geq 0$, except on a finite set. Likewise, $L_0^\downarrow = -L_0^\uparrow$ is the set of all functions such that $f(x) \leq 0$, except on a finite set. Furthermore, we have

$$I^\uparrow(f) = \lim_{n\to\infty} I_0(f_n) = \lim_{n\to\infty} \sum_{x\in X} f_n(x) = \sum_{x\in X} f(x),$$

which can be infinite (as in our example where $f \equiv 1$).

Finally, we can extend $L_0^\uparrow$ further to $\hat{L}$ by considering differences of functions $f_1, f_2 \in L_0^{\uparrow,\text{fin}}$: $f = f_1 - f_2$. This means that in this example, $\hat{L}$ is the space of real-valued functions on X. As such, we have a Daniell integral on this space $\hat{L}$ of the form

$$I(f) = I^\uparrow(f_1) - I^\uparrow(f_2),$$

where $f \in \hat{L}$ and $f_1, f_2 \in L_0^{\uparrow,\text{fin}}$. In particular, since $I(f) < \infty$, $\hat{L}$ is the space of real-valued functions on X that give rise to an absolutely convergent series:

$$\sum_{x \in X} |f(x)| = \sum_{x \in X} |f_1(x) - f_2(x)| \leq \sum_{x \in X} |f_1(x)| + \sum_{x \in X} |f_2(x)| = I^{\uparrow}(|f_1|) + I^{\uparrow}(|f_2|) < \infty.$$

2.4 Defining measures using Daniell integrals

Now that we have set up a framework for the Daniell integral, it is natural to consider how a measure could arise from such an integral. We consider two approaches to showcase the convenience of the complete Daniell approach in this setting. The first approach makes use of the properties of the extended vector lattice L together with the Daniell integral I as described in Section 2.2.3. Because of these properties, we can immediately construct a measure on a σ-algebra without having to invoke Carathéodory's Extension Theorem.

In contrast, the second approach only assumes the Daniell construction up to the extension to $L_0^{\uparrow}$. As a result, we can construct a set function on a lattice of sets (i.e., containing the empty set and being closed under finite union and intersection). However, to extend this set function to a measure on a σ-algebra, we first need to extend this lattice of sets to a ring or algebra of sets, after which we can finally invoke Carathéodory's Extension Theorem. This process turns out to be much more involved than the first approach. As a result, at some point we omit some of the arguments so as to not spend too much time on this second approach.

2.4.1 The approach of extending L_0 to L

This approach is based on the extension process of a Daniell integral on an elementary vector lattice L_0 to a Daniell integral on a larger vector lattice L that we have discussed in previous sections. We also for the most part follow Section 16.4 of [Roy88], though the major results are compared with Chapter 7 of [Bog06b].

Let us assume that we have some vector lattice L_0 of elementary functions on X together with a Daniell integral I_0 defined on L_0. By the extension procedure defined in Section 2.2, we then also have a Daniell integral I defined on the set L of integrable functions. We define the following sets Σ_L and $\mathcal{M}$ together with a set function μ on Σ_L:

$$\mathcal{M} := \{f : X \to \mathbb{R} : f \wedge g \in L \quad \forall g \in L\},$$
$$\Sigma_L := \{G \subset X : \mathbb{1}_G \in \mathcal{M}\},$$
$$\mu_L(G) := \begin{cases} I(\mathbb{1}_G) & \text{if } \mathbb{1}_G \in L, \\ \infty & \text{otherwise.} \end{cases}$$

We will call the functions in $\mathcal{M}$ *measurable* functions. Notice that all functions in L are measurable by the lattice properties of L. In what follows in this section, we will assume that $\mathbb{1}$ (the constant function 1) is measurable, unless specified otherwise. In other words, we have

$$\mathbb{1} \wedge f \in L \quad \forall f \in L.$$

Note that this means that all constant functions are measurable, by the vector space properties of L.

Lemma 2.23. *Let $f_n \uparrow f$ be a sequence of measurable functions. Then f is measurable.*

Proof. Let $f_n \uparrow f$ be a sequence of measurable functions and let $g \in L$. Then $f_n \wedge g \in L$ and Lemma 2.13 tells us that $f \wedge g \in L$ since

$$I(f \wedge g) = \lim_{n \to \infty} I(f_n \wedge g) \leq I(g) < \infty.$$

Since g was chosen arbitrarily, we conclude that f is measurable. □

Recall that $f^+ := f \vee 0$ and let us denote $f^{-1}(c, \infty)$ by $\{f > c\}$. We will now show that there are actually 'sufficiently many' sets in the class Σ_L for this definition to be non-trivial.

Lemma 2.24. *Let $f \in L$ and $c \in \mathbb{R}$. Then*

$$\mathbb{1}_{\{f>c\}} = \lim_{n\to\infty} \mathbb{1} \wedge (n(f - c\mathbb{1})^+) \in \mathcal{M}.$$

In particular, $\{f > c\} \in \Sigma_L$ for all $c \in \mathbb{R}$ and $f \in L$.

Proof. Let $f \in L$. We first show that $\mathbb{1}_{\{f>c\}} = \lim_{n\to\infty} \mathbb{1} \wedge (n(f - c\mathbb{1})^+)$. Note that the function $(n(f - c\mathbb{1})^+)$ is equal to $n(f - c\mathbb{1}) > 0$ on $\{f > c\}$ and 0 elsewhere. Since $n(f - c\mathbb{1})^+$ is guaranteed to be larger than 1 on $\{f > c + \frac{1}{n}\}$, we have

$$(\mathbb{1} \wedge (n(f-c)^+))(x) = \begin{cases} 0, & \text{if } x \in \{f \le c\} \\ 1, & \text{if } x \in \{f > c + \frac{1}{n}\} \\ n(f(x) - c) < 1, & \text{if } x \in \{c < f \le c + \frac{1}{n}\} \end{cases}$$

By letting n tend to infinity, it follows that $\mathbb{1}_{\{f>c\}} = \lim_{n\to\infty} \mathbb{1} \wedge (n(f - c\mathbb{1})^+)$.

Next, we show that $\mathbb{1} \wedge (n(f - c\mathbb{1})^+)$ is measurable for all n. Notice that $f + g = f \wedge g + f \vee g$ for all $f, g \in L$, since L is a vector lattice. This implies for all $c \neq 0$ that

$$\begin{aligned}(f - c\mathbb{1})^+ &= (f - c\mathbb{1}) \vee 0 = (f \vee c\mathbb{1}) - c\mathbb{1} = f + c\mathbb{1} - (f \wedge c\mathbb{1}) - c\mathbb{1} \\ &= f - (f \wedge c\mathbb{1}) = f - c(\frac{1}{c} f \wedge \mathbb{1}) \in L,\end{aligned}$$

since $\mathbb{1} \in \Sigma_L$ and L is a vector space. If $c = 0$, then $(f - c\mathbb{1})^+ = f^+ \in L$ holds as required. Since $\mathbb{1} \in \Sigma_L$, it follows that $\mathbb{1} \wedge (n(f - c\mathbb{1})^+) \in L \subset \mathcal{M}$.

Finally, from Lemma 2.23 we conclude that $\mathbb{1}_{\{f>c\}} = \lim_{n\to\infty} \mathbb{1} \wedge (n(f - c\mathbb{1})^+)$ is measurable. □

Next, we will show that μ_L is in fact a measure on the σ-algebra Σ_L by collecting some straightforward properties of Σ_L and μ_L.

Proposition 2.25. *The set function μ_L is a measure on the σ-algebra Σ_L.*

Proof. (1) We first prove that Σ_L is a σ-algebra. Since $\mathbb{1}$ is measurable, it is clear that $X \in \Sigma_L$. Furthermore, for any $G \in \Sigma_L$ and $f \in L$ we have

$$\mathbb{1}_{X\setminus G} \wedge f = (\mathbb{1} - \mathbb{1}_G) \wedge f = (\mathbb{1} \wedge f) - (\mathbb{1}_G \wedge f) + f^+ \in L,$$

which implies that $X \setminus G \in \Sigma_L$ and hence that Σ_L is closed under taking complements.

Since L is a lattice, for any $G_1, G_2 \in \Sigma_L$ and we have that $\mathbb{1}_{G_1\cup G_2} = \mathbb{1}_{G_1} \vee \mathbb{1}_{G_2}$ and $\mathbb{1}_{G_1\cap G_2} = \mathbb{1}_{G_1} \wedge \mathbb{1}_{G_2}$. For any $f \in L$, we then have

$$\begin{aligned}(\mathbb{1}_{G_1} \vee \mathbb{1}_{G_2}) \wedge f = (\mathbb{1}_{G_1} \wedge f) \vee (\mathbb{1}_{G_2}) \wedge f) \in L, \\ (\mathbb{1}_{G_1} \wedge \mathbb{1}_{G_2}) \wedge f = (\mathbb{1}_{G_1} \wedge f) \wedge (\mathbb{1}_{G_2}) \wedge f) \in L.\end{aligned}$$

This proves that Σ_L is closed under finite unions and intersections.

Now, let $(G_n)_{n\in\mathbb{N}}$ be a sequence of sets in Σ_L and define $G := \bigcup_{n=1}^\infty G_n$. Then for $H_m := \bigcup_{n=1}^m G_n \in \Sigma_L$ with $H_m \subset H_{m+1}$ we have

$$\mathbb{1}_G = \mathbb{1}_{\bigcup_{n=1}^\infty G_n} = \sup_{m\in\mathbb{N}} \mathbb{1}_{H_m} = \lim_{m\to\infty} \mathbb{1}_{H_m}.$$

Since H_m is measurable for all m by definition, it follows from Lemma 2.23 that $\mathbb{1}_G$ is measurable as well. This shows that Σ_L is closed under countable unions, and we conclude that Σ_L is a σ-algebra.

(2) Next, we prove that μ_L is a measure. Note that for all $G \in \Sigma_L$ we have $\mu_L(G) = I(\mathbb{1}_G) \geq I(0) = 0 = \mu_L(\emptyset)$ by the properties of the Daniell integral L, which shows that μ_L is monotone, non-negative, and zero under the empty set.

It remains to prove that μ_L is σ-additive. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in Σ_L and define $G := \bigcup_{n=1}^{\infty} G_n$. We may assume without loss of generality that $\mathbb{1}_{G_n} \in L$ for all n, since otherwise $\mathbb{1}_G \notin L$ and

$$\mu_L(G) = I(\mathbb{1}_G) = \infty = \sum_{n=1}^{\infty} I(\mathbb{1}_{G_n}).$$

Once again we define $H_m := \bigcup_{n=1}^{m} G_n \in \Sigma_L$ with $H_m \subset H_{m+1}$. Then $\mathbb{1}_{H_m} \uparrow \mathbb{1}_G$ and $H_m \in L$ for all m. Since all the G_n are pairwise disjoint, we have additivity:

$$\mu_L(H_m) = I(\mathbb{1}_{H_m}) = I\left(\sum_{n=1}^{m} \mathbb{1}_{G_n}\right) = \sum_{n=1}^{m} \mu_L(G_n),$$

which yields

$$\lim_{m\to\infty} I(\mathbb{1}_{H_m}) = \sum_{n=1}^{\infty} \mu_L(G_n)$$

Lemma 2.13 now implies that we have two cases. In the first case, $\mathbb{1}_G \notin L$ and $\lim_{m\to\infty} I(\mathbb{1}_{H_m}) = \sum_{n=1}^{\infty} \mu_L(G_n) = \infty$. It then immediately follows that $\mu_L(G) = \infty = \sum_{n=1}^{\infty} \mu_L(G_n)$. In the second case, $\mathbb{1}_G \in L$ and $\mu_L(G) = \sum_{n=1}^{\infty} \mu_L(G_n) < \infty$. In both cases we have $\mu_L(G) = \sum_{n=1}^{\infty} \mu_L(G_n)$, and so μ_L must be σ-additive.

We conclude that μ_L is a measure on the σ-algebra Σ_L. □

The following lemma describes the connection between the σ-algebra Σ_L and the other σ-algebras at play here.

Lemma 2.26. *Let L_0 and Σ_L be as above. Then*

$$\sigma(L_0) = \sigma(L_0^{\uparrow}) = \sigma(L_0^{\uparrow,fin}) = \sigma(\hat{L}) \subset \sigma(L) = \Sigma_L.$$

Proof. (1) To prove the first equation, let $f \in L_0^{\uparrow}$ and let $f_n \uparrow f$ be a sequence in L_0. Then for all n and $c \in \mathbb{R}$ we have

$$\{f_n > c\} \subset \{f_{n+1} > c\},$$

and so $\bigcup_{n=1}^{\infty}\{f_n > c\} = \{f > c\} \in \sigma(L_0)$, since the σ-algebra $\sigma(L_0)$ is closed under countable union. Hence,

$$\sigma(L_0^{\uparrow}) = \sigma\left(\{\{g > c\} : g \in L_0^{\uparrow}, c \in \mathbb{R}\}\right) \subset \sigma\left(\{\{f > c\} : f \in L_0, c \in \mathbb{R}\}\right) = \sigma(L_0).$$

The other inclusion follows directly from $L_0 \subset L_0^{\uparrow}$, and so it follows that $\sigma(L_0) = \sigma(L_0^{\uparrow})$.

(2) For the second and third equations, recall the definition $\hat{L} := L_0^{\uparrow,\text{fin}} - L_0^{\uparrow,\text{fin}}$. From this it is clear that $L_0^{\uparrow,\text{fin}} \subset \hat{L}$ and hence $\sigma(L_0^{\uparrow,\text{fin}}) \subset \sigma(\hat{L})$. For the other inclusion, let $f \in \hat{L}$ be written as $f = f_1 - f_2$, for $f_1, f_2 \in L_0^{\uparrow,\text{fin}}$. It follows for all $c \in \mathbb{R}$ that

$$\begin{aligned}\{f > c\} = \{f_1 - f_2 > c\} &= \bigcup_{r\in\mathbb{R}}\{f_1 > c + r\} \cap \{f_2 \leq r\} = \bigcup_{q\in\mathbb{Q}}\{f_1 > c + q\} \cap \{f_2 \leq q\} \\ &= \bigcup_{q\in\mathbb{Q}}\{f_1 > c + q\} \cap \{f_2 > q\}^c \in \sigma(L_0^{\uparrow,\text{fin}}),\end{aligned}$$

which shows that proves the third equation $\sigma(L_0^{\uparrow,\text{fin}}) = \sigma(\hat{L})$. Here we make use of the fact that any $r \in \mathbb{R}$ can be approximated arbitrarily well from above by a sequence of $(q_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ with $q_n \downarrow r$ so that

$$\{f > r\} = \bigcup_{n=1}^{\infty} \{f > q_n\},$$

which is a necessary step in the argument, since $\sigma(L_0^{\uparrow,\text{fin}})$ is only closed under countable unions.

The second equation now follows directly from the inclusions $\sigma(L_0) \subset \sigma(L_0^{\uparrow,\text{fin}}) = \sigma(\hat{L}) \subset \sigma(L_0^{\uparrow})$ and the first equation $\sigma(L_0) = \sigma(L_0^{\uparrow})$.

(3) The inclusion $\sigma(\hat{L}) \subset \sigma(L)$ follows directly from $\hat{L} \subset L$. For the final equation, consider Lemma 2.24. By the definition of Σ_L, this lemma implies that $\{f > c\} \in \Sigma_L$ for all $c \in \mathbb{R}$ and $f \in L$. As a result, all $f \in L$ are measurable with respect to Σ_L and we have $\sigma(L) \subset \Sigma_L$. On the other hand, let $G \in \Sigma_L$. Since $\mathbb{1}$ is measurable, this implies that $\mathbb{1} \wedge \mathbb{1}_G = \mathbb{1}_G \in L$. Since $G = \{\mathbb{1}_G > 0\} \in \sigma(L)$, we find that $G \in \sigma(L)$ and hence $\sigma(L) = \Sigma_L$.

