Variations of GIT

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Introduction

The study of invariant theory is indispensable for mathematicians seeking to classify various objects, and algebraic geometry is no exception. As a result, a lot of research has been dedicated to this subject. The foundations of invariant theory for algebraic varieties were initially laid by Hilbert in 1890 and later refined by Mumford, leading to the development of the geometric invariant theory we know today. One significant motivation behind this theory is the construction of moduli spaces.

However, as one might have seen in an introduction course on scheme theory, quotients by some group action do not generally give nice results. Even when we look at regular algebraic varieties, there should be no reason for the regular functions on a variety to be compatible with the group action. Moreover, we are faced with two different types of quotients that deserve our attention. The first type corresponds to categorical quotients, which are objects that have a specific universal property within the associated category. On top of that, in classical settings, there is often a desire for quotients that provide some insight into the structure of the orbit space, known as geometric quotients.

In this thesis, we aim to provide an introduction to geometric invariant theory while exploring its applications as seen in the literature. Specifically, we will look into the theory of variations of geometric invariant theory, investigating the implications of altering the quotient. As we would like it to be a complete introduction, anyone that has followed an introductory course in scheme theory should be able to follow the entire thesis. Although some concepts may not have been explicitly defined within such a course, they will not hinder the general view or main ideas presented. Nonetheless, we shall recall some definitions and notations if we feel that this is necessary.

In the first chapter we shall introduce the theory of group schemes, which are schemes endowed with a natural group structure. This gives us the tools to define group actions of group schemes on schemes, and then define what we mean by a quotient. As previously mentioned, it would be nice for the group action to have some induced structure on the sections of a scheme. Moreover, we aim to extend this concept to general quasi-coherent modules. This gives us the theory of equivariant structures and linearizations of our group on invertible sheaves. We end this chapter by introducing line bundles corresponding to invertible sheaves, providing an elegant reformulations of linearizations.

Chapter two will contain the main theory on GIT-quotients. After exploring the theory defined by Mumford in [5], we shall look at some alternative methods of expanding this theory. One way to do this is by considering something we call a GIT-fan. There are multiple versions of this fan, but we shall use the definition used by Ressayre in [21]. The final part shall be some theory on flips, the idea being that crossing a "wall" in our fan will "flip" a Cartier divisor.

The rest of the thesis shall be used to work towards variations of GIT theory. To build up to this theory, we will define triangulated categories, Verdier quotients and derived categories in chapter three. Chapter four shall be used to tell a short note on stacks, HKKN stratifications and factorizations all used in the final theory.

In the final two chapters we consider the variation of GIT theory as developed by Ballard, Favero and Katzarkov in [3]. We will give some definitions, theorems and explanations to be able to sketch the proof of this theorem. On top of this we have some applications firstly given by Favero, Kaplan and Kelly in [10]. We shall give some examples of important singularity categories that they haven't explored yet, and after that study their statements. As a last part, we give some other version of the main theorem as stated in this chapter.

1 Group actions and quotients

In this chapter we introduce the topic of group actions and quotients. We start off by defining this for general schemes, and slowly make our way towards varieties as it is our goal to study their quotients. At the end of the chapter we introduce some new way to view invertible sheaves on a variety, which gives us more tools to study G-equivariant structures on sheaves.

1.1 Conventions

There are quite a few conventions and notation we shall use throughout the thesis. We have listed all conventions here so that the reader has an extra reference just in case something is unclear.

- When talking about schemes X and Y over an affine base scheme $S = \mathrm{Spec}(A)$, we will usually denote the fiber product as $X \times_A Y := X \times_{\mathrm{Spec}(A)} Y$.
- If $X_1, ..., X_n$ are all schemes over some base scheme S, we shall write $p_i : X_1 \times_S X_2 \times_S \cdots \times_S X_n \to X_i$ for the i-th projection map.
- With a geometric point of a scheme X we shall mean a $\mathrm{Spec}(k)$-valued point for some algebraically closed field k.
- Whenever we have a map of schemes $f : X \to Y$ with $Y = \mathrm{Spec}(R)$ affine, the usual duality tells us that f corresponds to some ring map $R \to \mathcal{O}_X(X)$. We shall denote this ring map by $f^\vee := f^\#_{\mathrm{Spec}(R)}$, called the dual map.
- With k we shall always mean a field, for simplicity we shall assume this field to be algebraically closed and of characteristic 0. In some cases of thesis these last two properties are not necessary. We write $k^\times := k \setminus \{0\}$.
- A variety X will be a separated scheme of finite type over k that is geometrically integral. With a geometrically integral scheme over k we mean a scheme X such that the base change $\overline{X} := X \times_k \mathrm{Spec}(\overline{k})$ is integral. A closed (respectively open) subvariety shall be a closed (respectively open) subscheme that is also a variety.
- We let $\mathbb{A}^n_k := \mathrm{Spec}\ k[x_1, ..., x_n]$ and $\mathbb{P}^n_k := \mathrm{Proj}\ k[x_0, ..., x_n]$ denote the standard affine and projective varieties. For a construction of the projectification of a graded ring, we refer to the definition on page 76 of Hartshorne, see [13].
- With $e_i \in \mathbb{A}^n_k$ we shall mean the element given on points by a 1 on coordinate i and a 0 else. With $e_i \in \mathbb{P}^n_k$ we shall mean the element given on points by a 1 on coordinate $i + 1$ and 0 else.

1.2 Group schemes

Let S be a scheme. In this section, all schemes shall be defined as schemes over S, unless stated otherwise. In other words, with a scheme X we shall mean a scheme X together with a structure morphism $\pi : X \to S$.

Definition 1.1. A **group scheme** is a tuple (G, μ, ι, e), where G is a scheme and $\mu : G \times_S G \to G$, $\iota : G \to G$ and $e : S \to G$ are S-morphisms satisfying the following properties:

(i) The diagram

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{\mathrm{Id}_G \times \mu} & G \times_S G \\ \downarrow{\scriptstyle \mu \times \mathrm{Id}_G} & & \downarrow{\scriptstyle \mu} \\ G \times_S G & \xrightarrow{\mu} & G \end{array}$$

commutes.

(ii) Write i_1, i_2 for the isomorphisms $G \cong S \times_S G$ and $G \cong G \times_S S$ respectively. Then the compositions

$$\begin{array}{ccccc} & & S \times_S G & & \\ & \overset{i_1}{\nearrow} & & \overset{e \times \mathrm{Id}_G}{\searrow} & \\ G & & & & G \times_S G \xrightarrow{\mu} G \\ & \underset{i_2}{\searrow} & & \underset{\mathrm{Id}_G \times e}{\nearrow} & \\ & & G \times_S S & & \end{array}$$

both equal the identity on G.

(iii) Let $\Delta_G : G \to G \times_S G$ denote the diagonal morphism and let $\pi : G \to S$ denote the structure morphism. Then the compositions

$$G \xrightarrow{\Delta_G} G \times_S G \begin{array}{c} \xrightarrow{\iota \times \mathrm{Id}_G} G \times_S G \xrightarrow{\mu} \\ \xrightarrow{\mathrm{Id}_G \times \iota} G \times_S G \xrightarrow{\mu} \end{array} G$$

both equal $e \circ \pi$.

We shall usually leave out the S-morphisms and write G for the group scheme if these morphisms are clear from the context.

Example 1.2. There are two natural group schemes $\mathbb{G}_{m,S}$ and $\mathbb{G}_{a,S}$, constructed as follows.

- First for $S = \mathrm{Spec}(R)$ affine we have the multiplicative group scheme $\mathbb{G}_{m,S} := \mathrm{Spec}(R[x, x^{-1}])$. Its group action σ is defined by the R-algebra map $R[x, x^{-1}] \to R[x, x^{-1}] \otimes_R R[x, x^{-1}]$ given by $x \mapsto x \otimes x$. The inverse map ι will be given by the R-algebra map $R[x, x^{-1}] \to R[x, x^{-1}]$ defined by $x \mapsto x^{-1}$ and the identity element e is given by the R-algebra map $R[x, x^{-1}] \to R$ defined by $x \mapsto 1$.

  If S is not affine, we can define $\mathbb{G}_{m,S} := \mathbb{G}_{m,\mathbb{Z}} \times_{\mathbb{Z}} S$, where $S \mapsto \mathrm{Spec}(\mathbb{Z})$ is the canonical scheme map induced from the canonical ring map $\mathbb{Z} \to \mathcal{O}_S(S)$. Note that whenever S is affine, this second definition agrees with the first definition.

- Another group scheme would be the additive group scheme $\mathbb{G}_{a,S}$ over S. Similarly as for the multiplicative group we can first define it when $S = \mathrm{Spec}(R)$ is affine by $\mathbb{G}_{a,S} := \mathrm{Spec}(R[x])$. Its group action is additive; σ would now be given by the R-algebra map $R[x] \to R[x] \otimes_R R[x]$ given by $x \mapsto x \otimes 1 + 1 \otimes x$. With the same idea in mind, the inverse map ι would be defined by $x \mapsto -x$ and the identity element e by $x \mapsto 0$.

  When S is not affine, we again define $\mathbb{G}_{a,S} := \mathbb{G}_{a,\mathbb{Z}} \times_{\mathbb{Z}} S$. If we ignore the group structure, then a common notation will be $\mathbb{A}^1_S$, also known as the affine line over S.

Example 1.3. Another interesting example is the group scheme $\mathrm{GL}(n, k)$ over $S = \mathrm{Spec}(k)$, where $n \in \mathbb{Z}_{\geq 1}$ and k is some field. This group scheme will mimic the $n \times n$ invertible matrices as we are used to from linear algebra. To be precise, we define $\mathrm{GL}(n, k) := \mathrm{Spec}(k[x_{ij}, \det(x)^{-1}]_{1 \leq i,j \leq n})$, where $\det(x)$ is the polynomial given by the 'determinant' of the matrix with indices x_{ij}. The multiplication will be given by the ring map

$$k[x_{ij}, \det(x)^{-1}]_{1 \leq i,j \leq n} \to k[y_{ij}, \det(y)^{-1}]_{1 \leq i,j \leq n} \otimes_k k[z_{ij}, \det(z)^{-1}]_{1 \leq i,j \leq n};$$

$$x_{ij} \mapsto \sum_{l=1}^{n} y_{il} z_{lj}.$$

The inverse will be given by

$$k[x_{ij}, \det(x)^{-1}]_{1 \leq i,j \leq n} \to k[x_{ij}, \det(x)^{-1}]_{1 \leq i,j \leq n};$$

$$x_{ij} \mapsto x^{\star}_{ij} \det(x)^{-1},$$

where $x^{\star}_{ij}$ is the (i, j)-th element of the adjugate matrix of x. Finally, the identity element will be given by

$$k[x_{ij}, \det(x)^{-1}]_{1 \leq i,j \leq n} \to k;$$

$$x_{ij} \mapsto \delta_{ij},$$

where δ_{ij} is the Kronecker delta function.

Example 1.4. The product of two group schemes gives a canonical group scheme by choosing the multiplication, inverse and identity element coordinatewise.

If G is a group scheme, then the axioms above actually give us a natural group structure in a way. Let us recall the functor of points. Suppose T is some other scheme, then we write $G(T) := \mathrm{Hom}_S(T, G)$ which we call the set of T-points of G. We then get a very nice interpretation.

Lemma 1.5. *Let (G, μ, ι, e) be a group scheme and T some other scheme. The set $G(T)$ has a natural group structure induced by μ, ι and e.*

Proof. The universal property of the product show that $(G \times_S G)(T) = G(T) \times_{S(T)} G(T)$, and therefore we have an induced operation $\mu(T) : G(T) \times_{S(T)} G(T) \to G(T)$. We claim that this puts a group action on the set $G(T)$; the first axiom of a group scheme shows that this operation is associative, the second that $G(T)$ has an identity element given by the composition of the structure map $T \to S$ and the map $e : S \to G$, and the third that every element $g \in G(T)$ has an inverse element given by $\iota \circ g : T \to G$.

□

Unsurprisingly, it is even true that the functor of points is a group functor. In other words, the (contravariant) functor $T \mapsto G(T)$ factorizes through the category of groups!

Definition 1.6. Let G, H be group schemes. A **homomorphism** of group schemes will be a morphism $f : G \to H$ of schemes over S such that for all schemes T the induced map $f(T) : G(T) \to H(T)$ is a group homomorphism.

Definition 1.7. Let G be a group scheme and let X be some other scheme. A **group action** of G on X is an S-morphism $\sigma : G \times_S X \to X$ satisfying the following properties:

(i) The diagram

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{\mathrm{Id}_G \times \sigma} & G \times_S X \\ \downarrow{\scriptstyle \mu \times \mathrm{Id}_X} & & \downarrow{\scriptstyle \sigma} \\ G \times_S X & \xrightarrow{\quad \sigma \quad} & X \end{array}$$

commutes.

(ii) Let i denote the isomorphism $X \cong S \times_S X$, then the composition

$$X \xrightarrow{\ i\ } S \times_S X \xrightarrow{e \times \mathrm{Id}_X} G \times_S X \xrightarrow{\ \sigma\ } X$$

equals the identity on X.

Definition 1.8. Let G be a group scheme acting on a scheme X via an action $\sigma : G \times_S X \to X$. Let Y be any scheme. We call a morphism $\phi : X \to Y$ **G-invariant** if the diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\ \sigma\ } & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow{\ \phi\ } & Y \end{array}$$

commutes.

Definition 1.9. Let G be a group scheme acting on a scheme X via an action σ. A **categorical quotient** of X by G is a pair (Y, ϕ) where Y is a scheme and $\phi : X \to Y$ is a G-invariant S-morphism such that if Z is a scheme and $\psi : X \to Z$ is a G-invariant morphism, then there exists a unique S-morphism $f : Y \to Z$ such that $\psi = f \circ \phi$. In other words, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\ \psi\ } & Z \\ {\scriptstyle \phi}\downarrow & \nearrow{\scriptstyle \exists! f} & \\ Y & & \end{array}$$

If a categorical quotient exists, we often write X/G for the quotient instead of Y. Note that this is only defined up to isomorphism.

Remark 1.10. We make the very important remark that categorical quotients do not always exist, but it can be difficult to check if this is the case. The paper by A'Campo-Neuen and Hausen tackles this problem, see [1]. Luckily, if categorical quotients exist, then the quotient preserves some properties. For example, one can quickly see that if X is reduced, irreducible, or connected respectively, then Y is reduced, irreducible or connected respectively. Other properties, like being Noetherian, are not necessarily preserved.

Definition 1.11 (GIT, [5], Definition 0.6)**.** Let X be a scheme and let G be a group scheme acting on X. A **geometric quotient** of X by G is a pair (Y, ϕ) where Y is a scheme and $\phi : X \to Y$ is a G-invariant S-morphism such that:

(i) ϕ is surjective, and the image of $(\sigma, p_2) : G \times_S X \to X \times_S X$ equals $X \times_Y X$.

(ii) A subset $U \subseteq Y$ is open if and only if $\phi^{-1}(U) \subseteq X$ is open.

(iii) If $f \in \Gamma(U, \phi_*(\mathcal{O}_X)) = \Gamma(\phi^{-1}(U), \mathcal{O}_X)$ and if $F : \phi^{-1}(U) \to \mathbb{A}^1_k$ is the corresponding morphism then $f \in \Gamma(U, \mathcal{O}_Y)$ if and only if F is G-invariant (by restricting to $\phi^{-1}(U)$). In other words, the sheaf $\mathcal{O}_Y$ is the subsheaf of $\phi_*(\mathcal{O}_X)$ consisting of G-invariant functions.

As stated in GIT, [5], property (i) has an equivalent statement that one might find easier to understand. It states that "the geometric fibres of ϕ are exactly the orbits of the geometric points of X over an algebraically closed field of sufficiently high transcendence degree". Moreover, if G is of finite type over S and X is of finite type over Y, then this would be true for any algebraically closed field. This statement is a bit vague, and will not be used in this thesis, but it might give some insight on how geometric fibres of a geometric quotient act.

Proposition 1.12 (GIT, [5], Proposition 0.1)**.** *Let G be a group scheme acting on a scheme X. If (Y, ϕ) is a geometric quotient of X by G, then it is also a categorical quotient of X by G. In particular, geometric quotients are unique up to isomorphism.*

Definition 1.13. Let G be a group scheme. A closed (respectively open) subscheme H of G is called a closed (respectively open) **subgroup scheme** if for all schemes T the set $H(T) \subseteq G(T)$ is a closed (respectively open) subgroup. We call a subgroup scheme **normal** if for all schemes T the sets $H(T)$ are normal subgroups.

We shall refer to closed subgroup schemes when talking about subgroup schemes.

1.3 Geometric invariance

In this thesis we are interested in geometric objects called varieties, and we would like to translate the notion of a classical variety to the language of schemes. Our definition will be a separated scheme of finite type over a field k that is geometrically integral. In particular notice that varieties are Noetherian. As we are mostly interested in the case when k is algebraically closed, the last requirement only asks the scheme to be integral. However, this provides a definition that is applicable outside of this thesis. One of the reasons for this definition, is that they give the same intuition as classical varieties. For example, we have the following result without assuming k to be algebraically closed.

Proposition 1.14. *Let X, Y be varieties and let $f, g : X \to Y$ be morphisms of varieties. Then the following are equivalent:*

(i) $f = g$;

(ii) For any non-zero dense open subset $U \subseteq X$ we have $f|_U = g|_U$;

(iii) f and g agree on $\overline{k}$-points.

Proof. (i)⇒ (ii) follows immediately. For (ii)⇒(iii), U being dense implies that the underlying topological maps of f and g agree. For (iii)⇒ (i) we have to do a bit more work. Since the set of $\overline{k}$-points of X is dense, we again get that the underlying topological maps agree. Next, we notice that it suffices to show that the base changes $f_{\overline{k}}$ and $g_{\overline{k}}$ equal, so that we may assume $k = \overline{k}$. It suffices now to show that the induced maps $f^\#_V, g^\#_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$ are the same for each open $V \subseteq Y$. We know that to give a k-algebra map $k[x] \to \mathcal{O}_Y(V)$ is the same as to give a k-morphism $V \to \mathbb{A}^1_k$, so we can identify any section $v \in \mathcal{O}_Y(V)$ by the morphism $\phi_v : V \to \mathbb{A}^1_k$ given by the k-algebra map sending x to v. Now $f^\#_V(v) = g^\#_V(v)$ holds when $\phi_v \circ f|_{f^{-1}V} = \phi_v \circ g|_{f^{-1}V}$, so we may assume that $Y = \mathbb{A}^1_k$. Furthermore, by viewing the morphisms locally, we may assume X to be affine.

Write $X = \mathrm{Spec}(R)$, where R is now a reduced k-algebra of finite type. Notice that this last property implies that R is Jacobson, and therefore that the intersection of all maximal ideals is 0. Now consider two global sections $v, w \in \mathcal{O}_X(X)$, and suppose that $v(x) = w(x)$ for all k-points x of X. We would like to show that $v = w$. Since this is the same as assuming that $(v - w)(x) = 0$ for all such k-points, we may reduce to the case where $w = 0$. But $v(x) = 0$ for all k-points x implies that v is an element of all maximal ideals of R, and therefore of all prime ideals of R. Hence $v = 0$.

□

For the rest of the thesis, let k be an algebraically closed field of characteristic 0. There is a special type of variety that combines this definition with the definition of a group scheme. In the literature they may also be commonly known as algebraic groups, but we stuck with the term group varieties.

Definition 1.15. Let G be a group scheme over k. We call G a **group variety** if G is also a variety.

Example 1.16. The group scheme $\mathrm{GL}(n, k)$ as seen in Example 1.3 is also a group variety.

There is a very important example of a group variety which we will use throughout this thesis that deserves a definition on its own.

Definition 1.17. Define $\mathbb{G}_m := \mathbb{G}_{m,k}$ and let r be a positive integer. We call the group variety $\mathbb{G}_m^r$ an r**-torus** or simply torus when r is irrelevant for the context.

The most important torus will be the 1-torus $\mathbb{G}_m$. Actions of this group variety on an affine variety can be given by looking at the underlying rings.

Lemma 1.18. *Let $X := Spec(R)$ be an affine variety and let $G := \mathbb{G}_m$. Then, to give an action of G on X is the same as giving a $\mathbb{Z}$-grading on R with $k \subseteq R_0$.*

Proof. Consider the group variety $G := \mathbb{G}_m$ acting on some affine variety $X = \operatorname{Spec}(R)$ via an action $\sigma : \mathbb{G}_m \times_k \operatorname{Spec}(R) \to \operatorname{Spec}(R)$. Then σ corresponds to a k-algebra map $\sigma^\vee : R \to R \otimes_k k[x, x^{-1}] \cong R[x, x^{-1}]$. For $i \in \mathbb{Z}$ we define $R_i := \{r \in R \mid \sigma^\vee(r) = x^i \otimes r\}$ and we shall write $\mu : \mathbb{G}_m \times_k \mathbb{G}_m \to \mathbb{G}_m$ for the multiplication of $\mathbb{G}_m$.

Since G, X are both affine, we can rewrite the first axiom of Definition 1.7 to a diagram of ring maps:

$$\begin{array}{ccc} k[x,x^{-1}] \otimes_k k[x,x^{-1}] \otimes_k R & \xleftarrow{\mathrm{Id}_{k[x,x^{-1}]}\otimes\sigma^\vee} & k[x,x^{-1}] \otimes_k R \\ \uparrow{\scriptstyle \mu^\vee\otimes\mathrm{Id}_R} & & \uparrow{\scriptstyle \sigma^\vee} \\ k[x,x^{-1}] \otimes_k R & \xleftarrow[\sigma^\vee]{} & R \end{array}$$

Hence if we let $r \in R$ and define $r_i \in R$ for $i \in \mathbb{Z}$ such that $\sum_{i\in\mathbb{Z}} x^i \otimes r_i = \sigma^\vee(r)$, then the commutativity of the square gives us the equality:

$$\sum_{i\in\mathbb{Z}} x^i \otimes x^i \otimes r_i = \sum_{i\in\mathbb{Z}} x^i \otimes \sigma^\vee(r_i).$$

Since R is a domain, this implies that $\sigma^\vee(r_i) = x^i \otimes r_i$ and hence that $r_i \in R_i$. The second axiom tells us that the composition

$$R \xleftarrow{e^\vee\otimes\mathrm{Id}_R} k[x,x^{-1}] \otimes_k R \xleftarrow{\sigma^\vee} R$$

has to equal the identity on R, where $e^\vee : k[x, x^{-1}] \to k$ was defined by $x \mapsto 1$. With the same $r \in R$ and corresponding $r_i \in R$ defined as above, this simply states that $r = \sum_i r_i$. We conclude that $R = \oplus_{i\in\mathbb{Z}} R_i$. But now notice that the opposite is also true. That is, given an $\mathbb{Z}$-grading of R with $k \subseteq R_0$, the map $\sigma^\vee$ defined on some homogeneous element r_i by $\sigma^\vee(r_i) = x^i \otimes r_i$ agrees with the definition of an action by reversing the arguments above. Hence an action of $\mathbb{G}_m$ on X is the same as giving a $\mathbb{Z}$-grading on R with $k \subseteq R_0$.

□

Remark 1.19. It is actually possible to generalise this statement. That is, if $G = \operatorname{Spec}(k[\Lambda])$, where Λ is some finitely generated Abelian group, then a G action on $X = \operatorname{Spec}(R)$ will be the same as a Λ-grading of R. We refer to Theorem 2.12 in the paper by Craw, see [6].

As seen in Proposition 1.14, we can in general just give the map on points if we want to give an action. In a lot of standard cases, this map on points will look fairly familiar.

Example 1.20. Consider the group variety $G := \mathbb{G}_m = \operatorname{Spec}(k[t, t^{-1}])$ and variety $X := \mathbb{A}_k^n = \operatorname{Spec}(k[x_1, ..., x_n])$. Suppose that we have some action $\sigma : \mathbb{G}_m \times \mathbb{A}_k^n \to \mathbb{A}_k^n$ so that for all $i = 1, ..., n$ we have $\sigma^\vee(x_i) = x_i \otimes t^{a_i}$ for some $a_i \in \mathbb{Z}$. For simplicity we shall write $k[t, t^{-1}] \otimes_k k[x_1, ..., x_n] = k[t, t^{-1}, x_1, ..., x_n]$. A k-point of $\mathbb{A}_n^k$ will be an element $(y_1, ..., y_n) \in k^n$ and can be observed as the maximal ideal $(x_1 - y_1, x_2 - y_2, ..., x_n - y_n) \subset k[x_1, ..., x_n]$. Similarly a k-point of G will be an element $s \in k^\times$, which can be seen as the maximal ideal $(t - s) \subset k[t, t^{-1}]$. Using this notation, σ gives a map on points by $s \cdot (y_1, ..., y_n) \in k^n$ being the point which corresponds to the maximal ideal $(\sigma^\vee)^{-1}(t - s, x_1 - y_1, x_2 - y_2, ..., x_n - y_n)$. To calculate this maximal ideal, we do the following.

Given our action, it is fair to suggest that we may obtain the point $(s^{a_1} y_1, ..., s^{a_n} y_n)$ as answer, so we may try this. To do so; for any $i = 1, ..., n$ we have

$$\sigma^\vee(x_i - s^{a_i} y_i) = x_i t^{a_i} - s^{a_i} y_i = (x_i - y_i)t^{a_i} + y_i t^{a_i} - s^{a_i} y_i = (x_i - y_i)t^{a_i} + y_i(t_i^a - s^{a_i}).$$

If $a_i \geq 0$, then $t^{a_i} - s^{a_i}$ is a polynomial in t with a zero s, so that $t^{a_i} - s^{a_i} \in (t-s)$. If $a_i < 0$, then $1 - t^{-a_i}s^{a_i} \in (t-s)$ holds using the same logic so that $t^{a_i}(1 - t^{-a_i}s^{a_i}) = t^{a_i} - s^{a_i} \in (t-s)$ holds again. Hence we obtain $\sigma^\vee(x_i - s^{a_i}y_i) \subset (t-s, x_1 - y_1, ..., x_n - y_n)$, which shows that $\sigma^\vee(x_1 - s^{a_1}y_1, ..., x_n - s^{a_n}y_n) \subseteq (t-s, x_1 - y_1, ..., x_n - y_n)$. In particular, this shows that $(x_1 - s^{a_1}y_1, ..., x_n - s^{a_n}y_n) \subseteq (\sigma^\vee)^{-1}(t-s, x_1 - y_1, ..., x_n - y_n)$. But on the left of this inclusion we have a maximal ideal, which is contained inside some prime ideal. Therefore, the only possibility is $(\sigma^\vee)^{-1}(t-s, x_1 - y_1, ..., x_n - y_n) = (x_1 - s^{a_1}y_1, ..., x_n - s^{a_n}y_n)$, which is exactly what we wanted. We conclude that, on points, we are working with the map $s \cdot (y_1, ..., y_n) = (s^{a_1}y_1, ..., s^{a_n}y_n)$.

With the reverse statement you'd have to be a bit careful. Notice that giving the 'action' $s \cdot (y_1, ..., y_n) = (s^{a_1}y_1, ..., s^{a_n}y_n)$ only gives us the information that $(\sigma^\vee)^{-1}(t-s, x_1 - y_1, ..., x_n - y_n) = (x_1 - s^{a_1}y_1, ..., x_n - s^{a_n}y_n)$ holds for all $s \in k^\times$ and $y_1, ..., y_n \in k$. This makes it seem quite difficult to define $\sigma^\vee$. Luckily, Proposition 1.14 tells us that it must be given by $x_i \mapsto x_i \otimes t^{a_i}$ if it is any morphism at all.

And even more is true! If we consider $\mathbb{G}_m^r = \operatorname{Spec}(k[t_1, ..., t_r, t_1^{-1}, ..., t_r^{-1}])$ acting on $\mathbb{A}_k^n = \operatorname{Spec}(k[x_1, ..., x_n])$ given by $x_i \mapsto x_i \otimes t_1^{a_{1,i}} \cdots t_r^{a_{r,i}}$ and follow the process above similarly, we can show that this action can be given on points by $(s_1, ..., s_r) \cdot (y_1, ..., y_n) := (s_1^{a_{1,1}} \cdots s_r^{a_{r,1}} y_1, ..., s_1^{a_{1,n}} \cdots s_r^{a_{r,n}} y_n)$.

As a conclusion, we find a relation between an action of a group variety on a variety and the action given on points. In particular, in this case the action on points is enough to describe the entire action.

Definition 1.21. Let G be a group variety. We call G **solvable** if for all schemes T/k, the group $G(T)$ is solvable. The **radical** of G will be the identity component of its maximal solvable subgroup scheme. We call G **reductive** if its radical is isomorphic (as group schemes) to $\mathbb{G}_m^r$ for some $r \in \mathbb{Z}_{\geq 0}$.

Example 1.22. The r-torus $\mathbb{G}_m^r$ is trivially reductive. Some other examples include the group variety $\operatorname{GL}(n, k)$ and subgroup variety $\operatorname{SL}(n, k)$, the group of invertible $n \times n$ matrices over k and $n \times n$ matrices over k with determinant 1 respectively. Here, $\operatorname{SL}(n, k)$ is defined analogous to $\operatorname{GL}(n, k)$ as in Example 1.3. The radical of $\operatorname{GL}(n, k)$ is isomorphic to $\mathbb{G}_m^n$ by considering the diagonal matrices and the radical of $\operatorname{SL}(n, k)$ is trivial.

For a non-example, we can consider any positive integer n and the additive group $\mathbb{G}_a^n := \mathbb{G}_{a,k}^n$. This group is Abelian and therefore its own radical. However, it is not isomorphic to $\mathbb{G}_m^r$ for some r. There is also the non-example $B_n \subset \operatorname{GL}(n, k)$ being the subgroup variety consisting of all upper-triangular matrices that have only ones on the diagonal.

Definition 1.23. Let G be a group variety. A **representation** of G will be a group homomorphism $G \to \operatorname{GL}(n, k)$ for some $n \in \mathbb{Z}_{\geq 0}$. We call G **linearly reductive** if any representation of G is completely reducible. That is, for any representation with induced action on $\mathbb{A}_k^n$ we have the property that if G leaves some subspace $\mathbb{A}_k^m \subset \mathbb{A}_k^n$ invariant, then it leaves invariant a complementary subspace $\mathbb{A}_k^{n-m}$.

In general we don't have to worry about checking for the linear reductive property, as there is a nice equivalence when working over a field of characteristic 0.

Proposition 1.24 (GIT, [5], Appendix A page 191)**.** *Suppose k is of characteristic 0. Then a group variety G is reductive if and only if it is linearly reductive.*

Definition 1.25. Let G be a group variety acting on a variety X, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Write $p_{23} : G \times_k G \times_k X \to G \times_k X$ for the projection on the second and third coordinates. Note that we have equalities of morphisms:

$$\begin{aligned} p_2 \circ (\mu \times \mathrm{id}_X) = p_3 = p_2 \circ p_{23} &: G \times_k G \times_k X \to X; \\ \sigma \circ p_{23} = p_2 \circ (\mathrm{id}_G \times \sigma) &: G \times_k G \times_k X \to X; \\ \sigma \circ (\mu \times \mathrm{id}_X) = \sigma \circ (\mathrm{id}_G \times \sigma) &: G \times_k G \times_k X \to X. \end{aligned}$$

The last equality follows by the first property of a group action.

An **equivariant structure** of G on $\mathcal{F}$ is an isomorphism $\phi : \sigma^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F}$ of $\mathcal{O}_{G \times_k X}$-modules such that the diagram

$$\begin{array}{ccccccc} (\sigma \circ (\mathrm{Id}_G \times \sigma))^*\mathcal{F} & \xrightarrow{(\mathrm{Id}_G \times \sigma)^*\phi} & (p_2 \circ (\mathrm{Id}_G \times \sigma))^*\mathcal{F} & = & (\sigma \circ p_{23})^*\mathcal{F} & \xrightarrow{p_{23}^*\phi} & (p_2 \circ p_{23})^*\mathcal{F} \\ \| & & & & & & \| \\ (\sigma \circ (\mu \times \mathrm{Id}_X))^*\mathcal{F} & & & \xrightarrow{(\mu \times \mathrm{Id}_X)^*\phi} & & & (p_2 \circ (\mu \times \mathrm{Id}_X))^*\mathcal{F} \end{array}$$

commutes, also called the cocycle condition. If $\mathcal{F}$ is an invertible sheaf, then we will also call a G-equivariant structure on $\mathcal{F}$ a **G-linearization** of $\mathcal{F}$ or simply a **linearization** of $\mathcal{F}$ when there is no confusion. A **G-equivariant sheaf** on X (respectively **G-linearized invertible sheaf**) will be a quasi-coherent $\mathcal{O}_X$-module with a given G-equivariant structure (respectively an invertible sheaf with a given linearization).

