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THESIS

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The effects of massive gravity on the spectrum of gravitational waves in the early Universe

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Abstract

In this thesis, we will explore the effect that adding a mass to the normally massless graviton has on the tensor power spectrum of the primordial stochastic gravitational wave background during inflation and in the subsequent epochs of radiation and matter domination. We start by going over the background theory of gravitational waves, stochastic backgrounds and their power spectra, and the effective field theory of inflation. We will derive the tensor power spectrum due to quantum fluctuations that get stretched out to superhorizon scales during inflation, after which we will discuss what happens in massive gravity theories. We will then proceed to use the same procedure as for a massless graviton and derive the tensor power spectrum due to a massive graviton. This is not possible for a general time-dependent mass so we make some choices. The spectrum can be derived in the case of a constant mass for a de Sitter background, a general background with $s = -2$ and $s = -2(1 + 1/p)$ – with the graviton mass $m_\chi(\eta)$ given by $m_\chi^2(\eta) \propto a^s(\eta)$ with $a(\eta) \propto \eta^p$ the scale factor as a function of conformal time η – or a radiation-dominated background with $s = -1$. The power spectra are all seen to be blue, i.e. their spectral tilt $n_T > 0$, when $m_\chi > 0$. We end with a critical review of the current literature in which we note that authors have made questionable choices for describing the inflationary period, where they approximate it as a pure de Sitter Universe while in reality, it should be quasi-de Sitter.

Contents

Chapter 1

Introduction

Einstein's theory of General Relativity (GR) has predicted a number of phenomena that at the time of its publishing had not yet been observed. These phenomena include the existence of black holes (whose first direct observational evidence was only published as recently as 2019 [1]), gravitational lensing, and *gravitational waves* (GWs), which are the focus of this thesis. GWs in their most basic definition can be described as "ripples in spacetime". They are fluctuations in the background metric that describes the Universe, that can be produced by basically anything. Even the movement of my hand while writing this text produces GWs, although they are extremely small and impossible to measure. In order to be able to measure GWs, we need extremely large, cosmological sources such as the merging of black holes or neutron stars. The merger of a binary black hole system is ultimately what lead to the first detection of GWs in 2015 by the joint efforts of the Laser Interferometer Gravitational-Wave Observatory (LIGO) and the Virgo interferometer [2]. These detectors work by sending a laser beam through a beam splitter, into two mirrors, collecting the two combined beams again in a detector, and measuring the phase difference of the beams. Perturbations in the spacetime metric cause it to expand or contract so that such a phase difference can occur. From this, the strength of the perturbation (the GW) can be derived. See also Section 4 of [3] for more details on GW interferometers.

GWs are important to study because they can act as probes of the very early Universe. For the entire range of energy scales for which our current understanding of gravity holds, the GW interaction rate is smaller than the expansion rate of the Universe (the Hubble parameter), meaning that once GWs are produced, they do not interact with anything and they propagate freely through the Universe, eventually for us to potentially measure them. This is true for black hole mergers that produce a sudden burst of large GWs, but also for early-Universe events such as during inflation (which we will introduce shortly). By just using telescopes based on receiving electromagnetic radiation, we can only look back to the time of *photon decoupling*, which happened around a redshift of $z \sim 1100$, or about 370,000 years after the beginning of the Universe. After this event, the Universe became transparent for photons so they started to free-stream. We see the electromagnetic radiation from this period today as the *Cosmic Microwave Background* (CMB), so this is an imprint of the earliest time when photons just decoupled. Since GWs decouple immediately as soon as they are produced, in theory it is possible to look back far beyond the CMB to the very early Universe.

Moreover, the GWs contain unaltered information about the processes that produced them, and therefore about the conditions of the Universe at times that are unreachable by standard telescopes. In this thesis, we will focus on these early-Universe events and in particular on the early period of *inflation*. This is a period of rapid exponential expansion of the Universe that was introduced as a way to overcome the problems of the standard cosmological model (the ΛCDM model) [4] (see also Section 2.3). The inflationary period is expected to produce a background of GWs that can be described in a statistical way (see Section 2.2 for more details about why and how this works). These kinds of GW backgrounds are called *stochastic gravitational wave backgrounds* (SGWBs). Such backgrounds are measured through their *power spectrum*, which basically gives the strength at different length or frequency scales. The SGWB produced by inflation in our current understanding is expected to have specific characteristics which means that being able to measure such backgrounds will give us more information about this period. It is also possible that there are more sources during or after inflation that also produce SGWBs with different properties.

In order for the SGWB power spectra to correspond with observations, we require them to be *blue-tilted*. This means that at small wavelengths (large wavenumbers or large frequencies), the power spectrum is large so that we can measure GWs at these scales. At large wavelengths (small wavenumbers or small frequencies), i.e. at CMB scales, a blue spectrum is small.

Let us now go back to GR, which in our current understanding can be described by a massless spin-2 particle that we call the *graviton*. As for the theory of inflation, often the *single-field slow-roll* model is adopted, in which the exponential expansion of inflation is assumed to be caused by a single scalar field called the *inflaton*. The "slow-roll"-part is not important for now but we will come back to this in Section 2.3. We can write down the action corresponding to a massless graviton and the inflaton, which we can eventually show to give rise to the SGWB power spectrum during inflation.

A daring next step is to try to give the graviton a mass. Even though according to many, GR is a very beautiful and accurate theory, there is still a discrepancy between theory and observations. Looking at supernova data [5, 6], the Universe is currently going through a phase of accelerated expansion. If GR is indeed as accurate as promised, then there must be a dark energy density $\rho \sim 10^{-29}$ g cm$^{-3} \sim 10^{-47}$ GeV^4. The simplest explanation is that this energy density is due to the cosmological constant Λ, which gets an extremely small value. On the other hand, if we were to consider quantum field theory arguments, summing the zero-point energies of all normal modes of some field of mass m_f up to some cutoff $\Lambda_\mathrm{c} \gg m_\mathrm{f}$ gives a vacuum energy density $\langle \rho \rangle \simeq 2 \times 10^{71}$ GeV^4, assuming the cutoff to be the Planck scale, $\Lambda_\mathrm{c} \simeq (8\pi G)^{-1/2}$ [7]. There is a discrepancy of nearly 120 orders of magnitude between the two densities, and this is clearly a problem (called the *cosmological constant problem*).

This is why scientists have tried to come up with different models of modified gravity: it can be shown that an accelerating Universe can be produced in such models, without the need for dark energy. In particular, in this work, we will look at the theory of massive gravity, in which the graviton is given a mass m. We will not go into much detail here but we will see that this theory is plagued by inconsistencies such as *ghosts* – fields with a negative kinetic term – and discontinuities (the *vDVZ discontinuity)*. Even though these can be avoided or solved, it takes specific cases or regimes to do so. One way to look at it, which is also the order that we will follow in this work, is to first look at the simplest form of massive gravity, the Fierz-Pauli form [8], which is a Lorentz-invariant model. This comes with some problems that can be solved within the Lorentz-invariant regime, but it will turn out to also be beneficial to look at Lorentz-violating massive gravity. In this way, certain instabilities and discontinuities can be avoided. Moreover, some problems might be an artefact of flat Minkowski space, so it might also help if we take the cosmological Friedmann-Lemaître-Robertson-Walker (FLRW) metric.

Taking then for granted that the graviton can have a mass, we can add it to our inflation action and calculate the tensor power spectrum from it. The mass can be time-dependent, so we will go over different models in this work and see what happens to the spectrum in each case. This can then help with future observations of the inflationary GW background, as each time-dependence creates a different spectral tilt of the spectrum.

Structure. This thesis is structured as follows. In Chapter 2, we will set up the basic theoretical framework of gravitational waves and SGWBs and their power spectra. We will also go over the theory of inflation, how quantum fluctuations during inflation can create an SGWB, and other sources of such backgrounds. In Chapter 3, we will then add a mass to the graviton and discuss the advantages and problems this brings. Chapter 4 will then discuss the implications of Lorentz breaking massive gravity for the tensor power spectrum of the SGWB. Finally, we will make a comparison with the literature and conclude the research in Chapter 5. Then there are also three appendices. Appendix A will go over important cosmological concepts and relations and Appendix B will give some more information about the Bessel functions and their properties. Finally, Appendix C will show some of the more lengthy and tedious calculations that would not fit in the main body, but for completeness are still included.

Conventions. Throughout this thesis, we will work in units where $c = \hbar = 1$. We will use the reduced Planck mass $M_{\rm Pl}^2 = 1/8\pi G$ where G is the gravitational constant, with the value $M_{\rm Pl} \simeq 2.44 \times 10^{18}$ GeV. Greek indices $\mu, \nu, \alpha, \beta, ... = 0, 1, 2, 3$ are used for spacetime dimensions, while Latin indices $i, j, k, ... = 1, 2, 3$ will be used for spatial dimensions. We assume the Einstein convention for summing over repeated indices, and we use the mostly positive metric signature, i.e., $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. Specific spacetimes will be introduced when they are necessary. The metric perturbation is denoted by $h_{\mu\nu}$, sometimes with the prefactor $2/M_{\rm Pl}^2$, but this will be indicated. h_{ij} is the spatial part of the perturbation and χ_{ij} will be used to refer to the transverse and traceless part of $h_{\mu\nu}$.

Chapter 2

Theoretical framework

This chapter will go over the basic theoretical background needed for the rest of the thesis. We will start with the basics of gravitational waves in Section 2.1, in which we quickly go over the main concepts. Section 2.2 will then describe stochastic backgrounds of gravitational waves and how they can be characterised by their power spectrum. Finally, Section 2.3 describes the period of inflation in the early Universe. We will first go over the basics, then calculate the irreducible SGWB power spectrum due to quantum fluctuations, and finally go over some other sources of SGWB that may or may not have to do with inflation.

This chapter will mainly follow the review on cosmological backgrounds of gravitational waves by Caprini and Figueroa (2018) [3]: the three sections 2.1, 2.2, and 2.3, will follow parts of Sections 2, 3, and 5 of [3], respectively. Other sources will be referenced at the appropriate times.

2.1 Gravitational waves

2.1.1 The transverse and traceless gauge

Let us start our description of GWs in flat space with the Minkowski metric $\eta_{\mu\nu}$, and add a small symmetric perturbation $|h_{\mu\nu}(x)| \ll 1$ to it so that the general metric becomes

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \tag{2.1}$$

This allows for the usual calculation of the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = \frac{1}{M_{\rm Pl}^2}T_{\mu\nu}, \tag{2.2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. The steps that we follow for this are (1) the calculation of the Christoffel symbols $\Gamma^{\alpha}{}_{\mu\nu}$ from the metric $g_{\mu\nu}$, (2) the calculation of the Riemann tensor $R^{\alpha}{}_{\mu\nu\beta}$ from $\Gamma^{\alpha}{}_{\mu\nu}$, (3) the calculation of the Ricci tensor $R_{\mu\nu} = -R^{\alpha}{}_{\mu\nu\alpha}$, and (4) the calculation of the Ricci scalar $R = R^{\mu}{}_{\mu}$. The steps are not shown here but can be found e.g. in [3]; to first order in $h_{\mu\nu}$ (which is why it is called the *linearised theory*), the Einstein tensor $G_{\mu\nu}$ has the form

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\alpha\partial_\nu\bar{h}^{\alpha}{}_{\mu} + \partial^\alpha\partial_\mu\bar{h}_{\nu\alpha} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial^\beta\bar{h}^{\alpha}{}_{\beta}\right), \tag{2.3}$$

where we have introduced a new metric perturbation

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.4}$$

with $h = h^{\mu}{}_{\mu}$ the trace of $h_{\mu\nu}$. The trace of $\bar{h}_{\mu\nu}$ can be calculated to be $\bar{h} = -h$, so this is also called the *trace-reversed* metric perturbation. Because GR is invariant under general coordinate transformations $x^\mu \to x'^\mu(x)$, we can choose to make an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, under which the trace-reversed perturbation transforms as

$$\bar{h}'_{\mu\nu}(x') = \bar{h}_{\mu\nu}(x) + \eta_{\mu\nu}\partial_\alpha\xi^\alpha - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu. \tag{2.5}$$

We can therefore choose a transformation in such a way that

$$\partial^\mu\bar{h}_{\mu\nu}(x) = 0, \tag{2.6}$$

called the *Lorentz gauge*, with which the Einstein tensor (2.3) becomes

$$G^{(\mathrm{L})}_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} = \frac{1}{M^2_{\mathrm{Pl}}}T_{\mu\nu} \quad \text{and so} \quad \Box\bar{h}_{\mu\nu} = -\frac{2}{M^2_{\mathrm{Pl}}}T_{\mu\nu}. \tag{2.7}$$

This is simply a wave equation with a source, and it has a homogeneous solution

$$\bar{h}_{\mu\nu}(x) = \int \mathrm{d}^3\mathbf{k}\left[\bar{h}_{\mu\nu}(\mathbf{k})e^{ik_\alpha x^\alpha} + \bar{h}^*_{\mu\nu}(\mathbf{k})e^{-ik_\alpha x^\alpha}\right], \tag{2.8}$$

where from the Lorentz gauge (2.6) it follows that the functions $\bar{h}_{\mu\nu}(\mathbf{k})$ have to satisfy $k^\mu\bar{h}_{\mu\nu} = 0$. From the homogeneous solution, it also becomes clear why we talk about gravitational *waves*, since the solutions to the *wave* equation (2.7) are *waves*.

Let us then go over the degrees of freedom (DoF) of the perturbation $\bar{h}_{\mu\nu}$, from which we will ultimately arrive at the transverse-traceless gauge and the fact that GWs have two polarisations. The Lorentz gauge (2.6) represents 4 constraints on the perturbation $\bar{h}_{\mu\nu}$, so with $\bar{h}_{\mu\nu}$ being a symmetric tensor with 10 independent components (DoF), there are naively $10 - 4 = 6$ DoF left. However, the Lorentz gauge in fact does not completely fix the gauge freedom. From Equation (2.5), we can see that the Lorentz gauge condition also requires $\Box\xi_\mu = 0$. In the homogeneous case, the wave equation is invariant under Lorentz preserving infinitesimal coordinate transformations, $\Box'\bar{h}'_{\mu\nu} = \Box\bar{h}_{\mu\nu} = 0$, and this means that we can take 4 infinitesimal displacements ξ_μ (where $\Box\xi_\mu = 0$) that place 4 more constraints over the transformed metric perturbation $\bar{h}'_{\mu\nu}$. This removes 4 more DoF so we are left with $6 - 4 = 2$ propagating DoF.

In total, this amounts to 8 constraints which we can re-distribute over the tensor components. We can choose ξ_μ in Equation (2.5) such that the trace vanishes, $\bar{h} = 0$, which means that $\bar{h}_{\mu\nu} = h_{\mu\nu}$, and such that $h_{0i} = 0$. We have now used 4 conditions. Furthermore, the Lorentz condition combined with $h_{0i} = 0$ implies $\dot{h}_{00} = 0$ and so h_{00} is only a function of the spatial coordinates $\mathbf{x}$. Since GWs are the time-dependent part, we can simply set $h_{00} = 0$. Finally, the spatial-spatial part of the Lorentz condition gives the three conditions $\partial_i h_{ij} = 0$, so in total we have set

$$h_{\mu 0} = 0, \quad h = 0, \quad \text{and} \quad \partial_i h_{ij} = 0, \tag{2.9}$$

which is called the *transverse-traceless* (TT) *gauge*. With this, we can construct a general TT tensor. The tensor should be symmetric so that there are only 10 independent components. Setting the time components zero leaves us with the 6 h_{ij} components. The transverse condition $\partial_i h_{ij}$ then means that the components parallel to the direction of propagation of the GW also vanish, and setting this direction to the $\hat{z}$-direction gets rid of the h_{i3} and h_{3i} components. We have then h_{11}, h_{12}, h_{21}, and h_{22} left. The symmetry of the tensor means that $h_{12} = h_{21}$, and the traceless condition means that $h_{11} = -h_{22}$, so we really only have 2 independent components left. We call these components $h_+ \equiv h_{11} = -h_{22}$ and $h_\times \equiv h_{12} = h_{21}$, so that

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.10}$$

and the metric can be written as

$$\mathrm{d}s^2 = -dt^2 + (1 + h_+)\,\mathrm{d}x^2 + (1 - h_+)\,\mathrm{d}y^2 + 2h_\times\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}z^2. \tag{2.11}$$

The $+$ and $\times$ components are the two polarisations of the GW.

In the rest of this thesis, we will call the TT part of the perturbation tensor χ_{ij}, to distinguish it from the spatial-spatial part of the tensor, h_{ij}. This means that where-ever we write χ_{ij}, we mean that it satisfies the TT conditions $\chi^i{}_i = \chi = 0$ and $\partial_i \chi_{ij} = 0$.

2.1.2 Expanding space

Let us now turn to cosmology where we have a homogeneous and isotropic background. The most general metric in this case is given by the FLRW metric,

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{2.12}$$

where t is the cosmic time, $a(t)$ is the scale factor describing how much the Universe has expanded since the beginning, and x^i are now comoving coordinates. Adding the GW perturbation $h_{\mu\nu}$ to this, we can write it in the TT gauge as

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left(\delta_{ij} + \chi_{ij}\right)\mathrm{d}x^i\,\mathrm{d}x^j, \tag{2.13}$$

where $\partial_i \chi_{ij} = \chi^i{}_i = 0$. By linearising the Einstein equations with this metric, i.e., calculating the Christoffel symbols, Riemann tensor, and Ricci tensor and scalar, we can derive an equation of motion for the TT part χ_{ij} (see [3]),

$$\ddot{\chi}_{ij}(\mathbf{x},t) + 3H\dot{\chi}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\chi_{ij}(\mathbf{x},t) = 16\pi G \Pi^{\mathrm{TT}}_{ij}(\mathbf{x},t), \tag{2.14}$$

with $H \equiv \dot{a}/a$ the Hubble parameter, a dot ($\dot{}$) indicating a derivative with respect to the cosmic time t, and Π^{TT}_{ij} the TT part of the *anisotropic stress* Π_{ij}, which can be calculated from the spatial part of the energy-momentum tensor T_{ij} and the pressure p via $a^2\Pi_{ij} = T_{ij} - pa^2(\delta_{ij} + \chi_{ij})$. We can simplify the above equation of motion a bit by going to Fourier space, in which we decompose the TT perturbation χ_{ij} as

$$\chi_{ij}(\mathbf{x},t) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\chi_{ij}(\mathbf{k},t)e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{2.15}$$

and subsequently decompose the Fourier components $\chi_{ij}(\mathbf{k},t)$ into their polarisations $+$ and $\times$ as

$$\chi_{ij}(\mathbf{k},t) = \sum_{r=+,\times} \chi_r(\mathbf{k},t)e^r_{ij}(\hat{\mathbf{k}}), \tag{2.16}$$

where the polarisation tensors $e^r_{ij}(\hat{\mathbf{k}})$ can be taken to be real, to satisfy $e^r_{ij}(-\hat{\mathbf{k}}) = e^r_{ij}(\hat{\mathbf{k}})$, and they follow the orthonormal relation

$$e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}}) = 2\delta_{rr'}. \tag{2.17}$$

Finally, for χ_{ij} to be real, we require $\chi^*_r(\mathbf{k},t) = \chi_r(-\mathbf{k},t)$.

For the rest of the thesis, it will be convenient to go to conformal time, defined by $\mathrm{d}\eta = \mathrm{d}t/a(t)$. The FLRW metric (2.13) can then be written as

$$\mathrm{d}s^2 = a^2(\eta)\left[-\,\mathrm{d}\eta^2 + \left(\delta_{ij} + \chi_{ij}\right)\mathrm{d}x^i\,\mathrm{d}x^j\right], \tag{2.18}$$

and we can define $X_{ij}(\mathbf{k},\eta) = a\chi_{ij}(\mathbf{k},\eta)$ so that the equation of motion (2.14) becomes in Fourier space

$$X''_{ij}(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right)X_{ij}(\mathbf{k},\eta) = 16\pi G a^3 \Pi^{\mathrm{TT}}_{ij}(\mathbf{k},\eta). \tag{2.19}$$

Here, a prime ($'$) denotes a derivative with respect to η and $\Pi^{\mathrm{TT}}_{ij}(\mathbf{k},\eta)$ is the Fourier component of $\Pi^{\mathrm{TT}}_{ij}(\mathbf{x},\eta)$,

$$\Pi^{\mathrm{TT}}_{ij}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Pi^{\mathrm{TT}}_{ij}(\mathbf{k},\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{2.20}$$

Version of June 29, 2023– Created June 29, 2023 - 18:47

Next, we take the case with no sources so that $\Pi_{ij}^{\mathrm{TT}} = 0$, and we write the X_{ij} in terms of their polarisations X_r using the relation (2.16) or the definition $X_r(\mathbf{k}, \eta) = a\chi_r(\mathbf{k}, \eta)$, so that we obtain

$$X_r''(\mathbf{k}, \eta) + \left(k^2 - \frac{a''}{a}\right) X_r(\mathbf{k}, \eta) = 0. \tag{2.21}$$

For a scale factor of the form $a(\eta) = (\eta/\eta_0)^p$ with $p = -1$ for de Sitter, $p = 1$ for radiation domination, and $p = 2$ for matter domination (see Appendix A.2.1), the general solutions can be written as

$$\chi_r(\mathbf{k}, \eta) = \frac{A_r(\mathbf{k})}{a(\eta)} \sqrt{\eta} J_{p-1/2}(k\eta) + \frac{B_r(\mathbf{k})}{a(\eta)} \sqrt{\eta} Y_{p-1/2}(k\eta), \tag{2.22}$$

where J_n and Y_n are the Bessel functions of the first and second kind (see Appendix B) and A_r and B_r are constants to be determined from the initial conditions. We can actually look at two specific regimes in which the solutions take some physically interesting forms. The second term in the brackets in Equation (2.21) is proportional to $a''/a \propto \mathcal{H}^2$ where $\mathcal{H} = aH = a'/a$ is the conformal Hubble parameter. This means that we compare k and $\mathcal{H}$ and we can eliminate one of these two terms by looking in the sub- or superhorizon regimes, $k \gg \mathcal{H}$ or $k \ll \mathcal{H}$, respectively (see Appendix A.2.4).

Let us first look at the subhorizon case, $k \gg aH$. In this case, the first term in the equation of motion (2.21) dominates and the differential equation is simply $X_r''(\mathbf{k}, \eta) + k^2 X_r(\mathbf{k}, \eta) = 0$, which is a simple harmonic oscillator and therefore has solutions

$$\chi_r(\mathbf{k}, \eta) = \frac{A_r(\mathbf{k})}{a(\eta)} e^{ik\eta} + \frac{B_r(\mathbf{k})}{a(\eta)} e^{-ik\eta}, \tag{2.23}$$

where again A_r and B_r are constants to be determined from the initial conditions. The solutions are simply oscillating in time which is to be expected from GWs.

However, in the superhorizon limit, $k \ll aH$, the second term, a''/a, dominates and the solution is given by

$$\chi_r(\mathbf{k}, \eta) = A_r(\mathbf{k}) + B_r(\mathbf{k}) \int^{\eta} \frac{\mathrm{d}\eta'}{a^2(\eta')}, \tag{2.24}$$

which has a constant term and a term that decays with the expansion of the Universe (and therefore it decays in time). This solution will become important in Section 2.3, where we have quantum fluctuations that grow beyond the horizon due to inflation. The decaying mode quickly becomes negligible but the constant mode stays there indefinitely, until they finally cross the horizon again and start oscillating (the subhorizon solution), which means that they become the standard GWs that we could measure today (if we were to have sensitive enough interferometers).

2.2 Stochastic backgrounds of gravitational waves

2.2.1 Description

Now that we know how to describe GWs, we can move on to cosmological sources of GWs and explain why we can regard them as having a stochastic nature. First of all, we must note that a stochastic background means that we can statistically describe the GWs that are produced, and this requires a large number of realisations of the same system, with each system having the same initial conditions so that we can properly study the statistics. However, there is only one Universe and therefore only one copy of the system.

This problem can be overcome by using the *ergodic hypothesis*, which states that taking an ensemble average over the many realisations of the system, is the same as taking either the spatial average over a large enough region, or a temporal average of a region over a long enough time. A good approximation of the number of uncorrelated regions from which we are receiving GW signals today is given by $\sim \Theta_p^{-2}$, where Θ_p is the angle subtending the size of the source. Taking as an example the electroweak phase transition, it turns out that there are about 10^{24} uncorrelated signals that we receive today from this GW source, which is enough for us to describe them statistically. One could also think of trying to observe the individual sources, i.e., not describing them statistically but individually, but because of the number of regions this would require extremely sensitive GW detectors that we do not have and also will not have in the future.

There are a few properties that SGWBs have: they are statistically homogeneous and isotropic, and they are Gaussian. The statistical homogeneity and isotropy basically comes from the fact that the Universe can be described by the expanding FLRW metric. Because of the homogeneity and isotropy of the Universe, the sources that produce the GW background produce them everywhere at the same time, and therefore these properties are handed over to the SGWB as well. Gaussianity follows from the notion that the GW background is produced by uncorrelated regions that are identically distributed throughout the Universe, which according to the central limit theorem means that the signal that we receive today tends towards the normal (or Gaussian) distribution.

A fourth property of these backgrounds is the absence of a net polarisation. We will not go into much detail here, but it has to do with the absence of a significant source of parity violation in the Universe, which causes the two polarisations $+$ and $\times$ to be uncorrelated. For more information, see [3].

2.2.2 The power spectrum

An SGWB can be characterised entirely by its *power spectrum*, in which we incorporate the four assumptions (statistical homogeneity and isotropy, Gaussianity, and the absence of a net polarisation). The power spectrum $h_c^2(k,\eta)$ is given by

$$\langle \chi_r(\mathbf{k},\eta)\chi_{r'}(\mathbf{k}',\eta)\rangle = \frac{8\pi^5}{k^3}\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{rr'}\chi_c^2(k,\eta), \tag{2.25}$$

where the $\delta_{rr'}$ comes from the lack of a net polarisation, the $\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ and the fact that χ_c is a function of just $k = |\mathbf{k}|$ are a result of the statistical homogeneity and isotropy, and the effect of Gaussianity is that the above power spectrum contains all information about the polarisation modes χ_r.

In addition to this definition of the power spectrum, in this work, we will instead use a slightly different version of this, in terms of the expectation value of the Fourier modes $\chi_{ij}(\mathbf{k}, \eta)$, which are related to the polarisation modes $\chi_r(\mathbf{k}, \eta)$ through Equation (2.16). Using a redefinition of the power spectrum $\mathcal{P}_\chi(k) = 2\chi_c^2$, we can instead write

$$\langle \chi_{ij}(\mathbf{k}, \eta)\chi_{ij}^*(\mathbf{k}', \eta)\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_\chi(k)\delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{2.26}$$

Using the Fourier decomposition (2.15) for $\chi_{ij}(\mathbf{x}, \eta)$, this can be rewritten to[1]

$$\langle \chi_{ij}(\mathbf{x}, \eta)\chi_{ij}(\mathbf{x}, \eta)\rangle = \int \frac{\mathrm{d}k}{k}\mathcal{P}_\chi(k) = 2\int \frac{\mathrm{d}k}{k}\chi_c^2(k, \eta). \tag{2.27}$$

For the remainder of this thesis, this will be everything that we need to know about SGWBs and their power spectra. For more details, we refer the reader to [3, 9].

2.3 Inflation

Let us now discuss a very important mechanism for the production of an SWGB: *inflation*. Inflation is a phase of accelerated expansion in the very early Universe and it is used primarily to solve the horizon and flatness problems that arise in the Hot Big Bang model of the Universe [4, 10]. We will quickly go over the two problems but we will skip a lot of details as the reader is assumed to be familiar with these problems.

The horizon problem arises due to there being causally disconnected regions in the early Universe, which is assumed to be homogeneous and isotropic, according to the cosmological principle. This homogeneity and isotropy can only arise, however, if the regions are in causal contact with each other and therefore they seem to violate causality. Inflation solves this problem because of the exponential expansion in the very early Universe, which causes regions that were causally connected before inflation, to become causally disconnected after this period. Detailed calculations show that about 60 *e*-folds are needed for this problem to be solved.

The flatness problem is related to the total energy density of the Universe, which appears to be extremely close to the critical density ρ_{crit}, which corresponds to a flat Universe and a density parameter $\Omega = \rho/\rho_{\text{crit}} = 1$. However, this is an unstable equilibrium. In fact, it can be computed that at the time of Big Bang Nucleosynthesis (BBN), the electroweak phase transition (EWPT), and at the Planck time t_{Pl}, Ω has to take the values $|1 - \Omega(t_{\text{BBN}})| < 10^{-16}$, $|1 - \Omega(t_{\text{EWPT}})| < 10^{-30}$, and $|1 - \Omega(t_{\text{Pl}})| < 10^{-60}$ for Ω to be so close to 1 today. These values are extremely small, which therefore raises the question, why the energy density was so finely tuned for the Universe to be so close to being flat.

[1] See calculation **5.** in Appendix C.1. Note that this calculation is actually done for the inflationary case, in which we have quantum fluctuations and therefore promote the perturbations χ_{ij} to operators. The Fourier decomposition also has a different denominator $(2\pi)^{3/2}$ so that it works out.