□

Let L_0 be such that for all $f \in L_0$ one has

$$\mathbb{1} \wedge f \in L_0.$$

Then L_0 is said to satisfy the *Stone condition*.

Lemma 2.27. *If the Stone condition holds for a vector lattice of elementary functions L_0, then $\mathbb{1}$ is measurable. That is, the Stone condition is satisfied by L as well.*

Proof. Let $f \in L$. By the Stone condition, for any $\varphi \in L_0$ we have $\mathbb{1} \wedge \varphi \in L$. Let $g \in L_0^{\uparrow,\text{fin}}$ and consider a corresponding increasing sequence of functions $\varphi_n \uparrow g$ in L_0. By Lemma 2.13, we have that

$$\mathbb{1} \wedge g = \lim_{n\to\infty} \mathbb{1} \wedge \varphi_n \in L,$$

since $\lim_{n\to\infty} I(\mathbb{1} \wedge \varphi_n) \le I(g) < \infty$.

Furthermore, recall that $\bar{I}(f) = \inf_{g \ge f, g \in L_0^{\uparrow}} I(g)$. By definition of the infimum this implies that there exists a (not necessarily decreasing) sequence $(g_n)_{n\in\mathbb{N}} \in L_0^{\uparrow}$ such that $I(g_n) \downarrow \bar{I}(f)$ and $g_n \ge f$ for all n. To mold this sequence into a decreasing sequence, we use the following trick: define

$$\bar{g}_n = g_1 \wedge g_2 \wedge \cdots \wedge g_n.$$

Then $\bar{g}_n \ge f$ for all n, $\bar{g}_n \downarrow \bar{g}$ where $\bar{g} := \inf_n g_n$, so $\bar{g} \ge f$. Moreover, we have $\bar{g}_n \le g_n$ and consequently $I(\bar{g}_n) \le I(g_n)$, from which it follows that $I(\bar{g}_n) \downarrow \bar{I}(f)$. We can thus write $f = \bar{g} - h_f$, where $h_f = \bar{g} - f \in L$ with $I(|h_f|) = 0$, since $\bar{g}$ was constructed in a way that $I(\bar{g}) = I(f)$. Since $\bar{g}$ is the limit of a decreasing sequence of functions in $g \in L_0^{\uparrow,\text{fin}} \subset L$, we apply 2.13 for $-\bar{g}$ to find that $\mathbb{1} \wedge \bar{g} \in L$. Now

$$0 \le |(\mathbb{1} \wedge \bar{g}) - (\mathbb{1} \wedge f)| \le |h_f|,$$

and since $I(|h_f|) = 0$, it follows that $I(\mathbb{1} \wedge f) = I(\mathbb{1} \wedge \bar{g})$ is finite. Hence, $\mathbb{1} \wedge f \in L$. Since f was chosen arbitrarily, we conclude that $\mathbb{1}$ is measurable. □

We are now ready to prove the general integral representation theorem for Daniell integrals. Since it is typically easier to check if the Stone condition holds than to check if $\mathbb{1}$ is measurable, the following theorem assumes the Stone condition. Lemma 2.27 implies that $\mathbb{1}$ is measurable, so the results above still hold in this setting.

Theorem 2.28. *(Extension of Thrm. 7.8.7., [Bog06b], p. 99-101) Let L_0 be a vector lattice of real-valued bounded functions on a set X satisfying the Stone condition and I_0 a Daniell integral defined on L_0. Let I be the extension of I_0 to the vector lattice L of integrable functions. Then there exists a measure μ_L on the σ-algebra $\Sigma_L = \sigma(L)$ such that $L \subset \mathcal{L}^1(\mu_L)$ and*

$$I(f) = \int_X f d\mu_L,$$

for all $f \in L$.

Proof. Lemma 2.27 implies that $\mathbb{1}$ is measurable. Therefore, as described above, the Daniell integral I_0 on L_0 induces a measure μ_L on the σ-algebra Σ_L. The equation $\Sigma_L = \sigma(L)$ is a consequence of Lemma 2.26.

It remains to show that $L \subset \mathcal{L}^1(\mu_L)$ and that

$$I(f) = \int_X f d\mu_L,$$

for all $f \in L$.

Let $f \in L$ such that $0 \leq f \leq 1$. Define an increasing sequence of functions (taken from the proof of Theorem 7.8.1. on page 101 of [Bog06b])

$$f_n := \sum_{j=1}^{2^n-1} j2^{-n}\mathbb{1}_{\{j2^{-n}<f\leq(j+1)2^{-n}\}}.$$

To see that this sequence converges to f, note that for any n the sets $\{j2^{-n} < f \leq (j+1)2^{-n}\}$ are disjoint for all $1 \leq j \leq 2^n - 1$. As n tends to infinity, the sets $\{j2^{-n} < f \leq (j+1)2^{-n}\}$ cover all of $(0, 1]$. Since $f \leq 1$, it follows that $f_n \uparrow f$.

Furthermore, we have

$$\begin{aligned} &j\mathbb{1}_{\{j2^{-n}<f\leq(j+1)2^{-n}\}} + (j+1)\mathbb{1}_{\{(j+1)2^{-n}<f\leq(j+2)2^{-n}\}} \\ &= j\left(\mathbb{1}_{\{f>j2^{-n}\}} - \mathbb{1}_{\{f>(j+1)2^{-n}\}}\right) + (j+1)\left(\mathbb{1}_{\{f>(j+1)2^{-n}\}} - \mathbb{1}_{\{f>(j+2)2^{-n}\}}\right) \\ &= j\mathbb{1}_{\{f>j2^{-n}\}} + \mathbb{1}_{\{f>(j+1)2^{-n}\}} - (j+1)\mathbb{1}_{\{f>(j+2)2^{-n}\}}. \end{aligned}$$

Since j starts at 1, we can see by induction that

$$f_n = \sum_{j=1}^{2^n-1} j2^{-n}\mathbb{1}_{\{j2^{-n}<f\leq(j+1)2^{-n}\}} = 2^{-n}\sum_{j=1}^{2^n-1}\mathbb{1}_{\{f>j2^{-n}\}}.$$

By Lemma 2.24, it follows that $\mathbb{1}_{\{f>j2^{-n}\}}$ is measurable. We also have

$$\mathbb{1}_{\{f>j2^{-n}\}} = \mathbb{1}_{\{f>j2^{-n}\}} \wedge 2^n f j^{-1},$$

since $2^n f j^{-1} > 1$ on $\{f > j2^{-n}\}$. This implies that $\mathbb{1}_{\{f>j2^{-n}\}} \in L$, and hence $f_n \in L$ for all n. The fact that the f_n are increasing is now also clear, if we consider for example f_1 and f_2:

$$f_1 = \frac{1}{2}\mathbb{1}_{\{f>\frac{1}{2}\}},$$

$$f_2 = \frac{1}{4}\left(\mathbb{1}_{\{f>\frac{1}{4}\}} + \mathbb{1}_{\{f>\frac{1}{2}\}} + \mathbb{1}_{\{f>\frac{3}{4}\}}\right),$$

then clearly $f_1 \leq f_2$.

By Lemma 2.13, we must have $I(f) = \lim_{n\to\infty} I(f_n)$. On the other hand, we have

$$I(f_n) = 2^{-n}\sum_{j=1}^{2^n-1} I(\mathbb{1}_{\{f>j2^{-n}\}}) = 2^{-n}\sum_{j=1}^{2^n-1}\mu_L(\{f > j2^{-n}\}) = \int_X f_n d\mu_L.$$

By the monotone convergence for the Lebesgue integral we get

$$I(f) = \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} \int_X f_n d\mu_L = \int_X f d\mu_L,$$

for all $f \in L$ with $0 \le f \le 1$.

Now let $f \in L$ be non-negative. Then $(n\mathbb{1}) \wedge f \le n\mathbb{1}$ is a function in L such that $0 \le (n\mathbb{1}) \wedge f \le n\mathbb{1}$ for all $n \in \mathbb{N}$. Taking our argument from before and scaling by n yields

$$I((n\mathbb{1}) \wedge f) = \int_X (n\mathbb{1}) \wedge f d\mu_L,$$

for all n. Since $(n\mathbb{1}) \wedge f \uparrow f$, it follows from monotone convergence and Lemma 2.13 that

$$I(f) = \int_X f d\mu_L \tag{2}$$

for all non-negative $f \in L$.

Extending to all functions $f \in L$ is achieved by writing $f = f^+ - f^-$, for the non-negative functions $f^+, f^- \in L$. Linearity of the integrals and equation 2 above then immediately imply that

$$I(f) = I(f^+) - I(f^-) = \int_X f^+ d\mu_L - \int_X f^- d\mu_L = \int_X f d\mu_L,$$

for all $f \in L$, as desired. Since $|f| \in L$ for all $f \in L$, it immediately follows that $L \subset \mathcal{L}^1(\mu)$, and this finishes the proof. ☐

Corollary 2.29. *In the setting of Theorem 2.28, we have*

$$L/N \cong L^1(\mu_L).$$

The construction as discussed until now has been very general, but oftentimes one can work in spaces L_0 such that $\mathbb{1} \in L_0$ and $I_0(\mathbb{1}) = 1$. In this case, alongside the Lebesgue integral representation of the Daniell integral, we also acquire a unique probability measure.

Theorem 2.30. *Let L_0 be a vector lattice of real-valued bounded functions on a set X such that $\mathbb{1} \in L_0$ and I_0 a Daniell integral defined on L_0 such that $I_0(\mathbb{1}) = 1$. Let I be the extension of I_0 to the vector lattice L of integrable functions. Then there exists a unique probability measure μ_L on the σ-algebra $\Sigma_L = \sigma(L)$ such that $L \subset \mathcal{L}^1(\mu_L)$ and*

$$I(f) = \int_X f d\mu_L,$$

for all $f \in L$.

Proof. Since L_0 is a vector lattice, the assumption that $\mathbb{1} \in L_0$ implies the Stone condition, and hence the existence of the measure is a special case of Theorem 2.28. The fact that μ_L is now a probability measure is immediate from $\mu_L(X) = I(\mathbb{1}) = 1$.

It remains to show that μ_L is unique. Since $\mathbb{1} \in L$, we have that all functions $\mathbb{1}_G$ are in L for $G \in \Sigma_L$. Hence, $\mu_L(G) = I(\mathbb{1}_G)$ for all $G \in \Sigma_L$, and μ_L must be unique. ☐

The unique probability measure in Theorem 2.30 above can be turned into a unique finite measure if one assumes that $I_0(\mathbb{1}) < \infty$ instead. One can also assume that $\mathbb{1} \in L_0^\uparrow$ or $\mathbb{1} \in L$ (which is slightly more general again), and from the proof it is obvious that the result will remain the same.

Since the Lebesgue integral defines a Daniell integral on the vector lattice of simple functions (Section 2.3.1), in view of Theorem 2.28 we end this section by showing that every finite measure is induced by a Daniell integral. Given any arbitrary (finite) measure space (X, Σ, μ), we let L_0 be the set of all bounded measurable functions $f : X \to \mathbb{R}$ (here $\mathbb{R}$ is paired with the Borel sigma algebra generated by open sets). Then L_0 is a vector lattice. A proof of this fact follows from combining Propositions 2.6 and 2.7 of [Fol99], which say that L_0 is a vector space and that it is closed under taking infima and suprema, respectively.

Now consider the Daniell integral

$$I_0(f) = \int_X f d\mu,$$

on L_0. Note that the constant function $\mathbb{1}$ is measurable with respect to Σ, since $\mathbb{1} = \mathbb{1}_X$ and $X \in \Sigma$. Now while $\mathbb{1} \in L_0$ is indeed true, it is not necessarily the case that $I_0(\mathbb{1}) = \int_X \mathbb{1} d\mu = \mu(X) = 1$. It is however the case that $\mathbb{1} \wedge f \in L_0$ for all $f \in L_0$, since L_0 is a vector lattice. Therefore, we invoke Theorem 2.28 which yields a measure ν defined on $\sigma(L_0) = \Sigma$ such that $L_0 \subset \mathcal{L}^1(\nu)$ and

$$\int_X f d\mu = I_0(f) = \int_X f d\nu,$$

for all $f \in L_0$. For any $E \in \Sigma$, we clearly have that $\mathbb{1}_E \in L_0$. It now follows that

$$\mu(E) = I_0(\mathbb{1}_E) = \nu(E),$$

for all $E \in \Sigma$, and hence $\nu = \mu$. It follows that any arbitrary measure μ on X is induced by a Daniell integral on the set of measurable functions of X.

Of course, in this example we did not actually construct any new measure, since the definition of the Daniell integral depended directly on the measure that was being constructed. Chapter 3 of this thesis will be about examples of measures that are induced by Daniell integrals in a non-trivial manner and a strategy that can be used to construct such measures.

2.4.2 The (incomplete) approach of using Carathéodory's Extension Theorem

The approach we use here is essentially based on Section 7.8 of [Bog06b], although we take a slightly different approach. We also extend some of the results and regularly add additional details. In this section, we will only assume the Daniell construction up to the extension to $L_0^\uparrow$.

Let us assume that we again have some vector lattice L_0 of elementary functions on X together with a Daniell integral I_0 defined on L_0. For simplicity, we will restrict ourselves to the case that $\mathbb{1}$ is in L_0 and that $I_0(\mathbb{1}) = 1$. Hence, constant functions are in L_0. Furthermore, we define the following set $\mathcal{G}$ together with a set function μ on $\mathcal{G}$:

$$\mathcal{G} := \{G \subset X : \mathbb{1}_G \in L_0^\uparrow\},$$
$$\mu(G) := I^\uparrow(\mathbb{1}_G).$$

Note that by monotonicity of the integrals we have that $0 = I^\uparrow(0) \leq \mu(G) = I^\uparrow(\mathbb{1}_G) \leq I_0(\mathbb{1}) = 1$ for all $G \in \mathcal{G}$, which shows that μ is positive and finite. Moreover, it follows that μ is monotone. Of course, it remains to show that the class $\mathcal{G}$ contains 'sufficiently many' sets once again such that μ defines a (probability) measure on some suitable σ-algebra on X related to $\mathcal{G}$, which in this case will turn out to be $\sigma(\{\{f > c\} : f \in L_0, c \in \mathbb{R}\})$. Note that since $\{f > c\} = \{f - c\mathbb{1} > 0\}$ and L_0 is a vector lattice containing $\mathbb{1}$, we have

$$\mathcal{C} := \{\{f > 0\} : f \in L_0\} = \{\{f > c\} : f \in L_0, c \in \mathbb{R}\}.$$

Recall that $f^+ := f \vee 0$. The next result points to the connection between $\mathcal{C}$, $\mathcal{G}$, and $L_0^\uparrow$. It is also a justification for the definition of $\mathcal{G}$ being based on $L_0^\uparrow$ instead of just L_0. This lemma essentially states that $\mathcal{C} \subset \mathcal{G}$, and hence that $\mathcal{G}$ contains 'many' sets so long as the function space that we are working in is in some form closed under taking limits.

Lemma 2.31. *Let $f \in L_0^\uparrow$ and $c \in \mathbb{R}$. Then*

$$\mathbb{1}_{\{f>c\}} = \lim_{n\to\infty} \mathbb{1} \wedge (n(f - c\mathbb{1})^+) \in L_0^\uparrow.$$

In particular, $\{f > c\} \in \mathcal{G}$.

Proof. Let $f \in L_0^\uparrow$. By Lemma 2.6, it follows that $\mathbb{1} \wedge n(f - c\mathbb{1})^+ \in L_0^\uparrow$ for all n. The equation $\mathbb{1}_{\{f>c\}} = \lim_{n\to\infty} \mathbb{1} \wedge (n(f - c\mathbb{1})^+) \in L_0^\uparrow$ follows directly from Lemma 2.24. The result is then a consequence of Lemma 2.7. □

We once again have a lemma describing the connection between the σ-algebra $\sigma(L_0)$ and the other σ-algebras that are at play here.