The main objects of interest here will be the G-linearizations of invertible sheaves. If we consider two G-linearized invertible sheaves, the tensor product naturally provides a G-linearization on the tensor product of the invertible sheaves. Moreover, the trivial invertible sheaf $\mathcal{O}_X$ has a natural "trivial" G-linearization by viewing the canonical isomorphisms $\sigma^*\mathcal{O}_X \cong p_2^*\mathcal{O}_X \cong \mathcal{O}_{G\times_k X}$, and the inverse of a G-linearized invertible sheaf $\mathcal{L}$ carries a natural G-linearization which is "inverse" to the G-linearization of $\mathcal{L}$ up to isomorphism. Therefore, we get a group structure.

Definition 1.26. Let G be a group variety acting on a variety X. The group of isomorphism classes of G-linearized invertible sheaves is denoted by $\mathrm{Pic}^G(X)$.

If seeing this definition for the first time, it might be tempting to show that $\mathrm{Pic}(X) \subseteq \mathrm{Pic}^G(X)$ holds. But this is in general not true! It is completely possible that there exists no G-linearization for some invertible sheaf, but it would be a lot of effort to show that this is the case for a given sheaf. The other way around is also not necessarily true; we may find multiple G-linearizations of some invertible sheaf. Luckily, the following proposition helps us to put some restrictions on G concerning the amount of G-linearizations.

Proposition 1.27 (GIT,[5], Proposition 1.4)**.** *Let G be a connected group variety acting on a variety X. If there is no surjective homomorphism $G \times_k \mathit{Spec}(\overline{k}) \to \mathbb{G}_{m,\overline{k}}$, then each invertible sheaf $\mathcal{L}$ on X carries at most one G-linearization.*

At this point, it is not (yet) so easy to give examples. In the next section we shall see a different approach to invertible sheaves, called line bundles, which will also help us understand these linearizations in an easy manner.

Let G be a group variety acting on some variety X, and let $\mathcal{L}$ be a G-linearized invertible sheaf on X. Consider the following composition.

$$\Gamma(X,\mathcal{L}) \xrightarrow{\sigma^*} \Gamma(G\times_k X, \sigma^*\mathcal{L}) \xrightarrow{\phi} \Gamma(G\times_k X, p_2^*\mathcal{L}) \cong \Gamma(G\times_k X, p_1^*\mathcal{O}_G \otimes p_2^*\mathcal{L}) \xrightarrow{\sim} \Gamma(G,\mathcal{O}_G)\otimes_k \Gamma(X,\mathcal{L})$$

where the last map is the isomorphism following from the Künneth formula. Then we obtain the notion of a G-invariant section of $\mathcal{L}$.

Definition 1.28. Let $G, X, \mathcal{L}$ as above. We call a section $s \in \Gamma(X,\mathcal{L})$ **G-invariant** if the image of s under the composition above equals $1\otimes s$. The subset of $\Gamma(X,\mathcal{L})$ consisting of all G-invariant sections will be denoted by $\Gamma(X,\mathcal{L})^G$.

We need to be a bit careful here whenever we use the structure sheaf $\mathcal{L} = \mathcal{O}_X$. For example, if $X = \mathrm{Spec}(R)$ is affine, any action $G\times_k X \to X$ corresponds to a ring map $R \to R\otimes_k \mathcal{O}_G(G)$. This gives us another notion of an action, and it is at a first glance not clear whether it agrees with the definition above. Moreover, it does not agree with the definition in general! Notice that this action of G on $\mathcal{O}_X(X)$ does not depend on the linearization of $\mathcal{O}_X$, so ϕ does not play any role for this action. Luckily, we know what this ring map does correspond to, namely the trivial linearization. Hence, given the trivial linearization of $\mathcal{O}_X$ for some affine scheme X, the G-invariant sections $\Gamma(X,\mathcal{O}_X)^G \subseteq \Gamma(X,\mathcal{O}_X) = R$ form a subring. If no G-linearizations are mentioned, we shall mean the trivial linearization. Finally, we have a nice result on affine schemes over k.

Theorem 1.29 (GIT,[5], Theorem 1.1 & Amplification 1.3)**.** *Let X/k be an affine scheme and let G be a reductive group variety acting on X. Then the categorical quotient of X by G exists, and is isomorphic to the pair $(\mathit{Spec}(\mathcal{O}_X(X)^G), \pi)$ where π is the natural map induced by the inclusion $\mathcal{O}_X(X)^G \hookrightarrow \mathcal{O}_X(X)$. Moreover, this quotient is a geometric quotient if and only if the orbits of all geometric points of X by G are closed.*

1.4 Line bundles

There is a nice geometric interpretation possible for any invertible sheaf on a variety, called a line bundle, which have useful applications. Any G-linearization given on an invertible sheaf also directly translates with something we call a bundle action on this line bundle. The other way around will hold as well, so that we get a G-linearization given some bundle action. Therefore, instead of working with the rather difficult definition of a G-linearization, we can give a bundle action on a line bundle. We will also talk shortly about vector bundles, but as we do not use them, they will not be defined properly. We shall follow the process of Brion closely, see [4].

Definition 1.30. Let X be a variety. A **line bundle** on X is a scheme $\mathbf{L}$ over k together with a k-morphism $\pi : \mathbf{L} \to X$ such that X admits an open cover $\{U_i\}_{i\in I}$, called a trivializing cover of $\mathbf{L}$, and for all $i \in I$ an isomorphism $\phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times_k \mathbb{A}^1_k$ such that for all $i, j \in I$ the restriction of the composition $(\phi_i \circ$

$\phi_j^{-1})|_{(U_i\cap U_j)\times_k \mathbb{A}^1_k} : (U_i \cap U_j) \times_k \mathbb{A}^1_k \to (U_i \cap U_j) \times_k \mathbb{A}^1_k$ is given on points by $(x,t) \mapsto (x, a_{ij}(x)t)$ for some $a_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$.

If X, Y are varieties, and $f : Y \to X$ is a morphism, then the pullback $f^*\mathbf{L}$ of some line bundle $\mathbf{L}$ on X will be defined as the fiber product $Y \times_X \mathbf{L}$. A morphism $\mathbf{L} \to \mathbf{M}$, where $\mathbf{L}$ is a line bundle on X and $\mathbf{M}$ is a line bundle on Y will be a cartesian square

$$\begin{array}{ccc} \mathbf{M} & \longrightarrow & \mathbf{L} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Note that we can obtain $\mathbf{L}$ by gluing the so-called trivial line bundles $U_i \times_k \mathbb{A}^1_k$ together via the transition functions a_{ij}. This also gives the inspiration to get more line bundles by using different transition functions. Suppose $\mathbf{L}, \mathbf{M}$ are line bundles on a variety X. Let us choose a trivializing cover for both $\mathbf{L}$ and $\mathbf{M}$ simultaneously. For example, this can be done by choosing $\{U_i \cap V_j\}_{i\in I, j\in J}$, if $\{U_i\}_{i\in I}$ is a trivializing cover for $\mathbf{L}$ and $\{V_i\}_{i\in J}$ is a trivializing cover for $\mathbf{M}$.

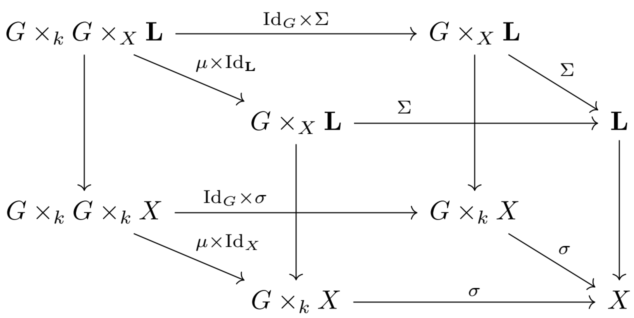
Let a_{ij} denote the transition functions of $\mathbf{L}$ and let b_{mn} denote the transition functions of $\mathbf{M}$. We define the tensor product $\mathbf{L} \otimes \mathbf{M}$ as the line bundle by gluing the $(U_i \cap V_m) \times_k \mathbb{A}^1_k$ via the transition functions $a_{ij}b_{mn}$. Similarly, we let $\mathbf{L}^\vee$ denote the line bundle by gluing the $U_i \times_k \mathbb{A}^1_k$ via the transition functions a_{ij}^{-1}. Note that $\mathbf{L} \otimes \mathbf{L}^\vee$ is isomorphic to the trivial line bundle $X \times_k \mathbb{A}^1_k \to X$. This puts a group structure on the set of line bundles up to isomorphism on X, we will denote this group by $\mathrm{Line}(X)$.

Suppose that $\mathcal{L}$ is an invertible sheaf. Choose a trivializing cover $\{U_i\}_{i\in I}$ of $\mathcal{L}$ and for all $i \in I$ choose a generator $s_i \in \Gamma(U_i, \mathcal{L})$ as $\mathcal{O}_X(U_i)$-module. Then, for all $i, j \in I$, there exist an element $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$ such that $s_i|_{U_i\cap U_j} = g_{ij} \cdot s_j|_{U_i\cap U_j}$. Define a map $f : \mathrm{Pic}(X) \to \mathrm{Line}(X)$ by sending an invertible sheaf to the line bundle obtained from gluing the trivial line bundles $U_i \times_k \mathbb{A}^1_k \to U_i$ via the transition functions g_{ij}^{-1}. We shall give a remark at the end of this construction as to why we have an inverse here. Note that, up to isomorphism, this does not depend on choice of the s_i or choice of representative $\mathcal{L}$. In other words, f is well-defined. Furthermore, if $\mathcal{L}, \mathcal{M}$ are invertible sheaves giving transition functions g_{ij} and h_{mn} respectively, we see that $\mathcal{L} \otimes \mathcal{M}$ will give the transition functions $g_{ij}^{-1}h_{mn}^{-1}$. So f is a group homomorphism.

Our function f is clearly injective, as we can obtain the trivial line bundle if and only if $g_{ij} = 1$ for all transition maps. Surjectivity follows readily as well. If $\mathbf{L}$ is a line bundle, we may define the invertible sheaf $\mathcal{L}$ by $\mathcal{L}(U) := \{k\text{-morphisms } \sigma : U \to \mathbf{L} \mid \pi \circ \sigma = \mathrm{id}_U\}$. Then $f(\mathcal{L}) = \mathbf{L}$. Hence f is an isomorphism, giving us a nice correspondence between line bundles and invertible sheaves (up to isomorphism).

Using this isomorphism it is natural to try to replicate definitions meant for invertible sheaves. In particular, G-linearizations of invertible sheaves have a nice interpretation on line bundles.

Suppose G is a group variety acting on a variety X, and let $\mathcal{L}$ be a G-linearized invertible sheaf given by the isomorphism $\phi : \sigma^*\mathcal{L} \xrightarrow{\sim} p_2^*\mathcal{L}$. Write $\pi : \mathbf{L} \to X$ for the corresponding line bundle (up to isomorphism). The cocycle condition of ϕ shows that a G-linearization of $\mathcal{L}$ corresponds to a bundle action of G on $\mathbf{L}$. In other words, we have a G-action $\Sigma : G \times_X \mathbf{L} \to \mathbf{L}$ such that the diagram



commutes, and that is fiberwise linear meaning that for k-points $x \in X$ the induced action of G on $\pi^{-1}(\{x\}) \cong \mathbb{A}^1_k$ is a linear (since ϕ is $\mathcal{O}_{G\times X}$-linear) automorphism.

Even more so, we can replicate this entire process whenever $\mathcal{E}$ is a locally free sheaf of finite rank n. That is, for a trivializing cover $\{U_i\}_{i\in I}$ of $\mathcal{E}$ we obtain morphisms $U_i \times_k \mathbb{A}^n_k \to U_i$ which may be glued via transition functions. In this case, any such $\mathcal{E}$ will give us a so-called **vector bundle** on X. These objects are interesting, and are used quite a lot in literature, but they do not play a massive role in this paper. As a result, we shall not dive more deeply into their definition.

There is another construction of the line bundle corresponding to an invertible sheaf, which is used a bit more widely in literature. Suppose we have any sheaf $\mathcal{E}$ on X. Then we can consider the relative spectrum of the symmetric algebra of this sheaf, namely $\underline{\mathrm{Spec}}(\mathrm{Sym}\ \mathcal{E})$. For anyone not familiar with these constructions, we shall refer to Algebraic Geometry by Hartshorne, see [13]. The notation for $\underline{\mathrm{Spec}}(\mathrm{Sym}\ \mathcal{E})$ might be $V(\mathcal{E})$ in some texts.

Notice that whenever $\mathcal{E}$ is a locally free sheaf of rank n on X, this process will give us a vector bundle of rank n on X (together with the natural map $\underline{\mathrm{Spec}}(\mathrm{Sym}\ \mathcal{E}) \to X$). Moreover, if G is a group variety acting on a variety X and $\mathcal{E}$ has some G-equivariant structure, then we get a natural G-action on $\underline{\mathrm{Spec}}(\mathrm{Sym}\ \mathcal{E})$. It is a nice exercise to check that for G-linearized invertible sheaves $\mathcal{E}$, the line bundle $\underline{\mathrm{Spec}}(\mathrm{Sym}\ \mathcal{E})$ will be isomorphic to the line bundle as obtained with the construction above, and that the G-linearization induces the same G-action on the line bundle. Hence we may shift between these constructions and choose whichever one we prefer.

Remark 1.31. Let $\mathcal{L}$ be an invertible sheaf on a variety X and let $\mathbf{L}$ be the corresponding line bundle. The main reason we defined $\mathbf{L}$ by gluing along transition functions g_{ij}^{-1} coming from the transition functions g_{ij} from $\mathcal{L}$ is that the process from invertible sheaves to bundles is a contravariant process. In the next chapter we will define a Mumford weight that comes with invertible sheaves and line bundles that have some kind of structure on them coming from G. We would like these weights to agree, which we can do with our choice of gluing. We could've also defined $\mathbf{L}$ by gluing along the g_{ij}, but we should then also invert the weights. So this will be our convention.

Example 1.32. Let $X := \mathbb{P}_k^n$ with homogeneous coordinates $(x_0 : ... : x_n)$ and consider the invertible sheaf $\mathcal{L} := \mathcal{O}_X(1)$. gluing some $D(x_i)$ to some $D(x_j)$, this sheaf is glued by

$$X_i \mapsto X_{ij} X_j$$

where X_{ij} represents X_i/X_j. Therefore, the corresponding line bundle $\mathbf{L}$ can locally be given by $D(x_i) \times \mathbb{A}_k^1$ for $i = 0, ..., n$, and it is obtained by gluing $D(x_i) \times \mathbb{A}_k^1$ to $D(x_j) \times \mathbb{A}_k^1$ via the maps

$$(v, s) \mapsto \left(v, \frac{X_j(v)}{X_i(v)} s \right).$$

Here $X_j(v), X_i(v)$ are the maps that send v to its value on the j-th coordinate and i-th coordinate respectively.

We end this section on line bundles by giving two general computations of $\mathrm{Pic}^G(X)$ where G is a torus making use of this construction. The first example will involve any variety with a trivial Picard group.

Example 1.33. Let $r \in \mathbb{Z}$ be a positive integer and consider some action of $G := \mathbb{G}_m^r$ on a variety X with trivial Picard group. Consider the structure sheaf $\mathcal{L} := \mathcal{O}_X$, with corresponding line bundle $\mathbf{L} := X \times \mathbb{A}_k^1$. A bundle action of G on $\mathbf{L}$ must be given by its action on $\mathbb{A}_k^1$, since it has to commute with the action on X on points. Therefore Example 1.20 shows that a linearization is simply given by an r-tuple of integers. Hence we get $\mathrm{Pic}^G(X) \cong \mathbb{Z}^r$.

Next, we shall see what happens on projective space.

Proposition 1.34. *Consider $G := \mathbb{G}_m$ acting on $X := \mathbb{P}_k^n$ by $t \cdot (x_0 : ... : x_n) = (t^{r_0} x_0 : ... : t^{r_n} x_n)$. Then $Pic^G(X) \cong \mathbb{Z}^2$.*

Proof. For $i = 0, ..., n$, write $U_i := D(x_i)$ for the standard affine open subsets of X, and consider some invertible sheaf $\mathcal{L} := \mathcal{O}_X(d)$ for $d \in \mathbb{Z}$. The corresponding line bundle $\mathbf{L}$ is given by gluing schemes $U_i \times \mathbb{A}_k^1$ by the isomorphism

$$\phi_{ij} : ((x_0 : ... : x_n), s) \mapsto \left((x_0 : ... : x_n), \frac{x_j^d}{x_i^d} s \right)$$

where we go from $U_i \times \mathbb{A}_k^1 \supset (U_i \cap U_j) \times \mathbb{A}_k^1 \to (U_i \cap U_j) \times \mathbb{A}_k^1 \subset U_j \times \mathbb{A}_k^1$. Now we want to give $\mathbf{L}$ a bundle action. Since this action has to commute with the action on $\mathbb{P}_k^n$ via the projection map, we only need to act on $\mathbb{A}_k^1$. As we've seen in Example 1.20, we get that for $i = 0, ..., n$ the bundle action on $U_i \times \mathbb{A}_k^1$ is given by $t \cdot ((x_0 : ... : x_n), s) = ((t^{r_0} x_0 : ... : t^{r_n} x_n), t^{a_i} s)$ for some $a_i \in \mathbb{Z}$. We do have some restrictions with this, namely that this action agrees on the overlap $(U_i \cap U_j) \times \mathbb{A}_k^1$ for all other j. The gluing map tells us exactly how this overlap should happen.

Consider some $0 \leq i, j \leq n$ and let $((x_0 : ... : x_n), s) \in (U_i \cap U_j) \times \mathbb{A}_k^1$. If we first act on this element with the bundle action, and then apply the gluing isomorphism, we will get;

$$\phi_{ij}(t \cdot ((x_0 : ... : x_n), s)) = \phi_{ij}((t^{r_0} x_0 : ... : t^{r_n} x_n), t^{a_i} s)$$

$$= \left((t^{r_0}x_0 : ... : t^{r_n}x_n), \frac{t^{dr_j}x_j^d}{t^{dr_i}x_i^d} t^{a_i} s \right) = \left((t^{r_0}x_0 : ... : t^{r_n}x_n), \frac{x_j^d}{x_i^d} t^{a_i + dr_j - dr_i} s \right).$$

If we first apply the gluing isomorphism, and then act via the bundle action, we will get;

$$t \cdot \phi_{ij}((x_0 : ... : x_n), s) = t \cdot \left((x_0 : ... : x_n), \frac{x_j^d}{x_i^d} s \right) = \left((t^{r_0}x_0 : ... : t^{r_n}x_n), t^{a_j} \frac{x_j^d}{x_i^d} s \right).$$

Hence, we should have $a_j = a_i + dr_j - dr_i$. In particular, the bundle action is characterized by choice of the element $a_0 \in \mathbb{Z}$ (or any other a_i). It should be noted that we have only computed what should happen if we try to define a bundle action, not that it actually exists. For this the equations $a_j = a_i + dr_j - dr_i$ should not contradict one another. Luckily, if we write $a_l = a_j + dr_l - dr_j$ for some l, then $a_l = (a_i + dr_j - dr_i) + dr_l - dr_j = a_i + dr_l - dr_i$ gives us the exact equation we wanted. So this bundle action is well-defined.

Let $d_1, d_2 \in \mathbb{Z}$. Suppose we have G-linearizations of $\mathcal{O}_X(d_1)$ given by $a \in \mathbb{Z}$ and G-linearization of $\mathcal{O}_X(d_2)$ given by $b \in \mathbb{Z}$ respectively, corresponding to the value of a_0 as in the construction above. Making use of the line bundle construction and their definition of the tensor product, it can be readily seen that the tensor product (and therefore the product of $\mathrm{Pic}^G(X)$) gives us the line bundle $\mathcal{O}_X(d_1 + d_2)$, with G-linearization given by the integer $a + b$ corresponding to the value of a_0. Since $\mathrm{Pic}(X) \cong \mathbb{Z}$, we conclude that $\mathrm{Pic}^G(X)$ is isomorphic to $\mathbb{Z}^2$ via the isomorphism $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X) \times \mathbb{Z}$ sending a G-linearized invertible sheaf $\mathcal{L}$ with value a to the pair $(\mathcal{L}, a)$ (where on the first coordinate we forget the linearization).

□

Remark 1.35. Note that this proposition can be improved by letting $\mathbb{G}_m^r$ act on $\mathbb{P}_k^n$ for some positive integer r. Similarly as in the proof above, we would get $(t_1, ..., t_r)$ acting on some $((x_0 : ... : x_n), s) \in U_i \times \mathbb{A}_k^1$, which will now be given by integers $a_{1,i}, ..., a_{r,i}$. The statement would then be that $\mathrm{Pic}^{\mathbb{G}_m^r}(\mathbb{P}_k^n) \cong \mathbb{Z}^{r+1}$ by the isomorphism sending some G-linearized invertible sheaf $\mathcal{L}$ to the underlying sheaf without the G-linearization, and the r-tuple $(a_{1,0}, ..., a_{r,0}) \in \mathbb{Z}^r$.

2 Geometric Invariant Theory

Our next goal is to build upon the theory of chapter one and look at the quotients we find interesting. We will look at something we call the GIT-quotient, and how to obtain it. In general this requires the use of semi-stable points; geometric points that have some kind of property. In the first section we will talk about how to obtain such points and what to do with them. The second section gives us a geometric interpretation of the structure of these points, and finally the last section describes shortly what happens within this interpretation.

2.1 The GIT-quotient

As we've stated before, categorical and geometric quotients do not always exist. This makes it natural to ask the question when they do exist, resulting in the definitions of semi-stable and stable points.

Definition 2.1. Let G be a reductive group variety acting on a variety X. Let x be a geometric point of X, and suppose we have an invertible sheaf $\mathcal{L}$ on X with a G-linearization ϕ. If s is any global section of $\mathcal{L}$, we let X_s denote the set of geometric points of X such that s_x generates $\mathcal{L}_x$ as $\mathcal{O}_{X,x}$-module.

(i) We call x **semi-stable** if there exists an $n \in \mathbb{Z}_{>0}$ and an invariant section $s \in H^0(X, \mathcal{L}^n)^G$ such that $s_x \neq 0$ and X_s is affine. The set of semi-stable points will be denoted by $X^{\mathrm{ss}}(\mathcal{L})$.

(ii) We call x **stable** if it is semi-stable, the orbits of all geometric points $y \in X_s$ are closed under the action of G, and the stabilizer of x is finite. The set of stable points will be denoted by $X^{\mathrm{s}}(\mathcal{L})$.

(iii) We call x **unstable** if it is not semi-stable. The set of unstable points will be denoted by $X^{\mathrm{us}}(\mathcal{L})$.

We make a quick remark on the definition of stable points, as they aren't exactly the same as classically used in [5]. Our definition of stable points will correspond to their definition of something they call properly stable points. But it seems that the properties of non-properly stable points are not desirable, which led modern texts to use the notion of properly stable points as if they were just stable points.

Following these definitions, we have a result from GIT using the semi-stable points. As a consequence, we may finally define GIT quotients.

Theorem 2.2 (GIT,[5], Theorem 1.10)**.** *Let X be a variety and let G be a reductive group variety acting on X. Assume we have a G-linearized invertible sheaf $\mathcal{L}$ on X. Then a categorical quotient of $X^{ss}(\mathcal{L})$ by G exists.*

Remark 2.3. Not only is this a strong theorem, it also explicitly computes the quotient for us. In general, the construction gives us a quotient of the form

$$\text{Proj}\left(\bigoplus_{n=0}^{\infty}\Gamma(X,\mathcal{L}^n)^G\right).$$

Notice that whenever $\mathcal{L} = \mathcal{O}_X$, this becomes Proj $\Gamma(X,\mathcal{O}_X)[z]^G$ where z keeps track of the grading. The action of G on z is determined by the bundle action when viewing $\mathcal{O}_X$ as a line bundle $X \times \mathbb{A}^1_k \to X$.

Definition 2.4. Let G be a reductive group variety acting on a variety X, and suppose we have a G-linearized invertible sheaf $\mathcal{L}$ on X. The **GIT-quotient** of X by G with respect to $\mathcal{L}$, will be defined (up to isomorphism) as a categorical quotient of $X^{\text{ss}}(\mathcal{L})$ by G. The notation will be $X/\!\!/_{\mathcal{L}}G := X^{\text{ss}}(\mathcal{L})/G$ or $X/\!\!/G$ when it is clear what G-linearized invertible sheaf is involved.

Example 2.5. Suppose we have some affine X over k and a reductive group variety G acting on X. Due to Theorem 1.29, it might be tempting to say that there is only one quotient $X/\!\!/_{\mathcal{L}}G$ up to isomorphism for any given line bundle $\mathcal{L}$. But this quotient actually also depends on the linearization that is chosen!

For an (easy) example, consider $X = \mathbb{A}^2_k = \text{Spec}(k[x,y])$ and let $G = \mathbb{G}_m$. We let G act on X on points by $t \cdot (x,y) := (tx,ty)$. The corresponding dual map $k[x,y] \to k[x,y] \otimes_k k[t]$ is given by $x \mapsto x \otimes t$ and $y \mapsto y \otimes t$. As every invertible sheaf on X is trivial, we let $\mathcal{L} = \mathcal{O}_X$. A G-linearization of $\mathcal{L}$ is now a bundle action of G on $X \times \mathbb{A}^1_k$. This corresponds to a (linear) action of G on $\mathbb{A}^1_k$, and therefore it is given by a choice of an integer r. From this we take two choices; if $r = 0$, then we are considering the trivial G-linearization. If $r = -1$, then we take the bundle action $t \cdot (x,y,z) = (tx,ty,t^{-1}z)$. We claim that these G-linearizations provide different quotients.

For $r = 0$, Theorem 1.29 tells us that $X/\!\!/_{\mathcal{O}_X}G \cong \text{Spec}(k[x,y]^G) = \text{Spec}(k)$ is a point. Notice that we use that all points are semi-stable here, which follows since $1 \in \mathcal{O}_X(X)$ is G-invariant. For $r = -1$ we get $k[x,y]^G = (x,y)$, since G acts on 1 in the same way the bundle action acts on the extra variable z. Looking at the definition, we see that the set of semi-stable points will be the set of points with either the first or second coordinate nonzero. Therefore, it is $X \setminus \{(0,0)\}$. Notice that this set is already different from the set of semi-stable points for $r = 0$ (which is X). The quotient can be given by Proj $k[x,y][z]^G =$ Proj $k[xz,yz] \cong$ Proj $k[x,y] \cong \mathbb{P}^1_k$, and therefore we get something different.

It now seems fair to want to compute the set of semi-stable points. However, in some more complex situations it might be difficult to compute it using our definition above. To help us be able to compute these sets, we will dive into the theory of the numerical criterion.

Definition 2.6. Let G be a group variety. A **one-parameter subgroup** of G or **1-PS** of G for short will be a non-trivial group homomorphism $\lambda : \mathbb{G}_m \to G$.

Definition 2.7. Assume that we have a group variety G acting on a variety X that is proper over k. Let $x \in X$ be a closed point and let $\lambda : \mathbb{G}_m \to G$ be a 1-PS. Let $\phi_x : G \to X$ denote the composition $G \cong G \times_k \text{Spec}(k) \to G \times_k X \to X$ given by $\sigma \circ (\text{Id}_G \times x)$. We get a map $\lambda \circ \phi_x : \mathbb{G}_m \to X$. Since X is proper, the valuative criterion for properness induces an extension to a map $\mathbb{A}^1_k \to X$. Let $y \in X$ denote the image of 0 and write $\lim_{t \to 0} \lambda(t) \cdot x := y$.

Next, we notice that y is fixed under the induced action of $\mathbb{G}_m$ on X via λ. If $\mathcal{L}$ is a G-linearized invertible sheaf with corresponding line bundle $\pi : \mathbf{L} \to X$, then $\mathbb{G}_m$ gives us a linear action on $\pi^{-1}(y) \cong \mathbb{A}^1_k$. Hence this will give us some integer, which we will denote by $\mu(\mathcal{L},\lambda,x)$. This is called the **Mumford weight** of x with respect to the 1-PS λ and invertible sheaf $\mathcal{L}$.

This weight has some functorial properties as seen in definition 2.2 of GIT [5]:

(i) For k-points α of G we have $\mu(\mathcal{L},\lambda,\sigma(\alpha,x)) = \mu(\mathcal{L},\alpha^{-1}\lambda\alpha,x)$;

(ii) If we fix x and λ, then the map $\text{Pic}^G(X) \to \mathbb{Z}$ given by $\mathcal{L} \mapsto \mu(\mathcal{L},\lambda,x)$ is a group homomorphism;

(iii) If $f : X \to Y$ is a G-invariant morphism of schemes on which G acts, and if $\mathcal{L}$ is a G-linearized invertible sheaf and x is a closed point of x, then $\mu(f^*\mathcal{L},\lambda,x) = \mu(\mathcal{L},\lambda,f(x))$;

(iv) If $\sigma(\lambda(\alpha),x) \to y$ as $\alpha \to 0$, then $\mu(\mathcal{L},\lambda,x) = \mu(\mathcal{L},\lambda,y)$.

Similarly to the notion of stable points, we don't exactly follow the classical text [5] here. Our number $\mu(\mathcal{L},\lambda,x)$ would equal $-\mu(\mathcal{L},\lambda,x)$ in that text. The difference is not massive, though the benefit to our definition is that we can interpret this number geometrically as a distance of some sort.

Remark 2.8. In practice, we are not just interested in actions on proper varieties. If X is not proper, then for any 1-PS λ we can define a subset $Y_\lambda \subseteq X(k)$ consisting of all points $x \in X(k)$ such that $\lim_{t\to 0} \lambda(t)\cdot x$ exists. With this restriction it makes sense to define $\mu(\mathcal{L}, \lambda, x)$ for any invertible sheaf $\mathcal{L}$ on X, any 1-PS $\lambda : \mathbb{G}_m \to G$ and any $x \in Y_\lambda$.

Recall the definition of a very ample invertible sheaf on a variety X. This will be an invertible sheaf $\mathcal{L}$ such that there exists an immersion $i : X \to \mathbb{P}^n_k$ for some n such that $\mathcal{L} \cong i^*(\mathcal{O}_{\mathbb{P}^n_k}(1))$. We call an invertible sheaf ample, if some positive power of the sheaf is very ample. Notice that the invertible sheaves $\mathcal{O}_{\mathbb{P}^n_k}(m)$ are (very) ample if and only if $m > 0$. See [13] for more details. The next theorem uses these sheaves for a very strong result.

Theorem 2.9 (GIT,[5], Theorem 2.1)**.** *Let G be a reductive group variety acting on a proper variety X. Suppose we have some ample G-linearized invertible sheaf $\mathcal{L}$ on X. Then for all closed points $x \in X$ we have:*

- *$x \in X^{ss}(\mathcal{L})$ if and only if for all 1-PS λ of G we have $\mu(\mathcal{L}, \lambda, x) \leq 0$;*
- *$x \in X^{s}(\mathcal{L})$ if and only if for all 1-PS λ of G we have $\mu(\mathcal{L}, \lambda, x) < 0$.*

We call this (Mumford's) ***numerical criterion****.*

Instead of using the numerical criterion right away, it would benefit us to generalize semi-stable and stable points using this criterion.

