The AdS_3/CFT_2 Correspondence

Camacho, Alexander

The AdS_3/CFT_2 correspondence

THESIS

submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
in
THEORETICAL PHYSICS

Author :	Alexander Camacho
Student ID :	3114201
Supervisor :	Koenraad Schalm
Second corrector :	Subodh Patil

Leiden, The Netherlands, August 10, 2023

The AdS_3/CFT_2 correspondence

Alexander Camacho

Instituut-Lorentz, Leiden University
P.O. Box 9500, 2300 RA Leiden, The Netherlands

August 10, 2023

Abstract

In the large N limit, quantum field theories organise themselves into string theories. The AdS/CFT correspondence is an important class of gauge/string dualities. In this paper, we provide a literature review of a precise AdS_3/CFT_2 duality. We calculate the spectrum for the symmetric product orbifold of $\mathbb{T}^4$ and show that is matches with that of the superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with one unit of NS-NS flux, based on the work of [36] [37]. Further support for the duality is obtained by matching the correlation functions at genus 0, based on [1] [39] [40]. Our analysis sheds light on why the two theories are so intimately related; it requires interpreting the worldsheet as the covering space over the boundary CFT. This is captured in a 'delta function localization' property of the vertex operator correlation function. When integrated over in worldsheet moduli space, it localizes onto points that holomorphically covers the boundary sphere thus reproducing features of the dual CFT.

Contents

Chapter 1

Introduction

Dualities in theoretical physics have laid at the heart of the subject for the last 100 years and have played a critical role in relating seemingly disconnected mathematical structures. They are riddled everywhere; from wave-particle duality in quantum mechanics to electro-magnetic duality in Maxwell theory, from bosonisation in quantum field theories (QFTs) to T and S dualities in string theory. While many are well understood, there exists dualities whose origins still remain mysterious, even after decades of investigation. The most counterintuitive of them all is the QFT/string duality (also referred to as the gauge/string duality) where, on many fronts, it appears impossible that these two theories can be related to one another. For instance, string theories exist in ten dimensions while renormalizable QFTs are required to exist in four dimensions or less. Also, string theories have many more states than QFTs due to the infinitely many modes at which the string can vibrate. One of these vibrational modes yields the graviton, whereas renormalizable QFTs cannot contain gravity. Given these superficial inconsistencies, one is motivated to find the underlying mechanism that connects the two frameworks.

The gauge/string duality was first realized in 't Hooft's original discussion [2] of the large N behaviour of gauge theories. In this limit, the leading order perturbative expansion is dominated by planar diagrams whose topological structure is identical to the genus expansion of string amplitudes. Following this discovery, only few examples of gauge/string dualities were found, such as the Kontsevich matrix model [3] dual to minimal topological string theory [4][5][6][7] and the Chern-Simons gauge theories in 3d dual to A-model topological strings [8]. These examples are topological and therefore have very little application to our physical universe. Also, there is no Einstein gravity limit in these theories, so one cannot explore interesting gravitational systems such as black holes.

The situation changed in 1997 when Maldacena discovered a certain set of gauge/string dualities broadly referred to as the AdS/CFT correspondence [9]. Formally, the AdS_{d+1}/CFT_d correspondence is a statement about the duality between a theory of gravity on a $d+1$ dimensional maximally symmetric (Anti-de Sitter) spacetime and a conformal field theory on a spacetime with one lower dimension. Although the correspondence holds in general, it's initial conjecture in the seminal paper [9] was stated within a string theoretic framework and has since been proven to be an extremely fruitful arena to test and understand certain gauge/string dualities. There are many examples of this correspondence which are based on decoupling limits of D-branes and string theories. For any gauge theory that can be constructed on a D-brane, one can take a decoupling limit that yields the standard superstring theory which is the stringy description of that particular large N gauge theory. In this set up, there is a powerful prescription for finding the string dual to a particular gauge theory.

One of the cases considered in [9] included an N parallel D3 brane system in type IIB string theory where in the low energy limit, the $U(N)$ gauge theory on the brane decouples from the bulk. The corresponding supergravity solution in this limit gives rise to a near horizon geometry that reduces to $AdS_5 \times S^5$. The supergravity solution can be trusted in the large N limit, since the curvature of the sphere and the AdS space is a positive power of $1/N$. For this particular system, the correspondence is manifested in that excitations in AdS_5 are included in the Hilbert space of a $4d$ $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. There is the following equivalence:

$$\text{Type IIB String theory on } AdS_5 \times S^5 \cong (\mathcal{N} = 4)\ SYM\ (4d)$$

$$(R, g_s) \cong (g_{\text{YM}}, N)$$

Where R is the radius of the AdS space, g_s is the string coupling constant, N is the rank of the gauge group and g_{YM} is the SYM theory coupling constant. The correspondence states that

$$g_s \sim \frac{1}{N}$$

$$\frac{R}{l_s} \sim g_{\text{YM}}^2 N = \lambda$$

where l_s is the string length scale and λ is the 't Hooft coupling constant. A natural interpretation then is that of a 'strong-weak' duality, where, for instance, a string theory at strong coupling is dual to a weakly coupled SYM theory, and vice versa. In this way, one can tune the couplings on either side to enter the perturbative regime on the other, where calculations become tractable. This feature has allowed applications of this correspondence to strongly coupled systems in black hole physics, nuclear physics and condensed matter systems, to name a few.

The AdS_5/CFT_4 correspondence is particularly interesting because the dual CFT lives in 4d and since our universe is 4 dimensional, we are motivated to understand the inner workings of this relationship. It was a hope that in knowing the string dual to $\mathcal{N} = 4$ SYM, one can deduce the string dual to ordinary (non-supersymmetric) Yang-Mills (YM) theories, since we have direct experimental evidence for a subset of these theories. However such a generalisation has not been made, which can largely be attributed to our lack of understanding of the string theory dual to the weakly coupled $\mathcal{N} = 4$ SYM. On the contrary, the dual to strongly coupled SYM is the well understood type IIB supergravity which describes the type IIB string theory on $AdS^5 \times S^5$ with large radius. At arbitrary radius however, one runs into trouble when quantizing a sigma model with $AdS_5 \times S^5$ target space due to the presence of the Ramond-Ramond (RR) 5-form flux fields so there has been little hope for the weakly coupled $\mathcal{N} = 4$ SYM that corresponds to small radius (highly curved) AdS space. For this reason, there is interest in studying the AdS_3/CFT_2 correspondence as it allows many advantages over it's higher dimensional analogue that are rooted in the fact that there exists $AdS_3 \times M^7$ constructions that circumvent the complications arising from the RR fields.

One conventional way for showing a duality is via a comparison of independent computations of correlation functions on both sides, however, one may wonder whether the AdS/CFT correspondence can be derived in a way that bypasses this step, that is, a way of transforming observables of one side to the other showing thus their equivalence. Given the aforementioned complications, there has been no such luck in deriving the correspondence for $d = 4$, however, conformal field theory in $d = 2$ is very well understood due to the large Virasoro symmetry. Also, there exists solvable worldsheet sigma models on AdS_3, thus AdS_3/CFT_2 becomes an attractive alternative. Indeed, the computational power in this setting was leveraged and a derivation of a particular AdS_3/CFT_2 duality was made in [1] where the structural reasoning underlying the relationship was made explicit. The precise duality is the following: *Type IIB superstring theory on* $AdS_3 \times S^3 \times \mathbb{T}^4$ *in the tensionless limit, with pure NS-NS flux, is dual to the symmetric product orbifold* $(\mathbb{T}^4)^{\otimes N}/S_N$*, in the large N limit*. Other non-trivial evidence such as a detailed matching of the spectra

[36][37] has provided concrete support for this claim. Up until this point there has been no other examples of a derivation of a gauge/string duality, so it will be interesting to translate the lessons learnt on AdS_3/CFT_2 to the higher dimensional analogue, in the hope that one can find the stringy duals on all backgrounds to 4d gauge theories, and more broadly to all gauge theories in general.

Given the significance of the findings in [1], it would be warranted to verify, validate and reproduce all claims that were made. To this end, this paper aims to provide a literature review of the AdS_3/CFT_2 correspondence placing a major focus on the calculations carried out in the quantum analysis of [1].

The AdS_3/CFT_2 dictionary relates correlation functions of the (primary) vertex operators inserted on the worldsheet to the correlation functions of the primary operators of the spacetime CFT:

$$\begin{aligned}\int_{\mathcal{M}_{g,n}} \langle \mathcal{V}_{h_1}^{w_1}(x_1;z_1)\mathcal{V}_{h_2}^{w_2}(x_2;z_2)...\mathcal{V}_{h_n}^{w_n}(x_n;z_n)\rangle_{\Sigma_{\mathrm{g},n}} \\ = \langle \mathcal{O}_{h_1}^{(w_1)}(x_1)\mathcal{O}_{h_2}^{(w_2)}(x_2)...\mathcal{O}_{h_n}^{(w_n)}(x_n)\rangle_{S^2}\Big|_{\mathrm{g}},\end{aligned} \tag{1.1}$$

where the vertex operators on the LHS are integrated over the moduli space of genus g worldsheet Riemann surfaces with n punctures, $\Sigma_{\mathrm{g},n}$. w and h label the states while x and z are the boundary and worldsheet coordinates respectively. In the large N limit, correlation functions of the spacetime CFT on the boundary sphere S^2 yield a genus expansion where the higher genus terms are increasingly suppressed. The dictionary in (1.1) thus relates the n-point correlatior of vertex operators evaluated on the genus g Riemann surface to the genus g term in the expansion of the n-point correlator of primary operators of the spacetime CFT. The main claim proposed in [1] is that the authors have found a structural reason that goes some way towards establishing a mapping between the two sides of (1.1). We aim to provide support for this claim by reproducing the calculations on both sides of (1.1) and highlighting the underlying machanism of the correspondence. We also find it necessary to motivate this calculation by explicitly showing the duality by first matching the spectra.

The outline of the paper is as follows. In chapter 2 we provide a brief historic account of the major milestones met along the way to discovering the precise AdS_3/CFT_2 duality. It starts with the prescription of Maldacena [9] and plods through the events that ultimately led to the exact matching of the spectra [36][37]. In chapter 3, the $(\mathbb{T}^4)^{\otimes N}/S_N$ theory is introduced in detail and we compute the n-point correlation function of the primary twist operators, making direct use of the methods outlined by Lunin & Mathur [45]. In chapter 4 we introduce the symmetry algebra and space of physical states of 2d CFTs and generalise to the physical spectrum of the $(\mathbb{T}^4)^{\otimes N}/S_N$ theory. In chapter 5 we describe the $SL(2,\mathbb{R}) \times SU(2)$ WZW sigma model which is used to describe strings propagating on $AdS_3 \times S^3$. We calculate the string spectrum in the tensionless limit and show that it matches on the nose with that of the dual CFT calculated in chapter 4. In chapter 6 we use a Ward identity analysis to constrain the form of the n-point correlation function of the vertex operators and show that, when integrated over in moduli space, it's structure matches precisely with that of the dual CFT as calculated in the large N limit in chapter 3. In chapter 7 we conclude by summarising our findings and provide a brief overview of the advances made in dualities and related topics which were directly inspired by this AdS_3/CFT_2 correspondence. We also provide potential future directions for research.

Chapter 2

Towards an exact AdS_3/CFT_2 correspondence

In this chapter, we highlight the major discoveries that helped in narrowing down towards the exact AdS_3/CFT_2 duality. We start with the conception of the correspondence using the D-brane construction of Maldacena. Following this, it was suspected that the dual CFT to string theory on $AdS_3 \times S^3 \times M^4$ was to lie on the same moduli space as the symmetric product orbifold of M^4. The challenge was to identify which point in string moduli space was dual to which point in the moduli space of the dual CFT. The obvious guess was that there should be a correspondence between the symmetric points in both spaces. We cover the many clues that suggested that the most symmetric point in string moduli space was at the tensionless point, which corresponds to a single unit of NS-NS flux. Indeed, this was confirmed by a detailed matching of the spectra.

From D-branes to tensionless strings

Consider a type IIB string theory compactified on a 4 dimensional manifold M^4 and 6 spacetime dimensions. Take a set of Q_5 D5 branes and compactify 4 of the directions on M^4, giving a string in 6 dimensions. Take a set of Q_1 D1 branes and orient them in parallel with the non-compactified time direction of the D5 branes. In the low energy limit, the decoupled field theory has a Higgs branch and a Coloumb branch. The supergravity solution corresponding to D1+D5 branes yields a metric of the near horizon geometry corresponding to the space $AdS_3 \times S^3$. Here, we can trust the supergravity solution when $N = Q_1Q_5$ is large. The conjecture in [9] states that the 1+1 dimensional CFT describing the Higgs branch of the D1/D5 brane system on M^4 is dual to type IIB string theory on $AdS_3 \times S^3 \times M^4$. We require that the theory has $\mathcal{N} = 4$ supersymmetry so we take M^4 to be $\mathbb{T}^4$ or $K3$.

We view the D1 branes that live on the D5 branes as instantons of the low energy SYM theory [10]. These instantons are translationally invariant and have a $SO(1,1)$ symmetry in the time and x^5 direction, where the x^5 direction is the non-compactified direction of the D5 brane. The instanton configuration has moduli that paramterize a continuous family of solutions with the same energy. Fluctuations in instanton configuration correspond to fluctuations of moduli in the time and x^5 direction. The low energy dynamics are then governed by a 1+1 dimensional sigma model whose target space is the moduli space of the instanton. It was argued in [11] and [12] that the instanton moduli space is a deformation of the symmetric product orbifold of N copies of M^4: $(M^4)^{\otimes N}/S^N$ (the details of this action will be shown in the following chapter). Deformations involve blowing up the orbifold fixed points and modifying the B field that live at the orbifold point. It was suspected then that the dual CFT of a type IIB string theory on $AdS_3 \times S^3 \times M^4$ on some background lives on the same moduli space as the symmetric product orbifold theory of M^4. To be more precise, the dual CFT is some marginal deformation away from the symmetric point in moduli space that preserves the superconformal symmetry.

Maldacena and Strominger [13] showed that an NS 5 Brane and fundamental string system is related to a D1/D5 brane system by an S duality transformation. In this set up, the complications of the Ramond/Ramond (RR) fields vanish leaving the Navier Schwarz - Navier Schwarz (NS-NS) B fields only. This is a unique feature of AdS_3 and theories in higher dimensional AdS spacetimes cannot be simplified in such a way. Within this system, there exists light strings that require finite cost of energy to escape to infinity [14]. These are the so called long string states that wind around the boundary of AdS_3 and are stabilized by two opposing forces: the tension that contracts the string and the NS-NS B field that forces expansion.

Maldacena and Ooguri studied bosonic string theory on AdS_3 using a $SL(2,\mathbb{R})$ WZW model in a 3 part series [15],[16] and [17]. Long string states were found to lie in 'spectrally flowed' continuous representations of the $SL(2,\mathbb{R})$ affine Kac-Moody algebra [15]. Here, spectral flow refers to an automorphism acting on the current algebra. Excitations of the long strings lie in a continuum of states [15] whereas the symmetric orbifold theory of $\mathbb{T}^4$ has a discrete spectrum so it was suspected then that the proposed duality may not hold using this worldsheet description.

In 2002, Vasiliev discovered an interesting class of AdS/CFT dualities by way of the higher spin (HS) theories in AdS [19], which do not have an obvious string theory embeddedment. These theories contain not only the massless spin $s = 2$ graviton, but an infinite tower of interacting massless fields with spin $s \geq 2$. It has long been suggested that these theories may be suitable for describing the large N behaviour of weakly coupled gauge theories [20] [21] [22]. The first realization of a HS/gauge duality was by Klebanov and Polyakov [23], who provided a general relation between higher spin theories on AdS_{d+1} and conformal field theories of $O(N)$ models in one lower dimension. In particular, they conjectured that the minimal bosonic theory on AdS_4 which contained massless gauge fields of even spin, is dual to a singlet sector of $O(N)$ vector model in the large N limit (in 3d). Evidence supporting this conjecture was elucidated in [24] [25].

Gaberdiel and Gopakumar (two of the authors of the paper of interest [1]), proposed a duality between a family of higher spin theories on AdS_3 and the 2d $\mathcal{W}_N$* minimal models in the large N 't Hooft limit, where the 2d CFT was described by a coset WZW model [26]. They showed that the spectra of the two theories match for all values of the 't Hooft coupling. This duality was, in some sense, the lower dimensional analogue of the Klebanov-Polyakov conjecture. The authors extended the class of 2d minimal model CFTs that was dual to HS theories on AdS_3 to theories with large $\mathcal{N} = 4$ superconformal symmetry [27].

Meanwhile, it was argued that a tensionless string is the correct starting point to study the high energy limit of string theory [28]. This is analagous to studying the massless field theories that are the high energy limit of massive theories. The broken gauge symmetries at finite tension are expected to be restored in the tensionless limit. The Vasiliev massless higher spin fields are then expected to be present in the tensionless limit of string theory, since within string theory there exists fields of arbitrarily high spin and the massless states correspond to zero tension [29].

Tensionless strings have been studied in a variety of cases on a flat background [30] [31] nevertheless quantisation of the string was proven to be problematic due to inconsistent interactions [28]. Since interactions between massless higher spin fields were known to be consistent on spacetimes with non-zero cosmological constant [32] [33], string theory on AdS

*The $\mathcal{W}$-algebra is a generalisation of the Virasoro algebra. It is generated by the meromorphic fields $W^{(h)}(z) = \sum_{n=\mathbb{Z}} W_n^{(h)} z^{-n-h}$. The Virasoro algebra is recovered when the conformal dimension is $h = 2$. In the minimal model case above the conformal dimension is $h = N$.

backgrounds can yield an interesting study of the tensionless string. The radius R of the AdS space provides an additional length scale, so the genuine tensionless limit can be considered when the radius of curvature becomes much smaller than the string length scale l_s, ie: $1 \gg R/l_s$. Using the usual AdS/CFT dictionary, this ratio corresponds to a coupling on the dual field theory side, so then in the tensionless limit where $R/l_s \to 0$, it is expected that the dual CFT becomes a free gauge theory equipped with a large gauge symmetry.

Gaberdiel and Gopakumar fitted the HS/gauge duality of [26] [27] in a string theoretic framework [34] involving tensionless strings: they showed that the large level limit of the $\mathcal{N} = 4$ cosets of [27] that are dual to the HS theory on AdS_3 describe a closed subsector of the symmetric product orbifold theory. The stringy symmetries of the tensionless limit are encoded in a large chiral algebra which can be described as an infinite sum of the $\mathcal{W}_\infty$ representations, exhibiting the extension of the conventional higher spin symmetry. Also, the full partition function of the symmetric orbifold theory was expressed in terms of the $\mathcal{W}_\infty$ extension of the higher spin algebra. The extended higher spin symmetries of the symmetric orbifold field theory is reminiscent of those present in the $4d$ free SYM theory, therefore there was reason to believe that the symmetric point is indeed it's lower dimensional analogue.

This picture was further developed in [35] where the Vasiliev higher spin theory of tensionless strings was directly studied from the worldsheet perspective. Using the $SL(2,\mathbb{R})$ WZW model with pure NS-NS flux, the states that make up the leading Regge trajectory were identified and showed that they fit in the $\mathcal{N} = 4$ Vasiliev higher spin theory. These set of states corresponded to the tensionless point in the worldsheet description and when the spectral flow parameter $w = 1$ †. At these values, the massless higher spin states sit at the bottom of a continuum of states that describe the excitations of a long string in AdS_3. This made manifest the idea that the precise string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ that is dual to the symmetric product orbifold is exactly the point in string moduli space that corresponds to the tensionless limit.

A particular set of states of an $SL(2,\mathbb{R})$ WZW description (in the RNS formalism) of tensionless (level $k = 1$) type IIB string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with pure NS-NS flux was calculated in [36] and shown to have the same spectrum as that of a single-cycle spectrum of a symmetric product orbifold CFT (see section 5.5). In doing so, one has to make the identi-

†The spectral flow parameter w can be identified with the number of times the long string winds around the boundary of AdS_3, as will be elucidated in chapter 5

fication between the the length of the single cycle twist w on the CFT side with the spectral flow parameter w on the string side. In particular, a focus was placed on the subsector that consisted of the lowest lying states in the spectrally flowed continuous representations ($w \geq 1$) corresponding to the long string continuum. This matching of the spectra between the two theories was not necessarily expected however, since type IIB string theory with purely NS-NS background is meant to describe a different point in moduli space. This reveals a certain universality property of the tensionless case, where both theories are shown to have a higher spin symmetry and have a partly matched spectrum [36].

Finally, a precise duality was proposed in [37] : type IIB string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ at the tensionless limit of $k = 1$ corresponding to minimal NS-NS flux is dual to the large N limit of the free symmetric product orbifold $(\mathbb{T}^4)^{\otimes N}/S_N$. As previously mentioned, the matching of the spectrum in [36] required looking at a particular set of states that lay at the bottom of the long string continuum. This worldsheet description was not well defined in the first place: taking the level of the theory $k = 1$ results in a negative level on the $\mathfrak{su}(2)_{-1}$ factor yielding a non-unitary theory. However there exists also a formalism for strings on $AdS_3 \times S^3 \times \mathbb{T}^4$ called the hybrid formalism, developed by Berkovits, Vafa and Witten [38]. In this formalism, the $AdS_3 \times S^3$ factor is replaced by its superspace analogue; the super group $PSU(1,1|2)$ thus making the spacetime supersymmetry of the model embedded. The supergroup contains the maximal bosonic subgroup: $SU(1,1) \times SU(2) \cong AdS_3 \times S^3$. This framework is particularly attractive because the R-R fluxes can be added to the background, which cannot be straightforwardly done in the RNS formalism. For some combination of NS-NS and R-R fluxes, the worldsheet theory is a sigma model with target space $PSU(1,1|2)$. For the case considered in [37], the R-R fluxes were switched off and in this limit the worldsheet theory reduces to a WZW sigma model with level k where k is the same level as in the RNS formalism, thus describing the amount of NS-NS flux in the background. There has been non-trivial evidence that the two formalisms are equivalent in this limit, at least for the lowest lying states [56][57][58]. The main advantage however is that the theory remains well defined for $k = 1$ unlike the RNS formalism. The main result is that a study of the representations at $k = 1$ tells us that the theory is only consistent for the spin $j = 1/2$ representations, that is, there is no continuum of states that lie on top of these states [37] (see subsection 5.7.1). The spectrum was then matched with the orbifold theory of $(\mathbb{T}^4)^N/S_N$ and the fusion rules of the worldsheet theory were identical to the those of the orbifold theory. Such non-trivial evidence provides support for the proposed duality.

Chapter 3

The Symmetric Product Orbifold: Correlation Functions

In this chapter, we give an overview of the theory of the symmetric product orbifold and evaluate the correlation functions of operators that perform the orbifold action i.e. the twist operators. In doing so we make critical use of the methods outlined by Lunin & Mathur [45] which involves computing the vacuum path integral on the covering space of S^2 with n marked points, accompanied by an exponentiated Liouville action. The covering space 'undoes' the action of the orbifold, and the conformal factor encodes information of the covering map. The end result is a discrete sum over coverings, where each covering has a particular genus given by the Riemann-Hurwitz formula. To compare to string theory, we take the large N limit which leads to the suppression of the higher genus terms. This chapter lays the foundation for computing the spectrum in chapter 4 and inspires the solution to the correlation functions of vertex operators of the dual string theory in chapter 6.

3.1 Correlation functions on symmetric product orbifolds.

Consider the following sigma model action:

$$S = \frac{1}{2\pi} \int d\tau d\sigma \, g_{ij}(X) \left(-\partial_\tau X^i \partial_\tau X^j + \partial_\sigma X^i \partial_\sigma X^j \right) + \dots , \tag{3.1}$$

where τ, σ are the timelike and spacelike coordinates on the base space and X^i is the coordinate on the target space $\mathcal{M}$ with metric $g_{ij}(X)$ where

$i, j \in \{1, 2, ...d\}$ and d is the dimensionality of $\mathcal{M}$. The "..." denotes possible fermionic terms. The internal structure of $\mathcal{M}$ is independent of this treatment of the correlation functions, however, for the case that we are interested in, $\mathcal{M} = \mathbb{T}^4$ and the theory (3.1) is $\mathcal{N} = (4, 4)$ superconformal. For the following, $\mathcal{M}$ will not be specified.

The symmetric product orbifold theory is obtained by acting on N copies of the target space $\mathcal{M}$ by the permutation group S_N:

$$Sym^N(\mathcal{M}) \equiv \mathcal{M}^{\otimes N}/S_N. \tag{3.2}$$

The operation of S_N is to permute the different copies of $\mathcal{M}$ into each other according to some element $h \in S_N$. In this orbifold geometry, an additional index on the coordinate X^i is used to denote the copy: X^i_I where $I \in \{1, 2, ...N\}$. Orbifold theories consist of untwisted sectors which contain states that are invariant under the orbifold action and twisted sectors containing states that do transform. We introduce the twist operators σ_g that create the twisted vacua, where $g \in S_N$. Without the twist fields, the X^i_I fields obey trivial boundary conditions. In the presence of the twist fields, the boundary conditions are non-trivial and are determined by g:

$$X^i_I(e^{2\pi i}x)\sigma_g(0) = X^i_{g(I)}(x)\sigma_g(0), \tag{3.3}$$

where we have moved to the complex plane $x = e^{\tau + i\sigma}$ and have inserted a twist operator at the origin. The boundary conditions of X^i_I are changed under the action of this operator such that after a circulation of 2π around it's insertion, the fields transform as $X^i_I \to X^i_{g(I)}$. It is clear that the operation of σ_g is not invariant under the action of S_N. Recall that the action of a group on itself is given by

$$h \cdot g := hgh^{-1}, \tag{3.4}$$

so acting $h \in S_N$ on some σ_g will give its conjugate $\sigma_{hgh^{-1}}$. However summing over all members of the conjugacy class of g will result in an operator that is S_N invariant, since each term will transform into another in the sum. The proper S_N invariant twist operators are obtained by summing over the group orbit:

$$\sigma_{[g]} \equiv \sum_{h \in S_N} \sigma_{h^{-1}gh} \tag{3.5}$$

where $[g]$ denotes the conjugacy class of g. As an example, consider the three conjugacy classes of S_3 acting on the set $\{1, 2, 3\}$:

1. No change: $(123) \to (123)$ (1 member)

2. Transposing two elements: $(123) \to (132)$, $(123) \to (213)$, $(123) \to (321)$ (3 members)

3. Cyclic permutation: $(123) \to (312)$, $(123) \to (231)$ (2 members).

In this example, there are three distinct S_N invariant twist operators that can be constructed. Note that the operator corresponding to no change produces states in the untwisted sector.

Each element of S_N can be expressed as disjoint cyclic permutations. For example, using Cauchy's two line notation, the permutations for $f, g \in S_5$,

$$f = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,2\,1\,5\,4 \end{pmatrix}, \quad g = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 5\,1\,4\,3\,2 \end{pmatrix}, \tag{3.6}$$

can simply be expressed as two cycles $f = (13)(45)$ and $g = (125)(34)$. Note that the the round brackets denote cyclic permutations of the form $(ijk) = i \to j,\ j \to k,\ k \to i$. Furthermore, two elements of S_N belong to the same conjugacy class if and only if they share the same cyclic shape. That is, f and g belong to two different conjugacy classes, but $h = (23)(15)$ and $k = (345)(12)$ belong to the same conjugacy class as f and g respectively. Therefore, in general, the conjugacy classes are characterized by the number of cycles and the length of each cycle of the permutations of the members in the class. We are particularly interested in twist operators that only perform single-cycle permutations because in the context of the AdS/CFT correspondence, these states correspond to single string states while more complicated cyclic structure corresponds to multiple strings. This is analagous to single trace operators in ordinary gauge theory corresponding to single particle states. Therefore the conjugacy classes of interest are characterized by a single number $w \in \{1, ..., N\}$, denoting the length of the single cycle permutation.

It is necessary that the theory distinguishes between conjugacy classes rather than individual group elements because we would like to calculate correlators of twist operators that are S_N invariant in direct analogue with calculating correlators that are gauge invariant in gauge theory. The object of interest is the n-point correlator:

$$\Big\langle \prod_{i=1}^{n} \sigma_{[g_i]}(x_i) \Big\rangle = \Big\langle \prod_{i=1}^{n} \sum_{h_i \in S_N} \sigma_{h_i^{-1} g_i h_i}(x_i) \Big\rangle, \tag{3.7}$$

where the operator's spacetime dependence on $\bar{x}_i$ is omitted to avoid cluttering. As emphasized before, operators that create single cycle permutations are of primary interest. By an abuse of notation, the correlator (3.7) can also be expressed in terms of the cyclic notation:

$$\Big\langle \prod_{i=1}^{n} \sigma_{[w_i]}(x_i) \Big\rangle = \Big\langle \prod_{i=1}^{n} \sum_{h_i \in S_N} \sigma_{(h_i(1)h_i(2)...h_i(w_i))}(x_i) \Big\rangle, \tag{3.8}$$

where $[w_i]$ denotes the conjugacy class for single cycle permutations of length w_i. In the remainder of the analysis, either of the two above notations will be used wherever it is considered convenient.

The calculation of gauge invariant correlators comes down to calculating individual correlators on the RHS of (3.7):

$$\Big\langle \sigma_{\hat{g}_1}(x_1)\sigma_{\hat{g}_2}(x_2)...\sigma_{\hat{g}_n}(x_n) \Big\rangle, \tag{3.9}$$

where $\hat{g}_i = h_i g_i h_i^{-1}$. By including points at infinity on the complex plane $\mathbb{C} \cup \{\infty\}$ base space can be taken to be the Riemann sphere $\mathbb{CP}^1 \cong S^2$ parameterized by the coordinates $(x, \bar{x})$. In direct analogue with time-ordering in flat space, correlators on the sphere are radially ordered such that $|x_1| < |x_2| < ... < |x_n|$.