Inflation solves this problem as well which can be seen by the scale factor dependence of $(1-\Omega)$, namely $(1-\Omega) \propto a^{-2}$. This means that as the Universe expands exponentially during inflation, $1-\Omega$ will tend towards zero and so Ω towards 1. Inflation causes the density parameter to be extremely close to 1 which means that today, we also measure it to be very close to 1. More detailed calculations show that about 60 to 70 *e*-folds are needed for this problem to be solved, in accordance with the number of *e*-folds needed for the horizon problem.

Apart from solving the horizon and flatness problems, inflation also seems to play an important role in the formation of large-scale structure (LSS) in the early Universe, and therefore is important for the existence of the current Universe. This is due to quantum fluctuations that get stretched to super-Hubble scales during the exponential expansion of inflation. At the end of Section 2.1.2, we have seen that for superhorizon scales, the solution has a decaying and a constant mode. The decaying mode quickly becomes negligible but the constant mode will stay there until at some point, after inflation has ended and the expansion has decelerated again, it re-enters the horizon and starts oscillating. A starting point for the formation of LSS has now been created, which will eventually lead to the LSS as we know it today.

In the following subsections, we will talk first about the basics of inflation in Section 2.3.1, where we take an effective field theory approach and assume that inflation is driven by a scalar field. We will go over the equations of motion and the quasi-exponential expansion of inflation, where we will skip a lot of the details as the reader is assumed to be familiar with this as well. In Section 2.3.2, we will then calculate the tensor power spectrum resulting from the quantum perturbations that get stretched by inflation. This will be an important basis for the calculations in Chapter 4, where we add a mass term to the inflaton action. Finally, Section 2.3.3 will cover some more sources of GW backgrounds during inflation, and also during later epochs.

2.3.1 Basics

In cosmology, we often describe inflation as being driven by a scalar field ϕ, called the *inflaton*. The inflaton has a simple kinetic term in its action, $-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$, and it rolls slowly down its potential $V(\phi)$ in the aptly-named *slow-roll model*. Including the Einstein-Hilbert term, the full inflaton action reads

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right]. \tag{2.28}$$

The fact that the inflaton is slowly rolling down its potential, can be translated to the fact that in this regime (the *slow-roll regime*) the kinetic energy is negligible compared to the potential energy, i.e., $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$.

From the action, we can calculate the energy-momentum tensor $T^{\mu\nu}$ via

$$T^{\mu\nu} = \frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\phi)}\partial^\nu\phi + g^{\mu\nu}\mathcal{L}_\phi = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi - g^{\mu\nu}V(\phi), \tag{2.29}$$

where we have used the Lagrangian density of the inflaton $\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$. From this, we can then calculate the energy density $\rho_\phi = T^{00}$ and pressure $p_\phi = \frac{1}{3}\sum_i T^{ii}$,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \text{and} \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{2.30}$$

where in the calculations we have assumed spatial homogeneity of the inflaton field so that $\nabla\phi = 0$. We can plug these expressions into the Friedmann equations (A.5) and (A.8) to obtain two equations of motion for ϕ,

$$3M_{\text{Pl}}^2 H^2 = V(\phi)\left(1 + \frac{\epsilon_\phi}{3}\right), \quad \text{and} \tag{2.31a}$$

$$\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 3H\dot{\phi}\left(\frac{\eta_\phi}{3} - 1\right), \tag{2.31b}$$

where we have defined the two slow-roll parameters $\epsilon_\phi \equiv 3\dot{\phi}^2/2V$ and $\eta_\phi \equiv -\ddot{\phi}/H\dot{\phi}$. The slow-roll relation $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ then translates to the simple regime $\epsilon_\phi \ll 1$, and we also have a second slow-roll condition, $|\ddot{\phi}| \ll H\dot{\phi}$, which means that the acceleration of the field is much smaller than the velocity of the field per Hubble time H^{-1}. This second condition translates directly into $\eta_\phi \ll 1$. Sometimes, it is also useful to define the potential slow-roll parameters $\epsilon_V \equiv (M_{\text{Pl}}^2/2)(V'/V)^2 \simeq \epsilon_\phi$ and $\eta_V \equiv M_{\text{Pl}}^2(V''/V) \simeq \eta_V - \epsilon_V$, where again the conditions $\epsilon_V \ll 1$ and $\eta_V \ll 1$ are sufficient for the slow-roll approximation.

Important for us is a third version of the slow-roll parameter ϵ_V (or ϵ_ϕ), which is defined by

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right), \tag{2.32}$$

and it represents the rate of change of the inflationary Hubble rate. To see how this is related to the other slow-roll parameters, we look at the equation of state parameter $w \equiv p_\phi/\rho_\phi$ (see Appendix A). With the pressure and energy density given by Equation (2.30), and using the slow-roll approximation $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$, we can write the equation of state parameter to linear order in ϵ_ϕ as

$$w \simeq -1 + \frac{\dot{\phi}^2}{V} = -1 + \frac{2}{3}\epsilon_\phi. \tag{2.33}$$

In Appendix A.2.2, we show that from the Friedmann equation it also follows that $w = 2\epsilon_H/3 - 1$, which means that we can approximately say $\epsilon_H \simeq \epsilon_V \simeq \epsilon_\phi$. In the rest of this work, we will refer to the collection of these parameters as simply ϵ, without a subscript.

From the expression (2.33) for w, we can see that when $\epsilon = 0$, then $w = -1$ and this corresponds to a de Sitter Universe or purely exponential expansion as then $p = -\rho$. For $\epsilon \neq 0$ (but still $\epsilon \ll 1$), there is a slight deviation from this pure de Sitter case and we call this a *quasi-de Sitter* Universe. Inflation is therefore described by a quasi-de Sitter Universe, with a nearly perfectly exponential expansion, but a slight deviation from this characterised by the slow-roll parameter ϵ. During exponential expansion, the Hubble parameter is approximately constant, so during inflation, i.e., quasi-exponential expansion, we can also approximately say that the Hubble parameter remains constant.

2.3.2 The irreducible inflationary SGWB power spectrum

Earlier, we mentioned quantum fluctuations that get stretched out to super-Hubble scales during inflation. They remain constant until they cross the horizon again after inflation has ended, they start oscillating, and they become the basis for LSS formation. The same principle applies to the tensor metric perturbations χ_{ij}, which also get stretched to super-Hubble scales, remain constant there until they cross the horizon after the end of inflation, after which they start oscillating and form an irreducible background of GWs whose power spectrum we could measure today. The reason that this background is a *stochastic* background, is because the metric perturbations are random variables because they are random quantum fluctuations. After they cross the horizon, they become effectively classical and this causes the metric perturbations to become equivalent to a stochastic variable, even though they were quantum in origin. After inflation, the modes re-enter the horizon and they keep this stochastic nature, meaning that the irreducible background of gravitational waves from the quantum fluctuations during inflation, constitutes a stochastic background and can therefore be described by a power spectrum. See [3, 11] for more in-depth discussions about the stochastic nature of the GW background from inflation.

In order to see this production of an SGWB from inflation, we can simply calculate this power spectrum from the inflaton action (2.28). Some calculations are quite lengthy and are therefore not included, but they can be found in Appendix C.1. At the appropriate times, we will refer to the specific calculations in this appendix. We follow approximately the steps in Section 5 of [3], but will sometimes deviate from this.

Action and equation of motion

Let us start at Equation (2.28) and expand the pure gravitational part with the perturbed FLRW metric in conformal time η (see Equation (2.18)),

$$\mathrm{d}s^2 = a^2(\eta)\left[-\,\mathrm{d}\eta^2 + \left(\delta_{ij} + \chi_{ij}\right)\mathrm{d}x^i\,\mathrm{d}x^j\right], \tag{2.34}$$

which gives us the second-order action

$$S_{\mathrm{g}}^{(2)} = -\frac{M_{\mathrm{Pl}}^2}{8}\int \mathrm{d}\eta\mathrm{d}^3\mathbf{x}a^2(\eta)\eta^{\mu\nu}\partial_\mu\chi_{ij}\partial_\nu\chi_{ij}. \tag{2.35}$$

Because χ_{ij} is the transverse and traceless part of the full perturbation $h_{\mu\nu}$, we can decompose it into its two polarisation states $r = +, \times$ (see Equations (2.15) and (2.16)) as

$$\chi_{ij}(\mathbf{x},\eta) = \sum_{r=+,\times}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\chi_r(\mathbf{k},\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}e^r_{ij}(\hat{\mathbf{k}}), \tag{2.36}$$

where the two polarisation tensors $e^r_{ij}(\hat{\mathbf{k}})$ are real and satisfy the conditions

$$e^r_{ij}(-\hat{\mathbf{k}}) = e^r_{ij}(\hat{\mathbf{k}}) \quad \text{and} \tag{2.37a}$$

$$e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}}) = 2\delta_{rr'} \tag{2.37b}$$

(the second relation is the orthonormal condition). Moreover, requiring that χ_{ij} be real translates to the condition

$$\chi_r^*(\mathbf{k},\eta) = \chi_r(-\mathbf{k},\eta). \tag{2.38}$$

Substituting all of this into Equation (2.35) and working through the steps, we arrive at[2]

$$S_{\rm g}^{(2)} = \frac{M_{\rm Pl}^2}{4}\sum_{r=+,\times}\int {\rm d}\eta \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}a^2(\eta)\left[\left|\chi_r'(\mathbf{k},\eta)\right|^2 - k^2\left|\chi_r(\mathbf{k},\eta)\right|^2\right]. \tag{2.39}$$

After this, we can go to *Mukhanov-Sasaki variables*, defined by

$$v_r(\mathbf{k},\eta) = \frac{M_{\rm Pl}}{\sqrt{2}}a(\eta)\chi_r(\mathbf{k},\eta). \tag{2.40}$$

Substitution of this into Equation (2.39) gives[3]

$$S_{\rm g}^{(2)} = \frac{1}{2}\sum_{r=+,\times}\int {\rm d}\eta \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\left[|v_r'|^2 - k^2|v_r| + \frac{a''}{a}|v_r|^2\right]. \tag{2.41}$$

This action describes the dynamics of two scalar fields $v_r(\mathbf{x},\eta)$ so we can find the equations of motion through the Euler-Lagrange formalism. Working through the steps again, we find[4]

$$v_r''(\mathbf{k},\eta) + \omega_k^2(\eta)v_r(\mathbf{k},\eta) = 0 \quad \text{with} \quad \omega_k^2(\eta) \equiv k^2 - \frac{a''(\eta)}{a(\eta)}. \tag{2.42}$$

Because the irreducible background is produced by vacuum quantum fluctuations of the χ_{ij} field, which are stretched by the quasi-exponential expansion of the Universe, we need to promote the real scalar fields $v_r(\mathbf{x},\eta)$ to quantum operators $\hat{v}_r$ that obey canonical commutation relations, as is often done in quantum field theory. The commutation relations are given by

$$\left[\hat{v}_r(\mathbf{x},\eta), \hat{\pi}_{r'}(\mathbf{x}',\eta)\right] = i\delta_{rr'}\delta^{(3)}(\mathbf{x}-\mathbf{x}') \quad \text{and} \tag{2.43a}$$
$$\left[\hat{v}_r(\mathbf{x},\eta), \hat{v}_{r'}(\mathbf{x}',\eta)\right] = \left[\hat{\pi}_r(\mathbf{x},\eta), \hat{\pi}_{r'}(\mathbf{x}',\eta)\right] = 0, \tag{2.43b}$$

where $\hat{\pi}_r$ is the conjugate momentum of the operator $\hat{v}_r$. We can decompose this operator into its mode expansion,

$$\hat{v}_r(\mathbf{x},\eta) = \int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\left[v_k(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}r} + v_k^*(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}r}^\dagger\right], \tag{2.44}$$

[2]See calculation **1.** in Appendix C.1.
[3]See calculation **2.** in Appendix C.1.
[4]See calculation **3.** in Appendix C.1.

where the mode functions $v_k(\eta)$ are just dependent on $k = |\mathbf{k}|$ since the background is spatially isotropic. $\hat{a}_{\mathbf{k}r}$ and $\hat{a}^\dagger_{\mathbf{k}r}$ are the annihilation and creation operators that satisfy the usual commutation relations

$$\left[\hat{a}_{\mathbf{k}r}, \hat{a}^\dagger_{\mathbf{k}'r'}\right] = (2\pi)^3 \delta_{rr'} \delta^{(3)}(\mathbf{k} - \mathbf{k}') \quad \text{and} \tag{2.45a}$$

$$\left[\hat{a}_{\mathbf{k}r}, \hat{a}_{\mathbf{k}'r'}\right] = \left[\hat{a}^\dagger_{\mathbf{k}r}, \hat{a}^\dagger_{\mathbf{k}'r'}\right] = 0. \tag{2.45b}$$

The mode functions $v_k(\eta)$ then satisfy the same equation of motion as the polarisation modes $v_r(\mathbf{k}, \eta)$, i.e.,

$$v_k''(\eta) + \omega_k^2(\eta) v_k(\eta) = 0 \quad \text{with} \quad \omega_k^2(\eta) = k^2 - \frac{a''(\eta)}{a(\eta)}. \tag{2.46}$$

Subhorizon solution

The differential equation (2.46) is not solvable for a general $\omega_k(\eta)$. Instead, we have to make an Ansatz for $a(\eta)$, but even before that, we can take a look at the subhorizon solution where $k \gg aH$ (see Equation (A.29b) in Appendix A), so that the term k^2 dominates over the term a''/a and the differential equation that needs to be solved is simply

$$v_k''(\eta) + k^2 v_k(\eta) = 0. \tag{2.47}$$

This is a simple harmonic oscillator with constant frequency $\omega_k^2 = k^2$ and it has the general solution

$$v_k(\eta) = c_+(k) e^{-ik\eta} + c_-(k) e^{ik\eta}. \tag{2.48}$$

We can associate the annihilation operator $\hat{a}_{\mathbf{k}r}$ to the positive frequency mode $e^{-ik\eta}$, and this operator is also used to define the vacuum state $|0\rangle$ via $\hat{a}_{\mathbf{k}r}|0\rangle = 0$. Because inflation can be described by a (quasi-)de Sitter Universe, this vacuum corresponds to the Bunch-Davies vacuum [12]. In this prescription, the solutions only have the positive frequency modes and so we set $c_-(k) = 0$. To determine the constant $c_+(k)$, we use the normalisation condition[5]

$$v_k v_k'^* - v_k^* v_k' = i, \tag{2.49}$$

from which we can quickly calculate that $c_+(k) = 1/\sqrt{2k}$ and so the subhorizon solution with the Bunch-Davies vacuum as the initial condition is given by

$$v_k(\eta) \simeq \frac{e^{-ik\eta}}{\sqrt{2k}} \quad \text{for} \quad k \gg aH. \tag{2.50}$$

This solution means that below the horizon, the solutions oscillate in time until the mode crosses the horizon, as we also saw at the end of Section 2.1.2.

[5]See calculation **4.** in Appendix C.1. For the derivation, we use the commutation relations (2.43a) and (2.45), as well as the mode expansion (2.44).

General solution

Instead of considering the superhorizon limit, $k \ll aH$, we choose to now find a general solution to the differential equation (2.46) for some Ansatz for the scale factor $a(\eta)$. In Section 2.3.1, we have shown that during inflation, the Universe expands quasi-exponentially (quasi-de Sitter) and this deviation from pure exponential expansion (de Sitter) is characterised by the slow-roll parameter ϵ. In Appendix A.2.2, it is shown that in this case, the scale factor approximately follows the relation

$$a(\eta) \simeq \frac{1}{\left[-(1-\epsilon)H_0\eta\right]^{1+\epsilon}}, \tag{2.51}$$

which can be written as $a(\eta) = (\eta/\eta_0)^p$ for some constant η_0, with in the quasi-de Sitter case, $p = -1/(1-\epsilon) \simeq -1-\epsilon$. We can then calculate

$$\frac{a''(\eta)}{a(\eta)} = \frac{p(p-1)}{\eta^2} \simeq \frac{1}{\eta^2}(1+\epsilon)(2+\epsilon) = \frac{1}{\eta^2}\left[2+3\epsilon+\mathcal{O}(\epsilon^2)\right]. \tag{2.52}$$

To linear order in ϵ, the differential equation (2.46) is then written as

$$v_k''(\eta) + \left[k^2 - \frac{1}{\eta^2}(2+3\epsilon)\right] v_k(\eta) = 0, \tag{2.53}$$

which is exactly in the form (B.11) (see Appendix B) with $a = k$, $b = 2+3\epsilon$, $t = \eta$, and $z = v_k$. The solutions are also given in the appendix as

$$v_k(\eta) = c_1(k)\sqrt{-\eta}H_\nu^{(1)}(-k\eta) + c_2(k)\sqrt{-\eta}H_\nu^{(2)}(-k\eta), \tag{2.54}$$

where the arguments all contain a minus sign because this will make it easier to take limits, as in a de Sitter Universe, super- and subhorizon are given by $-k\eta \ll 1$ and $-k\eta \gg 1$, respectively (see Appendix A.2.4). The order ν is given by

$$\nu = \frac{1}{2}\sqrt{4(2+3\epsilon)+1} = \frac{3}{2}\sqrt{1+\frac{4\epsilon}{3}} \simeq \frac{3}{2}\left(1+\frac{1}{2}\frac{4\epsilon}{3}\right) = \frac{3}{2}+\epsilon, \tag{2.55}$$

so this shows that $\nu = 3/2$ in pure de Sitter ($\epsilon = 0$), and again the slow-roll parameter ϵ denotes the deviation from this. To determine the constants $c_1(k)$ and $c_2(k)$, we use the subhorizon limit of the solution, i.e. the solution in the limit $-k\eta \gg 1$. The large-argument limits for the Hankel functions are given in Appendix B.2.2 and show that $c_2(k) = 0$ and $c_1(k)$ is calculated by comparing it with the subhorizon solution (2.50),

$$\begin{aligned}
v_k(\eta) \simeq \frac{e^{-ik\eta}}{\sqrt{2k}} &= c_1(k)\sqrt{-\eta}\sqrt{\frac{2}{-\pi k\eta}}\exp\left[i\left(-k\eta - \frac{\pi}{2}\left(\nu+\frac{1}{2}\right)\right)\right] \\
&= c_1(k)\sqrt{\frac{2}{\pi k}}e^{-ik\eta}\exp\left[-\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)\right] \\
\Rightarrow c_1(k) = \frac{1}{2k}\sqrt{\frac{\pi k}{2}}\exp\left[\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)\right] &= \frac{\sqrt{\pi}}{2}\exp\left[\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)\right].
\end{aligned} \tag{2.56}$$

With these constants, the full solution finally becomes

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} \exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{-\eta} H_\nu^{(1)}(-k\eta). \tag{2.57}$$

The power spectrum

To now calculate the irreducible inflationary power spectrum $\mathcal{P}_\chi(k)$ from this solution, we use the definition (2.26),

$$\langle 0|\hat{\chi}_{ij}(\mathbf{k},\eta)\hat{\chi}^*_{ij}(\mathbf{k}',\eta)|0\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_\chi(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{2.58}$$

which can be rewritten to (2.27)[6],

$$\langle 0|\hat{\chi}_{ij}(\mathbf{x},\eta)\hat{\chi}_{ij}(\mathbf{x},\eta)|0\rangle = \int \frac{\mathrm{d}k}{k}\mathcal{P}_\chi(k). \tag{2.59}$$

With the mode expansion

$$\hat{\chi}_{ij}(\mathbf{x},\eta) = \sum_{r=+,\times}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\chi_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}r} + \chi_k^*(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^\dagger_{\mathbf{k}r}\right] e^r_{ij}(\hat{\mathbf{k}}), \tag{2.60}$$

where χ_k follows from v_k via Equation (2.40), the power spectrum can be calculated to be[7]

$$\mathcal{P}_\chi(k) = -\frac{k^3\eta}{\pi M_{\rm Pl}^2 a^2(\eta)}\left|H_\nu^{(1)}(-k\eta)\right|^2 \quad \text{with} \quad \nu = \frac{3}{2} + \epsilon. \tag{2.61}$$

The superhorizon power spectrum

Physically, this form of the power spectrum does not say much, and it makes sense to take the sub- and superhorizon limits. Let us start with the subhorizon limit, i.e., the limit in which $-k\eta \gg 1$ or $k \gg aH$. We can use the large-argument limit of the Hankel function again, given in Appendix B.2.2, $|H_\nu^{(1)}(x \gg 1)|^2 \simeq 2/\pi x$, so then

$$\mathcal{P}_\chi(k) \simeq -\frac{k^3\eta}{\pi M_{\rm Pl}^2 a^2}\frac{2}{-\pi k\eta} = \frac{2H^2}{\pi^2 M_{\rm Pl}^2}\left(\frac{k}{aH}\right)^{n_T} \quad \text{where} \quad n_T = 2, \quad k \gg aH. \tag{2.62}$$

We can see that the spectrum has a spectral tilt $n_T = 2$,[8] and at the horizon, $k = aH$, the power spectrum is given by

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2}, \tag{2.63}$$

[6] See calculation **5.** in Appendix C.1.

[7] See calculation **6.** in Appendix C.1. This involves the commutation relations (2.45) and the definition of the vacuum state, $\hat{a}_{\mathbf{k}r}|0\rangle = 0$ or $\langle 0|\hat{a}^\dagger_{\mathbf{k}r} = 0$.

[8] This spectral tilt is not very important because it is the superhorizon scales that stay constant (see the end of Section 2.1.2), re-enter the horizon in the post-inflationary era, and then turn into normal GWs for us to measure. The subhorizon scales simply oscillate in time.

which is often regarded in the literature as the irreducible inflationary power spectrum.

Taking the superhorizon limit, $-k\eta \ll 1$ or $k \ll aH$, requires a bit more work and it is here that we see the effects of the quasi-de Sitter regime. We now use the small-argument limit of the Hankel function, given in Appendix B.2.1,

$$\left|H_\nu^{(1)}(x \ll 1)\right|^2 \simeq \frac{1}{\pi^2}\Gamma^2(\nu)\left(\frac{x}{2}\right)^{-2\nu}. \tag{2.64}$$

Substituting this into our power spectrum, we can rewrite it to[9]

$$\begin{aligned}\mathcal{P}_\chi(k) &\simeq \frac{2H^2}{\pi^3 M_{\rm Pl}^2} f^2(\epsilon)\left(\frac{k}{aH}\right)^{n_T}, \quad \text{where} \quad n_T = -2\epsilon, \quad k \ll aH, \\ \text{and where} \quad f(\epsilon) &= 2^{1+\epsilon}\Gamma\left(\frac{3}{2}+\epsilon\right)(1-\epsilon)^{1+\epsilon}.\end{aligned} \tag{2.65}$$

The function $f(\epsilon)$ can be Taylor expanded around $\epsilon = 0$ since $\epsilon \ll 1$, which gives $f(\epsilon) \simeq \sqrt{\pi}(1 - 0.27\epsilon)$. The quasi-de Sitter regime therefore causes a very slight deviation from the pure de Sitter case. It is for this reason that we approximate $f(\epsilon) \simeq f(0)$ in the following, where $f(0) = \sqrt{\pi}$. The power spectrum is then given by

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2}\left(\frac{k}{aH}\right)^{-2\epsilon}. \tag{2.66}$$

This therefore has a spectral tilt of $n_T = -2\epsilon$ due to the quasi-de Sitter nature of inflation. At the horizon $k = aH$, it again reduces to the inflationary power spectrum (2.63),

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2}, \tag{2.67}$$

so the two regimes (super- and subhorizon) converge to a common value at their intersection. Since $\epsilon > 0$, the spectral tilt $n_T = -2\epsilon$ is negative and this results in a slightly red-tilted spectrum, while in reality a blue-tilted spectrum (with $n_T > 0$) is desired (see Chapter 1).

2.3.3 Other sources of SGWBs

Now that we have discussed the irreducible SGWB power spectrum from quantum fluctuations during inflation, let us quickly point out some other sources that generate such a background. We will first restrict ourselves to inflation, and then move on to later times. All of these cases are discussed in more detail in the final sections of [3]. In most cases, the background is created due to a nonzero anisotropic stress which, according to Equation (2.14), actively sources GWs (in contrast to the inflation case that we just discussed, in which the background is created by quantum fluctuations).

[9] See calculation **7.** in Appendix C.1.

First of all, we can have additional fields during inflation, that can have interactions leading to particle production. These particles are produced by the inflaton rolling down its potential, which creates a time-dependent background that is needed for this particle production. Another method is by so-called *spectator fields*, which are fields that do not influence any background dynamics in any way (hence the name), but they can acquire scalar and/or tensor perturbations in the same way that the inflaton acquires them, through quantum fluctuations. Several authors (e.g. [13]; for others, see the references within [3]) have investigated the effects of spectator fields and it turns out that they can enhance the inflationary tensor modes. It is even possible in some cases to obtain a blue GW signal, which is desired.

These two options both add extra fields in addition to the massless inflaton, which is responsible for the quasi-exponential expansion of the inflationary Universe. Instead, we might look at modifications of the theory of gravity, in which we make the graviton massive. This is typically done by considering new symmetries in the inflationary sector, which is already known to break time-reparametrisations (because of the time-dependence of the scale factor in the FLRW metric), but it is invariant under space-reparametrisations. Breaking this symmetry, the tensor modes do not need to be massless anymore, as we will see in Chapter 3. It turns out that such a tensor mass term can affect the spectral tilt of the power spectrum in such a way that it could become blue, as desired. However, as we will see in Chapter 4, it is in fact not generally possible to obtain a quasi-de Sitter solution for massive gravity, but instead often in the literature, the inflationary era is approximated as pure de Sitter. In the next chapter, we will go into further detail about massive gravity and we will see where this graviton mass comes from. In Chapter 4 we will then see the implications of this graviton mass on the power spectrum, and all the problems that come with it.

Before we discuss massive gravity, however, let us quickly touch upon some more important mechanisms generating GW backgrounds after inflation (again, see [3] for more details). During the history of the Universe, several phase transitions have occurred, such as the QCD (quantum chromodynamics) or the electroweak phase transitions. These are examples of first-order phase transitions, in which there is a barrier in the order parameter driving the phase transition. The exact physics of such transitions is not important for our purposes (as we just want to mention it), but it causes a nonzero anisotropic stress and therefore these phase transitions are a source for GWs. Finally, one could also have GW backgrounds generated during the preheating era just after inflation, or due to so-called “cosmic defects”, but we will not go into any detail here.

Chapter 3

Review of massive gravity

In this chapter, we will discuss the reasons for and consequences of giving a mass to the graviton. We will start with a flat Minkowski background and discuss the most general action for a massive graviton, the Fierz-Pauli action, in Section 3.1. We then move on to a curved background and also add Lorentz breaking mass terms in Section 3.2.

3.1 Lorentz-invariant massive gravity on a flat background

Let us start with the discussion on a flat Minkowski background. We follow mainly the review article by Hinterbichler (2012) [14]. For a massless spin-2 particle (i.e. the massless graviton), carried by a symmetric tensor field $h_{\mu\nu}$, the linearised Einstein-Hilbert (EH) action can be written as

$$S_{\mathrm{EH}} = \frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R \tag{3.1}$$

$$= \int \mathrm{d}^4x \left[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h\right], \tag{3.2}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric, R is the Ricci scalar, and $h = h^\mu{}_\mu$ is the trace of $h_{\mu\nu}$. The prefactor disappears because of our choice to perturb the metric such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\mathrm{Pl}}}h_{\mu\nu}. \tag{3.3}$$

The terms in Equation (3.2) describe the massless, helicity-2 graviton and they are invariant under the gauge symmetry

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{3.4}$$

with $\xi_\mu(x)$ the gauge parameter, which depends on the spacetime coordinates x_μ. As we discussed in Chapters 1 and 2, we can try to add a mass m to the graviton, and we do this by adding the term

$$S_m = \int \mathrm{d}^4x \left[-\frac{1}{2}m^2 \left(h_{\mu\nu}h^{\mu\nu} - h^2 \right) \right] \tag{3.5}$$

to the EH action (3.2). This term breaks the gauge symmetry (3.4), and the full action,

$$S_{\mathrm{FP}} = S_{\mathrm{EH}} + S_m, \tag{3.6}$$

is called the *Fierz-Pauli* (FP) *action* [8]. The combination $h_{\mu\nu}h^{\mu\nu} - h^2$ is specifically chosen such that this linearised theory of massive gravity is healthy, i.e., it does not contain any "ghostly" DoF; this specific choice is called the *Fierz-Pauli tuning*. If we were to take any other combination $h_{\mu\nu}h^{\mu\nu} + ah^2$ where $a \neq -1$, then the theory does not anymore describe a massive spin-2 particle, and instead also includes a scalar with a negative kinetic energy, which is called a *ghost*. Because of their negative kinetic energy, ghosts are unphysical modes and we therefore want to avoid them in any theory.

3.1.1 The vDVZ discontinuity

This first problem of creating ghosts is readily solved by choosing the coefficient between the terms $h_{\mu\nu}h^{\mu\nu}$ and h^2 in the action (3.5) to be -1, i.e., choosing the Fierz-Pauli tuning. However, there is another problem that arises with this FP action, called the *vDVZ discontinuity*.