We follow the construction as in the paper by Ressayre, see [21]. Consider a maximal torus T of G. Write $\chi_*(T)$ and $\chi_*(G)$ for the sets of 1-PS's of T and G respectively. First let us consider the real vector space $\chi_*(T)_\mathbb{R} := \chi_*(T) \otimes \mathbb{R}$. If we let $W := N(T)/Z(T) = N(T)/T$ denote the Weyl group of T, then W is a finite group acting linearly on $\chi_*(T)_\mathbb{R}$. By finiteness, there exists a W-invariant Euclidean norm $||.||$ on our vector space. Now if λ is a 1-PS of G, then λ maps a torus into G. Therefore since T is a maximal torus, there exists a $g \in G$ such that $g \cdot \lambda \cdot g^{-1}$ is a 1-PS of T. Even more so, any two elements of $\chi_*(T)$ that are conjugated by some element of G are also conjugated by an element of $N(T)$. See [15]. Hence we may define a norm $||.||$ on $\chi_*(G)_\mathbb{R}$ by using the norm above, and using the formula $||\lambda|| = ||g\lambda g^{-1}||$.

Definition 2.10. For G-linearized invertible sheaves $\mathcal{L}$ we shall define

$$M(\mathcal{L}, x) := \sup_{\lambda \in \chi_*(G)} \frac{\mu(\mathcal{L}, \lambda, x)}{||\lambda||}.$$

It is known that these values are finite, see [7]. Using these functions, we may reformulate the numerical criterion.

Theorem 2.11 (Numerical criterion)**.** *Let G be a reductive group variety acting on a proper variety X. Suppose we have some ample G-linearized invertible sheaf $\mathcal{L}$ on X. Then for all closed points $x \in X$ we have:*

- *$x \in X^{ss}(\mathcal{L})$ if and only if $M(\mathcal{L}, x) \leq 0$;*
- *$x \in X^{s}(\mathcal{L})$ if and only if $M(\mathcal{L}, x) < 0$.*

Definition 2.12. Let G be a group variety acting on a variety X, and suppose we have two G-linearized invertible sheaves $\mathcal{L}, \mathcal{M}$. We call $\mathcal{L}_1, \mathcal{L}_2$ ***G*-algebraically equivalent** if there exists a connected variety Y, closed points $y_1, y_2 \in Y$, and a G-linearized invertible sheaf $\mathcal{L}$ on $Y \times_k X$ such that $\mathcal{L}|_{\{y_1\}\times_k X} = \mathcal{L}_1$ and $\mathcal{L}|_{\{y_2\}\times_k X} = \mathcal{L}_2$. This defines an equivalence relation on $\mathrm{Pic}^G(X)$, and we shall denote $\mathrm{NS}^G(X)$ for the quotient of $\mathrm{Pic}^G(X)$ by this relation. This group is called the **Néron-Severi group** of X with respect to the action of G.

Before we provide the following lemma which shows that the Mumford weight is locally constant with respect to the set of fixed points, we shall have to consider multi-dimensional Mumford weights. In particular, what happens when we replace the line bundle in the definition of the Mumford weight with a vector bundle of rank n? We can still follow the same procedure, but end up (locally) with a $\mathbb{G}_m$ action on $\mathbb{A}^n_k$. As seen in Example 1.20, this can be given on points by an n-tuple of integers. So we could have defined $\mu(\mathcal{L}, \lambda, x) \in \mathbb{Z}^n$ in this case! This is a nice construction, but does not provide any other applications in this thesis compared to the one-dimensional version. However, it is good to mention it for anyone interested in further research. The following lemma can also be extended to such vector bundles.

Lemma 2.13. *Suppose that G is a group variety acting on a variety X. Let λ be a one parameter subgroup of G, and write X^λ for the set of geometric points of X that are fixed under the induced action of λ. Let $\mathcal{E}$ be a G-equivariant sheaf of X that is locally free of finite rank n. For any two points $x, x' \in X^\lambda$ that lie in the same connected component we have*

$$\mu(\mathcal{E}, \lambda, x) = \mu(\mathcal{E}, \lambda, x').$$

Proof. Let $x \in X^\lambda$ and consider the restricition $\mathcal{E}|_{X^\lambda}$. This will again be a locally free sheaf of finite rank n. Choose a neighbourhood $U \subseteq X^\lambda$ of x so that the corresponding vector bundle is isomorphic to $U \times_k \mathbb{A}_k^n$. The induced $\mathbb{G}_m$ action on X^λ is trivial, hence the induced $\mathbb{G}_m$-structure on $\mathcal{E}|_{X^\lambda}$ will be given by an action of $\mathbb{G}_m$ on $U \times_k \mathbb{A}_k^n$ that is constant on U. Therefore, it is given by a (linear) action of $\mathbb{G}_m$ on $\mathbb{A}_k^n$. But then for any $x' \in U$ the weight $\mu(\mathcal{E}|_{X^\lambda}, \lambda, x') = \mu(\mathcal{E}, \lambda, x')$ is simply given by this $\mathbb{G}_m$ action. We conclude that $\mu(\mathcal{E}, \lambda, x') = \mu(\mathcal{E}, \lambda, x)$ for any $x' \in U$, so that the assignment $x' \mapsto \mu(\mathcal{E}, \lambda, x')$ is locally constant.

□

Proposition 2.14. *Let G be a reductive group variety acting on a complete variety X. Let $x \in X$ be a closed point and $\lambda \in \chi_*(G)$. Suppose we have $\mathcal{L}_1, \mathcal{L}_2 \in Pic^G(X)$ that are G-algebraically equivalent. Then*

$$\mu(\mathcal{L}_1, \lambda, x) = \mu(\mathcal{L}_2, \lambda, x).$$

Proof. Let $\mathcal{L}$ be a G-linearized invertible sheaf and Y be a connected variety with closed points $y_1, y_2 \in Y$ as in the definition of G-algebraic equivalence. We have been given the natural G-action on $Y \times_k X$ where G acts trivial on Y. Write x_0 for the limit $\lim_{t \to 0} \lambda(t)x \in X$ (which exists due to completeness). Then $x_0 \in X^\lambda$. By Lemma 2.13 we get that the assignment $y \mapsto \mu(\mathcal{L}, \lambda, (y, x_0))$ is locally constant on $Y^\lambda = Y$, and therefore constant since Y is connected. We conclude that

$$\mu(\mathcal{L}_1, \lambda, x) = \mu(\mathcal{L}, \lambda, (y_1, x)) = \mu(\mathcal{L}, \lambda, (y_2, x)) = \mu(\mathcal{L}_2, \lambda, x).$$

□

This allows us in particular to define $\mu(\mathcal{L}, \lambda, x)$ for $\mathcal{L} \in NS^G(X)$ by choosing a lift. By linear continuation we may define a map $\mu(\bullet, \lambda, x) : NS^G(X)_\mathbb{R} \to \mathbb{R}$. Therefore we may also define $M(\mathcal{L}, x)$ for $\in NS^G(X)_\mathbb{R}$ by using the same formula as before. This allows us to define semi-stable and stable sets for such elements. If $\mathcal{L} \in NS^G(X)_\mathbb{R}$, we let:

$$X^{\mathrm{ss}}(\mathcal{L}) := \{x \in X(\overline{k}) \mid M(\mathcal{L}, x) \leq 0\};$$
$$X^{\mathrm{s}}(\mathcal{L}) := \{x \in X(\overline{k}) \mid M(\mathcal{L}, x) < 0\}.$$

Definition 2.15. Let X be a variety acted on by a linearly reductive group variety G. A **variation** will be defined as a pair $(\mathcal{L}_-, \mathcal{L}_+)$ of ample quasi-coherent G-linearized invertible sheaves. For $t \in [-1, 1]$ we define

$$\mathcal{L}_t := \mathcal{L}_-^{\frac{1-t}{2}} \otimes \mathcal{L}_+^{\frac{1+t}{2}}$$

in $\mathrm{NS}^G(X)_\mathbb{R}$. We say that $(\mathcal{L}_-, \mathcal{L}_+)$ satisfies the **DHT** condition if the following properties hold:

(i) For any $s, t \in [-1, 0)$ we have $X^{\mathrm{ss}}(\mathcal{L}_s) = X^{\mathrm{ss}}(\mathcal{L}_t)$, we shall denote it by $X^{\mathrm{ss}}(-)$. Similarly for any $s, t \in (0, 1]$ we have $X^{\mathrm{ss}}(\mathcal{L}_s) = X^{\mathrm{ss}}(\mathcal{L}_t)$, which we shall denote by $X^{\mathrm{ss}}(+)$. Write $X^{\mathrm{ss}}(0) := X^{\mathrm{ss}}(\mathcal{L}_0)$.

(ii) For any $x \in X^{\mathrm{ss}}(0) \setminus (X^{\mathrm{ss}}(+) \cup X^{\mathrm{ss}}(-))$, the stabilizer of x is isomorphic to $\mathbb{G}_m$.

(iii) The set $X^{\mathrm{ss}}(0) \setminus (X^{\mathrm{ss}}(+) \cup X^{\mathrm{ss}}(-))$ is either empty or connected.

Example 2.16. Suppose that X is a variety acted on by a linearly reductive group variety G, and suppose we have an ample quasi-coherent G-linearized invertible sheaf $\mathcal{L}$. Then $(\mathcal{L}, \mathcal{L})$ is a variation satisfying the DHT condition.

Example 2.17. Consider the $G := \mathbb{G}_m$ action on $X := \mathbb{P}_k^1$ with homogeneous coordinates $(x : y)$ by $t \cdot (x : y) := (tx : t^{-1}y)$ and consider the corresponding line bundle to the sheaf $\mathcal{L} := \mathcal{O}_X(1)$. We may give a G-linearization on invertible sheaf by choice of an integer $a \in \mathbb{Z}$ as we did in Proposition 1.34. Let's recall the proof and result of this proposition in this particular case.

Our line bundle has a trivializing cover $D(x) \cup D(y)$, and is given by the gluing map $D(x) \times \mathbb{A}_k^1 \supset (D(x) \cap D(y)) \times \mathbb{A}_k^1 \to (D(x) \cap D(y)) \times \mathbb{A}_k^1 \subseteq D(y) \times \mathbb{A}_k^1$ given by $((x : y), s) \mapsto ((x : y), \frac{y}{x}s)$ (and the inverse is given similarly). A G-linearization of $\mathcal{L}$ is a G-action on this line bundle can therefore be given by two integers a, b so that $t \cdot ((x : y), s) = ((tx : t^{-1}y), t^a s)$ on $D(x)$ and $t \cdot ((x : y), s) = ((tx : t^{-1}y), t^b s)$ on $D(y)$. But, we have one more restriction, namely that this action agrees with the gluing map. This will give the relation $a = b + 1 \cdot 1 - 1 \cdot (-1) = b + 2$, see the proof for more details.

This integer also gives us a way to compute the semi-stable sets of X, which we will do via the numerical criterion. Since raising a 1-PS to a positive power does not change the limit, we notice that (in our case) we have

$\mu(\mathcal{L}, \lambda^n, x) = n\mu(\mathcal{L}, \lambda, x)$ for all $x \in X(k)$, 1-PS λ, and positive integers n. Therefore, it suffices to compute the Mumford weights for the 1-PS $\lambda : t \mapsto t$ and $\lambda^{-1} : t \mapsto t^{-1}$ of G.

First of all, a quick observation shows us that

$$\mu(\mathcal{L}, \lambda, (1:0)) = a = -\mu(\mathcal{L}, \lambda^{-1}, (1:0));$$

and

$$\mu(\mathcal{L}, \lambda, (0:1)) = a - 2 = -\mu(\mathcal{L}, \lambda^{-1}, (0:1)),$$

using that $(1:0), (0:1)$ are fixed points for the action of G. Notice that this shows that $(1:0)$ or $(0:1)$ can only be semi-stable points if $a = 0$ or $a = 2$ respectively. For some $(x:y) \in D(x) \cap D(y)$ we have

$$\mu(\mathcal{L}, \lambda, (x:y)) = \mu(\mathcal{L}, \lambda, (0:1)) = a - 2;$$

and

$$\mu(\mathcal{L}, \lambda^{-1}, (x:y)) = \mu(\mathcal{L}, \lambda^{-1}, (1:0)) = -a.$$

Here we used functorial property (iv) from the definition of the Mumford weight. We conclude that such $(x:y)$ is semi-stable if and only if $a - 2 \leq 0$ and $a \geq 0$, giving us the restriction $0 \leq a \leq 2$.

Consider G-linearized invertible sheaves $\mathcal{L}_+, \mathcal{L}_-$ on X, both being $\mathcal{O}_X(1)$ as sheaves, but $\mathcal{L}_+$ being linearized with $a = 1$ and $\mathcal{L}_-$ being linearized with $a = -1$. We claim that $(\mathcal{L}_-, \mathcal{L}_+)$ is a variation satisfying the DHT condition, let's show this. For $t \in [-1, 1]$, define:

$$\mathcal{L}_t := \mathcal{L}_-^{\frac{1-t}{2}} \otimes \mathcal{L}_+^{\frac{1+t}{2}} \in \mathrm{NS}^G(X)_{\mathbb{R}}.$$

As a matter of fact, for this example it will not be important to compute $\mathrm{NS}^G(X)$, as we can simply work with the equivalence classes of $\mathrm{Pic}^G(X)$. Our first task is to compute the Mumford weight for $\mathcal{L}_t$ with respect to $\lambda^{\pm 1}$ and all $x \in X(k)$. By definition the Mumford weight is extended linearly, so for $(1:0)$ using the computations above we get;

$$\mu(\mathcal{L}_t, \lambda, (1:0)) = -\frac{1-t}{2} + \frac{1+t}{2} = t = -\mu(\mathcal{L}_t, \lambda^{-1}, (1:0)),$$

and for $(0:1)$ we get;

$$\mu(\mathcal{L}_t, \lambda, (0:1)) = -3 \cdot \frac{1-t}{2} - 1 \cdot \frac{1+t}{2} = t - 2 = -\mu(\mathcal{L}_t, \lambda^{-1}, (0:1)).$$

For $(x:y) \in D(x) \cap D(y)$ we get;

$$\mu(\mathcal{L}_t, \lambda, (x:y)) = -3 \cdot \frac{1-t}{2} - \frac{1+t}{2} = t - 2;$$

and

$$\mu(\mathcal{L}_t, \lambda^{-1}, (x:y)) = \frac{1-t}{2} - \frac{1+t}{2} = -t.$$

Next, we have;

$$M(\mathcal{L}_t, (1:0)) = \sup_{\rho \in \chi_*(G)} \frac{\mu(\mathcal{L}_t, \rho, (1:0))}{||\rho||}.$$

In our case, G itself is a torus, so the Weyl-group acting on $\chi_*(G)$ is trivial. Therefore the Euclidean norm $||.||$ can be chosen to be the "natural" absolute value. We can identify $\chi_*(G)$ with $\mathbb{Z}$ by writing an element as a power of λ. Then we expand to $\chi_*(G)_{\mathbb{R}}$, which we can view as a copy of $\mathbb{R}$, with $||.||$ being the absolute value. Therefore, $||\lambda^n|| = |n|$. As discussed before, we have $n\mu(\mathcal{L}, \lambda, x) = \mu(\mathcal{L}, \lambda^n, x)$ for positive n, so this shows that

$$M(\mathcal{L}_t, (1:0)) = \max_{\rho \in \{\lambda, \lambda^{-1}\}} \mu(\mathcal{L}_t, \rho, (1:0)) = \max\{t, -t\} = |t|.$$

Similarly we get

$$M(\mathcal{L}_t, (0:1)) = |t-2|;$$

and for $(x:y) \in D(x) \cap D(y)$:

$$M(\mathcal{L}_t, (x:y)) = \max\{t-2, -t\}.$$

Recall the (improved) numerical criterion. Then we have $(1:0)$ as semi-stable point if and only if $t=0$. Since $t \in [-1,1]$, $(0:1)$ will never be a semi-stable point, and $(x:y) \in D(x) \cap D(y)$ will be a semi-stable point if and only if $t \geq 0$. Hence;

$$X^{\mathrm{ss}}(\mathcal{L}_t) = \begin{cases} \emptyset, & \text{if } t \in [-1,0); \\ D(x), & \text{if } t = 0; \\ D(x) \cap D(y), & \text{if } t \in (0,1]. \end{cases}$$

We conclude that this variation satisfies the DHT condition.

Variations satisfying the DHT condition carry a lot of structure. In particular, the paper by Ballard, Favero and Katzarkov covers a theorem similar to their main theorem on variations satisfying the DHT condition. We shall mention it in Chapter 4. We will also see that in some cases they correspond to something called "crossing a wall" in a setting we will define in the next section.

We have not really mentioned or used the stable points, and these points will also be left out of the rest of the theory used in this thesis. It is not that they are not interesting. As a matter of fact, they give us an analog to the use of semi-stable points in the sense of the following theorem.

Theorem 2.18 (GIT, [5], Theorem 1.10)**.** *Let X be a variety and G a reductive group variety acting on X. Let $\mathcal{L}$ be a G-linearized invertible sheaf on X. Write (Y, ϕ) for a categorical quotient of $X^{ss}(\mathcal{L})$ by G. Then, there exists an open subset $\widetilde{Y} \subseteq Y$ such that $X^s(\mathcal{L}) = \phi^{-1}(\widetilde{Y})$ and such that $(\widetilde{Y}, \phi|_{X^s(\mathcal{L})})$ is a geometric quotient of $X^{ss}(\mathcal{L})$ by G.*

In particular, we could also shift our attention to the study of geometric quotients instead of categorical quotients, but this will not be our goal in this thesis.

2.2 HKKN stratifications

The next bit of theory in the sense of GIT will be that of HKKN stratifications. These indicate an interesting subdivision of a variety.

Definition 2.19. Let G be a group variety, and let λ be a one parameter subgroup of G. We define

$$P(\lambda) := \{g \in G(\overline{k}) \mid \lim_{t \to 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1} \text{ exists}\}$$

and

$$U(\lambda) := \{g \in G(\overline{k}) \mid \lim_{t \to 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1} = e\}$$

where $e \in G(\overline{k})$ is the identity element. Also define $C(\lambda)$ to be the centralizer of λ in G.

Definition 2.20. Let G be a group variety acting on a variety X, and let $\lambda : \mathbb{G}_m \to G$ be a one parameter subgroup of G. Write X^λ for the set of closed points of X that are fixed under the induced action of λ. Consider some connected component Z_λ^0 of X^λ and define

$$Z_\lambda := \{x \in X(\overline{k}) \mid \lim_{t \to 0} \lambda(t) \cdot x \in Z_\lambda^0\}.$$

We will call this the **contracting variety** (with respect to Z_λ^0). On top of this, we define

$$S_\lambda^0 := G \cdot Z_\lambda^0, \qquad \text{and} \qquad S_\lambda := G \cdot Z_\lambda.$$

In general, there will not be a natural choice for Z^0_λ. Hence we shall always define this variety to avoid confusion.

Suppose we have the same situation as in the previous definition, and consider the action of $P(\lambda)$ on $G \times_k Z_\lambda$ given on points as $p \cdot (g, z) := (gp^{-1}, pz)$. Then we get a quotient stack $[(G \times_k Z_\lambda)/P(\lambda)]$, which we shall denote by $G \times^{P(\lambda)} Z_\lambda$. Thanks to the work of Thomason in [25], we know that this stack is in general an algebraic space, and we have a natural map $\tau_\lambda : G \times^{P(\lambda)} Z_\lambda \to S_\lambda$ given on points by $(g, z) \mapsto g \cdot z$.

Definition 2.21. Let G be a group variety acting on a variety X. An **HKKN stratification** of X, denoted by $\mathfrak{K}$, will be a sequence

$$X = X^{\mathfrak{K}}_0 \supseteq X^{\mathfrak{K}}_1 \supseteq ... \supseteq X^{\mathfrak{K}}_n$$

of G-invariant open subvarieties together with for each $1 \leq i \leq n$ a one parameter subgroup λ_i and a choice $Z^0_{\lambda_i}$ of connected component of $(X^{\mathfrak{K}}_{i-1})^{\lambda_i}$ such that:

- $X^{\mathfrak{K}}_i = X^{\mathfrak{K}}_{i-1} \setminus S_{\lambda_i}$;
- For all $1 \leq i \leq n$, the morphism τ_{λ_i} defined above is an isomorphism;
- For all $1 \leq i \leq n$, S_{λ_i} is a closed subvariety of $X^{\mathfrak{K}}_{i-1}$.

An **elementary HKKN stratification** will be a HKKN stratification with $n = 1$. In this case we shall often leave out $\mathfrak{K}$ from the notation and write $X = X_\lambda \sqcup S_\lambda$.

Even though HKKN stratifications give very nice results in dividing our space, only the elementary wall crossings will get our attention in this thesis as they provide intuition on crossing a wall.

Definition 2.22. Let G be a group variety acting on a variety X. An **elementary wall crossing** will be a pair of elementary HKKN stratifications $(X_+ \sqcup S_\lambda, X_- \sqcup S_{\lambda^{-1}})$ corresponding to the same one parameter subgroup λ together with the same choice of connected component. That is, $Z^0_\lambda = Z^0_{\lambda^{-1}}$.

For a nice use of HKKN stratifications as a whole, we will refer to the following theorem. If the reader is interested in the theory behind it, we recommend to look at Theorem 2.1.28 of the paper by Ballard, Favero and Katzarkov in [3].

Theorem 2.23 (BFK, [3], Theorem 2.1.28 and Corollary 2.1.29)**.** *Let G be a linearly reductive group variety acting on either a smooth projective variety X or the affine variety $X = \mathbb{A}^n_k$ for some integer n. Let $\mathcal{L}$ be a G-linearized ample invertible sheaf. Then, there is an HKKN stratifcation*

$$X = X^{\mathfrak{K}}_0 \supset X^{\mathfrak{K}}_1 \supset ... \supset X^{\mathfrak{K}}_m = X^{ss}(\mathcal{L})$$

for some non-negative integer m.

Now in the literature a couple of choices could be made for the next topic that we should approach. Variations satisfying the DHT condition have some very nice properties, and we could study them further. We could look at so-called GIT-fans, in which we study the space $\mathrm{NS}^G(X)_\mathbb{R}$ by dividing it into chambers and rooms. We are going to study the GIT-fans at first, and from the next chapter onwards we will build our way to the study of derived categories involving GIT.

2.3 The GIT-fan

The idea behind the GIT-fan is to divide the vector space $\mathrm{NS}^G(X)_\mathbb{R} := \mathrm{NS}^G(X) \otimes_\mathbb{Z} \mathbb{R}$ into walls and chambers in a specific manner. The semi-stable set of any two points in the same chamber will be the same, and crossing a wall will give the intuition of changing this set in a "small" way. We will first provide some definitions to let these sentences make sense. After that we give some explicit examples so that the intuition can stick.

Definition 2.24. Let G be a linearly reductive group variety acting on a variety X. Consider the Neron-Severi group $\mathrm{NS}^G(X)$.

- We shall write $\mathrm{NS}^G(X)^+_\mathbb{R}$ for the convex cone generated by the classes of ample G-linearized invertible sheaves in $\mathrm{NS}^G(X)$.
- We call a point $l \in \mathrm{NS}^G(X)_\mathbb{R}$ **ample** if it belongs to $\mathrm{NS}^G(X)^+_\mathbb{R}$.
- A **polyhedral cone** C of $\mathrm{NS}^G(X)^+_\mathbb{R}$ is a subset defined by a finite number of linear inequalities. We call C **rational** if the inequalities can be chosen to be rational.
- A **face** of a polyhedral cone C is a subset $F \subseteq C$ such that there exists a linear form f on $\mathrm{NS}^G(X)^+_\mathbb{R}$ that is non-negative on C and satisfies $f(c) = 0$ for all $c \in F$.

- We call a point $l \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$ **effective** if $X^{\mathrm{ss}}(l) \neq \emptyset$ and define the G**-ample cone** $C^G(X)$ as the set of effective points.
- We shall call two points $l, l' \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$ **GIT-equivalent** if $X^{\mathrm{ss}}(l) = X^{\mathrm{ss}}(l')$. The **GIT-class** of some point $l \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$ is the set of all $l' \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$ that are GIT-equivalent to l.

We will use the following definitions to indicate the structure of what our fan will be.

Definition 2.25. Let G be a linearly reductive group variety acting on a variety X.

- For any point $x \in X(k)$ we define the **stability set** of x to be

$$\Omega(x) := \{l \in \mathrm{NS}^G(X)^+_{\mathbb{R}} \mid x \in X^{\mathrm{ss}}(l)\}.$$

- We define a **wall** of $C^G(X)$ to be a stability set of codimension one in $C^G(X)$.
- We define a **chamber** to be a GIT-class of codimension 0 in $C^G(X)$.

So far we have not given any reason why we should call them walls and chambers. The next proposition shows that this is actually not so weird when looking at the G-ample cone.

Proposition 2.26. *Let $W \subseteq C^G(X)$ denote the union of the walls. The chambers are exactly the connected components of $C^G(X) \setminus W$.*

There is some detail missing now. We have defined walls and chambers, but not the structure they form. In a slightly more general sense we can define fans.

Definition 2.27. We define a **fan** Δ in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ to be a finite set of rational convex polyhedral cones in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ such that

(i) The face of each polyhedral cone in Δ is again a polyhedral cone in Δ;

(ii) The intersection of two polyhedral cones in Δ is a face of each of these two polyhedral cones.

Our walls and chambers give us a specific fan, which is the result of a theorem from the paper by Ressayre.

Theorem 2.28 (Ressayre, [21], Theorem 4)**.** *Let X be a normal projective variety and let G be a reductive group variety acting on X. Then:*

(i) For all $l \in C^G(X)$ the set

$$C(l) := \{l' \in C^G(X) \mid X^{ss}(l) \subseteq X^{ss}(l')\}$$

is a closed rational polyhedral convex cone in $C^G(X)$.

(ii) The convex cones $C(l)$ form a fan covering $C^G(X)$.

(iii) All GIT-classes are relative interiors for the cones $C(l)$.

We call this fan the ***GIT-fan*** *for the action of G on X.*

In this general sense, we have not yet said what the walls and chambers actually look like in the GIT-fan. We will give some examples that can hopefully clarify this a bit. We still refer to Ressayre in [21] if the reader is interested in a more broad explanation.

Example 2.29. Let's compute the GIT-fan of the action given in Example 2.17, recall that the action was $t \cdot (x : y) = (tx : t^{-1}y)$ given by $G := \mathbb{G}_m$ acting on $X := \mathbb{P}^1_k$. Write $\mathcal{O}(n, a)$ for the G-linearized invertible sheaf on X given by the invertible sheaf $\mathcal{O}_X(n)$ with linearization given by the integer a. Consider such invertible sheaf that is ample, which is in this case equivalent to $n > 0$. Let $\lambda : t \mapsto t$ denote the identity 1-PS. Then we observe;

$$\begin{aligned}
\mu(\mathcal{O}(n,a), \lambda, (1:0)) &= a;\\
\mu(\mathcal{O}(n,a), \lambda, (0:1)) &= a - 2n;\\
\mu(\mathcal{O}(n,a), \lambda^{-1}, (1:0)) &= -a;\\
\mu(\mathcal{O}(n,a), \lambda^{-1}, (0:1)) &= -a + 2n;
\end{aligned}$$

following mainly from how we computed a in the proof of Proposition 1.34 to begin with. We may also compute limit points;

$$\lim_{t\to 0} \lambda(t) \cdot (x:y) = \begin{cases} (0:1) & , y \neq 0; \\ (1:0) & , y = 0. \end{cases} \qquad \lim_{t\to 0} \lambda^{-1}(t) \cdot (x:y) = \begin{cases} (1:0) & , x \neq 0; \\ (0:1) & , x = 0. \end{cases}$$

Using the properties of the Mumford weight, we have now computed the Mumford weight for any point $(x : y)$. Using this, the numerical criterion tells us the following.

- $D(x) \cap D(y) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a - 2n \leq 0$ and $-a \leq 0$. Therefore this holds when $0 \leq a \leq 2n$.
- $D(x) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a, a - 2n \leq 0$ and $-a \leq 0$. Therefore this holds when $a = 0$.
- $D(y) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a - 2n \leq 0$ and $-a, -a + 2n \leq 0$. Therefore this holds when $a = 2n$.
- $X = X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a, a - 2n \leq 0$ and $-a, -a + 2n \leq 0$. Therefore this holds when $a = 0$ and $a = 2n$, which does not occur as $n > 0$.

It might seem weird to only consider these cases. However, all limit points rely only on whether some coordinate is zero or non-zero. Hence using the numerical criterion all semi-stable sets will be some intersection or union of $D(x)$ and $D(y)$.

We get;

$$X^{\mathrm{ss}}(\mathcal{O}(n,a)) = \begin{cases} D(x) \cap D(y) & \text{if} \quad 0 < a < 2n; \\ D(x) & \text{if} \quad a = 0; \\ D(y) & \text{if} \quad a = 2n; \\ \emptyset & \text{if} \quad a < 0 \text{ or } a > 2n. \end{cases}$$

Now we know all the GIT-classes of ample line bundles. Before we can actually start drawing the fan, note that $\mathrm{NS}^G(X)^+ = \mathrm{Pic}^G(X)^+$ by Proposition 2.14 (where the plus in $\mathrm{Pic}^G(X)^+$ means that we only consider the ample invertible sheaves). Indeed, if $\mathcal{O}(n,a)$ and $\mathcal{O}(m,b)$ give the same class in $\mathrm{NS}^G(X)^+$, then

$$a = \mu(\mathcal{O}(n,a), \lambda, (1:0)) = \mu(\mathcal{O}(m,b), \lambda, (1:0)) = b$$

and

$$a - 2n = \mu(\mathcal{O}(n,a), \lambda, (0:1)) = \mu(\mathcal{O}(m,b), \lambda, (0:1)) = b - 2m$$

so that $a = b$ and $n = m$. Since the Mumford weight is extended linearly to $\mathrm{NS}^G(X)_{\mathbb{R}}$, for any $r \in \mathbb{R}_{>0}$ and $x \in X(k)$ we get

$$M(r\mathcal{O}(n,a), x) = \max_{\rho \in \{\lambda, \lambda^{-1}\}} \mu(r\mathcal{O}(n,a), \rho, x) = r \max_{\rho \in \{\lambda, \lambda^{-1}\}} \mu(\mathcal{O}(n,a), \rho, x) = rM(\mathcal{O}(n,a), x).$$

Where we have used some observations on $M(\mathcal{O}(n,a), x)$ as in the example before. This tells us that the GIT-classes form actual convex cones. Now using the bullet points above, we can directly draw out the fan. See Figure 1.

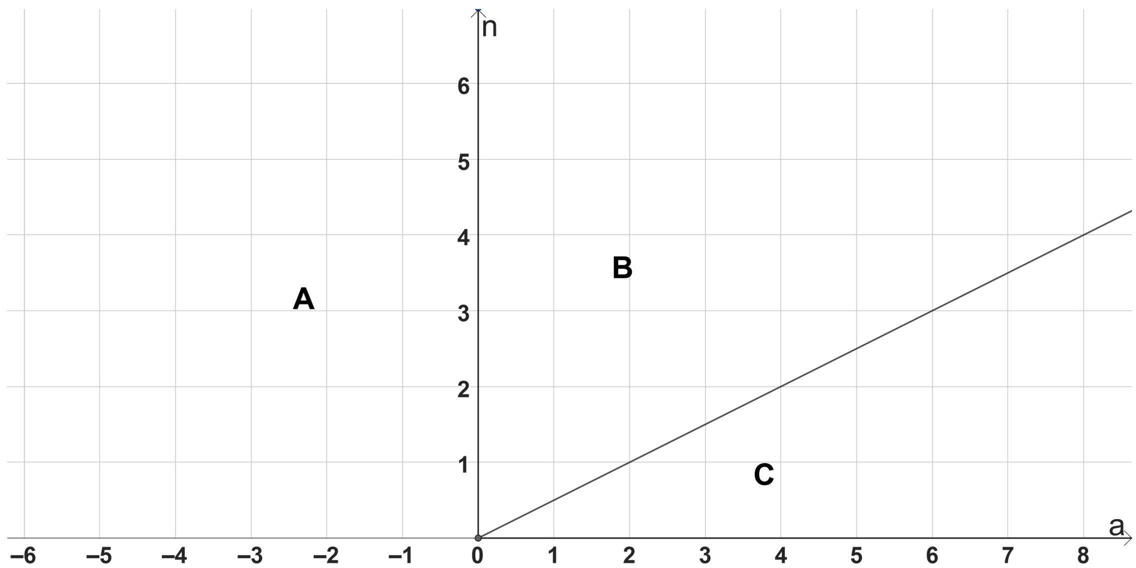


Figure 1: The GIT-fan for a $\mathbb{G}_m$-action on $\mathbb{P}^1_k$.