For two twist insertions $\sigma_{g_1}(x_1)$ and $\sigma_{g_2}(x_2)$, in the limit when $x_1 \sim x_2$, $\sigma_{g_1}(x_1)\sigma_{g_2}(x_2)$ and $\sigma_{g_1 g_2}(x_2)$ give the same boundary conditions [40] [41] [42] [43] [44]. States that lie in the conjugacy classes of g_1 and g_2 fuse to states in the conjugacy class of $g_1 g_2$. Take three conjugacy classes given by $[g_1]$, $[g_2]$ and $[g_3]$, the fusion rules can be written as [40]

$$[g_1] \times [g_2] = \sum_{[g_3]} N_{[g_1][g_2][g_3^{-1}]}[g_3], \tag{3.10}$$

where $\sigma_{g^{-1}}(x)$ is the conjugate field to $\sigma_g(x)$. The fusion coefficients take the form [44]

$$N_{[g_1][g_2][g_3]} = \big|\{(g_1, g_2, g_3)|\; g_1 g_2 g_3 = \mathbb{1}\}/\sim\big|, \tag{3.11}$$

where the equivalence relation is $(g_1, g_2, g_3) \sim (g g_1 g^{-1}, g g_2 g^{-1}, g g_3 g^{-1})$ for any $g \in S_N$. Therefore, in the OPE limit of (3.9), after successive fusions between the fields, we end up with a 1-point function of the form $\langle \sigma_{\hat{g}_1 \hat{g}_2 ... \hat{g}_n} \rangle$. The fusion rules of (3.10) and the form of the coefficient (3.11) imply the following constraint on the terms of the type (3.9) that contribute non-trivially to (3.7):

$$\hat{g}_1 \hat{g}_2 ... \hat{g}_n = \mathbb{1}, \tag{3.12}$$

where the sequence in (3.12) is determined by the radially ordered operators. For example, consider the gauge invariant correlator

$\langle\sigma_{[3]}(a)\sigma_{[2]}(b)\sigma_{[2]}(c)\sigma_{[3]}(d)\rangle$. A non-trivial term contributing to this correlator is $\langle\sigma_{(132)}(a)\sigma_{(24)}(b)\sigma_{(34))}(c)\sigma_{(124)}(d)\rangle$ because the permutation trajectory for each copy is

$$\begin{aligned} &1 \xrightarrow{a} 3 \xrightarrow{c} 4 \xrightarrow{d} 1\\ &2 \xrightarrow{a} 1 \xrightarrow{d} 2\\ &3 \xrightarrow{a} 2 \xrightarrow{b} 4 \xrightarrow{c} 3\\ &4 \xrightarrow{b} 2 \xrightarrow{d} 4, \end{aligned} \tag{3.13}$$

thus satisfying the requirement (3.12). Two correlators of type (3.9) are identical if their group element sequence (3.12) are related by a global S_N transformation:

$$(\hat{g}_1\hat{g}_2...\hat{g}_n) \sim (h\hat{g}_1h^{-1}h\hat{g}_2h^{-1}...h\hat{g}_nh^{-1}), \tag{3.14}$$

which amounts to simply relabelling the indices which should not be distinguished between. The sequences in (3.14) belong to the same equivalence class. Using the example from above, the term $\langle\sigma_{(132)}(a)\sigma_{(24)}(b)\sigma_{(34))}(c)\sigma_{(124)}(d)\rangle$ and the term $\langle\sigma_{(356)}(a)\sigma_{(68)}(b)\sigma_{(58))}(c)\sigma_{(368)}(d)\rangle$ belong to the same equivalence class. It is instructive then to only compute the correlator of one representative and multiply it by a numerical factor that counts the number of elements in it's same equivalence class.

Furthermore, it is possible that when computing terms on the RHS of (3.7) that there will be contributions coming from terms that generate a group of elements that form a direct product of two disjoint subgroups of S_N, resulting in a factorization into the product of two correlators. If the correlator was to be expressed as a sum over diagrams, these particular terms would correspond to disconnected diagrams. We can restrict to 'connected terms' by enforcing the following requirement on the types of correlators that appear on the RHS of (3.7):

The subgroup $\langle g_1, ..., g_n\rangle$ generated by the permutations $(g_1, g_2, ...g_n)$ is transitive, (3.15)

i.e a subgroup that is irreducible. The correlator (3.7) can be rewritten in terms of the following sum:

$$\Big\langle\prod_{i=1}^{n}\sigma_{[g_i]}(x_i)\Big\rangle_c = \sum_{\alpha}\mathcal{C}_\alpha(\{g_i\})\Big\langle\prod_{i=1}^{n}\sigma_{g_i^\alpha}(x_i)\Big\rangle, \tag{3.16}$$

where the subscript c denotes a correlator yielding connected terms, α denotes the equivalence class, the set $\{g_i^\alpha\}$ is it's representative and the numerical prefactor $\mathcal{C}_\alpha(\{g_i\})$ captures the multiplicity of each class. It is also implicitly assumed that only representatives that satisfy the requirements (3.12) and (3.15) appear.

3.2 The Lunin & Mathur Construction

In this section, we review the procedure for calculating correlators of the type on the RHS of (3.16) using the methods of Lunin & Mathur [45] which is based on a path integral approach. For simplicity, the 2-point function will be calculated with single cycle twists of length 2: $\langle \sigma_{[2]}(x_1)\sigma_{[2]}(x_2)\rangle$. In particular, a focus is placed on the representative $\langle \sigma_{(12)}(x_1)\sigma_{(12)}(x_2)\rangle$.

The effect of the twists on the boundary conditions of X_1 and X_2 (dropping the i index) is

$$X_1 \to X_2, \quad X_2 \to X_1, \tag{3.17}$$

after a circulation of 2π around the points x_1 and x_2. The two point correlator is given by

$$\langle \sigma_{(12)}(x_1)\sigma_{(12)}(x_2)\rangle = \frac{Z[\sigma_{(12)}(x_1)\sigma_{(12)}(x_2)]}{Z^N}, \tag{3.18}$$

where the numerator denotes the path integral with the boundary conditions (3.17). Construct a patch on S^2 near the insertions of the twist operators; there will exist two functions on this patch. However, each function on this patch cannot uniquely be defined as being X_1 or X_2 due to the boundary conditions (3.17) therefore the twist operators have introduced a multi-valuedness to the functions. This multi-valuedness makes computing path integrals like (3.18) complicated, so we aim to compute the path integral on a space where the functions become single valued. The problem of finding a space where multivalued functions become single valued is well known and it typically involves constructing a Riemann surface that plays the role of a covering space over a base space.

Let us briefly go through a useful example; consider the complex function x and it's inverse z, each defined on their respective complex plane:

$$x = \Gamma(z) = (z - z_0)^2, \quad \Rightarrow \quad z = \Gamma^{-1}(x) = z_0 \pm \sqrt{x}, \tag{3.19}$$

where $\Gamma(z)$ will soon be defined. In this case, the complex function z is multi-valued in that it has two values for every x with the exception of a special point $x = 0$, where it becomes single valued at $z = z_0$. To make sense of the 'z-space' requires constructing a higher dimensional surface, the Riemann surface, which is identified as the covering of the 'x-space'.

Monodromy around the point z_0 on the plane gives z corresponding to $z_0 \pm \sqrt{x} \to z_0 \mp \sqrt{x}$, therefore we aim to construct a two sheeted covering space Σ with the following property:

$$z = z_0 + \sqrt{x} \quad \text{(on sheet 1)}, \qquad z = z_0 - \sqrt{x} \quad \text{(on sheet 2)}, \tag{3.20}$$

so that in this sense, z is locally single valued on each sheet. Also, Σ should be constructed such that monodromy around z_0 gives:

$$\text{sheet 1} \xrightarrow{2\pi} \text{sheet 2} \xrightarrow{2\pi} \text{sheet 1}, \tag{3.21}$$

to replicate the behaviour of z on the plane. Constructing Σ involves making cuts on the plane at the point z_0 and cleverly folding and gluing the sheets together to obtain the desired geometric properties above. The unique point z_0 is called the 'ramification point' and in this example has a branching order of 2 (for the 2 sheets). The image of z_0 under the map $\Gamma(z)$ is referred to as the branch point (in this example is $x = 0$).

The above case can be generalized slightly to the map

$$x = \Gamma(z) = x_0 + a(z - z_0)^w, \tag{3.22}$$

where the pre-image of the branch point $x = x_0$ gives the ramification point $z = z_0$ with branching order w. 'a' is some complex valued prefactor. It is clear then that the local geometry of Σ is completely determined by the branch point and it's branching order which are both encoded in the map $\Gamma(z)$. We refer to $\Gamma(z)$ as the covering map of Σ over the base space x, where the number of pre-images of $\Gamma(z)$ in the neighbourhood of the branch point is given by the branching order at the ramification point. In general, if the base space is a connected Riemann surface then there exists a *holomorphic* covering map $\Gamma(z)$ from a covering space that is connected and a simply connected Riemann surface [48].

Now, we can return to the case of interest. For the following, the x, z notation will be used due to it's strong relation to the above example. We will first study the monodromy of $X_1(x)$ around the insertion point x_1. Circulating around x_1 by 4π gives the trajectory

$$X_1(x) \xrightarrow{2\pi} X_2(x) \xrightarrow{2\pi} X_1(x), \tag{3.23}$$

due to the boundary conditions (3.17). It is very tempting then to construct a Riemann surface Σ, parameterized by z, that has the same geometric properties as the one constructed in the previous example (see (3.21)).

That is, a two sheeted covering over the insertion x_1 can be constructed and on this covering define a function $X(z)$ (a single copy of $\mathcal{M}$) with the property:

$$X(z) = X_1(x) \quad \text{on sheet 1}, \quad X(z) = X_2(x) \quad \text{on sheet 2}, \tag{3.24}$$

so that now there is a space with a single valued function $X(z)$ that encodes the information of X_1 and X_2. The same arguments can also be applied at the x_2 insertion. As detailed before, the geometry of Σ is fully determined by the covering map $\Gamma(z)$. That $\Gamma(z)$ *is* the covering of S^2 with insertions at x_1 and x_2, it must have the following *local* behaviour near the ramification points z_1 and z_2:

$$\Gamma(z)|_{z\sim z_1} \sim (z - z_1)^2 \quad \text{and} \quad \Gamma(z)|_{z\sim z_2} \sim (z - z_2)^2. \tag{3.25}$$

To find the ramification point is straight forward: the multi-valuedness of the fields arises from circulating around the insertion point x_1 however it is impossible to circulate *at* x_1 so in some sense, the functions are single valued at this point. It is natural then to define the ramification point as:

$$z_1 \equiv \Gamma^{-1}(x_1). \tag{3.26}$$

In fact, after placing the insertions at $x_1 = 0$ and $x_2 = b$, the full covering map will be

$$x = \Gamma(z) = b\frac{z^2}{2z - 1}. \tag{3.27}$$

Expanding around $z = 0$ and $z = 1$ up to second order, $\Gamma(z)$ behaves as

$$\begin{aligned} \Gamma(z)|_{z\sim 0} &\approx -bz^2, \\ \Gamma(z)|_{z\sim 1} &\approx b + b(z-1)^2, \end{aligned} \tag{3.28}$$

(see (3.22)). Indeed, the ramification points are located at $z_1 = 0$ and $z_2 = 1$ with branching order 2.

The full construction of the *closed* covering surface Σ involves cutting open disks at the insertion points and near infinity and inserting the appropriate operators at the edges and gluing them back. These act as regulators and the final solution to the correlators is shown to be independent of the choices made here. Details on this procedure are outlined in [45].

The covering space 'undoes' the action of the twist operators. Computing the path integral on the base space with twist operators inserted is thus equivalent to calculating the path integral on the covering space with no operator insertion.

The two point function (3.18) can now be computed on Σ where there is only one copy of $\mathcal{M}$. The induced metric from the base space to the covering space is given by

$$\begin{aligned} ds^2 = dx d\bar{x} &= \left|\frac{\partial x}{\partial z}\right|^2 dz d\bar{z} \\ &= \left|\partial_z \Gamma\right|^2 dz d\bar{z}. \end{aligned} \tag{3.29}$$

The path integral with $X(z)$ can be computed using the above metric, however in order to keep track of the conformal anomaly * when moving to the covering space, we are instructed that if $ds^2 = e^{\phi} d\hat{s}^2$ (where $d\hat{s}^2$ is the metric on Σ), then the partition function $Z^{(s)}$ computed using the metric ds^2 is equivalent to the partition function $Z^{(\hat{s})}$ using the metric $d\hat{s}^2$ through:

$$Z^{(s)}[\sigma_{(12)}(x_1)\sigma_{(12)}(x_2)] = e^{S_L[\phi]} Z^{(\hat{s})}, \tag{3.30}$$

where e^{ϕ} is the conformal factor and ϕ is the Liouville field. The Louiville action $S_L[\phi]$ is given by:

$$S_L = \frac{c}{96\pi} \int d^2 z \sqrt{-g^{(\hat{s})}} \left(\partial_\mu \phi \partial_\nu \phi g^{(\hat{s})\mu\nu} + 2R^{\hat{s}} \phi\right), \tag{3.31}$$

where c is the central charge † of the CFT and $R^{\hat{s}}$ is the Ricci scalar. Finally, the two point correlation function of (3.18) goes as

$$\langle \sigma_{(12)}(x_1)\sigma_{(12)}(x_2) \rangle \sim e^{S_L[\phi]}. \tag{3.32}$$

All the spacetime information of the twist correlator (3.32) is encoded in the holomorphic covering map $\Gamma(z)$ since $\left|\partial_z \Gamma\right|^2 = e^{\phi}$.

The above treatment was based on a correlator with two twists insertions of length 2. There is only one equivalence class in this case, and the treatment was only carried out for one representative. We are interested in n-point functions in general that will likely have multiple equivalence classes with multiple members in each class. In particular, we are interested in solving the correlator on the LHS of (3.16). The question then becomes; how many ramified coverings over S^2, that has ramification

*The hallmark of a conformal field theory is the tracelessness of the stress tensor at the classical level. When moving to the quantum theory, it becomes finite. This is referred to as the conformal anomaly. The anomaly should still hold when moving to the covering space.

†Details of the origin of the central charge are provided in chapter 4.

of order w_i near each z_i corresponding to the insertion of twist operator $\sigma_{[w_i]}(x_i)$, exist for a gauge invariant n-point correlator?

Finding the number of ramified coverings over the sphere is a well defined problem in mathematics called the “Hurwitz Problem". The Hurwitz problem can be reformulated in terms of subgroups of the symmetric group [46][47]. The theorem in Hurwitz theory says that the number of different ramified coverings over the sphere with n ramification points of order $(w_1, w_2, ...w_n)$, the so-called *Hurwitz number*, is equal to the number of equivalence classes of the n-tuple $(\pi_1, \pi_2, ...\pi_n)$ where $\pi_i \in S_N$ is the group element corresponding to the single-cycle permutation of length w_i, such that

1. The equivalence relation is defined by $(\pi_1, \pi_2, ...\pi_n) \sim (h\pi_1 h^{-1}, h\pi_2 h^{-1}, ...h\pi_n h^{-1})$ for any $h \in S_N$,
2. The subgroup $\langle \pi_1, ..., \pi_n \rangle$ generated by the permutations $(\pi_1, \pi_2, ...\pi_n)$ is transitive,
3. $\pi_1 \pi_2 ... \pi_n = \mathbb{1}$,

which are precisely the requirements (3.12), (3.14) and (3.15) for computing the correlator (3.16)! The interpretation then is that each representative in the sum on the RHS of (3.16) is to be computed using the Lunin & Mathur approach on the distinct ramified covering of S^2 representing that equivalence class.

The genus of each covering is given by the *Riemann-Hurwitz* relation:

$$\mathrm{g} = \frac{1}{2}\sum_{i=1}^{n}(w_i - 1) - M + 1, \tag{3.33}$$

where M is the number of active copies taking part in the transformation, i.e the *degree* of the covering map. For example, the correlator $\langle \sigma_{(132)}\sigma_{(24)}\sigma_{(34)}\sigma_{(124)} \rangle$ contains four active copies. Acknowledging that there is now a correspondence between the equivalence classes of (3.14) and the ramified coverings of S^2, the correlation function (3.16) can be expressed as

$$\Big\langle \sigma_{[w_1]}(x_1)\sigma_{[w_2]}(x_2)...\sigma_{[w_n]}(x_n)\Big\rangle_c = \sum_{\mathrm{g}}\sum_{\Gamma_\mathrm{g}} C_{\Gamma_\mathrm{g}}(\{w_i\})\Big\langle \prod_{i=1}^{n} \sigma_{g_i{}^{\Gamma_\mathrm{g}}}(x_i)\Big\rangle_{\mathrm{g}}, \tag{3.34}$$

where, again, $[w_i]$ denotes the conjugacy class for single cycle permutations of length w_i. On the RHS, the sum was expanded in terms of the genus g of the covering spaces, and for each genus there is a sum over the coverings Γ_g of S^2, where each covering corresponds to an equivalence class. The numerical prefactor $\mathcal{C}_{\Gamma_g}(\{L_i\})$ counts the members of each equivalence class.

3.3 The large N limit

In this section, we begin to make contact with the dual string theory by taking the large N limit. In principle, this will allow us to set up a relation between N and the string coupling constant, g_s. Further details on this are provided in section 6.8, after we evaluate the correlation functions of the vertex operators.

In order to understand the large N behaviour of the twist correlators, the twist operators must first be normalized. The symmetries of conformal field theory restrict the normalized 2-point function to take the form

$$\left\langle \mathcal{O}_{[w_1]}(x_1)\mathcal{O}_{[w_2]}(x_2)\right\rangle = \frac{\delta_{w_1 w_2}}{|x_1 - x_2|^{2\Delta_1}}, \tag{3.35}$$

where the $\delta_{L_1 L_2}$ is to ensure the requirement (3.12) is satisfied for non-trivial contributions. In fact, the only non-trivial contribution comes from the sphere since $g = \frac{1}{2}2(2-1) - 2 + 1 = 0$. We first note that for the 2-point function, $M = w_1 = w_2 = w$. In general, there are $\frac{N!}{(N-M)!}$ possible ways of choosing the active copies, and for *each* twist operator acting with permutation length w there are $(N-w)!$ ways to permute the remaining copies. Additionally, within the single-cycle representative, we have the freedom to perform cyclic permutations of the active copies which will not affect the result. We can do this w many times, so there is an additional factor of w. The non-normalized 2-point correlator takes the form

$$\begin{aligned}\left\langle \sigma_{[w_1]}(x_1)\sigma_{[w_2]}(x_2)\right\rangle &= w\frac{N!}{(N-M)!}(N-w)!(N-w)!\frac{\delta_{w_1 w_2}}{|x_1 - x_2|^{2\Delta_{w_1}}} \\ &= w(N-w)!N!\frac{1}{|x_1 - x_2|^{2\Delta_w}},\end{aligned} \tag{3.36}$$

where we have used that $M = w_1 = w_2 = w$ in going to the second line. The normalized twist operators can be defined to be

$$\mathcal{O}_{[w]} \equiv \frac{1}{\sqrt{w(N-w)!N!}}\,\sigma_{[w]}, \tag{3.37}$$

and the connected, normalized, gauge invariant n-point correlation function:

$$\left\langle \mathcal{O}_{[L_1]}(x_1)\mathcal{O}_{[L_2]}(x_2)...\mathcal{O}_{[L_n]}(x_n)\right\rangle_c = \sum_{\mathrm{g}}\sum_{\Gamma_{\mathrm{g}}} \hat{\mathcal{C}}_{\Gamma_{\mathrm{g}}}(N,\{w_i\})\left\langle \prod_{i=1}^{n} \sigma_{g_i{}^{\Gamma_{\mathrm{g}}}}(x_i)\right\rangle_{\mathrm{g}}, \tag{3.38}$$

where the N dependence is encoded in the numerical prefactor $\hat{\mathcal{C}}_{\Gamma_{\mathrm{g}}}$

$$\hat{\mathcal{C}}_{\Gamma_{\mathrm{g}}}(N,\{w_i\}) = A(\{w_i\})\left[\prod_{k=1}^{n}\sqrt{\frac{(N-w_k)!}{w_k(N!)}}\right]\frac{N!}{(N-M)!}. \tag{3.39}$$

The factor $A(\{w_i\})$ is N independent and accounts for the freedom of cyclic reordering of each twist

$$A(\{w_i\}) = \prod_{i=1}^{n} w_i. \tag{3.40}$$

The term in the brackets comes from the normalization factors and the number of permutations of the copies not taking part in the given cycle. The last factor comes from the different ways of selecting the active copies. M is controlled by the genus via (3.33):

$$M = 1 - \mathrm{g} - \frac{n}{2} + \frac{1}{2}\sum_{k=1}^{n} w_k. \tag{3.41}$$

Focusing strictly on the factors depending on N, we can take the large N limit of $\hat{\mathcal{C}}_{\Gamma_{\mathrm{g}}}$

$$\begin{aligned}
\hat{\mathcal{C}}_{\Gamma_{\mathrm{g}}} &\sim \frac{N!}{(N-M)!}\prod_{k=1}^{n}\sqrt{\frac{(N-w_k)!}{N!}} \\
&\sim \binom{N}{M}\prod_{k=1}^{n}\binom{N}{w_k}^{-\frac{1}{2}} \\
&\sim N^M\left(a_0+\frac{a_1}{N}+...\right)\prod_{k=1}^{n}\left(N^{w_k}\right)^{-\frac{1}{2}} \qquad \text{(for large } N) \\
&\sim N^{M-\frac{1}{2}\sum_{k=1}^{n} w_k}\left(a_0+\frac{a_1}{N}+...\right) = N^{1-\mathrm{g}-\frac{n}{2}}\left(a_0+\frac{a_1}{N}+...\right),
\end{aligned} \tag{3.42}$$

where in going to the second line we have used the binomial coefficient notation and in going to the last equality we have used (3.41). The expansion in the third line comes from the $\sim e^{\mathcal{O}(1/N)}$ term in the large N limit of the binomial coefficient. In the large N limit, the higher genus terms in (3.38) are suppressed in the expansion however at a given genus, there is an infinite sum of subleading terms. This should be contrasted with the large N behaviour of normalized single traced operators of ordinary $U(N)$ gauge theory which is simply $\sim N^{2-2g-n}$.

Chapter 4

The Symmetric Product Orbifold: The Spectrum

In the previous chapter we found the structure of the correlation functions of the symmetric product orbifold theory. Before calculating the correlation functions of the string theory, it is necessary to find the precise duality. One way of showing that two theories are equivalent to one another is by matching their spectra. In this chapter, we calculate the energy spectrum of the w-cycle twisted sector of the $(\mathbb{T}^4)^{\otimes N}/S_N$ theory. We first lay out the foundations for calculating the space of physical states in 2d CFTs and then generalise to the orbifold theory. The theory developed in this chapter will help motivate the content in chapter 5 where the worldsheet spectrum is calculated.

4.1 The 2d CFT symmetry algebra

We start by discussing some basic facts about the symmetry algebras of $2d$ CFTs and the corresponding representations. In complex co-ordinates, the conserved stress-energy tensor splits into holomorphic $T(x)$ and anti-holomorphic $\bar{T}(\bar{x})$ parts and can be treated separately. In the radial quantization approach, physical states live on circles of constant radius (on the complex plane) and evolve in the radial direction. Restricting to the circle and assuming periodic boundary conditions, we can Laurent expand $T(x)$ in terms of it's fourier modes:

$$T(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}, \tag{4.1}$$

where the modes $L_n = -x^{n+1}\partial_x$ are interpreted as the generators of the conformal transformations. The -2 in the power is because the conformal weight of the stress tensor is 2. It is easy to show that the modes satisfy the *Witt Algebra*:

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{4.2}$$

Symmetries of the theory greatly constrain the possible physical states that can exist. We aim to identify the spectrum by understanding how the symmetry transformations act on these states. To do so, we would like to represent the algebra (4.2) as linear operators acting on vectors in a vector space, V, where the vectors represent the physical states. There is, however, a subtlety when moving to the quantum theory. Recall that quantum states live in a Hilbert space and that the states $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ are physically indistinguishable from one another. When we search for representations, it is often too strong a requirement to look for *true representations* where the states are uniquely defined up to a phase factor. Instead, we are interested in *projective representations*, i.e. representations where two states that differ by a phase factor are considered equivalent. In the following, we will work with some Lie group G to illustrate the effect of such a requirement. The arguments also apply to Lie algebras.

The projective representation is given by the homomorphism $\rho : G \to PGL(V)$ where $PGL(V) = GL(V)/A$. $GL(V)$ is the general linear group of invertible linear transformations on V and A is the normal subgroup associated with scalar multiplication. The projective representations $\rho(g)$, with $g \in G$, satisfies the homomorphism property:

$$\rho(gh) = c(g,h)\rho(g)\rho(h), \tag{4.3}$$

where $c(g,h) \in A$. These projective representations are difficult to work with precisely because of the ambiguity with the phase factor, so we can ask whether the projection can be "lifted" to some linear representation. Suppose there is a group E (where at this point, it is not clear whether E *is* G or not) that has a linear representation that can be projected onto $PGL(V)$. The lift ℓ must satisfy

$$\ell(gh) = c(g,h)\ell(g)\ell(h), \tag{4.4}$$

where $\ell \in E$ and $g, h \in G$. From the associative property of G, it is easy to show that c satisfies the cocycle equation

$$c(g,hk)c(h,k) = c(gh,k)c(g,h). \tag{4.5}$$

In fact, c in general is not unique: it depends on the choice of the lift ℓ. Define another lift $\ell'(g) = f(g)\ell(g)$, then by associativity we have a different cocycle

$$c'(g,h) = f(gh)c(g,h)f^{-1}(g)f^{-1}(h), \tag{4.6}$$

thus ℓ defines a unique class in the second cohomology group $H^2(G, A)$. If the classes are non-trivial, then $H^2(G, A)$ is in one to one correspondence with the *central extensions* of G by A. The central extension is the short exact sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1, \tag{4.7}$$

where $\text{im}(i) = ker(\pi)$. The group E is the central extension of G by elements of A. A is in the center of E, that is, each element of A commutes with each element of E. It should be clear by now that if the classes are trivial then E *is* G and the lifted linear representation is just $GL(V)$.

It follows from Schur's Lemma that the irreducible representations of the central extensions of G and the irreducible projective representations of G are the same, so it suffices to study the group E and it's representations when moving to the quantum theory. For the application to $2d$ CFT, we are thus interested in the central extension of the Witt Algebra and its representations. Without proof, we state the *unique* central extension of the Witt Algebra; the so-called *Virasoro Algebra*:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{4.8}$$

where the second term is the central term and c is the central charge: it commutes with all L_m's. In the language of $2d$ CFT, the central charge 'counts' the degrees of freedom in a theory. Note that the above treatment can be reproduced for the right-moving Virasoro modes $\bar{L}_m$ from the Laurent expansion of the anti-holomorphic stress tensor $\bar{T}(\bar{x})$. For the remainder of the chapter, we will continue to work with the left-moving Virasoro algebra and understand that there is an analogue statement for the right-moving modes.

4.2 Representation theory of 2d CFTs

One of the axioms of $2d$ CFT states that the representations of the Virasoro algebra decomposes into irreducible, factorizable representations and when acting on the spectrum, L_0 and $\bar{L}_0$ are diagonalizable and $L_0 + \bar{L}_0$ is bounded from below. Note that the operator $L_0 + \bar{L}_0$ corresponds to the dilation operator on the complex plane. However, after a conformal transformation to the cylinder (for which the CFT is invariant), $L_0 + \bar{L}_0$ corresponds to translations in the time-like direction so it is interpreted as the

Hamiltonian operator and it's eigenvalues in the representation are interpreted as the energy levels. Call this representation $\mathcal{R}$, and in $\mathcal{R}$ let the state $|h_0\rangle$ be the eigenvector of L_0 with the lowest eigenvalue (the ground state energy), h_0. Let h be the eigenvalue of the state $L_n|h_0\rangle$. Using the Virasoro algebra, it's easy to see that $h = h_0 - n$. We have already defined h_0 to be the lowest eigenvalue therefore the state $L_n|h_0\rangle$ *cannot* exist if $n > 0$. In other words, we have the condition that $L_n|h_0\rangle = 0 \quad \forall n > 0$. In fact, for this condition to be satisfied, $|h_0\rangle$ *must* be a primary state with conformal weight h_0. To see this, we first assume that $|h_0\rangle$ is indeed primary. Primary states can be generated through the *State-Operator Correspondence*:

$$\lim_{x\to 0}\phi(x)|0\rangle \equiv |h_0\rangle\,, \tag{4.9}$$

for some primary operator $\phi(x)$ with conformal weight h_0 acting on the vacuum $|0\rangle$. Now we can apply the Fourier transform of L_n and the usual OPE relations between the stress tensor $T(x)$ and the primary field $\phi(y)$ to show

$$\begin{aligned} L_n|h_0\rangle &= \lim_{y\to 0}\oint dx\, x^{n+1}T(x)\phi(y)|0\rangle \\ &= \oint dx\, x^{n+1}\left(\frac{h_0\phi(0)}{x^2}+\frac{\partial\phi(0)}{x}\right)|0\rangle \\ &= \oint dx\,\Big(x^{n-1}h_0\phi(0)+x^n\partial\phi(0)\Big)|0\rangle\,, \end{aligned} \tag{4.10}$$

which vanishes if $n > 0$ since in this case there are no residues. If $n = 0$ then we recover the eigenvalue equation $L_0|h_0\rangle = h_0|h_0\rangle$. The states in $\mathcal{R}$ can be created by acting on the primary state $|h_0\rangle$ with the negative Virasoro modes.