Degree of freedom count

To see what this is, where it comes from and how it is resolved, we start by looking at the number of DoF in the massless and massive theories. In Section 2.1 of [14], it is shown that for $m = 0$, there are two DoF, corresponding to the two polarisations of the massless graviton (see also Section 2.1.1). This is done by expanding the $h_{\mu\nu}$ into their components h_{00}, $h_{0i} = h_{i0}$, and h_{ij}. The h_{ij} have six DoF[1] but the h_{00} and h_{0i} appear only linearly so they can be considered Lagrange multipliers enforcing two constraints that reduce the number of DoF by four, leaving just two.

When $m \neq 0$, the h_{0i} do not create these constraints anymore and we only have the constraint from h_{00}, which reduces the number of DoF by one, leaving five. If, on the other hand, we have the more general combination $h_{\mu\nu}h^{\mu\nu} + ah^2$ where $a \neq -1$, then even the h_{00} term does not appear linearly anymore, it does not anymore enforce a constraint, and so the full six DoF of h_{ij} are active. This extra DoF is the ghostly DoF. Therefore, we again see that we can avoid an unhealthy theory (a theory with ghosts) if we choose the Fierz-Pauli tuning.

[1] In general, the 3×3 matrix h_{ij} has 9 independent coefficients and so 9 DoF, but since $h_{\mu\nu}$ is symmetric, h_{ij} must also be symmetric and we only have 6 independent terms.

Coupling to matter

Now, since gravity has an effect on all matter, it makes sense to couple the tensor $h_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}(x)$ of a source. This gives us the total action

$$\begin{aligned} S = \int \mathrm{d}^4x \Big[& -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \\ & - \frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) + \frac{1}{M_{\mathrm{Pl}}}h_{\mu\nu}T^{\mu\nu}\Big]. \end{aligned} \tag{3.7}$$

The equations of motion of $h_{\mu\nu}$ can now be calculated and from that, we can find a general solution for $h_{\mu\nu}(x)$ in the form of a Fourier transform of terms including the Fourier transformed energy-momentum tensor $T_{\mu\nu}(p)$. Solutions for a point source can then also be found; the details of these calculations will not be shown here but can be found in Section 3 of [14]. For the massless graviton, they find

$$\begin{aligned} h_{00}(x) &= \frac{M}{2M_{\mathrm{Pl}}}\frac{1}{4\pi r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{M}{2M_{\mathrm{Pl}}}\frac{1}{4\pi r}\delta_{ij}, \end{aligned} \tag{3.8}$$

and for the massive graviton, they show that the solution is gauge equivalent to[2]

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_{\mathrm{Pl}}}\frac{1}{4\pi}\frac{e^{-mr}}{r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{M}{3M_{\mathrm{Pl}}}\frac{1}{4\pi}\frac{e^{-mr}}{r}\delta_{ij}. \end{aligned} \tag{3.9}$$

In these equations, M is the mass of the point source (i.e., $T^{\mu\nu}(x) = M\delta_0^\mu\delta_0^\nu\delta^{(3)}(\mathbf{x})$) and r is the distance from this source.

Next, we can write (see e.g. Chapter 7 of [15]) $2h_{00}/M_{\mathrm{Pl}} = -2\Phi$, $2h_{ij}/M_{\mathrm{Pl}} = -2\Psi\delta_{ij}$, and $2h_{0i}/M_{\mathrm{Pl}} = 0$ with $\Phi(r)$ the Newtonian potential and $\Psi(r)$ some other function. If then $\Psi(x) = \gamma\Phi(x)$ for some constant γ, then the photon deflection angle α at impact parameter b around this source of mass M is given by

$$\alpha = 2(\gamma+1)\frac{GM}{b}. \tag{3.10}$$

For the massless case, we can see from Equation (3.8) that $\gamma = 1$ and so the deflection angle is $\alpha_{\mathrm{massless}} = 4GM/b$. If we do the same for the massive case, Equation (3.9) in

[2]Note that the massive solution (3.9) shows the Yukawa suppression factor e^{-mr}/r, characteristic for a massive field satisfying the Klein-Gordon equation $(\Box - m^2)\psi = 0$. Long-range forces, like gravity, should not have this exponential suppression and therefore must have $m = 0$.

the massless limit shows that $\gamma = 1/2$ and therefore $\alpha_{\text{massive}} = 3GM/b$. This difference in deflection angles between the massless theory and the massless limit of the massive theory,

$$\alpha_{\text{massive}} = \frac{3}{4}\alpha_{\text{massless}}, \tag{3.11}$$

is called the *van Dam-Veltman-Zakharov* (vDVZ) *discontinuity*, named after the people who first discovered it in 1970 [16, 17].

The origin of the vDVZ discontinuity

To see where the vDVZ discontinuity comes from, let us go back to the DoF count from before. We have seen that in the massless theory, there are two DoF while in the massive theory, there are five. In fact, if we were to take $m \to 0$ in the sourced action (3.7), then we should not even be able to end up at the massless theory, as three DoF are lost and the limit is not smooth. Instead, we would expect the other three DoF to still be present and one of them will turn out to be responsible for the vDVZ discontinuity.

To explicitly see this, we use the so-called *Stückelberg trick*, in which we make a replacement of the field in such a way that the new action, after this replacement, is dynamically equivalent to the old action but it now has a gauge symmetry. This is usually done by modelling the replacement after the would-be gauge symmetry in the massless case. For our purposes, we have the action (3.7) which, when $m = 0$, has the gauge symmetry (3.4), but the mass term breaks this gauge symmetry. Nevertheless, we can model our Stückelberg replacement after this gauge symmetry and replace

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu. \tag{3.12}$$

This new field A_μ is called a *Stückelberg field*. We see that because of this replacement, we now have a gauge symmetry by simultaneously replacing

$$\begin{aligned} &h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \delta h_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\nu \quad \text{and} \\ &A_\mu \to A'_\mu = A_\mu + \delta A_\mu = A_\mu - \xi_\mu. \end{aligned} \tag{3.13}$$

This replacement with the new field A_μ actually still has four DoF in the massless limit so we still lose one. To account for this, we introduce another Stückelberg field

$$A_\mu \to A_\mu + \partial_\mu \phi. \tag{3.14}$$

This creates another gauge symmetry if we simultaneously replace

$$\begin{aligned} &A_\mu \to A'_\mu = A_\mu + \delta A_\mu = A_\mu + \partial_\mu \Lambda \quad \text{and} \\ &\phi \to \phi' = \phi + \delta\phi = \phi - \Lambda. \end{aligned} \tag{3.15}$$

A rescaling $A_\mu \to A_\mu/m$ and $\phi \to \phi/m^2$, the assumption that the source is conserved ($\partial_\mu T^{\mu\nu} = 0$), and taking the limit $m \to 0$, gives us an action with all five DoF in the massless limit,

$$S = \int d^4x \Big[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \\ - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2\left(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi\right) + \frac{1}{M_{\rm Pl}}h_{\mu\nu}T^{\mu\nu}\Big]. \tag{3.16}$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the five DoF are the two from the massless graviton ($h_{\mu\nu}$; the top line is exactly the massless graviton action (3.2)), two from a massless vector (A_μ; equivalent to the two polarisations of the photon), and one from the scalar ϕ, which is kinetically mixed with the graviton.

As a final step, we can decouple the scalar ϕ from the tensor $h_{\mu\nu}$ by making a redefinition $h_{\mu\nu} = h'_{\mu\nu} + \phi\eta_{\mu\nu}$.[3] With this, the action becomes

$$S = \int d^4x \Big[-\frac{1}{2}\partial_\lambda h'_{\mu\nu}\partial^\lambda h'^{\mu\nu} + \partial_\mu h'_{\nu\lambda}\partial^\nu h'^{\mu\lambda} - \partial_\mu h'^{\mu\nu}\partial_\nu h' + \frac{1}{2}\partial_\lambda h' \partial^\lambda h' \\ - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 3\partial_\mu\phi\partial^\mu\phi + \frac{1}{M_{\rm Pl}}h'_{\mu\nu}T^{\mu\nu} + \frac{1}{M_{\rm Pl}}\phi T\Big], \tag{3.17}$$

where $T = T^\lambda{}_\lambda$ is the trace of the energy-momentum tensor $T_{\mu\nu}$ and $F_{\mu\nu}$ is defined as before. We can now manifestly see that we have five DoF: two for the massless graviton, two for a massless vector, and one for a massless scalar. This scalar is still coupled to the trace T, even in the massless limit, and this is exactly the origin of the vDVZ discontinuity.

From this result, we can understand the difference in the deflection angle of light between the massless theory and the massless limit of the massive theory as follows. For light the energy-momentum tensor is traceless, $T = 0$, so the scalar DoF does not affect the bending of light. However, for nonrelativistic masses, i.e., when $T \neq 0$, we do see a difference and therefore the Newtonian potential is altered. This alteration of the Newtonian potential exactly causes this discontinuity between the light bending in the massless limit of the massive theory and in the massless theory.

3.1.2 Resolution of the vDVZ discontinuity

To see how the vDVZ discontinuity is resolved, we first need to introduce the nonlinear theory of Lorentz-invariant massive gravity. We will see that this theory causes a ghost which we normally do not want. However, in this case, it plays a crucial role in the resolution of the discontinuity.

The nonlinear theory

In massless GR, we can expand the GR action (3.1) around the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}/M_{\rm Pl}$, after which we end up at the second order (linear) action (3.2). We can go a step further and expand to higher order in $h_{\mu\nu}$ so that we obtain an action with a lot of

[3] This $h'_{\mu\nu}$ is different from the $h'_{\mu\nu}$ in the gauge symmetry equation (3.13).

interaction terms of the form $\sim \partial^2 h^{n+2}$ where $n \geq 0$ is an integer. These higher powers are suppressed by higher and higher powers of $1/M_{\text{Pl}}$, which enters in our normalisation.

In massive gravity, we can do a similar expansion. For massive gravity, we add a mass term to the GR term. For simplicity, we can take this mass term to be the FP term, which is a term to second order in $h_{\mu\nu}$. Any nonlinearities therefore arise in the GR term in this case and the action can be written with a flat Minkowski background as[4]

$$S = \frac{M_{\text{Pl}}^2}{2} \int \mathrm{d}^4 x \left[(\sqrt{-g} R) - \frac{1}{4} m^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) \right]. \tag{3.18}$$

If we wanted to include a more general nonlinear mass term as well, then we would need to add some general potential $V(g, h)$ which is given by

$$V(g,h) = V_2(g,h) + V_3(g,h) + V_4(g,h) + \cdots , \tag{3.19}$$

where

$$V_2(g,h) = \langle h^2 \rangle - \langle h \rangle^2, \tag{3.20a}$$
$$V_3(g,h) = c_1 \langle h^3 \rangle + c_2 \langle h^2 \rangle \langle h \rangle + c_3 \langle h \rangle^3, \tag{3.20b}$$
$$V_4(g,h) = d_1 \langle h^4 \rangle + d_2 \langle h^3 \rangle \langle h \rangle + d_3 \langle h^2 \rangle^2 + d_4 \langle h^2 \rangle \langle h \rangle^2 + d_5 \langle h \rangle^4, \tag{3.20c}$$
$$\vdots$$

In these equations, the angled brackets are traces with the indices raised by the metric $g^{\mu\nu}$. The first term is simply the FP term with the minus sign indicating the FP tuning $(h_{\mu\nu} h^{\mu\nu} - h^2)$, and the subsequent terms are combinations of traces of higher and higher powers of $h_{\mu\nu}$. For our purposes, we will stick to the FP mass term and nonlinearities that only arise in the GR term (Equation (3.18)), with a flat Minkowski background.[5] In this case, we can calculate the equation of motion to have the form

$$\sqrt{-g} \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) + \frac{1}{2} m^2 \left(h^{\mu\nu} - \eta^{\mu\nu} h \right) = 0. \tag{3.21}$$

The Vainshtein radius

Before we explore the number of DoF in the nonlinear theory, we first need to see how a crucial radius arises if we look for spherical metric solutions to the above equation of motion. Taking the background to be the spherical flat Minkowski metric, $\eta_{\mu\nu}\, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega^2$ with $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\, \mathrm{d}\phi^2$, we can look for solutions of the form

$$g_{\mu\nu}\, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = -B(r)\, \mathrm{d}t^2 + C(r)\, \mathrm{d}r^2 + A(r) r^2\, \mathrm{d}\Omega^2. \tag{3.22}$$

[4]Notice that we perturb around this action with the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, after which we canonically normalise it as $h_{\mu\nu} \to 2h_{\mu\nu}/M_{\text{Pl}}$ to get rid of the prefactors. This expression is therefore slightly different from the expressions that we saw at the beginning of this chapter.

[5]For the more general case, see Section 6 of [14].

The fact that A, B, and C are only functions of r is due to the fact that we want the metric to be static (no time-dependence) and isotropic (no angular dependence). By plugging this metric into the equation of motion (3.21), we can find three equations that the variables A, B, and C must adhere to. We expand these variables using a small parameter $\delta \ll 1$ (i.e., $A(r) = A_0(r) + \delta A_1(r) + \delta^2 A_2(r) + \cdots$ and similar expressions for $B(r)$ and $C(r)$) and solve this iteratively for higher orders in δ each time. Going through the steps (see Section 6.3 of [14]), we end up in the limit $mr \ll 1$ with

$$B(r) - 1 = -\frac{8}{3}\frac{GM}{r}\left(1 - \frac{1}{6}\frac{GM}{m^4 r^5} + \cdots\right), \tag{3.23a}$$

$$C(r) - 1 = -\frac{8}{3}\frac{GM}{m^2 r^3}\left(1 - 14\frac{GM}{m^4 r^5} + \cdots\right), \quad \text{and} \tag{3.23b}$$

$$A(r) - 1 = \frac{4}{3}\frac{GM}{4\pi m^2 r^3}\left(1 - 4\frac{GM}{m^4 r^5} + \cdots\right), \tag{3.23c}$$

which we observe is an expansion in terms of the parameter r_{V}/r where

$$r_{\mathrm{V}} \equiv \left(\frac{GM}{m^4}\right)^{1/5} \tag{3.24}$$

is called the *Vainshtein radius* [18]. In the massless limit $m \to 0$, $r_{\mathrm{V}} \to \infty$ and the expansion breaks down, providing hope that the vDVZ is only a problem in the linear theory.

The Boulware-Deser ghost

As already mentioned when we did the DoF count in Section 3.1.1, when we do not choose the FP tuning so the mass term is given by $h_{\mu\nu}h^{\mu\nu} + ah^2$ with $a \neq -1$, then the full six DoF of h_{ij} are active and the sixth DoF is a ghost. This can also be seen if we write the metric components in terms of the shift N_i and lapse N, via $g_{00} = -N^2 + g^{ij}N_iN_j$, $g_{0i} = N_i$, and $g_{ij} = g_{ij}$. We will not go into much detail here (see Section 6.4 of [14]), but it boils down to the notion that for $m = 0$, N_i and N are Lagrange multipliers that remove 4 of the 6 DoF, leaving 2. These calculations have been done for the nonlinear theory so we see that in the massless theory, the nonlinear and linear action provide the same number of DoF.

When adding the FP mass term, we get terms that are quadratic in the lapse and shift so they do not act anymore as Lagrange multipliers. Instead, they are auxiliary fields whose equations of motion we can plug back into the action to obtain a nonvanishing Hamiltonian.[6] This now means that all 6 DoF are active, so we have one more than in the linear theory. As is discussed in Section 8 of [14], this sixth DoF is a ghost,[7] and it is called the *Boulware-Deser* (BD) *ghost* [19].

[6] In massless gravity, the Hamiltonian vanished which made sure that the lapse and shift appeared as Lagrange multipliers and we only had 2 DoF.

[7] The fact that this has to be a ghost has to do with the scalar action which has higher derivatives. The equations of motion are fourth order, the scalar Lagrangian propagates two degrees of freedom, and by "Ostrogradsky's theorem", one of these has to be a ghost.

Ghosts are nonphysical states that we want to avoid where possible, so it makes sense to try to find a way to remove this BD ghost. It turns out that there is a way to get rid of this instability by showing that in the full nonlinear theory, the lapse N leads to a Hamiltonian constraint, which in turn leads to a secondary constraint. These two are enough to remove one propagating mode and make sure that there are only five DoF remaining of which none are ghosts. For more details about this procedure, see [20].

The decoupling limit and the resolution of the vDVZ discontinuity

Finally, we are in the right place to see how the vDVZ discontinuity can be resolved in the nonlinear theory through the so-called *Vainshtein mechanism*. For this, we use the nonlinear Stückelberg formalism, which is similar to the linear formalism we described in Section 3.1.1, but now the replacement has a lot more terms,

$$\begin{aligned} h_{\mu\nu} \to H_{\mu\nu} = h_{\mu\nu} &+ \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu\partial_\nu\phi + \partial_\mu A^\alpha \partial_\nu A_\alpha + \partial_\mu A^\alpha \partial_\nu \partial_\alpha \phi \\ &+ \partial_\mu \partial^\alpha \phi \partial_\nu A_\alpha + \partial_\mu \partial^\alpha \phi \partial_\nu \partial_\alpha \phi + \cdots , \end{aligned} \tag{3.25}$$

where in the dots we include terms quadratic and higher in the fields and containing at least one power of h, but these are not important for us. We can compare this with our linear Stückelberg formalism, Equations (3.12) and (3.14), and see that the linear part constitutes the first four terms of the nonlinear part. Canonical normalisation requires us to replace $h_{\mu\nu} \to 2h_{\mu\nu}/M_{\mathrm{Pl}}$ as before, as well as $A_\mu \to 2A_\mu/mM_{\mathrm{Pl}}$ and $\phi \to 2\phi/m^2 M_{\mathrm{Pl}}$, which makes sure that the expressions in this formalism can be easily compared to the expressions from Section 3.1.1.

Making this Stückelberg replacement in the linear FP action gives a lot of interaction terms. For now, we want to focus on the scalar field ϕ, as it is the scalar ghost that we eventually want to use in the Vainshtein mechanism to get rid of the vDVZ discontinuity. The term suppressed by the smallest scale turns out to be the term $\sim (\partial^2\phi)^3/\Lambda_5^5$ where

$$\Lambda_5 = \left(M_{\mathrm{Pl}} m^4\right)^{1/5} \tag{3.26}$$

is the smallest cutoff scale of the effective field theory. Next, we take the *decoupling limit*, in which $m \to 0$, $M_{\mathrm{Pl}} \to \infty$, $T \to \infty$, but we keep Λ_5 and the ratio T/M_{Pl} fixed. In this limit, all of the above interaction terms go to zero except for the term $\sim (\partial^2\phi)^3/\Lambda_5^5$. Also applying the conformal transformation $h_{\mu\nu} = h'_{\mu\nu} + m^2\phi\eta_{\mu\nu}$ (see just before Equation (3.17)), we end up with the scalar action

$$S_\phi = \int \mathrm{d}^4x \left\{ -3(\partial\phi)^2 + \frac{2}{\Lambda_5^5}\left[(\Box\phi)^3 - (\Box\phi)(\partial_\mu\partial_\nu\phi)^2\right] + \frac{1}{M_{\mathrm{Pl}}}\phi T \right\}. \tag{3.27}$$

In addition to these terms, the free graviton coupled to the source and the free decoupled vector also survive this decoupling limit.

Around a massive point source (as is needed for the appearance of the vDVZ discontinuity), we can expand the scalar field ϕ into a background term $\Phi(r)$ and a fluctuation

$\varphi \equiv \phi - \Phi$. From the action, we can then derive a Lagrangian and from this it follows that the theory propagates two DoF of which one is a ghost with a mass (see [14, 21] for more details)

$$m^2_{\text{ghost}}(r) \sim \frac{\Lambda_5^5}{\partial^2\Phi(r)}. \tag{3.28}$$

Around a flat background, or far from the source, $\Phi \to 0$ and $m_{\text{ghost}} \to \infty$, which means that the ghost is not seen in the linear theory. Around nontrivial backgrounds, however, the ghost mass becomes finite and we observe the ghost. It can also be shown (see [14]) that at distances much below the Vainshtein radius $r_V \sim (1/\Lambda_5)(M/M_{\text{Pl}})^{1/5}$ (Equation (3.24) combined with (3.26)), the ghost mass becomes very small, which corresponds to it mediating a long-range force. Normally, a scalar field corresponds with an attractive force, but because we now have a scalar ghost with a negative kinetic energy, this actually now mediates a repulsive force. It turns out that it precisely cancels the attractive force due to the coupling of the scalar field to the trace T, which is the force responsible for the vDVZ discontinuity. This is more explicitly shown in [14], but for us it is important that the force responsible for the vDVZ discontinuity is cancelled in the nonlinear regime and therefore we do not see it anymore.

To summarise, the BD ghost that arises in the nonlinear massive gravity theory can be used to resolve the vDVZ discontinuity that arises in the linear massive gravity theory. Still, though, we require the ghostly DoF for this, which we want to try to avoid as it is nonphysical. Another way to overcome this is to look at curved backgrounds (the FLRW background that allows us to connect this massive gravity theory with our cosmological problem), and, going another step further, explicitly break Lorentz invariance.

3.2 Lorentz-violating massive gravity on a curved background

So far, we have discussed a model of massive gravity that exhibits Lorentz invariance, and we have just seen that this comes with some problems. In the linear approximation, when we add the diffeomorphism-breaking FP mass term to the linearised EH action, we propagate five DoF instead of the usual two in massless gravity. One of these couples to the energy-momentum tensor even in the massless limit and we see the vDVZ discontinuity. This discontinuity can be solved in the nonlinear regime through the Vainshtein mechanism, in which we use the BD ghost that only arises in the nonlinear regime. Although it solves the vDVZ discontinuity, it is still a ghostly DoF which we want to try to avoid.

The inconsistencies could potentially be solved if we look at Lorentz-violating massive gravity [22, 23]. Furthermore, it has been shown that some problems may be solved if we go to a curved background instead of the flat, Minkowski background we used in Section 3.1 (see e.g. [24, 25]). It is for these reasons that it might be useful to combine the two and look at Lorentz-violating massive gravity on a curved background, which is what was done by Blas et al. (2009) [23]. This is also the paper that we will follow in this section.

3.2.1 Action and SVT decomposition

We will adopt the flat FLRW metric in conformal time η, given by

$$\mathrm{d}s^2 = a^2(\eta)\left[-\,\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]. \tag{3.29}$$

The GR action with this background is given by

$$S_{\mathrm{GR}} = \int \mathrm{d}^4x\sqrt{-g}M_{\mathrm{Pl}}^2(R - 6H^2). \tag{3.30}$$

Here, H is the Hubble parameter given by $H = \dot{a}/a = a'/a^2$ where a dot ($\dot{}$) represents again a derivative with respect to the cosmic time t (which we will not use further) and a prime ($'$) is a derivative with respect to η. We might also sometimes use the conformal Hubble parameter $\mathcal{H} = a'/a = aH$.

The curved background also affects the masses that we can give to the perturbation h (which will be defined shortly); the general second-order action for the Lorentz-breaking mass terms is given by[8,9]

$$S_{\mathrm{LB}}^{(2)} = \frac{M_{\mathrm{Pl}}^2}{8}\int \mathrm{d}^4x\sqrt{-\bar{g}}\Big[m_0^2h_{00}^2 + 2m_1^2h_{0i}^2 - (m_2^2 - 4H'a^{-1})h_{ij}^2 + (m_3^2 - 2H'a^{-1})h_{ii}^2 - 2m_4^2h_{00}h_{ii}\Big]. \tag{3.31}$$

Notice that here we still have the prefactor $M_{\mathrm{Pl}}^2/8$ as we have not normalised $h_{\mu\nu}$ in the same way as we did in Equation (3.3). Instead, we have used the metric

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}) = \bar{g}_{\mu\nu} + a^2h_{\mu\nu} \tag{3.32}$$

(where $\bar{g}_{\mu\nu} = a^2\eta_{\mu\nu}$ is the FLRW metric (3.29)). The masses in Equation (3.31) are general time-dependent masses, $m_i = m_i(\eta)$ for $i = 0,1,2,3,4$, since there is no reason for preserving time-reparametrisations in an FLRW Universe. Notice that for the specific choice

$$m_0^2 = 0, \quad m_1^2 = m_2^2 - 4H'a^{-1} = m_3^2 - 2H'a^{-1} = m_4^2 = m^2, \tag{3.33}$$

the action (3.31) reduces back to the FP action (3.5) if we canonically normalise the perturbation according to $h_{\mu\nu} \to 2h_{\mu\nu}/M_{\mathrm{Pl}}$. Not making this specific choice for the different masses will in some cases create ghosts again, which we want to avoid.

To be able to analyse the Lorentz-breaking action (3.31) and see in which cases we can avoid ghosts, we should decompose it into a tensor (T), a vector (V), and a scalar (S) part

[8] Rotational and translational symmetries are still preserved because these are properties of the FLRW Universe.

[9] Notice the factor 1/2 difference with [23]. Different authors use different conventions that might not all be consistent with each other, but we have tried to keep the notation as consistent as possible throughout this thesis, meaning that we have a prefactor 1/8 here. Sometimes we just have to accept that these small inconsistencies exist.

and look at these sectors separately. This is known as the *SVT decomposition*. We start by decomposing the components of $h_{\mu\nu}$ into irreducible fields,

$$\begin{aligned} h_{00} &= \psi, \\ h_{0i} &= u_i + \partial_i v, \\ h_{ij} &= \chi_{ij} + \partial_i s_j + \partial_j s_i + \partial_i \partial_j \sigma + \delta_{ij}\tau, \end{aligned} \tag{3.34}$$

where we now have four scalars ψ, v, σ, τ, two transverse vectors u_i, s_i (with the constraints $\partial_i u_i = \partial_i s_i = 0$), and a transverse and traceless tensor χ_{ij} (in the same notation as in Chapter 2) which has $\partial_j \chi_{ij} = \delta_{ij}\chi_{ij} = 0$.

Tensor part

Taking just the tensor part that includes the TT tensor χ_{ij}, and also including the GR action (3.30) linearised in the metric perturbation $h_{\mu\nu}$, we end up with the action for tensor perturbations

$$S^{(T)} = \frac{M_{\text{Pl}}^2}{8} \int d^4x a^2 \left[-\eta^{\mu\nu} \partial_\mu \chi_{ij} \partial_\nu \chi_{ij} - a^2 m_2^2 \chi_{ij}^2 \right]. \tag{3.35}$$

This action is free from ghosts as there are no values for the mass m_2 that cause the kinetic term to become negative. We can write out the first term in the action and also write $m_2 = m_\chi$ as the mass of the χ field if we are merely looking at the tensorial part of the action. This will give us an action in a form that we will be able to use in Chapter 4,

$$S^{(T)} = \frac{M_{\text{Pl}}^2}{8} \int d^4x a^2(\eta) \left[(\chi_{ij}')^2 - (\partial_k \chi_{ij})^2 - a^2(\eta) m_\chi^2(\eta) \chi_{ij}^2 \right]. \tag{3.36}$$

We have also included the explicit η-dependencies in the terms. Notice that when $m_\chi = 0$, the above actions reduce to the massless action (2.35).

Because this part does not give any ghosts, we can move on for now to the vector and scalar actions, but we will come back to this action in Chapter 4.

Vector part

We can do a similar thing for the vector part, where we collect all terms with u_i and s_i in the action, after which we end up with the action

$$S^{(V)} = \frac{M_{\text{Pl}}^2}{4} \int d^4x a^2 \left[-(u_i - s_i') \nabla^2 (u_i - s_i') + a^2 \left(m_1^2 u_i^2 + m_2^2 s_j \nabla^2 s_j \right) \right]. \tag{3.37}$$

Because u_i has no time-derivatives, we can integrate this out using its equation of motion. The action then takes the form

$$S^{(V)} = \frac{M_{\text{Pl}}^2}{4} \int d^4x a^4 \left[m_1^2 s_i' \frac{\nabla^2}{\nabla^2 - a^2 m_1^2} s_i' + m_2^2 s_i \nabla^2 s_i \right], \tag{3.38}$$

from which we can see that there are no instabilities for $m_1^2 \geq 0$ and $m_2^2 \geq 0$. In the case $m_1 = 0$, there are no time-derivatives of s_i anymore so there is no propagating vector mode. In Section 3.2.2 we will also look at this case in a bit more detail in the context of the vDVZ discontinuity.