Here we see $\mathrm{NS}^G(X)_{\mathbb{R}} \cong \mathbb{R}^2$ drawn out. The variable a covers the horizontal axis and n covers the vertical axis. We have two lines, $a = 0$ and $a = 2n$, that make up the walls of our fan. The areas A and C consist of all classes with an empty semi-stable set. The area B consists of all classes with a semi-stable set equal to $D(x) \cap D(y)$. As a result, B together with the walls equals $C^G(X)$. These walls are the stability sets of $(1:0)$ and $(0:1)$.

Looking at this fan, we can see exactly what happened with our variation satisfying the DHT condition. We have $\mathcal{L}_- = \mathcal{O}(1,-1)$ and $\mathcal{L}_+ = \mathcal{O}(1,1)$. Then $\mathcal{L}_t$ crosses the wall $a = 0$ exactly when $t = 0$. This will give the intuïtion of an elementary wall crossing as we will see in the main theorem of Chapter 4.

Example 2.30. Consider a $G := \mathbb{G}_m$ action on $X := \mathbb{P}^2_k$ given by $t \cdot (x:y:z) = (tx : t^{-1}y : t^2 z)$. By Proposition 1.34 we can again write our invertible sheaves as $\mathcal{O}(n,a)$ corresponding to the invertible sheaf $\mathcal{O}_X(n)$ with linearization given by a. Consider a sheaf $\mathcal{O}(n,a)$ with $n > 0$, in other words, such that the sheaf is ample. Let $\lambda : t \mapsto t$ denote the identity 1-PS. We compute the Mumford weights similarly as in the previous example;

$$\begin{aligned} \mu(\mathcal{O}(n,a),\lambda,(1:0:0)) &= a; \\ \mu(\mathcal{O}(n,a),\lambda,(0:1:0)) &= a-2n; \\ \mu(\mathcal{O}(n,a),\lambda,(0:0:1)) &= a+n; \\ \mu(\mathcal{O}(n,a),\lambda^{-1},(1:0:0)) &= -a; \\ \mu(\mathcal{O}(n,a),\lambda^{-1},(0:1:0)) &= -a+2n; \\ \mu(\mathcal{O}(n,a),\lambda^{-1},(0:0:1)) &= -a-n. \end{aligned}$$

The limit points are also always one of these points:

$$\lim_{t\to 0} \lambda(t)\cdot(x:y:z) = \begin{cases} (0:1:0), & y \neq 0; \\ (1:0:0), & y = 0, x \neq 0; \\ (0:0:1), & x = y = 0. \end{cases} \qquad \lim_{t\to 0} \lambda^{-1}(t)\cdot(x:y:z) = \begin{cases} (0:0:1), & z \neq 0; \\ (1:0:0), & z = 0, x \neq 0; \\ (0:1:0), & x = z = 0. \end{cases}$$

Now we consider the different cases again, using the numerical criterion;

- $D(x) \cap D(y) \cap D(z) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a - 2n \leq 0$ and $-a - n \leq 0$. Which is the same as saying $-n \leq a \leq 2n$. Note that this is the same restriction as for the open set $D(y) \cap D(z)$.
- $D(x) \cap D(y) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a - 2n \leq 0$ and $-a, -a-n \leq 0$. Giving us the restriction $0 \leq a \leq 2n$.
- $D(x) \cap D(z) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a, a-2n \leq 0$ and $-a-n \leq 0$. Giving us $-n \leq a \leq 0$.
- $D(x) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a-2n, a \leq 0$ and $-a-n, -a \leq 0$. Therefore $a = 0$ is the only value making this hold.
- $D(y) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a - 2n \leq 0$ and $-a, -a+2n, -a-n \leq 0$. Giving us the restriction $a = 2n$.
- $D(z) \subseteq X^{\mathrm{ss}}(\mathcal{O}(n,a))$ if and only if $a, a-2n, a+n \leq 0$ and $-a-n \leq 0$. Thus $a = -n$ is the only value that works.
- $X^{\mathrm{ss}}(\mathcal{O}(n,a)) = \mathbb{P}^2_k$ will never hold, since then we need $a = 0$ and $a = 2n$ similarly to the previous example.

We get the following cases;

$$X^{\mathrm{ss}}(\mathcal{O}(n,a)) = \begin{cases} D(z) \cap (D(x) \cup D(y)) & \text{if} \quad -n < a < 0; \\ D(y) \cap (D(x) \cup D(z)) & \text{if} \quad 0 < a < 2n; \\ D(z) & \text{if} \quad a = -n; \\ D(x) & \text{if} \quad a = 0; \\ D(y) & \text{if} \quad a = 2n; \\ \emptyset & \text{if} \quad a < -n \text{ or } a > 2n. \end{cases}$$

Using the exact same logic as in the previous example shows that we can draw the fan directly by extending the GIT-classes linearly. See Figure 2 for the GIT-fan.

Now sections A and D correspond to empty semi-stable sets, and sections B and C correspond to different chambers of our fan. As a result, the walls are the stability sets of $(1:0:0), (0:1:0)$ and $(0:0:1)$. We can actually find a sweet pattern here.

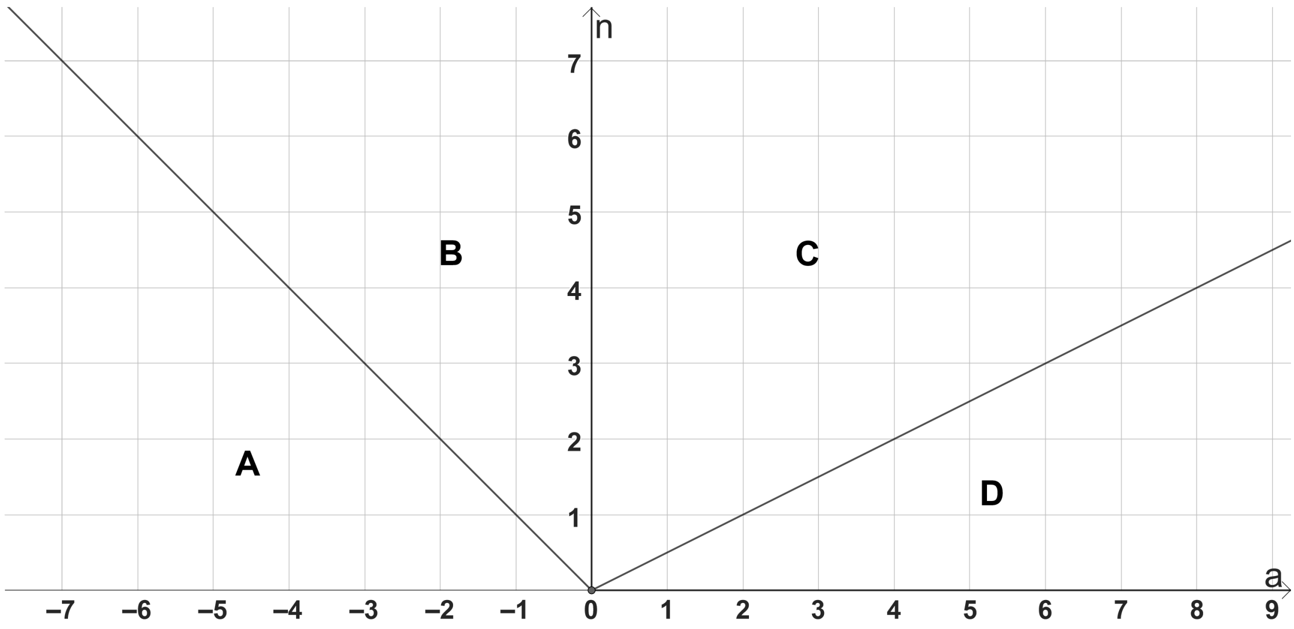


Figure 2: The GIT-fan for a $\mathbb{G}_m$-action on $\mathbb{P}^2_k$.

Proposition 2.31. *Consider a $G := \mathbb{G}_m$ action on $X := \mathbb{P}^n_k$ by $t \cdot (x_0, ..., x_n) = (t^{a_0}x_0 : t^{a_1}x_1 : ... : t^{a_n}x_n)$ for integers $a_1, ..., a_n \neq 1$ that are pairwise distinct. Then the GIT-fan of this action is given by $n+1$ walls of the form $a = (a_0 - a_i)n$ for $i = 0, 1, ..., n$, separating n chambers.*

Proof. We could change our action to the form $t \cdot (x_0, ..., x_n) = (tx_0 : t^{a_1}x_1 : ... : t^{a_n}x_n)$ by multiplying every coordinate by t^{1-a_0}. Then the result would follow directly by applying the method from the examples above. Alternatively, we can give a general proof.

Let $e_i \in \mathbb{P}^n_k(k)$ denote the point $(x_0 : ... : x_n)$ with $x_i = 1$ and $x_j = 0$ for $j \neq i$. Since all a_i are pairwise distinct, the set of fixed points is exactly $\{e_0, ..., e_n\}$. Let n be a positive integer and consider the invertible sheaf $\mathcal{O}_X(n)$ on X. We give it a G-linearization with the element $a \in \mathbb{Z}$ and denote this by $\mathcal{O}(n, a)$. Let λ denote the 1-PS given by $t \mapsto t$. Then for any $i = 0, ..., n$ the Mumford weights are observed to be;

$$\mu(\mathcal{O}(n, a), \lambda, e_i) = a - (a_0 - a_i)n;$$

and

$$\mu(\mathcal{O}(n, a), \lambda^{-1}, e_i) = -a + (a_0 - a_i)n.$$

As they are fixed points, the numerical criterion immediately tells us that $e_i \in X^{\text{ss}}(\mathcal{O}(n, a))$ if and only if $a - (a_0 - a_i)n \leq 0$ and $-a + (a_0 - a_i)n \leq 0$. In other words, this holds only when $a = (a_0 - a_i)n$. By linearity, the stability set $\Omega(e_i)$ is given by this line. This gives us $n+1$ walls.

Now let $x \in X(k)$ and consider the stability set $\Omega(x)$. We immediately know that $\lim_{t \to 0} \lambda(t)x = e_i$ and $\lim_{t \to 0} \lambda(t)^{-1}x = e_j$ for some $i, j \in \{0, 1, ..., n\}$. Therefore, the stability set is given by the inequalities

$$a \leq (a_0 - a_i)n; \qquad a \geq (a_0 - a_j)n.$$

This will be a line if and only if $i = j$, and therefore provides a wall if and only if $\Omega(x) = \Omega(e_i)$. Now consider two walls $w_1 : a = (a_0 - a_i)n$ and $w_2 : a = (a_0 - a_j)n$ that are next to each other. Without loss of generality assume that w_2 lies directly after w_1, looking at the walls clockwise. Then, this implies that $a_j < a_i$. The region between these walls is given by the inequalities

$$-a + (a_0 - a_i)n \leq 0; \qquad a - (a_0 - a_j)n \leq 0.$$

But using $a_j < a_i$, we can do the following. Let x denote the element $(x_0 : ... : x_n)$ with $x_i = x_j = 1$ and $x_l = 0$ for all $l \neq i, j$. Then $\lim_{t\to 0} \lambda(t)x = e_j$ and $\lim_{t\to 0} \lambda(t)^{-1}x = e_i$. Hence the stability set of x is given by the equations $a - (a_0 - a_j)n \leq 0$ and $-a + (a_0 - a_i)n \leq 0$, which was exactly the condition of the region between the walls! Therefore the region between these walls forms a chamber, as the semi-stable set of any point is non-empty. We obtain n chambers from this method.

Finally, we only have to show that all points of the final two regions which are closed in by a wall and the line $n = 0$ have empty semi-stable sets. First consider the most left wall when looking at the walls clockwise. Then in particular for any $j \neq i$ we must have $a_j < a_i$ and the region can be given by the inequalities $a < (a_0 - a_i)n$ and $n > 0$. Thus given some a, n that satisfy these parameters, the value $a - (a_0 - a_j)n$ is maximal whenever $j = i$ and therefore $-a + (a_0 - a_j)n$ is minimal for $j = i$. But $-a + (a_0 - a_i)n > -(a_0 - a_i)n + (a_0 - a_i)n = 0$ and hence the Mumford weight $\mu(\mathcal{O}(n, a), \lambda^{-1}, x)$ will always be strictly positive for such a, n. We conclude that the semi-stable sets are always empty in this region. The same follows for the most right region by flipping the argument with a_i being the minimal a_j.

□

Example 2.32. We finish this section on GIT-fans with a final example of $G := \mathbb{G}_m$ acting on a product of projective n-spaces. We shall consider an action on $X := \mathbb{P}^1_k \times \mathbb{P}^2_k$ by $t \cdot ((x_0 : x_1), (y_0 : y_1 : y_2)) = ((tx_0 : t^{-1}x_1), (ty_0 : t^{-1}y_1 : t^2 y_2))$ as we've seen the actions on the different factors in previous examples, but what we will do can be done in general similarly to how we got the previous corollary. As this example can become quite large if worked out exactly, we give a relatively short sketch.

The invertible sheaves on $\mathbb{P}^1_k \times \mathbb{P}^2_k$ are given by a pair of integers n, m via the pullbacks on the factors. That is, the sheaves $p^*_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k}(1))$ and $p^*_{\mathbb{P}^2_k}(\mathcal{O}_{\mathbb{P}^2_k}(1))$ generated $\mathrm{Pic}(X)$ making it isomorphic to $\mathbb{Z}^2$ (where $p_{\mathbb{P}^1_k}$ and $p_{\mathbb{P}^2_k}$ are the projection maps). We write such an invertible sheaf by $\mathcal{O}_X(n, m)$. These sheaves are obtained by gluing $D(x_{i_1}) \times D(y_{j_1}) \times \mathbb{A}^1_k$ to $D(x_{i_2}) \times D(y_{j_2}) \times \mathbb{A}^1_k$ by the map

$$((x_0 : x_1), (y_0 : y_1 : y_2), s) \mapsto \left((x_0 : x_1), (y_0 : y_1 : y_2), \frac{x^n_{i_2}}{x^n_{i_1}} \frac{y^m_{j_2}}{y^m_{j_1}} s \right).$$

This shows that again all actions of t on $D(x_i) \times D(y_j) \times \mathbb{A}^1_k$ are related similarly as before. That is, the linearization is again given by a single integer a. We denote such linearized invertible sheaf by $\mathcal{O}(n, m, a)$ and compute;

x	$\mu(\mathcal{O}(n, m, a), \lambda, x)$	$\mu(\mathcal{O}(n, m, a), \lambda^{-1}, x)$
$((1 : 0), (1 : 0 : 0))$	a	$-a$
$((1 : 0), (0 : 1 : 0))$	$a - 2m$	$-a + 2m$
$((1 : 0), (0 : 0 : 1))$	$a + m$	$-a - m$
$((0 : 1), (1 : 0 : 0))$	$a - 2n$	$-a + 2n$
$((0 : 1), (0 : 1 : 0))$	$a - 2n - 2m$	$-a + 2n + 2m$
$((0 : 1), (0 : 0 : 1))$	$a - 2n + m$	$-a + 2n - m$

The GIT-fan will reside in a 3-dimensional space, and since all limits are pairs of points in projective space with a 1 as a coordinate and a 0 everywhere else, the stability sets of these points make up all the walls. Therefore, we get a GIT-fan with walls given by equations $a = 0$, $a = 2m$, $a = -m$, $a = 2n$, $a = 2n + 2m$ and $a = 2n - m$. This shows that the GIT-fan of the product has walls given by mixing the walls of the GIT-fans by restricting to a factor!

As a final note, let's talk about variations satisfying the DHT condition. Since we know what the fan looks like, we can have some idea on how such variations look like. Suppose that $(\mathcal{L}_-, \_+)$ is a variation satisfying the DHT condition. Also assume that $\mathcal{L}_-, \mathcal{L}_+$ are not GIT-equivalent, making this a little bit more interesting. Define $\mathcal{L}_t$ as in the definition of the DHT condition. Since $\mathcal{L}_t$ for $t \in [-1, 0)$ is GIT-equivalent to any $\mathcal{L}_s$ for $s \in [-1, 0)$ and the same for $t \in (0, 1]$ with $s \in (0, 1]$, we can say for certain that $\mathcal{L}_0$ has to lie inside a wall of our GIT-fan. Moreover, by our choice $\mathcal{L}_-$ and $\mathcal{L}_+$ have to lie in neighbouring chambers. Hence, the only thing we have to do is check whether $X^{\mathrm{ss}}(0) \setminus (X^{\mathrm{ss}}(+) \cup X^{\mathrm{ss}}(-))$ is connected or empty, and check if $X^{\mathrm{ss}}(0) \setminus (X^{\mathrm{ss}}(+) \cup X^{\mathrm{ss}}(-))$ is contained in the fixed locus of $G = \mathbb{G}_m$.

Let's look at something concretely, like $(\mathcal{O}(1, 3, 1), \mathcal{O}(5, 5, 3))$ with $\mathcal{L}_0 = \mathcal{O}(3, 4, 2)$. These invertible sheaves are seperated by the wall $a = 2n - m$, which is the stability set of the point $((0 : 1), (0 : 0 : 1))$. It can be checked that these invertible sheaves are indeed in neighbouring chambers, an easy way of doing this is by filling in the table above in our general case, and then comparing it for these two invertible sheaves. Since the signs differ in exactly one row, namely the row of $a - 2n + m$ and $-a + 2n - m$, these must be neighbouring chambers. Now suppose we have some $x \in X^{ss}(0) \setminus (X^{ss}(+) \cup X^{ss}(-))$. If we fill in the table as above for $\mathcal{L}_0$, linearity will tell us that we get the same table, but with the bottom row containing only 0. Hence, we get a requirement on x that $\lim_{t\to 0} \lambda(t)x = ((0:1),(0:0:1))$ and $\lim_{t\to 0} \lambda(t)^{-1}x = ((0:1),(0:0:1))$. This can only be $x = \{((0:1),(0:0:1))\}$, which gives a connected set and the point in this set is a fixed point. Hence we have found a variation satisfying the DHT condition rather easily.

It can be checked that this is true more generally. That is, if in our example $\mathcal{L}_-$ and $\mathcal{L}_+$ reside in neighbouring chambers, such that $\mathcal{L}_0$ lies in exactly one wall, then we get the DHT condition for free. This follows since our action was quite "nice". All walls are given by the stability set of the so-called standard elements $((1:0),(1:0:0)),((1:0),(0:1:0)),((1:0),(0:0:1)),((0:1),(1:0:0)),((0:1),(0:1:0)),((0:1),(0:0:1))$. Therefore since $\mathcal{L}_0$ lies in exactly one wall, the set $X^{ss}(0) \setminus (X^{ss}(+) \cup X^{ss}(-))$ will the singleton given by the standard element that gives the wall by its stability set. All these standard elements are fixed, and therefore we get the DHT condition.

2.4 Flips

In this final section of this chapter we consider the notion of a flip. The idea is to explain the occurrence of change in GIT-quotient when crossing a wall. We shall talk shortly about the case when $G = \mathbb{G}_m$ acts on an affine variety $X = \mathrm{Spec}(R)$ and refer to Thaddeus' work in [24] for a detailed analysis of the general case.

There are two definitions here that might not have been given on an introductory course of scheme theory. A birational morphism will be similar to that of the case when talking about classical varieties, so we shall think about our varieties as in this classical sense. The definition of a Cartier divisor will take a lot of definitions and propositions to define properly and give some intuition on. However, in the general sense, we can think about invertible sheaves as the corresponding groups are isomorphic if X is integral or if X is projective. If the reader is still interested in these divisors, we refer to the definitions of Hartshorne, starting on page 140 of [13].

Definition 2.33. Let $f : X \to Y$ be a birational morphism of varieties. Let $E \subseteq Y$ be the smallest closed subset such that the restriction

$$f|_{X\setminus f^{-1}E} : X \setminus f^{-1}E \to Y \setminus E$$

is an isomorphism. If this exists, then we define the **exceptional set** of f to be $f^{-1}E$.

Definition 2.34. We shall call a birational morphism of varieties **small**, if the exceptional set has codimension greater than one.

Definition 2.35. Let $f : X \to S$ be a morphism of schemes. We call some invertible sheaf $\mathcal{L} \in \mathrm{Pic}(X)$ **relatively ample** if f is quasi-compact, and for all affine open subspaces $V \subseteq S$ the restriction $\mathcal{L}|_{f^{-1}(V)}$ is an ample line bundle.

Definition 2.36. Let $f : X \to Y$ be a small birational proper morphism of varieties. Let D be a $\mathbb{Q}$-Cartier divisor class on X such that $\mathcal{O}(-D)$ is relatively ample with respect to f. A **D-flip** is a variety Z with a small birational proper morphism $Z \to Y$ such that;

(i) $\mathcal{O}(D)$ is relatively ample over Y.

(ii) If $g : X \to Z$ is the induced birational map, then the divisor class g_*D is $\mathbb{Q}$-Cartier.

Example 2.37. Consider a $G := \mathbb{G}_m$ action on an affine variety $X = \mathrm{Spec}(R)$. As seen in Lemma 1.3, this is the same as providing a $\mathbb{Z}$-grading of the k-algebra R. Denote this by $R = \oplus_{i=1}^{\infty} R_i$. We shall consider linearizations of the structure sheaf $\mathcal{O}_X$, which has corresponding line bundle $X \times \mathbb{A}^1_k$. Therefore, a G-linearization of $\mathcal{O}_X$ will correspond to an action of G on $\mathbb{A}^1_k := \mathrm{Spec}(k[x])$, which corresponds to a choice of integer. By Remark 2.3, the GIT quotient of X by G with respect to this linearization will then be Proj $R[x]^G$, where the grading of x is the same as the grading given by the linearization. If we let $-n \in \mathbb{Z}$ denote the grading of x, then the quotient will be

$$\mathrm{Proj} \bigoplus_{i=0}^{\infty} R_{ni} x^i.$$

Actually, we do not need to consider many values of n. As seen in exercise 5.13 of section II in Hartshorne, see [13], we know that these constructions only depend on the sign of n. So we get three possible cases; $n = 0$, $n = 1$ and $n = -1$. Denote these quotients by $X/\!\!/0, X/\!\!/+$ and $X/\!\!/-$ respectively. Let $X^{\pm} \subseteq X$ denote the subvarieties given by the ideals $\langle R_i \mid \mp i > 0 \rangle$.

Proposition 2.38 (Thaddeus, [24], Propositions 1.1 & 1.6)**.** *Suppose $X/\!\!/+, X/\!\!/-$ are both nonempty. There is a natural birational map $f : X/\!\!/- \to X/\!\!/+$. If both $X^{\pm} \subset X$ have codimension greater or equal to 2, then f is a flip with respect to $\mathcal{O}(1)$.*

Proposition 2.39 (Thaddeus, [24], Proposition 1.7)**.** *Let Y be a normal and affine variety. Suppose we have a flip $f : Y_- \to Y_+$ of normal varieties over Y. Then there exists an affine variety X with a G-action such that $Y \cong X/\!\!/0$ and $Y_{\pm} \cong X/\!\!/\pm$.*

In particular, working with normal affine varieties gives us a way of generating flips with GIT-quotients and similarly generate GIT-quotients of such varieties using flips. There are generalizations of these propositions, explored in the paper by Thaddeus, see [24].

Example 2.40. Consider the action of $G := \mathbb{G}_m$ on $\mathbb{A}^4_k := \mathrm{Spec}(k[x, y, z, w])$ given on points by $t \cdot (x, y, z, w) = (tx, t^{-1}y, tz, t^{-1}w)$. Then we have the $\mathbb{Z}$-grading of $R := k[x, y, z, w]$ where x, z have grading 1 and y, w have grading -1. Therefore X^+ is given by the subvariety $\mathrm{Spec}(R/(y, w)) \cong \mathbb{A}^2_k$, which can also be given as the subvariety where $y = w = 0$. By symmetry $X^- \subset X$ has codimension 2 as well. We can also compute the quotients, namely;

$$X/\!\!/0 = \mathrm{Spec}(R_0) = \mathrm{Spec}(k[xy, xw, zy, zw]) = \mathrm{Spec}(k[a, b, c, d]/(ad - bc));$$

$$X/\!\!/+ = \mathrm{Proj}\ \bigoplus_{i=0}^{\infty} R_i n^i = \mathrm{Proj}\ k[xy, xw, zy, zw, xn, zn] = \mathrm{Proj}\ k[a, b, c, d, e, f]/(ad - bc, ed - bf);$$

$$X/\!\!/- = \mathrm{Proj}\ \bigoplus_{i=0}^{\infty} R_{-i} n^i = \mathrm{Proj}\ k[xy, xw, zy, zw, yn, wn] = \mathrm{Proj}\ k[a, b, c, d, e, f]/(ad - bc, ed - cf).$$

There is a clear natural isomorphism $f : X/\!\!/- \to X/\!\!/+$ here, and the proposition tells us that this is a flip.

The example above is also another way of showing why flips are interesting. Notice that the Néron Severi group is one-dimensional, and the fan is given by a wall in the origin separating two chambers. So what happens when we consider a quotient with respect to some invertible sheaf in one chamber, compared to that of an invertible sheaf in the other chamber? The example clearly shows that the quotients are isomorphic, so the answer could be "nothing". However another way of viewing it is that they are isomorphic quotients that are a flip away from one another, meaning that we can actually distinguish between them in this sense.

3 Categories and factorizations

In the next couple of chapters, we will introduce and apply the subject of variations of GIT in the sense of derived categories. The idea will be to apply main theorems of papers such as that of Ballard, Favero and Katzarkov in [3]. In this chapter we start with some simple notions of triangulated categories and derived categories, and eventually make our way up to factorizations.

3.1 Triangulated categories

Our first goal will be to define triangulated categories. For a reference, we shall follow closely the procedure of Huybrechts, see [16].

Definition 3.1. An **additive category** is a category $\mathcal{A}$ together with the structure of an Abelian group on $\mathrm{Hom}(A, B)$ for any pair of objects A, B of $\mathcal{A}$, such that:

(i) For any triple of objects A, B, C of $\mathcal{A}$ the composition map $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ is a bilinear map.

(ii) The category $\mathcal{A}$ contains a zero object 0.

(iii) For any pair of objects A, B of $\mathcal{A}$, their product and coproduct exist, and are isomorphic via the canonical map
$$A \oplus B \to A \times B.$$

A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ will be called **additive**, if for any pair of objects A, B of $\mathcal{A}$ the induced map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ is a group homomorphism.

Definition 3.2. Let $\mathcal{A}$ be an additive category. A **shift functor** of $\mathcal{A}$ will be an additive functor $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ which is an equivalence of categories. Given an object A of $\mathcal{A}$, a shift functor $\mathcal{F}$ and an non-negative integer n we shall write $A[n] := \mathcal{F}^n(A)$ if $\mathcal{F}$ is clear from the context. Here $\mathcal{F}^n$ denotes the n-times composition of $\mathcal{F}$. Similarly for such $\mathcal{F}$,n and an arrow $f : A \to B$ in $\mathcal{A}$ we shall write $f[n] := \mathcal{F}^n(f) : A[n] \to B[n]$.

Definition 3.3. Let $\mathcal{A}$ be an additive category, and suppose we have a shift functor $\mathcal{F}$ of $\mathcal{A}$. A **triangle** of $\mathcal{A}$ will be a diagram

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

of objects in $\mathcal{A}$. If we have two triangles $A_i \to B_i \to C_i \to A_i[1]$ for $i = 1, 2$, then a morphism between these triangles is a triple of morphisms $(f : A_1 \to A_2, g : B_1 \to B_2, h : C_1 \to C_2)$ such that the diagram

$$\begin{array}{ccccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & A_1[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & A_2[1] \end{array}$$

commutes. It will be an isomorphism when f, g, h are all isomorphisms.

Definition 3.4. Let $\mathcal{A}$ be an additive category. The **structure of a triangulated category** on $\mathcal{A}$ is a shift functor $\mathcal{F}$, and a set of triangles of $\mathcal{A}$ called the **distinguished triangles** such that the following axioms T1-T4 hold:

(T1) i. For any object A of $\mathcal{A}$, the triangle

$$A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

ii. Any triangle isomorphic to a distinguished triangle is distinguished.

iii. If we have a morphism $f : A \to B$ in $\mathcal{A}$, then f can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

(T2) A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if the triangle

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is distinguished.

(T3) Suppose we have a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow & & \downarrow & & & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

of arrows in $\mathcal{A}$ where the two horizontal triangles are distinguished. Then there exists an arrow $f : C \to C'$ in $\mathcal{A}$ such that f makes the diagram above into a morphism of triangles.

(T4) Suppose we have distinguished triangles

$$A \xrightarrow{f} B \longrightarrow C' \longrightarrow A[1]$$

$$B \xrightarrow{g} C \longrightarrow A' \longrightarrow B[1]$$

$$A \xrightarrow{g \circ f} C \longrightarrow B' \longrightarrow A[1]$$

Then there exists a distinguished triangle

$$C' \longrightarrow B' \longrightarrow A' \longrightarrow C'[1]$$

such that the following diagram commutes:

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \longrightarrow & C' & \longrightarrow & A[1] \\
\downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{A[1]}} \\
A & \xrightarrow{g\circ f} & C & \longrightarrow & B' & \longrightarrow & A[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_C} & & \downarrow & & \downarrow{\scriptstyle f[1]} \\
B & \xrightarrow{g} & C & \longrightarrow & A' & \longrightarrow & B[1] \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{A'}} & & \downarrow \\
C' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & C'[1]
\end{array}$$

We call this axiom the **octahedral axiom**. Even though this axiom looks quite intimidating, we shall not use it in this thesis. The purpose of including it here is purely for the completeness of the definition. Readers interested in this axiom are advised to read the book by Kashiwara and Shapira, see [17].

We call $\mathcal{A}$ a **triangulated category** if it is endowed with the structure of one.

There are a couple of definitions involving triangulated categories that we will see in future chapters. We mention them here as they are more related to the general theory of these categories.

Definition 3.5. Let $\mathcal{A}$ be a triangulated category, and let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$. We call $\mathcal{B}$ a **thick subcategory** if $\mathcal{B}$ is a triangulated subcategory (in other words, the triangulated structure of $\mathcal{A}$ naturally puts a triangulated structure on $\mathcal{B}$) and if $\mathcal{B}$ is closed under taking summands.

Definition 3.6. Let $\mathcal{A}$ be a triangulated category. A **semi-orthogonal decomposition of** $\mathcal{A}$ is a sequence $\mathcal{A}_1, ..., \mathcal{A}_r$ of full triangulated subcategories such that;

- For all $1 \leq i < j \leq n$ and all $A_i \in \mathcal{A}_i$, $A_j \in \mathcal{A}_j$, we have $\mathrm{Hom}(A_j, A_i) = 0$.
- The smallest full triangulated subcategory of $\mathcal{A}$ that contains all $\mathcal{A}_1, ..., \mathcal{A}_n$ equals $\mathcal{A}$.

In such situations we shall write $\mathcal{A} = \langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$.

For the rest of this section, we care to define tilting objects. We need some more definitions to be able to define them.

Definition 3.7. A **k-linear triangulated category** $\mathcal{A}$ is a triangulated category $\mathcal{A}$ together with the structure of a vector space over k on $\mathrm{Hom}(A, B)$ for any pair of objects A, B of $\mathcal{A}$.

Definition 3.8. Let $\mathcal{A}$ be a k-linear triangulated category such that for any pair of objects A, B of $\mathcal{A}$ the vector space $\mathrm{Hom}(A, B)$ is finite dimensional.