We will briefly give an overview of the structure of $\mathcal{R}$. Let $\mathfrak{V}$ denote the left moving Virasoro algebra. Since associative algebras are easier to work with, $\mathcal{R}$ should be a linear representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{V})$ of $\mathfrak{V}$, that is, the universal associative algebra generated by the Virasoro modes. Since $|h_0\rangle$ is a primary state, it is simpler to consider the algebra of the creation modes $\mathcal{U}(\mathfrak{V}^+)$. Representations of this type are the highest weight representations. The basis of $\mathcal{U}(\mathfrak{V}^+)$ is given by ordered p-uples $(-n_1, ..., -n_p)$ of decreasing negative integers by $\{L_{-n_1}...L_{-n_p}\}_{1\le n_1...\le n_p}$. Define the Verma module $\mathcal{V}_{h_0}$ to be the highest weight representation containing the primary state $|h_0\rangle$ with conformal weight h_0 obtained by the isomorphism $\Phi_{\mathcal{V}_{h_0}} : \mathcal{U}(\mathfrak{V}^+) \to \mathcal{V}_{h_0}$. The basis in $\mathcal{V}_{h_0}$ is given by the states $\{L_{-n_1}...L_{-n_p}|h_0\rangle\}$ therefore $\mathcal{V}_{h_0}$ is the largest possible highest weight representation with conformal weight h_0. Note that any state that is generated by acting on the primary state by negative Virasoro modes is referred to as a *descendant state*. The energy of a descendent state such as $L_{-n_1}...L_{-n_p}|h_0\rangle$ is clearly $h_0 + \sum_{i=1}^{p} n_i$.

The Verma module is reducible if it contains non-trivial sub-representations, however any non-trivial subrepresentation of a highest weight representatison is also a highest weight representation. By definition, the highest weight states of these sub-representations are simultaneously both primary states and descendants of $|h_0\rangle$. States of this kind are called *singular vectors*. In general, singular vectors are linear combinations of descendant states that are annihilated by the positive Virasoro modes. To obtain an irreducible highest weight representation $\mathcal{R}$, one must remove these singular vectors and their descendants: $\mathcal{R} = \mathcal{V}_{h_0}/\mathcal{R}'$ where R' is the Verma module associated with the singular vectors. Requiring that the representations are unitary and that the Hilbert space does not contain any negative norm states places additional constraints such as $h_0 \geq 0$ and $c > 0$.

4.3 The spectrum of the w-cycle twisted sector of $(\mathbb{T}^4)^{\otimes N}/S_N$

We can now generalize to the symmetric product orbifold theory, where we are particularly interested in computing the spectrum of the twisted sector. We denote the stress-energy tensor for each copy $\mathcal{M}$ by $T_I(x)$. In the presence of a twist insertion however, the boundary conditions of $T_I(x)$ change, since it is just a special case of (3.3):

$$T_I(e^{2\pi i}x)\sigma_g(0) = T_{I+1}(x)\sigma_g(0), \tag{4.11}$$

in the case of g being a single cycle permutation. We can diagonalize the action of the twist operators by defining a linear combination of the stress tensors (over the orbit of the cycle), weighted by a phase:

$$T_\ell(x) = \sum_{I=1}^{w} \exp\left(-\frac{2\pi i I\ell}{w}\right) T_I(x), \qquad \ell = \{1,2,...,w\}. \tag{4.12}$$

After one circulation around the twist insertion, it simply picks up a phase factor:

$$\begin{aligned} T_\ell(e^{2\pi i}x)\sigma_{(12\ldots w)}(0) &= \sum_{I=1}^{w} \exp\left(-\frac{2\pi i I\ell}{w}\right) T_I(e^{2\pi i}x)\sigma_{(12\ldots w)}(0) \\ &= \exp\left(\frac{2\pi i\ell}{w}\right) \sum_{I=1}^{w} \exp\left(-\frac{2\pi i(I+1)\ell}{w}\right) T_{I+1}(x)\sigma_{(12\ldots w)}(0) \\ &= \exp\left(\frac{2\pi i\ell}{w}\right) T_\ell(x)\sigma_{(12\ldots w)}(0), \end{aligned} \tag{4.13}$$

where we have identified $w+1 \cong 1$. The symmetry generator of this combination is given by it's Fourier transform:

$$L_n^{(12\ldots w)} = \oint dx\, x^{n+1} T_\ell(x). \tag{4.14}$$

For the modes to make sense in the presence of a twist operator, they must be fractionally moded. To see this, we make the insertion at $\sigma_{(12\ldots w)}(0)$ and take $x \to e^{2\pi i}x$:

$$\begin{aligned} L_n^{(12..w)} &= \oint dx\, (e^{2\pi i}x)^{n+1} T_\ell(e^{2\pi i}x) \\ &= e^{2\pi i(n+\frac{\ell}{w})} \oint dx\, x^{n+1} T_\ell(x) \\ &= \oint dx\, x^{n+1} T_\ell(x) \qquad \text{for } n \in \mathbb{Z} - \tfrac{\ell}{w}. \end{aligned} \tag{4.15}$$

As will soon be demonstrated, the fractional modes give rise to fractional excitations in the twisted sector. The 'twisted' Virasoro algebra is given by

$$[L_m^{(12\ldots w)}, L_n^{(12..w)}] = (m-n)L_{m+n}^{(12\ldots w)} + \frac{cw}{12} m(m^2-1)\delta_{m+n,0}, \tag{4.16}$$

where we multiplied the central charge by w because there are w many active copies of $\mathcal{M}$ in the orbifold action. The highest weight state in the w-cycle twisted sector is generated by the primary twist operator $\sigma_{[(12\ldots w)]}(x)$ (recall the twisted sectors are labelled by conjugacy classes) through the State-Operator Correspondence:

$$\lim_{x\to 0} \sigma_{[(12\ldots w)]}(x)\, |0\rangle \equiv |h_0\rangle_w\,, \tag{4.17}$$

where as per the previous discussion, the ground state energy is given by the eigenvalue equation $L_0^{(12\ldots w)}\, |h_0\rangle_w = h_0\, |h_0\rangle_w$. We fill out the spectrum by acting on $|h_0\rangle_w$ with the negative fractional modes.

We can compute the ground state energy by making direct use of the covering space. Recall that the covering space is a conformal transformation of the base space: $z = x^{\frac{1}{w}}$ see (section 3.2). Under conformal transformations, the stress energy tensor transforms as:

$$T^{\text{cover}}(z) = \left(\frac{\partial z}{\partial x}\right)^{-2} \left[T^{\text{base}}(x) - \frac{c}{12} S(z,x)\right], \tag{4.18}$$

where c is, again, the central charge for a single copy of $\mathcal{M}$ and $S(z,x)$ is the *Schwarzian*, defined by:

$$S(z,x) = \left(\frac{\partial^3 z}{\partial x^3}\right)\left(\frac{\partial z}{\partial x}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 z}{\partial x^2}\right)^2\left(\frac{\partial z}{\partial x}\right)^{-2}. \tag{4.19}$$

For the conformal transformation $z = x^{\frac{1}{w}}$ we have

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{1}{w} x^{\frac{1}{w}-1} \\ \frac{\partial^2 z}{\partial x^2} &= \frac{1}{w}\left(\frac{1}{w}-1\right) x^{\frac{1}{w}-2} \\ \frac{\partial^3 z}{\partial x^3} &= \frac{1}{w}\left(\frac{1}{w}-1\right)\left(\frac{1}{w}-2\right) x^{\frac{1}{w}-3} \\ S(z,x) &= -\left(\frac{1}{w^2}-1\right)\frac{x^{-2}}{2}, \end{aligned} \tag{4.20}$$

Substituting (4.20) into (4.18) we get the stress tensor on the covering space:

$$T^{\text{cover}}(z) = \left(wT^{\text{base}}(x)\, x^2 + c\frac{1-w^2}{24w}\right) w x^{-\frac{2}{w}}. \tag{4.21}$$

Taking $T^{\text{base}}(x) = \frac{1}{w} T_\ell(x)$, we can Laurent expand both sides in terms of modes:

$$\begin{aligned} \sum_n \hat{L}_n z^{-n-2} &= \left(\frac{w}{w}\left(\sum_m L_m^{(12\ldots w)} x^{-m-2}\right) x^2 + c\frac{1-w^2}{24w}\right) w x^{-\frac{2}{w}} \\ &= \sum_m \left(L_m^{(12\ldots w)} + c\frac{1-w^2}{24w}\delta_{m,0}\right) w x^{-m-\frac{2}{w}} \\ &= \sum_m \left(L_m^{(12\ldots w)} + c\frac{1-w^2}{24w}\delta_{m,0}\right) w z^{-mw-2} \end{aligned} \tag{4.22}$$

where we denote $\hat{L}_n$ as the Virasoro mode for a single copy of $\mathcal{M}$ on the covering space. After rearranging, we get the following isomorphism between modes:

$$L_m^{(12...w)} = \frac{1}{w}\hat{L}_{mw} + c\frac{w^2-1}{24w}\delta_{m,0}. \tag{4.23}$$

Finally, the ground state energy of the w-cycle twisted sector, with w *odd*, can be identified with the second term:

$$h_0 = c\frac{w^2-1}{24w}. \tag{4.24}$$

There is a slight subtlety when considering the case where w is *even*. Consider first the character of a single copy of $\mathcal{M}$:

$$\chi(\tau) = \mathrm{Tr}_{\mathcal{R}^{(1)}}\left(q^{L_0-\frac{c}{24}}\right) = q^{-\frac{c}{24}}(1+...), \tag{4.25}$$

where $q = e^{2\pi i\tau}$ and τ the modular parameter. The character over the full space $\mathcal{R}^{(N)}$ with the insertion of a twist operator σ_w is:

$$\mathrm{Tr}_{\mathcal{R}^{(N)}}\left(\sigma_w q^{L_0-\frac{c}{24}}\right) = \chi(w\tau)\chi(\tau)^{N-w}, \tag{4.26}$$

where we multiplied τ by w for the active copies due to the boundary conditions imposed by the twist operators. To obtain the proper character in the twisted sector, we need to take the S-modular transformation:

$$Z = \chi\left(\frac{\tau}{w}\right)\chi(\tau)^{N-w} = q^{-c\frac{1}{24w}}q^{-c\frac{N-w}{24}}(1+...). \tag{4.27}$$

The ground state energy of (4.24) can be recovered by taking the leading exponent of (4.27). The leading exponent minus $\frac{N}{24}$ gives the ground state energy

$$h_0 = \left(b+\frac{f}{2}\right)\frac{w}{24} - \left(b+\frac{f}{2}\right)\frac{1}{24w} = \left(b+\frac{f}{2}\right)\frac{w^2-1}{24w}, \tag{4.28}$$

where the central charge c was expressed in terms of the number of bosons, b, and fermions, f, on $\mathcal{M}$. Each boson contributes 1 to the central charge while each fermion contributes $\frac{1}{2}$. For w odd, (4.28) will do, however for w even, we take the character:

$$\mathrm{Tr}_{\mathcal{R}^{(N)}}\left(\sigma_w q^{L_0-\frac{c}{24}}\right) = \tilde{\chi}(w\tau)\chi(\tau)^{N-w}, \tag{4.29}$$

where $\tilde{\chi} = \mathrm{Tr}_{R^{(N)}}\left((-1)^F q^{L_0-\frac{c}{24}}\right)$ is the NS sector character with F the fermionic number operator. For w even, the fermionic states pick up a sign relative to the bosonic states. The S modular transformation of $\tilde{\chi}$ gives an R-sector like partition function thus for each fermion, the leading exponent is

$$q^{\frac{1}{24w}} q^{-\frac{N-w}{48}}, \tag{4.30}$$

so for w even, the ground state energy is

$$h_0 = b\frac{w^2-1}{24w} + \frac{f}{2}\frac{w^2+2}{24w}. \tag{4.31}$$

Now, the spectrum can be calculated; we start with w odd. In the following, the upper indices on the fractional Virasoro modes will be dropped for notational convenience. The energy of the excited state is

$$\begin{aligned} L_0\big(L_{-\frac{\tilde{N}}{w}} \left|h_0\right\rangle_w\big) &= \big([L_0, L_{-\frac{\tilde{N}}{w}}] + L_{-\frac{\tilde{N}}{w}} L_0\big) \left|h_0\right\rangle_w \\ &= \left(\frac{\tilde{N}}{w} + c\frac{w^2-1}{24w}\right)\big(L_{-\frac{\tilde{N}}{w}} \left|h_0\right\rangle_w\big) \end{aligned} \tag{4.32}$$

with excitation number $\tilde{N} \in \mathbb{N}$. Also, we have used $L_0 \left|h_0\right\rangle_w = h_0 \left|h_0\right\rangle_w$ and the algebra (4.16) in going to the second line. We are particularly interested in the case $\mathcal{M} = \mathbb{T}^4$. This theory consists of 4 free bosons and 4 free fermions, therefore the central charge is $c = 4(1) + 4(\frac{1}{2}) = 6$ *. The spectrum for the symmetric product orbifold theory $(\mathbb{T}^4)^{\otimes N}/S_N$ in the w-cycle (with w odd) twisted sector is thus

$$h = \frac{\tilde{N}}{w} + \frac{(w^2-1)}{4w} \tag{4.33}$$

where the first term is the set of fractional excitations while the second term is the ground state energy. A similar computation can be carried out for w even, and in this case, the spectrum is given by:

$$h = \frac{\tilde{N}}{w} + \frac{w}{4} \tag{4.34}$$

where the ground state energy was calculated using (4.31).

*Each boson contributes 1 and each fermion contributes $\frac{1}{2}$ to the central charge on each circle in $\mathbb{T}^4 = S^1 \times S^1 \times S^1 \times S^1$

Chapter 5

Type IIB String Theory on $AdS_3 \times S^3 \times \mathbb{T}^4$: The Spectrum

In this chapter, we introduce the Wess-Zumino-Witten (WZW) sigma model worldsheet description. We start by formulating the WZW model and define it's target space for our application. We study the representation theory of the spectrum generating algebra and introduce an automorphism, referred to as spectral flow, acting on the algebra. We compute the spectrum of the w spectrally flowed sector and show its equivalence with the spectrum of the w-cycle twisted sector of the $(\mathbb{T}^4)^{\otimes N}/S_N$ theory calculated in chapter 4, showing thus the correspondence.

5.1 The bosonic WZW Model

In the most general form of the WZW model, we consider a quantum field theory on some Riemann surface, however for our application we will only consider the Riemann sphere $\mathbb{CP}^1 \cong S^2$. The model is a principal chiral model on a Lie-group G (that we assume is semi-simple), and the fields are the elements of this group with the following mapping:

$$g : S^2 \to G. \tag{5.1}$$

Consider the following WZW action:

$$S = \frac{1}{4\lambda^2} \int_{S^2} d^2z \, \mathrm{Tr}(g^{-1}\partial_\mu g g^{-1}\partial^\mu g) - \frac{ik}{12\pi} \int_B d^3y \, \epsilon_{\alpha\beta\gamma} \, \mathrm{Tr}(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g). \tag{5.2}$$

This action requires some explanation. We take λ to be a constant which will soon be fixed. The g fields are to be thought of as matrices: they are elements of some faithful representation of G. The terms $g^{-1}\partial_\mu g$ are elements of the Lie algebra $\mathfrak{g}$, that is, elements of the tangent space at the identity of the G manifold, T_eG. This is because, in the language of differential geometry, $w_g := g^{-1}\mathrm{d}g$ is the *Maurer-Cartan form*. The Maurer-Cartan form is a globally defined one-form on G that linearly maps the tangent space T_gG for $\forall g \in G$ into T_eG (the Lie algebra). In particular, it is defined on vectors $v \in T_gM$

$$w_g(v) = (L_{g^{-1}})_* v \tag{5.3}$$

where $L_g(h) = gh$. $(L_g)_*$ is a *pushforward* along the left-translation of the group, that is, a mapping from one tangent space to another. The Maurer-Cartan form satisfies:

1. $w_e = id : T_eG \to \mathfrak{g}$
2. $\forall g \in G \quad w_g = Ad(g^{-1})(R^*_{g^{-1}} w_e)$,

where R^*_g is the *pullback* of forms along the right-translation in the group and $Ad(g)$ is the adjoint action on the Lie algebra.

The second term of (5.2) is called the Wess-Zumino (WZ) term and is required for the theory to be conformally invariant at the quantum level for non-Abelian Lie groups. The integral in the WZ term is over the volume of a ball B whose boundary is S^2. The map (5.1) is classified up to homotopy by the second homotopy group $\pi_2(G)$ and is thus homotopic to the constant map since the second fundamental group of every Lie group vanishes: $\pi_2(G) = 0$. Since the constant map can be continued into the interior of S^2 then so can g. The dimensionless coefficient k is called the *level* of the theory which for compact Lie Groups is required to be an integer for topological reasons. Furthermore, in order for the action to be positive definite so that path integrals converge, we require that the level is positive $k > 0$.

After moving to complex coordinates, variation of the action (5.2) yields the equations of motion:

$$\left(1 + \frac{\lambda^2 k}{\pi}\right) \partial(g^{-1}\bar{\partial} g) \; + \; \left(1 - \frac{\lambda^2 k}{\pi}\right) \bar{\partial}(g^{-1}\partial g) = 0. \tag{5.4}$$

Fixing $\lambda^2 = \pi/k$ leads to the vanishing of the second term in (5.4), giving a *conserved* anti-holomorphic current $\bar{J} \equiv k g^{-1}\bar{\partial} g$ (or the Lie algebra valued current):

$$\partial(g^{-1}\bar{\partial} g) = 0. \tag{5.5}$$

Note that a given choice of k parameterizes the entire WZW action. (5.5) also implies the presence of a holomorphic current $J \equiv -k\partial g g^{-1}$:

$$\bar{\partial}(\partial g g^{-1}) = g\partial(g^{-1}\bar{\partial}g)g^{-1} = 0, \tag{5.6}$$

where in going to the second equality we have used the identity of item 2 above. The (anti)holomorphicity is due to (5.2) having a local $G(z) \times G(\bar{z})$ symmetry acting as

$$g(z,\bar{z}) \to g_L(z)g(z,\bar{z})g_R^{-1}(\bar{z}), \tag{5.7}$$

where $g_L(z)$ is some holomorphic map $S^2 \to G$ and $g_R(\bar{z})$ is some anti-holomorphic map.

By a dimensional analysis, the current $J(z) = -k\partial g g^{-1}$ has left-moving conformal weight 1 (coming from the derivative) and since the current is holomorphic, the right-moving conformal weight is 0. We say that the conformal weight is $(1,0)$. Likewise the anti-holomorphic current $\bar{J}(\bar{z})$ has conformal weight (0,1). When moving to the quantum theory, we consider the OPEs of the components of the holomorphic currents $J^a(z)$ (the same treatment can be applied to the anti-holomorphic currents) where the indices a, b, c label the adjoint indices. Writing the OPE in the form

$$J^a(z)J^b(w) \sim \sum_p \frac{X_p(w)}{(z-w)^p}, \tag{5.8}$$

where $X_p(w)$ is some holomorphic field. The conformal weight on the LHS is $1+1=2$, so every term on the RHS must have the same conformal weight. In particular, the field $X_p(w)$ must have conformal weight $2-p$. Unitary constrains that there exists no fields with negative conformal dimension, therefore p must be at most order 2. Focusing on the singular terms, the OPE can be written as

$$J^a(z)J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{if_c^{ab}J^c(w)}{(z-w)} \tag{5.9}$$

where κ^{ab} and f_c^{ab} are some constants. Symmetry and associativity of the OPE constrains κ^{ab} to be a symmetric invariant tensor and f_c^{ab} to be the structure constant of a Lie algebra $\mathfrak{g}$ (it satisfies the Jacobi identity). In fact, since the currents encode the symmetry of the theory, the lie algebra $\mathfrak{g}$ is precisely that belonging to G. For simple Lie algebras, there is the Killing form which is the unique invariant 2-tensor, defined as $\mathrm{Tr}(\mathrm{Ad}\ T^a \cdot \mathrm{Ad}\ T^b)$ where $T^{a,b}$ are elements of the algebra and Ad is the adjoint action. κ^{ab}

is to be identified with the Killing form. The basis of the Lie algebra can be chosen such that $\kappa^{ab} = k\delta^{ab}$, where k is identified with the level of the theory. We finally have the OPE

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{ab}_c J^c(w)}{z-w}. \tag{5.10}$$

Since the currents are meromorphic functions, their Laurent expansion around the origin can be taken

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}, \tag{5.11}$$

where the -1 in the power comes from the current having conformal weight 1. By the usual methods of contour deformation and residues, the commutation relations between the modes using the OPEs (5.10) is given by

$$\begin{aligned}
\left[J^a_m, J^b_n\right] &= \oint \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} J^a(z)J^b(w) z^m w^n \\
&= \oint \frac{dw}{2\pi i} w^n \oint_{C_w} \frac{dz}{2\pi i}\left(\frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{ab}_c J^c(w)}{z-w}\right)(w^m + mw^{m-1}(z-w) + ...) \\
&= \oint \frac{dw}{2\pi i} w^n \,\mathrm{Res}\left[\frac{mk\delta^{ab} w^{m-1}}{(z-w)} + \frac{if^{ab}_c J^c(w) w^m}{(z-w)}\right] \\
&= km\delta^{ab} \oint \frac{dw}{2\pi i} w^{m+n-1} + if^{ab}_c \sum_k J^c_k \oint \frac{dw}{2\pi i} w^{m+n-k-1} \\
&= km\delta^{ab}\delta_{m+n,0} + if^{ab}_c \sum_k J^c_k \delta_{m+n,k} \\
&= km\delta^{ab}\delta_{m+n,0} + if^{ab}_c J^c_{m+n}.
\end{aligned} \tag{5.12}$$

This algebra is an example of a *Kac-Moody* algebra at level k, $\mathfrak{g}_k$; the infinite dimensional generalisation of a Lie algebra. The first term is the central term; k commutes with all J^a_m. It is important to note that the zero modes recover the original Lie algebra.

The stress-energy tensor of the theory is computed via the *Sugawara construction*:

$$T(z) \equiv \frac{1}{2(k+h^\vee)} : J^a J^a : (z), \tag{5.13}$$

where $h^\vee$ is the dual coxeter number and the colons denote normal ordering. With this construction, the OPE of the stress-energy tensor and the chiral current is given by

$$T(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}. \tag{5.14}$$

Note that the coefficient of the quadratic pole in an OPE between $T(z)$ and a primary field is the conformal weight of that field. Indeed, by construction, (5.14) reproduces that the current has conformal weight 1. The OPE of the stress energy tensor with itself gives:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \qquad c \equiv \frac{k \dim \mathfrak{g}}{k+h^\vee}, \tag{5.15}$$

where c is the central charge of the theory and dim $\mathfrak{g}$ is the dimension of the algebra $\mathfrak{g}$. From (5.15) we can read off the conformal weight of $T(z)$ to be 2. Also, this OPE structure confirms that the theory is conformal. Recall that the stress tensor can be expanded in terms of it's Fourier modes:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{5.16}$$

Using contour deformations and Cauchy's formula, (5.13), (5.14) and (5.15) can be expressed in terms of the modes of the fields

$$L_m = \frac{1}{2(k+h^\vee)} \sum_{n\in\mathbb{Z}} :J^a_n J^a_{m-n}:, \tag{5.17}$$

$$[L_m, J^a_n] = -nJ^a_{m+n}, \tag{5.18}$$

$$[L_m, L_n] = \frac{c}{12}m(m^2-1)\delta_{m+n,0} + (m-n)L_{m+n}. \tag{5.19}$$

The commutation relations of (5.19) is the familiar Virasoro algebra, which was already encountered in (4.8). It is important to note that through the Sugwara construction, the Virasoro algebra is embedded in the larger Kac-Moody algebra.

5.2 The supersymmetric case

We aim to study superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ so we must generalise the above treatment to include fermions. The $\mathcal{N}=1$ superconformal WZW model is generated by a bosonic Kac-Moody algebra $\mathfrak{g}_k$ (5.12) coupled to fermions in the adjoint representation. The OPEs are given by

$$J^a(z)\psi^b(w) \sim \frac{if^{ab}_c\psi^c(w)}{z-w}, \tag{5.20}$$

$$\psi^a(z)\psi^b(w) \sim \frac{k\delta^{ab}}{z-w}. \tag{5.21}$$

In terms of modes:

$$\left[J^a_m, \psi^b_r\right] = if^{ab}_c \psi^c_{m+r}, \tag{5.22}$$

$$\{\psi^a_r, \psi^b_s\} = k\delta^{ab}\delta_{r,-s}. \tag{5.23}$$

In fact, we can decouple the fermions from the bosons by defining a shifted current $\mathcal{J}^a$ as

$$\mathcal{J}^a(z) \equiv J^a(z) + \frac{i}{2k} f^a_{bc} : \psi^b\psi^c : (z). \tag{5.24}$$

The OPEs of the shifted currents become

$$\mathcal{J}^a(z)\mathcal{J}^b(w) \sim \frac{\kappa\delta^{ab}}{(z-w)^2} + \frac{if^{ab}_c J^c(z)}{z-w}, \tag{5.25}$$

$$\mathcal{J}^a(z)\psi^b(w) \sim 0, \tag{5.26}$$

where we have a shifted level $\kappa \equiv k - h^\vee$. In terms of the modes

$$\left[\mathcal{J}^a_m, \mathcal{J}^b_n\right] = \kappa m\delta^{ab}\delta_{m+n,0} + if^{ab}_c \mathcal{J}^c_{m+n}, \tag{5.27}$$

$$\left[\mathcal{J}^a_m, \psi^b_r\right] = 0. \tag{5.28}$$

Thus the superconformal algebra is isomorphic to the direct sum of the affine Kac-Moody algebra of (5.27) at shifted level κ and $\dim(\mathfrak{g})$ decoupled free fermions. Via the Sugawara construction, the stress tensor $T(z)$ is given by:

$$T(z) = \frac{1}{2(\kappa + h^\vee)}\Big[: \mathcal{J}^a\mathcal{J}^a : (z) - : \psi^a\partial\psi^a : (z)\Big]. \tag{5.29}$$

Taking the the OPE of (5.29) with itself, we can read off the central charge to be

$$c = \dim(\mathfrak{g})\left(\frac{\kappa}{\kappa + h^\vee} + \frac{1}{2}\right) = \dim(\mathfrak{g})\left(\frac{k - h^\vee}{k} + \frac{1}{2}\right). \tag{5.30}$$

5.3 The supersymmetric WZW model with target space $AdS_3 \times S^3 \times \mathbb{T}^4$

The geometry of AdS_3 can be thought of as the hypersurface:

$$X^2_{-1} + X^2_0 - X^2_1 - X^2_2 = l^2, \tag{5.31}$$

embedded in flat $R^{2,2}$ with co-ordinates X_{-1}, X_0, X_1, X_2 and l the radius of curvature. This parameterization can be encoded in a matrix g:

$$g = \frac{1}{l}\begin{pmatrix} X_{-1}+X_1 & X_0 - X_2 \\ -X_0 - X_2 & X_{-1} - X_1 \end{pmatrix}, \tag{5.32}$$

so that setting the determinant of g to unity gives (5.31). g is an element of the $SL(2,\mathbb{R})$ group manifold therefore for the worldsheet theory on the AdS_3 factor, the group elements of $SL(2,\mathbb{R})$ are interpreted as the fields in the WZW action (5.2) with the mapping

$$g(z,\bar{z}) : S^2 \to SL(2,\mathbb{R}), \tag{5.33}$$

where the worldsheet is taken to be the Riemann sphere S^2 and $(z,\bar{z})$ are the co-ordinates of S^2. In fact, the topology of AdS_3 is $S^1 \times R^2$ where the S^1 is timelike; it is periodic in time and contains timelike curves. We are particularly interested in the *universal cover* of $SL(2,\mathbb{R})$ since in this case the timelike direction is non-compact. So in (5.33) and all that follows, by '$SL(2,\mathbb{R})$' we are referring to its universal cover, unless otherwise stated. The symmetry group of the S^3 factor is $SU(2)$ so the worldsheet theory on $AdS_3 \times S^3$ is given by the supersymmetric $SL(2,\mathbb{R}) \times SU(2)$ WZW model. On $\mathbb{T}^4$ there are 4 free bosons and 4 free fermions so the entire symmetry algebra is given by the direct sum:

$$\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \oplus \mathfrak{su}(2)^{(1)}_{k'} \oplus (4\text{ bosons } + \text{ 4 fermions}) \tag{5.34}$$

where the superscript (1) is for the $\mathcal{N}=1$ superconformal theory, k and k' are the levels of the theory on the $SL(2,\mathbb{R})$ and $SU(2)$ factors respectively. The group $SL(2,\mathbb{R})$ is non-compact, so the topological constraint (see the discussion below (5.2)) on k does not apply so it can take on any positive value, whereas $SU(2)$ is compact so k' is quantized $k' \in \mathbb{Z}_{>0}$. However, in order to have a critical superstring theory on 10 spacetime dimensions, the total central charge is required to be $c = 15$. Thus criticality imposes the following constraint on the levels of the theory:

$$\begin{aligned} &c(\mathfrak{sl}(2,\mathbb{R})^{(1)}_k) \ + \ c(\mathfrak{su}(2)^{(1)}_{k'}) + 4 + \frac{4}{2} = 15 \\ &3\Big(\frac{k+2}{k} + \frac{1}{2}\Big) \ + \ 3\Big(\frac{k'-2}{k'} + \frac{1}{2}\Big) = 9 \quad \Longrightarrow \ k = k', \end{aligned} \tag{5.35}$$

where (5.30) was used in the second line. The above constraint now forces the level k to also be quantized: $k \in \mathbb{Z}_{>0}$. After decoupling the fermions, the supersymmetric algebra (5.34) reduces to the bosonic algebra with shifted levels $\kappa = k - h^{\vee}_{SL(2,\mathbb{R})}$ and $\kappa' = k - h^{\vee}_{SU(2)}$:

$$\mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus \mathfrak{su}(2)_{k-2} \oplus (4 \text{ bosons } + \text{ 10 fermions}), \qquad (5.36)$$

where we have used that the respective dual coxeter numbers are $h^{\vee}_{SL(2,\mathbb{R})} = -2$ and $h^{\vee}_{SU(2)} = 2$ in the shifted level. Also, both algebras are 3 dimensional so we get $3+3=6$ additional free fermions.