Lemma 2.32. *Let L_0, $\mathcal{C}$, and $\mathcal{G}$ be as above. Then*

$$\sigma(\mathcal{C}) = \sigma(L_0) = \sigma(L_0^{\uparrow}) = \sigma(\mathcal{G}).$$

Proof. We only have to show the first and last equations, for the rest has already been proven in Lemma 2.26.

(1) The first equation is the definition of a σ-algebra generated by a function set: the sets $\{f > c\}$ here are the inverse images of the Borel sets (c, ∞) of $\mathbb{R}$, which generate $\mathcal{B}(\mathbb{R})$ under f. By definition of $\mathcal{C}$, we find that $\sigma(\mathcal{C}) = \sigma(\{\{f > c\} : f \in L_0, c \in \mathbb{R}\}) = \sigma(L_0)$.

(2) For the last equation, consider Lemma 2.31. By definition of $\mathcal{G}$, this lemma implies that $\{f > c\} \in \mathcal{G} \subset \sigma(\mathcal{G})$ for all $f \in L_0^{\uparrow}$, $c \in \mathbb{R}$. As a result, all $f \in L_0^{\uparrow}$ are measurable with respect to $\sigma(\mathcal{G})$ and we have $\sigma(L_0^{\uparrow}) \subset \sigma(\mathcal{G})$. On the other hand, for any $G \in \mathcal{G}$ we have that $\mathbb{1}_G \in L_0^{\uparrow}$. Clearly, $\{\mathbb{1}_G > 0\} = G$, and hence $G \in \sigma(L_0^{\uparrow}) = \sigma(L_0)$. Combining these results with the other equations above, we find that $\sigma(L_0^{\uparrow}) = \sigma(\mathcal{G})$, as required.

□

Next, we will work towards showing that μ extends to a measure on $\sigma(L_0)$ (which we denote by the same symbol) by collecting some straightforward properties of $\mathcal{G}$ and μ. Recall that a *lattice of sets* is a family of sets containing the empty set that is closed under finite union and intersection.

Lemma 2.33. *$\mathcal{G}$ is a lattice of sets that is closed under countable union and contains X.*

Proof. Since $L_0^{\uparrow}$ is a lattice, for any $G_1, G_2 \in \mathcal{G}$ we have that $\mathbb{1}_{G_1 \cup G_2} = \mathbb{1}_{G_1} \vee \mathbb{1}_{G_2} \in L_0^{\uparrow}$ and $\mathbb{1}_{G_1 \cap G_2} = \mathbb{1}_{G_1} \wedge \mathbb{1}_{G_2} \in L_0^{\uparrow}$. Hence, $\mathcal{G}$ is closed under finite union and intersection. Furthermore, let $(G_n)_{n\in\mathbb{N}} \in \mathcal{G}$ be a sequence of sets in $\mathcal{G}$ and define $G := \bigcup_{n=1}^{\infty} G_n$. Then, for $H_m := \bigcup_{n=1}^{m} G_n \in \mathcal{G}$ with $H_m \subset H_{m+1}$ we have

$$\mathbb{1}_G = \mathbb{1}_{\bigcup_{n=1}^{\infty} G_n} = \sup_{m\in\mathbb{N}} \mathbb{1}_{H_m} = \lim_{m\to\infty} \mathbb{1}_{H_m}.$$

Since $\mathbb{1}_{H_m} \in L_0^{\uparrow}$ for all m and $\mathbb{1}_{H_m} \uparrow \mathbb{1}_G$, it follows that $\mathbb{1}_G \in L_0^{\uparrow}$ by Lemma 2.7 and hence that $G \in \mathcal{G}$, so that $\mathcal{G}$ is closed under countable unions. Finally, $0, \mathbb{1} \in L_0^{\uparrow}$, which implies that $\emptyset, X \in \mathcal{G}$. □

Note that $\mu(\emptyset) = I^{\uparrow}(0) = 0$, and $\mu(X) = I^{\uparrow}(\mathbb{1}) = I_0(\mathbb{1}) = 1$. Furthermore, if $G_1, G_2 \in \mathcal{G}$, then

$$\mu(G_1 \cup G_2) = I^{\uparrow}(\mathbb{1}_{G_1} \vee \mathbb{1}_{G_2}) \le I^{\uparrow}(\mathbb{1}_{G_1} + \mathbb{1}_{G_2}) = I^{\uparrow}(\mathbb{1}_{G_1}) + I^{\uparrow}(\mathbb{1}_{G_2}) = \mu(G_1) + \mu(G_2),$$

so that μ is finitely subadditive. From this we also deduce that μ is σ-subadditive (countably subadditive) on $\mathcal{G}$:

$$\mu(G) = I^{\uparrow}(\mathbb{1}_G) = \lim_{m\to\infty} I^{\uparrow}(\mathbb{1}_{H_m}) = \lim_{m\to\infty} \mu(H_m) \le \lim_{m\to\infty} \sum_{n=1}^{m} \mu(G_n) = \sum_{n=1}^{\infty} \mu(G_n).$$

If in particular G_1 and G_2 are disjoint, then $\mathbb{1}_{G_1} \vee \mathbb{1}_{G_2} = \mathbb{1}_{G_1} + \mathbb{1}_{G_2}$ and

$$\mu(G_1 \cup G_2) = \mu(G_1) + \mu(G_2),$$

proving that μ is an additive set function. By a similar argument, we find that μ is σ-additive.

Since the set $L_0^{\uparrow}$ is a vector lattice, we have the identity $f \vee g + f \wedge g = f + g$, which implies that we have the following nice property for μ on $\mathcal{G}$:

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B). \tag{3}$$

Together with the condition that $\mu(\emptyset) = 0$, this makes μ into a so-called *modular function* (cf. [Bog06a] Exercise 1.12.123.).

Next, we introduce several classes of sets commonly found in measure theory that will play a crucial role in the application of Carathéodory's Extension Theorem.

Definition 2.34. *(i) A family $\mathcal{R}$ of subsets of X is called a ring if it contains the empty set and the sets $A \cup B$, $A \cap B$, and $A \setminus B$ belong to $\mathcal{R}$ for all $A, B \in \mathcal{R}$.*

(ii) A family $\mathcal{S}$ of subsets of X is called a semiring if it contains the empty set, $A \cap B \in \mathcal{S}$ for all $A, B \in \mathcal{S}$, and for every pair of sets $A, B \in \mathcal{S}$ with $B \subset A$ the set $A \setminus B$ is the union of finitely many disjoint elements of $\mathcal{S}$.

(iii) A family $\mathcal{A}$ of subsets of X is called a (semi)algebra if it is a (semi)ring and $X \in \mathcal{A}$.

It is important to note that $\mathcal{G}$ is **neither** an algebra **nor** a ring, since for any $A, B \in \mathcal{G}$ with $B \subset A$ it is not guaranteed that $A \setminus B \in \mathcal{G}$. This becomes clear when we write out what this condition means: $A \setminus B \in \mathcal{G}$ iff $\mathbb{1}_{A \setminus B} = \mathbb{1}_A - \mathbb{1}_B \in L_0^\uparrow$. Since $\mathbb{1}_B \in L_0^\uparrow$, we have that $-\mathbb{1}_B \in L_0^\downarrow$ and hence the fact that $L_0^\uparrow$ is not a vector lattice prevents $\mathcal{G}$ from being closed under complementation in general. This presents a problem, since this means that we cannot directly apply the Carathéodory approach to extend the set function μ on $\mathcal{G}$ to a measure on the σ-algebra of Carathéodory measurable sets. This is also the main reason why in Section 2.4.1 we made use of the space L of integrable functions: L has both a vector lattice structure (like L_0) as well as a form of stability under taking limits (like $L_0^\uparrow$) by Lemma 2.13.

To solve this problem without using L, in view of Exercise 1.12.123 on page 94 of [Bog06a] we first extend $\mathcal{G}$ (and μ) to

$$\overline{\mathcal{G}} := \{A \setminus B : A, B \in \mathcal{G}, B \subset A\},$$

which will then be extended to

$$\widehat{\mathcal{G}} := \left\{ \bigcup_{n=1}^{N} \overline{G}_n : N \in \mathbb{N}, \overline{G}_n \in \overline{\mathcal{G}} \right\}.$$

It is clear that $\mathcal{G} \subset \overline{\mathcal{G}} \subset \widehat{\mathcal{G}}$, since $\emptyset \in \mathcal{G}$.

The necessity of the second extension comes from the fact that $\overline{\mathcal{G}}$ is not closed under unions, which is a problem that the introduction of $\widehat{\mathcal{G}}$ solves.

Lemma 2.35. *In view of Definition 2.34, we have the following categorization of $\overline{\mathcal{G}}$ and $\widehat{\mathcal{G}}$:*

(i) $\overline{\mathcal{G}}$ is a semialgebra. In particular, for any $\overline{G}_1, \overline{G}_2 \in \overline{\mathcal{G}}$ we have that $\overline{G}_1 \setminus \overline{G}_2 = C_1 \cup C_2$, for two disjoint sets $C_1, C_2 \in \overline{\mathcal{G}}$.

(ii) $\widehat{\mathcal{G}}$ is an algebra.

Proof. Since $\mathbb{1} \in L_0$ implies that both $X \in \overline{\mathcal{G}}$ and $X \in \widehat{\mathcal{G}}$, it remains to prove that $\overline{\mathcal{G}}$ is a semiring and that $\widehat{\mathcal{G}}$ is a ring.

(i) We first prove that $\overline{\mathcal{G}}$ is a semiring. Clearly, $\overline{\mathcal{G}}$ contains the empty set. Let $A, B, C, D \in \mathcal{G}$ such that $B \subset A$ and $D \subset C$. Using the set-theoretic identities $A \setminus (C \setminus D) = (A \cap D) \cup (A \setminus C)$ and $(A \setminus B) \cap D = (A \cap D) \setminus B$, we find that

$$\begin{aligned}(A \setminus B) \setminus (C \setminus D) &= ((A \setminus B) \cap D) \cup ((A \setminus B) \setminus C) = ((A \cap D) \setminus B) \cup (A \setminus (B \cup (A \cap C)) \\ &= ((A \cap D) \setminus (B \cap D)) \cup (A \setminus (B \cup (A \cap C)).\end{aligned}$$

Since $\mathcal{G}$ is closed under taking unions and (finite) intersections, this shows that for any $\overline{G}_1, \overline{G}_2 \in \overline{\mathcal{G}}$ we have that $\overline{G}_1 \setminus \overline{G}_2 = C_1 \cup C_2$, for two disjoint sets $C_1, C_2 \in \overline{\mathcal{G}}$.

Letting $A, B, C, D \in \mathcal{G}$ with $B \subset A$ and $D \subset C$ once again, it follows from De Morgan's laws and the identity $A \setminus B = A \cap B^c$ that

$$\begin{aligned}(A \setminus B) \cap (C \setminus D) &= A \cap C \cap B^c \cap D^c \\ &= (A \cap C) \cap (B \cup D)^c = (A \cap C) \setminus ((B \cup D) \cap (A \cap C)) \\ &= (A \cap C) \setminus ((B \cap (A \cap C)) \cup (D \cap (A \cap C))) = (A \cap C) \setminus ((B \cap C) \cup (A \cap D)) \in \overline{\mathcal{G}}.\end{aligned}$$

Hence, $\overline{\mathcal{G}}$ is closed under taking intersections and this proves that $\overline{\mathcal{G}}$ is a semiring.

(ii) We now prove that $\widehat{\mathcal{G}}$ is a ring of subsets of X. Let $A_1, A_2, \ldots, A_N, B_1, B_2, \ldots, B_M \in \overline{\mathcal{G}}$ for some $N, M \in \mathbb{N}$. Since $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$, we now find by part (i) that

$$\left(\bigcup_{n=1}^{N} A_n\right) \setminus \bigcup_{m=1}^{M} B_m = \bigcap_{m=1}^{M} \left(\bigcup_{n=1}^{N} A_n \setminus B_m\right) = \bigcap_{m=1}^{M} \left(\bigcup_{n=1}^{N} C_{n,1} \cup C_{m,2}\right),$$

for $C_{n,1}, C_{m,2} \in \overline{\mathcal{G}}$. Since $\widehat{\mathcal{G}}$ contains the empty set, is closed under taking unions, and $\overline{\mathcal{G}} \subset \widehat{\mathcal{G}}$, it remains to show that $\widehat{\mathcal{G}}$ is closed under taking intersections. By the distributive property of unions and intersections, we have that

$$\left(\bigcup_{n=1}^{N} A_n\right) \cap \left(\bigcup_{m=1}^{M} B_m\right) = \bigcup_{m=1}^{M} \left(\bigcup_{n=1}^{N} A_n \cap B_m\right),$$

which implies that it suffices to show that $A_1 \cap A_2 \in \widehat{\mathcal{G}}$ for all $A_1, A_2 \in \overline{\mathcal{G}}$. But we have already established in (i) that $\overline{\mathcal{G}}$ is closed under taking intersections. Therefore, $\widehat{\mathcal{G}}$ is closed under taking intersections and this proves that $\widehat{\mathcal{G}}$ is a ring.

We conclude that $\overline{\mathcal{G}}$ is a semialgebra and that $\widehat{\mathcal{G}}$ is an algebra. Furthermore, for any $\overline{G}_1, \overline{G}_2 \in \overline{\mathcal{G}}$ we have that $\overline{G}_1 \setminus \overline{G}_2 = C_1 \cup C_2$, for two disjoint sets $C_1, C_2 \in \overline{\mathcal{G}}$ by part (i). □

We now define the following non-negative set functions on $\overline{\mathcal{G}}$ and $\widehat{\mathcal{G}}$, respectively:

$$\bar{\mu}(A \setminus B) := \mu(A) - \mu(B),$$

for all $A, B \in \mathcal{G}$ with $B \subset A$, and

$$\hat{\mu}\left(\bigcup_{n=1}^{N} \overline{G}_n\right) := \sum_{n=1}^{N} \bar{\mu}\left(\overline{G}_n\right),$$

for $\overline{G}_n \in \overline{\mathcal{G}}$ pairwise disjoint and $N \in \mathbb{N}$.

Since $\hat{\mathcal{G}}$ allows non-disjoint unions of sets $\overline{G}_n \in \overline{\mathcal{G}}$, in view of the definition of $\hat{\mu}$ above we note that any element $\hat{G} = \bigcup_{n=1}^{N} \overline{G}_n \in \hat{\mathcal{G}}$ can be written as a disjoint union of sets in $\overline{\mathcal{G}}$. To see this, consider the sets $\overline{H}_k := \overline{G}_k \setminus \left(\bigcup_{n=1}^{k-1} \overline{G}_n\right)$ for all $1 \leq k \leq N$. By construction, the $\overline{H}_k$ are pairwise disjoint and

$$\bigcup_{k=1}^{N} \overline{H}_k = \bigcup_{n=1}^{N} \overline{G}_n.$$

By part (i) of Lemma 2.35, we know that $\overline{G}_k \setminus \overline{G}_n = C_{1,n} \cup C_{2,n}$, where $C_{1,n}, C_{2,n} \in \overline{\mathcal{G}}$ are disjoint. It now follows that

$$\overline{H}_k = \overline{G}_k \setminus \left(\bigcup_{n=1}^{k-1} \overline{G}_n\right) = \bigcap_{n=1}^{k-1} \left(\overline{G}_k \setminus \overline{G}_n\right) = \bigcap_{n=1}^{k-1} \left(C_{1,n} \cup C_{2,n}\right).$$

Since $\overline{\mathcal{G}}$ is closed under taking intersections, it follows from the distributive property of intersections and unions that $\overline{H}_k$ is a disjoint union of elements of $\overline{\mathcal{G}}$. This proves that any $\bigcup_{n=1}^{N} \overline{G}_n \in \hat{\mathcal{G}}$ can be written as a disjoint union of sets in $\overline{\mathcal{G}}$.