- We call an object E of $\mathcal{A}$ **exceptional** if $\mathrm{Hom}(E, E[n]) = 0$ for any $n \neq 0$ and $\mathrm{Hom}(E, E) = k \cdot \mathrm{id}_E$.
- We call a sequence $(E_1, ..., E_r)$ of exceptional objects of $\mathcal{A}$ an **exceptional collection** if for all $n \in \mathbb{Z}$ and all $1 \leq i < j \leq r$ we have $\mathrm{Hom}(E_j, E_i[n]) = 0$.
- We call an exceptional collection $(E_1, ..., E_r)$ of $\mathcal{A}$ **strong** if for all $n \neq 0$ and all i, j we have $\mathrm{Hom}(E_j, E_i[n]) = 0$.
- We call an exceptional collection $(E_1, ..., E_r)$ of $\mathcal{A}$ **full** if the smallest thick triangulated subcategory of $\mathcal{A}$ is $\mathcal{A}$ itself. In other words, if the E_i generate $\mathcal{A}$.
- We call an object T of $\mathcal{A}$ a **tilting object** if the following three conditions are satisfied:
  (i) T generates $\mathcal{A}$. In other words, $\mathcal{A}$ is the smallest thick subcategory of $\mathcal{A}$ that contains T.
  (ii) For any $n \neq 0$ we have $\mathrm{Hom}(T, T[n]) = 0$.
  (iii) The algebra $\mathrm{Hom}(T, T)$ has finite global dimension.

Finally, here we can find a reason why we should be interested in exceptional collections and semi-orthogonal decompositions if we want to look for tilting objects.

Lemma 3.9 (FKK,[10], Proposition 2.5)**.** *Let $\mathcal{A}$ be a k-linear triangulated category so that for any pair of objects A, B, the vector space Hom(A, B) is finite dimensional. If $\mathcal{A}$ admits a full and strong exceptional collection, then it admits a tilting object.*

Proof. Write $(E_1, ..., E_r)$ for the collection and define $T := \oplus_{i=1}^r E_i$. We claim that this is a tilting object. First of all, T generates $\mathcal{A}$ since all E_i generate $\mathcal{A}$. Secondly, if $n \neq 0$, then

$$\mathrm{Hom}(T, T[n]) = \mathrm{Hom}(\oplus_{i=1}^r E_i, \oplus_{j=1}^r E_j[n]) = \oplus_{i,j=1}^r \mathrm{Hom}(E_i, E_j[n]) = 0.$$

Finally, the finite global dimension is clear as $\mathrm{Hom}(T, T)$ has dimension n.

□

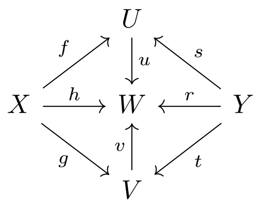
We shall see in chapter 5 why we care about tilting objects in our case.

3.2 Verdier quotients of triangulated categories

Our derived categories will formally be known as absolute derived categories. These mimic the properties of derived categories, but are obtained in a different way via something known as a Verdier quotient. We can think of these as a quotient of two categories. Since not all categories come with a naturally nice structure, we shall have to assume some properties before we can take a quotient.

We will follow the procedure used by the Stacks Project [23, Tag 05RA]. If the reader is not familiar with the notion of a multiplicative system of a category we also refer to the Stacks Project [23, Tag 04VB].

Definition 3.10. Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a multiplicative system of $\mathcal{C}$. Define $\mathcal{X}$ as the collection of pairs of morphisms $(f : X \to U, s : Y \to U)$ where $s \in S$ and f is some morphism in $\mathcal{C}$. We define a relation on $\mathcal{X}$ by stating $(f : X \to U, s : Y \to U) \sim (g : X \to V, t : Y \to V)$ if there exists another pair $(h : X \to W, r : Y \to W)$ and morphisms $u : U \to W, v : V \to W$ of $\mathcal{C}$ such that the diagram



commutes. The composition of equivalence classes of $(f : X \to U, s : Y \to U)$ and $(g : Y \to V, t : Z \to V)$ will be defined as the equivalence class of the pair $(h \circ f : X \to W, r \circ t : Z \to W)$ where $r \in S$ and h are chosen so that $r \circ g = h \circ s$. This choice is possible since S is multiplicative, and this equivalence class is actually well defined (it does not depend on the choice of r and h).

We define the category $\mathcal{S}^{-1}\mathcal{C}$ as the category with the same objects as $\mathcal{C}$, and a morphism $X \to Y$ being defined as a pair $(f : X \to U, s : Y \to U)$ with $s \in S$ and f in $\mathcal{C}$ up to the equivalence above. The identity morphism $X \to X$ will be the equivalence class of the pair $(\mathrm{id}_X, \mathrm{id}_X)$.

Definition 3.11. Let $\mathcal{C}$ be a triangulated category, and $\mathcal{B}$ a full triangulated subcategory. Define a set of arrows $\mathcal{S}$ consisting of all arrows $f : X \to Y$ in $\mathcal{C}$ such that there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{C}$ such that Z is isomorphic to an object of $\mathcal{B}$. Then $\mathcal{S}$ is a multiplicative system, and we define the **Verdier quotient** $\mathcal{C}/\mathcal{B} := \mathcal{S}^{-1}\mathcal{C}$.

3.3 Derived categories

On top of triangulated categories, derived categories will take a big part of the thesis. Our approach is to first define derived categories in general to give the idea behind the construction and afterwards give a definition that we shall work with. A slight knowledge of category theory is expected to understand this chapter.

Definition 3.12. Let $\mathcal{A}$ be an additive category. We call $\mathcal{A}$ **Abelian** if for any morphism $f : A \to B$ between objects of $\mathcal{A}$ we have the following two properties.

(i) Both its kernel and cokernel exist.

(ii) The natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

We note that the image $\mathrm{Im}(f)$ is a kernel of the natural map $B \to \mathrm{Coker}(f)$. Similarly $\mathrm{Coim}(f)$ is a cokernel of the natural map $\mathrm{Ker}\ (f) \to A$. Hence images and coimages exist by the first property, so that the second property makes sense to state.

Definition 3.13. Let $\mathcal{A}$ be an Abelian category. The category of (cochain) complexes $\mathrm{Kom}(\mathcal{A})$ is defined as follows.

- Its objects will be diagrams of the form

$$\cdots \longrightarrow A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \longrightarrow \cdots$$

with $d_A^n \circ d_A^{n-1} = 0$ for all $n \in \mathbb{Z}$. Such objects shall usually be denoted by $A^\bullet$.

- A morphism $f^\bullet : A^\bullet \to B^\bullet$ consists of morphisms $f^n : A^n \to B^n$ such that $f^n \circ d_A^{n-1} = d_B^{n-1} \circ f^{n-1}$ for all $n \in \mathbb{Z}$.

Proposition 3.14 (Huybrechts, [16], Proposition 2.3)**.** *The category of complexes of an Abelian category is again Abelian.*

Definition 3.15. Let $\mathcal{A}$ be an Abelian category and let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$. The **cohomology groups** of $A^\bullet$ are the quotients $H^n(\mathcal{A}^\bullet) := \mathrm{Ker}\ (d_A^n)/\mathrm{Im}(d_A^{n-1}) \in \mathcal{A}$. Note that a morphism $f^\bullet : A^\bullet \to B^\bullet$ induces a natural morphisms $H^n(f) : H^n(A) \to H^n(B)$ for all $n \in \mathbb{Z}$. We call such a morphism a **quasi-isomorphism** if these induced maps $H^n(f)$ are all isomorphisms.

Definition 3.16. Let $\mathcal{A}$ be an Abelian category, and let $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ be two morphisms between two complexes. We call $f^\bullet$ and $g^\bullet$ **homotopy equivalent** there exist homomorphisms $h^n : A^n \to B^{n-1}$ in $\mathcal{A}$ for all $n \in \mathbb{Z}$ such that

$$f^n - g^n = h^{n+1} \circ d_A^n + d_B^{n-1} \circ h^n.$$

The **homotopy category of complexes** $K(\mathcal{A})$ will be the category with the same objects as $\mathrm{Kom}(\mathcal{A})$, but with the morphisms being the morphisms up to homotopy equivalence.

Definition 3.17. Let $\mathcal{A}$ be an Abelian category.

(i) We call a complex $A^\bullet \in \mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$ **bounded below** if there exists an $n_0 \in \mathbb{Z}$ such that $A^n = 0$ for all $n \leq n_0$. Let $\mathrm{Kom}^+(\mathcal{A})$ be the full subcategory of $\mathrm{Kom}(\mathcal{A})$ consisting of complexes that are bounded below.

(ii) We call a complex $A^\bullet \in \mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$ **bounded above** if there exists an $n_0 \in \mathbb{Z}$ such that $A^n = 0$ for all $n \geq n_0$. Let $\mathrm{Kom}^-(\mathcal{A})$ be the full subcategory of $\mathrm{Kom}(\mathcal{A})$ consisting of complexes that are bounded above.

(iii) We call a complex $A^\bullet \in \mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$ **bounded** if there exists an $n_0 \in \mathbb{Z}$ such that $A^n = 0$ for all $|n| \geq n_0$. Let $\mathrm{Kom}^b(\mathcal{A})$ be the full subcategory of $\mathrm{Kom}(\mathcal{A})$ consisting of complexes that are bounded.

(iv) For $* = +, -, b$, we let $K^*(\mathcal{A})$ be the full subcategories of $K(\mathcal{A})$ of which the objects are the complexes bounded below, bounded above, and bounded respectively.

Theorem 3.18 (Huybrechts, [16], Theorem 2.10)**.** *Let $\mathcal{A}$ be an Abelian category. There exists a category $D(\mathcal{A})$, called the* ***derived category of*** *$\mathcal{A}$, and a functor $Q : Kom(\mathcal{A}) \to D(\mathcal{A})$ such that the following properties hold:*

(i) The image $Q(f^\bullet)$ of any quasi-isomorphism $f^\bullet : A^\bullet \to B^\bullet$ is an isomorphism.

(ii) Suppose we have another functor $\mathcal{F} : Kom(\mathcal{A}) \to D$ (where D is some category) satisfying (i). Then there exists a unique functor $\mathcal{G} : D(\mathcal{A}) \to D$, up to isomorphism, such that $\mathcal{G} \circ Q \simeq \mathcal{F}$.

Remark 3.19. Note that a derived category is a special case of Verdier quotient by considering the category $\mathcal{C} = \mathrm{Kom}(\mathcal{A})$ and $\mathcal{B}$ to be the full subcategory of complexes quasi-isomorphic to 0.

Definition 3.20. Let $\mathcal{A}$ be an Abelian category and let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism of complexes. The **mapping cone** of $f^\bullet$ is the complex $C(f)^\bullet$ with $C(f)^n = A^{n+1} \oplus B^n$ and

$$d_{C(f)}^n = \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}.$$

The derived category of some Abelian category $\mathcal{A}$ is in general not Abelian, but we do have a natural structure of a triangulated category on $D(\mathcal{A})$. This can be done similarly to a triangulated structure on $K(\mathcal{A})$. To do so, we shall first have to define the distinguished triangles.

Definition 3.21. Let $\mathcal{A}$ be an Abelian category. For a complex $A^\bullet$ of $\mathcal{A}$ we shall define the complex $A^\bullet[1]$ by $(A^\bullet[1])^n := A^{n+1}$ and $d_{A[1]}^n := -d_A^{n+1}$. This defines a shift functor. Then, a triangle

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$

in $K(\mathcal{A})$ or $D(\mathcal{A})$ shall be called **distinguished** if it isomorphic to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f)^\bullet \xrightarrow{\pi} A^\bullet[1]$$

in $K(\mathcal{A})$ or $D(\mathcal{A})$ respectively, where τ is given by the natural map $B^n \to A^{n+1} \oplus B^n = C(f)^n$ and π is given by the natural map $C(f)^n = A^{n+1} \oplus B^n \to A^{n+1} = (A^\bullet[1])^n$.

At last, Huybrechts showes that these nice constructions give us triangulated categories.

Proposition 3.22 (Huybrechts, [16], Proposition 2.24)**.** *Let $\mathcal{A}$ be an Abelian category. Then the distinguished triangles given in Definition 3.21 together with the shift functor $A^\bullet \mapsto A^\bullet[1]$ makes both $K(\mathcal{A})$ and $D(\mathcal{A})$ into triangulated categories.*

3.4 Stacks

Stacks will play a massive role in this thesis. These are somewhat nice categories to work with, making sure that categorical quotients always exist. Their constructions are very specific, and it is not necessary to understand them in their fullest to understand this thesis. Therefore, we shall not introduce stacks formally. For anyone interested in the definitions we are working with, we refer to [9].

In general, a stack is denoted by $[X]$ where we use the brackets to indicate that this is a stack. The description of a stack is in literature often denoted in the same fashion as the functor of points for a scheme. For example, the functor of points for a scheme X gives us a stack associated to X. Since stacks representing a scheme are representable, we denote these stacks by X instead of $[X]$. In our case we let X be some variety. As stated, quotients of a variety stack X by a group variety stack G exists in the category of stacks and will be denoted by $[X/G]$. To get some intuition on sheaves on these quotients, it would be nice to think about them as G-equivariant sheaves on X. For the space itself, note that it can be dangerous to think about the quotient as a quotient of a scheme by an action, as these quotient stacks cannot be representable in all cases (for example whenever the quotient of a variety X by a group variety G doesn't exist).

3.5 Factorizations of LG-models

The next terms we will discuss are those of factorizations of gauged Landau-Ginzburg models. These factorizations are designed to mimic the matrix factorizations first introduced by Eisenbud in [8]. In this section we shall follow the work of Ballard, Favero and Katzarkov. See [3] and [2].

Definition 3.23. Let X be a variety, and let G be a reductive group variety acting on X. Let $\mathcal{L}$ be a G-linearized invertible sheaf on X and $\omega \in \Gamma(X, \mathcal{L})^G$ a G-invariant global section of $\mathcal{L}$. A **gauged Landau-Ginzburg model**, or **gauged LG-model**, is the tuple $(X, G, \mathcal{L}, \omega)$. Instead of this tuple, we shall denote such models by $([X/G], \omega)$.

Definition 3.24. Let $([X/G], \omega)$ be a gauged LG-model. A **factorization** of this model is a diagram

$$\mathcal{E}_{-1} \xrightarrow{\phi^{\mathcal{E}}_{-1}} \mathcal{E}_0 \xrightarrow{\phi^{\mathcal{E}}_0} \mathcal{E}_{-1} \otimes \mathcal{L}$$

of G-equivariant morphisms of G-equivariant and quasi-coherent $\mathcal{O}_X$-modules such that the compositions $\phi^{\mathcal{E}}_0 \circ \phi^{\mathcal{E}}_{-1}$ and $(\phi^{\mathcal{E}}_{-1} \otimes \mathcal{L}) \circ \phi^{\mathcal{E}}_0$ are both given by mulitplication with ω. For such factorization we shall either write the tuple $(\mathcal{E}_{-1}, \mathcal{E}_0, \phi^{\mathcal{E}}_{-1}, \phi^{\mathcal{E}}_0)$ or $\mathcal{E}$ if there is no confusion.

A morphism $f : \mathcal{E} \to \mathcal{F}$ of factorizations will be a pair $(f_{-1} : \mathcal{E}_{-1} \to \mathcal{F}_{-1}, f_0 : \mathcal{E}_0 \to \mathcal{F}_0)$ of morphisms of quasi-coherent G-equivariant sheaves, so that the diagram

$$\begin{array}{ccccc}
\mathcal{E}_{-1} & \xrightarrow{\phi^{\mathcal{E}}_{-1}} & \mathcal{E}_0 & \xrightarrow{\phi^{\mathcal{E}}_0} & \mathcal{E}_{-1} \otimes \mathcal{L} \\
\downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}\otimes\mathcal{L}} \\
\mathcal{F}_{-1} & \xrightarrow{\phi^{\mathcal{F}}_{-1}} & \mathcal{F}_0 & \xrightarrow{\phi^{\mathcal{F}}_0} & \mathcal{F}_{-1} \otimes \mathcal{L}
\end{array}$$

commutes. This gives us a category of factorizations of the gauged LG-model, denoted by $\mathrm{Fact}(X, G, \omega)$. We can also consider the full subcategory where the $\mathcal{O}_X$-modules are coherent, which we shall denote by $\mathrm{fact}(X, G, \omega)$. We define a shift functor on $\mathrm{Fact}(X, G, \omega)$ (and $\mathrm{fact}(X, G, \omega)$) by $\mathcal{E}[1]$ being the factorization given by the quadruple $(\mathcal{E}_0, \mathcal{E}_{-1} \otimes \mathcal{L}, -\phi^{\mathcal{E}}_0, -\phi^{\mathcal{E}}_{-1} \otimes \mathcal{L})$.

For a morphism of factorizations, we have a natural cone construction.

Definition 3.25. Let $([X/G], \omega)$ be a gauged LG-model, and let $\mathcal{E}, \mathcal{F}$ be factorizations of this model. The **cone** of a morphism $f : \mathcal{E} \to \mathcal{F}$ is defined to be the factorization

$$C(f) := \left(\mathcal{E}_0 \oplus \mathcal{F}_{-1}, \mathcal{E}_{-1} \otimes \mathcal{L} \oplus \mathcal{F}_0, \begin{pmatrix} -\phi^{\mathcal{E}}_0 & 0 \\ f_0 & \phi^{\mathcal{F}}_{-1} \end{pmatrix}, \begin{pmatrix} -\phi^{\mathcal{E}}_{-1} \otimes \mathcal{L} & 0 \\ f_{-1} \otimes \mathcal{L} & \phi^{\mathcal{F}}_0 \end{pmatrix} \right).$$

Definition 3.26. Let

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{E}^{-n} \xrightarrow{g^{-n}} \mathcal{E}^{-n+1} \xrightarrow{g^{-n+1}} \cdots \xrightarrow{g^{-1}} \mathcal{E}^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

be a bounded complex of factorizations for some non-negative integer n. We define a new sequence of factorizations recursively. First, let us define $T^{-1} := C(g^{-1})$. Then, repeatedly apply the following procedure. We let $\widetilde{g^i}$ be the composition

$$\mathcal{E}^i[-1-i] \xrightarrow{g^i[-1-i]} \mathcal{E}^{i+1}[-1-i] \longrightarrow T^{i+1}$$

and define $T^i := C(\widetilde{g^i})$. Here, the last morphism follows from the natural morphism $\mathcal{E}^{i+1}[-2-i][1] \to C(\widetilde{g^{i+1}}) = T^{i+1}$ using $\mathcal{E}^{i+1}[-2-i][1] = \mathcal{E}^{i+1}[-1-i]$.

Then define the **totalization** of the complex of factorizations to be the factorization T^{-n}.

Definition 3.27. Let $([X/G], \omega)$ be a gauged LG-model. We call a factorization $\mathcal{E}$ of this model **acyclic**, if it lies in the smallest thick subcategory of $K(\mathrm{Fact}(X, G, \omega))$ containing the totalizations of all exact complexes from $\mathrm{Fact}(X, G, \omega)$. Define $\mathrm{Acyc}([X/G], \omega)$ as the thick subcategory of $K(\mathrm{Fact}(X, G, \omega))$ consisting of acyclic factorizations. The **(absolute) derived category of quasi-coherent factorizations** will be defined as the Verdier quotient

$$D([X/G], \omega) := K(\mathrm{Fact}(X, G, \omega))/\mathrm{Acyc}([X/G], \omega).$$

In analogy to the derived category as defined in the general case, we call a morphism in $\mathrm{Fact}(X, G, \omega)$ which becomes an isomorphism in $D([X/G], \omega)$ a **quasi-isomorphism**.

Instead of looking at factorizations of quasi-coherent $\mathcal{O}_X$-modules, it will benefit us to look at those factorizations that have coherent components. Using the same constructions as above we define the **derived category of coherent factorizations** as $D(\mathrm{coh}[X/G], \omega)$.

There is an extremely useful theorem given in the paper by Ballard, Favero and Katzarkov that we will use in the final chapters. The following corollary is an immediate consequence of this theorem.

Theorem 3.28 (BFK, [3], Corollary 2.3.12)**.** *There exists an equivalence*

$$D^b(coh[X/G]) \simeq D(coh[X/(G \times \mathbb{G}_m)], 0)$$

where $\mathbb{G}_m$ acts trivially on X.

Sketch. This theorem is a special case of Theorem 2.3.11 of [3] by choosing $\mathcal{E} = 0$ for some G-equivariant locally free sheaf $\mathcal{E}$. In this specific case, the proof goes as follows. Firstly, we define a functor $\mathfrak{J} : \mathrm{coh}[X/G] \to \mathrm{coh}([X/(G \times \mathbb{G}_m)], 0)$ by

$$\mathcal{F} \longmapsto 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathcal{F}$$

This functor extends to the category of cochain complexes $\mathrm{Kom}(\mathrm{coh}[X/G])$ by taking the totalization of each complex. Moreover, this functor descends to a functor $\mathfrak{J} : D^b(\mathrm{coh}[X/G]) \to D(\mathrm{coh}[X/(G \times \mathbb{G}_m)], 0)$ as it sends acyclics to acyclics. Thanks to the work of Mirković and Riche as in section 4.3 of [20], we obtain an equivalence of categories.

□

4 Variation of GIT

We can almost formulate the main idea of this thesis. The last bit of theory we should cover is that of grade restricted windows, and weights of factorizations induced by the windows. These subjects are not necessary to understand the theorem, but they are essential to help us sketch a proof of the main theorem. We shall be following the process of the paper from Ballard, Favero and Katzarkov, see [3].

With the variation of GIT, or VGIT for short, we mean the variation of the G-linearized invertible sheaves. As we have seen, varying these sheaves or even only the linearizations on the sheaves may give very different semi-stable sets. Therefore, it is interesting to see what the difference is between these semi-stable sets. In chapter 2 we saw the use of the GIT-fan, which helps us to visualize the these differences. In this chapter we shall be looking at an interesting use of elementary wall crossings, which gives us a theorem that relates derived categories of factorizations to one another.

4.1 Grade windows

In this paragraph, we shall let G denote a group variety, now acting on a smooth variety X. As usual, we let $\mathcal{L}$ be a G-linearized invertible sheaf, and let $\omega \in \Gamma(X, \mathcal{L})^G$.

The main theorem which we shall prove and use in this thesis makes a lot of use of something which are called windows. The definitions of these windows are quite technical and are therefore difficult to fully understand. As a consequence, we shall assume a couple of results following from the paper of Ballard, Favero and Katzarkov in [3]. These will help us use the power of grade windows without fully understanding them.

Let's first introduce some notation. Let λ be a 1-PS of G and suppose that λ induces an elementary HKKN stratification $\mathfrak{K}$ given by $X = X_\lambda \sqcup S_\lambda$, with a choice Z_λ^0 of connected component of X^λ. We shall assume that S_λ is a (non-empty) smooth closed subvariety of X, and let $\mathcal{N}_{S_\lambda/X}^\vee$ be the conormal sheaf of this closed subvariety. For a construction, see the definition on page 182 of Hartshorne [13]. Next, we define $\omega_{S_\lambda/X} := \wedge^{\text{codim} S_\lambda} \mathcal{N}_{S_\lambda/X}^\vee$. This will be an invertible sheaf on S_λ, also called the relative canonical sheaf of the embedding $S_\lambda \to X$. We define $t(\mathfrak{K}) := \mu(\omega_{S_\lambda/X}, \lambda, x)$, where x is any element of Z_λ^0. Note that this doesn't depend on x by Lemma 2.13. We observe that $t(\mathfrak{K}) < 0$ will hold (as long as $S_\lambda \neq \emptyset$) since by definition of Z_λ the normal vectors to S_λ must have negative weight with respect to λ along Z_λ^0.

Here we run into some very technical notation. We define $N_{S_\lambda^0/X} := V(\mathcal{N}_{S_\lambda^0/X}^\vee)$, using the notation $V(\mathcal{E}) := \underline{\text{Spec}}(\text{Sym } \mathcal{E})$ as in Section 1.4. Next, restrict $N_{S_\lambda^0/X}$ to Z_λ^0 and complete it along the zero section, which we shall denote by $\widehat{N^0}$. Finally, for any open subset $V \subseteq Z_\lambda^0$ we let $\widehat{N_V^0}$ denote the corresponding open subscheme of $\widehat{N^0}$.

Definition 4.1. Suppose we are in the situation described above. Let $\mathcal{E} \in \text{fact}(X, G, \omega)$ and let $I \subseteq \mathbb{Z}$ be a subset. We shall say that $\mathcal{E}$ has **weights along $\widehat{N^0}$ in** I if there exists an open affine cover $\{U_j\}_{j \in J}$ of Z_λ^0 such that $\mathcal{E}|_{\widehat{N_{U_j}^0}}$ is λ-equivariantly quasi-isomorphic to some λ-equivariant factorization with locally finite rank components $\mathcal{F}^n$ satisfying $\mu(\mathcal{F}^n, \lambda, x) \subseteq I$ for all $j \in J$, $n \in \mathbb{Z}$ and any $x \in Z_\lambda^0$.

Definition 4.2. We define the **I-window**, or **I-grade restricted window**, notation $\mathfrak{W}_{\lambda,I}(X, G, \omega)$ or just $\mathfrak{W}_{\lambda,I}$ when the context allows it, as the full subcategory of $D(\text{coh}[X/G], \omega)$ consisting of all factorizations that have weights along $\widehat{N^0}$ in I.

The first important lemma which we shall use gives us an idea of how these windows are given inside $D(\text{coh}[X/G], \omega)$, assuming some conditions. The proof of this lemma would require us to consider a lot more theory, relying on local hypercohomology. As this is not the point of this thesis, we shall not cover the proof of this lemma.

Lemma 4.3 (BFK, [3], Lemma 3.2.1)**.** *Let λ be a 1-PS of G and assume that λ induces an elementary HKKN stratification $\mathfrak{K}$ given by $X = X_\lambda \sqcup S_\lambda$ with choice Z_λ^0 of connected component of X^λ. Assume that S_λ^0 admits a G-invariant affine open cover and that $\mu(\mathcal{L}, \lambda, x) = 0$ for any $x \in Z_\lambda^0$. Let $I, I' \subset \mathbb{Z}$ be subsets and let $\mathcal{E} \in \mathfrak{W}_{\lambda,I}, \mathcal{F} \in \mathfrak{W}_{\lambda,I'}$.*

If $I' - I := \{u - v \in \mathbb{Z} \mid u \in I', v \in I\} \subseteq [t(\mathfrak{K}) + 1, \infty)$, then the pullback

$$i^* : \mathit{Hom}_{\mathit{fact}(X,G,\omega)}(\mathcal{E}, \mathcal{F}) \to \mathit{Hom}_{\mathit{fact}(X_\lambda,G,\omega|_{X_\lambda})}(\mathcal{E}|_{X_\lambda}, \mathcal{F}|_{X_\lambda})$$

induced by the inclusion $i : X_\lambda \to X$ is an isomorphism.

This fact immediately gives us a corollary for the derived categories by taking $I' = I$.

Corollary 4.4 (BFK,[3], Corollary 3.2.2)**.** *Suppose we are in the same situation as Lemma 4.3, and assume that* $\sup\{u - v \mid u, v \in I\} < -t(\mathfrak{K})$*. Then the restriction*

$$i^*|_{\mathfrak{W}_{\lambda,I}} : \mathfrak{W}_{\lambda,I} \to D(\mathit{coh}[X_\lambda/G], \omega|_{X_\lambda})$$

of the pullback induced by the inclusion $i : X_\lambda \to X$ is fully-faithful.

Of course it would be nice if this functor would be something more than fully-faithful. The proposition underneath tells us that we actually get an equivalence of categories, as long as we put a larger restriction on I. For integers u and v we shall write $[u, v] := \{n \in \mathbb{Z} \mid u \leq n \leq v\}$. Similarly to the lemma before, we will not give a proof of this statement.

Proposition 4.5 (BFK,[3], Proposition 3.3.2)**.** *Let λ be a 1-PS of G and assume that λ induces an elementary HKKN stratification $\mathfrak{K}$ given by $X = X_\lambda \sqcup S_\lambda$ with choice Z_λ^0 of connected component of X^λ. Assume that S_λ^0 admits a G-invariant affine open cover and that $\mu(\mathcal{L}, \lambda, x) = 0$ for any $x \in Z_\lambda^0$. Finally, fix $c \in \mathbb{Z}$. Then the restriction*

$$i^*|_{\mathfrak{W}_{\lambda,[c+t(\mathfrak{K})+1,c]}} : \mathfrak{W}_{\lambda,[c+t(\mathfrak{K})+1,c]} \to D(coh[X_\lambda/G], \omega|_{X_\lambda})$$

of the pullback induced by the inclusion $i : X_\lambda \to X$ *is essentially surjective.*

As a final part of this section, we will show some relation between windows obtained from different 1-PS's.

Proposition 4.6. *Suppose we are in the same situation as Proposition 4.5 with two different 1-PS's* λ *and* λ'*. Assume that* $Z^0_\lambda = Z^0_{\lambda'}$ *and* $S_\lambda = S_{\lambda'}$*. Fix* $d, d' \in \mathbb{Z}$*. Then we have an equivalence;*

$$\mathfrak{W}_{\lambda,[d,-t(\mathfrak{K}^\lambda)+d-1]} \simeq \mathfrak{W}_{\lambda',[d',-t(\mathfrak{K}^\lambda)+d'-1-\mu]}$$

where $\mu := -t(\mathfrak{K}^\lambda) + t(\mathfrak{K}^{\lambda'})$*. In other words, we can replace* λ *by* λ' *if we decrease the window size by* μ*.*

Proof. This follows from Corollary 4.4 and Proposition 4.5 as both windows are equivalent to the same derived category.

□

Remark 4.7. Notice that we also get a relation $\mathfrak{W}_{\lambda,I} = \mathfrak{W}_{\lambda^{-1},-I}$, by flipping all weights in the definition of a window. Therefore the previous proposition can be improved upon by including all 1-PS's λ' such that $S_{(\lambda')^{-1}} = S_\lambda$ and $Z^0_\lambda = Z^0_{(\lambda')^{-1}}$.

4.2 Weights of factorizations

The weights of factorizations will be necessary to help us to relate different graded windows with one another. The main ideas are Lemma 4.10, showing that the weights are related to something we know a bit better, and Proposition 4.11, which decomposes a window in a semi-orthogonal decomposition by making the window smaller.

Lemma 4.8. *Any object* $\mathcal{E}$ *of* $D([Z^0_\lambda/C(\lambda)], w|_{Z^0_\lambda})$ *can be split as a direct sum*

$$\mathcal{E} \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{E}_n$$

so that each $\mathcal{E}_n$ *is a factorization with quasi-coherent components that are locally isomorphic to* $\mathcal{O}_X(n)^{m_n}$ *for some* $m_n \in \mathbb{Z}$*. Moreover, this splitting can be chosen to be functorial and* λ*-equivariant.*

Proof. This is the contents of Lemma 3.4.2 of BFK [3].

□

As a direct consequence, we can define weights on these categories.

Definition 4.9. For any $d \in \mathbb{Z}$ we define

$$D([Z^0_\lambda/C(\lambda)], w|_{Z^0_\lambda}) := \{\mathcal{E} \in D([Z^0_\lambda/C(\lambda)], w|_{Z^0_\lambda}) \mid \mathcal{E} = \mathcal{E}_n\}$$

as the full subcategory of $D([Z^0_\lambda/C(\lambda)], w|_{Z^0_\lambda})$ consisting of those factorizations that have weight n.

These weights have quite interesting properties, and can be used to understand other categories. In particular the subcategory of weight 0 factorizations can be seen as the category of factorizations with respect to $[Z^0_\lambda/(C(\lambda)/\lambda)]$. On top of this, the weights have some periodic behavior.