Scalar part

For the scalar part of the action, we end up with the very long expression

$$\begin{aligned} S^{(S)} = \frac{M_{\text{Pl}}^2}{8} \int \mathrm{d}^4 x a^2 \Big\{ & -6(\tau' + \mathcal{H}\psi)^2 + 2(2\psi - \tau)\nabla^2\tau \\ & + 4(\tau' + \mathcal{H}\psi)\nabla^2(2v - \sigma') \\ & + a^2 \Big[m_0^2 \psi^2 - 2m_1^2 v \nabla^2 v - m_2^2(\sigma\nabla^4\sigma + 2\tau\nabla^2\sigma + 3\tau^2) \\ & + m_3^2(\nabla^2\sigma + 3\tau)^2 - 2m_4^2\psi(\nabla^2\sigma + 3\tau) \Big] \Big\}. \end{aligned} \tag{3.39}$$

We can see that ψ and v do not have any time-derivatives, so they can be regarded as Lagrange multipliers that provide constraints that we can substitute back into the action, getting rid of the ψ and v terms. This leaves us with a Lagrangian in terms of the two remaining variables $\Sigma \equiv \sigma/\nabla^2$ and τ, which we group into a vector $\varphi = (\Sigma, \tau)^T$ (see [23] for more details). From the Lagrangian, we can construct the Hamiltonian

$$\mathcal{H}_{\Sigma,\tau} = \frac{1}{2}\pi^T \mathcal{K}^{-1}\pi + \frac{1}{2}\varphi^T \mathcal{M}\varphi, \tag{3.40}$$

where π_i is the conjugate momentum ($\pi_i = \mathcal{K}_{ij}\varphi_j' - \mathcal{B}_{ij}\varphi_j$), $\mathcal{K}^{-1}$ is the kinetic term which has the form

$$\mathcal{K}^{-1} = \frac{1}{M_{\text{Pl}}^2 a^2} \begin{pmatrix} 3 - 4\nabla^2/a^2 m_1^2 & -2 \\ -2 & 2H^2/m_0^2 \end{pmatrix}, \tag{3.41}$$

$\mathcal{M} = \mathcal{A} + \mathcal{B}\mathcal{K}^{-1}\mathcal{B}$ is the potential term, and $\mathcal{A}$ and $\mathcal{B}$ are matrices whose explicit expressions are not important for us. From the kinetic term, we require that it is positive definite to avoid ghosts, which translates into positivity of the eigenvalues of $\mathcal{K}^{-1}$, and therefore the conditions $0 < m_0^2 \leq 6H^2$ and $m_1^2 > 0$. We can now have a healthy (ghost-free) theory for some choices of the masses, where in the linear flat case with a general mass term (without the FP tuning), or in the flat nonlinear case (c.f. Section 3.1.2) this was not possible. This also allows us to get rid of the vDVZ discontinuity in this curved theory, as we will see shortly.

Often, we also require that there are no so-called *gradient instabilities*, which are equivalent to ghosts (with a negative kinetic term), but for the spatial part. This translates to positive definiteness of the potential term $\mathcal{M}$ whose calculations are not given here or in [23], but work out to give the constraints

$$m_3^2 - m_2^2 < \frac{(m_1^2 - 2m_4^2)^2}{16m_0^2} \quad \text{and} \quad H'a^{-1} < -\left[\frac{m_1^2}{4} + \frac{(m_1^2 - 2m_4^2)^2}{16m_1^2}\right], \tag{3.42}$$

where we have assumed $m_0^2 > 0$, $m_1^2 > 0$, and $H^2 > 0$. This shows that for a theory completely free of any instabilities, there are constraints on all masses m_0^2 until m_4^2. These expressions also assume that $m_0, m_1 \neq 0$, but in [23] it is explored what happens when these masses are set to zero. The only thing that we will mention here is that at intermediate momenta and for $m_0 = 0$ (which is required if we want to go to the FP case), we obtain two constraints on the masses which reduce to the *Higuchi bound* [24] $2H^2 \leq m^2$ when we set all nonzero masses to the value m^2 (so we go back to the Lorentz-invariant case) and we take a de Sitter background. This shows that this theory is still compatible with the Lorentz-invariant theory when we take the appropriate limits.

3.2.2 The vDVZ discontinuity revisited

We will conclude this chapter by revisiting the vDVZ discontinuity in curved space for the Lorentz-breaking theory. We follow Section 7 of [23] in this part, in which the physics of the vDVZ discontinuity is captured by expanding the potentials Φ and Ψ (see the text just before Equation (3.10)) in powers of $1/\nabla^2$. In GR and for a point source, these potentials are given by $\Phi_{\text{GR}} = \Psi_{\text{GR}} = T_{00}/M_{\text{Pl}}^2\nabla^2$. In massive gravity, it can be shown that at small scales and provided that $m_2^2 \neq m_3^2$, there is no discontinuity, but at large scales and when $m_2^2 = m_3^2$, the relation between Ψ and Φ is given by

$$2m_3^2\Psi = m_4^2\Phi. \tag{3.43}$$

In the language of Section 3.1.1, this implies $\gamma = 1/2$ for massive gravity[10] and therefore we see the vDVZ discontinuity.

As is detailed in [23], in a curved background and for $m_1 = 0$, the potentials are shown to reduce back to their GR solutions in de Sitter space, and in a general FLRW background the corrections to Φ and Ψ vanish and we also reduce to the GR values, meaning that there is no vDVZ discontinuity in these cases. The authors also adopt a time-dependence of the masses and then show that still there is no discontinuity in the potentials. For the case $m_0 = 0$ similar arguments are true, just with different expressions for the potentials that still reduce to the GR potentials in the massless limit.

To summarise this part, the Lorentz-violating masses that we include in this theory, together with the fact that we now have a curved (FLRW) background, make sure that in specific cases for the masses we can resolve the vDVZ discontinuity. Moreover, ghosts and spatial inconsistencies can be avoided by constraining the different mass terms in various ways.

For now, however, we will focus on the tensorial part of this Lorentz-violating massive gravity theory (Equation (3.36)). We will accept that a graviton mass $m_2(\eta) = m_\chi(\eta)$ causes this action and investigate its implications in more detail for the period of inflation, and beyond that.

[10] Recall that γ was defined as $\Psi(x) = \gamma\Phi(x)$.

Chapter 4

Implications of massive gravity

In this chapter, we will explore the effects that adding a mass to the graviton has on the GW tensor power spectrum that we observe today. We will first derive the equation of motion from the action (3.36) in Section 4.1, after which we will study several possibilities of solving this equation of motion in Section 4.2. In Section 4.3, we will then use these solutions to derive the power spectra in general and in the superhorizon limit, so that we can see explicitly the spectral tilt of each spectrum.

4.1 Action and equation of motion

The following derivation of the equation of motion will follow very closely the derivation presented in Section 2.3.2, so some steps will be skipped over. For more details, see the derivations and calculations in Section 2.3.2 and Appendix C.1.

Let us start at the second order massive tensor action (3.36),

$$S^{(T)} = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x a^2(\eta)\left[-\eta^{\mu\nu}\partial_\mu\chi_{ij}\partial_\nu\chi_{ij} - a^2(\eta)m_\chi^2(\eta)\chi_{ij}\chi_{ij}\right] \tag{4.1a}$$

$$= \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x a^2(\eta)\left[(\chi'_{ij})^2 - (\partial_k\chi_{ij})^2 - a^2(\eta)m_\chi^2(\eta)\chi_{ij}^2\right]. \tag{4.1b}$$

As it turns out, we can make this a bit more general by adding derivative operators in such a way that we can write the second term as $c_T^2(\eta)(\partial_k\chi_{ij})^2$, where $c_T(\eta)$ is called the *tensor sound speed*. In GR, this is set to 1, i.e., the speed of light, but in massive gravity theories it can deviate from unity. We will not go into this in more detail but for a bit more explanation, see [26–28]. $c_T(\eta)$ is in general a function of time, but since this term is not the focus of our research, we will keep it constant during our calculations, $c_T(\eta) = c_T$ ([27, 28] explore some options with a time-dependent tensor sound speed). The complete action now reads

$$S^{(T)} = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x a^2(\eta)\left[(\chi'_{ij})^2 - c_T^2(\partial_k\chi_{ij})^2 - a^2(\eta)m_\chi^2(\eta)\chi_{ij}^2\right]. \tag{4.2}$$

We now follow the same procedure as in Section 2.3.2, that is, we first insert the Fourier expansion (2.36),

$$\chi_{ij}(\mathbf{x},\eta) = \sum_{r=+,\times} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \chi_r(\mathbf{k},\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} e^r_{ij}(\hat{k}), \tag{4.3}$$

with e^r_{ij} the usual polarisation tensors and where the modes χ_r have the usual reality condition (see Equations (2.37) and (2.38)). This will give the action (steps are not shown here as they are very similar as before)

$$S^{(T)} = \frac{M_{\mathrm{Pl}}^2}{4} \sum_{r=+,\times} \int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} a^2(\eta) \Big\{ |\chi'_r(\mathbf{k},\eta)|^2 - \left[c_T^2 k^2 + a^2(\eta) m_\chi^2(\eta) \right] |\chi_r(\mathbf{k},\eta)|^2 \Big\}. \tag{4.4}$$

Next, we go to Mukhanov-Sasaki variables, defined by Equation (2.40),

$$v_r(\mathbf{k},\eta) = \frac{M_{\mathrm{Pl}}}{\sqrt{2}} a(\eta) \chi_r(\mathbf{k},\eta), \tag{4.5}$$

after which we can rewrite the action to

$$S^{(T)} = \frac{1}{2} \sum_{r=+,\times} \int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} a^2(\eta) \Bigg\{ |v'_r(\mathbf{k},\eta)|^2 - \left[c_T^2 k^2 + a^2(\eta) m_\chi^2(\eta) - \frac{a''(\eta)}{a(\eta)} \right] |v_r(\mathbf{k},\eta)|^2 \Bigg\}. \tag{4.6}$$

From this, we can derive the equations of motion for $|v_r|$, using the standard Euler-Lagrange formalism, after which we can safely remove the absolute values, promote the functions v_r to operators $\hat{v}_r$, and use the mode expansion (2.44),

$$\hat{v}_r(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[v_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}r} + v_k^*(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}^\dagger_{\mathbf{k}r} \right], \tag{4.7}$$

where $v_k(\eta)$ only depends on $k = |\mathbf{k}|$ because of the isotropic background. The equations of motion are now found to be

$$\boxed{v''_k(\eta) + \omega_k^2(\eta) v_k(\eta) = 0, \quad \text{where} \quad \omega_k^2(\eta) = c_T^2 k^2 + a^2(\eta) m_\chi^2(\eta) - \frac{a''(\eta)}{a(\eta)}.} \tag{4.8}$$

Because $m_\chi(\eta)$ can be any general function of the conformal time η, this differential equation cannot in general be solved. Moreover, there are different models for the scale factor $a(\eta)$, e.g., a (quasi-)de Sitter Universe or radiation or matter domination, which all have a different time-dependence. It is for this reason that we have no choice but to make choices for both the functions $m_\chi(\eta)$ and $a(\eta)$. We will discuss these choices in the next section.

4.2 Solutions to the differential equation

Let us now take a look at the specific forms that we have to let the graviton mass $m_\chi(\eta)$ and the scale factor $a(\eta)$ take on. Starting with the scale factor, we can be quickly done as we can assume a general power-law Ansatz,

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^p, \tag{4.9}$$

where $p = 1$ for radiation domination, $p = 2$ for matter domination, $p = -1$ for a de Sitter background, and $p = -1/(1-\epsilon) \simeq -1-\epsilon$ for a quasi-de Sitter background (see Appendices A.2.1 and A.2.2). Substituting this into the equation of motion (4.8), we obtain

$$v_k''(\eta) + \left[c_T^2 k^2 + \left(\frac{\eta}{\eta_0}\right)^{2p} m_\chi^2(\eta) - \frac{p(p-1)}{\eta^2}\right] v_k(\eta) = 0. \tag{4.10}$$

Although this power-law scale factor simplifies the differential equation already, it is still not generally solvable if we do not specify p and $m_\chi(\eta)$. Let us therefore take a look at the various cases in which the differential equation *is* solvable.

Subhorizon solution

The first solvable option is to take the limit $c_T k \gg aH$,[1] i.e., the subhorizon limit, in accordance with our analysis in Section 2.3.2. In this case, the first term in the square brackets dominates and we simply have

$$v_k''(\eta) + c_T^2 k^2 v_k(\eta) = 0. \tag{4.11}$$

This has the solution $v_k(\eta) = Ce^{-ic_T k\eta}$, where the negative-frequency mode ($e^{ic_T k\eta}$) has already been removed to correspond with the Bunch-Davies vacuum. The constant C can be found from the condition $v_k v_k'^* - v_k^* v_k' = i$ (from [3]; see also calculation **4.** in Appendix C.1), and the solution in this limit is

$$v_k(\eta) \simeq \frac{e^{-ic_T k\eta}}{\sqrt{2c_T k}}, \quad c_T k \gg aH. \tag{4.12}$$

We see that by setting $c_T = 1$, we reduce to the subhorizon solution (2.50).

Using this subhorizon case as our solution for comparing the general solutions, we can then take into account the scale factor and graviton mass terms in the equation of motion (4.8) and try to solve it for specific cases. We note beforehand that the only analytic solutions are obtained when we also take the graviton mass to be a power-law of the form

$$m_\chi^2(\eta) = \mu_\chi^2 a^s(\eta), \tag{4.13}$$

[1]Notice that we now have to use $c_T k \gg aH$ (and $c_T k \ll aH$ for superhorizon) instead of $k \gg aH$, as the horizon is slightly different for waves not travelling at the speed of light. See also Appendix A.2.4.

where $a(\eta)$ is given by the power-law (4.9), μ_χ is constant and s is the degree that we can vary freely (as it will turn out, there are certain constraints on the values of s). With this in mind, let us first summarise the different cases and the reasons for them.

1. **(Quasi-)de Sitter background, constant mass.** We start the analysis by taking a constant mass ($s = 0$) and attempting to take a quasi-de Sitter background ($p = -1/(1-\epsilon) \simeq -1-\epsilon$), in order to try to simulate what happens to the power spectrum caused by a massive graviton during inflation. However, the combination of the second and third terms in square brackets in the differential equation (4.8), makes it so that the quasi-de Sitter case is now unsolvable. Instead, we make the approximation $-1-\epsilon \simeq -1$, in which case it *is* solvable, but it does not describe inflation anymore as the scale factor is now assumed to evolve as $a(\eta) \propto \eta^{-1}$ and not $\propto \eta^{-1-\epsilon}$ anymore. This approximation gives the general solution (4.22).

2. **General background, $s = -2$.** Moving beyond inflation, we cannot find solutions for general s and $p = -1, 1, 2$. Instead, there are two possibilities for s, which group the mass term in the differential equation (4.8) with either the first term ($c_T^2 k^2$) or the final term ($p(p-1)/\eta^2$). The first case corresponds with choosing $s = -2$, in which case the solutions depend on the value of p. For a de Sitter background ($p = -1$), the general solution is given by Equation (4.28), for radiation domination ($p = 1$), by Equation (4.30), and for matter domination ($p = 2$), by Equation (4.31). We can also have a quasi-de Sitter background here, in which case we just have the solution (4.27) with $p = -1/(1-\epsilon)$.

3. **General background, $s = -2(1 + 1/p)$.** Another option is to group the mass term with the final term ($p(p-1)/\eta^2$), which requires it to obtain a $\propto 1/\eta^2$-dependence. This is achieved by choosing $s = -2(1+1/p)$ and the solutions are given by Equations (4.33) and (4.37) (for $p < 0$ and $p > 0$, respectively), where the order ν is dependent on p and given by (4.35) and (4.39), respectively.

4. **Radiation-dominated background.** In the case of a radiation-dominated background, the final term in the differential equation (4.8) disappears so we have more freedom with our power-law graviton mass. Taking integer s, we can take $s = 0, -1, -2, -3, -4$; $s \geq 1$ gives infinite polynomial solutions and generally a nonvanishing graviton mass at late times, which we do not expect (at the present time, we want to reduce back to massless GR). Furthermore, $s \leq -5$ does not have analytic solutions, so we restrict ourselves to $-4 \leq s \leq 0$. The solutions are given in terms of parabolic cylinder functions ($s = 0$), Airy functions ($s = -1$), the subhorizon solution ($s = -2$), confluent hypergeometric functions ($s = -3$), and Bessel functions ($s = -4$). We note that the cases $s = -2$ and $s = -4$ correspond with the previous two cases. The only other solution for which we have also been able to find the integration constants is the case $s = -1$, and it is given by Equation (4.43). For $s = 0$ and $s = -3$, we will therefore not continue to try to find the power spectrum.

5. **Different mass functions.** We have also tried to adopt different functions for the graviton mass, other than a power-law, but we do not see any way in which the

differential equation can be solved in such cases. This is because of the power-law behaviour of the scale factor ($a \propto \eta^p$ for $p = -1, 1, 2$), which means that the differential equation is only solvable if the graviton mass also follows a power-law.

4.2.1 (Quasi-)de Sitter background with constant graviton mass

Let us first look at the inflation case, i.e., a quasi-de Sitter background like in Section 2.3.2, but this time with the added graviton mass term $a^2(\eta)m_\chi^2(\eta)$. During quasi-de Sitter expansion, the scale factor approximately evolves in time as (see Appendix A.2.2)

$$a(\eta) \simeq \left[-(1-\epsilon)H_0\eta\right]^{-1-\epsilon}, \tag{4.14}$$

which means that, like in Chapter 2 (Equation (2.52)),

$$\frac{a''(\eta)}{a(\eta)} \simeq \frac{1}{\eta^2}(2+3\epsilon) \tag{4.15}$$

to first order in ϵ. As for the mass term $m_\chi(\eta)$, a small time-dependence is expected proportional to ϵ [26], but for now we will ignore the time-dependence of the mass and simply take $m_\chi(\eta) = \mu_\chi$ constant. With these assumptions, the differential equation becomes

$$v_k''(\eta) + \left[c_T^2 k^2 + \frac{\mu_\chi^2}{\left[(1-\epsilon)H_0\eta\right]^{2+2\epsilon}} - \frac{2+3\epsilon}{\eta^2}\right] v_k(\eta) \simeq 0. \tag{4.16}$$

Already at this point, we notice that this equation is not solvable. This is because the second and third terms in the square brackets do not have the same η-dependence: the second term is $\propto 1/\eta^{2+2\epsilon}$ while the third term is $\propto 1/\eta^2$. The two terms cannot be combined and we cannot find any solutions.

One option is therefore to simply stop here and accept that massive gravity does not yield any analytic solutions during inflation (at least for a constant mass, as is often done in the literature), and this is the main point that we want to try to convey. However, several authors ([26, 29–35]; see also the discussion in Chapter 5) have opted to approximate the inflationary period as pure de Sitter simply in order to find analytic solutions to the differential equation (4.8). We cannot make clear enough that doing this defeats the entire purpose of inflation, namely that it evolves as a *quasi*-exponential Universe and therefore strictly speaking, it cannot be approximated by pure de Sitter.

Nevertheless, we will go on with a similar calculation, but whereas the references mentioned above simply set $\epsilon = 0$[2] in the differential equation (4.16), we will still keep some aspects of the quasi-de Sitter background. We will merely approximate the exponent $2 + 2\epsilon \simeq 2$ and keep the $(1-\epsilon)$ in the denominator of the second term and the

[2]Sometimes, the quasi-de Sitter effect is added later on by simply adding the spectral tilt of $n_T = -2\epsilon$ to the spectral tilt from massive gravity in pure de Sitter. We will see that the result that we obtain is the same as this and is therefore, again, not representative of inflation.

$(2+3\epsilon)$ in the numerator of the third.[3] In this way, we can combine the second and third terms and get it in the form

$$v_k''(\eta) + \left[c_T^2 k^2 - \frac{1}{\eta^2}\left(2+3\epsilon - \frac{\mu_\chi^2}{(1-\epsilon)^2 H_0^2}\right)\right] v_k(\eta) = 0, \tag{4.17}$$

but we stress again that this equation is not representative of inflation because it does not follow the quasi-exponential expansion that is assumed in that period. Following through with the calculations, from Appendix B.1.3 we can see that the general solution is now given by

$$v_k(\eta) = c_1(k)\sqrt{-\eta} H_\nu^{(1)}(-c_T k\eta) + c_2(k)\sqrt{-\eta} H_\nu^{(2)}(-c_T k\eta), \tag{4.18}$$

where

$$\nu = \frac{1}{2}\sqrt{4\left(2+3\epsilon - \frac{\mu_\chi^2}{(1-\epsilon)^2 H_0^2}\right)+1} = \frac{3}{2}\sqrt{1+\frac{4}{3}\epsilon - \frac{4}{9}\frac{\mu_\chi^2}{(1-\epsilon)^2 H_0^2}}. \tag{4.19}$$

We can look at this expression for ν in two regimes: for large graviton mass, $4\mu_\chi \gg 9H_0$, the final term dominates and we have an imaginary ν which we do not want. Instead, we take the small mass limit, $4\mu_\chi \ll 9H_0^2$ so that we can Taylor expand this expression. We also know that $\epsilon \ll 1$ so we can simultaneously Taylor expand this function to obtain the approximate form

$$\nu \simeq \frac{3}{2} + \epsilon - \frac{1}{3}\frac{\mu_\chi^2}{H_0^2}\left(1+\frac{4}{3}\epsilon\right). \tag{4.20}$$

We can see that in the massless limit, $\mu_\chi \to 0$, we have $\nu \simeq \epsilon + 3/2$ which is the inflationary solution (2.55). We also note that we have included the cross-terms between μ_χ^2 and ϵ, but have not gone to second order in μ_χ^2 and ϵ themselves. The cross-term does not add much further information but we will leave it in for completeness.

The next step is to find the constants $c_1(k)$ and $c_2(k)$, which can be done by applying the large-argument limit $-c_T k\eta \gg 1$, corresponding to the subhorizon limit in de Sitter space (see Appendix A.2.4), and comparing this with the subhorizon limit we found before (Equation (4.12)). Like in Chapter 2, we use the large-argument limits of the two Hankel functions (Appendix B.2.2) and from this we see that again $c_2(k) = 0$ and

$$c_1(k) = \frac{\sqrt{\pi}}{2}\exp\left[\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)\right]. \tag{4.21}$$

[3] If we want to be more consistent, we should also approximate $1-\epsilon \simeq 1$ and $2+3\epsilon \simeq 2$ so that we are entirely in pure de Sitter.

The general solution is then given by

$$\begin{aligned} v_k(\eta) &= \frac{\sqrt{\pi}}{2} \exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{-\eta} H_\nu^{(1)}(-c_T k\eta) \\ \text{with} \quad \nu &\simeq \frac{3}{2} + \epsilon - \frac{1}{3}\frac{\mu_\chi^2}{H_0^2}\left(1 + \frac{4}{3}\epsilon\right). \end{aligned} \tag{4.22}$$

4.2.2 General background with $s = -2$

We can then look at some cases where the graviton mass is not constant but instead follows the power-law $m_\chi^2(\eta) = \mu_\chi^2 a^s$ with $s \neq 0$ some constant. We also go back to the more general $a(\eta) = (H_0\eta/p)^p$ again (see Appendix A.2.1), so we can also see what happens during a radiation- or matter-dominated era, i.e., after the end of inflation with its quasi-exponential expansion.

In these cases, the differential equation (4.10) becomes

$$v_k''(\eta) + \left[c_T^2 k^2 + \mu_\chi^2 \left(\frac{H_0\eta}{p}\right)^{(2+s)p} - \frac{p(p-1)}{\eta^2}\right] v_k(\eta) = 0. \tag{4.23}$$

As before, this is not generally solvable for arbitrary s and p. We can therefore choose s and p such that the second term is grouped either with the first or with the final term, i.e., we have either an η^0 or η^{-2} power-law in the second term. We first focus on the first case where $s = -2$ so that the second term simplifies to μ_χ^2, and the differential equation becomes

$$v_k''(\eta) + \left[c_T^2 k^2 + \mu_\chi^2 - \frac{1}{\eta^2} p(p-1)\right] v_k(\eta) = 0. \tag{4.24}$$

This has the general solution

$$\begin{aligned} v_k(\eta) &= c_1(k)\sqrt{\pm\eta}\, H_{p-1/2}^{(1)}\left(\pm\eta\sqrt{c_T^2 k^2 + \mu_\chi^2}\right) \\ &\quad + c_2(k)\sqrt{\pm\eta}\, H_{p-1/2}^{(2)}\left(\pm\eta\sqrt{c_T^2 k^2 + \mu_\chi^2}\right), \end{aligned} \tag{4.25}$$

where the $+$ or $-$ signs depend on the value of p: if $p = -1$ (de Sitter), sub- and super-horizon is given by the limits $-c_T k\eta \gg 1$ and $-c_T k\eta \ll 1$, respectively. On the other hand, for $p = 1$ (radiation domination), we have $c_T k\eta \gg 1$ and $c_T k\eta \ll 1$, respectively, and the same holds for matter domination ($p = 2$), just with an extra factor $1/2$ (see Appendix A.2.4). We see that this therefore changes our definition of large- and small-argument limits of the Hankel functions, so we must change the arguments accordingly. Also, in order to even be able to take this limit, we have to assume that $\mu_\chi^2 \ll c_T^2 k^2$ so the argument approximately reduces to the desired $\pm c_T k\eta$. In order to continue, we must therefore treat the three cases $p = -1, 1, 2$ separately.

Moreover, we cannot compare our subhorizon limit (4.12) with the subhorizon limit of the above solution, because we now also have the mass in our argument. Instead, we have the subhorizon solution

$$v_k(\eta) \simeq \frac{1}{\sqrt{2\tilde{\omega}_k}} e^{-i\tilde{\omega}_k\eta} \quad \text{with} \quad \tilde{\omega}_k^2 = c_T^2 k^2 + \mu_\chi^2, \quad c_T k \gg aH. \tag{4.26}$$

The mass term has to be included in the subhorizon solution because this term does not have a time-dependence anymore, and this alters the frequency $\omega_k = c_T k$ to $\tilde{\omega}_k$. Let us then look at the three cases $p = -1, 1, 2$.

(Quasi-)de Sitter case

In the de Sitter case, $p = -1$, we take the minus signs in the solution (4.25). Assuming for the subhorizon limit that the arguments can be approximated as $\simeq -c_T k\eta$, we can take the large-argument limits of the Hankel functions again and see that $c_2(k) = 0$ and $c_1(k) = (\sqrt{\pi}/2)e^{i\pi p/2}$, so that the general solution becomes

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i\pi p/2} \sqrt{-\eta} H^{(1)}_{p-1/2}\left(-\eta\sqrt{c_T^2 k^2 + \mu_\chi^2}\right). \tag{4.27}$$

We can then plug in $p = -1$ and use $H^{(1)}_{-3/2}(x) = i\sqrt{2/\pi}e^{ix}(x+i)x^{-3/2}$ to obtain

$$\begin{aligned} v_k(\eta) &= \frac{\sqrt{\pi}}{2} e^{-i\pi/2} \sqrt{-\eta} \cdot i\sqrt{\frac{2}{\pi}} e^{-i\eta\tilde{\omega}_k}(-\eta\tilde{\omega}_k + i)(-\eta\tilde{\omega}_k)^{-3/2} \\ &= \frac{1}{\sqrt{2}} e^{-i\eta\tilde{\omega}_k} \frac{\eta\tilde{\omega}_k - i}{\eta\tilde{\omega}_k^{3/2}} \\ &= \frac{1}{\sqrt{2\tilde{\omega}_k}}\left(1 - \frac{i}{\eta\tilde{\omega}_k}\right) e^{-i\eta\tilde{\omega}_k}, \end{aligned} \tag{4.28}$$

where we have written $\tilde{\omega}_k^2 = c_T^2 k^2 + \mu_\chi^2$. Notice that we can also take a quasi-de Sitter background with $p = -1/(1-\epsilon)$, but in this case we cannot write the Hankel function out because the order $p - 1/2 \simeq -3/2 - \epsilon$ depends on ϵ. In this case, we simply have the solution (4.27) with $p = -1/(1-\epsilon)$.

Radiation-dominated case

In the radiation-dominated case, $p = 1$, we take the plus signs in the solution (4.25). Again approximating the argument as $\simeq c_T k\eta$, we can take the subhorizon limit $c_T k\eta \gg 1$, compare it with Equation (4.26), and obtain the constants $c_1(k) = 0$ and $c_2(k) = (\sqrt{\pi}/2)e^{-i\pi p/2}$, such that the full solution in this case becomes

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} e^{-i\pi p/2} \sqrt{\eta} H^{(2)}_{p-1/2}\left(\eta\sqrt{c_T^2 k^2 + \mu_\chi^2}\right). \tag{4.29}$$

Plugging in $p = 1$ and using $H^{(2)}_{1/2}(x) = i\sqrt{2/\pi x}e^{-ix}$, we have

$$\begin{aligned} v_k(\eta) &= \frac{\sqrt{\pi}}{2} e^{-i\pi/2} \sqrt{\eta} \cdot i \sqrt{\frac{2}{\pi \eta \tilde{\omega}_k}} e^{-i\eta\tilde{\omega}_k} \\ &= \frac{1}{\sqrt{2\tilde{\omega}_k}} e^{-i\eta\tilde{\omega}_k}, \end{aligned} \tag{4.30}$$

which is the same as the subhorizon solution (4.26). This is also logical, because when $p = 1$, then $p(p-1) = 0$, the final term is absent in the differential equation (4.10), and we go back to the subhorizon differential equation $v_k''(\eta) + \tilde{\omega}_k v_k(\eta) = 0$, which indeed has the solution (4.26).