5.4 The tensionless limit

Recall that we are interested in the tensionless limit of the worldsheet theory. The implications of this limit for our model will be described in the following. For the WZW action (5.2) to describe a string theory, the level k must be proportional to the tension T of the string $k \sim 2\pi T$. In fact since k is dimensionless, k is interpreted to be the dimensionless analogue of the tension:

$$k \;=\; 2\pi T R^2 \;=\; \frac{R^2}{\alpha'} \qquad (5.37)$$

where R is the size of the group manifold and α' is the inverse tension. Taking the tensionless limit of the worldsheet theory described by the bosonic $SL(2,\mathbb{R}) \times SU(2)$ WZW model (5.36) comes with some subtleties. Since k is quantized and positive, the smallest tension in the superstring theory corresponds to the level $k = 1$. However, after decoupling the fermions, this corresponds to a shifted level on the bosonic $\mathfrak{su}(2)$ algebra: $\kappa' = 1 - 2 = -1$ which appears to be a somewhat singular feature of the theory; recall that this leads to divergences in the path integral. In addition, the central charge of the theory on the S^3 factor is

$$c(\mathfrak{su}(2)_{\kappa'=-1}) = 3\Big(\frac{1-2}{1} + \frac{1}{2}\Big) = -\frac{3}{2} \qquad (5.38)$$

which corresponds to a non-unitary theory. However, in [36], it was argued that sense can be made of this theory by observing that $\mathfrak{su}(2)$ at level $\kappa' = -1$ has a free field construction in terms of four symplectic bosons that effectively behave as fermionic ghosts [50][51]. In this interpretation, the degrees of freedom that are left match with that of a $\mathbb{T}^4$ theory. The arguments are summarized as follows. There are 3 decoupled bosons and fermions in the worldsheet theory on the $\mathfrak{sl}(2,\mathbb{R})_3$ factor. After imposing the super-Virasoro constraints, 2 bosons and 2 fermions are eliminated, leaving 1 boson and 1 fermion on $\mathfrak{sl}(2,\mathbb{R})_3$. The second factor is the decoupled $\mathfrak{su}(2)_{-1} \oplus$ 3 fermions algebra. In the free field construction, the algebra $\mathfrak{su}(2)_{-1} \oplus \mathfrak{u}(1)$ yields 4 symplectic bosons where each carry a central charge $c = -\frac{1}{2}$ [36]. The 1 remaining boson on $\mathfrak{sl}(2,\mathbb{R})_3$ can be combined with $\mathfrak{su}(2)_{-1}$ to produce the algebra $\mathfrak{su}(2)_{-1} \oplus \mathfrak{u}(1)$ which corresponds to 4 symplectic bosons. What remains are the 1+3 fermions, 4 symplectic bosons and the degrees of freedom on the $\mathbb{T}^4$ factor. Since the 4 symplectic bosons effectively behave as fermionic ghosts, they cancel the 4 fermions and all that remains are the degrees of freedom on $\mathbb{T}^4$. At this point in our analysis, counting the degrees of freedom already demonstrates that the dual CFT must be a $\mathbb{T}^4$ theory. The detailed free field construction can be seen in section 4.2 of [36].

5.5 The representation theory of $\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \oplus \mathfrak{su}(2)^{(1)}_k$

The worldsheet conformal field theory on AdS_3 is characterized by its spectrum together with its symmetry algebra, thus we aim to understand the representation theory of the $\mathfrak{sl}(2,\mathbb{R})^{(1)}_k$ algebra and the accompanying physical constraints. The representations for the bosonic $\mathfrak{sl}(2,\mathbb{R})_\kappa$ WZW model was studied in [15] and the arguments will apply to our case after decoupling from the fermions. The relevant representations that appeared in [15] are the principal continuous representations, which under spectral flow describe the long string states. The other consistent set of representations are the principal discrete representations which describe the short strings, however, for our application we will focus on the states that lie in the continuum. In particular, as we will soon demonstrate, the spectra generated by a special set of states that lay at the bottom of the continuum can be shown to match with the single-particle spectrum of the symmetric product orbifold theory.

Recall that the zero modes of the $\mathfrak{sl}(2,\mathbb{R})^{(1)}_k$ algebra are the conserved charges of the chiral current (5.11):

$$J^a_0 = \oint dz\, J^a(z) \tag{5.39}$$

These global modes act on the entire boundary of AdS_3 (which we take to be the sphere S^2) so it is natural to identify them with the global Möbius generators of the boundary CFT. Thus we have the holographic dictionary:

$$\begin{aligned} L_0^{\text{CFT}} &= J_0^3 \\ L_1^{\text{CFT}} &= J_0^- \\ L_{-1}^{\text{CFT}} &= J_0^+, \end{aligned} \tag{5.40}$$

where the L_m^{CFT} with $m \in \{-1, 0, 1\}$ are the global Möbius generators of the boundary CFT. With this identification, one can show that it indeed satisfies the Witt algebra: $[L_m^{\text{CFT}}, L_n^{\text{CFT}}] = (m-n)L_{m+n}^{\text{CFT}}$ using (5.39) and the OPE relations (5.10). There is now a point of contact between the superstring theory on AdS_3 and it's CFT dual. We would like to use this dictionary to organise the string spectrum in terms of the energy of the dual CFT and in this way the spectra can be matched precisely. As such, we are interested in the representation content of the coupled currents J^a to make use of this dictionary.

In the NS sector, label the ground states by $|j,m\rangle$ where m is the J_0^3 eigenvalue and j is the spin of the representation:

$$C_2 \, |j,m\rangle = -j(j-1) \, |j,m\rangle \, , \quad J_0^3 \, |j,m\rangle = m \, |j,m\rangle \, , \tag{5.41}$$

where $-j(j-1)$ is the Casimir of the $\mathfrak{sl}(2,\mathbb{R})$ spin j representation obtained from:

$$C_2 = \frac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+) - J_0^3 J_0^3. \tag{5.42}$$

In order for the state $|j,m\rangle$ to be the ground state, it must be annihilated by the positive modes:

$$J_n^a \, |j,m\rangle = 0 \;\; \forall \, n \geq 1 \quad \text{and} \quad \psi_r^a \, |j,m\rangle = 0 \;\; \forall \, r \geq \tfrac{1}{2}, \tag{5.43}$$

where the fermionic modes are half integrally moded. (5.43) implies a direct agreement with the decoupled current modes on the ground state: $J_n^a \, |j,m\rangle = \mathcal{J}_n^a \, |j,m\rangle \;\; \forall n \geq 0$ since the positive fermionic modes of $\mathcal{J}_n^a$ annihilate $|j,m\rangle$ (see (5.24)).

In the R sector, there are additional subtleties since the fermionic generators are integrally moded and have a zero-mode, thus the zero mode coupled and decoupled bosonic generators act on different ground states. The ground states of the coupled generators are now characterized by an additional irreducible 2 dimensional spinor representation of the clifford algebra in (2+1) dimensions. Recall the decoupled bosonic zero mode:

$$\mathcal{J}_0^3 = J_0^3 - \frac{1}{k}(\psi^+\psi^-)_0. \tag{5.44}$$

The ground states in the R sector are equipped with another quantum number $s_0 = \pm 1$. The fermionic part of (5.44) acts as

$$(\psi^+\psi^-)_0 \, |j,m;s_0\rangle = \frac{1}{2}[\psi_0^+,\psi_0^-] \, |j,m;s_0\rangle = k\frac{\sigma^3}{2} \, |j,m;s_0\rangle = k\frac{s_0}{2} \, |j,m;s_0\rangle \,, \tag{5.45}$$

where σ^3 is the usual Pauli matrix. The eigenvalue of the J_0^3 mode on the ground state in the R sector is

$$J_0^3 \, |j,m;s_0\rangle = \left(m + \frac{s_0}{2}\right) |j,m;s_0\rangle \,. \tag{5.46}$$

Thus the ground state in the R sector can be interpreted as sitting in a representation that has a spin that is $\pm\frac{1}{2}$ relative to that of the decoupled algebra.

Which representations do the zero modes J_0^a belong to? Since the zero modes of the coupled and decoupled currents agree on the ground state (up to a shift in spin), we can borrow the representation theory of the bosonic algebra. Maldacena and Ooguri studied the representations of the bosonic $\mathfrak{sl}(2,\mathbb{R})_\kappa$ algebras in detail in [15] and the results will be summarized below.

The representation theory of ordinary bosonic $\mathfrak{sl}(2,\mathbb{R})$ i.e. not its universal cover is well understood but the universal cover representations can be built from those of the the ordinary algebra since the zero modes $J_0^{3,\pm}$ of the universal cover algebra recovers the ordinary algebra. There are five unitary representations of the zero mode algebra: The principal discrete representations (lowest and highest weight), continuous representations, complementary representations and identity representations.

The existence of these representations is a purely algebraic feature of $\mathfrak{sl}(2,\mathbb{R})$, however if we aim to identify the physical spectrum, only a subset of them will be relevant. Maldacena and Ooguri [15] studied the full set and found that under geometric considerations, in the large level limit, there are only two classes of relevant representations that follow: the discrete series (lowest weight) representations and the continuous series representations. Together they form a complete basis of square integrable functions on AdS_3.

The lowest weight discrete series representations are in the Hilbert Space of:

$$\mathcal{D}_j^+ = \{|j,m\rangle : \; m = j, \; j+1, \; j+2, ...\}, \tag{5.47}$$

where $J_0^- \, |j,j\rangle = 0$ i.e. J_0^- annihilates the lowest lying state in the series and j is real and $j > 0$ for unitarity. These states are lowest weight representations with respect to $\mathfrak{sl}(2,\mathbb{R})$.

The principal continuous series representations are in the Hilbert space of

$$\mathcal{C}^{\alpha}_j : \{|j,\alpha;m\rangle : \ m = \alpha, \alpha \pm 1, \alpha \pm 2, ...\}, \tag{5.48}$$

where $0 \leq \alpha < 1$ and the representation is unitary if $j = \frac{1}{2} + ip$ with $p \in \mathbb{R}$. The continuous series is neither highest weight nor lowest weight with respect to $\mathfrak{sl}(2,\mathbb{R})$.

Given a unitary representation $\mathcal{R}$ of the zero mode algebra, one can construct the representations of the universal covering by taking $\mathcal{R}$ to be the set of primary states that are annihilated by the positive modes $J^{3,\pm}_n$. The full representation space is generated by acting on $\mathcal{R}$ with the negative modes J^a_{-n} and ψ^a_{-n}. Thus the full current algebra representations have eigenvalues of L_0 that are bounded from below.

There are additional representations generated by a process known as *spectral flow* which is an outer automorphism that acts on the currents in the following way:

$$\sigma^w(J^{\pm}_n) = J^{\pm}_{n\mp w} \tag{5.49}$$

$$\sigma^w(J^3_n) = J^3_n + \frac{kw}{2}\delta_{n,0} \tag{5.50}$$

$$\sigma^w(\psi^3_r) = \psi^3_r \tag{5.51}$$

$$\sigma^w(\psi^{\pm}_r) = \psi^{\pm}_{r\mp w} \tag{5.52}$$

$$\sigma^w(L_n) = L_n - wJ^3_n - \frac{k}{4}w^2\delta_{n,0} \tag{5.53}$$

where $w \in \mathbb{N}$ parameterizes the spectral flow. The spectrally flowed representations are characterized by using the same underlying vector space but letting the $\sigma^w(J^a_n)$ and $\sigma^w(\psi^a_n)$ act on it instead. Recall that the positive modes annihilate the ground states. This implies that

$$\sigma^w(J^-_0)\,|j,m\rangle = J^-_w\,|j,m\rangle = 0 \tag{5.54}$$

for $w > 0$. In other words, there is a zero mode of the spectrally flowed current that annihilates the ground state because it corresponds to a positive mode of the unspectrally flowed current. Note that because of (5.54), the spectrally flowed continuous representations become lowest weight with respect to $\mathfrak{sl}(2,\mathbb{R})$. Also, both the continuous and discrete representations are no longer highest weight with respect to L_0 after spectral flow, i.e. they are no longer bounded from below due to the second term in (5.53). The authors of [15] found that the long string states which were predicted to exist in AdS_3 [14] actually live in the spectrally flowed variation of the continuous representations.

Recall that the superstring theory on S^3 is described by a WZW model based on $\mathfrak{su}(2)_k^{(1)}$. Similar to the case with $\mathfrak{sl}(2,\mathbb{R})^{(1)_k}$, the ground states transform in the representations of the decoupled zero mode algebra i.e. of the $\mathfrak{su}(2)$ algebra. Let j' label the spins of the representations, with $j' = 0, \frac{1}{2}, 1, \frac{3}{2}...$ and the different states in the spin j' representations are labelled by the quantum number $m' = -j', -j'+1, ..., j'-1, j'$. The Casimir of the zero mode algebra is given by:

$$C_2 \, |j', m'\rangle_{S^3} = j'(j'+1) \, |j', m'\rangle_{S^3} \, . \tag{5.55}$$

As before, the coupled and decoupled modes agree on the ground states in the NS sector but in the R sector the spins are shifted by $\pm\frac{1}{2}$ due to the fermionic contributions.

Finally, the $\mathbb{T}^4$ theory corresponds to 4 free bosons Y^i and 4 free fermions χ^i with $i \in \{1,2,3,4\}$. The ground states are characterised by the momentum vector $|p\rangle$.

$$(\partial Y^i)_0 \, |p\rangle = p^i \, |p\rangle \quad \text{and} \quad L_0 \, |p\rangle = \frac{1}{2} \sum_{i=1}^{4} (p^i)^2 \, |p\rangle \, . \tag{5.56}$$

For the compact torus, the left and right moving momenta can differ from each other by winding numbers, but for our application the momentas will be taken to zero, as we will soon see.

5.6 The string spectrum

As usual, in any R-NS formulation of string theory, there exists tachyonic states that need to be removed so they do not appear in the spectrum. We use the convention that

$$(-1)^F \, |j, m\rangle_{\text{NS}} = -1 \, |j, m\rangle_{\text{NS}} \, , \tag{5.57}$$

on the ground state.

Denote the excitation numbers in the $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$ and $\mathbb{T}^4$ sectors as N, N' and N'' respectively. On some excited state, the worldsheet parity operator will be

$$(-1)^F = (-1)^{2(N+N'+N'')+1}. \tag{5.58}$$

When all excitations are zero, we recover the parity on the ground state (5.57). The factor of 2 is to ensure that the power is always an integer. Since the tachyon has parity $(-1)^F = -1$, we impose the GSO projection as an extra constraint, that is, all physical states in the NS sector should have parity $(-1)^F = (-1)^{\bar{F}} = +1$ thus requiring $F \in 2\mathbb{N}$. In the following we will be particularly interested in the spectrally flowed representations of $\mathfrak{sl}(2,\mathbb{R})$. Each unit of spectral flow changes the fermionic number of the ground state by one. Thus the GSO projection places the following constraint

$$\begin{aligned} F &= 2(N + N' + N'') + 1 + w \in 2\mathbb{N} \\ &\Rightarrow \tilde{N} + \frac{w+1}{2} \in \mathbb{N}, \end{aligned} \tag{5.59}$$

where $\tilde{N}$ is the total excitation number. Clearly, $\tilde{N}$ is an integer in the NS sector if w odd, while it is a half integer if w even.

In worldsheet string theory, the negative-norm states are removed to enforce a positive definite spectrum by imposing the mass-shell condition:

$$L_0^{tot} - \nu = \overline{L}_0^{tot} - \overline{\nu} = 0, \tag{5.60}$$

where $\nu, \overline{\nu} = 0, \frac{1}{2}$ in the R and NS sectors respectively and $L_0^{tot} = L_0^{\mathfrak{sl}(2,\mathbb{R})} + L_0^{\mathfrak{su}(2)} + L_0^{\mathbb{T}^4}$.

Let $|j,m\rangle$ be the ground state in the (unflowed) representation with spin j of either $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$. The ground state energy (the conformal weight of the highest weight state) of the representation can be computed using the Sugawara construction:

$$\begin{aligned} L_0 |j,m\rangle &= \frac{1}{2k}\left(\sum_{n\leq -1} J_n^a J_{-n}^a + \sum_{n\geq 0} J_{-n}^a J_n^a\right)|j,m\rangle \\ &= \frac{1}{2k} J_0^a J_0^a |j,m\rangle \\ &= \frac{C_2(j)}{k} |j,m\rangle\,. \end{aligned} \tag{5.61}$$

Indeed, from (5.18) we conclude that the energy of the excited state is given by:

$$L_0\left(J^{a_1}_{-n_1}...J^{a_m}_{-n_m}|j,m\rangle\right) = \left[\frac{C_2(j)}{k} + \sum_{p=1}^{m} n_p\right]\left(J^{a_1}_{-n_1}...J^{a_m}_{-n_m}|j,m\rangle\right). \tag{5.62}$$

Thus the energy is the conformal weight of the ground state plus the total number of mode oscillations applied . Note that the negative fermionic modes can also act on the ground state to create excitations.

Contributions from each component in (5.60) is parameterized in the following way:

$$L_0^{\mathfrak{sl}(2,\mathbb{R})} = -\frac{j(j-1)}{k} + N$$
$$L_0^{\mathfrak{su}(2)} = \frac{j'(j'+1)}{k} + N' \tag{5.63}$$
$$L_0^{\mathbb{T}^4} = h^{\mathbb{T}} + N'',$$

where $h^{\mathbb{T}^4}$ is the ground state energy on the $\mathbb{T}^4$. An analagous paramaterisation can be made for the right-movers. The mass shell condition (5.60), in the NS sector, reads

$$-\frac{j(j-1)}{k} - w\Big(m + \frac{k}{4}w\Big) + \tilde{N} + h_{\text{rest}} = \frac{1}{2}, \tag{5.64}$$

where $\tilde{N}$ denotes the total number of excitations and h_{rest} is the sum of the ground state energies on the $SU(2)$ and $\mathbb{T}^4$ factors. Note we have applied the spectral flow automorphism of (5.53) on the $\mathfrak{sl}(2,\mathbb{R})$ algebra hence the second term. Thus the eigenvalue m is defined to be the quantum number that is physically constrained by the mass-shell condition. Furthermore, we would like to make use of the holographic dictionary to find the physical spectrum of the string theory from the perspective of the dual CFT in terms of its conformal weight h, i.e. the eigenvalue of L_0^{CFT}:

$$h = m + \frac{k}{2}w, \tag{5.65}$$

where we have used the dictionary (5.40) and the spectrally flowed J_0^3 of (5.50). Restricting to the spectrally flowed continuous representations, the Casimir reads $-j(j-1) = \frac{1}{4} + p^2$. Substituting (5.65) into the the mass shell condition we get

$$\frac{\frac{1}{4} + p^2}{k} - w\Big(h - \frac{k}{4}w\Big) + \tilde{N} + h_{\text{rest}} = \frac{1}{2}. \tag{5.66}$$

Taking the tensionless limit $k = 1$ the spectrum simplifies to

$$h = \frac{\tilde{N}}{w} + \frac{h_{\text{rest}} + p^2}{w} + \frac{w^2 - 1}{4w}. \tag{5.67}$$

An identical analysis can be done for the R sector i.e. when $\nu = 0$. In the R sector the spectrum is

$$h = \frac{\tilde{N}}{w} + \frac{h_{\text{rest}} + p^2}{w} + \frac{w^2 + 1}{4w}. \tag{5.68}$$

The partition function (subject to the GSO constraint (5.59)) was calculated in [36] and it was found that the free fermions were found to sit in the NS sector for w odd and in the R sector for w even (see eqn's (2.19) and (2.21) of [36]). Thus in the case for w odd, a special set of states in the spectrum exist at the bottom of the continuum $j = \frac{1}{2}$ $(p = 0)$ after setting the momentum and winding modes of the circle theory to zero i.e $h_{\text{rest}} = 0$ in (5.67). The spectrum of this special subsector is

$$h = \frac{\tilde{N}}{w} + \frac{w^2 - 1}{4w}, \tag{5.69}$$

which is precisely the spectrum of the w-cycle (with w odd) twisted sector of the $(\mathbb{T}^4)^N/S_N$ theory (see (4.33) and the surrounding discussion) once we identify the spectral flow parameter with the length of the single cycle permutation.

For w even, each symplectic boson lowers the R sector energy by $\delta h = -\frac{1}{16}$ and since we have 4 symplectic bosons we arrive at $h_{\text{rest}} = -\frac{1}{4}$ in (5.68). Thus for $j = \frac{1}{2}$ the spectrum reads

$$h = \frac{\tilde{N}}{w} + \frac{w}{4}, \tag{5.70}$$

which is the spectrum of the w-cycle (with w even) twisted sector of the $(\mathbb{T}^4)^N/S_N$ theory (see (4.34).

5.7 The $j = \frac{1}{2}$ representations

In the previous section, the spectrum of the symmetric product orbifold of $\mathbb{T}^4$ was found at the bottom of the continuum, i.e. in the spectrally flowed $j = \frac{1}{2}$ continuous representations. To see why these representations are the relevant ones, we introduce an alternative worldsheet description on $AdS_3 \times S^3 \times \mathbb{T}^4$ which is based on the so-called hybrid formalism of Berkovitz, Vafa and Witten [38]. In this description, the tensionless limit is well defined and the representations are tightly constrained.

The formalism employed in the previous section is known as the RNS (Ramond Neveu-Schwarz) formulation of string theory, where powerful methods of conformal field theory were available. The shortcoming however is that spacetime supersymmetry is not visible as a classical symmetry of the worldsheet action, rather, there is an explicit superconformal invariance on the worldsheet. On the other hand, there is a covariant formulation of string theory, introduced by Green and Schwarz [52], where the spacetime supersymmetry is made manifest, however, quantization in this regime is quite difficult except in the light-cone gauge.

It is possible to reformulate the RNS description in terms of the Green-Schwarz like variables in a way that makes manifest a portion of the spacetime symmetry group in either 4 or 6 dimensions of the 10 dimensional spacetime while the other dimensions are described in terms of the RNS formulation, see for example [53] and [54].

The original motivation for developing the hybrid formulation is due to the complexity that arises from the presence of the bosonic fields in the Ramond sector of the RNS formalism, which are represented by spin fields [55]. It is very difficult to describe backgrounds when RR fields are present in the vacuum; the worldsheet supercurrents are not local with respect to the RR field thus violating superconformal symmetry. In addition, it is not clear what is considered a satisfactory worldsheet action with RR fields. The covariance approach described above can alleviate some of these issues; if the RR fields only live in the 4 or 6 spacetime dimensions as in the constructions of [53] or [54] where the spacetime supersymmetry is explicit, then their vertex operators are ordinary untwisted worldsheet operators and can be added to the Lagrangian.

In [38], a conformal field theory description of the $AdS_3 \times S^3 \times \mathbb{T}^4$ background with RR fields was developed using the covariant Green-Schwarz like methods. The Type IIB string theory on $AdS_3 \times S^3$ is a sigma model with target space the supergroup $PSU(1,1|2)$ which depends on two parameters that correspond to the RR and NS fluxes. In this CFT description, RR flux can be added to the background in a straightforward way.

For our application, we will only consider the case with pure NS-NS flux. In this set-up, the sigma model on $PSU(1,1|2)$ becomes a WZW model at level k where k corresponds to the amount of NS-NS flux in the background. It was shown that there is an equivalence between this hybrid description and the RNS description for the low lying states [56] [57] [58]. The motivation for using the hybrid description is because the theory is well defined at $k = 1$; recall that after decoupling from the fermions, there was a negative level on the $\mathfrak{su}(2)$ factor at $k = 1$ (see section 5.4).

5.7.1 Representations of $\mathfrak{psu}(1,1|2)_{k=1}$

The representation theory for the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R})_k \oplus \mathfrak{su}(2)_k \subset \mathfrak{psu}(1,1|2)_k$ was outlined in the previous section, and will be directly applied in the following. In addition, there are 8 fermionic generators, corresponding to 4 creators and 4 annihilators, that transform in the bi-spinor representation of the bosonic subalgebra. The full (anti)commutation relations amongst the generators is listed in the appendix.

In the following, we will focus on the continuous representations of $\mathfrak{sl}(2,\mathbb{R})$ since the discrete series form subrepresentations. Let the highest weight state transform in the representation $(\mathcal{C}^j_\alpha, \mathbf{n})$ where $\mathcal{C}^j_\alpha$ is, as before, the continous representation series at spin j of the zero mode algebra $\mathfrak{sl}(2,\mathbb{R})$ (see section 5.5) and $\mathbf{n}$ is the dimensionality of the $\mathfrak{su}(2)$ algebra. 4 of the supercharges annihilate the states in $(\mathcal{C}^j_\alpha, \mathbf{n})$ while the other 4 act on the highest weight states to generate a Clifford module. The multiplet takes the form

$$\begin{array}{ccccc}
 & & (\mathcal{C}^j_\alpha, \mathbf{n}) & & \\
(\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n}+\mathbf{1}) \quad (\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n}-\mathbf{1}) & & & & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n}+\mathbf{1}) \quad (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n}-\mathbf{1}) \\
(\mathcal{C}^{j+1}_\alpha, \mathbf{n}) & (\mathcal{C}^j_\alpha, \mathbf{n}+\mathbf{2}) & 2\cdot(\mathcal{C}^j_\alpha, \mathbf{n}) & (\mathcal{C}^j_\alpha, \mathbf{n}-\mathbf{2}) & (\mathcal{C}^{j-1}_\alpha, \mathbf{n}) \\
(\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n}+\mathbf{1}) \quad (\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n}-\mathbf{1}) & & & & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n}+\mathbf{1}) \quad (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n}-\mathbf{1}) \\
 & & (\mathcal{C}^j_\alpha, \mathbf{n}) & &
\end{array} \tag{5.71}$$

where the states at the top of the module are the highest weight states and the action of the supercharges moves between the bosonic representations. These are called the long representations. If we only consider the cases $\mathbf{n} = \mathbf{1}$ and $\mathbf{n} = \mathbf{2}$, some shortenings occur, for example if $\mathbf{n} = 2$ then the representations with $\mathbf{n} - \mathbf{2}$ vanish:

$$\begin{array}{ccccc}
 & & (\mathcal{C}^j_\alpha, \mathbf{2}) & & \\
(\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{3}) & (\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{3}) & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1}) \\
(\mathcal{C}^{j+1}_\alpha, \mathbf{2}) & (\mathcal{C}^j_\alpha, \mathbf{4}) & 2\cdot(\mathcal{C}^j_\alpha, \mathbf{2}) & & (\mathcal{C}^{j-1}_\alpha, \mathbf{2}) \\
(\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{3}) & (\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{3}) & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1}) \\
 & & (\mathcal{C}^j_\alpha, \mathbf{2}) & &
\end{array} \tag{5.72}$$

When $\mathbf{n} = 1$, more shortenings occur;

$$\begin{array}{ccccc} & & (\mathcal{C}^j_\alpha, \mathbf{1}) & & \\ & (\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{2}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{2}) & \\ (\mathcal{C}^{j+1}_\alpha, \mathbf{1}) & (\mathcal{C}^j_\alpha, \mathbf{3}) & & (\mathcal{C}^j_\alpha, \mathbf{1}) & (\mathcal{C}^{j-1}_\alpha, \mathbf{1}) \\ & (\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{2}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{2}) & \\ & & (\mathcal{C}^j_\alpha, \mathbf{1}) & & \end{array} \tag{5.73}$$

Note that these representations also exist for the discrete series by replacing $\mathcal{C}^j_\alpha$ with $\mathcal{D}^j_\pm$.

Since we are interested in the affine algebra $\mathfrak{psu}(1,1|2)_{k=1}$, this leads to the affine subalgebra $\mathfrak{su}(2)_{k=1}$. However restricting to $k = 1$ imposes limitations on the available representations of $\mathfrak{su}(2)$, namely only the representations of $\mathbf{n} = 1$ and $\mathbf{n} = 2$ can occur [37]. In the case of (5.72), the multiplet reduces to the following shortened representation:

$$\begin{array}{ccc} & (\mathcal{C}^j_\alpha, \mathbf{2}) & \\ (\mathcal{C}^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1}) & & (\mathcal{C}^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1}) \end{array} \tag{5.74}$$

where we took that the two representations with $\mathbf{n} = 3$ in the second line of 5.72 are null which eliminates all other representations below in the multiplet. Reducing 5.73 in the same way leads also to (5.74) after redefining $j \to j \pm \frac{1}{2}$.

Under what conditions does this shortening occur? For the discrete series case, the algebra has the BPS bound $h = j \geq j'$ where j' is the spin of the $\mathfrak{su}(2)$ representation. The BPS states are the representations that saturate this bound i.e. the ones with $j = j'$. Since the dimension of the $\mathfrak{su}(2)$ representation is $\mathbf{n} = 2j' + 1$ and because of the restriction of $\mathbf{n} = 2$ then the shortening occurs for $j = \frac{1}{2}$. For the case $\mathbf{n} = 1$ the shortening condition is still $j = \frac{1}{2}$ because of the shift in j previously discussed.

Since there is no highest weight state in the continuous representations of $\mathfrak{sl}(2,\mathbb{R})$, we compute the shortening condition through a different route that involves the Casimir. For the following, it will be useful to state the anti-commutation relations of the fermionic generators, $S^{\alpha\beta\gamma}_m$:

$$\{S^{\alpha\beta\gamma}_m, S^{\mu\nu\rho}_n\} = km\epsilon^{\alpha\mu}\epsilon^{\beta\nu}\epsilon^{\gamma\rho} - \epsilon^{\beta\nu}\epsilon^{\gamma\rho}c_a\sigma^{\alpha\mu}_a J^a_{m+n} + \epsilon^{\alpha\mu}\epsilon^{\gamma\rho}\sigma^{\beta\nu}_a K^a_{m+n}, \tag{5.75}$$

where J^a_m and K^a_m are the $\mathfrak{sl}(2,\mathbb{R})_k$ and $\mathfrak{su}(2)_k$ generators respectively and α, β... are the spinor indices that take values in $\{+,-\}$. The third spinor index of the fermionic generators encode the transformation properties under the outer automorphism $\mathfrak{su}(2)$ of $\mathfrak{psu}(1,1|2)$. The adjoint index a is raised and lowered by the $\mathfrak{su}(2)$ invariant form

$$\eta_{+-} = \eta_{-+} = \tfrac{1}{2}, \quad \eta_{33} = 1. \tag{5.76}$$

The c_a is a constant and takes on values -1 for $a = -$, and $+1$ for $a = +, 3$. The σ matrices are given by

$$(\sigma^-)^+{}_- = 2, \quad (\sigma^3)^-{}_- = -1, \quad (\sigma^3)^+{}_+ = 1, \quad (\sigma^+)^-{}_+ = 2, \tag{5.77}$$

$$(\sigma_-)^{--} = 1, \quad (\sigma_3)^{-+} = 1, \quad (\sigma_3)^{+-} = 1, \quad (\sigma_+)^{++} = -1. \tag{5.78}$$

See, for example, chapter 3 of [36] for the full $\mathfrak{psu}(1,1|2)_k$ algebra, though it will not be needed in the following.