Lemma 4.10 (BFK, [3], Lemma 3.4.4)**.** *We have an equivalence*

$$D(coh[Z^0_\lambda/(C(\lambda)/\lambda)], \omega|_{Z^0_\lambda}) \simeq D(coh[Z^0_\lambda/C(\lambda)], \omega|_{Z^0_\lambda})_0$$

If we assume that there exists some character $\chi : C(\lambda) \to \mathbb{G}_m$ *such that* $\chi \circ \lambda$ *is given by* $t \mapsto t^r$ *for some integer* r*, then for any* $d \in \mathbb{Z}$ *we obtain an equivalence*

$$D(coh[Z^0_\lambda/C(\lambda)], \omega|_{Z^0_\lambda})_d \simeq D(coh[Z^0_\lambda/C(\lambda)], \omega|_{Z^0_\lambda})_{d+r}.$$

As a final proposition we can shorten a window in certain cases by compensating with a semi-orthogonal decomposition. In the next sections this together with the periodicy of the weights will give us the idea to compare different windows via such decompositions.

Proposition 4.11. *Fix $n \in \mathbb{Z}$ and assume that S^0_λ admits a G-invariant affine cover. There exists a functor*

$$\Upsilon_n : D(coh[Z^0_\lambda/C(\lambda)], \omega|_{Z^0_\lambda})_n \to D(coh[X/G], \omega)$$

that is fully-faithful and has essential image in $\mathfrak{W}_{\lambda,[n+t(\mathfrak{K}),n]}$.

If on top of this we consider $n, m \in \mathbb{Z}$ such that $n - m > -t(\mathfrak{K})$, then the functor Υ_n induces a natural semi-orthogonal decomposition

$$\mathfrak{W}_{\lambda,[m,n]} = \langle \Upsilon_n(D(coh[Z^0_\lambda/C(\lambda)], \omega|_{Z^0_\lambda})_n), \mathfrak{W}_{\lambda,[m,n-1]} \rangle.$$

Proof. This is the contents of Lemma 3.4.5, Lemma 3.4.6 and Proposition 3.4.7 of the paper by Ballard, Favero and Katzarkov. For the details, see [3].

□

4.3 Relations between elementary wall crossings

We will now move on to the main theorem of this thesis. The idea is to use the variations of GIT, namely elementary wall crossings, to provide either an equivalence or semi-orthogonal decomposition between derived categories of factorizations. Instead of proving the main theorem, we shall provide a sketch based on the proof as given in the paper. The theorem is stated as follows.

Theorem 4.12 (BFK, [3], Theorem 3.5.2)**.** *Let G be a linearly reductive group variety acting on a smooth, quasi-projective variety X. Suppose we have a G-linearized invertible sheaf $\mathcal{L}$ on X and let $\omega \in \Gamma(X, \mathcal{L})^G$. Let λ be a 1-PS of G and choose some connected component Z^0_λ of X^λ. Fix $d \in \mathbb{Z}$.*

Assume:

- *λ induces an elementary wall crossing $(\mathfrak{K}^+, \mathfrak{K}^-) := (X_+ \sqcup S_\lambda, X_- \sqcup S_{\lambda^{-1}})$;*
- *for any $x \in Z^0_\lambda$ we have $\mu(\mathcal{L}, \lambda, x) = 0$;*
- *S^0_λ admits a G-invariant affine open cover.*

Then;

(a) If $t(\mathfrak{K}^+) < t(\mathfrak{K}^-)$, there exist:

a fully faithful functor

$$\Phi^+_d : D(\mathrm{coh}[X_-/G], \omega|_{X_-}) \to D(\mathrm{coh}[X_+/G], \omega|_{X_+});$$

for all $-t(\mathfrak{K}^-) + d \le j \le -t(\mathfrak{K}^+) + d - 1$, fully faithful functors

$$\Upsilon^+_j : D(\mathrm{coh}[Z^0_\lambda/C(\lambda)], \omega|_{Z^0_\lambda})_j \to D(\mathrm{coh}[X_+/G], \omega|_{X_+});$$

and a semi-orthogonal decomposition

$$D(\mathrm{coh}[X_+/G], \omega|_{X_+}) = \langle \Upsilon^+_{-t(\mathfrak{K}^-)+d}, ..., \Upsilon^+_{-t(\mathfrak{K}^+)+d}, \Phi^+_d \rangle.$$

(b) If $t(\mathfrak{K}^+) = t(\mathfrak{K}^-)$, there exists an exact equivalence

$$\Phi^+_d : D(\mathrm{coh}[X_-/G], \omega|_{X_-}) \to D(\mathrm{coh}[X_+/G], \omega|_{X_+}).$$

(c) If $t(\mathfrak{K}^+) > t(\mathfrak{K}^-)$, there exist:

a fully faithful functor

$$\Phi^-_d : D(\mathrm{coh}[X_+/G], \omega|_{X_+}) \to D(\mathrm{coh}[X_-/G], \omega|_{X_-});$$

for all $-t(\mathfrak{K}^+) + d \le j \le -t(\mathfrak{K}^-) + d - 1$, fully faithful functors

$$\Upsilon^-_j : D(\mathrm{coh}[Z^0_\lambda/C(\lambda)], \omega|_{Z^0_\lambda})_j \to D(\mathrm{coh}[X_-/G], \omega|_{X_-});$$

and a semi-orthogonal decomposition

$$D(\mathrm{coh}[X_-/G], \omega|_{X_-}) = \langle \Upsilon^-_{-t(\mathfrak{K}^+)+d}, ..., \Upsilon^-_{-t(\mathfrak{K}^-)+d}, \Phi^-_d \rangle.$$

Sketch. Observe that (c) is the same statement as (a), but flipped. Hence we may assume that $t(\mathfrak{K}^+) \leq t(\mathfrak{K}^-)$.

Next, we use Corollary 4.4 and Propostion 4.5 to get equivalences of $D(\text{coh}[X_+/G], \omega|_{X_+})$ and $D(\text{coh}[X_-/G], \omega|_{X_-})$ to windows. The main difference between these windows is that the first window is a window with respect to the 1-PS λ, and the second is a window with respect to the 1-PS λ^{-1}. To relate these windows, we make the observation that $\mathfrak{W}_{\lambda^{-1},I} = \mathfrak{W}_{\lambda,-I}$ holds for any subset $I \subseteq \mathbb{Z}$ by flipping all weights. If we chose a suitable c as in Proposition 4.5, this will provide us with an inclusion of windows. This inclusion of windows induces the fully faithful functor $\Phi_d^+ : D(coh[X_-/G], \omega|_{X_-}) \to D(coh[X_+/G], \omega|_{X_+})$, which is an exact equivalence if $t(\mathfrak{K}^+) = t(\mathfrak{K}^-)$.

The inclusion of windows will now make repeated use of Proposition 4.11. We can repeatedly 'shrink' the window corresponding to $D(\text{coh}[X_+/G], \omega|_{X_+})$ exactly $-t(\mathfrak{K}^+) + t(\mathfrak{K}^-)$ times until the last subcategory in the semi-orthogonal decomposition is the window that corresponds to $D(\text{coh}[X_-/G], \omega|_{X_-})$. The fully faithful functors Υ_j^+ are given by the same functors as the ones from this proposition.

□

This theorem can be difficult to understand, and even more difficult to use. The following proposition helps us reduce certain problems so that the theorem above can be used more freely.

Proposition 4.13 (FKK, [10], Proposition 2.6)**.** *Let X be a smooth variety over $k := \mathbb{C}$ and consider the action of an affine group variety G on X. Let $\mathcal{L}$ be a G-linearized invertible sheaf on X, and let $\omega \in \Gamma(X, \mathcal{L})^G$. Let U be any G-invariant open subvariety of X containing the singular locus of ω, and write $i : U \hookrightarrow X$ for the open immersion. Finally, assume that $[X/G]$ has enough locally free sheaves. Then i induces an equivalence*

$$i^* : D(\text{coh}[X/G], \omega) \to D(\text{coh}[U/G], \omega|_U)$$

of categories.

Proof. Consider a factorization $\mathcal{E} = (\mathcal{E}_{-1}, \mathcal{E}_0, \phi_{-1}^{\mathcal{E}}, \phi_0^{\mathcal{E}})$ of the gauged LG-model $([X/G], \omega)$ where $\mathcal{E}_0$ and $\mathcal{E}_{-1}$ are locally free. In particular $\phi_0^{\mathcal{E}} \circ \phi_{-1}^{\mathcal{E}}$ is given as multiplication by ω. Therefore, the Leibniz rule shows that $d\omega$ can be written as $d\phi_0^{\mathcal{E}} \circ \phi_{-1}^{\mathcal{E}} + \phi_0^{\mathcal{E}} \circ d\phi_{-1}^{\mathcal{E}}$, which will (locally) be a map $\mathcal{E} \to \mathcal{E} \otimes \Omega_X$. Now writing $d\omega = d\omega - 0$, we get a homotopy from $d\omega$ to 0 given by the maps $d\phi_0^{\mathcal{E}}$ and $d\phi_{-1}^{\mathcal{E}}$. By definition, morphisms that are homotopic will be the same in the dervied category, showing that $d\omega$ annihilates $\mathcal{E}$. If $\mathcal{E}$ has support on some non-singular point of ω, then $d\omega$ provides a bijective map at this stalk, which is impossible by our earlier discovery. Therefore since $[X/G]$ has enough locally free sheaves, any factorization must be supported on the critical locus of ω.

Now suppose $\mathcal{E}$ is any factorization of the model above, and consider the unit of the adjunction $f : \mathcal{E} \to i_* i^* \mathcal{E}$ evaluated at $\mathcal{E}$. By axiom T1 of a triangulated category, we obtain a factorization $C(f)$ so that $\mathcal{E} \to i_* i^* \mathcal{E} \to C(f) \to \mathcal{E}[1]$ is a distinguished triangle (this object $C(f)$ is also called the cone of f). One can check that some stalk of $C(f)$ is trivial if and only if f induces an isomorphism at that same stalk. We explicitely get that f induces an isomorphism at some stalk x if $x \in U$, and equals the zero map if $x \notin U$, which shows that $C(f)$ is supported on $X \setminus U$. Hence $C(f)$ is supported on $X \setminus U$ and the critical locus of ω. Since they do not intersect, we get $C(f) = 0$. But then f must be an isomorphism. Hence the unit $\text{id} \to i_* \circ i^*$ is a natural isomorphism.

Since i is an open immersion, the counit always gives a natural isomorphism $i^* \circ i_* \to \text{id}$. We conclude that i^* is an equivalence of categories.

□

Given a gauged LG-model $([X/G], \omega)$, an interesting question could be whether or not the derived category $D(\text{coh}[X/G], \omega)$ has an exceptional collection. With the help of Theorem 4.12 and Proposition 4.13, we can answer this question with a yes in some cases. For example, if $\omega \in k[x_1, ..., x_n]$ is a so-called invertible polynomial in the sense of the paper by Favero, Kaplan and Kelly, then we can already show that the singularity category $D(\text{coh}[\mathbb{A}_k^n/\Gamma_\omega], \omega)$ has an exceptional collection. Even more so, the length of this exceptional collection will be given by some number associated to the dual polynomial ω^T. For more information, see [10].

5 Applications of VGIT

In this chapter we finish off the thesis by providing some explicit applications of the main theorem. In the first couple of sections we will follow the process of Favero, Kaplan and Kelly in [10]. After this we consider some other application given by Ballard, Favero and Katzarkov and talk about any direction future research could go into.

5.1 Invertible polynomials

In this section we shall give some examples with inspiration coming from the paper by Favero, Kaplan and Kelly, see [10]. Before we do this, we should introduce some definitions and notation.

Definition 5.1. Fix some positive integer n and let $\omega \in k[x_1, ..., x_n]$.

- We shall call ω **quasi-homogeneous**, if there exist positive weights q_i for x_i making ω homogeneous. In other words, if there exist $q_1, ..., q_n \in \mathbb{Z}_{>0}$ such that $\omega(x_1^{q_1}, ..., x_n^{q_n})$ is a homogeneous polynomial.
- We shall call ω **quasi-smooth**, if the induced map $\omega : \mathbb{A}_k^n \to \mathbb{A}_k^1$ has the origin as only singular point.

Definition 5.2. Fix some positive integer n and consider a polynomial $\omega \in k[x_1, ..., x_n]$ of the form

$$\omega = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}},$$

and consider the matrix $A_\omega := (a_{ij})_{1 \leq i,j \leq n}$. We shall call ω **invertible**, if ω is quasi-homogeneous, quasi-smooth and if A_ω is invertible over $\mathbb{Q}$.

As has been talked about in the paper by Favero, Kaplan and Kelly, we are mostly interested in invertible polynomials because of Kontsevich's Homological Mirror Symmetry Conjecture. If ω is an invertible polynomial as above, then we define the transpose polynomial to be

$$\omega^T := \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ji}}.$$

The conjecture then predicts that the Fukaya-Seidel category of ω^T, see [22], is equivalent to $D(\mathrm{coh}[\mathbb{A}_k^n/\Gamma_\omega], \omega)$ (where Γ_ω is some affine group we shall consider in the next section). The case where $n = 1$ has been proven by Futaki and Ueda in [11]. The case where $n = 2$ has been proven by Habermann and Smith in [12]. Both methods involed matching tilting objects of the corresponding categories, so therefore it is desirable to obtain tilting objects for $D(\mathrm{coh}[\mathbb{A}_k^n/\Gamma_\omega], \omega)$.

After talking about the properties of invertible polynomials in this section, the next section will provide some examples of equivalences and decompositions of these categories following the same procedures as Favero, Kaplan and Kelly. Here they analysed the derived categories of factorizations using only methods from VGIT, and found tilting objects in specific cases. In particular, the case when $n = 1$ can easily be proven. See section 3.1 of [10]. On top of that we show some examples of what happens when we take polynomials that are not invertible.

Remark 5.3. When considering invertible polynomials, it is important that the form condition is satisfied to rule out duplicates. For example, the polynomial $\omega = xy \in k[x, y]$ is (quasi-)homogeneous and quasi-smooth, but it is not invertible. Even if we wanted n terms, we could write $\omega = xy/2 + xy/2$. This does still not comply to the form condition, and it would be a hassle to have to check every way a polynomial can be written.

If we remove the form condition and the condition that A_ω needs to be invertible, we could get examples like $\omega = x^5y^6 + y^8z^5 + x^6z^6 + x^{10} + y^{12} + z^{15} \in k[x, y, z]$ which is quasi-homogeneous of degree $(6, 5, 4, 60)$ and quasi-smooth, but not invertible (it has too many terms).

Remark 5.4. It is at a first glance not obvious that the polynomial $\omega = x^5y^6 + y^8z^5 + x^6z^6 + x^{10} + y^{12} + z^{15} \in k[x, y, z]$ is quasi-smooth. Therefore we would like to provide a method of showing this.

The first idea is to argue that all singular points (x, y, z) that has 0 as one of its coordinates needs to be the origin. This can be done very generally by taking one coordinate equal to 0, and then use the partials to show that the other two must be zero. For the next part we assume that for some singular point (x, y, z) all coordinates are non-zero. In this case, if we substitute $y = rx$ and $z = srx$ for some $s, r \in k^\times$, we can use the equations to obtain two polynomials in the variable s^6x and use the resultant that these polynomials do not share a zero over an algebraically closed field of characteristic zero.

This method can become quite a lot of work. Therefore it would be nice to have some kind of classification. While this doesn't exist for quasi-smooth polynomials on their own, it does exist for invertible polynomials, which we call the Kreuzer-Skarke classfication. Before we can introduce this, let us introduce the polynomials of atomic type.

Definition 5.5. We shall call a polynomial $\omega \in k[x_1, ..., x_n]$ of **atomic type**, if it can be written as one of the following types:

- **Fermat type**: $\omega = x_1^r$ for some $r \in \mathbb{Z}_{\geq 0}$ and $n = 1$.

- **Chain type**: $\omega = x_1^{a_1}x_2 + x_2^{a_2}x_3 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$ for $a_1, ..., a_n \in \mathbb{Z}_{\geq 1}$ and $n \geq 2$.
- **Loop type**: $\omega = x_1^{a_1}x_2 + x_2^{a_2}x_3 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$ for $a_1, ..., a_n \in \mathbb{Z}_{\geq 1}$ and $n \geq 2$.

Definition 5.6. Let X, Y be affine varieties and consider sections $\omega \in \Gamma(X, \mathcal{O}_X)$ and $\upsilon \in \Gamma(Y, \mathcal{O}_Y)$. We define the **Thom-Sebastiani sum** of ω and υ as the element $\omega + \upsilon := \omega \otimes 1 + 1 \otimes \upsilon \in \Gamma(X \times_k Y, \mathcal{O}_{X \times_k Y}) \cong \Gamma(X, \mathcal{O}_X) \otimes_k \Gamma(Y, \mathcal{O}_Y)$.

As referenced in the paper by Favero, Kaplan and Kelly, the Kreuzer-Skarke classification as in [18] tells us that any invertible polynomial, up to a permutation of its variables, is a Thom-Sebastiani sum of polynomials of atomic type, assuming that $k = \mathbb{C}$. However, it can be observed that the reverse is not true, for example by considering the Fermat type polynomial $x \in k[x]$. By explicit computation, one can show the following.

Lemma 5.7. *We have the following reverse statement of the Kreuzer-Skarke classification.*

- *A Fermat type polynomial is invertible if and only if $r \geq 2$.*
- *A chain type polynomial is invertible if and only if $a_n \geq 2$.*
- *A loop type polynomial is invertible if and only if n is odd, or n is even and we have $a_i, a_j \geq 2$ for some $1 \leq i, j \leq n$ with i even and j odd.*

Proof.

- The statement on Fermat polynomials is trivial.
- Consider a polynomial of chain type $\omega = x_1^{a_1}x_2 + x_2^{a_2}x_3 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$ for some $a_1, ..., a_n \in \mathbb{Z}_{\geq}1$ and $n \geq 2$. Let A_ω denote the corresponding matrix, and observe that A_ω is invertible over $\mathbb{Q}$. Suppose that ω is quasi-homogeneous with weights $q_1, ..., q_n$ for $x_1, ..., x_n$ respectively, then there exists some integer r so that

$$A_\omega \cdot \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix}$$

Hence we get a general formula;

$$\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = A_\omega^{-1} \cdot \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix}$$

A simple calculation shows that we have

$$A_\omega^{-1} = \begin{pmatrix} \frac{1}{a_1} & \frac{-1}{a_1a_2} & \frac{1}{a_1a_2a_3} & \cdots & \frac{(-1)^{n-2}}{a_1\cdots a_{n-1}} & \frac{(-1)^{n-1}}{a_1\cdots a_n} \\ 0 & \frac{1}{a_2} & \frac{-1}{a_2a_3} & \cdots & \frac{(-1)^{n-3}}{a_2\cdots a_{n-1}} & \frac{(-1)^{n-2}}{a_2\cdots a_n} \\ 0 & 0 & \frac{1}{a_3} & \cdots & \frac{(-1)^{n-4}}{a_3\cdots a_{n-1}} & \frac{(-1)^{n-3}}{a_3\cdots a_n} \\ \vdots & & \ddots & \ddots & & \\ 0 & \cdots & 0 & 0 & \frac{1}{a_{n-1}} & \frac{-1}{a_{n-1}a_n} \\ 0 & \cdots & 0 & 0 & 0 & \frac{1}{a_n} \end{pmatrix}$$

Since A_ω is invertible, there is exactly one solution (in $\mathbb{Q}$) for the q_i given any r. Therefore we may take $r = \det(A_\omega) = a_1 \cdots a_n$ to get a solution for the weights. For any $i = 1, ..., n$ this will give

$$q_i = a_1 \cdots a_{i-1}(a_{i+1} \cdots a_n - a_{i+2} \cdots a_n + ... + (-1)^{n-i}).$$

Recall that we required all weights to be positive. If any value for r gives a non-positive q_i, then that q_i will stay non-positive for all other r with the same sign. So $q_{n-1} = a_1 \cdots a_{n-2}(a_n - 1)$ shows that $a_n \geq 2$ must hold for ω to be quasi-homogeneous. If we let $a_n \geq 2$, we can readily see that for all i we have $q_i > 0$.

Therefore it suffices to show that ω is quasi-smooth given $a_n \geq 2$. For the partials we get;

$$\frac{d\omega}{dx_1} = a_1x_1^{a_1-1}x_2, \qquad \frac{d\omega}{dx_n} = x_{n-1}^{a_{n-1}} + a_nx_n^{a_n-1}$$

and for $1 < i < n$;

$$\frac{d\omega}{dx_i} = x_{i-1}^{a_{i-1}} + a_i x_i^{a_i - 1} x_{i+1}.$$

Consider some tuple $(x_1, ..., x_n)$ in the singular locus. Note that if $x_i = 0$ for some $2 < i \leq n$, then the partial with respect to x_{i-1} shows that $x_{i-2} = 0$ holds as well. Now as $a_n \geq 2$, the n-th partial shows that if $x_n = 0$, then $x_{n-1} = 0$. So $x_n = 0$ now implies that for all i we have $x_i = 0$. This also shows that the reverse is true. If $x_n \neq 0$, then $x_{n-1} \neq 0$ so that $x_i \neq 0$ for all i holds by considering the other partials. But the first partial shows that either $x_2 = 0$ or $x_1^{a_1 - 1} = 0$, and therefore we cannot have $x_i \neq 0$ for all i. Hence the singular locus consists only of the origin, showing that ω is invertible.

- Now consider a polynomial of loop type $\omega = x_1^{a_1} x_2 + x_2^{a_2} x_3 + ... + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$ for some $a_1, ..., a_n \in \mathbb{Z}_{\geq 1}$ and $n \geq 2$. The matrix A_ω is now equal to

$$\begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & a_n \end{pmatrix}$$

which has determinant $a_1 \cdots a_n + (-1)^{n+1}$. Therefore A_ω is invertible if and only if n is odd or n is even and for some i we have $a_i \geq 2$.

For the homogeneity, we compute the inverse again, we get;

$$\det(A_\omega) A_\omega^{-1} = \begin{pmatrix} a_2 \cdots a_n & -a_3 \cdots a_n & a_4 \cdots a_n & \cdots & (-1)^{n-2} a_n & (-1)^{n-1} \\ (-1)^{n-1} & a_1 a_3 \cdots a_n & -a_1 a_4 \cdots a_n & \cdots & (-1)^{n-3} a_1 a_n & (-1)^{n-2} a_1 \\ (-1)^{n-2} a_2 & (-1)^{n-1} & a_1 a_2 a_4 \cdots a_n & \cdots & (-1)^{n-4} a_1 a_2 a_n & (-1)^{n-3} a_1 a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_2 \cdots a_{n-2} & -a_3 \cdots a_{n-2} & a_4 \cdots a_{n-2} & \cdots & a_1 \cdots a_{n-2} a_n & -a_1 \cdots a_{n-2} \\ -a_2 \cdots a_{n-1} & a_3 \cdots a_{n-1} & -a_4 \cdots a_{n-1} & \cdots & (-1)^{n-1} & a_1 \cdots a_{n-1} \end{pmatrix}$$

In particular, for some $1 \leq i \leq n$ we get the weight:

$$q_i = a_1 \cdots a_{i-1} a_{i+1} \cdots a_n - a_1 \cdots a_{i-1} a_{i+2} \cdots a_n + ... + (-1)^{n-i} a_1 \cdots a_{i-1}$$

$$+ (-1)^{n-i-1} a_2 \cdots a_{i-1} + ... + (-1)^{n-1}.$$

If we pair up the $2j$-th term with the $(2j+1)$-term in this sum for $j \geq 0$, then we can readily see that this weight must be non-negative as all these pairs add up to something non-negative. If n is odd, it is clear that it should always be positive because of the final term.

If n is even, we can only get 0 for q_i if all these pairing add up to 0. If we look at this case, then the difference between two members of the same pair as above is the exclusion of some a_j where j has the same parity as i. Moreover, the pairs run over all such j. Therefore, in this case we get a sum of zero if and only if $a_j = 1$ for all j with the same parity as i. So whenever n is even, we conclude that ω is quasi-homogeneous if and only if there are $a_i, a_j \geq 2$ for some i even and j odd.

For the partials we now get;

$$\frac{d\omega}{dx_1} = a_1 x_1^{a_1 - 1} x_2 + x_n^{a_n}, \qquad \frac{d\omega}{dx_n} = x_{n-1}^{a_{n-1}} + a_n x_n^{a_n - 1} x_1$$

and for $1 < i < n$;

$$\frac{d\omega}{dx_i} = x_{i-1}^{a_{i-1}} + a_i x_i^{a_i - 1} x_{i+1}.$$

Consider some $(x_1, ..., x_n)$ in the singular locus. Looking at the i-th partial, we get that $x_{i+1} = 0$ implies that $x_{i-1} = 0$. Furthermore the n-th partial shows that $x_1 = 0 \Rightarrow x_{n-1} = 0$ and the first partial shows that $x_2 = 0 \Rightarrow x_n = 0$. Hence if n is odd, then $x_i = 0$ for some i implies that for all j we have $x_j = 0$. If i is even, it only implies that for j having the same parity as i we have $x_j = 0$. However, we are assuming there is some $a_j \geq 2$ with j having the same parity as i. Hence we get that $x_{j-1} = 0$ (or $x_n = 0$ if $j = 1$) by looking at the j-th partial, which implies that for j having a different parity we also get $x_j = 0$. It now suffices to check that there are no solutions when for all i we have $x_i \neq 0$.

The n-th partial shows that $x_{n-1}^{a_{n-1}} = -a_n x_n^{a_n - 1} x_1$. We can substitute this into the $(n-1)$-th partial to obtain;

$$x_{n-2}^{a_{n-2}} = -a_{n-1} x_{n-1}^{a_{n-1}-1} x_n = \frac{a_{n-1} a_n x_n^{a_n} x_1}{x_{n-1}}.$$

Doing the same for the $(n-2)$-th partial gives

$$x_{n-3}^{a_{n-3}} = \frac{-a_{n-2} a_{n-1} a_n x_n^{a_n} x_1}{x_{n-2}},$$

so we can repeatedly substitute this until we get

$$x_1^{a_1} = \frac{(-1)^{n-1} a_2 \cdots a_n x_n^{a_n} x_1}{x_2}$$

Finally, the first partial gives

$$x_n^{a_n} = \frac{(-1)^n a_1 a_2 \cdots a_n x_n^{a_n} x_1}{x_1} = (-1)^n a_1 \cdots a_n x_n^{a_n}$$

and therefore we have the equation

$$1 = (-1)^n a_1 \cdots a_n.$$

This will never hold if n is odd or when n is even and for some i we have $a_i \geq 2$. Hence the singular locus consists only of the origin, showing that ω is invertible.

□

Remark 5.8. Notice that the Thom-Sebastiani sum of invertible polynomials is again invertible, and similarly that all Thom-Sebastiani summands of an invertible polynomial must be invertible polynomials. Therefore the lemma above helps us to classify and understand any invertible polynomial.

5.2 Examples of singularity categories

For all examples in this section, we shall fix some positive integer n, and consider some polynomial $\omega \in k[x_1, ..., x_n]$. The group we are considering shall be

$$\Gamma_\omega := \{(t_1, .., t_{n+1}) \in \mathbb{G}_m^{n+1} \mid \omega(t_1 x_1, ..., t_n x_n) = t_{n+1} \omega(x_1, ..., x_n)\}.$$

We let Γ_ω act naturally on $X := \mathbb{A}_k^n$ via the homomorphism $\Gamma_\omega \to \mathrm{GL}(n, k)$ given by projection on its first n coordinates and then mapping to the diagonal matrices. Our main object of interest will be the singularity categories.

Definition 5.9. For any polynomial $w \in k[x_1, ..., x_n]$, the **singularity category of** w is defined to be the category $D(\mathrm{coh}[\mathbb{A}_k^n/\Gamma_w], w)$.

Example 5.10. Let's start off by showing why we are in the case of Theorem 4.12.

- Firstly, since Γ_ω is Abelian, it is solvable. Hence it is linearly reductive if and only if $\Gamma_\omega \cong \mathbb{G}_m^r$ for some $r \in \mathbb{Z}_{\geq 0}$. We will assume this for now. Also notice that $C(\lambda) = \Gamma_\omega$, which we shall use later.
- The invertible sheaf we will choose is going to be the structure sheaf $\mathcal{O}_X$, but (in general) with a non-trivial linearization. Writing $X = \mathbb{A}_k^n = \mathrm{Spec}(k[x_1, ..., x_n])$, we can consider the linearization given by $x_i \mapsto x_i \otimes t_i$ and $1 \mapsto 1 \otimes t_{n+1}^{-1}$. Then the definition of Γ_ω shows that ω is Γ_ω-invariant.
- As a quick observation, the linearization above corresponds to the Γ_ω action on $X \times \mathbb{A}_k^1$ given on points by $(t_1, ..., t_{n+1}) \cdot (x_1, ..., x_n, y) = (t_1 x_1, ..., t_n x_n, t_{n+1} y)$. Therefore, if $\lambda : \mathbb{G}_m \to \Gamma_\omega$ is a 1-PS given by $t \mapsto (t^{r_1}, ..., t^{r_{n+1}})$, we must have $r_{n+1} = 0$ to get the requirement of $\mu(\mathcal{O}_X, \lambda, x) = 0$.

- Using some 1-PS $\lambda : t \mapsto (t^{r_1}, ..., t^{r_n}, 1)$ as above, we can already calculate S_λ very generally. First of all, the fixed locus X^λ will be the set of $(x_1, ..., x_n)$ where $x_i = 0$ if $r_i \neq 0$. Since this is connected, we get $Z^0_\lambda = Z(x_i \mid r_i \neq 0)$. We notice that $S^0_\lambda = \Gamma_\omega \cdot Z^0_\lambda$ is affine. Moreover, $Z_\lambda := \{x \in X \mid \lim_{t\to 0} \lambda(t) \cdot x \in Z^0_\lambda\} = Z(x_i \mid r_i < 0)$ and hence $S_\lambda = Z(x_i \mid r_i < 0)$. We claim that S_λ gives an elementary HKKN stratification.
- S_λ is now a closed subvariety of X given by the ideal generated by all x_i such that $r_i < 0$. Let us denote $x_{i_1}, ..., x_{i_s}$ for these generators. This shows that $\omega_{S_\lambda/X}$ is an invertible sheaf on S_λ generated by the section $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_s}$. Hence this generator has Mumford weight the sum of the Mumford weights of the x_{i_l}. Since the induced action of λ on x_i is given by $x_i \mapsto x_i \otimes t^{r_i}$, we may conclude that $t(\mathfrak{K}^+) = \sum_{r_i<0} r_i$. Totally analogously we get that $t(\mathfrak{K}^-) = \sum_{r_i>0} -r_i$.

As the reader might have noticed, Theorem 4.12 gives us an equivalence or semi-orthogonal decomposition of derived categories of factorizations with respect to the subspaces $X_\pm \subset X$. If $X_\pm = \mathbb{A}^n_k \setminus Z(x_i)$, we can actually reduce $X_\pm$ to $\mathbb{A}^{n-1}_k$.

Lemma 5.11 (FKK, [10], Lemma 2.2)**.** *Consider a quasi-homogeneous polynomial ω. Then we have an isomorphism*

$$[(\mathbb{A}^n_k \setminus Z(x_i))/\Gamma_\omega] \cong [\mathbb{A}^{n-1}_k/\Gamma_{\omega_i}]$$

of stacks induced by the natural inclusion, where $\omega_i := \omega(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_n)$.

Using our knowledge of the theorem so far, it is possible to find an exceptional collection in the case when $n = 1$. It has been shown in the paper by Favero, Kaplan and Kelly in [10], but we shall show the method here again in some more detail.