Matter-dominated case

Lastly, we have the matter-dominated case, $p = 2$, for which we also take the plus signs in Equation (4.25). The general solution is therefore given by Equation (4.29) and we can plug in $p = 2$ and use $H^{(2)}_{3/2}(x) = \sqrt{2/\pi}e^{-ix}(i - x)x^{-3/2}$ to find

$$\begin{aligned} v_k(\eta) &= \frac{\sqrt{\pi}}{2} e^{-i\pi} \sqrt{\eta} \cdot \sqrt{\frac{2}{\pi}} e^{-i\eta\tilde{\omega}_k} \left(i - \eta\tilde{\omega}_k\right) \left(\eta\tilde{\omega}_k\right)^{-3/2} \\ &= \frac{1}{\sqrt{2}} e^{-i\eta\tilde{\omega}_k} \frac{\eta\tilde{\omega}_k - i}{\eta\tilde{\omega}_k^{3/2}} \\ &= \frac{1}{\sqrt{2\tilde{\omega}_k}} \left(1 - \frac{i}{\eta\tilde{\omega}_k}\right) e^{-i\eta\tilde{\omega}_k}, \end{aligned} \tag{4.31}$$

which we observe to be the same solution as in the de Sitter case. This is because $p(p-1) = 2$ both for $p = -1$ and for $p = 2$, so the solutions must be identical as well.[4]

4.2.3 General background with $s = -2(1 + 1/p)$

Having gone over all possibilities for the case $s = -2$, we can look at the other case presented at the beginning of the previous section, $s = -2(1 + 1/p)$. This ensures that the mass-term has a $1/\eta^2$-dependence so it can be grouped with the final term in the differential equation (4.23), and we write

$$v_k''(\eta) + \left[c_T^2 k^2 - \frac{1}{\eta^2}\left(p(p-1) - \frac{\mu_\chi^2 p^2}{H_0^2}\right)\right] v_k(\eta) = 0. \tag{4.32}$$

The case in Section 4.2.1 looks like a specific case of this if we take $p \simeq -1 - \epsilon \simeq -1$, although this would imply that $s = -2\epsilon$, while before we took it to be zero and approximated the exponent $2 + 2\epsilon \simeq 2$. Solving this more general case goes analogously to that

[4]Also, the minus signs all cancel in such a way that the solutions are identical, making stronger the claim that we can freely add such minus signs to make it easier to take super- and subhorizon limits, as long as they are added consistently.

case, but again we have to distinguish between the + and − signs inside the arguments, which depends again on if we take (quasi-)de Sitter or radiation or matter domination.

(Quasi-)de Sitter case

For $p < 0$ (i.e., $p = -1$ or $p \simeq -1 - \epsilon$), we have

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} \exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{-\eta} H_\nu^{(1)}(-c_T k\eta), \tag{4.33}$$

where now we have, since $p < 0$,

$$\begin{aligned} \nu &= \frac{1}{2}\sqrt{4\left(p(p-1) - \frac{\mu_\chi^2 p^2}{H_0^2}\right) + 1} \\ &= \frac{1}{2}\sqrt{(2p-1)^2 - \left(\frac{2\mu_\chi p}{H_0}\right)^2} \\ &= \frac{1-2p}{2}\sqrt{1 - \left(\frac{2\mu_\chi p}{(1-2p)H_0}\right)^2}. \end{aligned} \tag{4.34}$$

For $\mu_\chi \gg H_0$, this gives an imaginary order, which we do not want, so we take $\mu_\chi \ll H_0$ so that

$$\nu \simeq \frac{1}{2} - p - \frac{\mu_\chi^2 p^2}{(1-2p)H_0^2}. \tag{4.35}$$

In the quasi-de Sitter case, with $p \simeq -1 - \epsilon$, this would give

$$\nu \simeq \frac{3}{2} + \epsilon - \frac{1}{3}\frac{\mu_\chi^2}{H_0^2}\left(1 + \frac{4}{3}\epsilon\right), \tag{4.36}$$

which is the same as we found before in Equation (4.20). Again, we stress that this quasi-de Sitter case is not the same as the one described in Section 4.2.1, as $s = -2(1 + 1/p)$ with $p \simeq -1 - \epsilon$ implies that $s = -2\epsilon$, which now also depends on ϵ, and so m_χ does as well. The mass is therefore not constant, as we assumed in Section 4.2.1, but still, it shows that if we take a slowly-varying graviton mass we can still find a solution. A constant mass during inflation is still not in general solvable.

Radiation- and matter-dominated case

For $p > 0$ (i.e., $p = 1$ or $p = 2$), we have instead the solution

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} \exp\left[-\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{\eta} H_\nu^{(2)}(c_T k\eta), \tag{4.37}$$

where since $p > 0$ now, we have

$$\nu = \frac{1}{2}\sqrt{4\left(p(p-1) - \frac{\mu_\chi^2 p^2}{H_0^2}\right) + 1}$$
$$= \frac{1}{2}\sqrt{(2p-1)^2 - \left(\frac{2\mu_\chi p}{H_0}\right)^2}$$
$$= \frac{2p-1}{2}\sqrt{1 - \left(\frac{2\mu_\chi p}{(2p-1)H_0}\right)^2}. \tag{4.38}$$

$\mu_\chi \gg H_0$ again gives an imaginary ν, so we take $\mu_\chi \ll H_0$ so that

$$\nu \simeq p - \frac{1}{2} - \frac{\mu_\chi^2 p^2}{(2p-1)H_0^2}. \tag{4.39}$$

4.2.4 Radiation-dominated background with general power-law mass

For our final viable option, let us return to a more general power-law graviton mass, with, at least for now, unspecified s. We choose a radiation-dominated background with $p = 1$, because then $p(p-1) = 0$ and $a(\eta) = H_0\eta$ (see Appendix A.2.1), and the differential equation (4.8) is reduced to

$$v_k''(\eta) + \left[c_T^2 k^2 + \mu_\chi^2 (H_0\eta)^{2+s}\right] v_k(\eta) = 0. \tag{4.40}$$

Once again, this is not solvable for any s so we have to make some choices. At least without the final term, proportional to $1/\eta^2$, we will now obtain solutions in terms of different functions than the Bessel functions. In the following, we will restrict ourselves to integer choices for s. Also, it is worth noting that we take $s \leq 0$, which makes sure that the mass decreases in time and we reduce to massless GR at late times. It also turns out that taking $s \geq 1$ will give polynomial solutions with an infinite amount of terms, so it is not worth looking in this regime.

$s = 0$

Taking $s = 0$ in Equation (4.40), corresponding with a constant graviton mass, the solution is given by the parabolic cylinder function $D_n(x)$ (see Chapter 12 of [36]),

$$v_k(\eta) = c_1(k) D_{-n-1/2}\left[(i+1)\sqrt{\mu_\chi H_0}\eta\right] + c_2(k) D_{n-1/2}\left[(i-1)\sqrt{\mu_\chi H_0}\eta\right], \tag{4.41}$$

with $n = ic_T^2 k^2 / 2\mu_\chi H_0$. Because of the form of the large-argument expansions of the function D_ν, we were not able to find expressions for the constants $c_1(k)$ and $c_2(k)$, so we will keep the general solution as above. We will therefore also not be able to find the power spectrum in this case.

$s = -1$

For $s = -1$, the solution is of the form

$$v_k(\eta) = c_1(k)\,\mathrm{Ai}\left(-\frac{c_T^2k^2 + \mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right) + c_2(k)\,\mathrm{Bi}\left(-\frac{c_T^2k^2 + \mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right), \tag{4.42}$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions (see Chapter 9 of [36]). Using the large-argument expansions of these functions, the approximation $\mu_\chi^2 H_0\eta \ll c_T^2k^2$, some Taylor expansions to order $\mathcal{O}(\eta)$, and some rewriting, finally gives us the values $c_2(k) = 0$ and $c_1(k)$ such that[5]

$$v_k(\eta) \simeq \left(\frac{4\pi^2}{\mu_\chi^2 H_0}\right)^{1/4}\left(-\mu_\chi^2 H_0\right)^{1/12}\exp\left[\frac{2ic_T^3k^3}{3\mu_\chi^2 H_0}\right]\mathrm{Ai}\left(-\frac{c_T^2k^2 + \mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right). \tag{4.43}$$

$s = -2$

With $s = -2$, we completely revert back to the subhorizon solution (4.30),

$$v_k(\eta) = \frac{1}{\sqrt{2\tilde{\omega}_k}}e^{-i\eta\tilde{\omega}_k}, \quad \text{with} \quad \tilde{\omega}_k^2 = c_T^2k^2 + \mu_\chi^2. \tag{4.44}$$

This case has been discussed in Section 4.2.2, so we will move on.

$s = -3$

Taking $s = -3$ gives solutions in terms of the so-called "confluent hypergeometric functions", $M(a,b,x)$ and $U(a,b,x)$ (see Chapter 13 of [36]),

$$\begin{aligned} v_k(\eta) = {} & c_1(k)\frac{\eta}{H_0}e^{ic_Tk\eta}M\left(1 - \frac{\mu_\chi^2}{2ic_TkH_0}, 2, 2ic_Tk\eta\right) \\ & + c_2(k)\frac{\eta}{H_0}e^{ic_Tk\eta}U\left(1 - \frac{\mu_\chi^2}{2ic_TkH_0}, 2, 2ic_Tk\eta\right) \end{aligned} \tag{4.45}$$

Because of the form of the large-argument expansions for the functions M and U, we cannot easily find equations for the constants $c_1(k)$ and $c_2(k)$, so we will keep these as constants and the general solution as above. Because of this, we will also not be able to calculate the power spectrum in this case.

$s = -4$

Finally, we can take $s = -4$, which gives us solutions in terms of the familiar Hankel functions,

$$v_k(\eta) = c_1(k)\sqrt{\eta}H_\nu^{(1)}(c_Tk\eta) + c_2(k)\sqrt{\eta}H_\nu^{(2)}(c_Tk\eta), \tag{4.46}$$

[5] See Calculation **6.** in Appendix C.2.

where $\nu = \frac{1}{2}\sqrt{1 - 4\mu_\chi^2/H_0^2}$. Like before, we can use the large-argument expansions of the Hankel functions, which gives the solution

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} \exp\left[-\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{\eta} H_\nu^{(2)}(c_T k\eta). \tag{4.47}$$

This is the same as the solution (4.37), since $s = -2(1 + 1/p)$ with $p = 1$ gives $s = -4$.

For $s \leq -5$, we cannot find analytic solutions anymore so we will stop here. In theory, we could go on and keep defining new solutions with specific large- and small-argument properties, power series, derivatives, and all other properties, but this is beyond the scope of this project and probably more of a mathematical physics problem.

In the same spirit, it is also not worth it to look in more detail into the matter-dominated background solutions. In this case, we have the third term again, proportional to $\propto 1/\eta^2$, so we cannot find solutions for a general s anymore. The two cases for which we *can* still find solutions are $s = -2$ and $s = -3$, and they have been discussed in Sections 4.2.2 and 4.2.3 (with $p = 2$), respectively.

4.2.5 Different graviton mass functions

Let us now move on from the power-law graviton masses, and look at other possible forms. A promising Ansatz is the decaying exponential, $m_\chi^2(\eta) = \mu_\chi^2 e^{-c\eta}$ (for c some constant with dimensions $1/[\eta]$), since this reduces to massless gravity at late times. On a power-law background, $a(\eta) = (\eta/\eta_0)^p$, the differential equation (4.8) now becomes

$$v_k''(\eta) + \left[c_T^2 k^2 + \mu_\chi^2 \left(\frac{\eta}{\eta_0}\right)^p e^{-c\eta} - \frac{p(p-1)}{\eta^2}\right] v_k(\eta) = 0. \tag{4.48}$$

However, it is not possible to analytically solve this. Even setting $p = 1$ so that the final term disappears still does not make it solvable. The same is true if we take any other function for $m_\chi^2(\eta)$: because of the power-law behaviour of the scale factor, we can only really solve this in some specific cases if we also have a power-law for the graviton mass. The only possibility to potentially have a solution for an exponentially decaying graviton mass would be if we are in the superhorizon limit, i.e. the first term, $c_T^2 k^2$ is absent, and $p = 1$ so that the final term is zero as well. But even then, we have a differential equation of the form

$$v_k''(\eta) + \mu_\chi^2 \eta_0^{-1} \eta e^{-c\eta} v_k(\eta) = 0, \tag{4.49}$$

which still is not solvable. So unfortunately, we have to let this possibility rest here.

4.3 The power spectrum

From the solutions for $v_k(\eta)$ that we have just found in the different cases, it is fairly straightforward to calculate the power spectrum $\mathcal{P}_\chi(k)$, as this follows the same steps as Chapter 2. We first calculate the modulus $|v_k(\eta)|$ and from there $|\chi_k(\eta)|$, and then via $\mathcal{P}_\chi(k) = 2k^3|\chi_k(\eta)|^2/\pi^2$ (see Equation C.41) the power spectrum. With Equation (4.5), we can write $|\chi_k(\eta)| = \sqrt{2}|v_k(\eta)|/M_{\rm Pl}a(\eta)$ and so the power spectrum is calculated directly from the solutions in Section 4.2 via

$$\mathcal{P}_\chi(k) = \frac{4k^3}{\pi^2 M_{\rm Pl}^2 a^2(\eta)}|v_k(\eta)|^2. \tag{4.50}$$

4.3.1 The (quasi-)de Sitter case

Let us start with the (quasi-)de Sitter case, which has the solution (4.22),

$$\begin{aligned} v_k(\eta) &= \frac{\sqrt{\pi}}{2}\exp\left[\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)\right]\sqrt{-\eta}H_\nu^{(1)}(-c_T k\eta) \\ \text{with} \quad \nu &\simeq \frac{3}{2}+\epsilon-\frac{1}{3}\frac{\mu_\chi^2}{H_0^2}\left(1+\frac{4}{3}\epsilon\right). \end{aligned} \tag{4.51}$$

The power spectrum is then calculated to be[6]

$$\mathcal{P}_\chi(k) = -\frac{k^3\eta}{\pi M_{\rm Pl}^2 a^2}\left|H_\nu^{(1)}(-c_T k\eta)\right|^2, \tag{4.52}$$

which in the superhorizon limit $-c_T k\eta \ll 1$ and with $\eta = -1/(1-\epsilon)aH$ can be calculated to be

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^{n_T} \quad \text{with} \quad n_T = -2\epsilon + \frac{2\mu_\chi^2}{3H_0^2}\left(1+\frac{4}{3}\epsilon\right). \tag{4.53}$$

As we have already said before, we stress that this is not the exact quasi-de Sitter solution, but for a more thorough discussion about this, see Section 4.2.1 and Chapter 5. Nevertheless, we can see from the spectral tilt n_T that we can have the desired blue-tilted spectrum if we have a graviton mass $\mu_\chi^2 > 0$, provided the slow-roll parameter is small enough that it does not cancel this contribution. The pure de Sitter case ($\epsilon = 0$) will always have a blue spectrum for positive graviton mass, as then the spectral tilt is simply $n_T = 2\mu_\chi^2/3H_0^2$.

4.3.2 General background, $s = -2$

The case in which we have a general background with the scale factor dependence of the graviton mass parametrised by choosing $s = -2$, has the solutions (4.27) and (4.29),

$$v_k(\eta) = \frac{\sqrt{\pi}}{2}e^{i\pi p/2}\sqrt{-\eta}H_{p-1/2}^{(1)}\left(-\eta\sqrt{c_T^2k^2+\mu_\chi^2}\right), \quad \text{for} \quad p<0, \quad \text{and} \tag{4.54a}$$

$$v_k(\eta) = \frac{\sqrt{\pi}}{2}e^{-i\pi p/2}\sqrt{\eta}H_{p-1/2}^{(2)}\left(\eta\sqrt{c_T^2k^2+\mu_\chi^2}\right), \quad \text{for} \quad p>0. \tag{4.54b}$$

[6]See Calculation **1.** in Appendix C.2.

Version of June 29, 2023– Created June 29, 2023 - 18:47

Here, we write $\tilde{\omega}_k^2 \equiv c_T^2 k^2 + \mu_\chi^2$. We can consider the three components $p = -1, 1, 2$ (de Sitter, radiation domination, matter domination) separately, as well as the quasi-de Sitter case $p = -1/(1-\epsilon) \simeq -1-\epsilon$.

Quasi-de Sitter

$p = -1/(1-\epsilon)$ gives the solution

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} e^{-i\pi/2(1-\epsilon)} \sqrt{-\eta} H^{(1)}_{-1/(1-\epsilon)-1/2} \left(-\eta \sqrt{c_T^2 k^2 + \mu_\chi^2} \right), \tag{4.55}$$

from which the power spectrum in the superhorizon limit $-c_T k \eta \ll 1$ can be calculated to be[7]

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{9\pi^2 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{aH} \right)^3 \left(\frac{\sqrt{c_T^2 k^2 + \mu_\chi^2}}{aH} \right)^{(3-\epsilon)/(1-\epsilon)} . \tag{4.56}$$

We can then look in the two regimes $c_T k \gg \mu_\chi$ and $c_T k \ll \mu_\chi$; the first one gives

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{9\pi^2 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{aH} \right)^{n_T} \quad \text{with} \quad n_T = \frac{6-4\epsilon}{1-\epsilon} \simeq 6 + 2\epsilon, \tag{4.57}$$

and $c_T k \ll \mu_\chi$ gives

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{9\pi^2 M_{\rm Pl}^2 c_T^3} \left(\frac{m}{aH} \right)^{(3-\epsilon)/(1-\epsilon)} \left(\frac{c_T k}{aH} \right)^3 \tag{4.58}$$

In both cases, we have a blue spectrum ($n_T > 0$) but for small graviton mass (or large wavenumbers), it is about twice as steep as for large mass (or small wavenumbers).

De Sitter and matter domination

$p = -1$ and $p = 2$ give the equal solutions (4.28) and (4.31),

$$v_k(\eta) = \frac{1}{\sqrt{2\tilde{\omega}_k}} \left(1 - \frac{i}{\eta \tilde{\omega}_k} \right) e^{-i\eta\tilde{\omega}_k}. \tag{4.59}$$

This solution gives a power spectrum dependent on the value of p,[8]

$$\mathcal{P}_\chi(k) = \frac{2H^2 (c_T k)^3}{\pi^2 M_{\rm Pl}^2 c_T^3 (aH)^2} \left(1 + \frac{1}{p^2} \frac{(aH)^2}{c_T^2 k^2 + \mu_\chi^2} \right) \frac{1}{\sqrt{c_T^2 k^2 + \mu_\chi^2}}. \tag{4.60}$$

[7] See Calculation **2.** in Appendix C.2.
[8] See Calculation **3.** in Appendix C.2.

We take the superhorizon limit, $c_T k \ll aH$, so that the second term in brackets dominates over the first. Also, we can split it up again into two cases where $c_T k \gg \mu_\chi$ or $c_T k \ll \mu_\chi$. The first gives

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{p^2 \pi^2 M_{\rm Pl}^2 c_T^3}, \tag{4.61}$$

which is the same as before (Equation (4.53)) in pure de Sitter ($p = -1$ and $\epsilon = 0$) and with $\mu_\chi \ll H$. In the case where $c_T k \ll \mu_\chi$, we have

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{p^2 \pi^2 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{\mu_\chi} \right)^3, \tag{4.62}$$

so we see that we now compare the wave number $c_T k$ with the graviton mass μ_χ instead of the conformal Hubble parameter aH. We also notice that this is not scale-invariant and has a spectral tilt $n_T = 3 > 0$, which means that the spectrum is blue-tilted. For the matter-dominated power spectrum, the only difference is that we divide by $p^2 = 4$ (since $p = 2$ then), which does not alter the scale-dependence of the power spectrum.

Radiation domination

For $p = 1$, we have the solution (4.30),

$$v_k(\eta) = \frac{1}{\sqrt{2\tilde{\omega}_k}} e^{-i\eta\tilde{\omega}_k}, \tag{4.63}$$

which gives the power spectrum[9]

$$\mathcal{P}_\chi(k) = \frac{2H^2 (c_T k)^3}{\pi^2 M_{\rm Pl}^2 c_T^3 (aH)^2} \frac{1}{\sqrt{c_T^2 k^2 + \mu_\chi^2}}. \tag{4.64}$$

For $c_T k \gg \mu_\chi$, this becomes

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{aH} \right)^2, \tag{4.65}$$

which also has a spectral tilt $n_T = 2 > 0$ so that it is blue-tilted. For $c_T k \ll \mu_\chi$, we have instead

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3} \frac{aH}{\mu_\chi} \left(\frac{c_T k}{aH} \right)^3, \tag{4.66}$$

which has a steeper tilt $n_T = 3$, the same as the $c_T k \ll \mu_\chi$ case for (quasi-)de Sitter and matter domination.

[9]See Calculation **4.** in Appendix C.2.

Version of June 29, 2023– Created June 29, 2023 - 18:47

4.3.3 General background, $s = -2(1+1/p)$

The other general background case that was solvable was the one where $s = -2(1+1/p)$ so that the mass term could be grouped with the a''/a-term. The solution depends again on the sign of p and is given by Equations (4.33) and (4.37),

$$v_k(\eta) = \frac{\sqrt{\pi}}{2}\exp\left[\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)\right]\sqrt{-\eta}H_\nu^{(1)}(-c_T k\eta), \quad \text{for} \quad p<0, \quad \text{and} \tag{4.67a}$$

$$v_k(\eta) = \frac{\sqrt{\pi}}{2}\exp\left[-\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)\right]\sqrt{\eta}H_\nu^{(2)}(c_T k\eta), \quad \text{for} \quad p>0. \tag{4.67b}$$

The order ν can be approximated as

$$\nu \simeq \mp\left(p-\frac{1}{2}-\frac{\mu_\chi^2 p^2}{(2p-1)H_0^2}\right) \tag{4.68}$$

where the minus sign is for $p<0$ and the plus sign for $p>0$. With the small-argument limit of the Hankel functions, we calculate the power spectrum, where the only dependence on the sign of p is in the order ν,[10]

$$\mathcal{P}_\chi(k) \simeq \frac{2^{2\nu}H^2}{\pi^3 M_{\text{Pl}}^2 c_T^3}\Gamma^2(\nu)p^{1-2\nu}\left(\frac{c_T k}{aH}\right)^{3-2\nu}, \tag{4.69}$$

We can calculate this explicitly for $p = -1, 1, 2$ (or $s = 0, -4, -3$),

$$\mathcal{P}_\chi(k) = \frac{2H^2}{\pi^2 M_{\text{Pl}}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^{n_T}, \quad \text{where} \quad n_T = \frac{2\mu_\chi^2}{3H_0^2}, \quad p=-1, \tag{4.70a}$$

$$\mathcal{P}_\chi(k) = \frac{2H^2}{\pi^2 M_{\text{Pl}}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^{n_T}, \quad \text{where} \quad n_T = 2+\frac{2\mu_\chi^2}{H_0^2}, \quad p=1, \tag{4.70b}$$

$$\mathcal{P}_\chi(k) = \frac{H^2}{2\pi^2 M_{\text{Pl}}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^{n_T}, \quad \text{where} \quad n_T = \frac{8\mu_\chi^2}{3H_0^2}, \quad p=2. \tag{4.70c}$$

We see that in each case, if $\mu_\chi > 0$, we have a blue-tilted spectrum. The first case corresponds to the approximately quasi-de Sitter case (Equation (4.53)) if we set $\epsilon = 0$ for pure de Sitter.

For the exact quasi-de Sitter case $p = -1/(1-\epsilon) \simeq -1-\epsilon$, the solution reduces to the solution of the constant mass so the power spectrum will be precisely given by the solution (4.53). As already noted before, the difference between the two cases is that the mass is now time-dependent, since $s = -2\epsilon$ now, whereas it was constant before.

[10] See Calculation **5.** in Appendix C.2.

4.3.4 Radiation-dominated background, $s = -1$

The final case that we can look at is the case for a radiation-dominated background with other values for s. We have done $s = -2$ and $s = -4$ (these were the radiation-dominated cases in Sections 4.3.2 and 4.3.3), and $s = 0$ and $s = -3$ did not give satisfactory solutions (they still have constants that we cannot calculate), but for $s = -1$ we did obtain a solution, namely Equation (4.43),

$$v_k(\eta) \simeq \left(\frac{4\pi^2}{\mu_\chi^2 H_0}\right)^{1/4} \left(-\mu_\chi^2 H_0\right)^{1/12} \exp\left[\frac{2ic_T^3 k^3}{3\mu_\chi^2 H_0}\right] \mathrm{Ai}\left(-\frac{c_T^2 k^2 + \mu_\chi^2 H_0 \eta}{(-\mu_\chi^2 H_0)^{2/3}}\right). \tag{4.71}$$

$\mathrm{Ai}(x)$ is the Airy function which can be converted to a Hankel function. From this, we can take the superhorizon limit and rewrite the equation a lot, after which we eventually end up with the general power spectrum,[11]

$$\mathcal{P}_\chi(k) = \frac{2H^2}{\pi^3 M_{\mathrm{Pl}}^2 c_T^3} \left(\frac{c_T k}{aH}\right)^3 \left(\frac{aH^2}{3\mu_\chi^2}\right)^{1/3} \Gamma^2\left(\frac{1}{3}\right) \tag{4.72a}$$

$$\simeq 1.58 \frac{2H^2}{\pi^2 M_{\mathrm{Pl}}^2 c_T^3} \left(\frac{aH^2}{\mu_\chi^2}\right)^{1/3} \left(\frac{c_T k}{aH}\right)^3. \tag{4.72b}$$

We have also used the Friedmann equation during the radiation-dominated era, $H/H_0 = 1/a^2$ if $\Omega_\gamma = 1$. The above power spectrum has a spectral tilt $n_T = 3 > 0$, which means that this is also a blue-tilted spectrum. In fact, the tilt is the same as for the case $s = -2$ when $c_T k \ll \mu_\chi$.

Table 4.1 on the next page shows a summary of the various spectral tilts for the different values of p and s and for the different regimes. We note that we have put the approximate spectral tilt for quasi-de Sitter with $s = 0$ in the table, even though strictly speaking this is not possible. The tilt for $s = -2\epsilon$ is the same as this approximate tilt.

[11] See Calculation **7.** in Appendix C.2.

Table 4.1: *The spectral tilt n_T for power-law scale factors $a(\eta) \propto \eta^p$ and graviton masses $m^2(\eta) = \mu_\chi^2 a^s(\eta)$. "NP" means "not possible", "NC" means "not calculated".*

Graviton mass degree s	n_T	**Scale factor degree p**: $p = -1/(1-\epsilon)$	$p = -1$	$p = 1$	$p = 2$
	$s = 0$	$\sim -2\epsilon + \frac{2\mu_\chi^2}{3H_0^2}\left(1 + \frac{4}{3}\epsilon\right)$	$\frac{2\mu_\chi^2}{3H_0^2}$	NC	NP
	$s = -1$	NP	NP	3	NP
	$s = -2$	$\begin{cases} \sim 6 + 2\epsilon, & c_T k \gg \mu_\chi \\ 3, & c_T k \ll \mu_\chi \end{cases}$	$\begin{cases} 0, & c_T k \gg \mu_\chi \\ 3, & c_T k \ll \mu_\chi \end{cases}$	$\begin{cases} 2, & c_T k \gg \mu_\chi \\ 3, & c_T k \ll \mu_\chi \end{cases}$	$\begin{cases} 0, & c_T k \gg \mu_\chi \\ 3, & c_T k \ll \mu_\chi \end{cases}$
	$s = -3$	NP	NP	NC	$\frac{8\mu_\chi^2}{3H_0^2}$
	$s = -4$	NP	NP	$2 + \frac{2\mu_\chi^2}{H_0^2}$	NP
	$s = -2\epsilon$	$-2\epsilon + \frac{2\mu_\chi^2}{3H_0^2}\left(1 + \frac{4}{3}\epsilon\right)$	NP	NP	NP

Chapter 5

Discussion and conclusion

In the previous chapter, we have shown that in some specific cases, we can find analytic expressions for the tensor power spectrum of the stochastic gravitational wave background due to an extra graviton mass. We will now compare our results with the literature and explain why the calculations that different authors perform are not always possible.

Let us start with the time-dependence of the graviton mass. There is not much to be found in the literature, but there are a few examples. First of all, let us note the power-law time-dependence used in [23], which we also used in Chapter 4. Although the authors of [23] do not use this to calculate the tensor power spectrum, but instead use it for the analysis of the different gravitational potentials and equations of motion, we have still decided to use such a time-dependence in our analysis.

The only actual time-dependence that we could find is adopted in Fujita et al. (2019, 2020) [34, 35], where the mass is assumed to be constant, $m(\eta) = m$, up until some time η_m in the radiation-dominated era, and zero afterwards. The scale factor is taken to be $a(\eta) = -1/H\eta$ during inflation which corresponds with a de Sitter background. After inflation (during reheating and further after that), the scale factor is taken as $a(\eta) = a_r\eta/\eta_r$, which corresponds to a radiation-dominated Universe, as expected. The power spectrum is then calculated in [34] in three phases, namely a massless phase and a mass-dominant phase after inflation (which are not as important for this discussion), but also what they call an "inflation phase", where the mass is constant and the scale factor evolves according to a de Sitter Universe. The calculation of the power spectrum in this case is equivalent to the calculations in Sections 4.2.1 and 4.3.1 of this work, but in reality, they do not apply at all to inflation because the background is not taken to be quasi-de Sitter, as it should, but it has to be approximated as pure de Sitter for us to be able to find analytic solutions. The authors of [34, 35] do mention that they approximate inflation as pure de Sitter, but they fail to mention the validity of this approximation, which does not exist.