The Casimir for $\mathfrak{psu}(1,1|2)$ decomposes into its bosonic and fermionic components:

$$\begin{aligned} C_2^{\mathfrak{psu}(1,1|2)} &= C_{2(\text{bos})}^{\mathfrak{psu}(1,1|2)} + C_{2(\text{ferm})}^{\mathfrak{psu}(1,1|2)} \\ C_{2(\text{bos})}^{\mathfrak{psu}(1,1|2)} &= C_2^{\mathfrak{sl}(2,\mathbb{R})} + C_2^{\mathfrak{su}(2)} \\ C_{2(\text{ferm})}^{\mathfrak{psu}(1,1|2)} &= -\frac{1}{2}\epsilon_{\lambda\mu}\epsilon_{\beta\nu}\epsilon_{\gamma\rho}S_0^{\lambda\beta\gamma}S_0^{\mu\nu\rho}. \end{aligned} \tag{5.79}$$

We can compute the Casimir on the highest weight states $(\mathcal{C}_\alpha^j, \mathbf{2})$ which we denote by $|m,\uparrow\rangle$, where m labels the state in the continuous series of $\mathfrak{sl}(2,\mathbb{R})$ and $\uparrow$ denotes the state in the 2 dimensional ($j' = \frac{1}{2}$) $\mathfrak{su}(2)$ representation:

$$\begin{aligned} C_{2(\text{ferm})}^{\mathfrak{psu}(1,1|2)} |m,\uparrow\rangle &= -\frac{1}{2}\epsilon_{\lambda\mu}\epsilon_{\beta\nu}\epsilon_{\gamma\rho}S_0^{\lambda\beta\gamma}S_0^{\mu\nu\rho}|m,\uparrow\rangle \\ &= -\frac{1}{2}\epsilon_{\lambda\mu}\epsilon_{\gamma\rho}\{S_0^{\lambda+\gamma}, S_0^{\mu-\rho}\}|m,\uparrow\rangle \\ &= -\frac{1}{2}\epsilon_{\lambda\mu}\epsilon_{\gamma\rho}(-\epsilon^{\gamma\rho}c_a\sigma_a^{\lambda\mu}J_0^a + \epsilon^{\lambda\mu}\epsilon^{\gamma\rho}\sigma_a^{+-}K_0^a)|m,\uparrow\rangle \\ &= -2K_0^3|m,\uparrow\rangle = -|m,\uparrow\rangle. \end{aligned} \tag{5.80}$$

So we have

$$C_{2(\text{ferm})}^{\mathfrak{psu}(1,1|2)}(\mathcal{C}_\alpha^j, \mathbf{2}) = -1, \quad C_{2(\text{ferm})}^{\mathfrak{psu}(1,1|2)}(\mathcal{C}_\alpha^j, \mathbf{1}) = 0, \tag{5.81}$$

where the second equality is obtained by a similar computation. The Casimir $C_2^{\mathfrak{psu}(1,1|2)}$ must be the same on all representations in the multiplet.

We have the condition:

$$\begin{aligned} &\big(C_2^{\mathfrak{sl}(2,\mathbb{R})} + C_2^{\mathfrak{su}(2)} + C_{2(\text{ferm})}^{\mathfrak{psu}(1,1|2)}\big)(\mathcal{C}_\alpha^j, \mathbf{2}) = \big(C_2^{\mathfrak{sl}(2,\mathbb{R})} + C_2^{\mathfrak{su}(2)} + C_{2(\text{ferm})}^{\mathfrak{psu}(1,1|2)}\big)(\mathcal{C}_{\alpha+\frac{1}{2}}^{j\pm\frac{1}{2}}, \mathbf{1}) \\ &-j(j-1) + \tfrac{1}{2}(\tfrac{1}{2}+1) - 1 = -(j \pm \tfrac{1}{2})(j \pm \tfrac{1}{2} - 1) \qquad \Rightarrow j = \tfrac{1}{2} \end{aligned} \tag{5.82}$$

Thus the shortening condition for the continuous representations occurs for spin $j = \frac{1}{2}$! In fact, this is true for the all the spectrally flowed images as well. This means that there is in fact *no continuum* at the tensionless limit $k = 1$; there are only the spin $j = \frac{1}{2}$ representations of $\mathfrak{sl}(2, \mathbb{R})$.

Chapter 6

Type IIB String Theory on $AdS_3 \times S^3 \times \mathbb{T}^4$: Correlation Functions

In the previous chapter, the correspondence between the string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ at the tensionless point and the symmetric product orbifold of $\mathbb{T}^4$ was confirmed by showing that the respective energy spectra were equivalent. In this chapter, the worldsheet analysis is completed by evaluating the form of the correlation functions of the vertex operators on the worldsheet, which is taken to be the sphere. In particular, a focus is placed on vertex operators that create ground states in the w spectrally flowed sector. A Ward identity analysis of the spectrally flowed primaries of the (bosonic) $SL(2,\mathbb{R})_{k+2}$ model is carried out and a set of recursion relations that must be satisfied by the correlators are found. An ansatz that solves these recursion relations is provided, and it typically requires interpreting the worldsheet as a covering space over the boundary sphere. An argument is made that the analysis generalizes to correlators on higher genus worldsheets. After integrating over moduli space, the vertex correlator shares all the same features as the twist correlator of the dual CFT calculated in chapter 3, thus providing further support for the duality. Furthermore, the proposed solution illuminates the mechanism that forces both sides of (1.1) to be equal to one another.

6.1 The basic worldsheet CFT set up

In this section, the OPE relations of the spectrally flowed currents with the vertex operators of the (decoupled) bosonic $SL(2,\mathbb{R})_{k+2}$ model are studied. These are the fundamental building blocks for carrying out the Ward identity analysis in the following section. In particular, a focus is placed on the vertex operators that create the physical ground state $|j,m\rangle^{(w)}$ in the w-spectrally flowed sector, since the detailed matching of the spectrum suggests a correspondence with the w-cycle twisted sector of the dual orbifold theory. As such, the contributions from the $S^3 \times \mathbb{T}^4$ factors will not be needed since the ground state does not receive contributions from these factors. This $\mathfrak{sl}(2,\mathbb{R})$ vertex operator is denoted by $V^w_h(x;z)$ to differentiate it from the full string theory vertex operator $\mathcal{V}^w_h(x;z)$.

The vertex operator has a (boundary) spacetime dependence denoted by the co-ordinate x while the co-ordinate z denotes the location of it's insertion on the worldsheet. We would like to know how these vertex operators transform under the action of the global Möbius generators of the spacetime CFT. Recall the action of the Virasoro modes on primary fields,

$$\begin{aligned}\left[L^{\text{CFT}}_n, V^w_h(x;z)\right] &= \oint \frac{dy}{2\pi i} y^{n+1} T^{\text{CFT}}(y) V^w_h(x;z) \\ &= h(n+1)x^n V^w_h(x;z) + x^{n+1}\frac{\partial}{\partial x} V^w_h(x;z),\end{aligned} \tag{6.1}$$

where h is the conformal dimension of the state that the vertex operator creates, from the perspective of the spacetime CFT (see (5.65)). Restricting to the global modes corresponding to $n \in \{-1,0,1\}$, the holographic dictionary (5.40) can be used to suggest the commutation relations,

$$[J^a_0, V^w_h(x;z)] = -\mathcal{D}^a V^w_h(x;z), \tag{6.2}$$

where the $\mathcal{D}^a$ are the differential operators

$$\mathcal{D}^+ = -\frac{\partial}{\partial x}, \quad \mathcal{D}^3 = -h - x\frac{\partial}{\partial x}, \quad \mathcal{D}^- = -2hx - x^2\frac{\partial}{\partial x}. \tag{6.3}$$

Note that the notation $\mathcal{J}^a$ for the decoupled bosonic currents of the decoupled affine bosonic $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ WZW model is replaced with J^a.

Recall that the mode $L^{\text{CFT}}_{-1} = -\partial_x$ is the momentum operator, so the global mode J^+_0 is the generator of translations on the boundary. Likewise, the L_{-1} Virasoro mode generates translations on the worldsheet. The vertex operators can be translated in both spaces by conjugation with these two generators,

$$V^w_h(x+y;z+\zeta) = e^{yJ^+_0} e^{\zeta L_{-1}} V^w_h(x;z) e^{-\zeta L_{-1}} e^{-yJ^+_0}, \tag{6.4}$$

where there is no ordering ambiguity since the two generators commute, recalling (5.18);

$$[L_m, J^a_n] = -nJ^a_{m+n} \quad \rightarrow \quad [L_{-1}, J^+_0] = 0. \tag{6.5}$$

From the worldsheet perspective, x can be thought of as the different states in a given $\mathfrak{sl}(2,\mathbb{R})$ representation since the $\mathfrak{sl}(2,\mathbb{R})$ modes change it's value. This viewpoint is referred to as the 'x-basis'. The Casimir of this representation is given by:

$$C = -(\mathcal{D}^3)^2 + \tfrac{1}{2}(\mathcal{D}^+\mathcal{D}^- + \mathcal{D}^-\mathcal{D}^+) = -h(h-1). \tag{6.6}$$

For the Ward identity analysis that follows in section 6.2, it is important to understand the OPE structure of the vertex operators with the affine $\mathfrak{sl}(2,\mathbb{R})$ currents. The fields and states are identified at $(x;z) = (0;0)$ and on these states, the spectral flow automorphism acts on the $\mathfrak{sl}(2,\mathbb{R})$ algebra as defined in (5.49) and (5.50), except with shifted level $k \rightarrow k+2$.

In the unflowed representations, the positive J^a_n modes annihilate the ground states while the negative modes produce excitations, however after spectral flow, the situation is different. For example, after spectral flow there are w many J^+_n positive modes that do not annihilate the ground state:

$$J^+_n V^w_h(0;0)\,|0\rangle = J^+_n\,|j,m\rangle^{(w)} = 0 \quad \text{for } n > w, \tag{6.7}$$

due to the shift in mode number after the automorphism (5.49): recall that the spectrally flowed modes are acting on the original vector space. On the other hand there are $w-1$ many J^-_n negative modes that do annhilate this state:

$$J^-_n V^w_h(0;0)\,|0\rangle = J^-_n\,|j,m\rangle^{(w)} = 0 \quad \text{for } n > -w, \tag{6.8}$$

again, due to (5.49). Finally, the positive J^3_n modes still annihilate the ground state however it's zero mode gets an additional term:

$$J^3_0 V^w_h(0;0)\,|0\rangle = J^3_0\,|j,m\rangle^{(w)} = \big(m + \tfrac{(k+2)w}{2}\big)\,|j,m\rangle^{(w)}, \tag{6.9}$$

due to (5.50).

The OPEs of the currents with the spectrally flowed vertex operators can be evaluated by taking the mode expansions. Consider first the OPE with $J^+(z)$,

$$
\begin{aligned}
J^+(z)V_h^w(0;0) &= \sum_{n=-\infty}^{\infty} \frac{(J_n^+ V_h^w)(0;0)}{z^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{(J_n^+ V_h^w)(0;0)}{z^{n+1}} + \frac{(J_0^+ V_h^w)(0;0)}{z} + \sum_{m=1}^{\infty} \frac{(J_{-m}^+ V_h^w)(0;0)}{z^{-m+1}} \\
&= \sum_{n=1}^{w} \frac{(J_n^+ V_h^w)(0;0)}{z^{n+1}} + \frac{[J_0^+, V_h^w](0;0)}{z} + \text{regular terms} \\
&\sim \sum_{p=2}^{w+1} \frac{(J_{p-1}^+ V_h^w)(0;0)}{z^p} + \frac{\partial_x V_h^w(0;0)}{z},
\end{aligned}
\tag{6.10}
$$

where in going to the third line, it was used that the operators will fit inside a correlator thus allowing to write the zero mode action as the commutator (6.2) (since $J_0^+ |0\rangle = 0$). Also, in the third line (6.7) was used to fix the upper limit on the first term to w, giving a pole of order $w+1$. Now, the OPE with $J^3(z)$,

$$
\begin{aligned}
J^3(z)V_h^w(0;0) &= \sum_{n=-\infty}^{\infty} \frac{(J_n^3 V_h^w)(0;0)}{z^{n+1}} \\
&= \frac{[J_0^3, V_h^w](0;0)}{z} + \text{regular terms} \\
&\sim \frac{hV_h^w(0;0)}{z},
\end{aligned}
\tag{6.11}
$$

where again, (6.2) was used in going to the third line. Finally, the $J^-(z)$ OPE is given by

$$
J^-(z)V_h^w(0;0) \sim \mathcal{O}(z^{w-1}), \tag{6.12}
$$

where (6.8) was used to show that the first $w-1$ regular terms vanish.

The above relations specify the OPEs for states inserted at $(0;0)$. The OPEs can be generalized to $(x;z)$ by applying the translation operation of (6.4),

$$
\begin{aligned}
J^a(\zeta)V_h^w(x;z) &= J^a(\zeta)e^{xJ_0^+}V_h^w(0;z)e^{-xJ_0^+} \\
&= e^{xJ_0^+}\left[J^{a(x)}(\zeta)V_h^w(0;z)\right]e^{-xJ_0^+},
\end{aligned}
\tag{6.13}
$$

where in the second line the currents obtain an x dependence on a after commuting with $e^{xJ_0^+}$. Before calculating each x dependence explicitly, it will be useful to recall the commutation relations of the $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ Kac-Moody algebra in the $a \in \pm, 3$ basis,

$$\begin{aligned} [J^+_m, J^-_n] &= -2J^3_{m+n} + (k+2)m\delta_{m+n,0} \\ [J^3_m, J^\pm_n] &= \pm J^\pm_{m+n}. \end{aligned} \tag{6.14}$$

The x dependence of the $J^{+(x)}$ current is given by

$$\begin{aligned} J^+(\zeta)e^{xJ_0^+}V_h^w(0;z)e^{-xJ_0^+} &= [J^+(\zeta), e^{xJ_0^+}]V_h^w(0;z)e^{-xJ_0^+} + e^{xJ_0^+}J^+(\zeta)V_h^w(0;z)e^{-xJ_0^+} \\ &= \sum_{n\in\mathbb{Z}} \frac{[J_n^+, e^{xJ_0^+}]}{\zeta^{n+1}} V_h^w(0;z)e^{-J_0^+} + e^{xJ_0^+}J^+(\zeta)V_h^w(0;z)e^{-xJ_0^+} \\ &= \sum_{n\in\mathbb{Z}} \frac{[J_n^+, J_0^+]}{\zeta^{n+1}} xe^{xJ_0^+}V_h^w(0;z)e^{-J_0^+} + e^{xJ_0^+}J^+(\zeta)V_h^w(0;z)e^{-xJ_0^+} \\ &= e^{xJ_0^+}J^+(\zeta)V_h^w(0;z)e^{-xJ_0^+}, \end{aligned} \tag{6.15}$$

where the commutator in the third line vanishes. Clearly the x dependence is trivial. The x dependence of the $J^{3(x)}$ current is

$$\begin{aligned} J^3(\zeta)e^{xJ_0^+}V_h^w(0;z)e^{-xJ_0^+} &= [J^3(\zeta), e^{xJ_0^+}]V_h^w(0;z)e^{-xJ_0^+} + e^{xJ_0^+}J^3(\zeta)V_h^w(0;z)e^{-xJ_0^+} \\ &= \sum_{n\in\mathbb{Z}} \frac{[J_n^3, e^{xJ_0^+}]}{\zeta^{n+1}} V_h^w(0;z)e^{-J_0^+} + e^{xJ_0^+}J^3(\zeta)V_h^w(0;z)e^{-xJ_0^+} \\ &= \sum_{n\in\mathbb{Z}} \frac{[J_n^3, J_0^+]}{\zeta^{n+1}} xe^{xJ_0^+}V_h^w(0;z)e^{-J_0^+} + e^{xJ_0^+}J^3(\zeta)V_h^w(0;z)e^{-xJ_0^+} \\ &= \sum_{n\in\mathbb{Z}} x\frac{[J_n^+, e^{xJ_0^+}]}{\zeta^{n+1}} V_h^w(0;z)e^{-J_0^+} + e^{xJ_0^+}\Big(J^3(\zeta) + xJ^+(\zeta)\Big)V_h^w(0;z)e^{-xJ_0^+} \\ &= e^{xJ_0^+}\Big(J^3(\zeta) + xJ^+(\zeta)\Big)V_h^w(0;z)e^{-xJ_0^+}, \end{aligned} \tag{6.16}$$

and finally, the $J^{-(x)}$ current,

$$\begin{aligned} J^-(\zeta)e^{xJ_0^+}V_h^w(0;z)e^{-xJ_0^+} &= [J^-(\zeta), e^{xJ_0^+}]V_h^w(0;z)e^{-xJ_0^+} + e^{xJ_0^+}J^-(\zeta)V_h^w(0;z)e^{-xJ_0^+} \\ &= \sum_{n\in\mathbb{Z}} \frac{[J_n^-, e^{xJ_0^+}]}{\zeta^{n+1}} V_h^w(0;z)e^{-J_0^+} + e^{xJ_0^+}J^-(\zeta)V_h^w(0;z)e^{-xJ_0^+}. \end{aligned} \tag{6.17}$$

It is worth treating the commutator in the second line separately; the identity $[A, e^B] = \int_0^1 ds\, e^{(1-s)B}[A, B]e^{sB}$ will be used to solve it:

$$\begin{aligned}
[J_n^-, e^{xJ_0^+}] &= \int_0^1 ds\, e^{(1-s)xJ_0^+} x[J_n^-, J_0^+] e^{sxJ_0^+} \\
&= \int_0^1 ds\, e^{(1-s)xJ_0^+}\, 2xJ_n^3\, e^{sxJ_0^+} \\
&= e^{xJ_0^+} 2xJ_n^3 + \int_0^1 ds\, e^{(1-s)xJ_0^+}\, 2x\, [J_n^3, e^{sxJ_0^+}] \\
&= e^{xJ_0^+} 2xJ_n^3 + \int_0^1 ds\, e^{(1-s)xJ_0^+}\, 2sx^2 [J_n^+, e^{sxJ_0^+}] \\
&\quad + e^{xJ_0^+} 2x^2 J_n^+ \int_0^1 ds\, s \\
&= e^{xJ_0^+}\left(2xJ_n^3 + x^2 J_n^+\right).
\end{aligned} \tag{6.18}$$

Substituting (6.18) back into (6.17), the x dependence of $J^{-(x)}$ is

$$J^-(\zeta) e^{xJ_0^+} V_h^w(0; z) e^{-xJ_0^+} = e^{xJ_0^+} \Big(J^-(\zeta) + 2xJ^3(\zeta) + x^2 J^+(\zeta) \Big) V_h^w(0; z) e^{-xJ_0^+}. \tag{6.19}$$

Putting everything together, the x dependence of each current can be summarized as

$$J^{+(x)}(z) = J^+(z) \tag{6.20a}$$

$$J^{3(x)}(z) = J^3(z) + xJ^+(z) \tag{6.20b}$$

$$J^{-(x)}(z) = J^-(z) + 2xJ^3(z) + x^2 J^+(z). \tag{6.20c}$$

Finally, note that the the vertex operators also depend on the spin j of the representation but is suppressed in the notation. To see this, first recall the zero modes before spectral flow $\tilde{J}_0^a$,

$$\tilde{J}_0^+ = J_w^+, \quad \tilde{J}_0^3 = J_0^3 - \tfrac{(k+2)w}{2}, \quad \tilde{J}_0^- = J_{-w}^-, \tag{6.21}$$

form a zero-mode algebra. These modes act on highest weight representations (before spectral flow) with spin j. We will work with the conventions that

$$J_w^+ \left|j, m\right\rangle^{(w)} = (m + j) \left|j, m + 1\right\rangle^{(w)} \tag{6.22a}$$

$$\left(J_0^3 - \tfrac{(k+2)w}{2}\right) \left|j, m\right\rangle^{(w)} = m \left|j, m\right\rangle^{(w)} \tag{6.22b}$$

$$J_{-w}^- \left|j, m\right\rangle^{(w)} = (m - j) \left|j, m - 1\right\rangle^{(w)}. \tag{6.22c}$$

Using the holographic dictionary and relations (6.21), the eigenvalue m is related to the conformal weight h by

$$h = m + \tfrac{(k+2)w}{2}. \tag{6.23}$$

Finally, the worldsheet conformal dimension takes the form

$$\Delta = -\frac{j(j-1)}{k} - wh + \frac{(k+2)w^2}{4}, \tag{6.24}$$

though it will not play a significant role in the remainder of the paper.

6.2 The Ward Identities

In this section, the aim is to determine the Ward identities for the n-point correlator of the vertex operators treated in section 6.2,

$$\langle V^{w_1}_{h_1}(x_1; z_1) V^{w_2}_{h_2}(x_2; z_2) ... V^{w_n}_{h_n}(x_n; z_n) \rangle, \tag{6.25}$$

which are calculated on the sphere. If all w_i's are set to zero, the Ward identities with the J^a currents are well known:

$$\left\langle J^a(z) \prod_{i=1}^{n} V^0_{h_i}(x_i; z_i) \right\rangle = -\sum_{i=1}^{n} \frac{\mathcal{D}^a_i}{z - z_i} \left\langle J^a(z) \prod_{j=1}^{n} V^0_{h_j}(x_j; z_j) \right\rangle, \tag{6.26}$$

where all $V^0_{h_i}(x_i; z_i)$ are highest weight operators. Since there is no spectral flow, the OPEs have only a first order pole. To see why, consider the following OPE

$$\begin{aligned} J^a(z) V^0_h(0;0) &= \sum_{n=-\infty}^{\infty} \frac{(J^a_n V^0_h)(0;0)}{z^{n+1}} \\ &= \frac{\left[J^a_0, V^0_h\right](0;0)}{z} + \text{regular terms} \\ &\sim -\frac{\mathcal{D}^a V^0_h(0;0)}{z}, \end{aligned} \tag{6.27}$$

where all positive modes annihilate the primary state. But as was outlined in section section 6.1, once spectral flow is switched on, this is no longer the case. A generalisation of (6.26) to the case where the w_i's are switched on is required. The idea is that the behaviour of the OPEs for the spectrally flowed case (6.10) (6.11) (6.12) will effectively constrain the form that the correlator (6.25) can take so that it satisfies the generalised Ward identities. Using these additional constraints, a solution for (6.25) can be proposed.

6.2.1 The constraint equations

The generalised analogue of (6.26) for $a = +$ is

$$\left\langle \oint \frac{dz}{2\pi i} J^+(z) \prod_{i=1}^{n} V_{h_i}^{w_i}(x_i; z_i) \right\rangle$$
$$= \sum_{i=1}^{n} \left\langle V_{h_1}^{w_1}(x_1; z_1) ... \left(\oint \frac{dz}{2\pi i} J^+(z) V_{h_i}^{w_i}(x_i; z_i) \right) ... V_{h_n}^{w_n}(x_n; z_n) \right\rangle$$
$$= \sum_{i=1}^{n} \left\langle V_{h_1}^{w_1}(x_1; z_1) .... \left(\oint \frac{dz}{2\pi i} \sum_{\ell=1}^{w_i} \frac{(J_\ell^+ V_{h_i}^{w_i})(x_i; z_i)}{(z - z_i)^{\ell+1}} + \frac{\partial_x V_{h_i}^{w_i}(x_i; z_i)}{z - z_i} \right) .... V_{h_n}^{w_n}(x_n; z_n) \right\rangle$$

$$\Rightarrow \oint \frac{dz}{2\pi i} \left\langle J^+(z) \prod_{i=1}^{n} V_{h_i}^{w_i}(x_i; z_i) \right\rangle = \oint \frac{dz}{2\pi i} \left[\sum_{i=1}^{n} \frac{\partial_{x_i}}{z - z_i} \left\langle \prod_{l=1}^{n} V_{h_l}^{w_l}(x_l; z_l) \right\rangle \right.$$
$$\left. + \sum_{i=1}^{n} \sum_{\ell=1}^{w_i} \frac{1}{(z - z_i)^{\ell+1}} \left\langle (J_\ell^+ V_{h_i}^{w_i})(x_i; z_i) \prod_{l \neq i} V_{h_l}^{w_l}(x_l; z_l) \right\rangle \right]$$

$$\Rightarrow \left\langle J^+(z) \prod_{i=1}^{n} V_{h_i}^{w_i}(x_i; z_i) \right\rangle = \sum_{i=1}^{n} \frac{\partial_{x_i}}{z - z_i} \left\langle \prod_{l=1}^{n} V_{h_l}^{w_l}(x_l; z_l) \right\rangle$$
$$+ \sum_{i=1}^{n} \sum_{\ell=1}^{w_i} \frac{1}{(z - z_i)^{\ell+1}} \left\langle (J_\ell^+ V_{h_i}^{w_i})(x_i; z_i) \prod_{l \neq i} V_{h_l}^{w_l}(x_l; z_l) \right\rangle, \tag{6.28}$$

where (6.20a) was used when acting $J^3(z)$ on the translated vertex operator $V_{h_i}^{w_i}(x_i; z_i)$ (although, again, it is worth noting the x dependence of $J^{+(x)}(z)$ is trivial). Also, countour deformations around each vertex insertion were used to yield the sum over each insertion when going to the second line. In going to the third line the OPE (6.10) was used. The above formulation of the correlator is not too useful since the terms

$$\hat{F}_\ell^i = \left\langle (J_\ell^+ V_{h_i}^{w_i})(x_i; z_i) \prod_{l \neq i} V_{h_l}^{w_l}(x_l; z_l) \right\rangle \tag{6.29}$$

that appear in the last line are not known. In order to determine them, the correlator with $J^3(z)$ or $J^-(z)$ inserted must first be calculated. Starting with $J^3(z)$, care must be taken when acting $J^3(z)$ on $V_{h_i}^{w_i}(x_i; z_i)$ due to the

x dependence in (6.20b):

$$\begin{aligned} J^3(z)V^{w_i}_{h_i}(x_i;z_i) &= e^{x_iJ^+_0}\left(J^{3(x)}(z)\right)V^{w_i}_{h_i}(0;z_i)e^{-x_iJ^+_0} \\ &= e^{x_iJ^+_0}\left(J^3(z)+x_iJ^+(z)\right)V^{w_i}_{h_i}(0;z_i)e^{-x_iJ^+_0} \\ &\sim e^{x_iJ^+_0}\left(\frac{(J^3_0V^{w_i}_{h_i})(0;z_i)}{z-z_i}+\sum_{\ell=1}^{w_i}\frac{x_i(J^+_\ell V^{w_i}_{h_i})(0;z_i)}{(z-z_i)^{\ell+1}}+\frac{x_i(J^+_0V^{w_i}_{h_i})(0;z_i)}{z-z_i}\right)e^{-x_iJ^+_0} \\ &\sim e^{x_iJ^+_0}\left(\frac{h_iV^{w_i}_{h_i}(0;z_i)}{z-z_i}+\frac{x_i\partial_{x_i}V^{w_i}_{h_i}(0;z_i)}{z-z_i}+x_i\sum_{\ell=1}^{w_i}\frac{(J^+_\ell V^{w_i}_{h_i})(0;z_i)}{(z-z_i)^{\ell+1}}\right)e^{-x_iJ^+_0} \\ &\sim \frac{-\mathcal{D}^3_iV^{w_i}_{h_i}(x_i;z_i)}{z-z_i}+\sum_{\ell=1}^{w_i}\frac{x_i}{(z-z_i)^{\ell+1}}(J^+_\ell V^{w_i}_{h_i})(x_i;z_i), \end{aligned} \tag{6.30}$$

where, as usual, in going to the fourth line the zero mode commutation relations (6.2) were used on the first and third term. The definition of the differential operator (6.3) was used in going to the fifth line. Using again the method of contour deformations, the correlator with $J^3(z)$ inserted gives

$$\begin{aligned} \left\langle J^3(z)\prod_{i=1}^{n}V^{w_i}_{h_i}(x_i;z_i)\right\rangle &= -\sum_{i=1}^{n}\frac{\mathcal{D}^3_i}{z-z_i}\left\langle\prod_{l=1}^{n}V^{w_l}_{h_l}(x_l;z_l)\right\rangle \\ &\quad+\sum_{i=1}^{n}\sum_{\ell=1}^{w_i}\frac{x_i}{(z-z_i)^{\ell+1}}\left\langle(J^+_\ell V^{w_i}_{h_i})(x_i;z_i)\prod_{l\neq i}V^{w_l}_{h_l}(x_l;z_l)\right\rangle. \end{aligned} \tag{6.31}$$

Similar reasoning as in (6.30) for the x dependence of $J^{-(x)}(z)$ can be applied, giving the correlator

$$\begin{aligned} \left\langle J^-(z)\prod_{i=1}^{n}V^{w_i}_{h_i}(x_i;z_i)\right\rangle &= -\sum_{i=1}^{n}\frac{\mathcal{D}^-_i}{z-z_i}\left\langle\prod_{l=1}^{n}V^{w_l}_{h_l}(x_l;z_l)\right\rangle \\ &\quad+\sum_{i=1}^{n}\sum_{\ell=1}^{w_i}\frac{x_i^2}{(z-z_i)^{\ell+1}}\left\langle(J^+_\ell V^{w_i}_{h_i})(x_i;z_i)\prod_{l\neq i}V^{w_l}_{h_l}(x_l;z_l)\right\rangle. \end{aligned} \tag{6.32}$$

It is important to note that all of the correlators contain the same unknown terms $\hat{F}^i_l$.