Example 5.12. When $n = 1$, there is only one polynomial, and that is $\omega := x^r \in k[x]$ for $r \in \mathbb{Z}_{\geq 2}$. Notice that this is a Fermat polynomial, and observe that it would not be invertible if $r = 0, 1$. We consider the polynomial $W := x^r y \in k[x, y]$. Then $\Gamma_W \cong \mathbb{G}^2_m$ via the projection on the first two coordinates. Let λ denote the 1-PS given by $t \mapsto (t, t^{-r}, 1)$. As computed in our first general example, we get $Z^0_\lambda = \{(0, 0, 0)\}$, $S_\lambda = Z(y)$ and $S_{\lambda^{-1}} = Z(x)$ together with an elementary HKKN stratification $(\mathfrak{K}^+, \mathfrak{K}^-)$. This will give us $t(\mathfrak{K}^+) = -r$ and $t(\mathfrak{K}^-) = -1$. We obtain the following semi-orthogonal decomposition using $d = 0$.

$$D(\mathrm{coh}[D(y)/\Gamma_W], W|_{D(y)}) = \langle \Upsilon^+_1, ..., \Upsilon^+_{r-1}, \Phi^+_0(D(\mathrm{coh}[D(x)/\Gamma_W], W|_{D(x)}))\rangle.$$

The key observation is that $[D(y)/\Gamma_W] \cong [\mathbb{A}^1_k/\Gamma_{x^r}]$ and $[D(x)/\Gamma_W] \cong [\mathbb{A}^1_k/\Gamma_x]$ using Lemma 5.11. But the singular locus of x is empty, so Proposition 4.13 implies that $D(\mathrm{coh}[\mathbb{A}^1_k/\Gamma_x], x) = 0$. We are left to identify the subcategories Υ^+_j for $1 \leq j \leq r - 1$. Note that the character $\chi : \Gamma_W \to \mathbb{G}_m$ given by projection on the first coordinates implies that the weights of factorizations have period 1 by Lemma 4.10. Hence all these subcategories are equivalent to the category $D(\mathrm{coh}[Z^0_\lambda/(\Gamma_W/\lambda)], W|_{Z^0_\lambda})$. It is clear that $[\Gamma_W/\lambda] \cong \mathbb{G}_m$ by simply showing an isomorphism of groups. Now since Z^0_λ consists only of the origin, and $W|_{(0,0,0)} = 0$, we get an equivalence

$$D(\mathrm{coh}[Z^0_\lambda/(\Gamma_W/\lambda)], W|_{Z^0_\lambda}) \simeq D(\mathrm{coh}[(0, 0, 0)/\mathbb{G}_m], 0).$$

Using Theorem 3.28, this category is equivalent to $D^b(\mathrm{coh}[\mathrm{Spec}(k)])$, since $(0, 0, 0)$ is a k-point. This bounded derived category is classically known to have an exceptional object k, let's denote it by E_j taking into account that we started off with Υ^+_j. Hence we get an exceptional collection;

$$D(\mathrm{coh}[\mathbb{A}^1_k/\Gamma_{x^r}], x^r) = \langle E_1, ..., E_{r-1}\rangle.$$

In the same paper by Favero, Kaplan and Kelly they looked for and found some exceptional collections in specific cases. In general they first considered Fermat polynomials as we have, and after that looked at the other two polynomials of atomic type with parameters $a_i \geq 2$. However, as we've seen in Lemma 5.7, these are not the only invertible polynomials that exist. Therefore it is interesting to see what happens in the cases when we have $a_i = 1$ for some i. As an example, we get the following.

Example 5.13. Fix some positive odd integer n and consider the polynomial

$$W = x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1x_{n+1} \in k[x_1, ..., x_{n+1}].$$

Next, let $\omega := W_{n+1} = x_1x_2 + ... + x_{n-1}x_n + x_nx_1$, which is an invertible polynomial of loop type, and $W_n = x_1x_2 + ... + x_{n-1} + x_1x_{n+1}$ which is a non-invertible polynomial. Let's compute the singular locus of W.

We get $Z(dW) = Z(x_2 + x_n x_{n+1}, x_1 + x_3, x_2 + x_4, ..., x_{n-1} + x_{n+1}x_1, x_n x_1)$. By the last condition, either $x_n = 0$ or $x_1 = 0$. Since n is odd, we can follow the other equations to get

$$x_n = 0 \Rightarrow x_{n-2} = 0 \Rightarrow ... \Rightarrow x_1 = 0$$

But from the first equation we also obtain $x_2 = 0$, showing that

$$x_2 = 0 \Rightarrow x_4 = 0 \Rightarrow ... \Rightarrow x_{n-1} = 0.$$

If $x_1 = 0$, the equations give

$$x_1 = 0 \Rightarrow x_3 = 0 \Rightarrow ... \Rightarrow x_n = 0 \Rightarrow x_2 = 0 \Rightarrow ...$$

In the end, we find that $Z(dW) = Z(x_1, ..., x_n)$. Next, define the 1-PS λ by $t \mapsto (t^{-1}, t, t^{-1}, t, ..., t^{-1}, t^2, 1)$. This will give us $t(\mathfrak{K}^+) = -(n+1)/2$ and $t(\mathfrak{K}^-) = -(n+1)/2 - 1$. We also get $X_+ = \cup_{i \text{ odd}} D(x_i)$ and $X_- = \cup_{i \text{ even}} D(x_i)$. Hence $X_+ \cap Z(dW) = \emptyset$ and $X_- \cap Z(dW) = D(x_{n+1})$. In particular Proposition 4.13 shows that $D(\text{coh}[X_+/\Gamma_W], W|_{X_+}) = 0$. By Theorem 4.12 and Lemma 5.11, we get a semi-orthogonal decomposition

$$D(\text{coh}[\mathbb{A}^n_k/\Gamma_\omega], \omega) \simeq \langle \Upsilon_0^+ \rangle$$

where $\Upsilon_0^+ \cong D(\text{coh}[Z^0_\lambda/\Gamma_W], W|_{Z^0_\lambda})_0 \cong D(\text{coh}[Z^0_\lambda/(\Gamma_W/\lambda)], W|_{Z^0_\lambda}) \cong D(\text{coh}[\text{Spec}(k)/\mathbb{G}_m^n], 0) \cong D^b(\text{coh B}\mathbb{G}_m^{n-1})$ using the same lemmas and proposition as before. We conclude that $\omega = x_1x_2 + ... + x_{n-1}x_n + x_n x_1$ gives

$$D(\text{coh}[\mathbb{A}^n_k/\Gamma_\omega], \omega) \simeq D^b(\text{coh B}\mathbb{G}_m^{n-1}).$$

However, our goal will not be to consider all cases, as there are simply too many similar cases to the ones presented in the paper. On the other hand, it could be interesting to study cases even when the polynomials are not invertible as it could provide us with some intuition on what these categories look like.

Example 5.14. Consider the polynomial $W = x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_n x_{n+1}^2$. Then W_n, W_{n+1} are both non-invertible, but we will still get something interesting as we will see at the end of this example.

Define the 1-PS λ by $t \mapsto (t^{2(-1)^n}, t^{2(-1)^{n+1}}, ..., t^{-2}, t, 1)$. In particular, if n is odd, we get $t(\mathfrak{K}^+) = -n - 1$ and $t(\mathfrak{K}^-) = -n$. If n is even, we get $t(\mathfrak{K}^+) = -n$ and $t(\mathfrak{K}^-) = -n - 1$. Similarly to the previous example, we get $X_+ = D(x_n) \cup D(x_{n-2}) \cup ...$ and $X_- = D(x_{n+1}) \cup D(x_{n-1}) \cup ...$. With a quick calculation we get $Z(dW) = Z(x_2, x_1 + x_3, x_2 + x_4, ..., x_{n-1} + x_{n+1}^2, 2x_n x_{n+1}) \subset Z(x_2, x_4, x_6, ...)$.

For odd n we get $Z(dW) \cap X_+ \subseteq Z(x_2, x_4, x_6, ...) \setminus \{(0, ..., 0)\} \subseteq D(x_n)$ and $Z(dW) \cap X_- = \emptyset$. For even n we get $Z(dW) \cap X_+ = \emptyset$ and $Z(dW) \cap X_- \subseteq D(x_{n+1})$.

Putting everything together in the same way as in the previous example will give us two results. For odd n we have an equivalence

$$D(\text{coh}[\mathbb{A}^n_k/\Gamma_{W_n}], W_n] \simeq D^b(\text{coh B}\mathbb{G}_m^{n-1})$$

and for even n we have an equivalence

$$D(\text{coh}[\mathbb{A}^n_k/\Gamma_{W_{n+1}}], W_{n+1}] \simeq D^b(\text{coh B}\mathbb{G}_m^{n-1}).$$

Therefore the derived categories of factorizations of these polynomials are also equivalent. It is very interesting to see that it is apparently not very rare to have equivalent categories for different polynomials. Moreover, using the last example we have the category of an invertible polynomial, which was also equivalent to $D^b(\text{coh B}\mathbb{G}_m^{n-1})$ (for odd n). Hence being invertible does not mean the corresponding category is different compared to the category corresponding to a non-invertible polynomial.

Example 5.15. Let $W = x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_n^2 x_{n+1}$. Then $\omega := W_{n+1}$ is an invertible polynomial of chain type. Consider the 1-PS $\lambda : t \mapsto (t^{(-1)^n}, t^{(-1)^{n+1}}, ..., t^{-1}, t^2, 1)$. For odd n we get $t(\mathfrak{K}^+) = -(n+1)/2$ and $t(\mathfrak{K}^-) = -(n+1)/2 - 1$ and for even n we get $t(\mathfrak{K}^+) = -n/2$ and $t(\mathfrak{K}^-) = -n/2 - 2$. We see that

$$Z(dW) = Z(x_2, x_1 + x_3, x_2 + x_4, ..., x_{n-2} + x_n, x_{n-1} + 2x_n x_{n+1}, x_n^2).$$

Let's compute this. Via the last equation we get $x_n = 0$. Then the other equations give

$$x_n = 0 \Rightarrow x_{n-2} = 0 \Rightarrow ...$$

By the second to last equation we get $x_n = 0 \Rightarrow x_{n-1} = 0$, so that

$$x_{n-1} = 0 \Rightarrow x_{n-3} = 0...$$

Hence $Z(dW) = Z(x_1, x_2, ..., x_n)$. We observe that $X_+ = D(x_n) \cup D(x_{n-2}) \cup ...$ and $X_- = D(x_{n+1}) \cup D(x_{n-1}) \cup ...$. Therefore $Z(dW) \cap X_+ = \emptyset$ and $Z(dW) \cap X_- \subseteq D(x_{n+1})$. We conclude that we get a different decomposition based on the parity of n, namely;

$$D(\text{coh}[\mathbb{A}^n_k/\Gamma_\omega], \omega) \simeq \begin{cases} D^b(\text{coh } \mathrm{B}\mathbb{G}_m^{n-1}), & \text{if } n \text{ is odd;} \\ \langle D^b(\text{coh } \mathrm{B}\mathbb{G}_m^{n-1}), D^b(\text{coh } \mathrm{B}\mathbb{G}_m^{n-1}) \rangle, & \text{if } n \text{ is even.} \end{cases}$$

It is quite unique to see the bounded derived categories of classifying stacks pop up in a lot of examples. Even though it wasn't obvious at first, having $Z^0_\lambda = \{(0, 0, ..., 0)\}$ almost guarantees that the semi-orthogonal decomposition contains subcategories equivalent to such categories.

As the Thom-Sebastiani sum of two invertible polynomials is again invertible, we could take a look on the effect by taking a Thom-Sebastiani sum of an invertible polynomial and a non-invertible polynomial. Let us consider the following example.

Example 5.16. Consider the polynomial $w = xy + z^2u + u^2 \in k[x, y, z, u]$ with char $k \neq 2$. Observe that w is quasi-homogeneous of degree $(2, 2, 1, 2)$. The singular locus is given by $Z(y, x, 2zu, z^2 + 2u)$. If we have (x, y, z, u) in this singular locus, then $zu = 0$, so that either $z = 0$ or $u = 0$. Hence by $z^2 + 2u = 0$ the other variable must be zero as well. We conclude that w is quasi-smooth, however we may note that w cannot be invertible due to a lack of terms. Note that w is the Thom-Sebastiani sum of the non-invertible polynomial xy and the chain type polynomial $z^2u + u^2$.

Now we fix $r \in \mathbb{Z}_{\geq 4}$ and let $W := xy + z^2u + u^2v^r \in k[x, y, z, u, v]$. Define the 1-PS $\lambda : \mathbb{G}_m \to \Gamma_W$ by $t \mapsto (t, t^{-1}, t^{-a}, t^{2a}, t^{-4}, 1)$. Using $Z^0_\lambda = \{(0, 0, 0, 0, 0)\}$ the induced HKKN stratifications have $t(\mathfrak{K}^+) = -1 - a - 4 = -(a + 5)$ and $t(\mathfrak{K}^-) = -1 - 2a = -(2a + 1)$. We have $t(\mathfrak{K}^+) \geq t(\mathfrak{K}^-)$ with equality if and only if $a = 4$. The projection homomorphism $\Gamma_W \to \mathbb{G}_m$ on the first coordinate together with Lemma 4.10 gives us a period of 1. The same computations as before give us $D(\text{coh}[Z^0_\lambda/\Gamma_W], W|_{Z^0_\lambda})_0 \simeq D^b(B\mathbb{G}^3_m)$. Hence Theorem 4.12 shows that we have an equivalence

$$D(\text{coh}[(D(x) \cup D(u))/\Gamma_W], W) \simeq \langle D^b(B\mathbb{G}^3_m), ..., D^b(B\mathbb{G}^3_m), D(\text{coh}[(D(y) \cup D(z) \cup D(v))/\Gamma_W], W) \rangle$$

where in the brackets we have $a-4$ times the term $D^b(B\mathbb{G}^3_m)$. The singular locus of W is equal to $Z(y, x, 2zu, z^2 + 2uv^r, ru^2v^{r-1})$. Notice that for $(x, y, z, u, v) \in Z(dW) \cap Z(v)$ we have $z = 0$, due to the fourth term. Moreover, $x, y = 0$ will always hold in the singular locus. This shows that $Z(dW) \cap (D(x) \cup D(u)) \subseteq D(u)$ and $Z(dW) \cap (D(y) \cup D(z) \cup D(v)) \subseteq D(v)$. Hence Proposition 4.13 shows that

$$D(\text{coh}[D(u)/\Gamma_W], W) \simeq \langle D^b(B\mathbb{G}^3_m), ..., D^b(B\mathbb{G}^3_m), D(\text{coh}[D(v)/\Gamma_W], W) \rangle$$

Finally, using Lemma 5.11, we get the statement;

$$D(\text{coh}[\mathbb{A}^4_k/\Gamma_{xy+z^2+v^r}], xy + z^2 + v^r) \simeq \langle D^b(B\mathbb{G}^3_m), ..., D^b(B\mathbb{G}^3_m), D(\text{coh}[\mathbb{A}^4_k/\Gamma_w], w) \rangle$$

We would get the opposite statement if we let $0 < r < 4$. We are stretching new examples here, as both polynomials are a Thom-Sebastiani sum of the polynomial xy (which is quasi-homogeneous, quasi-smooth, but not invertible), and an invertible polynomial. However if we considered the polynomial xy on its own, the statement we would get would be trivial. This may also be observed from the following corollary.

Corollary 5.17. *Let $w, w' \in k[x_1, ..., x_n]$ and $v, v' \in k[y_1, ..., y_m]$ be polynomials. If Theorem 4.12 provides semi-orthogonal decompositions*

$$D(\mathit{coh}[\mathbb{A}^n_k/\Gamma_w], w) \simeq \langle \mathcal{A}_1, ..., \mathcal{A}_r, D(\mathit{coh}[\mathbb{A}^n_k/\Gamma_{w'}], w') \rangle$$

and

$$D(\mathit{coh}[\mathbb{A}^m_k/\Gamma_v], v) \simeq \langle \mathcal{B}_1, ..., \mathcal{B}_s, D(\mathit{coh}[\mathbb{A}^m_k/\Gamma_{v'}], v') \rangle,$$

then it also provides a semi-orthogonal decomposition of the form

$$D(\mathit{coh}[\mathbb{A}^{n+m}_k/\Gamma_{w+v}], w + v) \simeq \langle \mathcal{C}_1, ..., \mathcal{C}_{r+s}, D(\mathit{coh}[\mathbb{A}^{n+m}_k/\Gamma_{w'+v'}], w' + v') \rangle.$$

Proof. Let $\lambda_w : \mathbb{G}_m \to \Gamma_w, \lambda_v : \mathbb{G}_m \to \Gamma_v$ denote 1-PS's which imply the decompositions via the theorem. Then λ_w must be of the form $t \mapsto (t^{a_1}, ..., t^{a_n}, 1)$ and λ_v must be of the form $t \mapsto (t^{b_1}, ..., t^{b_m}, 1)$ for some integers $a_1, ..., a_n, b_1, ..., b_m$. Define $\lambda : \mathbb{G}_m \to \Gamma_{w+v}$ by $t \mapsto (t^{a_1}, ..., t^{a_n}, t^{b_1}, ..., t^{b_m}, 1)$.

Let $\mathfrak{K}^+, \mathfrak{K}^-$ be the HKKN-stratifications obtained from λ and note that these are well-defined as λ_w, λ_v give HKKN-stratifications. Then by assumption $-t(\mathfrak{K}^+)+t(\mathfrak{K}^-) = r+s$, so we obtain the semi-orthogonal decomposition.

□

Remark 5.18. We assumed that the semi-orthogonal decompositions in the previous corollary are given by Theorem 4.12, but it would be logical for this corollary to be true without assuming this. However, we have not proved this so far.

Remark 5.19. To support the previous example, we note that we get a similar equivalence between just $z^2 + v^4$ and $z^2u + u^2$. The equivalence between xy and itself is trivial, and also follows from Theorem 4.12.

Next we will go back to our analysis of $D(\text{coh}[\mathbb{A}^n_k/\Gamma_W], W)$ for general polynomials W. We show some interesting examples which might shed some light on the way these categories present themselves.

Example 5.20. Our first example will concern the polynomial $\omega = x^2yz \in k[x, y, z]$. Choose the 1-PS λ given by $t \mapsto (t, t^{-1}, t^{-1}, 1)$ so that $t(\mathfrak{K}^+) = -2$ and $t(\mathfrak{K}^-) = -1$. We can quickly see that $Z^0_\lambda = \{(0, 0, 0)\}$, and that $S_\lambda = Z(y, z)$, $S_{\lambda^{-1}} = Z(x)$. Choosing $d = t(\mathfrak{K}^{-1}) = -1$, Theorem 4.12 implies that we get a semi-orthogonal decomposition

$$D(\text{coh}[X_+/\Gamma_\omega], \omega|_{X_+}) = \left\langle \Upsilon^+_0(D(\text{coh}[Z^0_\lambda/C(\lambda)], \omega|_{Z^0_\lambda})_0), \Phi^+_{-1}(D(\text{coh}[X_-/\Gamma_\omega], \omega|_{X_-})) \right\rangle.$$

Observe that $C(\lambda) = \Gamma_\omega \cong \mathbb{G}^3_m$, $[\Gamma_\omega/\lambda] \cong \mathbb{G}^2_m$ and that $\omega|_{(0,0,0)} = 0$. Using Theorem 3.28 and Lemma's 5.11 and 4.10 we get equivalences;

$$D(\text{coh}[X_+/\Gamma_\omega], \omega|_{X_+}) \simeq \left\langle D(\text{coh}[Z^0_\lambda/(\Gamma_\omega/\lambda)], \omega|_{Z^0_\lambda}), D(\text{coh}[\mathbb{A}^2_k/\Gamma_{xy}], xy) \right\rangle$$

$$\simeq \left\langle D(\text{coh}[\text{Spec}(k)/\mathbb{G}^2_m], 0), D(\text{coh}[\mathbb{A}^2_k/\Gamma_{xy}], xy) \right\rangle \simeq \left\langle D^b(\text{coh}[\text{Spec}(k)/\mathbb{G}_m]), D(\text{coh}[\mathbb{A}^2_k/\Gamma_{xy}], xy) \right\rangle$$

$$\simeq \left\langle D^b(\text{coh } \text{B}\mathbb{G}_m), D(\text{coh}[\mathbb{A}^2_k/\Gamma_{xy}], xy) \right\rangle.$$

We should of course not forget the isomorphisms that give this decomposition meaning, since the bounded derived category of a classifying stack doesn't naturally lie inside of $D(\text{coh}[X_+/\Gamma_\omega], \omega|_{X_+})$. However, we do get a small idea as to what the category looks like.

An interesting question might be if we can lengthen this decomposition in some way. As we know from Theorem 4.12, the length of the decomposition will be $-t(\mathfrak{K}^+) + t(\mathfrak{K}^-) + 1$, assuming $t(\mathfrak{K}^+) \le t(\mathfrak{K}^-)$. So since these numbers were only 1 apart in our previous case, the question will be if we can get $t(\mathfrak{K}^+)$ a lot smaller than $t(\mathfrak{K}^-)$. This is possible in more ways than one by raising the weights. We will give two ways of doing this in the next example.

Example 5.21. We come back to the polynomial $\omega = x^2yz \in k[x, y, z]$, and fix some positive integer n. Choose 1-PS λ given by $t \mapsto (t^n, t^{-n}, t^{-n}, 1)$, which is the 1-PS of our previous example raised to the n-th power. Compared to this example the only real change is $t(\mathfrak{K}^+) = -2n$ and $t(\mathfrak{K}^-) = -n$. Choosing $d := t(\mathfrak{K}^-) = -n$ again, Theorem 4.12 gives us a semi-orthogonal decomposition

$$D(\text{coh}[X_+/\Gamma_\omega], \omega|_{X_+}) \simeq \left\langle \Upsilon^+_0, ..., \Upsilon^+_{n-1}, D(\text{coh}[\mathbb{A}^2_k/\Gamma_{xy}], xy) \right\rangle$$

where we have used Lemma 5.11 to get the derived category on the right. The triangulated subcategories Υ^+_i are now quite easy to understand. In this case we have $\Gamma_\omega/\lambda \cong \mathbb{G}^2_m$, using that k is algebraically closed. Therefore, as we've done before, for weight 0 we get the following equivalences by using Theorem 3.28 and Lemma 4.10:

$$\Upsilon^+_0 \simeq D(\text{coh}[Z^0_\lambda/\Gamma_\omega], \omega|_{Z^0_\lambda})_0 \simeq D(\text{coh}[Z^0_\lambda/(\Gamma_\omega/\lambda)], \omega|_{Z^0_\lambda})$$

$$\simeq D(\text{coh}[\text{Spec}(k)/(\mathbb{G}_m \times \mathbb{G}_m)], 0) \simeq D^b(\text{coh } \text{B}(\mathbb{G}_m)).$$

However, Lemma 4.10 doesn't tell us more, as any character χ such that $\chi \circ \lambda$ is given by $t \mapsto t^r$ makes r into a multiple of n. Luckily, we can argue that $\Upsilon_v^+ = 0$ for $v = 1, ..., n-1$. We take note that λ gives us the weights of the factorizations, by considering the induced weights on the components of the factorization. But we can write λ as the composition of the map $\mathbb{G}_m \to \mathbb{G}_m : t \mapsto t^n$ and some 1-PS. Hence the induced weights will always be multiples of n. This shows that raising the power of λ will give the same conclusion as λ itself.

As seen, we don't really get to our goal of obtaining a larger decomposition. Therefore, we consider the following example.

Example 5.22. Consider $\lambda : t \mapsto (t^n, t^{-2n+1}, t^{-1}, 1)$. Then almost everything stays the same, including the numbers $t(\mathfrak{K}^{\pm})$. The main difference is that $\Gamma_\omega/\lambda \cong \mathbb{G}_m^2$ now holds, and that Lemma 4.10 gives a period of 1 by letting χ be the projection on the third coordinate. So we will again get a decomposition

$$D(\mathrm{coh}[X_+/\Gamma_\omega], \omega|_{X_+}) \simeq \left\langle \Upsilon_0^+, ..., \Upsilon_{n-1}^+, D(\mathrm{coh}[\mathbb{A}_k^2/\Gamma_{xy}], xy) \right\rangle$$

but this time all $\Upsilon_i^+ \cong D^b(\mathrm{coh}\ \mathrm{B}\mathbb{G}_m)$. Observe that we have a decomposition that is larger than the one we found in Example 5.20, but it contains the same ingredients. The decomposition in that example consisted of one copy of $D^b(\mathrm{cohB}\ \mathbb{G}_m)$ and a copy of $D(\mathrm{coh}[\mathbb{A}_k^2/\Gamma_{xy}], xy)$. In this example, we see the same happening, but with more copies of $D^b(\mathrm{cohB}\ \mathbb{G}_m)$. This is a weird occurrence! We now get a larger decomposition, and it is not quite obvious where the extra bounded derived categories of classifying stacks come from. We do not understand what is happening here, but we can guess that something is happening here similar to a decomposition of $\mathbb{Z}$ into copies $n\mathbb{Z} + v$ for $v = 0, ..., n-1$.

Example 5.23. This example will be something very similar, in the sense that the decomposition will be exactly the same. This should give us an idea that similar decompositions for different polynomials are not (necessarily) rare.

Fix some positive integer n and let $\omega = x^{n+1}yz$. Define the 1-PS λ by $t \mapsto (t, t^{-n}, t^{-1}, 1)$. Then X_+ and X_- are the same as in the previous examples. Furthermore, $t(\mathfrak{K}^+) = -n-1$ and $t(\mathfrak{K}^-) = -1$. Using Lemma 5.11 and Theorem 4.12, we obtain a semi-orthogonal decomposition

$$D(\mathrm{coh}[X_+/\Gamma_\omega], \omega|_{X_+}) \simeq \left\langle \Upsilon_0^+, ..., \Upsilon_{n-1}^+, D(\mathrm{coh}[\mathbb{A}_k^2/\Gamma_{xy}], xy) \right\rangle.$$

We note that Lemma 4.10 and Theorem 3.28 now tell us that $\Upsilon_i^+ \cong D^b(\mathrm{coh}\ \mathrm{B}\mathbb{G}_m)$. But we have seen this exact decomposition in the previous example! This means that the derived categories of factorizations of x^2yz and those of x^nyz have a similar decomposition when restricted to $X_+ = D(y) \cup D(z)$.

Finally, we have one last example which will showcase what the effect is of lowering the power of a variable on the derived category.

Example 5.24. Consider any polynomial that can be written as

$$\omega = \sum_{i=1}^{m} \prod_{j=1}^{n} x_j^{a_{ij}}.$$

Choose $l \in \{1, ..., n\}$ and define $r_l := \gcd_{i \in \{1,...,m\}} a_{il}$. In words, we are choosing r_l to be the greatest common divisor of the powers of x_l that we have in ω. Define

$$W := \sum_{i=1}^{m} x_{n+1}^{\frac{a_{il}}{r_l}} \prod_{j=1}^{n} x_j^{a_{ij}} \in k[x_1, ..., x_{n+1}].$$

Our idea will now be to replace x_l by x_{n+1}. To do this, define $\lambda : t \mapsto (1, ...1, t, 1, ..., 1, t^{-r_l}, 1)$, where the t sits on spot l. We can compute that $Z_\lambda^0 = Z(x_l, x_{n+1})$, $X_+ = D(x_{n+1})$ and $X_- = D(x_l)$. Furthermore, $t(\mathfrak{K}^+) = -r_l$ and $t(\mathfrak{K}^-) = -1$. Hence, by Lemma 5.11 and Theorem 4.12 we get a decomposition

$$D(\mathrm{coh}[\mathbb{A}_k^n/\Gamma_{W_{n+1}}], W_{n+1}) \simeq \left\langle \Upsilon_0^+, ..., \Upsilon_{r_l-2}^+, D(\mathrm{coh}[\mathbb{A}_k^n/\Gamma_{W_l}], W_l) \right\rangle$$

where we can quickly check that $W_{n+1} = \omega$ and $W_l = \omega(x_1, ..., x_{l-1}, x_l^{1/r_l}, x_{l+1}, ..., x_n)$. The period on the weights is 1, so Lemma 4.10 shows that for each i we have $\Upsilon_i^+ \cong D(\mathrm{coh}[Z(x_l, x_{n+1})/(\Gamma_\omega/\lambda)], W|_{Z(x_l, x_{n+1})}) \cong D(\mathrm{coh}[\mathbb{A}_k^{n-1}/\mathbb{G}_m^n], \nu)$ where $\nu := \omega(x_1, ..., x_{l-1}, 0, x_{l+1}, ..., x_n)$. In particular, if ω is a multiple of x_l, then $\nu = 0$ so that Theorem 3.28 gives $\Upsilon_i^+ \cong D^b(\mathrm{coh}[\mathbb{A}^{n-1}/\mathbb{G}_m^{n-1}])$.

As we have now seen, given such polynomial, we can deconstruct the derived category we started off with into a decomposition containing the derived category of the polynomial where we lower the powers. This gives a similar idea as the main idea of Theorem 4.12 where we consider a window and compare it to a different window by making one of the two shorter. Notice that we could shift the analysis of all $D(\mathrm{coh}[\mathbb{A}^n_k/\Gamma_W], W)$ to that of all those categories where any variable x_i of W has coprime powers in the polynomial.

5.3 Exceptional collections for invertible singularity categories

We will now study the paper by Favero, Kaplan and Kelly a bit more. The goal of the paper is to prove an interesting statement on singularity categories of invertible polynomials. For this section, all results are computed over the field $k = \mathbb{C}$, which we shall assume.

Theorem 5.25 (FKK,[10], Theorem 1)**.** *For any invertible polynomial ω, the singularity category of ω has an exceptional collection with size the Milnor number of ω^T.*

This was a conjecture from the paper by Hirano and Ouchi in [14], but we can show it using techniques from VGIT. We first need a definition.

Definition 5.26. Suppose $w \in k[x_1, ..., x_n]$ has isolated singularities. Then the **Milnor number** of w is defined to be

$$\mu(w) := \dim(k[x_1, ..., x_n]/(\partial_{x_1} w, ..., \partial_{x_n} w)).$$

To support the theorem we should be able to compute some Milnor numbers. In particular, Milnor and Orlik showed the following in the paper [19].

Theorem 5.27. *If $w = x^r$ is a Fermat polynomial, then*

$$\mu(w^T) = \mu(w) = r - 1.$$

If $w = x_1^{a_1} x_2 + ... + x_n^{a_n}$ is a chain polynomial, then

$$\mu(w^T) = \sum_{i=0}^{n} (-1)^{n-i} \prod_{j=1}^{i} a_j.$$

If $w = x_1^{a_1} x_2 + ... + x_n^{a_n} x_1$ is a loop polynomial, then

$$\mu(w^T) = \prod_{i=1}^{n} a_i.$$

Now we have the ability to give these numbers for polynomials of atomic type, so the following lemma is useful to compute them in general. The proof is a rather simple observation on tensor products, so we will leave this out. It can be read in the paper.

Lemma 5.28 (FKK,[10], Lemma 2.10)**.** *Suppose $w \in k[x_1, ..., x_n]$ and $v \in k[y_1, ..., y_m]$ have isolated singularities. Then $\mu(w + v) = \mu(w)\mu(v)$.*

Next we consider the two remaining cases of atomic type polynomials. We shall show a statement on loop type polynomials, and then claim a similar statement on chain type polynomials. Both techniques can be found in the paper by Favero, Kaplan and Kelly.

Example 5.29. We consider the polynomial $W := x_1^{a_1} x_2 + ... + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1 x_{n+1}^b \in k[x_1, ..., x_{n+1}]$ for $a_1, ..., a_n, b \in \mathbb{Z}_{\geq 2}$. Notice that W_{n+1} is a loop type polynomial and W_n is a chain type polynomial. Next, we define the following integers:

$$d_i := (-1)^{i+n+1} \cdot b \cdot \prod_{j=1}^{i-1} a_i, \qquad \text{for } 1 \leq i \leq n;$$

$$d_{n+1} := a_1 \cdots a_n + (-1)^{n+1}.$$

These d_i are chosen specifically so that $(-1)^{i+n+1} d_i$ is the determinant of the i-th maximal minor of A_W i.e. of the matrix obtained from A_W after removing the $(n+1)$-th row (which is a row full of zeroes) and the i-th column. For $i = 1, ..., n+1$ we define $c_i := d_i / \gcd(d_1, ..., d_{n+1})$. As we will see, this will be useful for our 1-PS to give us a period of 1 in the weights.