It seems that we single out a small group of authors, but in fact, this problem is widespread amongst several authors and research groups. Sticking for now to a graviton with a constant mass, there are more examples, such as [26, 30–33], that all come

Version of June 29, 2023– Created June 29, 2023 - 18:47

to the same result. Focusing first on Bartolo et al. (2016a, 2016b) [30, 31], they find again the solution for $\chi_k(\eta)$ in terms of the Hankel function of the first kind, with the order $\nu = 9/4 - m_\chi^2/H^2$. This leads eventually to a power spectrum with a spectral tilt $n_T = 2m_\chi^2/3H^2$. However, this is again the product of a pure de Sitter background, where $a^2 \propto \eta^{-2}$ so that we can find the analytic Hankel function solution. Where [34, 35] explain that they have to assume a de Sitter approximation during inflation, [30] does not make any such assumption and it looks like they simply take this for granted. Luckily, [31] does explain that we need a quasi-de Sitter background during inflation and they even mention that "analytical considerations can be carried out in a quasi-de Sitter space, $\epsilon \ll 1$ for c_T and m_χ only mildly depending on time", although the authors then continue to take pure de Sitter and constant c_T and m_χ, and end up again with the spectral tilt $n_T = 2m_\chi^2/3H^2$. In fact, we have shown in Section 4.3.3 that when $m_\chi^2 \propto a^{-2\epsilon}$, it is possible to find analytic solutions so it is unclear why the authors of [31] have not included this case.

Let us next quickly consider the other three references. Riccardione et al. (2018) [33] again states that inflationary tensor modes can be approximated by using a de Sitter background and end up with the same result for the spectral tilt. Domènech et al. (2017) [32] even goes a step further and adds the spectral tilt for the quasi-de Sitter Universe (with a massless graviton) to the pure de Sitter Universe with a massive graviton, obtaining the spectral tilt $n_T \simeq -2\epsilon + 2m_\chi^2/3H^2$, the same that we found in Section 4.3.1. [30] does the same, but we stress again that this method cannot be used because we cannot add a quasi-de Sitter solution to a massive pure de Sitter one. Finally, Cannone et al. (2015) [26] finds a slightly different action for tensor fluctuations and therefore also a slightly different spectral tilt n_T, which also includes an interaction between the mass-term m_χ^2/H^2 and the quasi-de Sitter slow-roll parameter ϵ. They do not explain if they use a pure de Sitter approximation here and do not show any steps, simply saying that it is "easy to derive the expression for the power spectrum" from the given action. Also, they mention that "a (small) time-dependence for [the masses] would instead be expected, proportional to slow-roll parameters quantifying the departure from an exact de Sitter phase during inflation", but they neglect this time-dependence. As noted in the previous paragraph, adding such a time-dependence proportional to the slow-roll parameter does in fact help.

These references have all made use of a pure de Sitter background and a constant graviton mass, and sometimes also a tensor sound speed $c_T \neq 1$, and they all found the same results. However, there is also a class of authors that take a more general time-dependence of the graviton mass. Going in chronological order, we start with probably the most general analysis in Bessada et al. (2009) [37], who decide to numerically solve the differential equation. However, they do this not in a (quasi-)de Sitter background but rather with a Universe only containing radiation and matter. The same arguments apply however, and we have seen in Chapter 4 that also in these regimes, we cannot solve the differential equation for a general time-dependence of the graviton mass. Myung et al. (2014) [29] adopts a completely different massive gravity model (the "Einstein-Chern-Simons-Weyl" model), but the idea is the same. Inflation is again approximated by de Sitter expansion and the power spectra for scalar, vector, and tensor perturbations are calculated in this model. It seems that we cannot compare it with our research because of the different massive gravity model, but at the very end they take the general massive gravity model that we adopt in this work, and again obtain the same spectral index $n_T \simeq 2m_\chi^2/3H^2$, because of the pure de Sitter approximation.

And then there are two papers by Kuroyanagi et al. (2018) [38] and Hiramatsu et al. (2018) [39], that solve the differential equation without making any assumptions about the scale factor and take a general time-dependence for the graviton mass. They end up with a power spectrum

$$P_\chi(k) \propto \exp\left[-\int \frac{2m_\chi^2}{3H}\,\mathrm{d}t\right], \tag{5.1}$$

where the integral bounds are not important for us right now. The only assumption that was made here is that we are on superhorizon scales, $k \ll aH$, so that the only surviving term in the differential equation[1] is the mass term. [38] neatly explains the breakdown of the de Sitter approximation and therefore we have to take the superhorizon limit. A general solution outside of this limit is still not possible, however, the authors of [38] still end up at the spectral tilt $n_T = -2\epsilon + 2m_\chi^2/3H^2$, which is the quasi-de Sitter massless tilt added to the pure de Sitter massive tilt, as we saw before.

[39] remains more general and only takes in their calculations (which are put in the appendix as it is not the main goal of the paper) the superhorizon limit and $m_\chi^2 \ll H^2$, after which they arrive at the general power spectrum (5.1). They then take a specific time-dependence

$$\frac{2m_\chi^2}{3H^2} = n_{T*} + \beta\alpha\frac{\sinh(\alpha n)}{\cosh^2(\alpha n)} \tag{5.2}$$

(where $\mathrm{d}n = H\,\mathrm{d}t$ and α and β are some constants with $\alpha \lesssim 1$), after which they end up with a power-law $(k/k_f)^{n_{T*}}$ multiplied with a peaked factor due to the second term in the above time-dependence. With e.g. $\alpha = 0$ or $\beta = 0$, the power spectrum again has a spectral tilt $2m_\chi^2/3H^2$ which shows that this, once again, corresponds to a de Sitter background because the mass is then constant.

To summarise this discussion and conclude the research, we stress again that it is not possible to find an analytic solution to the differential equation (4.8) in the quasi-de Sitter background that describes inflation, if the graviton mass is taken to be constant. The best we can do is (1) to assume inflation can be approximated by a pure de Sitter Universe [26, 29–35], which is really not an option as it defeats the whole purpose of inflation, namely that it does not exactly follow exponential expansion, (2) to take the superhorizon limit [38, 39] and not worry about what happens in the general case around the horizon $k \sim aH$, (3) to numerically solve the differential equation [37] and accept that we cannot find analytic solutions, or (4) adopt a specific time-dependence so that $m_\chi^2 \propto a^{-2}$ or $\propto a^{-2\epsilon}$, after which we *can* find analytic solutions.

[1] The differential equation in question is slightly different from our Equation (4.8), in that the authors of [38, 39] divide by a^2 so that in the mass term, there is no scale factor anymore.

In this thesis, we have shown that in addition to the above solutions, we can also find analytic solutions in other specific cases, namely when looking at pure de Sitter or radiation or matter domination. For a power-law $m_\chi^2(\eta) = \mu_\chi^2 a^s(\eta)$, we can have a blue GW spectrum by taking $s = -2$ in de Sitter, radiation domination, and matter domination, by taking $s = -2(1 + 1/p)$ (with $p = -1$ for de Sitter, $p = 1$ for radiation domination, and $p = 2$ for matter domination) and a mass $\mu_\chi^2 > 0$, or in a radiation-dominated background with $s = -1$. In addition to the constant mass quasi-de Sitter case of Sections 4.2.1 and 4.3.1 (which is not completely valid because of the reasons explained in the discussion above), these constitute all the cases where a power-law graviton mass can lead to an analytical solution for the GW power spectrum.

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Appendices

Appendix A

Important concepts in cosmology

In this appendix, we will go over some concepts from cosmology that are relevant for this thesis. We will start with a short overview of the Friedmann equations coming from the Einstein equations in an FLRW Universe and derive the scale factor-dependence of the energy density. Then, we will derive some useful properties relating the scale factor, conformal time, and comoving wavenumber in different single-component Universes.

A.1 The Friedmann equations

A good starting point for this appendix will be Einstein's theory of General Relativity, which can be described by the Einstein-Hilbert action with an additional matter term,

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}_\mathrm{m} \right], \tag{A.1}$$

with $R = R^{\mu\nu} g_{\mu\nu}$ the Ricci scalar, $R^{\mu\nu}$ the Ricci tensor, $g = \det(g_{\mu\nu})$ the determinant of the metric tensor $g_{\mu\nu}$, and $\mathcal{L}_\mathrm{m}$ the Lagrangian density for matter. Varying this action with respect to the metric (see Chapter 4 of [15]), we arrive at the well-known Einstein equation,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \text{or} \quad G_{\mu\nu} = \frac{T_{\mu\nu}}{M_\mathrm{Pl}^2}. \tag{A.2}$$

Here, $T_{\mu\nu}$ is the energy-momentum tensor, given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}}, \tag{A.3}$$

with S_m the action for matter, $S_\mathrm{m} = \int \mathrm{d}^4x \sqrt{-g} \mathcal{L}_\mathrm{m}$. The next step is to adopt a metric. The most general metric describing a homogeneous and isotropic Universe is the FLRW

Version of June 29, 2023– Created June 29, 2023 - 18:47

metric[1]

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right], \quad \text{where} \quad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2. \tag{A.4}$$

From this metric, we can calculate the Christoffel symbols and from those the nonzero Ricci tensor components and the Ricci scalar (see Chapter 8 of [15] for more details). Finally, we choose to model energy and matter by a perfect fluid, so that the energy-momentum tensor is simply given by $T_{00} = \rho$, $T_{ij} = g_{ij}p$, and $T_{0i} = T_{i0} = 0$, where ρ is the energy density and p the pressure.

We can then plug everything into the Einstein equation (A.2); the $\mu\nu = 00$ equation gives the first *Friedmann equation*,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \tag{A.5}$$

The $\mu\nu = ij$ equations give the equation

$$-2\frac{\ddot{a}}{a} = 8\pi G p + \left(\frac{\dot{a}}{a}\right)^2, \tag{A.6}$$

which, after substitution of the first Friedmann equation, gives the second Friedmann equation, also known as the *acceleration equation*,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right). \tag{A.7}$$

A second form of the acceleration equation comes from requiring the conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$. Taking $\nu = 0$ and working out the steps we then arrive at the *fluid equation*,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{A.8}$$

This equation can also be derived using the first law of thermodynamics, $\mathrm{d}Q = \mathrm{d}E + p\,\mathrm{d}V$, but this method is beyond the scope of this appendix. A third way is to differentiate the first Friedmann equation and substitute the acceleration equation.

A.1.1 The equation of state

From the Friedmann equations, it is possible to derive the scale factor-dependence of the energy density ρ, if we assume some relation between ρ and the pressure p. This relation is known as the equation of state and is parametrised by the equation of state parameter w,

$$p = w\rho. \tag{A.9}$$

[1] For our purposes the flat FLRW metric is enough, but we can also derive the Friedmann equations with an extra curvature κ, which manifests itself as an extra factor $-\kappa/a^2$ in the first Friedmann equation.

Substituting this into the fluid equation (A.8) and getting rid of the time differentials, we can quickly obtain the relation

$$\rho(a) = \rho_0 a^{-3(1+w)}. \tag{A.10}$$

The value of w depends on the component: if we look at nonrelativistic matter, then due to the thermal energy being negligible compared to the rest mass energy, we have $w_{\mathrm{m}} = 0$. For relativistic matter, including photons, it can be shown that $w_\gamma = 1/3$. Adding a cosmological constant Λ, we can see that because the energy density should be constant, we should have $w_\Lambda = -1$ and the equation of state is $p = -\rho$.

This all means that we can break up the Universe into different components that all have a different evolution of the energy density. For nonrelativistic matter, we have $\rho_{\mathrm{m}} \propto a^{-3}$, for relativistic matter, $\rho_\gamma \propto a^{-4}$, and for the cosmological constant, $\rho_\Lambda \propto$ const. We can also rewrite the first Friedmann equation (A.5) into these components if we define the critical density, $\rho_{\mathrm{crit}} \equiv 3H^2/8\pi G$, and the density parameter for a single component, $\Omega_i \equiv \rho_i/\rho_{\mathrm{crit}}$. Using the subscript $_0$ for present-day values, we then rewrite

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\gamma,0}}{a^4} + \frac{\Omega_{\mathrm{m},0}}{a^3} + \Omega_{\Lambda,0}. \tag{A.11}$$

A.2 Scale factor, conformal time, and comoving wavenumber

With the rewritten form of the Friedmann equation, we can derive some relations for the scale factor as a function of conformal time, and other properties related to this. First of all, we note that in the previous section, we have exclusively worked with the cosmic time t, but in this section, we will quickly move to conformal time as the relations that we derive will have nicer forms.

A.2.1 Time-dependence of the scale factor

Let us look at a single-component Universe, with a general equation of state parameter w. The energy density evolves as $\rho = \rho_0 a^{-3(1+w)}$ and the Friedmann equation for this Universe reads

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_0 H_0^2}{a^{3(1+w)}}. \tag{A.12}$$

We can rewrite this by separation of variables where we bring all a to the left-hand side and the rest to the right-hand side,

$$a^{(1+3w)/2}\,\mathrm{d}a = \sqrt{\Omega_0 H_0^2}\,\mathrm{d}t. \tag{A.13}$$

We then move to conformal time, which is defined by $\mathrm{d}\eta \equiv \mathrm{d}t/a(t)$. Moving this extra scale factor to the left and integrating from $a(\eta = 0) = 0$ to $a(\eta)$, we obtain

$$a^{(3w-1)/2}\,\mathrm{d}a = \sqrt{\Omega_0 H_0^2}\,\mathrm{d}\eta \Rightarrow \frac{2}{3w+1} a^{(3w+1)/2} = \eta\sqrt{\Omega_0 H_0^2}, \tag{A.14}$$

which can be rewritten to

$$a(\eta) = \left[\frac{3w+1}{2}\eta\sqrt{\Omega_0 H_0^2}\right]^{2/(3w+1)}. \tag{A.15}$$

Usually, we will take $\Omega_0 = 1$, corresponding to a flat Universe, and so this gives

$$a(\eta) = \left[\frac{3w+1}{2}H_0\eta\right]^{2/(3w+1)}. \tag{A.16}$$

We can consider the three components that we saw before: (nonrelativistic) matter with $w_\mathrm{m} = 0$, relativistic matter or radiation with $w_\gamma = 1/3$, and the cosmological constant, with $w_\Lambda = -1$ (a Universe with just a cosmological constant is called a *de Sitter* Universe). Substituting these values for w gives the relations

$$a(\eta) = \left(\frac{H_0\eta}{2}\right)^2 \quad \text{(matter)}, \tag{A.17a}$$

$$a(\eta) = H_0\eta \quad \text{(radiation)}, \tag{A.17b}$$

$$a(\eta) = (-H_0\eta)^{-1} \quad \text{(de Sitter)}. \tag{A.17c}$$

In Chapter 4, we parametrise the scale factor as $a(\eta) = (\eta/\eta_0)^p$, so we see from Equation (A.16) that in general, we have

$$a(\eta) = \left(\frac{H_0\eta}{p}\right)^p, \quad p = \frac{2}{3w+1}. \tag{A.18}$$

For matter, we then have $p = 2$, for radiation $p = 1$, and for a de Sitter Universe $p = -1$.

A.2.2 A quasi-de Sitter Universe

In addition to the *pure* de Sitter case, we can also have a *quasi-de Sitter* Universe, which is the Universe that is typically adopted during inflation. The deviation from pure de Sitter is characterised by the slow-roll parameter ϵ (see Section 2.3.1). This parameter is defined in various ways, but the one that we will use here is

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right). \tag{A.19}$$

We know from solving Equation (A.13) for the cosmic time t, that the scale factor depends on t as $a \propto t^{2/(3w+3)}$, so the Hubble parameter is

$$\begin{aligned}\frac{1}{H} = \frac{a}{\dot{a}} &= t^{2/(3w+3)} \cdot \left[\frac{2}{3(w+1)}t^{2/(3w+3)-1}\right]^{-1} \\ &= \frac{3}{2}(w+1)t^{2/(3w+3)-2/(3w+3)+1} = \frac{3}{2}(w+1)t. \end{aligned} \tag{A.20}$$

Then the time-derivative of this gives

$$\epsilon = \frac{3}{2}(w+1) \Longleftrightarrow w = \frac{2}{3}\epsilon - 1. \tag{A.21}$$

Finally, using $p = 2/(3w+1)$, we can write, to linear order in ϵ,[2]

$$p = \frac{2}{2\epsilon - 3 + 1} = \frac{2}{2\epsilon - 2} = -\frac{1}{1-\epsilon} \simeq -(1+\epsilon). \tag{A.22}$$

Applying this to Equation (A.18), we have for a quasi-de Sitter Universe,

$$a(\eta) \simeq \left[-(1-\epsilon)H_0\eta\right]^{-1-\epsilon} = \frac{1}{\left[-(1-\epsilon)H_0\eta\right]^{1+\epsilon}} \qquad \text{(quasi-de Sitter)}. \tag{A.23}$$

A.2.3 η in terms of a and H

We can rewrite Equation (A.16) so we obtain an equation for η in terms of the scale factor a and the Hubble parameter H, which is given by Equation (A.12). Setting $\Omega_0 = 1$ again, we can rewrite this to $H_0 = Ha^{3(1+w)/2}$, and rewriting Equation (A.16) to

$$\eta = \frac{2}{3w+1}\frac{a^{(3w+1)/2}}{H_0} \tag{A.24}$$

then gives

$$\eta = \frac{2}{3w+1}\frac{1}{aH} = \frac{p}{aH}. \tag{A.25}$$

We see that $\eta \propto 1/aH$ for all components, and they just have a different prefactor based on the value for w (or p). For the three components and the quasi-de Sitter case, we have

$$\eta = \frac{2}{aH} \qquad \text{(matter)}, \tag{A.26a}$$

$$\eta = \frac{1}{aH} \qquad \text{(radiation)}, \tag{A.26b}$$

$$\eta = -\frac{1}{aH} \qquad \text{(de Sitter)}, \tag{A.26c}$$

$$\eta = -\frac{1}{(1-\epsilon)aH} \qquad \text{(quasi-de Sitter)}. \tag{A.26d}$$

A.2.4 Conformal time and comoving wavenumber

We might also sometimes use a comoving wavenumber k. To see the relation to the conformal time, we start at the Hubble law, $v = Hr$. The horizon is the distance photons

[2]Since $\epsilon \ll 1$ in the slow-roll approximation, we can use a Taylor expansion. However, this Taylor expansion is only really used if p is used as a power.

can have maximally travelled, so we set $v = c = 1$ and obtain the horizon $R_{\mathrm{H}} = 1/H$. The physical distance at this horizon is $r = R_{\mathrm{H}} = 1/H$, and the comoving distance x, defined by $r = ax$ with a the scale factor, is then $x = 1/aH$. Finally, we can define the comoving wavenumber simply as the inverse of the comoving distance, $k = 1/x$, so that the horizon is given by

$$k = aH. \tag{A.27}$$

We can then also already see the conditions for super- and subhorizon modes. The superhorizon regime has $r \gg R_H$, so $x \gg 1/aH$, and thus $k \ll aH$. Subhorizon is defined by $k \gg aH$.

Next, we can relate the horizon $k = aH$ to the conformal time η via Equation (A.25),

$$\eta = \frac{p}{k}. \tag{A.28}$$

For matter ($p = 2$), radiation ($p = 1$), de Sitter ($p = -1$), and quasi-de Sitter ($p = -1/(1-\epsilon)$), we therefore have $\eta = 2/k$, $\eta = 1/k$, $\eta = -1/k$, and $\eta = -1/(1-\epsilon)k$, respectively. We can again look at the super- and subhorizon regimes. Taking for the superhorizon case $k \ll aH = p/\eta$, we see that we have $k\eta/p \ll 1$, and the subhorizon limit $k \gg p/\eta$ gives $k\eta/p \gg 1$. Summarised in two equations, we can write

$$k \ll aH \quad \text{or} \quad \frac{k\eta}{p} \ll 1 \qquad \text{(superhorizon)}, \tag{A.29a}$$

$$k \gg aH \quad \text{or} \quad \frac{k\eta}{p} \gg 1 \qquad \text{(subhorizon)}. \tag{A.29b}$$

An important case is the de Sitter Universe, where $p = -1$, so that $-k\eta \ll 1$ and $-k\eta \gg 1$ for the super- and subhorizon limits, respectively.

Also worth noting is the modification that happens when the horizon is caused by waves not travelling at the speed of light. Following the same procedure as before, we set $v = c_T$ (in the same notation used in Chapter 4) with c_T some sound speed not equal to the speed of light. Following through, the horizon is now given by $c_T k = aH$ and the super- and subhorizon limits are given by

$$c_T k \ll aH \quad \text{or} \quad \frac{c_T k\eta}{p} \ll 1 \qquad \text{(superhorizon)}, \tag{A.30a}$$

$$c_T k \gg aH \quad \text{or} \quad \frac{c_T k\eta}{p} \gg 1 \qquad \text{(subhorizon)}. \tag{A.30b}$$

Setting $c_T = 1$ again, we reduce to the previous results.

Appendix B

Selected properties of Bessel functions

In this appendix, we will go over some important properties of the Bessel functions used in this thesis, including definitions, derivatives, and small- and large-argument limits. All properties shown here come from Chapter 10 of the *Digital Library of Mathematical Functions* (DLMF) [36].

B.1 Basic properties

B.1.1 Bessel and Hankel functions

The different Bessel functions are all solutions of *Bessel's differential equation*,

$$x^2 \frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + x \frac{\mathrm{d}y(x)}{\mathrm{d}x} + (x^2 - \nu^2) y(x) = 0, \tag{B.1}$$

and there are four relevant solutions, namely the Bessel functions of the first and second kind, $J_\nu(x)$ and $Y_\nu(x)$, respectively, and the Hankel functions of the first and second kind, $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$, respectively. The main functions J_ν and Y_ν have the general form

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu} \quad \text{and} \tag{B.2a}$$

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{B.2b}$$

Any solution to Bessel's equation can be written as a linear combination of these two solutions, i.e.,

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x) \tag{B.3}$$

with c_1 and c_2 some constants, determined from the initial conditions. The Hankel functions of the first and second kind comprise a second set of linearly independent solutions to Bessel's equation. They are defined via the Bessel functions as

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \quad \text{and} \tag{B.4a}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x), \tag{B.4b}$$

thus we can also write the solutions as

$$y(x) = c_1 H_\nu^{(1)}(x) + c_2 H_\nu^{(2)}(x), \tag{B.5}$$

where again c_1 and c_2 are constants determined from the initial conditions.

B.1.2 Derivatives

If we look at the derivatives of the Bessel and Hankel functions, it is convenient to group the four functions J_ν, Y_ν, $H_\nu^{(1)}$, and $H_\nu^{(2)}$ into one function, $\mathcal{C}_\nu(x)$. This will make the following more clear as the derivatives have the same form for each of these functions. In general, the k-th derivative can be written in two ways,

$$\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left(x^\nu \mathcal{C}_\nu(x)\right) = x^{\nu-k}\mathcal{C}_{\nu-k}(x) \quad \text{or} \tag{B.6a}$$

$$\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left(x^{-\nu} \mathcal{C}_\nu(x)\right) = (-1)^k x^{-\nu-k}\mathcal{C}_{\nu+k}(x). \tag{B.6b}$$

For our purposes, only the first derivatives are important. Taking $k = 1$, it can easily be shown that

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{C}_\nu(x) = \mathcal{C}_{\nu-1}(x) - \frac{\nu}{x}\mathcal{C}_\nu(x) = \frac{\nu}{x}\mathcal{C}_\nu(x) - \mathcal{C}_{\nu+1}(x). \tag{B.7}$$

B.1.3 A different form of Bessel's equation

In Chapters 2 and 4, we are often confronted with a slightly different form of the differential equation (B.1). To arrive at this form, we make the substitution

$$y(x) = \frac{1}{\sqrt{x}} z(x), \tag{B.8}$$

which modifies Bessel's equation to the form[1]

$$x^2 \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \left[x^2 - \left(\nu^2 - \frac{1}{4}\right)\right] z(x) = 0. \tag{B.9}$$

[1] This derivation is trivial and left as an exercise to the reader. It involves some simple derivatives and collecting and cancelling of terms, after which the differential equation (B.9) is quickly found.

Version of June 29, 2023– Created June 29, 2023 - 18:47

The solutions to this differential equation are still given by $y(x)$, and with the substitution (B.8) this means that the solutions are given by

$$z(x) = c_1\sqrt{x}J_\nu(x) + c_2\sqrt{x}Y_\nu(x) \tag{B.10a}$$
$$= c_1\sqrt{x}H_\nu^{(1)}(x) + c_2\sqrt{x}H_\nu^{(2)}(x). \tag{B.10b}$$

Again, the constants c_1 and c_2 can be determined from the initial conditions, as usual, and the solutions can be checked by differentiating twice and substituting into Equation (B.9), making clever use of the two possibilities for the derivative, Equation (B.7), and collecting and cancelling terms. A final set of alterations that can be performed is to replace $x = at$ for some constant a, divide by t^2, and write $\nu^2 - 1/4 \equiv b$, so that the differential equation reads

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \left(a^2 - \frac{b}{t^2}\right) z(t) = 0. \tag{B.11}$$

The solutions for this are the same as before, but with $\nu^2 - 1/4 = b$ or $\nu = \sqrt{b + 1/4} = \sqrt{4b+1}/2$. Also, we replace $x = at$ in the Bessel functions[2] so that the solutions read

$$z(t) = c_1\sqrt{t}J_{\sqrt{4b+1}/2}(at) + c_2\sqrt{t}Y_{\sqrt{4b+1}/2}(at) \tag{B.12a}$$
$$= c_1\sqrt{t}H^{(1)}_{\sqrt{4b+1}/2}(at) + c_2\sqrt{t}H^{(2)}_{\sqrt{4b+1}/2}(at). \tag{B.12b}$$

The second solution (B.12b) is usually used in Chapters 2 and 4.

B.2 Limiting properties

Also useful are the limiting cases, i.e. the cases for very small and very large arguments, $x \ll 1$ and $x \gg 1$. Let us look at the four functions in these two limits.

B.2.1 Small-argument limits

In the limit $x \ll 1$, the $k = 0$-term in Equation (B.2a) dominates and we can write

$$J_\nu(x \ll 1) \simeq \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu. \tag{B.13}$$

The definition (B.2b) then also gives us the small-argument limit of the second Bessel function,

$$Y_\nu(x \ll 1) \simeq -\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu}. \tag{B.14}$$

[2]Notice that the $\sqrt{x}$ should also become $\sqrt{at}$, but the $\sqrt{a}$ can be absorbed in the constants c_1 and c_2.

In the same way, the definitions (B.4a) and (B.4b) give the small-argument limits of the Hankel functions,

$$H_\nu^{(1)}(x \ll 1) \simeq -\frac{i}{\pi}\Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu} \quad \text{and} \tag{B.15a}$$

$$H_\nu^{(2)}(x \ll 1) \simeq \frac{i}{\pi}\Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu}. \tag{B.15b}$$

B.2.2 Large-argument limits

In the limit $x \gg 1$, we have slightly different expressions. The Bessel functions of the first and second kind are given by

$$J_\nu(x \gg 1) \simeq \sqrt{\frac{2}{\pi x}}\cos\left[x - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \quad \text{and} \tag{B.16a}$$

$$Y_\nu(x \gg 1) \simeq \sqrt{\frac{2}{\pi x}}\sin\left[x - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right]. \tag{B.16b}$$

Again, using the definitions (B.4a) and (B.4b), the Hankel functions have the approximation

$$H_\nu^{(1)}(x \gg 1) \simeq \sqrt{\frac{2}{\pi x}}\exp\left[i\left(x - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right)\right] \quad \text{and} \tag{B.17a}$$

$$H_\nu^{(2)}(x \gg 1) \simeq \sqrt{\frac{2}{\pi x}}\exp\left[-i\left(x - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right)\right]. \tag{B.17b}$$

Appendix C

Lengthy calculations

In this appendix, we will show the more lengthy calculations that do not fit in the main body. Often, they are not very insightful but for completeness they are still included here.

C.1 Irreducible inflationary power spectrum

Below are calculations related to the irreducible SGWB power spectrum during inflation for massless gravity (i.e., Chapter 2).