The OPE (6.12) can be leveraged in this analysis as it imposes clear constraints on the first $w-1$ regular terms. Consider first the following specific combination of x dependent currents,

$$\left(J^{-(x_j)}(z)-2x_jJ^{3(x_j)}(z)+x_j^2J^{+(x_j)}(z)\right)=J^-(z), \tag{6.33}$$

see (6.20). The OPE of this combination with the vertex operator gives

$$\begin{aligned}\left(J^-(z) - 2x_jJ^3(z) + x_j^2J^+(z)\right)V_{h_j}^{w_j}(x_j;z_j) \\ &= e^{x_jJ_0^+}\left(J^{-(x_j)}(z) - 2x_jJ^{3(x_j)}(z) + x_j^2J^{+(x_j)}(z)\right)V_{h_j}^{w_j}(0;z_j)e^{-x_jJ_0^+} \\ &= e^{x_jJ_0^+}J^-(z)V_{h_j}^{w_j}(0;z_j)e^{-x_jJ_0^+} \\ &\sim \mathcal{O}((z-z_j)^{w_j-1}),\end{aligned} \tag{6.34}$$

where in going to the fourth line the OPE (6.12) was used. This automatically implies the following constraint equations:

$$\left\langle \left(J^-(z) - 2x_jJ^3(z) + x_j^2J^+(z)\right)\prod_{i=1}^n V_{h_i}^{w_i}(x_i;z_i)\right\rangle = \mathcal{O}((z-z_j)^{w_j-1}), \tag{6.35}$$

when evaluating the LHS at $z \sim z_j$. Using the correlators from above, the LHS of (6.35) can be calculated explicitly to be

$$\begin{aligned}\left\langle \left(J^-(z) - 2x_jJ^3(z) + x_j^2J^+(z)\right)\prod_{i=1}^n V_{h_i}^{w_i}(x_i;z_i)\right\rangle \\ &= \sum_{i=j}^n \frac{-\mathcal{D}_i^- + 2x_j\mathcal{D}_i^+ + x_j^2\partial_{x_i}}{z-z_i}\left\langle \prod_{l=1}^n V_{h_l}^{w_l}(x_l;z_l)\right\rangle + \sum_{i=j}^n\sum_{\ell=1}^{w_i}\frac{x_i^2 - 2x_ix_j + x_j^2}{(z-z_i)^{\ell+1}}\hat{F}_\ell^i \\ &= \sum_{i\neq j}^n \frac{2(x_i-x_j)h_i + (x_i-x_j)^2\partial_{x_i}}{z-z_i}\left\langle \prod_{l=1}^n V_{h_l}^{w_l}(x_l;z_l)\right\rangle + \sum_{i\neq j}^n\sum_{\ell=1}^{w_i}\frac{(x_i-x_j)^2}{(z-z_i)^{\ell+1}}\hat{F}_\ell^i,\end{aligned} \tag{6.36}$$

where it is clear from the factorization in the last line that the terms with $i = j$ would have vanished. In the limit $z \sim z_j$, the LHS of (6.36) is regular as per (6.35). In fact, there are $w_j - 1$ equations by requiring that the coefficients of the terms

$$1, \quad (z-z_j), \quad (z-z_j)^2, \ldots, \quad (z-z_j)^{w_j-2}, \tag{6.37}$$

vanish. This provides $\sum_{j=1}^n (w_j - 1)$ constraints. These equations can be solved in terms of the $\sum_{i=1}^n w_i$ unknowns $\hat{F}_\ell^i$. While a closed form solution is not uniquely picked out of these constraints, they can be evaluated for any choice of the n-point correlators.

6.2.2 The fusion rules

For the simple case of a 3-point function for generic w_j where $j \in \{1,2,3\}$, the constraint equations (6.35) only have a solution provided that

$$w_i + w_j \geq w_l - 1, \tag{6.38}$$

where $\{i, j, l\}$ are all mutually disjoint, as was experimentally found by the authors of [1]. The 3-point function

$$\langle V_{h_1}^{w_1}(x_1; z_1) V_{h_2}^{w_2}(x_2; z_2) V_{h_3}^{w_3}(x_3; z_3) \rangle \tag{6.39}$$

is only non-zero provided that (6.38) is satisfied, in other words, (6.38) is part of the fusion rules. This was already predicted in [37] but here it is directly deduced from a Ward identity analysis on the worldsheet. The condition (6.38) has a natural generation to higher point functions

$$\sum_{i \neq j} w_i \geq w_j - 1 \tag{6.40}$$

for all j. There is no analytic proof however it was extensively tested by the authours of [1] and found to be true.

6.2.3 The Recusion Relations

The terms that correspond to the states before spectral flow can also be leveraged in this analysis. It is worth restating $J_w^+ |j, m\rangle = (m + j) |j, m+1\rangle$ from (6.22a) and the quantum number $m = h - \frac{(k+2)w}{2}$ from (6.23). Putting these two together is equivalent to

$$(J_{w_i}^+ V_{h_i}^{w_i})(x_i; z_i) = \left(h_i - \frac{(k+2)w_i}{2} + j_i \right) (V_{h_i+1}^{w_i})(x_i; z_i). \tag{6.41}$$

Note the shift in the conformal dimension on the vertex operator on the RHS. This is because $V_{h_i+1}^{w_i}(x_i; z_i)$ creates the state $|j, m+1\rangle$ via (6.23). Setting $\ell = w_i$, one of the unknown terms $\hat{F}_\ell^i$ can be written as

$$\hat{F}_{w_i}^i = \left(h_i - \frac{(k+2)w_i}{2} + j_i \right) \left\langle V_{h_i+1}^{w_i} \prod_{\ell \neq i}^{n} V_{h_\ell}^{w_\ell}(x_\ell; z_\ell) \right\rangle. \tag{6.42}$$

In other words, it can be written as a correlator of spectrally flowed highest weight states with shifted value of h_i. The same reasoning can be applied for the state (6.22c):

$$(J^-_{-w_j} V^{w_j}_{h_j})(x_j; z_j) = \left(h_j - \frac{(k+2)w_j}{2} - j_j\right)(V^{w_i}_{h_j-1})(x_j; z_j). \tag{6.43}$$

Let us now consider the term of order $(z - z_j)^{w_j - 1}$ in (6.35):

$$\begin{aligned}
&\left\langle \left(J^-(z) - 2x_j J^3(z) + x_j^2 J^+(z)\right) \prod_{i=1}^{n} V^{w_i}_{h_i}(x_i; z_i) \right\rangle \\
&= \left\langle V^{w_1}_{h_1}(x_1; z_1).....e^{x_j J^+_0} \left(\sum_{n=-\infty}^{-w_j} \frac{(J^-_n V^{w_j}_{h_j})(0; z_j)}{(z - z_j)^{n+1}} \right) e^{-x_j J^+_0}.....V^{w_n}_{h_n}(x_n; z_n) \right\rangle \\
&= \left\langle V^{w_1}_{h_1}(x_1; z_1).....e^{x_j J^+_0} \left(\frac{(J^-_{-w_j} V^{w_j}_{h_j})(0; z_j)}{(z - z_j)^{-w_j+1}} + \mathcal{O}\left((z - z_j)^{w_j}\right) \right) e^{-x_j J^+_0}.....V^{w_n}_{h_n}(x_n; z_n) \right\rangle \\
&= \left(h_j - \tfrac{(k+2)w_j}{2} - j_j\right)(z - z_j)^{w_j - 1} \left\langle V^{w_j}_{h_j - 1} \prod_{i \neq j}^{n} V^{w_i}_{h_i}(x_i; z_i) \right\rangle + \mathcal{O}\left((z - z_j)^{w_j}\right)
\end{aligned} \tag{6.44}$$

for z near each z_j. In going to the fifth line (6.43) was used. Using Cauchy's formula, the first term on the RHS (6.35) can be extracted:

$$\oint_{z_j} \frac{dz}{(z - z_j)^{w_j}} \left\langle \left(J^-(z) - 2x_j J^3(z) + x_j^2 J^+(z)\right) \prod_{i=1}^{n} V^{w_i}_{h_i}(x_i; z_i) \right\rangle, \tag{6.45}$$

since using (6.45) yields that the first term is the only term with a residue. Note that (6.44) can also be computed in terms of the correlators from (6.36).

Now that (6.45) can be forced to be equal to the first term on the RHS of (6.44), this provides n additional relations (1 for each index j). We went from having $\sum_{j=1}^{n}(w_j - 1)$ conditions on $\hat{F}^i_\ell$ to now $\sum_{j=1}^{n} w_j$ equations for $\sum_{i=1}^{n} w_i$ many unknowns $\hat{F}^i_\ell$. Given that $\hat{F}^i_{w_i}$ can be expressed as (6.42) then there are the truly unknown coefficients $\hat{F}^i_\ell$ with $1 \leq \ell \leq w_i - 1$, which if eliminated, gives us n "recursion relations" involving only correlation functions of spectrally flowed affine highest weight states. It was checked experimentally by the authors of [1], that n linearly independent equations can be derived given that (6.40) is satisfied.

Let us provide one such recursion relation for the case in which $w_i = 1$ for all i (these values for w_i satisfies (6.40)). The recursion relation corresponding to $\hat{F}^i_{w_i}$ will be chosen so that (6.42) can be used. Applying the contour integral of (6.45) to (6.44) under this setting gives:

$$\begin{aligned}
&\oint_{z_j} \frac{dz}{(z-z_j)} \left\langle \left(J^-(z) - 2x_j J^3(z) + x_j^2 J^+(z)\right) \prod_{i=1}^{n} V^1_{h_i}(x_i;z_i) \right\rangle \\
&\qquad = \left(h_j - \tfrac{(k+2)}{2} - j_j\right) \left\langle V^1_{h_j-1}(x_j;z_j) \prod_{\ell \neq j}^{n} V^1_{h_\ell}(x_\ell;z_\ell) \right\rangle
\end{aligned} \tag{6.46}$$

On the other hand, applying the contour integral of (6.45) to (6.36) gives

$$\begin{aligned}
&\oint_{z_j} \frac{dz}{(z-z_j)} \left\langle \left(J^-(z) - 2x_j J^3(z) + x_j^2 J^+(z)\right) \prod_{i=1}^{n} V^1_{h_i}(x_i;z_i) \right\rangle \\
&= \sum_{i \neq j} \frac{(x_j - x_i)^2}{(z_j - z_i)^2} \left(h_i - \frac{k+2}{2} + j_i\right) \left\langle V^1_{h_i+1}(x_i;z_i) \prod_{\ell \neq i}^{n} V^1_{h_\ell}(x_\ell;z_\ell) \right\rangle \\
&- \sum_{i \neq j} \frac{2(x_i - x_j)h_i + (x_i - x_j)^2 \partial_{x_i}}{z_i - z_j} \left\langle \prod_{\ell=1}^{n} V^1_{h_\ell}(x_\ell;z_\ell) \right\rangle
\end{aligned} \tag{6.47}$$

where we substituted in (6.42) for $w_i = 1$. The contour integral 'picked out' the $z = z_j$. Finally, bringing (6.46) and (6.47) together gives the recursion relation:

$$\begin{aligned}
&\left(h_j - \tfrac{(k+2)}{2} - j_j\right) \left\langle V^1_{h_j-1}(x_j;z_j) \prod_{\ell \neq j}^{n} V^1_{h_\ell}(x_\ell;z_\ell) \right\rangle \\
&= \sum_{i \neq j} \Bigg[\frac{(x_j - x_i)^2}{(z_j - z_i)^2} \left(h_i - \frac{k+2}{2} + j_i\right) \left\langle V^1_{h_i+1}(x_i;z_i) \prod_{\ell \neq i}^{n} V^1_{h_\ell}(x_\ell;z_\ell) \right\rangle \\
&- \frac{2(x_i - x_j)h_i + (x_i - x_j)^2 \partial_{x_i}}{z_i - z_j} \left\langle \prod_{\ell=1}^{n} V^1_{h_\ell}(x_\ell;z_\ell) \right\rangle \Bigg].
\end{aligned} \tag{6.48}$$

The Möbius symmetries that act on the z_i and x_i independently can be exploited to set $x_1 = z_1 = 0$, $x_2 = z_2 = 1$ and $x_3 = z_3 = \infty$. This sets the correlators to $\left\langle V^{w_1}_{h_1}(0;0) V^{w_2}_{h2}(1;1) V^{w_3}_{h_3}(\infty;\infty) \prod_{i=4}^{n} V^{w_i}_{h_i}(x_i;z_i) \right\rangle$. Let us illustrate what happens for one such recursion relation after applying the Möbius transformations (the coordinate dependence of the correlators will be suppressed in the following to avoid clutter). For example, take the recursion relation for index $j = 1$ (do not confuse this with the spin j):

$$\left(h_1 - \tfrac{(k+2)}{2} - j_1\right)\left\langle V^1_{h_1-1} V^1_{h_2} V^1_{h_3} \prod_{i=4}^{n} V^1_{h_i}\right\rangle$$
$$= \left[h_1 - h_2 - h_3 + \sum_{i=4}^{n} \frac{h_i(z_i - 2x_i) + x_i(z_i - x_i)\partial_{x_i}}{z_i}\right]\left\langle V^1_{h_1} V^1_{h_2} V^1_{h_3} \prod_{i=4}^{n} V^1_{h_i}\right\rangle$$
$$+ \sum_{i=2} \frac{x_i^2}{z_i^2}\left(h_i - \tfrac{k+2}{2} + j_i\right)\left\langle V^1_{h_1} V^1_{h_2} V^1_{h_3} V^1_{h_i+1} \prod_{\ell \neq 1,2,3,i}^{n} V^1_{h_\ell}\right\rangle. \tag{6.49}$$

It is clear that the third line comes from the second line of (6.48). The second line requires some explanation. First, it will be useful to derive the global Ward identities for the zero modes J_0^a that satisfy $\langle 0| J_0^a = 0$ and $J_0^a |0\rangle = 0$:

$$\begin{aligned} 0 &= \langle 0| J_0^a V_{h_1}^{w_1} ... V_{h_n}^{w_n} |0\rangle \\ &= \langle 0| [J_0^a, V_{h_1}^{w_1}] ... V_{h_n}^{w_n} |0\rangle + ... + \langle 0| V_{h_1}^{w_1} ... [J_0^a, V_{h_n}^{w_n}] |0\rangle . \end{aligned} \tag{6.50}$$

The commutation relations (6.2) can be substituted into (6.50) to obtain the following three global Ward identities,

$$\sum_{i=1}^{n} \partial_{x_i} \langle V_{h_1}^{w_1} ... V_{h_n}^{w_n} \rangle = 0, \quad a = + \tag{6.51}$$

$$\sum_{i=1}^{n} (x_i \partial_{x_i} + h_i) \langle V_{h_1}^{w_1} ... V_{h_n}^{w_n} \rangle = 0, \quad a = 3 \tag{6.52}$$

$$\sum_{i=1}^{n} (x_i^2 \partial_{x_i} + 2h_i x_i) \langle V_{h_1}^{w_1} ... V_{h_n}^{w_n} \rangle = 0, \quad a = -. \tag{6.53}$$

Take the third line of (6.48), set the index $j = 1$ and recall that $x_1 = z_1 = 0$. The global identity (6.52) can be added freely since it vanishes:

$$\begin{aligned} &\sum_{i=2}^{n} -\frac{2h_i x_i + x_i^2 \partial_{x_i}}{z_i} \langle \prod_{\ell=1}^{n} V_{h_\ell^1} \rangle + \sum_{i=1}^{n} (x_i \partial_{x_i} + h_i) \langle \prod_{\ell=1}^{n} V_{h_\ell^1} \rangle \\ &= \left(h_1 + \sum_{i=2}^{n} \frac{z_i x_i \partial_{x_i} + z_i h_i - 2h_i x_i - x_i^2 \partial_{x_i}}{z_i}\right) \langle \prod_{\ell=1}^{n} V_{h_\ell^1} \rangle \\ &= \left(h_1 - h_2 - h_3 + \sum_{i=4}^{n} \frac{h_i(z_i - x_i) + x_i(z_i - x_i)\partial_{x_i}}{z_i}\right) \langle \prod_{\ell=1}^{n} V_{h_\ell^1} \rangle, \end{aligned} \tag{6.54}$$

where in going to the third line the $i = \{2, 3\}$ terms were separated and it was used that $x_2 = z_2$ and $x_3 = z_3$, hence the second line of (6.49). Note the convention that $\frac{0}{0} = \frac{\infty}{\infty} = 1$ was also used. Following the above procedure, the recursion relations for higher values of w_i's can be found.

The data from a covering map over the sphere will soon be used as a solution to (6.25), so we will assume a stronger condition than (6.40):

$$\sum_{i \neq j} (w_i - 1) \geq w_j - 1, \qquad \text{for all } j, \tag{6.55}$$

where $w_k - 1$ will eventually be identified with the ramification order of the covering map. This is a necessary condition for a covering map with a given ramification over each branch point to exist. It was suspected in [1] that the solutions to the recursion relations for the cases where only (6.40) were satisfied but not (6.55) are somewhat pathological.

6.2.4 Comparison to Maldacena-Ooguri

In this section we will apply the above methods to the example of a 3-point function involving two unflowed representations $w_1 = w_2 = 0$ and one spectrally flowed representation $w_3 = 1$. A calculation for this was already done in [17] using a different method, so we would like to compare our answer to theirs (see eq. 5.38 of that paper). For the J^+ correlator we will make use of (6.28):

$$\begin{aligned}
&\langle J^+(z) V^0_{h_1}(x_1; z_1) V^0_{h_2}(x_2; z_2) V^1_{h_3}(x_3; z_3) \rangle \\
&\quad = \sum_{i=1}^{3} \frac{1}{z - z_i} \partial_{x_i} \langle V^0_{h_1}(x_1; z_1) V^0_{h_2}(x_2; z_2) V^1_{h_3}(x_3; z_3) \rangle \\
&\quad + \frac{1}{(z - z_3)^2} \langle J^+_1 V^1_{h_3}(x_3; z_3) V^0_{h_1}(x_1; z_1) V^0_{h_2}(x_2; z_2) \rangle \\
&\quad + \sum_{i=1,2} \frac{1}{(z - z_i)^2} \langle J^+_1 V^0_{h_i}(x_i; z_i) V^0_{h_k}(x_k; z_k) V^1_{h_3}(x_3; z_3) \rangle \\
&\qquad = \sum_{i=1}^{3} \frac{1}{z - z_i} \partial_{x_i} \langle V^0_{h_1}(x_1; z_1) V^0_{h_2}(x_2; z_2) V^1_{h_3}(x_3; z_3) \rangle \\
&\qquad + \frac{1}{(z - z_3)^2} \big(h_3 - \tfrac{\hat{k}}{2} + j_3\big) \langle V^0_{h_1}(x_1; z_1) V^0_{h_2}(x_2; z_2) V^1_{h_3+1}(x_3; z_3) \rangle,
\end{aligned} \tag{6.56}$$

where in going to the last equality $J^+_1 V^0_{h_i} = 0$ was used since all positive modes of J^+_n annihilate the highest wight state created by a spectrally unflowed ($w_i = 0$) primary operator $V^0_{h_i}$, and this is obviously the case for both $i \in \{1,2\}$. On the other hand, $J_1^+ V_{h_3}^1$ gives (6.41) hence the last equality. Also note that the level $\hat{k} = k+2$ is the level that appears in the bosonic analysis of [17]. The same logic as above can be applied when calculating the correlators with J^3 (6.31) and J^- (6.32):

$$\begin{aligned}
&\langle J^3(z) V_{h_1}^0(x_1;z_1) V_{h_2}^0(x_2;z_2) V_{h_3}^1(x_3;z_3)\rangle \\
&\quad = \sum_{i=1}^{3} \frac{1}{z-z_i}\big(h_i + x_i\partial_{x_i}\big)\langle V_{h_1}^0(x_1;z_1) V_{h_2}^0(x_2;z_2) V_{h_3}^1(x_3;z_3)\rangle \\
&\quad + \frac{x_3}{(z-z_3)^2}\big(h_3 - \tfrac{\hat{k}}{2} + j_3\big)\langle V_{h_1}^0(x_1;z_1) V_{h_2}^0(x_2;z_2) V_{h_3+1}^1(x_3;z_3)\rangle
\end{aligned} \tag{6.57}$$

and

$$\begin{aligned}
&\langle J^-(z) V_{h_1}^0(x_1;z_1) V_{h_2}^0(x_2;z_2) V_{h_3}^1(x_3;z_3)\rangle \\
&\quad = \sum_{i=1}^{3} \frac{1}{z-z_i}\big(2h_i x_i + x_i^2\partial_{x_i}\big)\langle V_{h_1}^0(x_1;z_1) V_{h_2}^0(x_2;z_2) V_{h_3}^1(x_3;z_3)\rangle \\
&\quad + \frac{x_3^2}{(z-z_3)^2}\big(h_3 - \tfrac{\hat{k}}{2} + j_3\big)\langle V_{h_1}^0(x_1;z_1) V_{h_2}^0(x_2;z_2) V_{h_3+1}^1(x_3;z_3)\rangle.
\end{aligned} \tag{6.58}$$

In the following, the coordinate dependence of the vertex operators will again be suppressed. The following combination can be made using the above correlators:

$$\begin{aligned}
&\left\langle \Big(J^-(z) - 2x_3 J^3(z) + x_3^2 J^+(z)\Big) V_{h_1}^0 V_{h_2}^0 V_{h_3}^1 \right\rangle \\
&= \sum_{i=1,2,3}\left(\frac{2h_i x_i + x_i^2\partial_{x_i}}{z-z_i} - 2x_3\frac{h_i + x_i\partial_{x_i}}{z-z_i} + x_3^2\frac{\partial_{x_i}}{z-z_i}\right)\langle V_{h_1}^0 V_{h_2}^0 V_{h_3}^1\rangle \\
&= \sum_{i=1,2}\left(\frac{(x_i-x_3)^2\partial_{x_i} + 2h_i(x_i-x_3)}{z-z_i}\right)\langle V_{h_1}^0 V_{h_2}^0 V_{h_3}^1\rangle,
\end{aligned} \tag{6.59}$$

where it should be clear by the last factorization that the terms with $i=3$ will vanish. Cauchy's formula can be used to pick out the regular term of order $(z-z_i)^0$:

$$\begin{aligned}
&\oint_{z_3}\frac{dz}{z-z_3}\left\langle \Big(J^-(z) - 2x_3 J^3(z) + x_3^2 J^+(z)\Big) V_{h_1}^0 V_{h_2}^0 V_{h_3}^1 \right\rangle \\
&= \sum_{i=1,2}\left(\frac{(x_i-x_3)^2\partial_{x_i} + 2h_i(x_i-x_3)}{z_3-z_i}\right)\langle V_{h_1}^0 V_{h_2}^0 V_{h_3}^1\rangle \\
&= \sum_{i=1,2}\left(\frac{x_3-x_i}{z_3-z_i}\Big(-2h_i + (x_3-x_i)\partial_{x_i}\Big)\right)\langle V_{h_1}^0 V_{h_2}^0 V_{h_3}^1\rangle.
\end{aligned} \tag{6.60}$$

The above expression can be simplified further by using the Möbius symmetry to set $x_1 = z_1,\ x_2 = z_2,\ x_3 = z_3$ and adding the global Ward identity of (6.52),

$$\begin{aligned}
&\left\langle \left(J^-(z) - 2x_3 J^3(z) + x_3^2 J^+(z) \right) V^0_{h_1} V^0_{h_2} V^1_{h_3} \right\rangle \Big|_{(z-z_3)^0} \\
&= \sum_{i=1,2} \left(-2h_i + (x_3 - x_i)\partial_{x_i} \right) \langle V^0_{h_1} V^0_{h_2} V^1_{h_3} \rangle + \sum_{i=1,2,3} (h_i + x_i \partial_{x_i}) \langle V^0_{h_1} V^0_{h_2} V^1_{h_3} \rangle \\
&= \left((h_3 - h_2 - h_1) + x_3 \sum_{i=1,2,3} \partial_{x_i} \right) \langle V^0_{h_1} V^0_{h_2} V^1_{h_3} \rangle \\
&= (h_3 - h_2 - h_1) \langle V^0_{h_1} V^0_{h_2} V^1_{h_3} \rangle,
\end{aligned} \tag{6.61}$$

where the global Ward identity of (6.51) was used in going to the last line. Using (6.44) and (6.45), the regular term of order $(z - z_3)^0$ must take the form

$$\begin{aligned}
&\left\langle \left(J^-(z) - 2x_3 J^3(z) + x_3^2 J^+(z) \right) V^0_{h_1} V^0_{h_2} V^1_{h_3} \right\rangle \Big|_{(z-z_3)^0} \\
&= \left(h_3 - \tfrac{\hat{k}}{2} + j_3 \right) \langle V^0_{h_1} V^0_{h_2} V^1_{h_3 - 1} \rangle.
\end{aligned} \tag{6.62}$$

Before setting the above two expressions equal to one another, there is a dimensional correction that needs to be calculated first. Recall that the x dependence of the 3-point function is constrained by conformal symmetry to be

$$\langle V^{w_1}_{h_1} V^{w_2}_{h_2} V^{w_3}_{h_3} \rangle \sim \frac{1}{(x_1 - x_2)^{h - 2h_3} (x_2 - x_3)^{h - 2h_1} (x_1 - x_3)^{h - 2h_2}}, \tag{6.63}$$

where $h = \sum_i h_i$. If we take $V^{w_3}_{h_3} \to V^{w_3}_{h_3 - 1}$ as in the correlator on the RHS of (6.62) then we have the ratio

$$\frac{\langle V^{w_1}_{h_1} V^{w_2}_{h_2} V^{w_3}_{h_3 - 1} \rangle}{\langle V^{w_1}_{h_1} V^{w_2}_{h_2} V^{w_3}_{h_3} \rangle} \sim \frac{(x_2 - x_3)(x_1 - x_3)}{(x_1 - x_2)}. \tag{6.64}$$

Setting (6.61) and (6.62) equal to each other and accounting for the dimensionality gives the condition that

$$\begin{aligned}
&\langle V^0_{h_1}(x_1; z_1) V^0_{h_2}(x_2; z_2) V^1_{h_3 - 1}(x_3; z_3) \rangle \frac{(x_1 - x_2)}{(x_2 - x_3)(x_1 - x_3)} \\
&= \frac{h_1 + h_2 - h_3}{h_3 - \frac{\hat{k}}{2} - j_3} \langle V^0_{h_1}(x_1; z_1) V^0_{h_2}(x_2; z_2) V^1_{h_3}(x_3; z_3) \rangle,
\end{aligned} \tag{6.65}$$

which is compatible with eq. (5.38) of [17] where instead we used that $h_1 = j_3,\ h_2 = j_4,\ J = h_3$ and $j_1^{\mathrm{MO}} = 1 - j_3$.

6.3 A simple solution - the covering map

The most general solution to the recursion relations is not yet known however there exists a simple solution that works very generally, which involves a covering map. Details of the covering map have been covered in section 3.2 however in this section it will be adapted to the worldsheet case, where the covering surface is the Riemann sphere; recall that we are computing the correlation function on the sphere.

Let the coordinates of the vertex operators $\{z_i\}_{i=1,\ldots,n}$ and $\{x_i\}_{i=1,\ldots,n}$ denote two collections of n marked points, each collection on a Riemann sphere *. Let the set of spectral flow paramaters for the vertex operators $\{w_i\}_{i=1,\ldots,n}$ denote the ramification indices near the z_i. We call $\Gamma(z)$ the covering map of this configuration if $\Gamma(z)$ is an analytic function satisfying

$$\Gamma(z) = x_i + a_i^{\Gamma}(z - z_i)^{w_i} + \mathcal{O}((z - z_i)^{w_i+1}) \tag{6.66}$$

near $z = z_i$ (see also (3.22) and the surrounding discussion). Additionally, it is assumed that $\Gamma(z)$ does not have any other critical points, i.e. the only points where $\partial\Gamma(z) = 0$ is for $z = z_i$ with $i = 1, ..., n$.

Consider the case when $n = 3$. By composing the covering map with Möbius transformations, we may always take $x_i = z_i$ for all $i = 1, 2, 3$. If the ramification indices satisfy the selection rule,

$$w_i + w_j \geq w_k + 1, \quad w_1 + w_2 + w_3 \in 2\mathbb{Z} + 1, \tag{6.67}$$

a unique covering map exists. Note that the selection rule agrees with the conditions under which the recursion relations have a solution (see (6.55)).

In the more general case when $n \geq 4$, a covering map generically does not exist. The analogue of (6.67) for $n \geq 4$ takes the form

$$\sum_{i \neq j}(w_i - 1) \geq w_j - 1, \quad \sum_i (w_i - 1) \in 2\mathbb{Z}, \tag{6.68}$$

however does not guarantee the existence of the covering map. We can use the example of the 4-point case to illustrate this. Take $w_1 = w_2 = w_3 = w_4 = 2$ and use the Möbius symmetry to set $x_1 = z_1 = 0$, $x_2 = z_2 = 1$, $x_3 = z_3 = \infty$. The covering map in this case is given explicitly by

$$\Gamma(z) = \frac{z^2\left(\pm z\sqrt{z_4^2 - z_4 + 1} - zz_4 - z + 3z_4\right)}{\pm(3z-2)\sqrt{z_4^2 - z_4 + 1} + 3zz_4 - 3z - z_4 + 2}, \tag{6.69}$$

*Both the worldsheet and the boundary of AdS_3 are Riemann spheres.

where the number of pre-images is given by the Riemann-Hurwitz relation:

$$\begin{aligned} M &= 1 + \frac{1}{2}\sum_{i=1}^{4}(w_i - 1) \\ &= 3, \end{aligned} \tag{6.70}$$

where the genus was taken to be zero (see (3.33)).

In this case, there are two possible covering maps corresponding to the different signs. That $\Gamma(z)$ is indeed the actual covering map forces the additional constraint that

$$x_4 = \Gamma(z_4) = z_4\Big(\pm 2(1 - z_4)\sqrt{1 - z_4 + z_4^2} + 2z_4^2 - 3z_4 + 2\Big). \tag{6.71}$$

In general, for $n \geq 4$ there are $n - 3$ additional constraints that $\Gamma(z)$ must satisfy, namely; $x_i = \Gamma(z_i)$ for $i = 4, ..., n$. This constraint will play a significant role in the form of the solution to the n-point function, in particular it forces a delta function localization property in worldsheet moduli space, as will soon be shown. For the simpler 3-point case, $\Gamma(z) = z$ (after imposing the Möbius transformations) so $x_i = \Gamma(z_i)$ is always satisfied. Since the covering map plays an integral role in the correlation functions, the difference between $n = 3$ and $n \geq 4$ will also be manifested in the solution. As such, we will treat the simpler 3-point function first.