Lemma 2.36. *The set functions $\bar{\mu}$ and $\hat{\mu}$ are well-defined.*

Proof. (1) To check that $\bar{\mu}$ is well-defined, suppose that $A \setminus B = C \setminus D$, with $A, B, C, D \in \mathcal{G}$ such that $B \subset A$ and $D \subset C$. Then $A \cup D = B \cup C$ as well as $A \cap D = B \cap C$. To see this, note that $A = B \cup (C \setminus D)$ and $A \cup D = B \cup (C \setminus D) \cup D = B \cup C$, since $D \subset C$. Furthermore, we also have $C = D \cup (A \setminus B)$, and hence by distributivity of intersections and unions we get

$$A \cap D = (B \cap D) \cup ((C \setminus D) \cap D) = B \cap D = D \cap B = (D \cap B) \cup ((A \setminus B) \cap B) = B \cap C.$$

Because of the modularity of μ, it follows from equation (3) that

$$\mu(A)+\mu(D)=\mu(A\cup D)+\mu(A\cap D)=\mu(B\cup C)+\mu(B\cap C)=\mu(B)+\mu(C).$$

Hence,

$$\bar{\mu}(A\setminus B)=\mu(A)-\mu(B)=\mu(C)-\mu(D)=\bar{\mu}(C\setminus D),$$

which proves that $\bar{\mu}$ is well-defined.

(2) To show that $\hat{\mu}$ is well-defined, assume that for some $N, M \in \mathbb{N}$ we have $\bigcup_{n=1}^{N} A_n = \bigcup_{m=1}^{M} B_m \in \widehat{\mathcal{G}}$ with $A_n \in \overline{\mathcal{G}}$ pairwise disjoint for all n and $B_m \in \overline{\mathcal{G}}$ pairwise disjoint for all m. We can partition each A_n through the B_m by considering that $A_n = \bigcup_{m=1}^{M}(A_n \cap B_m)$ for all $1 \le n \le N$, since the B_m are pairwise disjoint and $\bigcup_{n=1}^{N} A_n = \bigcup_{m=1}^{M} B_m \in \widehat{\mathcal{G}}$. By symmetry, we also write $B_m = \bigcup_{n=1}^{N}(A_n \cap B_m)$ for all $1 \le m \le M$. The additivity of $\bar{\mu}$ on $\overline{\mathcal{G}}$ now implies for all n, m that

$$\bar{\mu}(A_n)=\sum_{m=1}^{M}\bar{\mu}(A_n\cap B_m),$$

$$\bar{\mu}(B_m)=\sum_{n=1}^{N}\bar{\mu}(A_n\cap B_m).$$

As a result, we find that

$$\begin{aligned}\hat{\mu}\left(\bigcup_{n=1}^{N}A_n\right)&=\sum_{n=1}^{N}\bar{\mu}\left(A_n\right)=\sum_{n=1}^{N}\sum_{m=1}^{M}\bar{\mu}(A_n\cap B_m)\\&=\sum_{m=1}^{M}\bar{\mu}\left(B_m\right)=\hat{\mu}\left(\bigcup_{m=1}^{M}B_m\right),\end{aligned}$$

where the sums can be interchanged since they are sums over finitely many elements. We conclude that $\hat{\mu}$ is well-defined.

□

In order to apply Carathéodory's Extension Theorem to $\hat{\mu}$ on the algebra $\hat{\mathcal{G}}$, it remains to show that $\hat{\mu}$ is a pre-measure, i.e., that $\hat{\mu}(\emptyset)=0$ and that $\hat{\mu}$ is σ-additive. The first condition is immediate, but showing that $\hat{\mu}$ is σ-additive is more involved. By definition, it is clear that $\hat{\mu}$ is additive. Hence, Proposition 3 on page 24 of [KS87] tells us that $\hat{\mu}$ is σ-additive if and only if it is σ-subadditive. It can be shown that σ-subadditivity of $\hat{\mu}$ follows from σ-subadditivity of $\bar{\mu}$, but this latter property is already difficult to show.

For example, let $(\overline{G}_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\overline{\mathcal{G}}$ such that $\overline{G} := \bigcup_{n=1}^{\infty} \overline{G}_n \in \overline{\mathcal{G}}$. Then for every n we have $\overline{G}_n = A_n \setminus B_n$ by definition, for some $A_n, B_n \in \mathcal{G}$ with $B_n \subset A_n$. However, it appears that we know too little about the sets A_n and B_n in particular to say much about $\mu(\overline{G})$.

Assuming that $\hat{\mu}$ is shown to be σ-additive, however, we can apply Carathéodory's Extension Theorem and consider the outer measure

$$\hat{\mu}^*(A) := \inf\left\{\sum_{n=1}^{\infty}\hat{\mu}(G_n) : G_n \in \hat{\mathcal{G}}, A \subset \bigcup_{n=1}^{\infty} G_n\right\}.$$

By Theorem 1.11.8. of [Bog06a], it then follows that $\hat{\mu}^*$ is a probability measure (remember that we have assumed that $\hat{\mu}(X) = I_0(\mathbb{1}) = 1$) on the σ-algebra

$$\mathcal{B} := \{B \subset X : \mu^*(B) + \mu^*(X \setminus B) = 1\} \supset \hat{\mathcal{G}},$$

that coincides with $\hat{\mu}$ on $\hat{\mathcal{G}}$. From here onwards, let μ denote $\hat{\mu}^*$ for convenience.

Under the assumptions $\mathbb{1} \in L_0$ and $I_0(\mathbb{1}) = 1$ of this section, we can prove a less general version of Theorem 2.30 which says that that $I(f) = \int_X f d\mu$ for all $f \in L_0$.

Theorem 2.37. *Let L_0 be a vector lattice of real-valued bounded functions on a set X and I_0 a Daniell integral defined on L_0. If $\mathbb{1} \in L_0$ and $I_0(\mathbb{1}) = 1$, then there exists a unique probability measure μ on the σ-algebra $\sigma(L_0) = \sigma(\hat{\mathcal{G}})$ generated by L_0 such that $L_0 \subset \mathcal{L}^1(\mu)$ and*

$$I(f) = \int_X f d\mu,$$

for all $f \in L_0$.

Proof. As described above, the Daniell integral I_0 on L_0 induces a probability measure μ on the σ-algebra $\mathcal{B}$, which contains $\sigma(\hat{\mathcal{G}})$. It follows from the definitions of $\overline{\mathcal{G}}$ and $\hat{\mathcal{G}}$ that

$$\sigma(\mathcal{G}) = \sigma(\overline{\mathcal{G}}) = \sigma(\hat{\mathcal{G}}),$$

since $\sigma(\mathcal{G})$ contains all unions of relative complements of elements of $\mathcal{G}$ by the σ-algebraic properties and $\mathcal{G} \subset \overline{\mathcal{G}} \subset \hat{\mathcal{G}}$. Hence, the equation $\sigma(L_0) = \sigma(\hat{\mathcal{G}})$ is a result of Lemma 2.31.

It remains to show that $L_0 \subset \mathcal{L}^1(\mu)$ and that

$$I(f) = \int_X f d\mu,$$

for all $f \in L_0$ and that μ is unique. The proof of Theorem 2.30 already yields the uniqueness result. On the other hand, the arguments in the proof of Theorem 2.28 give us that

$$I(f) = \int_X f d\mu,$$

for all non-negative $f \in L_0^{\uparrow}$. This then also yields the result for all non-negative $f \in L_0$, and we make the jump to all $f \in L_0$ by writing

$$f = f^+ - f^-,$$

with $f^+, f^- \in L_0$ non-negative functions. Once again $L_0 \subset \mathcal{L}^1(\mu)$ is now immediate from the fact that L_0 is a vector lattice. □

3 Measures induced by Daniell integrals

In this section, which is divided into two parts, we consider examples of measures that are induced by Daniell integrals. Here we choose to ignore the trivial case from the end of section 2.4.1. In the first part, we consider examples where the chosen Daniell integrals do not necessarily follow a pattern. In the second part, we introduce the notion of projective systems and use these systems to formulate a strategy for constructing measures induced from certain Daniell integrals that do in fact follow a pattern. We will also revisit some examples from the first part, to see that these examples were actually special cases of projective systems all along.

3.1 Examples

The following subsections will consist of examples of measures, induced by some Daniell integral, without invoking projective systems directly.

3.1.1 Finite product spaces

We start with a straightforward example. Suppose that we have $n \in \mathbb{N}$ finite measure spaces (X_i, Σ_i, μ_i), where the X_i are topological spaces, the Σ_i are the Baire σ-algebras of the X_i, and $1 \leq i \leq n$. Here the X_i can be for example locally compact Hausdorff spaces, such as Euclidean space $\mathbb{R}^n$ together with their Baire σ-algebras $\Sigma_i = \mathcal{B}a(\mathbb{R}^n)$. Consider the finite product space (considered as a topological space endowed with the product topology)

$$X := \prod_{i=1}^{n} X_i,$$

together with the σ-algebra $\Sigma := \bigotimes_{i=1}^{n} \Sigma_i$: the σ-algebra generated by subsets of the form $\prod_{i=1}^{n} A_i$ for $A_i \in \Sigma_i$. One can then define a product measure $\mu := \bigotimes_{i=1}^{n} \mu_i$ on X defined by

$$\mu(A) = \left(\bigotimes_{i=1}^{n} \mu_i \right)(A) = \prod_{i=1}^{n} \mu(A_i),$$

for all $A \in \Sigma$ and $A_i \in \Sigma_i$. The fact that μ is a measure follows from Theorem 3.3.1. in [Bog06a].

Now consider the space $C_b(X)$ of bounded continuous functions from X to $\mathbb{R}$. It is immediate that this is a vector lattice for the usual pointwise partial ordering, and hence we can define the following Daniell integral on $C_b(X)$:

$$I_0(f) := \int_{X_1} \int_{X_2} \cdots \int_{X_n} f(x_1, x_2, \ldots, x_n) \mu_n(dx_n) \ldots \mu_2(dx_2) \mu_1(dx_1),$$

for $f \in C_b(X)$. Since $\mathbb{1} \in C_b(X)$, and $\mathbb{1} \wedge f \in C_b(X)$ for all $f \in C_b(X)$, we invoke Theorem 2.28 which yields a measure ν on $\sigma(C_b(X))$ together with the integral equation

$$I^{\uparrow}(f) = \int_X f d\nu,$$

for all $f \in C_b(X)^{\uparrow}$ in particular. Take $A = \prod_{i=1}^{n} A_i$ and let $f_n \uparrow \mathbb{1}_A$ be a sequence of functions in $C_b(X)$. This sequence exists, since $\mathbb{1}_A \in C_b(X)^{\uparrow}$ for all $A \in \Sigma$ (see Lemma 2.31 and observe that Σ is contained in the Baire σ-algebra $\sigma(C_b(X))$ of X by Lemma 6.4.1. of [Bog06b]). Since $\mathbb{1}_A(x) = \prod_{i=1}^{n} \mathbb{1}_{A_i}(x_i)$, it follows for all $A \in \Sigma$ by dominated convergence that

$$\nu(A) = I^{\uparrow}(\mathbb{1}_A) = \lim_{n \to \infty} I_0(f_n) = \lim_{n \to \infty} \int_{X_1} \int_{X_2} \cdots \int_{X_n} f_n(x_1, x_2, \ldots, x_n) \mu_n(dx_n) \ldots \mu_2(dx_2) \mu_1(dx_1)$$

$$= \int_{X_1} \int_{X_2} \cdots \int_{X_n} \mathbb{1}_{A_1}(x_1) \mathbb{1}_{A_2}(x_2) \ldots \mathbb{1}_{A_n}(x_n) \mu_n(dx_n) \ldots \mu_2(dx_2) \mu_1(dx_1)$$

$$= \prod_{i=1}^{n} \int_{X_i} \mathbb{1}_{A_i}(x_i)\mu_i(dx_i) = \prod_{i=1}^{n} \mu_i(A_i) = \mu(A).$$

This proves that $\nu = \mu$, and since $I_0(f)$ was not defined by $\int_X f d\mu$, we conclude that μ is induced by a Daniell integral.

Since in this example we have chosen $C_b(X)$ as our space of elementary functions, the measure ν is defined on the σ-algebra $\sigma(C_b(X))$, which is precisely the Baire σ-algebra $\mathcal{B}a(X)$ of X. If we wanted to construct a measure that is defined on the larger Borel σ-algebra $\mathcal{B}(X)$ instead, then we could have assumed that the Σ_i were Borel σ-algebras and then let the elementary functions be the bounded measurable functions. Under suitable conditions (for example when X is a metric space), the Baire and Borel σ-algebras coincide (by for example [Bog06b] Proposition 6.3.4. and Corollary 6.3.5). If we also assume that all spaces X_i are separable metric spaces (for example when $X_i = \mathbb{R}^n$ for all i), then [Bog06b] Lemma 6.4.2.(ii) implies that

$$\mathcal{B}a(X) = \mathcal{B}a(\prod_{i=1}^{n} X_i) = \bigotimes_{i=1}^{n} \mathcal{B}a(X_i).$$

Both these observations will become relevant in Section 3.1.2.

3.1.2 Wiener process and Wiener measure

In this section, we will take a look at an example of a probability measure induced by a Daniell integral. This example comes from Norbert Wiener's work *Differential Space* [Wie23] from 1923, in which he defined what we now call the Wiener process and the Wiener measure. We will follow Wiener's notation here.

Definition 3.1. *[Wie23] Let $\omega : [0, 1] \to \mathbb{R}^n$ be a continuous stochastic process. We call such ω a Wiener process if the following conditions are met:*

(1) Starting at zero: $\omega(0) = 0$.

(2) Gaussian increments: For $0 \leq t_1 < t_2$, the increment $\omega(t_2) - \omega(t_1)$ has a Gaussian distribution with mean 0 and variance $t = t_2 - t_1$.

(3) Independent increments: For any $0 \leq t_1 < t_2$, the increment $\omega(t_2) - \omega(t_1)$ is independent of all values $\omega(s)$ for $s \leq t_1$.

These conditions come from the physical phenomenon that Wiener was studying at the time: the Brownian motion of small particles suspended in a liquid when viewed under a microscope. In this liquid, the particles constantly experience impulses from all sides by colliding with the liquid molecules that it is suspended in. As a result, we get a seemingly random movement path $\omega([0, 1])$ in $n = 3$ dimensions for the physical case.

In this context the three conditions from Definition 3.1 arise as follows:

(1) The first condition comes down to picking a starting point and origin for the Wiener process.

(2) The second condition is an approximation for the impulses the particle experiences in a certain time frame $[t_1, t_2]$, by partitioning $[t_1, t_2]$ into smaller intervals of displacement of the particle and then adding all these displacements together to form displacement $\omega(t_2) - \omega(t_1)$.

(3) The third condition comes from Einstein's theory, verified by Perrin in [PS05], which says that the initial velocity of a particle on any time interval (which is needed to predict its future motion) is negligible when compared to the amount of impulses the particle receives in that time interval. This is to say that any new increment $\omega(t_2) - \omega(t_1)$ is independent of the past values $\omega(s)$ for $s \leq t_1$.

Since contemporary probability theory was not fully developed yet in 1923, from now on we will follow a more modern approach found in Chapter 16 of Michael Taylor's 2006 book *Measure Theory and Integration* [Tay06].