1. Action in terms of Fourier modes

We start from the action (2.35),

$$S_{\rm g}^{(2)} = -\frac{M_{\rm Pl}^2}{8}\int {\rm d}\eta {\rm d}^3\mathbf{x} a^2(\eta)\eta^{\mu\nu}\partial_\mu\chi_{ij}\partial_\nu\chi_{ij}. \tag{C.1}$$

Writing out the terms using the Einstein convention, we have

$$\begin{aligned} S_{\rm g}^{(2)} &= -\frac{M_{\rm Pl}^2}{8}\int {\rm d}\eta {\rm d}^3\mathbf{x} a^2(\eta)\left[-\left(\partial_\eta\chi_{ij}\right)^2 + \left(\partial_m\chi_{ij}\right)^2\right] \\ &= \frac{M_{\rm Pl}^2}{8}\int {\rm d}\eta {\rm d}^3\mathbf{x} a^2(\eta)\left[\left(\chi_{ij}'\right)^2 - \left(\partial_m\chi_{ij}\right)^2\right]. \end{aligned} \tag{C.2}$$

Next, we substitute the Fourier expansion into the two polarisations, Equation (2.36),

$$\chi_{ij}(\mathbf{x},\eta) = \sum_{r=+,\times}\frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\chi_r(\mathbf{k},\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}e_{ij}^r(\hat{\mathbf{k}}), \tag{C.3}$$

where $e^r_{ij}(-\hat{\mathbf{k}}) = e^r_{ij}(\hat{\mathbf{k}})$, $e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}}) = 2\delta_{rr'}$, and $\chi^*_r(\mathbf{k},\eta) = \chi_r(-\mathbf{k},\eta)$. Performing this substitution, we observe that a derivative ∂_m simply gives us a factor $-i\mathbf{k}$, and we have

$$\begin{aligned} S^{(2)}_{\rm g} &= \frac{M^2_{\rm Pl}}{8}\sum_{r,r'=+,\times}\int\frac{\mathrm{d}\eta\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^6}\Big\{\Big[\chi'_r(\mathbf{k},\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}e^r_{ij}(\hat{\mathbf{k}})\Big]\Big[\chi'_{r'}(\mathbf{k}',\eta)e^{-i\mathbf{k}'\cdot\mathbf{x}}e^{r'}_{ij}(\hat{\mathbf{k}}')\Big] \\ &\qquad -\Big[-i\mathbf{k}\chi_r(\mathbf{k},\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}e^r_{ij}(\hat{\mathbf{k}})\Big]\cdot\Big[-i\mathbf{k}'\chi_{r'}(\mathbf{k}',\eta)e^{-i\mathbf{k}'\cdot\mathbf{x}}e^{r'}_{ij}(\hat{\mathbf{k}}')\Big]\Big\}a^2(\eta) \\ &= \frac{M^2_{\rm Pl}}{8}\sum_{r,r'=+,\times}\int\frac{\mathrm{d}\eta\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^6}a^2(\eta)e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}}')\times \\ &\qquad \times\left[\chi'_r(\mathbf{k},\eta)\chi'_{r'}(\mathbf{k}',\eta)+\mathbf{k}\cdot\mathbf{k}'\chi_r(\mathbf{k},\eta)\chi_{r'}(\mathbf{k}',\eta)\right]. \end{aligned} \tag{C.4}$$

Next, we get rid of the integral over $\mathbf{x}$ by using the definition of the three-dimensional Dirac delta, $\delta^{(3)}(\mathbf{k}-\mathbf{k}') = \int\frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^3}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}$, so then

$$\begin{aligned} S^{(2)}_{\rm g} &= \frac{M^2_{\rm Pl}}{8}\sum_{r,r'=+,\times}\int\frac{\mathrm{d}\eta\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}a^2(\eta)e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}}')\delta^{(3)}(\mathbf{k}+\mathbf{k}')\times \\ &\qquad \times\left[\chi'_r(\mathbf{k},\eta)\chi'_{r'}(\mathbf{k}',\eta)+\mathbf{k}\cdot\mathbf{k}'\chi_r(\mathbf{k},\eta)\chi_{r'}(\mathbf{k}',\eta)\right]. \end{aligned} \tag{C.5}$$

This can now be integrated over $\mathbf{k}'$, which because of the Dirac delta sets all $\mathbf{k}' = -\mathbf{k}$. The polarisation tensors do not change under this change, as $e^r_{ij}(\hat{\mathbf{k}}) = e^r_{ij}(-\hat{\mathbf{k}})$. With $\mathbf{k}\cdot\mathbf{k} = k^2$, we then have

$$\begin{aligned} S^{(2)}_{\rm g} &= \frac{M^2_{\rm Pl}}{8}\sum_{r,r'=+,\times}\int\mathrm{d}\eta\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}a^2(\eta)e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}})\times \\ &\qquad \times\Big[\chi'_r(\mathbf{k},\eta)\chi'_{r'}(-\mathbf{k},\eta)-k^2\chi_r(\mathbf{k},\eta)\chi_r(-\mathbf{k},\eta)\Big] \end{aligned} \tag{C.6}$$

We can then use the orthonormal relation for the polarisation tensors and the reality condition for the perturbation χ_r, which we both mentioned before, such that

$$S^{(2)}_{\rm g} = \frac{M^2_{\rm Pl}}{4}\sum_{r,r'=+,\times}\int\mathrm{d}\eta\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}a^2(\eta)\delta_{rr'}\Big[\chi'_r(\mathbf{k},\eta)\chi'^*_{r'}(\mathbf{k},\eta)-k^2\chi_r(\mathbf{k},\eta)\chi^*_r(\mathbf{k},\eta)\Big], \tag{C.7}$$

and then we can remove the $\delta_{rr'}$ with the summation over r' so that we end up with

$$\begin{aligned} S^{(2)}_{\rm g} &= \frac{M^2_{\rm Pl}}{4}\sum_{r=+,\times}\int\mathrm{d}\eta\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}a^2(\eta)\Big[\chi'_r(\mathbf{k},\eta)\chi'^*_r(\mathbf{k},\eta)-k^2\chi_r(\mathbf{k},\eta)\chi^*_r(\mathbf{k},\eta)\Big] \\ &= \frac{M^2_{\rm Pl}}{4}\sum_{r=+,\times}\int\mathrm{d}\eta\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}a^2(\eta)\Big[\left|\chi'_r(\mathbf{k},\eta)\right|^2-k^2\left|\chi_r(\mathbf{k},\eta)\right|^2\Big]. \end{aligned} \tag{C.8}$$

This final equation is Equation (2.39).

2. Action in terms of Mukhanov-Sasaki variables

We start at Equation (2.39),

$$S_{\mathrm{g}}^{(2)} = \frac{M_{\mathrm{Pl}}^2}{4} \sum_{r=+,\times} \int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} a^2(\eta) \left[\left|\chi_r'(\mathbf{k},\eta)\right|^2 - k^2 \left|\chi_r(\mathbf{k},\eta)\right|^2 \right], \tag{C.9}$$

and substitute the Mukhanov-Sasaki variables (2.40),

$$v_r(\mathbf{k},\eta) = \frac{M_{\mathrm{Pl}}}{\sqrt{2}} a(\eta)\chi_r(\mathbf{k},\eta) \Longleftrightarrow \chi_r(\mathbf{k},\eta) = \frac{\sqrt{2}}{M_{\mathrm{Pl}} a(\eta)} v_r(\mathbf{k},\eta). \tag{C.10}$$

For clarity, we omit the $\mathbf{k}$- and η-dependencies in the following; the derivative is

$$\chi_r' = \frac{\sqrt{2}}{M_{\mathrm{Pl}}} \left[\frac{v_r'}{a} - \frac{v_r a'}{a^2} \right] = \frac{\sqrt{2}}{M_{\mathrm{Pl}} a} \left[v_r' - \frac{a'}{a} v_r \right]. \tag{C.11}$$

Then

$$|\chi_r'|^2 = \frac{2}{M_{\mathrm{Pl}}^2 a^2} \left[|v_r'|^2 + \left(\frac{a'}{a}\right) |v_r|^2 - \frac{a'}{a} \left(v_r' v_r^* + v_r v_r'^* \right) \right]. \tag{C.12}$$

We now have this third term that we have to rewrite. This is done by considering that the term appears inside an integral over $\mathrm{d}\eta$, so we can partially integrate it, assuming the boundary terms are zero, as

$$\begin{aligned} \int \mathrm{d}\eta \left[-\frac{a'}{a} \left(v_r' v_r^* + v_r v_r'^* \right) \right] &= -\int \mathrm{d}\eta \left[\left(\frac{a'}{a}\right) (v_r v_r^*)' \right] = \int \mathrm{d}\eta \left[\left(\frac{a'}{a}\right)' v_r v_r^* \right] \\ &= \int \mathrm{d}\eta \left[\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 \right] |v_r|^2. \end{aligned} \tag{C.13}$$

The second term in this equation then cancels with the second term in Equation (C.12) and we can effectively write

$$|\chi_r'|^2 = \frac{2}{M_{\mathrm{Pl}}^2 a^2} \left[|v_r'|^2 + \frac{a''}{a} |v_r|^2 \right]. \tag{C.14}$$

Substituting this into Equation (C.9), we finally obtain

$$\begin{aligned} S_{\mathrm{g}}^{(2)} &= \frac{M_{\mathrm{Pl}}^2}{4} \sum_{r=+,\times} \int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} a^2 \left[\frac{2}{M_{\mathrm{Pl}}^2 a^2} \left(|v_r'|^2 + \frac{a''}{a} |v_r|^2 \right) - k^2 \frac{2}{M_{\mathrm{Pl}}^2 a^2} |v_r|^2 \right] \\ &= \frac{1}{2} \sum_{r=+,\times} \int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[|v_r'|^2 - k^2 |v_r| + \frac{a''}{a} |v_r|^2 \right]. \end{aligned} \tag{C.15}$$

This final equation is Equation (2.41).

3. Equation of motion

We start at the action (2.41) and apply the Euler-Lagrange equation of motion to the Lagrangian density

$$\mathcal{L}(|v_r|, |v_r'|) = \frac{1}{2}|v_r'|^2 - \frac{1}{2}\left(k^2 - \frac{a''}{a}\right)|v_r|^2 = \frac{1}{2}|v_r'|^2 - \frac{1}{2}\omega_k^2(\eta)|v_r|^2. \tag{C.16}$$

The relevant derivatives are

$$\frac{\partial \mathcal{L}}{\partial |v_r|} = -\omega_k^2(\eta)|v_r| \quad \text{and} \tag{C.17a}$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu |v_r|)}\right) = \partial_\eta\left(\frac{\partial \mathcal{L}}{\partial |v_r'|}\right) = \partial_\eta |v_r'| = |v_r''|, \tag{C.17b}$$

where in the second line, we have expanded the indices with the Einstein convention and gotten rid of the spatial derivative ∂_m as the term $\partial_m|v_r|$ is not represented in the Lagrangian density. The equations of motion now become

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu |v_k|)}\right) - \frac{\partial \mathcal{L}}{\partial |v_r|} = |v_r''| + \omega_k^2(\eta)|v_r| = 0. \tag{C.18}$$

At this point, we can safely remove the absolute values because the complex conjugate of this equation of motion will effectively be the same equation. We then have

$$v_r''(\mathbf{k}, \eta) + \omega_k^2(\eta)v_r(\mathbf{k}, \eta) = 0 \quad \text{with} \quad \omega_k^2(\eta) = k^2 - \frac{a''(\eta)}{a(\eta)}. \tag{C.19}$$

This is Equation (2.42).

4. Normalisation condition

We want to derive the normalisation condition (2.49),

$$v_k v_k'^* - v_k^* v_k' = i. \tag{C.20}$$

To do so, we start at the commutation relation (2.43a),

$$\mathcal{C} \equiv \left[\hat{v}_r(\mathbf{x}, \eta), \hat{\pi}_{r'}(\mathbf{x}', \eta)\right] = i\delta_{rr'}\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{C.21}$$

and for this, we require the conjugate momentum $\pi_r(\mathbf{x}, \eta) = \partial\mathcal{L}/\partial v_r'(\mathbf{x}, \eta)$. From Equation (C.17b), we can see that $\pi_r(\mathbf{x}, \eta) = v_r'(\mathbf{x}, \eta)$, and therefore the same holds when we promote them to operators. We can then use the mode expansion (2.44),

$$\hat{v}_r(\mathbf{x}, \eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[v_k(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}r} + v_k^*(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}r}^\dagger\right], \tag{C.22}$$

where the annihilation and creation operators $\hat{a}_{\mathbf{k}r}$ and $\hat{a}^\dagger_{\mathbf{k}r}$ follow the commutation relations (2.45),

$$\left[\hat{a}_{\mathbf{k}r}, \hat{a}^\dagger_{\mathbf{k}'r'}\right] = (2\pi)^3 \delta_{rr'} \delta^{(3)}(\mathbf{k}-\mathbf{k}') \quad \text{and} \quad [\hat{a}_{\mathbf{k}r}, \hat{a}_{\mathbf{k}'r'}] = \left[\hat{a}^\dagger_{\mathbf{k}r}, \hat{a}^\dagger_{\mathbf{k}'r'}\right] = 0. \tag{C.23}$$

Plugging the mode expansion into the commutation relation (C.21), and using the identity $[\hat{A}+\hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$ of the commutator, we obtain

$$\begin{aligned} \mathcal{C} &\equiv \left[\hat{v}_r(\mathbf{x},\eta), \hat{\pi}_{r'}(\mathbf{x}',\eta)\right] = \left[\hat{v}_r(\mathbf{x},\eta), \hat{v}'_{r'}(\mathbf{x}',\eta)\right] \\ &= \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^6} \left[\left(v_k e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}r} + v_k^* e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}^\dagger_{\mathbf{k}r}\right), \left(v'_{k'} e^{-i\mathbf{k}'\cdot\mathbf{x}'} \hat{a}_{\mathbf{k}'r'} + v'^*_{k'} e^{i\mathbf{k}'\cdot\mathbf{x}'} \hat{a}^\dagger_{\mathbf{k}'r'}\right)\right] \\ &= \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^6} \Big\{ v_k v'_{k'} e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}')} [\hat{a}_{\mathbf{k}r}, \hat{a}_{\mathbf{k}'r'}] + v_k^* v'^*_{k'} e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}')} \left[\hat{a}^\dagger_{\mathbf{k}r}, \hat{a}^\dagger_{\mathbf{k}'r'}\right] \\ &+ v_k v'^*_{k'} e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')} \left[\hat{a}_{\mathbf{k}r}, \hat{a}^\dagger_{\mathbf{k}'r'}\right] + v_k^* v'_{k'} e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')} \left[\hat{a}^\dagger_{\mathbf{k}r}, \hat{a}_{\mathbf{k}'r'}\right] \Big\}. \end{aligned} \tag{C.24}$$

In this equation, we see the canonical commutation relations (C.23); the first and second terms have commutators that are zero so we are just left with the third and fourth terms. We can then continue the calculation,

$$\mathcal{C} = \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} \left[v_k v'^*_{k'} e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')} - v_k^* v'_{k'} e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')}\right] \delta_{rr'} \delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{C.25}$$

$$= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[v_k v'^*_k e^{-i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}} - v_k^* v'_k e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}}\right] \delta_{rr'} \tag{C.26}$$

$$= \left(v_k v'^*_k - v_k^* v'_k\right) \delta_{rr'} \delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{C.27}$$

In the first line, we have reversed the final commutator so we obtain the minus sign, and we have substituted the delta functions of the commutators. In the second line, we have integrated over $\mathbf{k}'$ to get rid of the delta function, and in the third line, we have used again the definition of the Dirac delta to get rid of the final integral, and used that $\delta^{(3)}(\mathbf{x}-\mathbf{x}') = \delta^{(3)}(\mathbf{x}'-\mathbf{x})$. Comparing this result with the requirement for $\mathcal{C}$, Equation (C.21), we see that indeed

$$v_k v'^*_k - v_k^* v'_k = i. \tag{C.28}$$

This is the normalisation condition (2.49).

5. General equation for the power spectrum

We want to show Equation (2.59) from Equation (2.58), so we use the latter equation,

$$\langle 0 | \hat{\chi}_{ij}(\mathbf{k},\eta) \hat{\chi}^*_{ij}(\mathbf{k}',\eta) | 0 \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_\chi(k) \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{C.29}$$

and the Fourier decomposition

$$\hat{\chi}_{ij}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\chi}_{ij}(\mathbf{k},\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{C.30}$$

to write

$$\begin{aligned}
\langle 0|\hat{\chi}_{ij}(\mathbf{x},\eta)\hat{\chi}_{ij}(\mathbf{x},\eta)|0\rangle &= \left\langle 0 \middle| \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} \hat{\chi}_{ij}(\mathbf{k},\eta)\hat{\chi}^*_{ij}(\mathbf{k}',\eta) e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \middle| 0 \right\rangle \\
&= \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} \langle 0|\hat{\chi}_{ij}(\mathbf{k},\eta)\hat{\chi}^*_{ij}(\mathbf{k}',\eta)|0\rangle e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
&= \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} \frac{2\pi^2}{k^3} \mathcal{P}_\chi(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}') e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{2\pi^2}{k^3} \mathcal{P}_\chi(k),
\end{aligned} \tag{C.31}$$

where in the last step we integrated over $\mathbf{k}'$ to get rid of the delta function and the exponential. The next step is to go to spherical coordinates since the power spectrum is simply a function of the modulus $k = |\mathbf{k}|$. We have $\mathrm{d}^3\mathbf{k} = 4\pi k^2\,\mathrm{d}k$ and therefore

$$\langle 0|\hat{\chi}_{ij}(\mathbf{x},\eta)\hat{\chi}_{ij}(\mathbf{x},\eta)|0\rangle = \frac{2\pi^2\cdot 4\pi}{8\pi^3} \int \frac{k^2\,\mathrm{d}k}{k^3} \mathcal{P}_\chi(k) = \int \frac{\mathrm{d}k}{k} \mathcal{P}_\chi(k), \tag{C.32}$$

which is exactly Equation (2.59).

6. The power spectrum from the irreducible SGWB during inflation

We now want to calculate the specific form of the power spectrum from the exact solution (2.57). We first write it with Equation (2.40) to

$$\begin{aligned}
\chi_k(\eta) &= \frac{\sqrt{2}}{M_{\mathrm{Pl}}a(\eta)} \frac{\sqrt{\pi}}{2} \exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{-\eta} H_\nu^{(1)}(-k\eta) \\
&= \sqrt{-\frac{\eta\pi}{2}} \exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \frac{H_\nu^{(1)}(-k\eta)}{M_{\mathrm{Pl}}a(\eta)}.
\end{aligned} \tag{C.33}$$

As it will turn out, we will only need the absolute value of this, i.e.,

$$|\chi_k(\eta)| = \sqrt{-\frac{\eta\pi}{2}} \frac{1}{M_{\mathrm{Pl}}a(\eta)} \left| H_\nu^{(1)}(-k\eta) \right|. \tag{C.34}$$

With this in mind, we calculate the quantity $\langle \chi^2 \rangle \equiv \langle 0|\hat{\chi}_{ij}(\mathbf{x},\eta)\hat{\chi}_{ij}(\mathbf{x},\eta)|0\rangle$, from which we can then read out the power spectrum with Equation (2.59). Let us substitute the Fourier expansion

$$\hat{\chi}_{ij}(\mathbf{x},\eta) = \sum_{r=+,\times} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[\chi_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}r} + \chi^*_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}^\dagger_{\mathbf{k}r}\right] e^r_{ij}(\hat{\mathbf{k}}) \tag{C.35}$$

into $\langle\chi^2\rangle$:

$$\begin{aligned}\langle\chi^2\rangle &= \sum_{r,r'=+,\times}\int\frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^6}\Big\langle 0\,\Big|\Big[\chi_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}r}+\chi_k^*(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^\dagger_{\mathbf{k}r}\Big]\times\\ &\qquad\times\Big[\chi_{k'}(\eta)e^{i\mathbf{k}'\cdot\mathbf{x}}\hat{a}_{\mathbf{k}'r'}+\chi_{k'}^*(\eta)e^{-i\mathbf{k}'\cdot\mathbf{x}}\hat{a}^\dagger_{\mathbf{k}'r'}\Big]\Big|\,0\Big\rangle\\ &= \sum_{r,r'=+,\times}\int\frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^6}\Big[\chi_k\chi_{k'}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}\langle 0|\hat{a}_{\mathbf{k}r}\hat{a}_{\mathbf{k}'r'}|0\rangle\\ &\qquad+\chi_k\chi_{k'}^*e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\langle 0|\hat{a}_{\mathbf{k}r}\hat{a}^\dagger_{\mathbf{k}'r'}|0\rangle\\ &\qquad+\chi_k^*\chi_{k'}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\langle 0|\hat{a}^\dagger_{\mathbf{k}r}\hat{a}_{\mathbf{k}'r'}|0\rangle\\ &\qquad+\chi_k^*\chi_{k'}^*e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}\langle 0|\hat{a}^\dagger_{\mathbf{k}r}\hat{a}^\dagger_{\mathbf{k}'r'}|0\rangle\Big]e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}}').\end{aligned} \tag{C.36}$$

These four terms all involve ladder operators operating on the vacuum state $|0\rangle$, and with the definition $\hat{a}_{\mathbf{k}r}|0\rangle = 0$ (or, equivalently, $\langle 0|\hat{a}^\dagger_{\mathbf{k}r} = 0$), we can see that the first, third, and fourth terms are all zero, and we just have the second term. With the commutation relation (2.45a), written as $\hat{a}_{\mathbf{k}r}\hat{a}^\dagger_{\mathbf{k}r} = (2\pi)^3\delta_{rr'}\delta^{(3)}(\mathbf{k}-\mathbf{k}') + \hat{a}^\dagger_{\mathbf{k}r}\hat{a}_{\mathbf{k}r}$, we can then also get rid of the final ladder operators and end up with

$$\langle\chi^2\rangle = \sum_{r,r'=+,\times}\int\frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\chi_k\chi^*_{k'}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\delta_{rr'}\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}}'). \tag{C.37}$$

Next, we perform the integral over $\mathbf{k}'$, getting rid of the Dirac delta while setting $k' = k$ and $\mathbf{k}' = \mathbf{k}$ everywhere. We can also get rid of the $\delta_{rr'}$ by summing over r', which sets all $r' = r$. We then simply have

$$\langle\chi^2\rangle = \sum_{r=+,\times}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}|\chi_k|^2e^r_{ij}(\hat{\mathbf{k}})e^r_{ij}(\hat{\mathbf{k}}). \tag{C.38}$$

Using the normalisation condition for the polarisation tensors, $e^r_{ij}(\hat{\mathbf{k}})e^{r'}_{ij}(\hat{\mathbf{k}})$, we get a factor $2\delta_{rr}$ since $r' = r$ in our case. We subsequently get rid of the final summation over r which combined with δ_{rr} gives us another factor 2, so that

$$\langle\chi^2\rangle = 4\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}|\chi_k|^2. \tag{C.39}$$

χ_k is only dependent on the modulus $k = |\mathbf{k}|$, so we can go to spherical coordinates for which $\mathrm{d}^3\mathbf{k} = 4\pi k^2\,\mathrm{d}k$, and so

$$\langle\chi^2\rangle = 4\cdot\frac{4\pi}{8\pi^3}\int k^2\,\mathrm{d}k|\chi_k(\eta)|^2 = \int\frac{\mathrm{d}k}{k}\frac{2k^3}{\pi^2}|\chi_k(\eta)|^2 = \int\frac{\mathrm{d}k}{k}\mathcal{P}_\chi(k), \tag{C.40}$$

where in the final step, we used Equation (2.59). From this, we can see that in general, the power spectrum is given by

$$\mathcal{P}_\chi(k) = \frac{2k^3}{\pi^2}|\chi_k(\eta)|^2. \tag{C.41}$$

We then finally substitute our solution for $|\chi_k(\eta)|$, Equation (C.34), so that

$$\begin{aligned} \mathcal{P}_\chi(k) &= \frac{2k^3}{\pi^2}\left(-\frac{\eta\pi}{2}\right)\frac{1}{M_{\rm Pl}^2 a^2(\eta)}\left|H_\nu^{(1)}(-k\eta)\right|^2 \\ &= -\frac{k^3\eta}{\pi M_{\rm Pl}^2 a^2(\eta)}\left|H_\nu^{(1)}(-k\eta)\right|^2, \quad \text{with} \quad \nu = \frac{3}{2}+\epsilon, \end{aligned} \tag{C.42}$$

which is Equation (2.61).

7. The superhorizon power spectrum

We have the power spectrum (2.61),

$$\mathcal{P}_\chi(k) = -\frac{k^3\eta}{\pi M_{\rm Pl}^2 a^2(\eta)}\left|H_\nu^{(1)}(-k\eta)\right|^2 \quad \text{with} \quad \nu = \frac{3}{2}+\epsilon, \tag{C.43}$$

and we want to write this in the superhorizon limit $-k\eta \ll 1$ or $k \ll aH$, where we use the small-argument limit of the Hankel function as shown in Appendix B.2.1,

$$\left|H_\nu^{(1)}(x \ll 1)\right|^2 \simeq \frac{1}{\pi^2}\Gamma^2(\nu)\left(\frac{x}{2}\right)^{-2\nu}. \tag{C.44}$$

With $x = -k\eta$ and $\eta = -1/(1-\epsilon)aH$ in quasi-de Sitter (see Appendix A.2.3), the power spectrum becomes

$$\begin{aligned} \mathcal{P}_\chi(k) &\simeq -\frac{k^3\eta}{\pi M_{\rm Pl}^2 a^2(\eta)}\frac{1}{\pi^2}\Gamma^2\left(\frac{3}{2}+\epsilon\right)\left(-\frac{k\eta}{2}\right)^{-3-2\epsilon} \\ &= \frac{H^2}{\pi^3 M_{\rm Pl}^2 (aH)^2}\frac{k^3}{(1-\epsilon)aH}\Gamma^2\left(\frac{3}{2}+\epsilon\right)2^{3+2\epsilon}\left(\frac{k}{(1-\epsilon)aH}\right)^{-3-2\epsilon} \\ &= \frac{2H^2}{\pi^3 M_{\rm Pl}^2}2^{2+2\epsilon}\Gamma^2\left(\frac{3}{2}+\epsilon\right)(1-\epsilon)^{2+2\epsilon}\left(\frac{k}{aH}\right)^{3-3-2\epsilon} \\ &= \frac{2H^2}{\pi^3 M_{\rm Pl}^2}\left[2^{1+\epsilon}(1-\epsilon)^{1+\epsilon}\Gamma\left(\frac{3}{2}+\epsilon\right)\right]^2\left(\frac{k}{aH}\right)^{-2\epsilon}. \end{aligned} \tag{C.45}$$

If we now set $f(\epsilon) \equiv 2^{1+\epsilon}\Gamma(3/2+\epsilon)(1-\epsilon)^{1+\epsilon}$, then we arrive at

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^3 M_{\rm Pl}^2}f^2(\epsilon)\left(\frac{k}{aH}\right)^{-2\epsilon}, \quad \text{for} \quad k \ll aH, \tag{C.46}$$

which is Equation (2.65).

C.2 The power spectrum from massive gravity

Below are calculations related to the power spectrum arising from massive gravity, in the different cases presented in Chapter 4. In each calculation, we derive the power spectrum $\mathcal{P}_\chi(k)$ from the solutions $v_k(\eta)$ via Equation (4.50),

$$\mathcal{P}_\chi(k) = \frac{4k^3}{\pi^2 M_{\rm Pl}^2 a^2(\eta)} |v_k(\eta)|^2. \tag{C.47}$$

1. The (quasi-)de Sitter power spectrum

We start at the solution (4.51),

$$\begin{aligned} v_k(\eta) &= \frac{\sqrt{\pi}}{2} \exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{-\eta} H_\nu^{(1)}(-c_T k\eta) \\ \text{with} \quad \nu &\simeq \frac{3}{2} + \epsilon - \frac{\mu_\chi^2}{3H_0^2}\left(1 + \frac{4}{3}\epsilon\right), \end{aligned} \tag{C.48}$$

and want to calculate the power spectrum (4.53) in the superhorizon limit from this. To do so, we simply substitute the solution into Equation (C.47) so that the exponential disappears because of the absolute signs, and the general power spectrum is

$$\begin{aligned} \mathcal{P}_\chi(k) &= \frac{4k^3}{\pi^2 M_{\rm Pl}^2 a^2} \frac{\pi}{4}(-\eta) \left| H_\nu^{(1)}(-c_T k\eta) \right| \\ &= -\frac{k^3 \eta}{\pi M_{\rm Pl}^2 a^2} \left| H_\nu^{(1)}(-c_T k\eta) \right|. \end{aligned} \tag{C.49}$$

We can then take the superhorizon limit $-c_T k\eta \ll 1$ (see Appendix A.2.4) in which

$$H_\nu^{(1)}(x \ll 1) \simeq -\frac{i}{\pi}\Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu} \tag{C.50}$$

(see Appendix B.2.1) and the power spectrum is written as

$$\begin{aligned} \mathcal{P}_\chi(k) &\simeq -\frac{k^3\eta}{\pi M_{\rm Pl}^2 a^2} \left| -\frac{i}{\pi}\Gamma(\nu)\left(-\frac{c_T k\eta}{2}\right)^{-\nu} \right|^2 \\ &= -\frac{k^3\eta}{\pi^3 M_{\rm Pl}^2 a^2} \Gamma^2(\nu) \left(-\frac{c_T k\eta}{2}\right)^{-2\nu}. \end{aligned} \tag{C.51}$$

Next, we substitute the expression for η in terms of aH during the quasi-de Sitter period, $\eta = -1/(1-\epsilon)aH$ (see Appendix A.2.3), so that

$$\begin{aligned} \mathcal{P}_\chi(k) &= \frac{H^2}{\pi^3 M_{\rm Pl}^2 (aH)^2 c_T^3} (c_T k)^3 \frac{1}{(1-\epsilon)aH} \Gamma^2(\nu) \cdot 2^{2\nu} \left(\frac{c_T k}{aH}\right)^{-2\nu} (1-\epsilon)^{2\nu} \\ &= \frac{H^2\Gamma^2(\nu)}{\pi^3 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{aH}\right)^{3-2\nu} 2^{2\nu}(1-\epsilon)^{2\nu}. \end{aligned} \tag{C.52}$$

As before, we can approximate the terms $2^{2\nu} \simeq 2^3 = 8$ and $(1-\epsilon)^{2\nu} \simeq (1-\epsilon)^3 \simeq 1$, so basically we set for these terms $\epsilon, \mu_\chi \simeq 0$ (we take the massless de Sitter approximation). For the gamma function, we do the same, so that $\Gamma^2(\nu) \simeq \Gamma^2(3/2) = \pi/4$. The spectral tilt is given by

$$n_T = 3 - 2\nu = 3 - 3 - 2\epsilon + \frac{2\mu_\chi^2}{3H_0^2}\left(1 + \frac{4}{3}\epsilon\right) = -2\epsilon + \frac{2\mu_\chi^2}{3H_0^2}\left(1 + \frac{4}{3}\epsilon\right), \tag{C.53}$$

so the power spectrum with all these approximations is finally given by

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^{n_T} \quad \text{with} \quad n_T = -2\epsilon + \frac{2\mu_\chi^2}{3H_0^2}\left(1 + \frac{4}{3}\epsilon\right), \tag{C.54}$$

which is Equation (4.53).