We now observe that the 1-PS given by $\lambda : \mathbb{G}_m \to \Gamma_W$ defined by

$$t \mapsto (t^{c_1}, ..., t^{c_{n+1}}, 1)$$

is well-defined. The c_i alternate sign, with c_{n+1} being a multiple of $a_1 \cdots a_{n+1} + (-1)^{n+1}$. Since we made the choice to have $a_i \geq 2$ for all i, we see that $c_{n+1}, c_{n-1}, c_{n-3}, ... > 0$ and $c_n, c_{n-2}, c_{n-4}, ... < 0$. Therefore, as a consequence of Example 5.10, we get $S_\lambda = Z(x_n, x_{n-2}, x_{n-4}, ...)$ and $S_{\lambda^{-1}} = Z(x_{n+1}, x_{n-1}, x_{n-3}, ...)$. Another consequence gives us $t(\mathfrak{K}^+) = c_n + c_{n-2} + c_{n-4} + ...$ and $t(\mathfrak{K}^-) = -c_{n+1} - c_{n-1} - c_{n-3} - ....$ Define $t_0(\mathfrak{K}^+) := d_n + d_{n-2} + d_{n-4} + ...$ and $t_0(\mathfrak{K}^-) := -d_{n+1} - d_{n-1} - d_{n-3} - ...$ and notice that $t(\mathfrak{K}^+) < t(\mathfrak{K}^-)$, $t(\mathfrak{K}^+) > t(\mathfrak{K}^-)$ or $t(\mathfrak{K}^+) = t(\mathfrak{K}^-)$ hold if and only if $t_0(\mathfrak{K}^+) < t_0(\mathfrak{K}^-)$, $t_0(\mathfrak{K}^+) > t_0(\mathfrak{K}^-)$ or $t_0(\mathfrak{K}^+) = t_0(\mathfrak{K}^-)$ hold respectively. By Theorem 5.27 we get;

$$\mu(W_{n+1}^T) - \mu(W_n^T) = a_1 \cdots a_n - \big((-1)^n + (-1)^{n+1}b + (-1)^{n+2}ba_1 + \cdots + ba_1 \cdots a_{n-1}\big)$$

$$= a_1 \cdots a_n + (-1)^{n+1} + (-1)^{n+2}b + (-1)^{n+3}ba_1 + ... + ba_1 \cdots a_{n-1} = \sum_{i=1}^{n+1} d_i = t_0(\mathfrak{K}^+) - t_0(\mathfrak{K}^-).$$

More specifically, we have $\mu(W_{n+1}^T) < \mu(W_n^T)$ if and only if $t_0(\mathfrak{K}^+) < t_0(\mathfrak{K}^-)$ if and only if $t(\mathfrak{K}^+) < t(\mathfrak{K}^-)$. The other two statements for $\mu(W_{n+1}^T) > \mu(W_n^T)$ and $\mu(W_{n+1}^T) = \mu(W_n^T)$ hold naturally as well.

The singular locus of W can also be computed, this equals:

$$Z(a_1 x_1^{a_1} x_2 + x_n^{a_n} x_{n+1}^b, x_1^{a_1} + a_2 x_2^{a_2}, ..., x_{n-1}^{a_{n-1}} + a_n x_n^{a_n - 1} x_1 x_{n+1}^b, b x_n^{a_n} x_1 x_{n+1}^{b-1}).$$

Observe that for any $x = (x_1, ..., x_{n+1})$ in this singular locus we have $x_n = 0 \Rightarrow x_{n-2} = 0 \Rightarrow x_{n-4} = 0 \Rightarrow ...$ and similarly that $x_{n+1} = 0 \Rightarrow x_{n-1} = 0 \Rightarrow x_{n-3} = 0 \Rightarrow ...$. Therefore we conclude that $Z(dW) \cap X_\lambda \subseteq D(x_n)$ and $Z(dW) \cap X_{\lambda^{-1}} \subseteq D(x_{n+1})$. Hence $D(\text{coh}[X_\lambda/\Gamma_W], W) \simeq D(\text{coh}[\mathbb{A}_k^n/\Gamma_{W_n}], W_n)$ and $D(\text{coh}[X_{\lambda^{-1}}/\Gamma_W], W) \simeq D(\text{coh}[\mathbb{A}_k^n/\Gamma_{W_{n+1}}], W_{n+1})$ by the same methods we've used before. Before we use the main theorem, we can already discuss what the categories Υ_j^+ will look like. By the period of 1 for the weights, following from our definition of λ, they will all be equivalent to one another. So by our main methods we get $\Upsilon_j^+ \simeq D(\text{coh}[\text{Spec}(k)/(\Gamma_W/\lambda)], 0)$ where we identify the origin with $\text{Spec}(k)$, as this is the only point in Z_λ^0.

We shall now assume a black box. By Lemma 2.8 of the paper by Favero, Kaplan and Kelly, the category $D(\text{coh}[\text{Spec}(k)/(\Gamma_W/\lambda)], 0)$ admits an exceptional collection. The length of this exceptional collection is given by $\gcd(d_1, ..., d_{n+1})$ as can be seen in the proof of Theorem 3.4 of this same paper. The proof uses some terms we have not mentioned, and it is not the goal of this thesis to go into depth on this.

The amount of objects Υ_j^+ in the decomposition will be the difference $|t(\mathfrak{K}^+) - t(\mathfrak{K}^-)| = |c_1 + ... + c_{n+1}|$ so that in total we have $\gcd(d_1, ..., d_{n+1}) \cdot |c_1 + ... + c_{n+1}| = |d_1 + ... + d_{n+1}| = |\mu(W_{n+1}^T) - \mu(W_n^T)|$ exceptional objects in the decomposition. We obtain the following result.

Theorem 5.30. *Let* $W := x_1^{a_1} x_2 + ... + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1 x_{n+1}^b$ *with* $a_1, ..., a_n, b \geq 2$.

If $\mu(W_{n+1}^T) < \mu(W_n^T)$*, then we have a semi-orthogonal decomposition*

$$D(coh[\mathbb{A}_k^n/\Gamma_{W_n}], W_n) \simeq \langle E_1, ..., E_{\mu(W_n^T) - \mu(W_{n+1}^T)}, D(coh[\mathbb{A}_k^n/\Gamma_{W_{n+1}}], W_{n+1})\rangle$$

where each E_i *is an exceptional object.*

If $\mu(W_{n+1}^T) = \mu(W_n^T)$*, then we have an equivalence*

$$D(coh[\mathbb{A}_k^n/\Gamma_{W_n}], W_n) \simeq D(coh[\mathbb{A}_k^n/\Gamma_{W_{n+1}}], W_{n+1}).$$

If $\mu(W_{n+1}^T) > \mu(W_n^T)$*, then we have a semi-orthogonal decomposition*

$$D(coh[\mathbb{A}_k^n/\Gamma_{W_{n+1}}], W_{n+1}) \simeq \langle E_1, ..., E_{\mu(W_{n+1}^T) - \mu(W_n^T)}, D(coh[\mathbb{A}_k^n/\Gamma_{W_n}], W_n)\rangle$$

where each E_i *is an exceptional object.*

As previously mentioned, we will not consider the other case in depth. The main idea of the paper is to show that the example above can always be manipulated so that on the left we have some singularity category of a loop polynomial and on the right we have the singularity category of a chain polynomial. Then the paper shows via induction (and an example on loop polynomials) that for any loop polynomial w, the singularity category of w has an exceptional collection of length $\mu(w^T)$. Together with the statement above and the trivial statement on Fermat polynomials, we get that this holds true for any invertible polynomial, proving Theorem 5.25.

Let's consider the result of the other case. This is given by the polynomial

$$W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_{n+1}^b.$$

The result is the following.

Theorem 5.31. *Let* $W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_{n+1}^b$ *with* $a_1, ..., a_n, b \geq 2$.
If $\mu(W_{n+1}^T) < \mu(W_n^T)$, *then we have a semi-orthogonal decomposition*

$$D(coh[\mathbb{A}_k^n/\Gamma_{W_n}], W_n) \simeq \langle E_1, ..., E_{\mu(W_n^T)-\mu(W_{n+1}^T)}, D(coh[\mathbb{A}_k^n/\Gamma_{W_{n+1}}], W_{n+1})\rangle$$

where each E_i *is an exceptional object.*
If $\mu(W_{n+1}^T) = \mu(W_n^T)$, *then we have an equivalence*

$$D(coh[\mathbb{A}_k^n/\Gamma_{W_n}], W_n) \simeq D(coh[\mathbb{A}_k^n/\Gamma_{W_{n+1}}], W_{n+1}).$$

If $\mu(W_{n+1}^T) > \mu(W_n^T)$, *then we have a semi-orthogonal decomposition*

$$D(coh[\mathbb{A}_k^n/\Gamma_{W_{n+1}}], W_{n+1}) \simeq \langle E_1, ..., E_{\mu(W_{n+1}^T)-\mu(W_n^T)}, D(coh[\mathbb{A}_k^n/\Gamma_{W_n}], W_n)\rangle$$

where each E_i *is an exceptional object.*

Notice that this is exactly the same statement, but now we are comparing different polynomials. The polynomial W_{n+1} is a chain polynomial, and W_n is a Thom-Sebastiani sum of a chain polynomial and a Fermat polynomial. Since we already know Theorem 5.25 for Fermat polynomials, this gives us a reason to try to show the same for chain polynomials using induction.

Example 5.32. Consider any chain polynomial $w = x_1^{a_1}x_2 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$ with all $a_i \geq 2$ and $n \geq 2$. Define the polynomial $W := x_1^{a_1}x_2 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_{n+1}^{a_n}$ so that $w = W_{n+1}$. We claim that $\mu(W_{n+1}^T) > \mu(W_n^T)$. To see this, we have to do a bit of algebra.

By Theorem 5.27 we get;

$$\mu(W_{n+1}^T) = \sum_{i=0}^{n}(-1)^{n-i}\prod_{j=1}^{i} a_j$$

and

$$\mu(W_n^T) = (a_n - 1) \cdot \sum_{i=0}^{n-1}(-1)^{n-i-1}\prod_{j=1}^{i} a_j$$

since W_n is the Thom-Sebastiani sum of the Fermat polynomial $x_{n+1}^{a_n}$ and a chain polynomial given by $a_1, ..., a_{n-1}$ in that order. Therefore, we get

$$\mu(W_n^T) = \sum_{i=0}^{n-1}(-1)^{n-i-1}a_n\prod_{j=1}^{i} a_j + \sum_{i=0}^{n-1}(-1)^{n-i}\prod_{j=1}^{i} a_j.$$

Notice that the second sum equals the first $n-1$ terms from $\mu(W_{n+1}^T)$. So to show $\mu(W_{n+1}^T) > \mu(W_n^T)$, it suffices to show that

$$a_1 \cdots a_n > \sum_{i=0}^{n-1}(-1)^{n-i-1}a_n\prod_{j=1}^{i} a_j.$$

By dividing both sides by a_n and writing out the right side, we can reduce to the case of showing

$$a_1 \cdots a_{n-1} > a_1 \cdots a_{n-1} - a_1 \cdots a_{n-2} + a_1 \cdots a_{n-3} - a_1 \cdots a_{n-4} + ...$$

where the last term is $(-1)^{n-1}$. This inequality follows from our assumption that all $a_i \geq 2$ must hold. In particular, we have;

$$-a_1 \cdots a_{n-2} + a_1 \cdots a_{n-3} - a_1 \cdots a_{n-4} + ... = -(a_1 \cdots a_{n-3})(a_{n-2} - 1) + -(a_1 \cdots a_{n-5})(a_{n-4} - 1) + ...$$

Therefore each of these terms is strictly negative, showing that the entire sum is negative. Notice that this cannot go wrong at the last term, since when n is odd, a $(-1)^n = 1$ would pair with the negative term in front of it (and we are assuming $n > 1$). We conclude that $\mu(W_{n+1}^T) > \mu(W_n^T)$. Now we can apply Theorem 5.31 to divide the singularity category of $w = W_{n+1}$ into a semi-orthogonal decomposition with $\mu(W_{n+1}^T) - \mu(W_n^T)$ exceptional objects and the singularity category of W_n. However, W_n is the Thom-Sebastiani sum of a Fermat polynomial and a chain polynomial of length $n-1$, so inductively this singularity category has a full exceptional collection of length $\mu(W_n^T)$. Hence the singularity category of w has a full exceptional collection of length $\mu(w^T)$.

Example 5.33. Now let $w = x_1^{a_1} x_2 + ... + x_{n-2}^{a_{n-2}} x_{n-1} + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$ be a loop type polynomial with all $a_i \geq 2$ and define the polynomial $W = x_1^{a_1} x_2 + ... + x_{n-2}^{a_{n-2}} x_{n-1} + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1 x_{n+1}^{a_n}$ so that $w = W_{n+1}$. In a similar fashion as to the previous example, we can see that $\mu(W_{n+1}^T) > \mu(W_n^T)$. Therefore Theorem 5.30 now holds, showing us that the singularity category of w has a semi-orthogonal decomposition with $\mu(W_{n+1}^T) - \mu(W_n^T)$ exceptional objects and the singularity category of the chain type polynomial W_n. But by the previous example we know that the singularity category of the chain type polynomial W_n has a full exceptional collection of length $\mu(W_n^T)$, and therefore we conclude that the singularity category of w has a full exceptional collection of length $\mu(w^T)$.

We are now not actually done with Theorem 5.25, as Lemma 5.7 shows that there are invertible polynomials that have $a_i = 1$ for some i. However, as our examples in the previous have shown, the theorem is at least true for the examples we have shown. For example when n is odd the polynomial $x_1 x_2 + ... + x_{n-1} x_n + x_n x_1$ has Milnor number 1. We saw that it was equivalent to the category $D^b(\text{coh } \mathrm{B}\mathbb{G}_m^{n-1})$ which has a full exceptional collection of length 1.

Similarly the example on the polynomial $x_1 x_2 + x_2 x_3 + ... + x_{n-1} x_n + x_n^2$ has a full exceptional collection of length 1 if n is odd, and length 2 if n is even. This agrees with Milnor number. Indeed, by Theorem 5.27 the Milnor number equals $(-1)^n + (-1)^{n-1} + ... + 1 - 1 + 2$ which equals 1 if n is odd and 2 if n is even.

Now we will consider any invertible chain polynomial with some $a_i = 1$ with a similar idea in mind.

Example 5.34. Let $w = x_1^{a_1} x_2 + ... + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$ be a chain polynomial so that for some i we have $a_i = 1$. Consider the polynomial $W := x_1^{a_1} x_2 + ... + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_{n+1}$ and define the integers:

$$d_i := (-1)^{n+i-1} \cdot \prod_{j=1}^{i-1} a_i, \qquad \text{for } 1 \leq i \leq n;$$

$$d_{n+1} := a_1 \cdots a_n.$$

Define the 1-PS $\lambda : \mathbb{G}_m \to \Gamma_W$ by $t \mapsto (t^{d_1}, ..., t^{d_{n+1}}, 1)$. Notice that projection on the first coordinate will give us a character with a period of 1, since $d_1 = (-1)^n$. We can also make the following observation.

$$\sum_{i=1}^{n+1} d_i = a_1 \cdots a_n + \sum_{i=1}^{n} (-1)^{n+i-1} \prod_{j=1}^{i-1} a_j = a_1 \cdots a_n + \sum_{i=0}^{n-1} (-1)^{n+i} \prod_{j=1}^{i} a_j = \mu(w^T)$$

making use of Theorem 5.27. With the help of Example 5.10 we can see that $X_+ = D(x_n) \cup D(x_{n-2}) \cup ...$ and $X_- = D(x_{n+1}) \cup D(x_{n-1}) \cup ...$, so we can see what happens when taking the intersection with $Z(dW)$. First we compute that:

$$Z(dW) = (a_1 x_1^{a_1 - 1} x_2, x_1^{a_1} + a_2 x_2^{a_2 - 1} x_3, ..., x_{n-1}^{a_{n-1}} + a_n x_n^{a_n - 1} x_{n+1}, x_n^{a_n}).$$

So for $x = (x_1, ..., x_n) \in Z(dW)$ we get the implications $x_n = 0 \Rightarrow x_{n-2} = 0 \Rightarrow x_{n-4} = 0 \Rightarrow ...$ and $x_{n+1} = 0 \Rightarrow x_{n-1} = 0 \Rightarrow x_{n-3} = 0 \Rightarrow ....$ In particular the condition $x_n^{a_n} = 0$ implies that $x_n = 0$, so $Z(dW) \cap X_+ = \emptyset$ and

$Z(dW) \cap X_- \subseteq D(x_{n+1})$. We can also observe that $t(\mathfrak{K}^-) = \sum_{d_i > 0} -d_i < \sum_{d_i<0} d_i = t(\mathfrak{K}^+)$ and hence Theorem 4.12 gives us a semi-orthogonal decomposition

$$D(\text{coh}[\mathbb{A}^n_k/\Gamma_w], w) = \langle \Upsilon_1^-, ..., \Upsilon_{\mu(w^T)}^- \rangle$$

where we have used Proposition 4.13. Each object Υ_j^- is equivalent to one another by the period of 1, and therefore they are all equivalent to $D^b(\text{coh} B\mathbb{G}_m^n)$. This category has a full exceptional collection of size 1, so the singularity category of ω has a full exceptional collection of size $\mu(w^T)$.

Example 5.35. We can now look at the final case, where $w = x_1^{a_1}x_2 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$ is a loop polynomial where for some i we have $a_i = 1$. We can start this example totally analogous to the previous example, let's apply a trick to find a suitable polynomial W. We would like to construct it so that $w = W_{n+1}$, and so that W_n is a chain polynomial. However, we would need some $a_i \geq 2$ for this. Luckily, if for all i we have $a_i = 1$, then we are in the context of Example 5.13. This gave us the correct conclusion as we've talked about before, so we may assume there is some i with $a_i \geq 2$. By switching variables, or looping the variables, we may assume $a_{n-1} \geq 2$. Now do the following.

Consider the polynomial $W := x_1^{a_1}x_2 + ... + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1x_{n+1}$ and define the integers:

$$d_i := (-1)^{n+i+1} \cdot \prod_{j=1}^{i-1} a_i, \qquad \text{for } 1 \leq i \leq n;$$

$$d_{n+1} := a_1 \cdots a_n + (-1)^{n+1}.$$

Define the 1-PS λ again using the formula $t \mapsto (t^{d_1}, ..., t^{d_{n+1}}, 1)$. There is one key observation necessary here. Since we are assuming $a_{n-1} \geq 2$, then w_n is a chain type polynomial. Hence following the same process gives us a semi-orthogonal decomposition

$$D(\text{coh}[\mathbb{A}^n_k/\Gamma_w], w) = \langle E_1, ..., E_{\mu(w^T)-\mu(W_n^T)}, D(\text{coh}[\mathbb{A}^n_k/\Gamma_{W_n}], W_n) \rangle$$

where each E_i is an exceptional object. The previous example shows the statement is true for chain type polynomials, so therefore the decomposition above gives us a full exceptional collection for the singularity category of w assuming that $a_{n-1} \geq 2$.

We conclude that Theorem 5.25 is true in general! That is, the singularity category of any invertible polynomial w has a full exceptional collection of length $\mu(w^T)$.

The main goal of finding these collections was the construction of tilting objects. Keen readers may have noticed that we are not done in finding these objects, as we have not shown whether or not our exceptional collections are strong. There is a small section in the paper by Favero, Kaplan and Kelly showcasing that we can actually get strong cases when looking at something called the Gorenstein case. Unfortunately, these cases are limited, and we do not know if the result holds more broadly. For readers interested we strongly recommend to read section 3.4 of [10].

5.4 Elementary wall crossings and GIT quotients

In this last section, we consider an immediate consequence of the main theorem and provide an example. The theorem shows that in the case of a variation satisfying the DHT condition, the derived category of factorizations with respect to the GIT quotients have a similar relation to that of the main theorem. Our goal here is to showcase another strong theorem resulting from the main theorem.

Theorem 5.36 (BFK, [3], Theorem 4.1.5 & Theorem 4.2.1)**.** *Let G be a linearly reductive group variety acting on a smooth and projective variety X. Suppose that $(\mathcal{L}_-, \mathcal{L}_+)$ is a variation satisfying the DHT condition. Then there exists a 1-PS $\lambda : \mathbb{G}_m \to G$ and a choice of connected component Z_λ^0 of $X^{ss}(0)^\lambda$ inducing an elementary wall crossing*

$$X^{ss}(0) = X^{ss}(+) \sqcup S_\lambda;$$
$$X^{ss}(0) = X^{ss}(-) \sqcup S_{\lambda^{-1}}.$$

Write $(\mathfrak{K}^+, \mathfrak{K}^-)$ for this elementary wall crossing and set $X/\!\!/\pm := [X^{ss}(\pm)/G]$. Let $\mathcal{L}$ be a G-linearized invertible sheaf, let $\omega \in \Gamma(X, \mathcal{L})^G$ and fix $d \in \mathbb{Z}$. If we also assume that $\mu(\mathcal{L}, \lambda, x) = 0$ for all $x \in Z_\lambda^0$, then;

(a) If $t(\mathfrak{K}^+) < t(\mathfrak{K}^-)$, there exist:

a fully faithful functor

$$\Phi_d^+ : D(\text{coh } X/\!\!/-, \omega|_{X_-}) \to D(\text{coh } X/\!\!/+, \omega|_{X_+});$$

for all $-t(\mathfrak{K}^-) + d \leq j \leq -t(\mathfrak{K}^+) + d - 1$, fully faithful functors

$$\Upsilon_j^+ : D(\text{coh}[Z_\lambda^0/C(\lambda)], \omega|_{Z_\lambda^0})_j \to D(\text{coh } X/\!\!/+, \omega|_{X_+});$$

and a semi-orthogonal decomposition

$$D(\text{coh } X/\!\!/+, \omega|_{X_+}) = \langle \Upsilon^+_{-t(\mathfrak{K}^-)+d}, ..., \Upsilon^+_{-t(\mathfrak{K}^+)+d}, \Phi_d^+ \rangle.$$

(b) If $t(\mathfrak{K}^+) = t(\mathfrak{K}^-)$, there exists an exact equivalence

$$\Phi_d^+ : D(\text{coh } X/\!\!/-, \omega|_{X_-}) \to D(\text{coh } X/\!\!/+, \omega|_{X_+}).$$

(c) If $t(\mathfrak{K}^+) > t(\mathfrak{K}^-)$, there exist:

a fully faithful functor

$$\Phi_d^- : D(\text{coh } X/\!\!/+, \omega|_{X_+}) \to D(\text{coh } X/\!\!/-, \omega|_{X_-});$$

for all $-t(\mathfrak{K}^+) + d \leq j \leq -t(\mathfrak{K}^-) + d - 1$, fully faithful functors

$$\Upsilon_j^- : D(\text{coh}[Z_\lambda^0/C(\lambda)], \omega|_{Z_\lambda^0})_j \to D(\text{coh } X/\!\!/-, \omega|_{X_-});$$

and a semi-orthogonal decomposition

$$D(\text{coh } X/\!\!/-, \omega|_{X_-}) = \langle \Upsilon^-_{-t(\mathfrak{K}^+)+d}, ..., \Upsilon^-_{-t(\mathfrak{K}^-)+d}, \Phi_d^- \rangle.$$

Example 5.37. Let's use an example of a variation satisfying the DHT condition as we have seen before. We can consider a $G := \mathbb{G}_m$ action on $\mathbb{P}^2_k$ given by $t \cdot (x : y : z) := (tx : t^{-1}y : t^2 z)$. We know what the GIT-fan looks like, as we have seen in Example 2.30. Our variation will be $(\mathcal{L}_-, \mathcal{L}_+)$ with $\mathcal{L}_+ = \mathcal{O}(4,1)$ and $\mathcal{L}_- = \mathcal{O}(2,-1)$. A quick observation shows that this indeed satisfies the DHT condition. We cross a wall of the fan at $\mathcal{L}_0 := \mathcal{O}(3,0)$, and using our computations we already see that;

$$X^{\text{ss}}(0) = D(x); \qquad X^{\text{ss}}(+) = D(y) \cap (D(x) \cup D(z)); \qquad X^{\text{ss}}(-) = D(z) \cap (D(x) \cup D(y)).$$

Now let's first focus on the first statement of the theorem. We can immediately compute what S_λ and $S_{\lambda^{-1}}$ should look like. Notice that $X^{\text{ss}}(0) \cap X^{\text{ss}}(+) = D(x) \cap D(y)$ and $X^{\text{ss}}(0) \cap X^{\text{ss}}(-) = D(x) \cap D(z)$. Therefore

$$S_\lambda = X^{\text{ss}}(0) \setminus X^{\text{ss}}(+) = \{(x : y : z) \in \mathbb{P}^2_k(k) \mid x \neq 0, y = 0\};$$

and

$$S_{\lambda^{-1}} = X^{\text{ss}}(0) \setminus X^{\text{ss}}(-) = \{(x : y : z) \in \mathbb{P}^2_k(k) \mid x \neq 0, z = 0\}.$$

We know that for any $w \in X(k)$, and any 1-PS λ the limit point $\lim_{t \to 0} \lambda(t) \cdot w$ is either $e_0 := (1 : 0 : 0), e_1 := (0 : 1 : 0)$ or $e_2 := (0 : 0 : 1)$. These will always be the set of fixed points, unless λ is not injective, but this is not a case we are interested in. Notice that e_0 is the only point that is in S_λ and $S_{\lambda^{-1}}$. Therefore it is natural to let $Z_\lambda^0 = \{e_0\}$.

Our computations on limits from the example come in handy now, as it immediately shows that the sets S_λ and $S_{\lambda^{-1}}$ can be given by the 1-PS $\lambda : t \mapsto t$. For our G-linearized invertible sheaf we want the property that $\mu(\mathcal{L}, \lambda, (1 : 0 : 0)) = 0$, so our only option is to linearize it using $a = 0$. That doesn't give us a lot of choice, so let's choose the G-linearized sheaf $\mathcal{L} := \mathcal{L}_0 = \mathcal{O}(3,0)$. Recall that the global sections of $\mathcal{L}$ are $k[X_0, X_1, X_2]_3$. Therefore, for ω we have two very natural choices, namely the sections X_0^3 and $X_1 X_2^2$. We consider the section $\omega := X_1 X_2^2$, as this section is only possible if we choose the first parameter of $\mathcal{L}$ to be a multiple of three.

If we identify $D(x)$ with $\mathbb{A}^2_k$, then S_λ corresponds to the subvariety given by $y = 0$ and its sheaf of ideals is therefore generated by the section y. The Mumford weight at $(0,0)$ corresponding to $(1 : 0 : 0)$ equals -2, and hence $t(\mathfrak{K}^+) = -2$. Similarly we obtain $t(\mathfrak{K}^-) = -1$. The theorem gives us a semi-orthogonal decomposition

$$D(\text{coh } X/\!\!/+, \omega|_{X_+}) = \langle \Upsilon_1, D(\text{coh } X/\!\!/-, \omega|_{X_-}) \rangle$$

by choosing $d = 1$. First of all, let's consider Υ_1. As a category, it is given by

$$D(\mathrm{coh}[(1:0:0)/\mathbb{G}_m], 0)_1 \simeq D(\mathrm{coh}[(1:0:0)/\mathbb{G}_m], 0)_0 \simeq D^b(\mathrm{coh}(\mathrm{Spec}(k))) \simeq \langle E \rangle$$

where E is the exceptional object. The global sections of $\mathcal{O}(4,1)$ are generated by sections of the form $X_0^{i_0} X_1^{i_1} X_2^{i_2}$ where i_0, i_1, i_2 are non-negative integers such that $i_0 + i_1 + i_2 = 4$. Here we have given X_0 weight 1, X_1 weight -1 and X_2 weight 2. Therefore $\Gamma(X, \mathcal{O}(4,1))^G$ is generated by $X_0^2 X_1^2$ (any G-invariant section containing x_2 needs at least two terms X_1, and after that it is impossible to get a total weight of 0). The GIT quotient is now computed as follows, using Remark 2.3:

$$X/\!\!/+ = \mathrm{Proj}\ \left(\bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}(4,1))^G \right) = \mathrm{Proj}\ k[X_0^2 X_1^2] = \mathrm{Proj}\ k[x] = \mathrm{Spec}(k).$$

Where we use that $\mathcal{O}(4,1)^n = \mathcal{O}(4n, n)$. Similarly, the invariant global sections of $\mathcal{O}(2,-1)$ are generated by X_2^2, and therefore

$$X/\!\!/- = \mathrm{Proj}\ k[X_2^2] = \mathrm{Proj}\ k[x] = \mathrm{Spec}(k).$$

The semi-orthogonal decomposition is therefore not immediately something you would expect. The difference between the categories can only be seen by the difference of factorizations of the section $X_1 X_2^2$ in $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ respectively.

6 Discussion

We will now discuss everything that might be interesting for future research in this topic. First of all, in a very general sense we do not know everything about GIT-quotients. When looking at the GIT-fan, you can imagine some curve going through it. Then we could put some restriction on this curve. For example, what is the result of restricting this curve so that it may not give an induced flip? Or what if we restrict this curve so that it may only cross walls when it does induce a flip? Some other questions rely on the structure of the GIT-fan and its walls. It seemed like in our cases the walls were specifically equal to the stability sets of the fixed points under the action. This raises the following question.

Question 6.1. *Let X be a projective variety and G a reductive group variety acting on X. What are the minimal conditions on this action so that the walls of the corresponding GIT-fan are explicitly given by the stability sets of fixed points?*

Of course, this could simply be a coincidence as we were working with the action of a torus on our variety, and this occurence might be rare whenever G is not a torus. Another interesting idea is whether there is a more advanced version of the GIT-fan. There have been multiple fans, like the GKZ fan described in Chapter 5 of [3]. Maybe there is some more detailed version of the fan that not only describes the semi-stable sets, but also the quotients obtained from the action.

An obvious different choice would be the choice of semi-stable points. We considered these points to obtain categorical quotients we call the GIT-quotients. However it could be interesting to look at the difference when only considering the stable points and obtain geometric quotients. As any stable point is also semi-stable, we could overlap such a fan (if it is actually a fan) inside of the GIT-fan. Moreover, we can look at the effect of a point that is purely semi-stable in comparison with a point that is stable to the stability sets. Some of these properties are already known as shown by Ressayre in [21], but there could be some new techniques of identifying quotients here.

One final point on such point is the exclusion of unstable points. In general the semi-stable points are constrained to have the properties so that the quotient can be nice enough to exist as a variety. So then we get another question about the unstable points.

Question 6.2. *Let X be a variety and G a reductive group variety acting on X. Suppose $x \in X(k)$ is a geometric point which is unstable with respect to any G-linearized invertible sheaf on X.*

Is there a variety Y with a G-action together with a G-linearized invertible sheaf $\mathcal{L}$ on Y and a G-invariant morphism $f : X \to Y$ sending x to a geometric point $f(x) \in Y(k)$ so that $f(x)$ is not unstable with respect to $\mathcal{L}$? And if so, what kind of restrictions can we put on f and Y for this to still be true?

We used the term stacks in this thesis, but we have not actually done a lot using the stacks in particular. Moreover, we know that quotients of the semi-stable points already give us categorical quotients. So is it possible to replicate the methods used and proof a similar main theorem where we instead do not look at stacks? If this is not possible in the case of semi-stable sets, could we do it for the stable sets?

In the final chapters we discussed some applications of variation of GIT, but these are not all of them. For example, the paper by Ballard, Favero and Katzarkov goes on to talk about K-equivalences and D-equivalences. These are special properties of smooth projective varieties, and they gave some thoughts on a nice conjecture in our setting of elementary wall crossings. Research in this area could be quite interesting. On top of this the paper by Favero, Kaplan and Kelly gives us some nice examples to work with, but maybe this can be generalized. Can we give a similar statement for polynomials ω where $\mu(\omega^T)$ is finite? And could we get tilting objects for such general categories $D(\text{coh}[X/G], \omega)$? Or maybe if we gave the action, variety and group some restrictions?

In conclusion, the field of GIT-quotients offers a lot possibilities for future research. The outstanding questions and unexplored aspects hold great potential for new understandings of this area of mathematics. Moreover, the insights gained from previous studies have already proven valuable to numerous branches of mathematics, showing the significance and impact of further exploration in this field.

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