In order to define the Wiener measure, we must first explore some properties of the Wiener process. By condition (2) of Definition 3.1, if $t_1 \in [0, 1]$ and A is some Borel set of $\mathbb{R}^n$, then we have the probability equation

$$P(\omega(t_1) \in A) = P(\omega(t_1) - \omega(0) \in A) = \int_A p(t_1, x_1) dx_1,$$

where $x_1 = \omega(t_1)$ and

$$p(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

Now let $0 < t_1 < t_2 < \cdots < t_m = 1$ be given together with corresponding Borel sets $A_i \subset \mathbb{R}^n$ for $1 \leq i \leq m$. Denote by B the event that $\omega(t_i) \in A_i$ for all $1 \leq i \leq m$, then

$$P(B) = \int_{A_1} \int_{A_2} \cdots \int_{A_m} p(t_1, x_1) \cdots p(t_m - t_{m-1}, x_m - x_{m-1}) dx_m \cdots dx_1.$$

Note the use of increments $p(t_m - t_{m-1}, x_m - x_{m-1})$ in the integrand, since if $\omega(t_i) = x_i$, then the probability of $\omega(t_{i+1})$ being in A_{i+1} is conditional on the event that $\omega(t_i) = x_i$ and hence by property (3) of the definition it suffices to look at the increment $[t_i, t_{i+1}]$.

It is important to consider that in the previous discussion, there are infinitely many continuous paths $\omega([0,1])$ such that $\omega(t_i) \in A_i$ for all $1 \leq i \leq m$, as long as the Borel sets A_i are all non-empty. Intuitively, the *Wiener measure* W of the set $\{\omega : \omega(t_i) \in A_i \text{ for all } 1 \leq i \leq m\}$ is then equal to the value $P(B)$ above. We will characterise a path $\omega([0,1])$ by all the values it takes for *rational* $t \in [0,1]$, so that we get a set of 'paths'

$$\mathfrak{P} = \prod_{t \in \mathbb{Q} \cap [0,1]} \mathbb{R}^n.$$

Here we equip $\mathfrak{P}$ with the product topology. Since the product is countable and $\mathbb{R}^n$ is a separable metric space for all $n \in \mathbb{N}$, it follows from [Bog06b] Lemma 6.4.2.(ii) that

$$\mathcal{B}(\mathfrak{P}) = \bigotimes_{t \in \mathbb{Q} \cap [0,1]} \mathcal{B}(\mathbb{R}^n).$$

This is the (Borel) σ-algebra that we will define the Wiener measure on.

Taking only rational values in the definition of $\mathfrak{P}$ is justified, since the rationals are dense in the reals and so by continuity of the paths one can approximate ω for irrational $t \in \mathbb{Q}^c$ by $\omega(t) = \lim_{n\to\infty} \omega(q_n)$, where $(q_n)_{n \in \mathbb{N}}$ is a rational sequence approaching t.

Since W will be defined on subsets of $\mathfrak{P}$, in light of Theorem 2.30 it is natural to consider a space of functions on $\mathfrak{P}$ that will play the role of the elementary functions L_0. Consider the space $C_\#$ of bounded and continuous functions $\varphi : \mathfrak{P} \to \mathbb{R}$ that only depend on finitely many values $\omega(t_i)$, i.e., functions of the form

$$\varphi(\omega) = F(\omega(t_1), \ldots, \omega(t_m)),$$

where $t_1 < \cdots < t_m$ are rationals in $[0,1]$ and F is a continuous function on $\prod_{i=1}^m \mathbb{R}^n$. It is then a fact that $C_\#$ is a vector lattice. Furthermore, we define a Daniell integral E on $C_\#$ as follows:

$$E(\varphi) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} p(t_1, x_1) \cdots p(t_m - t_{m-1}, x_m - x_{m-1}) F(x_1, \ldots, x_m) dx_m \cdots dx_1.$$

These two facts are special cases of results for so-called *projective systems*, a topic which will be discussed in Section 3.2. The key assumption here is the independence of increments, which will become apparent in Section 3.2.2.

Since the constant function $\mathbb{1}$ is indeed dependent on a finite amount of $\omega(t_i)$ (none in fact), it is a function in $C_\#$ and hence we can take its Daniell integral

$$E(\mathbb{1}) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} p(t_1, x_1) \cdots p(t_m - t_{m-1}, x_m - x_{m-1}) dx_m \cdots dx_1 = 1,$$

since $p(t,x)$ is the density of a (multivariate) normal distribution and hence integrates to 1 over its domain $\mathbb{R}^n$.

Putting everything together, we have all the conditions necessary to apply Theorem 2.30 to E on $C_\#$. The representing measure for the Daniell integral E will be defined on $\sigma(C_\#)$, which is the smallest σ-algebra that contains all finite products of $\mathcal{B}a(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^n)$. By definition, this is precisely $\bigotimes_{t\in\mathbb{Q}\cap[0,1]} \mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathfrak{P})$. From this observation and the results in this section, the following theorem is immediate.

Theorem 3.2. *There exists a unique Borel probability measure W defined on $\mathcal{B}(\mathfrak{P})$ called the* ***Wiener measure*** *such that for all $\varphi \in C_\#$ we have*

$$E(\varphi) = \int_{\mathfrak{P}} \varphi dW.$$

From the results in this section we conclude that the Wiener measure is induced by a Daniell integral.

It is important to note that in the representation $\mathfrak{P}$, we actually describe *all* continuous functions from $[0, 1]$ to $\mathbb{R}^n$, and not necessarily the trajectories of the continuous Wiener processes ω. In [Fol99], Theorem 10.28 can be found. This theorem says that the Wiener measure W constructed here is in fact concentrated on $C([0, 1], \mathbb{R}^n)$, which is precisely the set of all Wiener processes. Hence, we can 'throw away' all the additional functions in $\mathfrak{P}$ that are not Wiener processes, since they will not be detected by the Wiener measure.

3.1.3 Gaussian measures

It turns out that the Wiener measure that we encountered in Section 3.1.2 is actually a special case of a so-called *Gaussian measure*. In this section, we will explore this concept and later on in Section 3.2.3 we will attempt to apply our theory on Daniell integration and projective systems in order to construct Gaussian measures on infinite dimensional Hilbert spaces. This section is based on [Eld16] and [O'C18]. We start by considering the simplest case of a Borel probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For our purposes, it is sufficient to only consider Gaussian measures with mean zero.

Definition 3.3. *A Borel probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called a Gaussian measure with variance $\sigma^2 > 0$ if*

$$\mu(B) = \int_B g(x)dx,$$

where $B \subset \mathbb{R}$ is a Borel set, and g is the density of a one-dimensional Gaussian distribution with mean 0 and variance σ^2:

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}.$$

In the case that $\sigma = 0$, we write $\mu = \delta_0$ as the Dirac measure concentrated at 0.

Intuitively, a Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is the probability measure assigning to each subset $B \in \mathbb{R}$ the area underneath the Gaussian curve for points in B. It is also possible to characterise a Gaussian measure by its characteristic function, in the same way that a characteristic function of a probability distribution is completely characteristic.

Proposition 3.4. *[O'C18] A Gaussian measure μ on $\mathbb{R}$ with variance σ^2 has characteristic function of the form*

$$\varphi(t) = \exp\left\{-\frac{1}{2}\sigma^2 t^2\right\}.$$

Proof. This follows directly from the definition of the characteristic function as the expectation of the function e^{itX} for X a real random variable:

$$\begin{aligned}\varphi(t) &= \int_{\mathbb{R}} \exp\{itx\} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(x^2 - 2\sigma^2 itx\right)\right\} dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(x^2 - 2\sigma^2 itx + (\sigma^2 it)^2 - (\sigma^2 it)^2\right)\right\} dx \\
&= \exp\left\{-\frac{1}{2\sigma^2}\left(\sigma^4 t^2\right)\right\} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(x - (\sigma^2 it)\right)^2\right\} dx \\
&= \exp\left\{-\frac{1}{2}\sigma^2 t^2\right\} \int_{\mathbb{R}-\sigma^2 it} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}x^2\right\} dx.
\end{aligned}
$$

A simple application of Cauchy's integral theorem to the holomorphic function $z \to \exp\{-\frac{1}{2\sigma^2}z\}$ on $\mathbb{C}$ now yields that

$$
\begin{aligned}
\varphi(t) &= \exp\left\{-\frac{1}{2}\sigma^2 t^2\right\} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}x^2\right\} dx \\
&= \exp\left\{-\frac{1}{2}\sigma^2 t^2\right\},
\end{aligned}
$$

since the integral evaluates to 1 as an integral of a density function over the entire domain $\mathbb{R}$. □

Of course, since Daniell's approach is so general, it would be useful to extend this notion of Gaussian measures to more general spaces than $\mathbb{R}$. We will first extend our definition to $\mathbb{R}^n$.

Definition 3.5. *A Borel probability measure μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is called a Gaussian measure with positive definite covariance matrix M if*

$$
\mu(B) = \int_B g(x)dx,
$$

where $B \subset \mathbb{R}^n$ is a Borel set, and g is the density of an n-dimensional Gaussian distribution with mean 0 and positive definite covariance matrix M:

$$
g(x) = \frac{1}{\sqrt{(2\pi)^n |M|}} \exp\left\{-\frac{1}{2}x^T M^{-1} x\right\},
$$

where $|M|$ denotes the determinant of M.

There is also a generalization of Proposition 3.4 for the case that $X = \mathbb{R}^n$, which we shall state here without proof.

Proposition 3.6. *[Eld16] A Borel probability measure μ on $\mathbb{R}^n$ is Gaussian if and only if for all $t \in \mathbb{R}^n$ we have*

$$
\int_{\mathbb{R}^n} e^{it^T x} \mu(dx) = e^{-\frac{1}{2}t^T M t},
$$

where M is some positive semidefinite (possibly) symmetric $n \times n$ matrix: the covariance matrix.

To abstract further, what we essentially require from our space X is structure that enables us to translate the Gaussian measure from the simple case $\mathbb{R}$ to the more general case X. It turns out that the continuous linear functionals $f \in X^*$ enable this translation, and their existence depends on the topological and vector space structure of X. Therefore, we assume that X is a topological vector space, i.e., a vector space such that the addition and scalar multiplication operators are continuous with respect to the topology on X.

Definition 3.7. *Let X be a topological vector space and μ a Borel probability measure on X. We say that μ is a Gaussian measure if for each continuous linear functional $f \in X^*$ the pushforward $\mu \circ f^{-1}$ is a Gaussian measure on $\mathbb{R}$.*

Note the rather strong requirement that this holds for *all* continuous linear functionals on X. There is once more a generalization of Proposition 3.4 for the case that X is a topological vector space.

Proposition 3.8. *[Eld16] A Borel probability measure μ on a topological vector space X is Gaussian if and only if for each $f \in X^*$ we have that*

$$\int_X e^{if(x)}\mu(dx) = e^{-\frac{1}{2}q(f,f)},$$

where q is some positive semidefinite symmetric bilinear form on X^*.

We finish this section with a proof that any one-dimensional Gaussian measure is induced by a Daniell integral.

Hence, we assume that $X = \mathbb{R}$ so that a Gaussian measure is directly defined by Definition 3.3. On the vector lattice of continuous bounded functions on $\mathbb{R}$, $L_0 := C_b(\mathbb{R})$, we define the following Daniell integral:

$$E(f) := \int_{-\infty}^{\infty} f(x)g(x)dx,$$

for all $f \in C_b(\mathbb{R})$ and g some Gaussian density centered around 0. Note the use of E for this integral, as this is essentially just the expectation of the random variable $f(X)$, where X is a random variable with law the given Gaussian distribution. The fact that $E : C_b(\mathbb{R}) \to \mathbb{R}$ is a Daniell integral follows immediately from the fact that it is defined by an improper Riemann integral and that the Riemann integral possesses all the defining properties of the Daniell integral. Since $\mathbb{1} \in C_b(\mathbb{R})$ and

$$E(\mathbb{1}) = \int_{-\infty}^{\infty} g(x)dx = 1,$$

we can apply Theorem 2.30 and acquire a unique Borel probability measure μ on $\mathbb{R}$ such that

$$E(f) = \int_{\mathbb{R}} f(x)\mu(dx),$$

for all $f \in C_b(\mathbb{R})$.

Proposition 3.9. *The probability measure μ on $\mathbb{R}$ induced by*

$$E(f) = \int_{\mathbb{R}} f(x)\mu(dx),$$

for $f \in C_b(\mathbb{R})$ is a Gaussian measure on $\mathbb{R}$.

Proof. By Theorem 2.30, we have

$$E^{\uparrow}(f) = \int_{\mathbb{R}} f(x)\mu(dx),$$

for all $f \in L_0^{\uparrow}$. This includes $\mathbb{1}_G$ for all $G \in \mathcal{G} \subset \sigma(C_b(\mathbb{R})) = \mathcal{B}(\mathbb{R})$, since any set $\{f > c\} \in \sigma(C_b(\mathbb{R}))$ will be open by continuity of f and the fact that (c, ∞) is open for all $c \in \mathbb{R}$. All of this is to say that we can now evaluate $\mu(B)$ for all $B \subset \mathbb{R}$ by evaluating $E(\mathbb{1}_B)$. Since $\mathbb{1}_B \in L_0^{\uparrow}$, there exists some increasing sequence $f_n \uparrow \mathbb{1}_B$ of functions in $C_b(\mathbb{R})$. Consider the non-negative sequence

$$f_n^+ := f_n \vee 0.$$

Since $\mathbb{1}_B(x) \geq 0$ for all $x \in \mathbb{R}$, it follows that $f_n^+ \uparrow \mathbb{1}_B$ and so that $0 \leq (f_n^+ \cdot g) \uparrow (\mathbb{1}_B \cdot g)$. Therefore, using the monotone convergence theorem for the Lebesgue integral, we can evaluate $\mu(B)$ as

$$\mu(B) = \int_{\mathbb{R}} \mathbb{1}_B \mu(dx) = E^{\uparrow}(\mathbb{1}_B) = \lim_{n\to\infty} E(f_n^+) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n^+(x)g(x)dx = \int_{-\infty}^{\infty} \mathbb{1}_B g(x)dx = \int_B g(x)dx.$$

Comparing this expression to Definition 3.3, we see that this implies that the constructed μ is in fact a Gaussian measure. □

It follows from Proposition 3.9 that every Gaussian measure on $\mathbb{R}$ is induced by a Daniell integral.

3.2 Projective systems

In Section 3.1.2, we constructed the Wiener measure on the set of all 'paths' $\mathfrak{P} = \prod_{t\in\mathbb{Q}\cap[0,1]} \mathbb{R}^n$, an infinite product which can be seen as a so-called *projective limit* of sets of the form $\prod_{t_i\in\mathbb{Q}\cap[0,1]} \mathbb{R}^n$ of spaces of finite products where $0 \leq t_1 \leq \cdots \leq t_m \leq 1$ for some $m \in \mathbb{N}$. This corresponds to only considering a finite amount of values $\omega(t_1), \ldots, \omega(t_m)$ of any path ω, which we used in the previous section to define $C_\#$ together with a Daniell integral on $C_\#$ which induced the Wiener measure. In this section, we will explore the general idea behind this approach in a topological setting, so that we may apply it to construct measures induced by Daniell integrals on topological spaces. For a more general theory of projective systems, see [Zde72]. In Section 3.2.2, we will return to the Wiener measure example and review it from the perspective of projective systems.

Let $\mathcal{F}$ be a directed set, which is to say that there exists a reflexive and transitive binary relation $\leq$ on $\mathcal{F}$ such that for every pair of elements $I, J \in \mathcal{F}$ there exists an upper bound $K \in \mathcal{F}$ such that $I \leq K$ and $J \leq K$. For each $I \in \mathcal{F}$, let X_I be a topological Hausdorff space. We would like to take limits of such X_I, so that we may transfer measure theoretic information from the sets X_I to some more complicated set X. For every $I, J \in \mathcal{F}$ with $I \leq J$, let $\pi_{J,I} : X_J \to X_I$ be continuous surjective projection maps. We have the following definition of a projective system.