6.4 The 3-point case

In this section, a solution to the 3-point function is proposed. In the following, let a_i^Γ with $i = 1, 2, 3$ denote the coefficients that appear in (6.66). The main claim is that a solution to the recursion relations, for the 3-point function, is of the form

$$\begin{aligned} &\langle V_{h_1}^{w_1}(x_1; z_1) V_{h_2}^{w_2}(x_2; z_2) V_{h_3}^{w_3}(x_3; z_3)\rangle \\ &\qquad = C(j_1, j_2, j_3)\prod_{i=1}^{3}(a_i^\Gamma)^{-h_i}\prod_{i\neq j}(z_i - z_j)^{\Delta_\ell^0 - \Delta_i^0 - \Delta_j^0}, \end{aligned} \tag{6.72}$$

where ℓ labels the index not equal to i or j and $C(j_1, j_2, j_3)$ is the normalisation constant. It was assumed that the spins j_i satisfy the condition

$$\sum_{i=1}^{3} j_i = \frac{k+2}{2}, \tag{6.73}$$

when proposing the solution (6.72). The motivation for this condition will be elaborated in the proof of the 3-point function in the following section. Additionally, the Δ_i^0's are defined to be

$$\Delta_j^0 \equiv \Delta_j + w_j h_j, \tag{6.74}$$

where Δ_j is the worldsheet conformal dimension of the j'th vertex operator. It should be noted that the solution in (6.72) also depends on the x_i's and z_i's. In fact, we have that [45]

$$a_i^\Gamma = \frac{\begin{pmatrix} \frac{1}{2}(w_i + w_{i+1} + w_{i+2} - 1) \\ \frac{1}{2}(-w_i + w_{i+1} + w_{i+2} - 1) \end{pmatrix}}{\begin{pmatrix} \frac{1}{2}(-w_i + w_{i+1} - w_{i+2} - 1) \\ \frac{1}{2}(w_i + w_{i+1} - w_{i+2} - 1) \end{pmatrix}} \frac{(x_i - x_{i+1})(x_{i+2} - x_i)(z_{i+1} - z_{i+2})^{w_i}}{(x_{i+1} - x_{i+2})(z_i - z_{i+1})^{w_i}(z_{i+2} - z_i)^{w_i}}, \tag{6.75}$$

where the indices are mod 3. Let the first factor in (6.75) be denoted by $a_{i,0}^\Gamma$. The product in (6.72) can be written as

$$\begin{aligned} \prod_{i=1}^{3} (a_i^\Gamma)^{-h_i} &= \prod_{i=1}^{3} (a_{i,0}^\Gamma)^{-h_i} \frac{(x_i - x_{i+1})^{-h_i}(x_{i+2} - x_i)^{-h_i}(z_{i+1} - z_{i+2})^{-h_i w_i}}{(x_{i+1} - x_{i+2})^{-h_i}(z_i - z_{i+1})^{-h_i w_i}(z_{i+2} - z_i)^{-h_i w_i}} \\ &= \prod_{i=1}^{3} (a_{i,0}^\Gamma)^{-h_i} \prod_{i \neq j} (x_i - x_j)^{h_\ell - h_i - h_j} (z_i - z_j)^{-h_\ell w_\ell + h_i w_i + h_j w_j}. \end{aligned} \tag{6.76}$$

Substituting (6.76) back into the solution (6.72) gives

$$\begin{aligned} &\langle V_{h_1}^{w_1}(x_1; z_1) V_{h_2}^{w_2}(x_2; z_2) V_{h_3}^{w_3}(x_3; z_3) \rangle \\ &= C(j_1, j_2, j_3) \prod_{i=1}^{3} (a_{i,0}^\Gamma)^{-h_i} \prod_{i \neq j} (z_i - z_j)^{\Delta_\ell^0 - h_\ell w_\ell - \Delta_i^0 + h_i w_i - \Delta_j^0 + h_j w_j} (x_i - x_j)^{h_\ell - h_i - h_j} \\ &= C(j_1, j_2, j_3) \prod_{i=1}^{3} (a_{i,0}^\Gamma)^{-h_i} \prod_{i \neq j} (z_i - z_j)^{\Delta_\ell - \Delta_i - \Delta_j} (x_i - x_j)^{h_\ell - h_i - h_j}, \end{aligned} \tag{6.77}$$

which is precisely the x_i and z_i dependence that is expected from a 3-point correlation function of primaries that depend on both co-ordinates. It is also convenient to use the Möbius symmetry to fix the x_i's and z_i's in the usual way: $z_1 = x_1 = 0, \ z_2 = x_2 = 1, \ z_3 = x_3 = \infty$. Under this setting the solution simplifies to:

$$\langle V_{h_1}^{w_1}(0; 0) V_{h_2}^{w_2}(1; 1) V_{h_3}^{w_3}(\infty; \infty) \rangle = C(j_1, j_2, j_3) \prod_{i=1}^{3} (a_{i,0}^\Gamma)^{-h_i}. \tag{6.78}$$

6.5 Proof of the 3-point solution

In this section, we aim to explicitly show that the solution (6.72) is indeed a valid one and that it satisfies the recursion relations. For the following treatment, the x_i's and z_i's will be taken to be generic i.e. they will not be fixed by symmetry.

Let us start by emphasizing again that near $z = z_j$ the constraint takes the form of (6.35) which means the m'th order regular term vanishes, where $m = 0, 1, 2 ... w_j - 2$ and $j = 1, 2 ... n$. This implies that the m'th order regular term of the RHS of (6.36) near $z = z_j$ also vanishes. Let the RHS of (6.36) be denoted as $f(z)$ for the moment. A Taylor expansion around $z = z_j$ gives

$$\sum_{n=0}^{\infty} \frac{\partial_z^n f(z)|_{z=z_j}}{n!} (z - z_j)^n, \tag{6.79}$$

therefore the m'th order regular term in this expansion is given by

$$\frac{\partial_z^m f(z)|_{z=z_j}}{m!} (z - z_j)^m, \tag{6.80}$$

where the coefficient $\partial_z^m f(z)|_{z=z_j}$ vanishes. This can be written explicitly in the form of (6.36) but here we will also divide throughout by the correlator,

$$\begin{aligned} 0 =& \frac{\left\langle \left(J^-(z) - 2x_j J^3(z) + x_j^2 J^+(z)\right) \prod_{i=1}^3 V_{h_i}^{w_i}(x_i; z_i) \right\rangle}{\left\langle \prod_{l=1}^3 V_{h_l}^{w_l}(x_l; z_l) \right\rangle} \Bigg|_{\mathcal{O}(z-z_j)^m} \\ =& \frac{1}{m!} \partial_z^m \sum_{i \neq j} \left(-\frac{2(x_j - x_i) h_j}{z - z_i} + \sum_{\ell=0}^{w_i} \frac{(x_j - x_i)^2}{(z - z_i)^{\ell+1}} F_\ell^i \right) \Bigg|_{z=z_j}, \end{aligned} \tag{6.81}$$

where F_ℓ^i is defined as

$$F_\ell^i \equiv \frac{\hat{F}_\ell^i}{\langle \prod_{l=1}^3 V_{h_l}^{w_l}(x_l; z_l) \rangle} = \frac{\langle (J_\ell^+ V_{h_i}^{w_i}(x_i; z_i) \prod_{l \neq i} V_{h_l}^{w_l}(x_l; z_l) \rangle}{\langle \prod_{l \neq i} V_{h_l}^{w_l}(x_l; z_l) \rangle}. \tag{6.82}$$

Note that the term with the partial derivative in (6.36) is the $\ell = 0$ term in the sum over ℓ on the RHS of (6.81). To see this more clearly, consider the $\hat{F}_0^i$ term:

$$\begin{aligned} \hat{F}_0^i &= \langle (J_0^+ V_{h_i}^{w_i}(x_i; z_i) \prod_{l \neq i} V_{h_l}^{w_l}(x_l; z_l) \rangle \\ &= \partial_{x_i} \langle \prod_{l=1}^3 V_{h_l}^{w_l}(x_l; z_l) \rangle \end{aligned} \tag{6.83}$$

where the global properties of $\mathfrak{sl}(2,\mathbb{R})$ (see (6.2) and (6.3)) were used in going to the second line. The RHS of (6.81) can be expressed in terms of the covering map $\Gamma(z)$,

$$\frac{1}{m!}\partial_z^m \sum_{i=1}^{3}\left(-\frac{2(\Gamma(z)-x_i)h_j}{z-z_i}+\sum_{\ell=0}^{w_i}\frac{(\Gamma(z)-x_i)^2}{(z-z_i)^{\ell+1}}F_\ell^i\right)\Bigg|_{z=z_j}, \tag{6.84}$$

where we have replaced $x_j = \Gamma(z_j)$ and extended the sum to include $i = j$. The first operation only has a non-trivial effect at order $m = w_j$. To see why, let us focus on the first term and set the order to be $m = w_j - 1$ to show that the transformation is acceptable only up to this order:

$$\begin{aligned}
&\sim \frac{1}{(w_j-1)!}\partial_z^{w_j-1}\frac{\Gamma(z)}{z-z_i}\Bigg|_{z=z_j}\\
&= \frac{1}{(w_j-1)!}\left[\frac{\partial_z^{w_j-1}\Gamma(z)}{z-z_i}+\Gamma(z)\partial_z^{w_j-1}(z-z_i)^{-1}\right]\Bigg|_{z=z_j} + \text{vanishing terms}\\
&\sim \left[\frac{\partial_z^{w_j-1}\left(x_j+a_j^\Gamma(z-z_j)^{w_j}+\mathcal{O}((z-z_j)^{w_j+1})\right)}{z-z_i}+c\frac{\Gamma(z)}{(z-z_i)^{w_j}}\right]\Bigg|_{z=z_j}\\
&\sim \left[\frac{ba_j^\Gamma(z-z_j)+\mathcal{O}((z-z_j)^2)}{z-z_i}+c\frac{\Gamma(z)}{(z-z_i)^{w_j}}\right]\Bigg|_{z=z_j}\\
&= c\frac{x_j}{(z_j-z_i)^{w_j}},
\end{aligned} \tag{6.85}$$

which is precisely the result we would have gotten in the original expression of (6.81). Here, b and c are numerical prefactors and in going to the third line $\Gamma(z)$ was expanded around $z = z_j$ to make the effect clear. On the other hand, if the order was taken to be $m = w_j$, one would get an additional term proportional to $\frac{a_j^\Gamma}{z_j-z_i}$. Since m only goes up to $w_j - 2$, this additional term will not appear. Similar arguments can be applied for the second term of (6.85) to show that it is an applicable transformation. The second operation of extending the sum to $i = j$ only has an effect up to order $m = w_j - 1$, as will be demonstrated below.

Define the function $G(z)$ as:

$$G(z) \equiv \sum_{i=1}^{3}\left(-\frac{2(\Gamma(z)-x_i)h_i}{z-z_i}+\sum_{\ell=0}^{w_i}\frac{(\Gamma(z)-x_i)^2}{(z-z_i)^{\ell+1}}F_\ell^i\right). \tag{6.86}$$

The constraint equations is equivalent to the requirement:

$$\partial_z^m G(z = z_j) = 0, \tag{6.87}$$

where, again, $m = 0, 1, 2...w_j - 2$ and $j = 1, 2...n$. Assuming these constraints are solved, $G(z)$ is a rational function with the following properties:

1. $G(z)$ has poles of second order at z_a^*, $a = 1, ..., M$, where z_a^* are the poles of $\Gamma(z)$. These poles are simply the pre-images of $\Gamma(z_a^*) = \infty$. There are M such poles because there are M pre-images, as calculated from the Riemann-Hurwitz relation. The poles are second order because of the quadratic term of the covering map in $G(z)$.
2. $G(z)$ has zeros of order $w_i - 1$ at the insertion points.
3. $G(z)$ behaves asymptotically as $\mathcal{O}(z^{-2})$

To explain the last property, consider the asymptotic expansion of $G(z)$ as $z \to \infty$, where the sum is expanded over ℓ to get

$$\begin{aligned} G(z) &= \frac{1}{z}\sum_{i=1}^{3}\left(-2(\Gamma(\infty) - x_i)h_i + (\Gamma(\infty) - x_i)^2 F_0^i\right) + O(z^{-2}) \\ &= \frac{1}{z}\sum_{i=1}^{3}\left(\Gamma(\infty)^2 F_0^i - 2\Gamma(\infty)(h_i + x_i F_0^i) + (2x_i h_i + x_i^2 F_0^i)\right) + O(z^{-2}). \end{aligned} \tag{6.88}$$

Substitute $\hat{F}_0^i$ of (6.83) into the global Ward identities (for $n = 3$) and divide again throughout by the 3-point function to get

$$\sum_{i=1}^{3} F_0^i = 0, \quad \sum_{i=1}^{3}(h_i + x_i F_0^i) = 0, \quad \sum_{i=1}^{3}(2x_i h_i + x_i^2 F_0^i) = 0, \tag{6.89}$$

Substituting the identities of (6.89) into (6.88) shows that the $\frac{1}{z}$ term vanishes.

The above three properties of $G(z)$ implies that it must take the form

$$G(z) = A\frac{\prod_{i=1}^{3}(z - z_i)^{w_i - 1}}{\prod_{a=1}^{M}(z - z_a^*)^2}, \tag{6.90}$$

where A is the normalisation constant. M in the denominator comes from the Riemann-Hurwitz formula (at genus 0):

$$M = 1 + \frac{1}{2}\sum_{i=1}^{3}(w_i - 1). \tag{6.91}$$

The derivative of the covering map has exactly the same properties as $G(z)$, therefore we propose that $G(z)$ must be proportional to $\partial\Gamma(z)$:

$$G(z) = \alpha\partial\Gamma(z), \tag{6.92}$$

where α is a constant of proportionality (however it still depends on x_i, z_i, j_i and k). This constant can be computed by summing over the M poles of $\Gamma(z)$. Take the following quantity:

$$-\oint_{z_a^*} dz \frac{\partial\Gamma(z)}{\Gamma(z)}, \tag{6.93}$$

which counts the number of times $\Gamma(z)$ winds around $\Gamma(z_a^*)$ as z winds around the pole z_a^* once. The winding number is 1, so if we sum the windings over M such poles we get M. Now, we have an expression for α to solve:

$$\begin{aligned} M &= -\sum_{a=1}^{M} \oint_{z_a^*} dz \frac{\partial\Gamma(z)}{\Gamma(z)} \\ \implies \alpha M &= -\sum_{a=1}^{M} \oint_{z_a^*} dz \frac{\alpha\partial\Gamma(z)}{\Gamma(z)} = -\sum_{a=1}^{M} \oint_{z_a^*} dz \frac{G(z)}{\Gamma(z)}. \end{aligned} \tag{6.94}$$

Substituting the expression for $G(z)$ gives:

$$\alpha M = -\sum_{a=1}^{M}\sum_{i=1}^{3} \oint_{z_a^*} dz \left(-\frac{2(\Gamma(z)-x_i)h_i}{(z-z_i)\Gamma(z)} + \sum_{\ell=0}^{w_i} \frac{(\Gamma(z)-x_i)^2}{(z-z_i)^{\ell+1}\Gamma(z)} F_\ell^i \right). \tag{6.95}$$

Since $\Gamma(z)$ is singular at $z = z_a^*$, the first term has no such poles (i.e. there is no $\Gamma(z)$ in the numerator) so it vanishes in the contour integral around z_a^*. After expanding the second term, the quadratic term $\Gamma^2(z)$ in the numerator is the only term with a pole, so we have the simplified expression:

$$\alpha M = -\sum_{a=1}^{M}\sum_{i=1}^{3}\sum_{\ell=0}^{w_i} \oint_{z_a^*} dz \left(\frac{\Gamma(z)}{(z-z_i)^{\ell+1}} F_\ell^i \right). \tag{6.96}$$

Note that there are other poles at $z = z_i$ that lie outside the contours. We can deform the contours to localize at these poles instead $-\sum_{a=1}^{N} \oint_{z_a^*} dz = \oint_{z_i} dz + \oint_\infty dz$. After the contour deformation:

$$\alpha M = \sum_{i=1}^{3}\sum_{\ell=0}^{w_i} \left(\oint_{z_i} dz \frac{\Gamma(z)}{(z-z_i)^{\ell+1}} F_\ell^i + \oint_\infty dz \frac{\Gamma(z)}{(z-z_i)^{\ell+1}} F_\ell^i \right). \tag{6.97}$$

We will compute the first term first and start by expanding $\Gamma(z)$ around $z = z_i$ (since the contour is around z_i):

$$\begin{aligned}
&\sum_{\ell=0}^{w_i}\left(\oint_{z_i} dz \frac{x_i + a_i^\Gamma (z-z_i)^{w_i} + \dots}{(z-z_i)^{\ell+1}} F_\ell^i\right) \\
&= \sum_{\ell=0}^{w_i}\left(\oint_{z_i} dz \frac{x_i}{(z-z_i)^{\ell+1}} F_\ell^i + \oint_{z_i} dz \frac{a_i^\Gamma (z-z_i)^{w_i}}{(z-z_i)^{\ell+1}} F_\ell^i\right) \\
&= \oint_{z_i} dz \frac{x_i}{(z-z_i)} F_0^i + \oint_{z_i} dz \frac{a_i^\Gamma}{(z-z_i)} F_{w_i}^i \\
&= x_i F_0^i + a_i^\Gamma F_{w_i}^i .
\end{aligned} \tag{6.98}$$

Now, the second term of (6.97):

$$\begin{aligned}
&\sum_{\ell=0}^{w_i}\left(\oint_{\infty} dz \frac{\Gamma(z)}{(z-z_i)^{\ell+1}} F_\ell^i\right) \\
&= \oint_{\infty} dz \frac{\Gamma(z)}{(z-z_i)} F_0^i + \sum_{\ell=1}^{w_i}\left(\oint_{\infty} dz \frac{\Gamma(z)}{(z-z_i)^{\ell+1}} F_\ell^i\right) \\
&= -\Gamma(\infty) F_0^i .
\end{aligned} \tag{6.99}$$

Substituting (6.98) and (6.99) into the RHS of (6.97) gives

$$\sum_{i=1}^{3} x_i F_0^i - \Gamma(\infty)\sum_{i=1}^{3} F_0^i + \sum_{i=1}^{3} a_i^\Gamma F_{w_i}^i = \sum_{i=1}^{3}(a_i^\Gamma F_{w_i}^i - h_i), \tag{6.100}$$

where $\sum_{i=1}^3 F_0^i = 0$ and $\sum_{i=1}^3 x_i F_0^i = -\sum_{i=1}^3 h_i$ from the first and second global Ward identities of (6.89) respectively. We now have an expression for α:

$$\alpha = \frac{1}{M}\sum_{i=1}^{3}(a_i^\Gamma F_{w_i}^i - h_i). \tag{6.101}$$

Thus the constraints imposed by (6.35) is equivalent to imposing that $G(z)$ has the following form:

$$G(z) = \alpha\partial\Gamma(z) = \frac{2}{w_1 + w_2 + w_3 - 1}\sum_{i=1}^{3}(a_i^\Gamma F_{w_i}^i - h_i)\partial\Gamma(z) \tag{6.102}$$

where the Riemann-Hurwitz formula was used to express M. (6.102) and (6.86) can be equated to each other and in principle, this allows us to determine the F_ℓ^i unknowns in terms of h_i, $F_{w_i}^i$ and the covering map $\Gamma(z)$ for each $\ell = 0, 1, ...w_i - 1$, thus solving the constraint equations.

Now we can make contact with the recursion relations. Expanding $G(z)$ of (6.86) around z_j and picking out the term of order $(w_j - 1)$:

$$\begin{aligned}\frac{\partial_z^{w_j-1} G(z = z_j)}{(w_j - 1)!} &= \frac{1}{(w_j-1)!}\partial_z^{w_j-1} \sum_{i \neq j} \left(-\frac{2(x_j - x_i)h_j}{z - z_i} + \sum_{\ell=0}^{w_i} \frac{(x_j - x_i)^2}{(z - z_i)^{\ell+1}} F_\ell^i \right)\Bigg|_{z=z_j} \\ &+ \frac{1}{(w_j-1)!}\partial_z^{w_j-1} \left(-\frac{2(\Gamma(z) - x_j)h_j}{z - z_j} + \sum_{\ell=0}^{w_j} \frac{(\Gamma(z) - x_j)^2}{(z - z_j)^{\ell+1}} F_\ell^j \right)\Bigg|_{z=z_j}\end{aligned} \tag{6.103}$$

where the term corresponding to $i = j$ was separated. Recall the discussion below (6.85) where it was mentioned that extending the sum to $i = j$ will only have an effect at order $m = w_j - 1$. This effect will now be computed; starting with the first term in the second line of (6.103)

$$\begin{aligned}&\frac{1}{(w_j-1)!}\partial_z^{w_j-1} \left(-\frac{2(\Gamma(z) - x_j)h_j}{z - z_j} \right)\Bigg|_{z=z_j} \\ &\quad = \frac{1}{(w_j-1)!}\partial_z^{w_j-1} \left(-\frac{2(x_j + a_j^\Gamma (z - z_j)^{w_j} + \mathcal{O}(z - z_j)^{w_j+1} - x_j)h_j}{z - z_j} \right)\Bigg|_{z=z_j} \\ &\quad = \frac{1}{(w_j-1)!}\partial_z^{w_j-1} \left(-2(a_j^\Gamma (z - z_j)^{w_j-1} + \mathcal{O}(z - z_j)^{w_j})h_j \right)\Bigg|_{z=z_j} \\ &\quad = -2a_j^\Gamma h_j\end{aligned} \tag{6.104}$$

where all terms with $(z - z_j)^k$ with $k > (w_j - 1)$ vanish after taking the derivative of order $(w_j - 1)$ because we evaluate it at $z = z_j$. Very similar arguments apply to the second term of the second line of (6.103) but care must be taken because the only non-vanishing term in the sum is at $\ell = w_j$ because $\frac{\Gamma^2(z)}{(z-z_j)^{w_j+1}} \sim (a_j^\Gamma)^2 \frac{(z-z_j)^{2w_j}}{(z-z_j)^{w_j+1}} = (a_j^\Gamma)^2 (z - z_j)^{w_j-1}$. We can rewrite (6.103) with these two extra terms as:

$$\begin{aligned}\frac{\partial_z^{w_j-1} G(z = z_j)}{(w_j - 1)!} &= \frac{1}{(w_j-1)!}\partial_z^{w_j-1} \sum_{i \neq j} \left(-\frac{2(x_j - x_i)h_j}{z - z_i} + \sum_{\ell=0}^{w_i} \frac{(x_j - x_i)^2}{(z - z_i)^{\ell+1}} F_\ell^i \right)\Bigg|_{z=z_j} \\ &- 2a_j^\Gamma h_j + (a_j^\Gamma)^2 F_{w_j}^i.\end{aligned} \tag{6.105}$$

Substituting (6.44) and (6.41) into the RHS of (6.105) gives:

$$\frac{\partial_z^{w_j-1} G(z=z_j)}{(w_j-1)!} = \left(h_j - \tfrac{(k+2)w_j}{2} - j_j\right) \frac{\left\langle V^{w_j}_{h_j-1}(x_j;z_j) \prod_{i\neq j}^3 V^{w_i}_{h_i}(x_i;z_i)\right\rangle}{\left\langle \prod_{i=1}^3 V^{w_i}_{h_i}(x_i;z_i)\right\rangle}$$
$$- 2a_j^\Gamma h_j + (a_j^\Gamma)^2 \left(h_j - \tfrac{(k+2)w_j}{2} + j_j\right) \frac{\left\langle V^{w_j}_{h_j+1}(x_j;z_j) \prod_{i\neq j}^3 V^{w_i}_{h_i}(x_i;z_i)\right\rangle}{\left\langle \prod_{i=1}^3 V^{w_i}_{h_i}(x_i;z_i)\right\rangle}. \tag{6.106}$$

But the LHS of (6.106) can be evaluated directly using (6.102),

$$\frac{\partial_z^{w_j-1} G(z=z_j)}{(w_j-1)!} = \frac{2}{w_1+w_2+w_3-1} \sum_{i=1}^3 (a_i^\Gamma F^i_{w_i} - h_i) \frac{\partial_z^{w_j-1} \partial\Gamma(z)}{(w_j-1)!}\Bigg|_{z=z_j}$$
$$= \frac{2}{w_1+w_2+w_3-1} \sum_{i=1}^3 (a_i^\Gamma F^i_{w_i} - h_i) \frac{\partial_z^{w_j-1}\left(w_j a_j^\Gamma (z-z_j)^{w_j-1} + \mathcal{O}(z-z_j)^{w_j}\right)}{(w_j-1)!}\Bigg|_{z=z_j}$$
$$= \frac{2w_j a_j^\Gamma}{w_1+w_2+w_3-1} \sum_{i=1}^3 (a_i^\Gamma F^i_{w_i} - h_i). \tag{6.107}$$

If the RHS of (6.106) and (6.107) are equated then we get one such recursion relation for the 3-point function. We are ready to insert our ansatz (6.72) into both sides of the recursion relation to confirm that it satisfies it. Starting with (6.106):

$$\frac{\partial_z^{w_j-1} G(z=z_j)}{(w_j-1)!} = \left(h_j - \tfrac{(k+2)w_j}{2} - j_j\right) \frac{C(a_j^\Gamma) \prod_{i=1}^3 (a_i^\Gamma)^{-h_i} \prod_{i\neq j} (z_i-z_j)^{\Delta^0_\ell - \Delta^0_i - \Delta^0_j}}{C \prod_{i=1}^3 (a_i^\Gamma)^{-h_i} \prod_{i\neq j} (z_i-z_j)^{\Delta^0_\ell - \Delta^0_i - \Delta^0_j}}$$
$$- 2a_j^\Gamma h_j + (a_j^\Gamma)^2 \left(h_j - \tfrac{(k+2)w_j}{2} + j_j\right) \frac{C(a_j^\Gamma)^{-1} \prod_{i=1}^3 (a_i^\Gamma)^{-h_i} \prod_{i\neq j} (z_i-z_j)^{\Delta^0_\ell - \Delta^0_i - \Delta^0_j}}{C \prod_{i=1}^3 (a_i^\Gamma)^{-h_i} \prod_{i\neq j} (z_i-z_j)^{\Delta^0_\ell - \Delta^0_i - \Delta^0_j}}$$

$$= \left(h_j - \tfrac{(k+2)w_j}{2} - j_j\right) a_j^\Gamma - 2a_j^\Gamma h_j + (a_j^\Gamma)^2 \left(h_j - \tfrac{(k+2)w_j}{2} + j_j\right) (a_j^\Gamma)^{-1}$$
$$= h_j a_j^\Gamma - \tfrac{(k+2)w_j}{2} a_j^\Gamma - j_j a_j^\Gamma - 2a_j^\Gamma h_j + a_j^\Gamma h_j - \tfrac{(k+2)w_j}{2} a_j^\Gamma + j_j a_j^\Gamma$$
$$= -a_j^\Gamma (k+2) w_j. \tag{6.108}$$

And the RHS of (6.107):

$$\frac{\partial_z^{w_j-1} G(z=z_j)}{(w_j-1)!} = \frac{2w_j a_j^\Gamma}{w_1+w_2+w_3-1}\sum_{i=1}^{3}\Bigg(a_i^\Gamma (h_i - \tfrac{(k+2)w_i}{2} + j_i)...$$
$$...\frac{C(a_i^\Gamma)^{-1}\prod_{j=1}^{3}(a_j^\Gamma)^{-h_j}\prod_{i\neq j}(z_i-z_j)^{\Delta_\ell^0-\Delta_i^0-\Delta_j^0}}{C\prod_{i=1}^{3}(a_i^\Gamma)^{-h_i}\prod_{i\neq j}(z_i-z_j)^{\Delta_\ell^0-\Delta_i^0-\Delta_j^0}} - h_i\Bigg)$$

$$\begin{aligned} &= \frac{2w_j a_j^\Gamma}{w_1+w_2+w_3-1}\sum_{i=1}^{3}\Big(a_i^\Gamma (h_i - \tfrac{(k+2)w_i}{2} + j_i)(a_i^\Gamma)^{-1} - h_i\Big) \\ &= \frac{2w_j a_j^\Gamma}{w_1+w_2+w_3-1}\sum_{i=1}^{3}\Big(j_i - \tfrac{(k+2)w_i}{2}\Big) \\ &= 2w_j a_j^\Gamma \frac{(k+2)}{2}\frac{\sum_{i=1}^{3}(1-w_i)}{\sum_{i=1}^{3}(w_i-1)} = -a_j^\Gamma (k+2) w_j, \end{aligned} \tag{6.109}$$

where in going to the last line we have used the assumption that $\sum_{i=1}^{3} j_i = \frac{k+2}{2}$ from (6.73). The above two expressions are exactly equal to each other, showing that the ansatz (6.72) indeed solves the recursion relations under the condition that (6.73) is satisfied.