2. Quasi-de Sitter for $s = -2$

We start at the solution (4.55),

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} e^{-i\pi/2(1-\epsilon)} \sqrt{-\eta} H^{(1)}_{-1/(1-\epsilon)-1/2}\left(-\eta\sqrt{c_T^2 k^2 + \mu_\chi^2}\right), \tag{C.55}$$

and want to calculate from this the general power spectrum (4.56). To do so, we substitute the solution again in Equation (C.47) so that the exponential disappears and we have

$$\begin{aligned} \mathcal{P}_\chi(k) &= \frac{4k^3}{\pi^2 M_{\rm Pl}^2 a^2}\frac{\pi}{4}(-\eta)\left|H^{(1)}_{-1/(1-\epsilon)-1/2}\left(-\eta\tilde{\omega}_k\right)\right|^2 \\ &= -\frac{k^3\eta}{\pi M_{\rm Pl}^2 a^2}\left|H^{(1)}_{-1/(1-\epsilon)-1/2}\left(-\eta\tilde{\omega}_k\right)\right|^2, \end{aligned} \tag{C.56}$$

where we have written $\tilde{\omega}_k^2 \equiv c_T^2 k^2 + \mu_\chi^2$ for simplicity. If we approximate the argument as $\simeq -c_T k\eta$, we can take the superhorizon limit $-c_T k\eta \ll 1$, but we substitute the full $\tilde{\omega}_k$ back after this. We use the small-argument limit of the Hankel function to calculate

$$\begin{aligned} \mathcal{P}_\chi(k) &\simeq -\frac{k^3\eta}{\pi^3 M_{\rm Pl}^2 a^2}\Gamma^2\left(-\frac{1}{1-\epsilon} - \frac{1}{2}\right)\left(-\frac{\eta\tilde{\omega}_k}{2}\right)^{2/(1-\epsilon)+1} \\ &= \frac{H^2}{\pi^3 M_{\rm Pl}^2 (aH)^2 c_T^3}\frac{(c_T k)^3}{(1-\epsilon)aH}\Gamma^2\left(\frac{-3+\epsilon}{2-2\epsilon}\right)\left(\frac{\sqrt{c_T^2 k^2 + \mu_\chi^2}}{2(1-\epsilon)aH}\right)^{(3-\epsilon)/(1-\epsilon)} \end{aligned} \tag{C.57}$$

where in the second line we substituted $\eta = -1/(1-\epsilon)aH$ in a quasi-de Sitter Universe. Let us then approximate $2^{-(3-\epsilon)/(1-\epsilon)} \simeq 2^{-3} = 1/8$, $(1-\epsilon)^{-(3-\epsilon)/(1-\epsilon)-1} \simeq 1$, and

$$\Gamma^2\left(\frac{-3+\epsilon}{2-2\epsilon}\right) \simeq \Gamma^2\left(-\frac{3}{2}\right) = \frac{16\pi}{9}. \tag{C.58}$$

The power spectrum then becomes

$$\mathcal{P}_\chi(k) = \frac{2H^2}{9\pi^2 M_{\rm Pl}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^3 \left(\frac{\sqrt{c_T^2 k^2 + \mu_\chi^2}}{aH}\right)^{(3-\epsilon)/(1-\epsilon)}, \tag{C.59}$$

which is Equation (4.56). We can go a step further and take the limits $c_T k \ll \mu_\chi$ and $c_T k \gg \mu_\chi$. In the second case, we approximate $\tilde{\omega}_k \simeq c_T k$ and the power spectrum becomes

$$\mathcal{P}_\chi(k) = \frac{2H^2}{9\pi^2 M_{\rm Pl}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^{n_T} \quad \text{with} \quad n_T = 3 + \frac{3-\epsilon}{1-\epsilon} = \frac{6-4\epsilon}{1-\epsilon} \simeq 6 + 2\epsilon, \tag{C.60}$$

which is Equation (4.57). For $c_T k \ll \mu_\chi$, we instead approximate $\tilde{\omega}_k \simeq \mu_\chi$ and we have simply

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{9\pi^2 M_{\rm Pl}^2 c_T^3}\left(\frac{m}{aH}\right)^{(3-\epsilon)/(1-\epsilon)} \left(\frac{c_T k}{aH}\right)^3, \tag{C.61}$$

which is Equation (4.58).

3. De Sitter and matter domination for $s = -2$

We have the solution (4.59) for $s = -2$ in a de Sitter or a matter-dominated background,

$$v_k(\eta) = \frac{1}{\sqrt{2\tilde{\omega}_k}}\left(1 - \frac{i}{\eta\tilde{\omega}_k}\right) e^{-i\eta\tilde{\omega}_k}, \tag{C.62}$$

with $\tilde{\omega}_k \equiv c_T^2 k^2 + \mu_\chi^2$, and want to calculate the general power spectrum (4.60). To do so, we substitute the solution into Equation (C.47). The exponential disappears and we use $|1 - Ai|^2 = 1 + A^2$ so that

$$\mathcal{P}_\chi(k) = \frac{4k^3}{\pi^2 M_{\rm Pl}^2 a^2}\frac{1}{2\tilde{\omega}_k}\left(1 + \frac{1}{\eta^2\tilde{\omega}_k^2}\right). \tag{C.63}$$

Using $\eta = p/aH$ in general (but noting that this is still only valid for $p = -1$ and $p = 2$), this can be written to

$$\mathcal{P}_\chi(k) = \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3}\frac{(c_T k)^3}{(aH)^2}\left(1 + \frac{1}{p^2}\frac{(aH)^2}{c_T^2 k^2 + \mu_\chi^2}\right)\frac{1}{\sqrt{c_T^2 k^2 + \mu_\chi^2}}, \tag{C.64}$$

which is Equation (4.60). Taking the superhorizon limit $c_T k \ll aH$, we see that the second term in brackets dominates over the first. We can then also take the limit $c_T k \gg \mu_\chi$, in which case we approximate $\tilde{\omega}_k \simeq c_T k$ and so

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{p^2\pi^2 M_{\rm Pl}^2 c_T^3}, \tag{C.65}$$

which is Equation (4.61). For $c_T k \ll \mu_\chi$, we instead end up with

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{p^2 \pi^2 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{\mu_\chi}\right)^3, \tag{C.66}$$

which is Equation (4.62).

4. Radiation domination for $s = -2$

We have the solution (4.63) for $s = -2$ in a radiation-dominated background,

$$v_k(\eta) = \frac{1}{\sqrt{2\tilde{\omega}_k}} e^{-i\eta\tilde{\omega}_k}, \tag{C.67}$$

with $\tilde{\omega}_k^2 \equiv c_T^2 k^2 + \mu_\chi^2$, and want to derive the general power spectrum (4.64) from this, which is again done by substituting the solution into Equation (C.47). This gets rid of the exponential and gives the power spectrum

$$\mathcal{P}_\chi(k) = \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3} \frac{(c_T k)^3}{(aH)^2} \frac{1}{\sqrt{c_T^2 k^2 + \mu_\chi^2}}, \tag{C.68}$$

which is immediately the general power spectrum (4.64). Taking then the limit $c_T k \gg \mu_\chi$ allows us to approximate $\tilde{\omega}_k \simeq c_T k$ so that

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{aH}\right)^2, \tag{C.69}$$

which is Equation (4.65). For $c_T k \ll \mu_\chi$, we instead approximate $\tilde{\omega}_k \simeq \mu_\chi$ and obtain the power spectrum

$$\begin{aligned} \mathcal{P}_\chi(k) &\simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3} \frac{(c_T k)^3}{\mu_\chi (aH)^2} \\ &= \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3} \frac{aH}{\mu_\chi} \left(\frac{c_T k}{aH}\right)^3, \end{aligned} \tag{C.70}$$

which is Equation (4.66).

5. General background with $s = -2(1 + 1/p)$

We start at the solutions (4.67) for $s = -2(1 + 1/p)$ in a general background,

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} \exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{-\eta} H_\nu^{(1)}(-c_T k\eta), \quad \text{for} \quad p < 0, \quad \text{and} \tag{C.71a}$$

$$v_k(\eta) = \frac{\sqrt{\pi}}{2} \exp\left[-\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right] \sqrt{\eta} H_\nu^{(2)}(c_T k\eta), \quad \text{for} \quad p > 0, \tag{C.71b}$$

where ν is approximated by Equation (4.68),

$$\nu \simeq \mp\left(p - \frac{1}{2} - \frac{\mu_\chi^2 p^2}{(2p-1)H_0^2}\right). \tag{C.72}$$

From this, we want to calculate the power spectrum for general p (Equation (4.69)), and we do so by plugging the solutions into Equation (C.47), as usual. We can combine the calculations of the two cases $p < 0$ and $p > 0$ by writing the solution as

$$v_k(\eta) = \frac{\sqrt{\pi}}{2}\exp\left[\pm\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right]\sqrt{\mp\eta}H_\nu^{(1,2)}(\mp c_T k\eta), \tag{C.73}$$

where the case $(+,-,(1),-)$ is for $p < 0$ and $(-,+,(2),+)$ for $p > 0$. The power spectrum can then in general be written as

$$\mathcal{P}_\chi(k) = \mp\frac{k^3\eta}{\pi M_{\mathrm{Pl}}^2 a^2}\left|H_\nu^{(1,2)}(\mp c_T k\eta)\right|^2. \tag{C.74}$$

We can then use the superhorizon limit, given by $\sim \mp c_T k\eta \ll 1$ (see Appendix A.2.4) for $p < 0$ and $p > 0$, in which case the Hankel functions are written as

$$\begin{aligned} &H_\nu^{(1,2)}(x \ll 1) \simeq \mp\frac{i}{\pi}\Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu} \\ \Longrightarrow &\left|H_\nu^{(1,2)}(x \ll 1)\right|^2 \simeq \frac{\Gamma^2(\nu)}{\pi^2}\left(\frac{x}{2}\right)^{-2\nu}. \end{aligned} \tag{C.75}$$

In this limit, the power spectrum is given by

$$\mathcal{P}_\chi(k) \simeq \mp\frac{k^3\eta}{\pi M_{\mathrm{Pl}}^2 a^2}\frac{\Gamma^2(\nu)}{\pi^2}\left(\mp\frac{c_T k\eta}{2}\right)^{-2\nu}. \tag{C.76}$$

Substituting then the general $\eta = p/aH$ (see Appendix A.2.3), gives

$$\begin{aligned} \mathcal{P}_\chi(k) &= (\mp 1)^{1-2\nu}\frac{H^2\Gamma^2(\nu)}{\pi^3 M_{\mathrm{Pl}}^2(aH)^2 c_T^3}\frac{p(c_T k)^3}{aH}\left(\frac{pc_T k}{2aH}\right)^{-2\nu} \\ &= \frac{2^{2\nu}H^2}{\pi^3 M_{\mathrm{Pl}}^2 c_T^3}\Gamma^2(\nu)p^{1-2\nu}\left(\frac{c_T k}{aH}\right)^{3-2\nu}, \end{aligned} \tag{C.77}$$

where the ∓ 1 disappears because

$$\begin{aligned} 1 - 2\nu &= 1 + 2p + 1 + \frac{2\mu_\chi^2 p^2}{(2p-1)H_0^2} \\ &= 2\left[1 + p + \frac{\mu_\chi^2 p^2}{(2p-1)H_0^2}\right] \quad \text{for} \quad p < 0, \end{aligned} \tag{C.78}$$

and

$$1-2\nu = 1-2p-1-\frac{2\mu_\chi^2 p^2}{(2p-1)H_0^2}$$
$$= 2\left[-p-\frac{\mu_\chi^2 p^2}{(2p-1)H_0^2}\right] \quad \text{for} \quad p>0. \tag{C.79}$$

In both cases, we can factor a 2 so $(\mp 1)^2 = 1$ then makes this entire term disappear. The above power spectrum (Equation (C.77)) is exactly the general power spectrum (4.69).

Next, we need to look at the three different cases $p = -1, 1, 2$ (we do not look at the quasi-de Sitter case as explained in Section 4.3.3). Taking $p = -1$, we calculate

$$\nu = \frac{3}{2} - \frac{\mu_\chi^2}{3H_0^2} \simeq \frac{3}{2} \quad \text{and} \quad -2\nu = -3 + \frac{2\mu_\chi^2}{3H_0^2} \simeq -3, \tag{C.80}$$

where we use the approximations only for terms that do not include the comoving wavenumber $c_T k$ or Hubble parameter aH, as these are the terms that determine the spectral tilt. Using therefore $\Gamma^2(\nu) \simeq \Gamma^2(3/2) = \pi/4$, $2^{2\nu} \simeq 2^3 = 8$, and $p^{1-2\nu} \simeq (-1)^{-2} = 1$, the power spectrum in this case becomes

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^{n_T} \quad \text{where} \quad n_T = \frac{2\mu_\chi^2}{3H_0^2}, \tag{C.81}$$

which is Equation (4.70a). For $p = 1$, we use

$$\nu = \frac{1}{2} - \frac{\mu_\chi^2}{H_0^2} \simeq \frac{1}{2} \quad \text{and} \quad -2\nu = -1 + \frac{2\mu_\chi^2}{H_0^2} \simeq -1, \tag{C.82}$$

and use again the full expressions only for terms including $c_T k$ and aH. We approximate $2^{2\nu} \simeq 2^1 = 2$ and $\Gamma^2(\nu) \simeq \Gamma^2(1/2) = \pi$, so the power spectrum becomes

$$\mathcal{P}_\chi(k) \simeq \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3}\left(\frac{c_T k}{aH}\right)^{n_T} \quad \text{where} \quad n_T = 2 + \frac{2\mu_\chi^2}{H_0^2}, \tag{C.83}$$

which is Equation (4.70b). Finally, for $p = 2$ we calculate

$$\nu = \frac{3}{2} - \frac{4\mu_\chi^2}{3H_0^2} \simeq \frac{3}{2} \quad \text{and} \quad -2\nu = -3 + \frac{8\mu_\chi^2}{3H_0^2} \simeq -3, \tag{C.84}$$

where we approximate in the same way as before $2^{2\nu} \simeq 2^3 = 8$, $\Gamma^2(\nu) \simeq \Gamma^2(3/2) = \pi/4$, and $p^{1-2\nu} \simeq 2^{-2} = 1/4$. The power spectrum in this case becomes

$$\mathcal{P}_\chi(k) \simeq \frac{H^2}{2\pi^2 M_{\rm Pl}^2 c_T^2}\left(\frac{c_T k}{aH}\right)^{n_T} \quad \text{where} \quad n_T = \frac{8\mu_\chi^2}{3H_0^2}, \tag{C.85}$$

which is Equation (4.70c).

6. Solution for $p = 1$ and $s = -1$

We have the general solution (4.42),

$$v_k(\eta) = c_1(k)\,\mathrm{Ai}\left(-\frac{c_T^2k^2 + \mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right) + c_2(k)\,\mathrm{Bi}\left(-\frac{c_T^2k^2 + \mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right), \tag{C.86}$$

and want to calculate the constants c_1 and c_2 so that we end up with the full general solution (4.43). To do so, we take the subhorizon limit $c_T k\eta \gg 1$, but since we do not have this combination inside the Airy functions, we simply take this argument to be $\gg 1$. From Chapter 9.7 of [36] it becomes clear that the large argument limits of the Airy functions are given by

$$\mathrm{Ai}(x \gg 1) \simeq \frac{e^{-\zeta}}{2\sqrt{\pi}x^{1/4}} \quad \text{and} \quad \mathrm{Bi}(x \gg 1) \simeq \frac{e^{\zeta}}{\sqrt{\pi}x^{1/4}}, \tag{C.87}$$

where $\zeta = 2x^{3/2}/3$ and in our case $x = -(c_T^2k^2 + \mu_\chi^2 H_0\eta)/(-\mu_\chi^2 H_0)^{2/3}$. Let us first focus on the Airy function of the first kind ($\mathrm{Ai}(x)$) and calculate the constant $c_1(k)$ from that. The subhorizon limit gives

$$v_k^{(1)}(\eta) \simeq \frac{c_1(k)}{2\sqrt{\pi}} \exp\left[-\frac{2}{3}\left(-\frac{c_T^2k^2 + \mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right)^{3/2}\right]\left(-\frac{(-\mu_\chi^2 H_0)^{2/3}}{c_T^2k^2 + \mu_\chi^2 H_0\eta}\right)^{1/4} \tag{C.88a}$$

$$= \frac{1}{\sqrt{2c_T k}} e^{-ic_T k\eta}, \tag{C.88b}$$

where in the second line we compare the expression with the subhorizon solution (4.12). Let us now focus on the exponent in the above expression, which we want to eventually reduce to something like $-ic_T k\eta$, as in the second line. We can write out the power $3/2 = 1 + 1/2$, and use $(-1)^{1/2} = i$ so that

$$\begin{aligned} -\frac{2}{3}x^{3/2} &= -\frac{2}{3}\left(-\frac{c_T^2k^2 + \mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right)^{3/2} \\ &= -\frac{2i}{3}\left(-\frac{c_T^2k^2 + \mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right)\left(\frac{\sqrt{c_T^2k^2 + \mu_\chi^2 H_0\eta}}{(-\mu_\chi^2 H_0)^{1/3}}\right) \\ &= -\frac{2i}{3}\left(\frac{\left(c_T^2k^2 + \mu_\chi^2 H_0\eta\right) c_T k\sqrt{1 + \mu_\chi^2 H_0\eta/c_T^2k^2}}{\mu_\chi^2 H_0}\right). \end{aligned} \tag{C.89}$$

We now assume that $\mu_\chi^2 H_0\eta \ll c_T^2k^2$ so that we can Taylor expand the square root to order $\mathcal{O}(\eta)$,

$$\sqrt{1 + \frac{\mu_\chi^2 H_0\eta}{c_T^2k^2}} \simeq 1 + \frac{\mu_\chi^2 H_0\eta}{2c_T^2k^2}. \tag{C.90}$$

Then we continue the calculation,

$$\begin{aligned} -\frac{2}{3}x^{3/2} &\simeq -\frac{2i}{3\mu_\chi^2 H_0}\left[c_T k\left(c_T^2k^2+\mu_\chi^2 H_0\eta\right)\left(1+\frac{\mu_\chi^2 H_0\eta}{2c_T^2k^2}\right)\right] \\ &= -\frac{2i}{3\mu_\chi^2 H_0}\left(c_T^2k^2+\mu_\chi^2 H_0\eta\right)\left(c_T k+\frac{\mu_\chi^2 H_0\eta}{2c_T k}\right) \\ &= -\frac{2i}{3\mu_\chi^2 H_0}\left[c_T^3k^3+\frac{1}{2}\mu_\chi^2 H_0 c_T k\eta+\mu_\chi^2 H_0 c_T k\eta+\mathcal{O}(\eta^2)\right] \\ &\simeq -\frac{2i}{3\mu_\chi^2 H_0}\left(\frac{3}{2}\mu_\chi^2 H_0 c_T k\eta+c_T^3k^3\right) \\ &= -ic_T k\eta-\frac{2ic_T^3k^3}{3\mu_\chi^2 H_0}. \end{aligned} \tag{C.91}$$

We can now finally see that the exponent contains a factor $-ic_T k\eta$, just like the subhorizon solution (C.88b). We therefore also immediately see that because $\mathrm{Bi}(x \gg 1) \propto e^{\zeta}$ while $\mathrm{Ai}(x \gg 1) \propto e^{-\zeta}$, this minus sign causes the appearance of the $-ic_T k\eta$, so for the Airy function of the second kind we will find the exponent $ic_T k\eta$, which is not seen in the subhorizon solution (C.88b). Therefore, we can conclude already that $c_2(k) = 0$. For $c_1(k)$, we substitute the exponent (C.91) into Equation (C.88a) to find

$$\begin{aligned} \frac{e^{-ic_T k\eta}}{\sqrt{2c_T k}} &\simeq \frac{c_1(k)}{2\sqrt{\pi}}\exp\left[-ic_T k\eta-\frac{2ic_T^3k^3}{3\mu_\chi^2 H_0}\right]\left(-\frac{(-\mu_\chi^2 H_0)^{2/3}}{c_T^2k^2+\mu_\chi^2 H_0\eta}\right)^{1/4} \\ \Longrightarrow c_1(k) &= \sqrt{\frac{2\pi}{c_T k}}\left(-\frac{c_T^2k^2+\mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right)^{1/4}\exp\left[\frac{2ic_T^3k^3}{3\mu_\chi^2 H_0}\right]. \end{aligned} \tag{C.92}$$

Let us then focus on the second term with the power $1/4$, and write out some distinct terms. Also including the square root at the beginning, we can write

$$\begin{aligned} \sqrt{\frac{2\pi}{c_T k}}\left(-\frac{c_T^2k^2+\mu_\chi^2 H_0\eta}{(-\mu_\chi^2 H_0)^{2/3}}\right)^{1/4} &= \sqrt{\frac{2\pi}{c_T k}}\frac{\sqrt{c_T k}(-1)^{1/4}}{(-\mu_\chi^2 H_0)^{1/6}}\left(1+\frac{\mu_\chi^2 H_0\eta}{c_T^2k^2}\right)^{1/4} \\ &= \left(\frac{-4\pi^2}{-\mu_\chi^2 H_0}\right)^{1/4}\frac{1}{(-\mu_\chi^2 H_0)^{-1/12}}\left(1+\mathcal{O}(\eta)\right) \\ &\simeq \left(\frac{4\pi^2}{\mu_\chi^2 H_0}\right)^{1/4}\left(-\mu_\chi^2 H_0\right)^{1/12}, \end{aligned} \tag{C.93}$$

where in the second line we broke up the denominator using the powers $1/6 = 1/4 - 1/12$, and we included the $\sqrt{2\pi} = (4\pi^2)^{1/4}$ in the first term as well. Also, we have Taylor expanded again the final term in the first line, using $\mu_\chi^2 H_0\eta \ll c_T^2k^2$ again, but since we do not want a η-dependence in the constant $c_1(k)$, we omit all terms to first order and higher in η. Substituting this now into Equation (C.92), we find

$$c_1(k) = \left(\frac{4\pi^2}{\mu_\chi^2 H_0}\right)^{1/4} \left(-\mu_\chi^2 H_0\right)^{1/12} \exp\left[\frac{2ic_T^3 k^3}{3\mu_\chi^2 H_0}\right], \tag{C.94}$$

and so the full solution finally becomes

$$v_k(\eta) = \left(\frac{4\pi^2}{\mu_\chi^2 H_0}\right)^{1/4} \left(-\mu_\chi^2 H_0\right)^{1/12} \exp\left[\frac{2ic_T^3 k^3}{3\mu_\chi^2 H_0}\right] \mathrm{Ai}\left(-\frac{c_T^2 k^2 + \mu_\chi^2 H_0 \eta}{(-\mu_\chi^2 H_0)^{2/3}}\right), \tag{C.95}$$

which is Equation (4.43).

7. Radiation domination with $s = -1$

We start at the general solution (4.71),

$$v_k(\eta) \simeq \left(\frac{4\pi^2}{\mu_\chi^2 H_0}\right)^{1/4} \left(-\mu_\chi^2 H_0\right)^{1/12} \exp\left[\frac{2ic_T^3 k^3}{3\mu_\chi^2 H_0}\right] \mathrm{Ai}\left(-\frac{c_T^2 k^2 + \mu_\chi^2 H_0 \eta}{(-\mu_\chi^2 H_0)^{2/3}}\right), \tag{C.96}$$

and want to calculate from this the general power spectrum (4.72a). To do so, we start by substituting the solution into Equation (C.47), as usual. The absolute values cause the exponential to disappear, and we also have $|-1|^{2/12} = 1$, so the power spectrum then becomes

$$\begin{aligned}\mathcal{P}_\chi(k) &= \frac{4k^3}{\pi^2 M_{\mathrm{Pl}}^2 a^2} \frac{2\pi}{(\mu_\chi^2 H_0)^{1/3}} \left|\mathrm{Ai}\left(-\frac{c_T^2 k^2 + \mu_\chi^2 H_0 \eta}{(-\mu_\chi^2 H_0)^{2/3}}\right)\right|^2 \\ &= \frac{8H^2}{\pi M_{\mathrm{Pl}}^2 c_T^3} \frac{(c_T k)^3}{(aH)^2} \frac{1}{(\mu_\chi^2 H_0)^{1/3}} \left|\mathrm{Ai}\left(-\frac{c_T^2 k^2 + \mu_\chi^2 H_0 \eta}{(-\mu_\chi^2 H_0)^{2/3}}\right)\right|^2,\end{aligned} \tag{C.97}$$

where we have combined the $\mu_\chi^2 H_0$-terms again, resulting in the power $1/3$. If we now want to apply the superhorizon limit to this, we cannot continue as we have no expression for the small-argument limit of the Airy function. Instead, we can write it as a Hankel function via (see Chapter 9.6 of [36])

$$\mathrm{Ai}(x) = \frac{1}{2}\sqrt{\frac{x}{3}} e^{2i\pi/3} H_{1/3}^{(1)}\left(\frac{2}{3}x^{3/2}e^{i\pi/2}\right) \tag{C.98a}$$

$$\Longrightarrow |\mathrm{Ai}(x)|^2 = \frac{|x|}{12}\left|H_{1/3}^{(1)}\left(\frac{2}{3}x^{3/2}e^{i\pi/2}\right)\right|^2. \tag{C.98b}$$

In the same way as in the previous calculation, we can take the superhorizon limit by assuming $x \ll 1$, for which we use the expression (see Appendix B.2.1)

$$\left|H_\nu^{(1)}(x \ll 1)\right|^2 \simeq \frac{\Gamma^2(\nu)}{\pi^2}\left(\frac{|x|}{2}\right)^{-2\nu}. \tag{C.99}$$

With $\nu = 1/3$ and $x = -(c_T^2 k^2 + \mu_\chi^2 H_0 \eta)/(-\mu_\chi^2 H_0)^{2/3}$, the power spectrum is then

$$\begin{aligned}
\mathcal{P}_\chi(k) &\simeq \frac{8H^2}{\pi M_{\rm Pl}^2 c_T^3} \frac{(c_T k)^3}{(aH)^2} \frac{1}{(\mu_\chi^2 H_0)^{1/3}} \frac{1}{12} \left| -\frac{c_T^2 k^2 + \mu_\chi^2 H_0 \eta}{(-\mu_\chi^2 H_0)^{2/3}} \right| \frac{\Gamma^2(1/3)}{\pi^2} \times \\
&\times \left[\frac{1}{3} \left(\frac{c_T^2 k^2 + \mu_\chi^2 H_0 \eta}{(\mu_\chi^2 H_0)^{2/3}} \right)^{3/2} \right]^{-2/3} \\
&= \frac{2H^2}{3\pi^3 M_{\rm Pl}^2 c_T^3} \frac{(c_T k)^3}{(aH)^2} \frac{1}{(\mu_\chi^2 H_0)^{1/3}} \Gamma^2\left(\frac{1}{3}\right) \left(\frac{1}{3}\right)^{-2/3} \\
&= \frac{2H^2}{\pi^3 M_{\rm Pl}^2 c_T^3} \frac{(c_T k)^3}{(aH)^2} \left(\frac{1}{3\mu_\chi^2 H_0} \right)^{1/3} \Gamma^2\left(\frac{1}{3}\right). \qquad \text{(C.100)}
\end{aligned}$$

Next, we can write the Friedmann equation during the radiation-dominated era with $\Omega_\gamma = 1$ as

$$\left(\frac{H}{H_0}\right)^2 = \frac{1}{a^4} \quad \Longleftrightarrow \quad H_0 = a^2 H, \qquad \text{(C.101)}$$

which means that we can write the power spectrum as

$$\begin{aligned}
\mathcal{P}_\chi(k) &= \frac{2H^2}{\pi^3 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{aH}\right)^3 aH \left(\frac{1}{3\mu_\chi^2 H a^2}\right)^{1/3} \Gamma^2\left(\frac{1}{3}\right) \\
&= \frac{2H^2}{\pi^3 M_{\rm Pl}^2 c_T^3} \left(\frac{c_T k}{aH}\right)^3 \left(\frac{aH^2}{3\mu_\chi^2}\right)^{1/3} \Gamma^2\left(\frac{1}{3}\right), \qquad \text{(C.102)}
\end{aligned}$$

which is Equation (4.72a). As a final step, we can numerically approximate some terms, leaving as the prefactor the irreducible inflationary constant power spectrum $\mathcal{P}_\chi^{\rm inf}(k) = 2H^2/\pi^2 M_{\rm Pl}^2$. We therefore calculate

$$\Gamma^2\left(\frac{1}{3}\right) \frac{3^{-1/3}}{\pi} \simeq 1.58, \qquad \text{(C.103)}$$

so that we end up with

$$\mathcal{P}_\chi(k) \simeq 1.58 \frac{2H^2}{\pi^2 M_{\rm Pl}^2 c_T^3} \left(\frac{aH^2}{\mu_\chi^2}\right)^{1/3} \left(\frac{c_T k}{aH}\right)^3, \qquad \text{(C.104)}$$

which is Equation (4.72b).