Definition 3.10. *Consider a family $(X_I)_{I\in\mathcal{F}}$ of topological Hausdorff spaces, together with continuous surjective projection maps $\pi_{J,I} : X_J \to X_I$ for all $I \leq J$ and $I, J \in \mathcal{F}$. If for all $I, J, K \in \mathcal{F}$ with $I \leq J \leq K$ we have*

$$\pi_{J,I} \circ \pi_{K,J} = \pi_{K,I},$$

and

$$\pi_{I,I} = id_{X_I},$$

then the pair $((X_I)_{I\in\mathcal{F}}, (\pi_{J,I})_{I\leq J\in\mathcal{F}})$ is called a (topological) projective system over $\mathcal{F}$. Furthermore, we can define the projective limit space X as

$$X := \varprojlim_{I\in\mathcal{F}} X_I = \{(x_I)_{I\in\mathcal{F}} : x_I \in X_I, \pi_{J,I}(x_J) = x_I \text{ for every } I \leq J \in \mathcal{F}\}.$$

The projective limit X comes equipped with projections $\pi_I : X \to X_I$ defined by $\pi_I((x_I)_{I\in\mathcal{F}}) := x_I$ and we equip X with the coarsest topology such that all of these projections π_I are continuous. In fact, X is a subset of $\prod_{I\in\mathcal{F}} X_I$, and we are equipping X with the relative topology here.

Since X and X_I for all $I \in \mathcal{F}$ have defined topologies, it is possible to talk about Borel and Baire measures on these sets. Given finite Borel or Baire measures that are 'consistent' with the projective system on all the X_I, the idea is to define a Borel or Baire measure on X using the Daniell approach as a basis. Let us now properly define this consistency requirement.

Definition 3.11. *Assume that for each $I \in \mathcal{F}$ we have a finite Borel or Baire measure μ_I on X_I. We call the collection $\{\mu_I\}$ consistent with respect to the projective system $((X_I)_{I\in\mathcal{F}}, (\pi_{J,I})_{I\leq J\in\mathcal{F}})$ if for all $I, J \in \mathcal{F}$ with $I \leq J$ the pushforward of μ_J under $\pi_{J,I}$ is μ_I:*

$$\pi_{J,I}\#\mu_J = \mu_I,$$

where the pushforward is defined by

$$(\pi_{J,I}\#\mu_J)(A) := \mu_J(\pi_{J,I}^{-1}(A)),$$

for any Borel or Baire set $A \in \mathcal{B}(X_I)$.

Note that since all projections $\pi_{J,I}$ are continuous for all $I \leq J \in \mathcal{F}$, the set $\pi_{J,I}^{-1}(A)$ is indeed measurable, and hence the pushforward is well-defined. We now introduce the prototypical space of elementary functions that we will define a Daniell integral on. Assume that $\mathbb{R}$ comes equipped with its Borel σ-algebra $\mathcal{B}(\mathbb{R})$. For the projective limit X, we define

$$C_\#(X) := \{f : X \to \mathbb{R} : \exists I_f \in \mathcal{F}, F \in C_b(X_{I_f}) : f(x) = F(\pi_{I_f}(x)), \forall x \in X\}.$$

Note how this set is a generalization of the set $C_\#$ found in Section 3.1.2. A reformulation of the Wiener measure in the context of projective systems will be considered in Section 3.2.2. It might be the case that f can be represented by two pairs (I_f, F) and (I'_f, F'). The following lemma tells us that we can either project from one pair to the other in a natural way, or $F \equiv F' \equiv f$ is a constant function.

Lemma 3.12. *Let (I_f, F) and (I'_f, F') be two pairs that both define f. Then the following holds:*

(1) If $I_f \leq I'_f$, then for all $y' \in X_{I'_f}$ we have

$$F'(y') = F(\pi_{I'_f, I_f}(y')).$$

If $I'_f \leq I_f$, then for all $y \in X_{I_f}$ we have

$$F(y) = F'(\pi_{I_f, I'_f}(y)).$$

(2) If neither $I_f \leq I'_f$ nor $I'_f \leq I_f$ holds, then $F \equiv F' \equiv f$ is a constant function.

Proof. Let (I_f, F) and (I'_f, F') both be pairs that define $f \in C_\#(X)$.

(1) Assume that $I_f \leq I'_f$. Let $y' = \pi_{I'_f}(x) \in X_{I'_f}$ for some $x \in X$. We have by the transitivity property of the projections that

$$F'(\pi_{I'_f}(x)) = f(x) = F(\pi_{I_f}(x)) = F(\pi_{I'_f, I_f}(\pi_{I'_f}(x)),$$

which implies that

$$F'(y') = F(\pi_{I'_f, I_f}(y')),$$

and similarly for the case that $I'_f \leq I_f$.

(2) Assume that neither $I_f \leq I'_f$ nor $I'_f \leq I_f$ holds. Then for all $x \in X$ we have

$$f(x) = F(\pi_{I_f}(x)) = F'(\pi_{I'_f}(x)).$$

Let $y \in X_{I_f}$. It follows that for all $x \in X$ with $\pi_{I_f}(x) = y$ we have

$$F(y) = F'(\pi_{I'_f}(x)) = F'(y'),$$

for all $y' \in X_{I'_f}$, since such x can be chosen to project under $\pi_{I'_f}$ to all values of $X_{I'_f}$ while retaining the property that $\pi_{I_f}(x) = y$. Since this holds for all $y \in X_{I_f}$, it follows that $F(y) = F'(y')$ for all $y \in X_{I_f}$ and $y' \in X_{I'_f}$. As a result, F, F' must both be constant functions equal to the same value on their domain, and hence induce the same (constant) f.

□

Equipped with pointwise ordering of functions in $C_\#(X)$, we show that we indeed have a vector lattice on our hands.

Lemma 3.13. *The function space $C_\#(X)$ is a vector lattice.*

Proof. Let $f, g \in C_\#(X)$ be functions with representations $f(x) = F(\pi_{I_f}(x))$ and $g(x) = G(\pi_{I_g}(x))$, respectively. Define $I := I_f \cup I_g$ so that

$$\tilde{F}(y) := F(\pi_{I, I_f}(y))$$
$$\tilde{G}(y) := G(\pi_{I, I_g}(y))$$

are continuous bounded functions on X_I, arriving very naturally from the proposition in Lemma 3.12(1). For all $\alpha, \beta \in \mathbb{R}$ and $x \in X$, we have

$$(\alpha f+\beta g)(x)=\alpha f(x)+\beta g(x)=\alpha F(\pi_{I_f}(x))+\beta G(\pi_{I_g}(x))=\alpha F(\pi_{I,I_f}(\pi_I(x)))+\beta G(\pi_{I,I_g}(\pi_I(x)))$$
$$=\alpha\tilde{F}(\pi_I(x))+\beta\tilde{G}(\pi_I(x))=(\alpha\tilde{F}+\beta\tilde{G})(\pi_I(x))\in C_\#(X),$$

since $I\in\mathcal{F}$ and $\alpha\tilde{F}+\beta\tilde{G}\in C_b(X_I)$. Similarly, we have

$$(f\vee g)(x)=\max(f(x),g(x))=\max(F(\pi_{I_f}(x)),G(\pi_{I_g}(x)))=\max(F(\pi_{I,I_f}(\pi_I(x))),G(\pi_{I,I_g}(\pi_I(x))))$$
$$=\max(\tilde{F}(\pi_I(x)),\tilde{G}(\pi_I(x)))=(\tilde{F}\vee\tilde{G})(\pi_I(x))\in C_\#(X),$$

since $I\in\mathcal{F}$ and $\tilde{F}\vee\tilde{G}\in C_b(X_I)$. An analogous argument proves the statement for $f\wedge g$. We conclude that $C_\#(X)$ is a vector lattice. □

By definition, all $f\in C_\#(X)$ are bounded and real-valued. We are thus ready to define a Daniell integral on $C_\#(X)$. For a projective system $((X_I)_{I\in\mathcal{F}},(\pi_{J,I})_{I\leq J\in\mathcal{F}})$ together with consistent measures μ_I on X_I we define

$$E(f)=E(F(\pi_{I_f}(x))):=\int_{X_{I_f}}F(y)\mu_{I_f}(dy),$$

if f is defined by (I_f,F).

Lemma 3.14. *The functional $E:C_\#(X)\to\mathbb{R}$ is a well-defined Daniell integral.*

Proof. First note that since all F are bounded and all μ_I are finite, E is indeed real-valued. We proceed by proving that E is well-defined. Assume that both (I_f,F) and (I_f',F') define f. By Lemma 3.12, there are two cases to consider.

(1) If $I_f\leq I_f'$, then

$$E(f)=\int_{X_{I_f'}}F'(y')\mu_{I_f'}(dy')=\int_{X_{I_f'}}F(\pi_{I_f',I_f}(y'))\mu_{I_f'}(dy')$$
$$=\int_{X_{I_f}}F(y)\mu_{I_f}(dy).$$

where the last equality follows from a change-of-variables formula

$$\int_{X_I}F(\pi_{I,I_f}(y))\mu_I(dy)=\int_{X_{I_f}}F(y)(\pi_{I,I_f}\#\mu_I)(dy)=\int_{X_{I_f}}F(y)\mu_{I_f}(dy)$$

which is justified by Theorem 3.6.1. found in [Bog06a], the fact that all the projections π_{I_f',I_f} are continuous and hence measurable, and because all measures μ_I are consistent. The case of $I_f'\leq I_f$ is similar and hence it follows that E is well-defined in this case.

(2) If neither $I_f\leq I_f'$ nor $I_f'\leq I_f$ holds, then $F\equiv F'\equiv f$ is a constant function and it immediately follows that E is well-defined.

Next, we show that E is a Daniell integral. Let $f,g\in C_\#(X)$ be functions with representations $f(x)=F(\pi_{I_f}(x))$ and $g(x)=G(\pi_{I_g}(x))$, respectively. For all $\alpha,\beta\in\mathbb{R}$ and $x\in X$, by the proof of Lemma 3.13 we have that

$$(\alpha f+\beta g)(x)=(\alpha\tilde{F}+\beta\tilde{G})(\pi_I(x)),$$

and hence

$$E(\alpha f + \beta g)) = \int_{X_I} \alpha \tilde{F}(y) + \beta \tilde{G}(y) \mu_I(dy) = \int_{X_I} \alpha \tilde{F}(y) \mu_I(dy) + \int_{X_I} \beta \tilde{G}(y) \mu_I(dy)$$

$$= \alpha \int_{X_I} F(\pi_{I,I_f}(y)) \mu_I(dy) + \beta \int_{X_I} G(\pi_{I,I_g}(y)) \mu_I(dy) = \alpha \int_{X_{I_f}} F(y) \mu_{I_f}(dy) + \beta \int_{X_{I_g}} G(y) \mu_{I_g}(dy)$$

$$= \alpha E(f) + \beta E(g),$$

where the second-to-last equality again follows the same change-of-variables formula as mentioned above in (1).

Positivity of E follows immediately from the fact that the Lebesgue integral is a positive functional. If $f_n \downarrow 0$ is a sequence in $C_{\#}(X)$, then $F_n \circ \pi_{I_{f_n}} \downarrow 0$ implies that $F_n \downarrow 0$, and since the Lebesgue integral is continuous with respect to monotone convergence, the same must hold for E. We conclude that E is a well-defined Daniell integral on $C_{\#}(X)$. □

It remains to check that $\mathbb{1} \in C_{\#}(X)$, so that we obtain a unique measure on X using Theorem 2.30. We have already seen in the proof of Lemma 3.14 that since $f \equiv 1$ is constant, any representative F on some set X_I of f (the specific set $I \in \mathcal{F}$ is irrelevant here, in fact any non-empty I will do) is also the constant function 1, which is clearly in $C_b(X_I)$. From Theorem 2.30 it now follows that there exists a unique finite measure μ on $\sigma(C_{\#}(X))$ such that for all $f \in C_{\#}(X)$ we have

$$E(f) = \int_X f d\mu.$$

It is worth noting that if we assume that all measures μ_I are Borel or Baire probability measures instead, then it follows that

$$E(\mathbb{1}) = \int_{X_I} \mathbb{1}(y) \mu_I(dy) = \mu_I(X_i) = 1,$$

and hence Theorem 2.30 yields a unique probability measure μ defined on $\sigma(C_{\#}(X))$. The following lemma characterizes $\sigma(C_{\#}(X))$.

Lemma 3.15. *Let $C_{\#}(X)$ be as above. Then*

$$\sigma(C_{\#}(X)) = \sigma\left(\{\{f > c\} : f \in C_{\#}(X), c \in \mathbb{R}\}\right) = \bigotimes_{I \in \mathcal{F}} \mathcal{B}a(X_I).$$

If each X_I has a countable base (e.g., if all X_I are separable metric spaces), then

$$\sigma(C_{\#}(X)) = \mathcal{B}a(\prod_{I \in \mathcal{F}} X_I).$$

Proof. The first equation is by definition. We have $\sigma\left(\{\{f > c\} : f \in C_{\#}(X), c \in \mathbb{R}\}\right) = \bigotimes_{I \in \mathcal{F}} \mathcal{B}a(X_I)$, since $\bigotimes_{I \in \mathcal{F}} \mathcal{B}a(X_I)$ is by definition the σ-algebra in $\prod_{I \in \mathcal{F}} X_I$ generated by the cylinder sets

$$A := A_{I_f} \times \prod_{I \in \mathcal{F}, I \neq I_f} X_I,$$

where $A_{I_f} \in \mathcal{B}a(X_{I_f})$ for some $f \in C_{\#}(X)$. The second statement is a consequence of Lemma 6.4.2. of [Bog06b]. □

To one well acquainted with the fundamentals of the theory of stochastic processes, the process outlined in this section may be considered very similar to the Daniell-Kolmogorov Theorem. For more information on this theorem, see for example Section 2.4 of [Tao13].

In summary, we have outlined a strategy to construct a Daniell (probability) measure on the projective limit X of a given projective system $((X_I)_{I \in \mathcal{F}}, (\pi_{J,I})_{I \leq J \in \mathcal{F}})$. In concrete cases, one might consider using this strategy to construct measures on spaces Y that are not necessarily equal to a projective limits X, but can be embedded in X such that the measure μ on X concentrates on Y and where the spaces X_I of the projective system may arise naturally. In any case, it is necessary to find a suitable set $\mathcal{F}$ and show that each X_I indeed has a Borel or Baire (probability) measure μ_I defined on it such that the collection of all μ_I forms a consistent set of measures with respect to the relevant projective system.

3.2.1 Infinite product spaces

The problem of existence of a product measure on an infinite product space is not as straightforward as the finite dimensional case encountered in Section 3.1.1. See for example [Sam22], where the various problems and known solutions can be found. Let $\{(X_i, \Sigma_i, \mu_i)\}_{i\in\mathbb{N}}$ be a family of probability spaces. The task is to construct a measure μ on the infinite product measurable space $(\prod_{i=1}^{\infty} X_i, \bigotimes_{i=1}^{\infty} \Sigma_i)$, where $\bigotimes_{i=1}^{\infty} \Sigma_i$ is the σ-algebra generated by the so-called *cylinder sets* of the form

$$A = \prod_{i=1}^{N} A_i \times \prod_{N+1}^{\infty} X_i \in \bigotimes_{i=1}^{\infty} \Sigma_i,$$

where $N \in \mathbb{N}$. Let $\mathcal{F}$ be the directed set of all non-empty finite subsets of $\mathbb{N}$, ordered by inclusion. If one considers the projective limit $\mathcal{X}$ of sets $X_I := \prod_{i\in I} X_i$ for $I \in \mathcal{F}$, then the cylinder sets are precisely the preimages of the projections π_I from $\mathcal{X}$ to X_I. In fact, this is precisely what we will use later to prove the existence of the measure μ. In [Sam22], the following result by S. Kakutani appears (although with a slightly different notation).

Theorem 3.16. *[Sam22] Let $\{(X_i, \Sigma_i, \mu_i)\}_{i\in\mathbb{N}}$ be a family of probability spaces. Then there exists a unique probability measure μ on the measurable space $(\prod_{i=1}^{\infty} X_i, \bigotimes_{i=1}^{\infty} \Sigma_i)$ such that for every cylinder set $A = \prod_{i=1}^{N} A_i \times \prod_{N+1}^{\infty} X_i$ we have the identity*

$$\mu(A) = \prod_{i=1}^{N} \mu_i(A_i).$$

We will proceed by proving this result for **Baire probability spaces** in the context of this thesis by constructing a probability measure on $(\prod_{i=1}^{\infty} X_i, \bigotimes_{i=1}^{\infty} \Sigma_i)$ using the projective systems approach from Section 3.2. Since we need topologies on the spaces X_i in order to define continuous functions for our vector lattice $C_{\#}(\mathcal{X})$, we will only be able to prove the weaker result where $\{(X_i, \mathcal{B}a(X_i), \mu_i)\}_{i\in\mathbb{N}}$ is a family of Baire probability spaces. The restriction to Baire spaces instead of Borel spaces here comes from the fact that

$$\sigma(C_{\#}(\mathcal{X})) = \bigotimes_{I\in\mathcal{F}} \mathcal{B}a(X_I),$$

as we have seen in Lemma 3.15.

Proof. (For Baire σ-algebras) For $i \in \mathbb{N}$, let $(X_i, \mathcal{B}a(X_i), \mu_i)$ be a family of Baire probability spaces. Recall that $\mathcal{F}$ is the directed set of all non-empty finite subsets of $\mathbb{N}$, ordered by inclusion. Furthermore, for each $I \in \mathcal{F}$ we defined

$$X_I := \prod_{i\in I} X_i.$$

This yields a natural projective system $((X_I)_{I\in\mathcal{F}}, (\pi_{J,I})_{I\subset J\in\mathcal{F}})$, where the projections are defined by sending $x \in X_J$ to X_I by

$$\pi_{J,I}(x) = \pi_{J,I}((x_{j_1}, x_{j_2}, \ldots, x_{j_n})) = (x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \in X_I,$$

if $J = \{j_1, \ldots, j_n\}$ and $I = \{i_1, \ldots, i_m\}$ and $n, m \in \mathbb{N}$.

Each X_I also naturally obtains a Baire probability measure μ_I as the product of the measures of X_i. For all sets $A \in \bigotimes_{i\in I} \Sigma_i$ (the σ-algebra on X_I generated by the Cartesian product of measurable subsets of the X_i) we have

$$\mu_I(A) := \bigotimes_{i \in I} \mu_i(A) = \mu_I\left(\prod_{i\in I} A_i\right) = \prod_{i \in I} \mu_i(A_i).$$

For any $I, J \in \mathcal{F}$ with $I \subset J$ and A a Baire set of X_I we now have

$$\mu_J(\pi_{J,I}^{-1}(A)) = \mu_J\left(\pi_{J,I}^{-1}\left(\prod_{i\in I} A_i\right)\right) = \mu_J\left(\prod_{i\in I} A_i \times \prod_{j \in J\setminus I} X_j\right)$$
$$= \prod_{i\in I}\mu_i(A_i)\cdot \prod_{j\in J\setminus I}\mu_j(X_j) = \prod_{i \in I}\mu_i(A_i) = \mu_I(A),$$

since $\mu_j(X_j) = 1$ for all $j \in J \setminus I$. It follows that we have a consistent system of measures μ_I on X_I. Hence, using the strategy outlined in the previous section, we can construct a unique Baire probability measure ν on the projective limit $\mathcal{X}$ (equipped with the Baire σ-algebra $\bigotimes_{I\in\mathcal{F}} \mathcal{B}a(X_I)$) of the X_I by defining the following Daniell integral E on the vector lattice $C_{\#}(\mathcal{X})$:

$$E(f) := \int_{X_{I_f}} F(y)\mu_{I_f}(dy) = \int_X f d\nu.$$

In particular, this result also holds for functions $f \in C_{\#}(\mathcal{X})^{\uparrow}$, which includes the indicator functions of Baire measurable subsets $A \subset \mathcal{X}$ as we have seen in Section 3.2 and Lemma 2.31. While $\mathcal{X}$ is not equal to $\prod_{i=1}^{\infty} X_i$, we do have a homeomorphism φ from $\mathcal{X}$ to $\prod_{i=1}^{\infty} X_i$ defined by

$$\varphi((x_I)_{I\in\mathcal{F}}) := (x_1, x_2, \dots),$$

where each x_i is the coordinate found in the thread $(x_I)_{I\in\mathcal{F}}$ corresponding to X_i. This also means that we have a unique pushforward Baire probability measure μ on $\bigotimes_{i=1}^{\infty}\mathcal{B}a(X_i)$ defined by

$$\mu(A) := (\varphi\#\nu)(A) = \nu(\varphi^{-1}(A)).$$

Let us consider the cylinder sets $A = \prod_{i=1}^{N} A_i \times \prod_{N+1}^{\infty} X_i \in \bigotimes_{i=1}^{\infty}\mathcal{B}a(X_i)$, where $A_i \in \mathcal{B}a(X_i)$. We let $f_n \uparrow \mathbb{1}_{\varphi^{-1}(A)}$ be a sequence of functions in $C_{\#}(\mathcal{X})$. For each n, we can take f_n such that

$$f_n(\varphi^{-1}((x_1, x_2, \dots))) = F_n(\pi_{I_n}(\varphi^{-1}((x_1, x_2, \dots)))) = F_{n,1}(x_1)F_{n,2}(x_2)\cdots F_{n,n}(x_n),$$

where $I_n = \{1, 2, \dots, n\} \in \mathcal{F}$, $x_i \in X_i$, and each $F_{n,i} \uparrow \mathbb{1}_{A_i}$ is a continuous non-negative function on X_i. Recall that we can choose f_n like this because the Daniell integral $E^{\uparrow}(f_n)$ is unique, and hence the exact choices of approximating functions do not affect the outcome of the limit. We now express $\mu(A)$ as follows:

$$\mu(A) = E^{\uparrow}(\mathbb{1}_{\varphi^{-1}(A)}) = \lim_{n\to\infty} E(f_n) = \lim_{n\to\infty}\int_{X_I} F_n(y)\mu_I(dy).$$

By Fubini's theorem, we now have that

$$\lim_{n\to\infty}\int_{X_I} F_n(y)\mu_I(dy) = \lim_{n\to\infty}\int_{X_{i_1}}\int_{X_{i_2}}\cdots\int_{X_{i_n}} F_{n,i_n}(x_{i_n})\mu_{i_n}(dx_{i_n})\cdots F_{n,i_1}(x_{i_1})\mu_{i_1}(dx_{i_1})$$
$$= \lim_{n\to\infty}\prod_{k=1}^{n}\int_{X_{i_k}} F_{n,i_k}(x_{i_k})\mu_{i_k}(dx_{i_k}).$$

This quantity converges to some a iff

$$\lim_{n\to\infty}\sum_{k=1}^{n}\log\left(\int_{X_{i_k}} F_{n,i_k}(x_{i_k})\mu_{i_k}(dx_{i_k})\right) = \log a, \tag{4}$$

which is just a consequence of the properties of the logarithm. Now notice that

$$\int_{X_{i_k}} F_{n,i_k}(x_{i_k})\mu_{i_k}(dx_{i_k}) \leq \int_{X_{i_k}} \mathbb{1}_{A_{i_k}}(x_{i_k})\mu_{i_k}(dx_{i_k}) = \mu_{i_k}(A_{i_k}) \leq 1.$$

Hence, the function

$$g_n(k) := \begin{cases} \log\left(\int_{X_{i_k}} F_{n,i_k}(x_{i_k})\mu_{i_k}(dx_{i_k})\right) & \text{if } 1 \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

is bounded in the following manner:

$$|g_n(k)| \leq -\log(\mu_{i_k}(A_{i_k})) =: h(k),$$

for all $k \in \mathbb{N}$ (here we let $h(k) := 0$ for any $k > n$). Since $A = \prod_{i=1}^N A_i \times \prod_{N+1}^\infty X_i$ is a cylinder set, it follows that

$$\sum_{k=1}^\infty |h(k)| = \sum_{k=1}^\infty h(k) < -\log\left(\prod_{i=1}^\infty \mu_i(A_i)\right) < \infty,$$

as long as $\mu_i(A_i) > 0$ for all i. But if $\mu_{i_k}(A_{i_k}) = 0$ for some k, then by non-negativity of F_{n,i_k} and the inequality $\int_{X_{i_k}} F_{n,i_k}(x_{i_k})\mu_{i_k}(dx_{i_k}) \leq \mu_{i_k}(A_{i_k}) = 0$ we have

$$\int_{X_{i_k}} F_{n,i_k}(x_{i_k})\mu_{i_k}(dx_{i_k}) = 0,$$

implying that $\mu(A) = 0 = \prod_{i=1}^N \mu_i(A_i)$ as required.

We continue on, assuming that $\mu_i(A_i) > 0$ for all i. Since h is integrable with respect to the counting measure, we can apply dominated convergence twice to the left side of Equation 4: once to the counting measure, and once to the Lebesgue integral inside the logarithm (since $|F_{n,i_k}| \leq \mathbb{1}_{A_{i_k}}$). This yields

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^\infty \log\left(\int_{X_{i_k}} F_{n,i_k}(x_{i_k})\mu_{i_k}(dx_{i_k})\right) &= \sum_{k=1}^\infty \log\left(\mu_{i_k}(A_{i_k})\right) \\ &= \log\left(\prod_{i=1}^\infty \mu_{i_k}(A_{i_k})\right). \end{aligned}$$

We deduce that $a = \prod_{i=1}^\infty \mu_{i_k}(A_{i_k})$ and hence that

$$\mu(A) = \lim_{n\to\infty} \prod_{k=1}^n \int_{X_{i_k}} F_{n,i_k}(x_{i_k})\mu_{i_k}(dx_{i_k}) = \prod_{i=1}^N \mu_i(A_i).$$

We conclude that there exists a unique Baire probability measure on the measurable space $(\prod_{i=1}^\infty X_i, \bigotimes_{i=1}^\infty \Sigma_i)$ such that for every cylinder set $A = \prod_{i=1}^N A_i \times \prod_{N+1}^\infty X_i$ we have the identity

$$\mu(A) = \prod_{i=1}^N \mu_i(A_i).$$

□

The above proof in fact tells us a bit more than the result of the theorem: we have $\mu(A) = \prod_{i=1}^\infty \mu_i(A_i)$ for any $A = \prod_{i=1}^\infty A_i$ such that the product of measures converges. In [Sam22] Theorem 1.3, Theorem 3.16 is extended to general measures, which is the main result of the paper. We will not proceed to extend the previous result further here, since our use of the projective systems in the proof above critically relies on the property that the measures are of probability, for otherwise the measures μ_I would no longer form a consistent system. As such, we shall now leave the example of measures on infinite product spaces behind us, and instead move on to the next section.

3.2.2 Wiener measure revisited

Let us now return to the setting of the construction of the Wiener measure of Section 3.1.2. Let $\mathcal{F}$ be the set of finite partitions

$$0 < t_1 < \cdots < t_m = 1,$$

of $[0,1]$ where $t_i \in \mathbb{Q} \cap [0,1]$ and $m \in \mathbb{N}$. We then acquire a natural ordering: for $I, J \in \mathcal{F}$ we define $I \leq J$ if $I \subset J$, which in words means that J is a refinement of I. The projections $\pi_{J,I}$ for $I \leq J$ are then natural as well. We now see that the set

$$\mathfrak{P} = \prod_{t \in \mathbb{Q} \cap [0,1]} \mathbb{R}^n$$

used in the construction of the Wiener measure can be identified with the projective limit X of the Hausdorff spaces X_I of the form

$$X_I = \prod_{t_i \in \mathbb{Q} \cap [0,1], i \in I} \mathbb{R}^n,$$

for $I \in \mathcal{F}$. In essence, every X_I consists of $|I|$ copies of $\mathbb{R}^n$, where the k-th copy encodes the value of a path ω at time t_{i_k}.

Since the Wiener process is defined to have independent *increments*, it is natural to consider tuples of time steps $\Delta t_m := t_m - t_{m-1}$. Each $I \in \mathcal{F}$ can then be identified with

$$I' := \psi(I) := (\Delta t_1, \Delta t_2, \ldots, \Delta t_m),$$

where $\sum_{i=1}^m \Delta t_i = 1$. Here ψ is clearly a bijection, since $t_i = \sum_{j=1}^i \Delta t_j$. Similarly, each $x = (x_1, x_2, \ldots, x_m) \in X_I$ can be identified under ψ with

$$y = (y_1, y_2, \ldots, y_m) := \psi(x) := (x_1, x_2 - x_1, \ldots, x_m - x_{m-1}).$$

Furthermore, we define $Y_{I'} := \psi(X_I)$, $\mathcal{F}' := \psi(\mathcal{F})$, and we introduce on $\mathcal{F}'$ the partial ordering of $\mathcal{F}$ (which retains the same symbol $\leq$) by the bijection ψ. It follows that any projective system $((X_I)_{I \in \mathcal{F}}, (\pi_{J,I})_{I \leq J \in \mathcal{F}})$ corresponds (under ψ) to a projective system $((Y_{I'})_{I' \in \mathcal{F}'}, (\pi_{J',I'})_{I' \leq J' \in \mathcal{F}'})$ with

$$\pi_{J',I'} = \psi \circ \pi_{J,I} \circ \psi^{-1},$$

if $I \leq J$.

Let $I \in \mathcal{F}$ be denoted by $\{t_1 < t_2 < \cdots < t_m\}$ and consider Borel sets $A_k \subset \mathbb{R}^n$ for all $1 \leq k \leq m$ together with the set $A_I = \prod_{k=1}^m A_k \subset X_I$. Back in Section 3.1.2, we essentially introduced the probability measure μ_I on X_I by

$$\mu_I(A_I) = \int_{A_1} \int_{A_2} \cdots \int_{A_m} p(t_1, x_1) \cdots p(t_m - t_{m-1}, x_m - x_{m-1}) dx_m \cdots dx_1.$$

Lemma 3.17. *The measures μ_I defined above form a consistent family of measures with respect to the projective system $((X_I)_{I \in \mathcal{F}}, (\pi_{J,I})_{I \leq J \in \mathcal{F}})$.*

Proof. If we write $B := B_1 \times B_2 \times B_3 \times \cdots \times B_m \subset Y_{I'}$, for $B_1 = A_1$ and $B_i := A_i - A_{i-1}$ (the set of increments that leads from A_{i-1} to A_i) for $2 \leq i \leq m$, then for $I' \in \mathcal{F}'$ we have the pushforward probability measure $\mu_{I'}$ on $Y_{I'}$ defined by

$$\mu_{I'}(B) := (\psi \# \mu_I)(B) = \int_{B_1} \int_{B_2} \cdots \int_{B_m} p(\Delta t_1, y_1) \cdots p(\Delta t_m, y_m) dy_m \cdots dy_1,$$

which by independence of the increments can now be conveniently written as

$$\mu_{I'}(B) = \prod_{i=1}^m \int_{B_i} p(\Delta t_i, y_i) dy_i.$$

It remains to show that the $\mu_{I'}$ form a consistent set of measures with respect to the projective system $((Y_{I'})_{I'\in\mathcal{F}'}, (\pi_{J',I'})_{I'\leq J'\in\mathcal{F}'})$. Let us assume that $I' \leq J'$. Then also

$$I = \psi^{-1}(I') \leq \psi^{-1}(J') = J.$$

It suffices to show that $\mu_{J'}$ and $\mu_{I'}$ are consistent when only a single point is added in the refinement J of I. Hence, for $I = \{t_1 < t_2 < \cdots < t_m\}$ we let

$$J = I \cup \{t_j^*\},$$

where $t_k < t_j^* < t_{k+1}$ for some $1 \leq k \leq m-1$. Then J' and I' are identical, except at the locations $k+1$ and $k+2$ where we have

$$J'_{k+1} = t_j^* - t_k, \quad J'_{k+2} = t_{k+1} - t_j^*,$$

which together adds up to the previous increment $I'_{k+1} = t_{k+1} - t_k = J'_{k+1} + J'_{k+2}$.

Now consider $B \subset Y_{I'}$ like before, but now B_{k+1} is replaced by $B_1^* \times B_2^*$, where $B_1^*, B_2^* \subset \mathbb{R}^n$ are Borel sets such that $y_1^* + y_2^* = y_{k+1}$ for all $y_1^* \in B_1^*$ and $y_2^* \in B_2^*$. In other words, we have $B_2^* = B_{k+1} - B_1^*$. Denote by C this new set (so C is a product of $m+1$ spaces), so that $\pi_{J',I'}^{-1}(B) = C$. It follows that

$$\mu_{J'}(\pi_{J',I'}^{-1}(B)) = \mu_{J'}(C) = \prod_{i\neq k+1, 1\leq i\leq m} \int_{B_i} p(\Delta t_i, y_i)dy_i \cdot \int_{B_1^*} p(t_j^* - t_k, y_1^*)dy_1^* \cdot \int_{B_2^*} p(t_{k+1} - t_j^*, y_2^*)dy_2^*$$

$$= \prod_{i=1}^{m} \int_{B_i} p(\Delta t_i, y_i)dy_i = \mu_{I'}(B),$$

which proves the consistency of the measures $\mu_{I'}$, and under the bijection ψ, also the consistency of the measures μ_I. □

Now, following Section 3.2, we have a vector lattice $C_\#(\mathfrak{P})$ corresponding to $C_\#$ defined by

$$C_\#(\mathfrak{P}) = \{\varphi : \mathfrak{P} \to \mathbb{R} : \exists I_\varphi \in \mathcal{F}, F \in C_b(X_{I_\varphi}) : \varphi(x) = F(\pi_{I_\varphi}(x)) = F(\omega(t_{i_1}), \omega(t_{i_2}), \ldots, \omega(t_{i_m})), \forall x \in X\}.$$

On $C_\#(\mathfrak{P})$, we have introduced the functional

$$E(\varphi) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} p(t_1, x_1) \cdots p(t_m - t_{m-1}, x_m - x_{m-1}) F(x_1, \ldots, x_m) dx_m \cdots dx_1$$

$$= \int_{X_{I_\varphi}} F(y) \mu_{I_\varphi}(dy).$$

Note that the theory of projective systems from Section 3.2 now proves why the functional E defined in Section 3.1.2 is a Daniell integral: the independence of increments implies that we have a consistent family of measures, which in turn implies that E has all the defining properties of a Daniell integral.

Finally, Theorem 2.30 yields the Wiener measure W on $\mathcal{B}(\mathfrak{P})$ and the relation

$$E(\varphi) = \int_{\mathfrak{P}} \varphi dW,$$

for all $\varphi \in C_\#(\mathfrak{P})$.

3.2.3 Gaussian measure revisited

In this section we return to the topic of Gaussian measures. Let H be an infinite dimensional separable real Hilbert space with Hilbert basis $\mathcal{E} = \{e_1, e_2, \dots\}$. Armed with the concept of projective systems, we construct a probability measure on H induced by a Daniell integral. For our set $\mathcal{F}$, consider the set of natural numbers $\mathbb{N}$ (excluding zero) with the standard ordering. Then every $I \in \mathcal{F}$ is just some natural number $n \in \mathbb{N}$, and we define for each $I \in \mathcal{F}$ the space H_I spanned by the basis vectors $\{e_1, \dots, e_I\} \subset \mathcal{E}$. This forms the basis for a projective system: for $I \leq J$ we can naturally project from H_J to H_I through the function $\pi_{J,I}$ defined by

$$\pi_{J,I}(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_J e_J) = (\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_I e_I),$$

where $\lambda_i \in \mathbb{R}$. Since each H_I is finite dimensional, these subspaces of H are in fact isomorphic to $\mathbb{R}^I$. One way to see this is to consider the function $f_I : H_I \to \mathbb{R}^I$ defined by

$$f_I(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_I e_I) = (\lambda_1, \lambda_2, \dots, \lambda_I),$$

for which it is not hard to see that this forms a continuous isomorphism between H_I and $\mathbb{R}^I$. This shows that the problem of constructing Gaussian measures on all H_I is equivalent to the problem of constructing Gaussian measures on all $\mathbb{R}^I$, in that a Gaussian measure on $\mathbb{R}^I$ induces a Gaussian measure on H_I and vice versa.

Now let M be a symmetric positive definite infinite matrix with real entries that is such that it defines a linear operator on H (which we denote by the same symbol), given by

$$\langle Me_i, e_j \rangle = M_{ij}.$$

We assume that M defines a bounded trace-class operator on H by also assuming that

$$\mathrm{Tr}(M) = \sum_{i=1}^{\infty} \langle Me_i, e_i \rangle < \infty.$$

From this matrix M we will construct Gaussian measures on all H_I by means of projections. That is, we define according to Definition 3.5 the Gaussian measures μ_I on H_I with covariance matrix M_I, which is the restriction of the infinite matrix M to the upper left $I \times I$ entries. By construction, these Gaussian measures form a consistent system of measures with respect to the projective system $((H_I)_{I\in\mathcal{F}}, (\pi_{J,I})_{I\leq J\in\mathcal{F}})$, and hence the approach from Section 3.2 yields a probability measure μ on the projective limit $\mathcal{H}$ of the sets H_I together with the corresponding σ-algebra Σ.

It is important to note that $\mathcal{H}$ is not the same set as H. Namely, $\mathcal{H}$ can be identified with *all* sequences of elements of H_I consistent with the projections $\pi_{I,J}$, whereas H is isomorphic to $\ell^2(\mathbb{N})$: the square summable sequences. To see the latter result, consider the mapping $\varphi : H \to \ell^2(\mathbb{N})$ defined by

$$\varphi(h)(n) := \hat{h}(n) = \langle h, e_n \rangle.$$

We now apply a form of Parseval's identity, known as the *Plancherel formula* (proven in for example Theorem 4.13.(f) in [Con94]), which states that

$$\|h\|^2 = \sum_{n=1}^{\infty} |\langle h, e_n \rangle|^2.$$

In particular, this shows that $\hat{h}$ defined above is square summable and hence that φ is well-defined and surjective. Furthermore, the linearity and injectivity φ are immediate, which shows that φ is indeed an isomorphism.

It is in fact the case that $\mathcal{H}$ is 'larger' than H, in the sense that there exists an embedding $\iota : H \to \mathcal{H}$ of H in $\mathcal{H}$ defined by

$$h \mapsto (h_I)_{I\in\mathcal{F}} := \left(\sum_{n=1}^{I} \langle h, e_n \rangle e_n \right)_{I\in\mathcal{F}}.$$

With this embedding, we can now define a set function μ_H on H by

$$\mu_H(E) := \mu(\iota(E)),$$

where $E \in \Sigma_H := \iota^{-1}(\Sigma)$. The following theorem shows that this process leads to a well-defined σ-algebra Σ_H and a probability measure μ_H on H, which completes our goal of this section.

Theorem 3.18. *The triple (H, Σ_H, μ_H) defined above is a probability space.*

Proof. First, we will show that Σ_H is a σ-algebra. Since Σ is already an established σ-algebra, we have for all $E \in \Sigma$ and $(E_i)_{i\in\mathbb{N}} \subset \Sigma$ the following properties:

- $\bigcup_{i=1}^{\infty} \iota^{-1}(E_i) = \bigcup_{i=1}^{\infty}\{x \in H : \iota(x) \in E_i\} = \iota^{-1}(\bigcup_{i=1}^{\infty} E_i) \in \iota^{-1}(\Sigma)$, which implies that Σ_H is closed under countable unions.
- $H \setminus \iota^{-1}(E) = \{x \in H : \iota(x) \notin E\} = \{x \in H : \iota(x) \in \iota(H) \setminus E\} = \iota^{-1}(\iota(H) \setminus E) \in \iota^{-1}(\Sigma)$, which implies that Σ_H is closed under taking complements so long as $\iota(H) \in \Sigma$.

This last point tells us that we only need to show that $\iota(H) \in \Sigma$, from which it will immediately follow that Σ_H is a σ-algebra. In order to do this, we first need to characterize $\iota(H)$ using functions in $C_\#(\mathcal{H})$. For any $N \in \mathbb{N}$, we define

$$\varphi_N((h_I)_{I\in\mathcal{F}}) = \left(\sum_{I=1}^{N} |\langle h_I, e_I\rangle|^2\right) \vee N \in C_\#(\mathcal{H}).$$

This forms an increasing sequence of functions in N, and so we define the limit function

$$\varphi_\infty((h_I)_{I\in\mathcal{F}}) := \lim_{n\to\infty} \varphi_N((h_I)_{I\in\mathcal{F}}) \in C_\#(\mathcal{H})^\uparrow.$$

Since the Hilbert space H is isomorphic to the square summable sequences, we can describe $\iota(H)$ as

$$\iota(H) = \{(h_I)_{I\in\mathcal{F}} : \varphi_\infty((h_I)_{I\in\mathcal{F}}) < \infty\} = \{\varphi_\infty < \infty\}.$$

Since $\varphi_\infty \in C_\#(\mathcal{H})^\uparrow$, it is measurable and since $\{\infty\} \in \mathcal{B}_{\overline{\mathbb{R}}}$ (the Borel σ-algebra on the extended real number line $\overline{\mathbb{R}}$), it follows that $\varphi_\infty^{-1}(\{\infty\}) \in \Sigma$. In particular, we have $\iota(H) = \mathcal{H} \setminus \varphi_\infty^{-1}(\{\infty\}) \in \Sigma$ as required.

Second, we will show that μ_H is a probability measure on Σ_H. That it is a measure follows directly from the definition and the fact that Σ_H is a σ-algebra. To show that μ_H is a probability measure, it suffices to show that μ is concentrated on $\iota(H) \subset \mathcal{H}$.

Consider the function $\varphi_\infty \in C_\#(\mathcal{H})^\uparrow$ from before and define $\|h_N\|_N := \left(\sum_{i=1}^{N} |\langle h_N, e_i\rangle|^2\right)^{\frac{1}{2}}$ for $h_N \in H_N$. Then we have that

$$\int_{\mathcal{H}} \varphi_\infty d\mu = \lim_{N\to\infty} \int_{H_N} \varphi_N d\mu_N = \sup_{N\in\mathbb{N}} \mathbb{E}_{\mu_N}\left[\|h_N\|_N^2\right] = \sup_{N\in\mathbb{N}} \sum_{i=1}^{N} \mathbb{E}_{\mu_N}\left[|\langle h_N, e_i\rangle|^2\right].$$

Now note that the function $\langle \cdot, e_i\rangle : H_N \to \mathbb{R}$ is a Gaussian random variable with mean 0 and variance M_{ii} (the i-th diagonal entry of the infinite covariance matrix M) for all $1 \le i \le N$. It follows that

$$\sum_{i=1}^{N} \mathbb{E}_{\mu_N}\left[|\langle h_N, e_i\rangle|^2\right] = \sum_{i=1}^{N} \mathrm{Var}_{\mu_N}(\langle \cdot, e_i\rangle) = \sum_{i=1}^{N} M_{ii} = \mathrm{Tr}(M_N).$$

Hence, since we have assumed that $\sup_{N\in\mathbb{N}} \mathrm{Tr}(M_N) < \infty$, which is simply a condition on the initial choice of covariance matrix M (take for a concrete example variances $M_{ii} = \frac{1}{i^2}$ going sufficiently quickly to zero so that the supremum of the traces is finite), we get that

$$\int_{\mathcal{H}} \varphi_\infty d\mu < \infty.$$

Note that we can write this integrability condition as

$$\int_{\mathcal{H}} \varphi_\infty d\mu = \int_{\{\varphi_\infty=\infty\}} \varphi_\infty d\mu + \int_{\{\varphi_\infty<\infty\}} \varphi_\infty d\mu = \infty \cdot \mu(\{\varphi_\infty = \infty\}) + \int_{\{\varphi_\infty<\infty\}} \varphi_\infty d\mu < \infty,$$

which together with $\varphi_\infty \geq 0$ implies that $\mu(\mathcal{H} \setminus \iota(H)) = \mu(\{\varphi_\infty = \infty\}) = 0$. Equivalently, this implies that $\mu(\iota(H)) = 1$, as required. We conclude that the triple (H, Σ_H, μ_H) forms a probability space. □

Looking at the proof of Theorem 3.18, we see that the condition on the Gaussian measures can be generalized to any consistent sequence of measures $(\mu_N)_{N\in\mathbb{N}}$ such that $\lim_{N\to\infty} \int_{H_N} \varphi_N d\mu_N < \infty$. The Gaussian measures chosen above are simply an example of such a consistent sequence of measures that works for these purposes.

Finally, it is natural to expect that μ_H, as a measure constructed from a process regarding Gaussian measures, must itself be a Gaussian measure. That this intuition is correct is the final result of this thesis.

Theorem 3.19. *The probability measure μ_H defined on the measurable space (H, Σ_H) is a Gaussian measure.*

Proof. To prove that μ_H is a Gaussian measure, by Proposition 3.8 we have to show that for each $f \in H^*$ we have that

$$\int_X e^{if(x)} \mu(dx) = e^{-\frac{1}{2}q(f,f)},$$

where q is some positive semidefinite symmetric bilinear form on H^*. Since H is a Hilbert space, the Riesz Representation Theorem (see for example Theorem 3.4. on page 13 of [Con94]) tells us that every $f \in H^*$ is of the form

$$f(x) = \langle x, h \rangle_H,$$

for some $h \in H$. Every $h \in H$ can be written in the form $h = \sum_{n=1}^\infty \langle h, e_n \rangle e_n$, so that we can write

$$f(x) = \langle x, \sum_{n=1}^\infty \langle h, e_n \rangle_H e_n \rangle_H = \sum_{n=1}^\infty \langle h, e_n \rangle_H \langle x, e_n \rangle_H = \sum_{n=1}^\infty f_n \langle x, e_n \rangle_H,$$

with $f_n := \langle h, e_n \rangle_H$. Now for any $I \in \mathcal{F} = \mathbb{N}$, we define

$$f_I(x) := \sum_{n=1}^I f_n \langle x, e_n \rangle_{H_I},$$

which is clearly a continuous linear functional on H_I, with corresponding element h_I such that $f_I(x) = \langle x, h_I \rangle_{H_I}$. Since $\lim_{I\to\infty} f_I(x) = f(x)$, by Lebesgue's Dominated Convergence Theorem and Proposition 3.6 we find that

$$\int_H e^{if(x)} \mu_H(dx) = \lim_{I\to\infty} \int_H e^{if_I(x)} \mu_H(dx) = \lim_{I\to\infty} \int_{H_I} e^{if_I(x)} \mu_I(dx) = \lim_{I\to\infty} \exp\{-\frac{1}{2}\langle h_I, M_I h_I \rangle_{H_I}\}.$$

We define the following bilinear form q on H^*:

$$q(f, g) := \langle h_f, M h_g \rangle_H,$$

where h_f and h_g are the Hilbert space elements corresponding to the continuous linear functionals f and g, respectively. This bilinear form is positive semidefinite and symmetric by the properties of M and the inner product on H. Since the exponential and inner product functions are continuous (with the latter following from the Cauchy-Schwarz inequality), it follows from before that

$$\int_H e^{if(x)} \mu_H(dx) = \lim_{I\to\infty} \exp\{-\frac{1}{2}\langle h_I, M_I h_I \rangle_{H_I}\} = \exp\{-\frac{1}{2}\langle h, Mh \rangle_H\} = \exp\{-\frac{1}{2}q(f, f)\},$$

as required. We conclude by Proposition 3.8 that μ is a Gaussian measure. □

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