6.6 The solution in the general case

The solution of the 3-point function can be generalized to that of an n-point function:

$$\Big\langle V_{h_1}^{w_1}(0;0)V_{h_2}^{w_2}(1;1)V_{h_1}^{w_1}(\infty;\infty)\prod_{i=4}^{n} V_{h_i}^{w_i}(x_i;z_i)\Big\rangle = \sum_\Gamma \prod_{i=1}^{n}(a_i^\Gamma)^{-h_i}\prod_{i=4}^{n}\delta(x_i-\Gamma(z_i))W_\Gamma(z_4,...,z_m), \tag{6.110}$$

that solves the recursion relations. The W_Γ is a function that depends on the remaining cross-ratios on the covering sphere, the spins j_i and k. It is the weight of the covering Γ contribution to the correlator. The condition on the spins in (6.73) is also generalised to

$$\sum_{i=1}^{n} j_i = \frac{(k+2)}{2}(n-2) - (n-3), \tag{6.111}$$

which must be satisfied in order for (6.110) to be a valid solution. The second term arises because there are $(n-3)$ many delta functions, as we will soon see. Note that for $n = 3$ (6.73) is recovered. The delta function on the RHS of (6.110) is to ensure that the analytic function $\Gamma(z)$ satisfies $\Gamma(z_i) = x_i$ i.e that it is indeed the relevant covering over the boundary sphere with marked points $\{x_i\}_{i=1,..,n}$ (see the discussion surrounding (6.71). Additionally there may be multiple coverings over the boundary sphere, as in the 4-point example in (6.71), hence the discrete sum.

The recursion relations that were explicitly given in (6.49) for the case when $w_i = 1$ for all i can be used to test the solution (6.110). In this case there is one (unique) covering map $\Gamma(z) = z$ and $a_i^\Gamma = 1$ for all i. Using the distributional identity,

$$(x_i - z_i)\delta'(x_i - z_i) = -\delta(x_i - z_i), \tag{6.112}$$

we use the general solution (6.110) applied to (6.49):

$$\begin{aligned}
&(h_1 - \frac{k+2}{2} - j_1)\prod_{i=4}^{n}\delta(x_i - z_i)W_\Gamma(z_4, ...z_n) \\
&= \left[h_1 - h_2 - h_3 + \sum_{i=4}^{n}\frac{h_i(z_i - 2x_i) + x_i(z_i - x_i)\partial_{x_i}}{z_i}\right]\prod_{i=4}^{n}\delta(x_i - z_i)W_\Gamma(z_4, ...z_n) \\
&\quad + \sum_{i=2}\frac{x_i^2}{z_i^2}\left(h_i - \tfrac{k+2}{2} + j_i\right)\prod_{i=4}^{n}\delta(x_i - z_i)W_\Gamma(z_4, ...z_n) \\
&= \left[h_1 - h_2 - h_3 + \sum_{i=4}^{n}\frac{h_i(z_i - 2x_i) + x_i}{z_i}\right]\prod_{i=4}^{n}\delta(x_i - z_i)W_\Gamma(z_4, ...z_n) \\
&\quad + \sum_{i=2}\frac{x_i^2}{z_i^2}\left(h_i - \tfrac{k+2}{2} + j_i\right)\prod_{i=4}^{n}\delta(x_i - z_i)W_\Gamma(z_4, ...z_n) \\
&= \left(\sum_{i=2}^{n}(h_i - \frac{k+2}{2} + j_i) + (n-3) + h_1 - \sum_{i=2}^{n}h_i\right)\prod_{i=4}^{n}\delta(x_i - z_i)W_\Gamma(z_4, ..., z_n),
\end{aligned} \tag{6.113}$$

where in going to the second equality the distributional identity was used and in going to the last equality we simplified the coefficient by taking $x_i = z_i$, which is otherwise forced by the delta function. Clearly, if the condition(6.111) is obeyed then the RHS becomes

$$\left(\sum_{i=2}^{n} h_i - \sum_{i=2}^{n} h_i - \frac{k+2}{2}(n-1) + (n-3) + h_1 + \sum_{i=1}^{n} j_i - j_1\right) \prod_{i=4}^{n} \delta(x_i - z_i) W_\Gamma(z_4, ..., z_n)$$
$$= (h_1 - \frac{k+2}{2} - j_1) \prod_{i=4}^{n} \delta(x_i - z_i) W_\Gamma(z_4, ...z_n), \tag{6.114}$$

thus satisfying the LHS of (6.113). The arguments outlined in section 6.5 for the 3 point function can be generalised to the $n \geq 4$ case to prove that (6.110) is always a solution to the recursion relations, the details of which are outlined in appendix A of [1].

6.7 The j condition

A possible solution to the condition on the spins j_i (6.111) that holds in general is:

$$j_1 = 1 - \frac{k}{2}, \quad j_i = \frac{k}{2}, \quad \text{for } 2 \leq i \leq n \tag{6.115}$$

The spin of the representation $j = \frac{k}{2}$ is precisely the representation of the worldsheet that corresponds to the twisted sector of the ground states in the dual symmetric product orbifold and $j = 1 - \frac{k}{2}$ is the conjugate representation.

In chapter 5 the spectrum of the tensionless ($k = 1$) superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ was matched precisely with that of the symmetric product orbifold theory of $\mathbb{T}^4$. It was also shown that for $k = 1$, the only allowed worldsheet representation is $j = \frac{1}{2}$. This tells us that our analysis on correlation functions is consistent since at $k = 1$, the condition (6.111) becomes particularly simple; it is always satisfied for all $j_i = \frac{1}{2}$.

6.8 String perturbation theory

So far we have only found the structure of the n-point correlation functions at tree-level. It delta function localizes onto worldsheet configurations in worldsheet moduli space, at genus 0, that act as the covering space over the boundary sphere. When we integrate over the moduli space of worldsheets as in the RHS of (1.1), the integral transforms into a discrete sum over coverings.

The above analysis generalizes to worldsheets at higher genus, and the correlation functions exhibit a similar delta function localisation property; take for example the worldsheet theory defined on a higher genus Riemann surface Σ_{g}, we state without proof the condition on the spins j_i in this case is

$$\sum_{i=1}^{n} j_i = \frac{(k+2)}{2}(n-2+2\text{g}) - \dim(\mathcal{M}_{\text{g},n}) \tag{6.116}$$

where $\dim(\mathcal{M}_{g,n}) = 3g - 3 + n$ is the dimension of the moduli space of genus g Riemann surfaces with n punctures[†]. Note that this is simply the generalisation of the constraint (6.111). Again, the second term arises because there will be $\dim(\mathcal{M}_{g,n})$ many delta functions, i.e. the integral completely localizes onto worldsheet configurations of genus g that covers the boundary sphere. At $k = 1$ (6.118) is satisfied for all $j_i = \frac{1}{2}$, independent of the genus, which is consistent with what we expect from a tensionless theory on AdS_3.

Now is an opportune time to recall the the findings of chapter 3, where the structure of the correlation functions of the symmetric product orbifold was found. In the large N limit, the gauge invariant n-point correlator takes the form

$$\langle \mathcal{O}_1 \mathcal{O}_2 ... \mathcal{O}_n \rangle_{S^2} \sim \sum_{\text{g}} N^{1-\text{g}-\frac{n}{2}} \sum_{\Gamma_{\text{g}}} e^{S_L[\phi]} \Big|_{\Gamma_{\text{g}}}, \tag{6.117}$$

where we have ignored the sub-leading terms of (3.42). The RHS was evaluated using the Lunin & Mathur approach, where computing the correlation function of twist operators on the base sphere is equivalent to computing the vacuum path integral on the covering space, with no operator insertion, accompanied by the exponentiated Liouville action (see section 3.2). Since there is actually an infinite series of subleading terms for each genus contrbution, it is difficult to express the correspondence in terms of N. If we identify $g_s \sim \frac{1}{\sqrt{N}}$ then the correlator goes as

$$\langle \mathcal{O}_1 \mathcal{O}_2 ... \mathcal{O}_n \rangle_{S^2} \sim g_s^{2-2\text{g}-n}, \tag{6.118}$$

so the genus of the covering map should be identified with the genus of the corresponding worldsheet. In fact, what our analysis has shown is that we may identify the worldsheet as the covering space, which guarantees that all higher $1/N$ corrections are reproduced from the worlsheet theory. This was already predicted in [39] [45].

We can now go further and make some claims about the worldsheet

[†] see section 8.1 of [1] for a derivation using the conservation of the anomalous charge.

correlators using what we know about the twist correlators. Take the Riemann-Hurwitz relation

$$2 - 2\mathrm{g} = 2M - \sum_{i=1}^{n}(w_i - 1), \tag{6.119}$$

where M is the degree of the covering map (see for example (3.33)). Using the fact that $M \geq \max(w_i)$ we have the relation

$$\begin{aligned} 2 - 2\mathrm{g} &\geq 2\max(w_i) - \sum_{i=1}^{n}(w_i - 1) \\ \Rightarrow \mathrm{g} &\leq 1 - \max(w_i) + \sum_{i=1}^{n}\frac{w_i - 1}{2}, \end{aligned} \tag{6.120}$$

thus the n-point correlation function vanishes on sufficiently high enough genera, hence the discrete sum over genera in the correlation functions is finite. Furthermore, for the 2-point function case, we know from the constraint (3.12) that $M = w_1 = w_2$, thus we have $\mathrm{g} = 0$. So for the corresponding two point function on the worldsheet, there are only tree level contributions, which is consistent with the analysis in [37] where the spectrum was matched exactly at tree level and did not require any higher genus corrections.

Chapter 7

Conclusions

In this particular instance of the gauge/string duality, we were able to make progress toward demystifying the underlying mechanism that relates both sides of the dictionary (1.1). The first step was to find which string theory on AdS_3 was dual to which CFT. Using the powerful prescription of Maldacena, it became clear that the CFT dual to type IIB string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ was to lie on the same moduli space as the symmetric product orbifold of $\mathbb{T}^4$. The best possible guess for finding the string theory dual to the symmetric product CFT was at the corresponding symmetric point on string moduli space i.e. the tensionless point. If two theories can be shown to have identical energy spectra, then this provides very strong evidence that they are dual to one another, and this is precisely what we've shown. Now that there was an example of a precise duality between a string theory and a gauge theory, we were able to probe the inner workings of the relationship by revealing the structure of their respective correlation functions. We were able to explicitly show from a worldsheet analysis that the correspondence is most natural by identifying the covering space of the boundary CFT with the worldsheet. This is captured in the delta function localisation property of the worldsheet correlators, which converts the integral over moduli space into a discrete sum over worldsheet coverings over the boundary sphere. This reproduces all of the features of the dual symmetric product orbifold.

The quantum analysis on the worldsheet correlators was carried out in chapter 6. This was the first prong of the worldsheet analysis in [1] and in section 7 of that paper, the classical solutions of string theory on AdS_3 were studied. A family of solutions that describe the ground state of the w spectrally flowed sector with $j = \frac{1}{2}$ was found by taking a suitable limit that localises them to the boundary of AdS_3. At $k = 1$, the size of AdS_3 is microscopic and highly curved, however this localisation property allowed for the semi-classical study of the worldsheet and the same solutions emerge from the quantum correlators developed in **??**. The worldsheet action becomes identical to the Liouville action picked up in the correlator analysis of the symmetric product orbifold.

As mentioned in section 6.8 there is a natural generalisation of the quantum analysis to correlation functions on higher genus Riemann surfaces. In [59] this generalisation was realized using a Ward identity analysis of the $SL(2,\mathbb{R})_{k+2}$ WZW model at higher genus. The Ward identities constrains the form of the correlation functions to possess the delta function localisation property, as for genus 0, such that the only contributions to the correlator come from worldsheets that holomorphically cover the boundary of AdS_3, on the condition that (6.118) is satisfied. This ensures that the integral over moduli space reduces to a finite sum. As discussed in section 6.8, this ensures that all $1/N$ corrections of the dual CFT are reproduced from the worldsheet perspective.

State of the art

Since the discovery of the proposed duality in the papers [1][36][37], there has been many significant advances in closely related topics to the AdS/CFT correspondence and gauge/string dualities more generally. The exact correspondence has also allowed to study adjacent subjects such as string scattering amplitudes and partition functions with greater precision. In this section we aim to briefly introduce these advances in the last few years and some potential future directions for research.

In [60], DDF operators for string theory on AdS_3 with pure NS-NS flux were constructed and the symmetry algebra for the spacetime CFT could be directly read off. After carrying out an analysis for the spacetime spectrum, it was found that the CFT dual to superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with general NS-NS flux is the symmetric product orbifold of $(\mathcal{N} = 4 \text{ Liouville theory}) \times \mathbb{T}^4$. In the special case where $k = 1$, corresponding to one unit of NS-NS flux, then the Liouville factor vanishes thus confirming the findings in [1][36][37].

String theory on $AdS_3 \times S^3 \times S^3 \times S^1$ was studied with minimal NS-NS flux [61]. On this background the spacetime spectrum and the symmetry generating operators matches exactly with that of the symmetric product orbifold of $S^3 \times S^1$ in the large N limit, thus giving strong support that these two theories are dual to one another.

The bosonic analogue of the duality proposed in this paper was studied [62]. In particular, the three point functions on $AdS_3 \times X$ and the symmetric orbifold of Liouville $\times$ X were compared, since it is believed that 2d CFT's are uniquely characterized by the structure constants of three point functions (together with the spectrum). It was showed the the low lying null vectors in Liouville theory correspond to the BRST exact states from the worldsheet perspective. The BRST exact states yield the expected BPZ equations for the dual CFT correlators. This provides support for the existence of the Liouville factor in the dual CFT, since the structure constants of Liouville theory are (uniquely) fixed by these constraints. Furthermore, in [63] a closed form formula for worldsheet three-point correlation functions of spectrally flowed vertex operators were proposed for the first time, on euclidean AdS_3 with pure NS-NS flux. In fact the analysis makes critical use of the recursion relations developed in chapter subsection 6.2.3. The structure constants in three point functions are completely fixed by the global conformal symmetry of the boundary sphere of AdS_3 and the worldsheet. Higher point functions contain much more dynamical information. In [64] a map was found that transforms bosonic worldsheet spectrally unflowed four-point functions into spectrally flowed four point functions, at genus 0. The singularities of the four point function was further studied in [65].

The correlation functions of string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with unit NS-NS flux were constrained in the free field realisation of $\mathfrak{psu}(1,1|2)$ [66]. The delta function localisation of the correlators was reproduced from this perspective, and was derived from the constraints of the OPE of the symplectic bosons with the spectrally flowed vertex operators, captured in a special 'incidence relation'.

In section 5.5 the holographic dictionary $L_0^{\mathrm{CFT}} = J_0^3$ was critical in matching the spectra. In [67] the stress-tensor of the symmetric product orbifold theory was identified with a specific vertex operator on the worldsheet. The 3-point function of this vertex operator with the spectrally flowed vertex operators that corresponds to the w-cycle twisted ground states of the dual CFT were calculated and the conformal dimension was directly read off. The authors showed that this alternative method of calculating the conformal dimension leads to the same result.

The aim of the analysis in the previous chapters was to make progress towards uncovering the underlying mechanism that relates both sides of the dictionary (1.1) to each other for the AdS_3/CFT_2 duality. Previously [68] [69] [70], a proposal was made for what this underlying mechanism might be for gauge/string duality more generally, and it typically involved reorganising the sum over the worldline trajectories that the Feynman diagrams represent in terms of a sum over the worldsheets. It is a direct prescription for going from the LHS to the RHS of (1.1) which associates specific points in worldsheet moduli space to individual feynman diagrams. It involves organising genus g feynman diagrams into 'skeletal graphs' [69] and identifying them with the so-called critical graphs associated with the Strebel differential defined on the worldsheet Riemann surface. By using the known AdS_3/CFT_2 duality derived in this paper as a laboratory, it was tested and confirmed in [71] that the correct worldsheet theory can be reconstructed using this prescription on the n-point correlation function of twist operators.

In chapter 1 it was mentioned that the conception of the AdS/CFT correspondence was exemplified with the conjectured duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory [9]. However it was never known which two points in moduli space between the two theories are exactly matched. Inspired by the precise duality between the tensionless string theory on AdS_3 and the free symmetric product orbifold, in [72] it was proposed that the tensionless string theory on $AdS_5 \times S^5$ is exactly dual to free $\mathcal{N} = 4$ SYM theory in 4d. A worldsheet theory was proposed and like it's lower dimensional analogue, it contains a free field realization and w spectrally flowed representations and is closely related to an ambi-twistor string theory. The main claim in [72] was that the physical sector of the oscillator Fock space, for each w, is identical to the space of all single trace operators built from w SYM fields and their derivatives. A set of residual gauge constraints were used to identify the physical fock space. In [73] the equivalence was derived analytically on the $PSU(2,2|4)$ sigma model; the planar operator spectrum of SYM agrees with the string theory after imposing certain gauge fixing conditions for quantisation of the sigma model.

In [74] a similar correspondence as [72] [73] described above was found for gauge theories with less supersymmetry. A family of $\mathcal{N} = 2$ superconformal field theories arise form taking the $\mathbb{Z}_k$ orbifold of the free $\mathcal{N} = 4$ SYM theory. In particular, this theory can be produced by taking a stack of D3 branes on a $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_k$ orbifold singularity with near horizon geometry of $AdS_5 \times S^5/\mathbb{Z}_k$. The resulting field content of the $\mathcal{N} = 2$ theory can be described by a circular quiver. The worldsheet theory is fixed by the orbifold construction and by following the prescriptions of [72] [73] to find the physical states, the spectrum was matched precisely with the $\mathcal{N} = 2$ quiver gauge theory.

The study of the CFT dual to strings on AdS_3 with pure NS-NS flux was revisited in [75]. The worldsheet theory gives a prediction for the correlation functions of vertex operators. It was investigated whether these correlators can be used to define the dual CFT at least perturbatively. In particular it was shown that the correct three-point functions of the CFT were derived from the worldsheet. Bosonic string theory on AdS_3 was considered, although it suffers from the usual tachyon instability, the AdS/CFT duality remains valid at tree level. The CFT is a symmetric orbifold of $\mathbb{R}_Q \times X$ where $\mathbb{R}_Q$ is a linear dilation direction and X is the internal CFT of the compactification. The constructed CFT is a deformation away from the symmetric point, in the linear dilation direction, using the marginal operator $e^{-\sqrt{k-2}\phi}\sigma_2$ where ϕ is the linear dilation and σ_2 is the twist 2 operator. The 'coloumb gas' formalism of [76] was used to derive string like 3 point functions up to fourth order in the deformation parameter μ. Additionally, in the supersymmetric case, the corresponding perturbation on the worldsheet side was identified and the correct conformal dimension of the symmetric product orbifold currents were reproduced from the worldsheet perspective [77].

The tensionless string theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ can be described as a free field realisation in the hybrid formalism. There are two families of maximal $SL(2,\mathbb{R})$ branes that can be constructed in this model: the AdS_2 and the spherical branes. The worldvolumes of the spherical branes are localised at a fixed time thus describing instantonic branes in AdS_3. The spherical brane extends all the way to the boundary of AdS_3 and creates a circular one dimensional defect on the two dimensional boundary. In [78] it was found that this defect is to be identified with certain boundary states of the symmetric product orbifold theory. It was also shown that the disk amplitudes of the D-branes localise in worldsheet moduli space to points that holomorphically covers the spacetime disk.

For AdS_3, the analyses on the worldsheet thus far has only involved correlation functions of vertex operators that are dual to the twisted sector ground states. This was generalised in [79] to include correlation functions of operators that are dual to the BPS states of the symmetric product orbifold. This was motivated by the interest of eventually generalising to all single particle states. The corresponding worldsheet states were identified and using the techniques of [66] the correlators were found up to an unfixed prefactor, which matched precisely with the BPS correlators produced in [80].

In chapter section 3.2 we made use of the approach of Lunin & Mathur to calculate correlation functions in the symmetric product orbifold theory. In [81], the same results for the correlation functions were obtained via a short and elegant method using the symmetry algebra. On the other hand, the correlation functions were fixed via differential equations hence the normalisation prefactor could not be calculated from the null vectors.

Instead, it was computed by factorisation of the four point functions into three point functions.

The partition function of the tensionless string on different backgrounds was considered in [82]. In particular, the partition function on thermal AdS_3 was computed and was found to be independent of the bulk geometry and only dependent on the geometry of the conformal boundary. Conventionally, to compute the boundary partition function, one has to sum over different bulk geometries but in fact it agrees with a single string partition function with the appropriate boundary. The Hawking-Page transition [83] can be seen from this point of view. Recall that in AdS spacetimes, Hawking and Page found a minimum temperature below which thermal radiation is stable and does not collapse to form a black hole but above which a black hole is in thermodynamic equilibrium with the radiation. The symmetric product orbifold posseses this Hawking-Page temperature, T_{HP}. Conventionally, when computing the gravitational path integral we are expected to sum over all bulk geometries: at temperatures below T_{HP}, the sum is dominated by thermal AdS while above T_{HP}, the sum is dominated by the BTZ black hole geometry. However in [82] this sum over geometries is instead replaced by a sum over worldsheet configurations, in particular the ones that holomorphically cover the boundary CFT. This is captured by the delta function in the partition function which for low temperatures, the sphere (vacuum) dominates while for high temperatures, the torus dominates. In the latter scenario, the string configuration consists of a gas of spherical worldsheets and a singular long string that winds around the boundary N many times (with N large). Thus the BTZ black hole has a dual description as a singular perturbative winding string. This is a concrete realisation of the black hole/string transition [84] [85] which states that when the curvature of a black hole is of the order of the string length scale, there is a correspondence between the black hole and a single excited fundamental string. The analysis of [82] was further strengthened in [86] where it was showed that the string partition functions around a fixed bulk background already includes a sum over geometries, and large stringy corrections are associated with various semi-classical geometries. Also, it was shown that the string partition function of a wormhole geomtry factorizes into the boundaries of spacetime. Again, the delta function localization property of the worldsheet moduli space onto covering spaces is central to this analysis.

Future directions

One of the biggest problems is that the hybrid formalsim which was developed in [38] and used in [37] to match the spectra is terribly ineffective; a more efficient worldsheet description should be developed.

From the Ward identity analysis, we conjectured a solution to the correlators that contains the delta function localisation property. It works very generally, however it would be desirable to find a simple argument for why this has to happen.

Since the BTZ black hole is related to the $PSU(1,1|2)$ WZW model by a $\mathbb{Z}$ orbifold, it would be interesting to compute correlation functions on this background. The partition function was already considered in [82]. This has the potential to probe black hole evaporation. One may consider 2-point functions where there are only genus zero contributions to start and then consider correlation functions yielding higher genera contributions. Taking the theory to the tensionless point makes the problem sufficiently simpler since we know the CFT dual.

Finally, it would be interesting to consider a worldsheet analysis for superstring theory on $AdS_3 \times S^3 \times S^3 \times S^1$, which is claimed to be dual to the symmretic product orbifold of $S^3 \times S^1$ with one unit of flux on one of the S^3's. It is not yet known how to related correlation functions to that of the symmetric product orbifold.

Chapter 8

Bibliography

[1] L. Eberhardt, M.R. Gaberdiel, R. Gopakumar, "Deriving the AdS_3/CFT_2 Correspondence," (2020), arXiv:1911.00378v2 [hep-th]

[2] G. 't Hooft, "A planar diagram theory for strong interactions", (1974), Nucl. Phys. B72 (1974) 461.

[3] M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function ", (1992) Commun. Math. Phys. 147, 1-23.

[4] E. Witten, "On The Structure Of The Topological Phase Of Two-dimensional Gravity ", (1990) Nucl. Phys. B340, 281-332

[5] J. Distler, "2-d Quantum Gravity, Topological Field Theory And The Multicritical Matrix Models", (1990) Nucl. Phys. B342, 523-538

[6] E. P. Verlinde, H. L. Verlinde, "A Solution Of Two-dimensional Topological Quantum Gravity", (1991), Nucl. Phys. B348, 457-489

[7] D. Gaiotto, L. Rastelli, "A Paradigm of open / closed duality: Liouville D-branes and the Kontsevich model", (2005) JHEP 0507, 053. [hep-th/0312196].

[8] R. Gopakumar, C. Vafa, "On the gauge theory / geometry correspondence", (1999), Adv. Theor. Math. Phys. 3, 1415-1443. [hep-th/9811131].

[9] J. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," (1998), hep-th/9711200.

[10] M. R. Douglas, "Branes within branes," (1995), hep-th/9512077.

[11] C. Vafa, “Instantons on D-branes,” Nucl. Phys. B463 (1996) 435â442, hep-th/9512078.

[12] E. Witten, “On the conformal field theory of the Higgs branch,” JHEP 07 (1997) 003, hep-th/9707093.

[13] J. Maldacena, A. Strominger, “AdS_3 Black Holes and a Stringy Exclusion principle,” (1998), hep-th/9804085

[14] N. Seiberg, E. Witten, “The D1/D5 System and Singular CFT,”(1999), hep-th/9903224

[15] J. Maldacena, H. Ooguri, “Strings in AdS_3 and the $SL(2, R)$ WZW model. Part 1: The Spectrum” (2000), arXiv:hep-th/0001053v

[16] J. Maldacena, H. Ooguri, J. Son, “Strings in AdS_3 and the $SL(2, R)$ WZW model. Part 2: Euclidean Black Hole” (2000), arXiv:hep-th/0005183

[17] J. Maldacena, H. Ooguri, “Strings in AdS_3 and the $SL(2, R)$ WZW model. Part 3: Correlation Functions” (2001), arXiv:hep-th/0111180

[18] G. Bonelli, “On the Tensionless Limit of Bosonic Strings, Infinite Symmetries and Higher Spins," (2003), arXiv:hep-th/0305155

[19] M.A. Vasiliev, “Non-linear equations for symmetric massless higher sping fields in (A)dS(d)", (2003) [arXiv:hep-th/0304049].

[20] B. Sundborg, “Stringy gravity, interacting tensionless strings and massless higher spins", (2001), [arXiv:hep-th/0103247]

[21] A. Mikhailov, “Notes on higher spin symmetries", (2002), arXiv:hep-th/0201019.

[22] E. Sezgin and P. Sundell, “Massless higher spins and holography", (2003), arXiv:hep-th/0205131

[23] I.R. Klebanov, A.M. Polyakov, “AdS dual of the critical O(N) vector model", (2002), [arXiv:hep-th/0210114].

[24] S. Giombi, X. Yin, “Higher spin gauge theory and holography: the three-point functions", (2009), arXiv:0912.3462.

[25] S. Giombi, X. Yin, “Higher spins in AdS and twistorial holography", (2010), arXiv:1004.3736.

[26] M. R. Gaberdiel, R. Gopakumar, "An AdS_3 Dual for Minimal Model CFTs", (2012), arXiv:1011.2986v4

[27] M. R. Gaberdiel, R. Gopakumar, "Large $\mathcal{N} = 4$ Holography", (2014) arXiv:1305.4181v2.

[28] J. Isberg, U. Lindstrom, B. Sundborg and G. Theodoridis, "Classical and Quantised Strings", Nucl. Phys. B 411, 122 (1994) [hep-th/9307108].

[29] U. Lindstrom, M. Zabine, "Tensionless strings, WZW models at critical level and massless higher spin fields", (2004), Phys. Lett. **584B**, 178.

[30] A. Karlhede and U. Lindstrom, "The classical bosonic string in the zero tension limit", Class.Quant.Grav. 3 (1986) L73.

[31] U. LindstrÂšom, B. Sundborg and G. Theodoridis, "The zero tension limit of the superstring", Lett. 253B (1991) 319.

[32] E. S. Fradkin and M. A. Vasiliev, "On the gravitational interaction of massless higher-spin fields ", Phys. Lett. B 189, 89 (1987).

[33] E. S. Fradkin and M. A. Vasiliev, "Consistent equations for interacting gauge fields of all spins in 3+1 dimensions", Nucl. Phys. B 291, 141 (1987).

[34] M.R. Gaberdiel, R. Gopakumar, "Higher Spins & Strings" (2014) arXiv:1406.6103v2.

[35] K. Ferreira, M. R. Gaberdiel, J.I. Jottar, "Higher spins on AdS_3 from the worldsheet," (2017), arXiv:1704.08667v1 [hep-th]

[36] M. R. Gaberdiel, R. Gopakumar, "Tensionless String Spectra on AdS_3" (2018), arXiv:1803.04423v1 [hep-th].

[37] L. Eberhardt, M.R. Gaberdiel, R. Gopakumar, "The Worldsheet Dual of the Symmetric Product CFT", (2018), arXiv:1812.01007v1 [hep-th]

[38] N. Berkovits, C. Vafa and E. Witten, "Conformal field theory of AdS background with Ramond-Ramond ux," JHEP 9903 (1999) 018 [hep-th/9902098]

[39] A. Pakman, L. Rastelli, S. Razamat, "Diagrams for Symmetric Product Orbifolds", 2009 JHEP 0910:034, arXiv:0905.3448v2

[40] A. Dei, L. Eberhardt. “Correlators of the symmetric product orbifold" , (2020) JHEP01(2020)108

[41] G.E. Arutyunov and S.A. Frolov, “Virasoro amplitude from the S^N R^{24} orbifold sigma-model", (1998) Theor. Math. Phys. 114 43 [hep-th/9708129].

[42] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, “The Conformal Field Theory of Orbifolds", (1987) Nucl. Phys. B 282 13 [INSPIRE].

[43] S. Hamidi and C. Vafa, “Interactions on Orbifolds", Nucl. Phys. B 279 (1987) 465

[44] R. Dijkgraaf, C. Vafa, E.P. Verlinde and H.L. Verlinde, “The Operator Algebra of Orbifold Models", Commun. Math. Phys. 123 (1989) 485

[45] O. Lunin and S.D. Mathur, “Correlation functions for M^N/S_N orbifolds", (2001) Commun. Math. Phys. 219 p 399

[46] S. K. Lando, “Ramified coverings of the two-dimensional sphere and the intersection theory in spaces of meromorphic functions on algebraic curves", (2002) Russ. Math. Surv. 57 463â533

[47] R. Cavalieri and E. Miles, London Mathematical Society Student Texts. Vol. 87: “Riemann surfaces and algebraic curves", Cambridge University Press, Cambridge U.K. (2016)

[48] O. Forster, “Lectures on Riemann surfaces. MÃŒnchen: Springer Berlin" (1991) ISBN 978-3-540-90617-9

[49] L. Eberhardt, “Wess-Zumino-Witten Models" at the YRISW PhD school in Vienna.

[50] P. Goddard, D.I. Olive and G. Waterson, “Superalgebras, Symplectic Bosons and the Sugawara Constructionâ, (1987) Commun. Math. Phys. **112**, 591.

[51] C. Beem, W. Peelaers, L. Rastelli and B.C. van Rees, “Chiral algebras of class Sâ, (2015) JHEP 1505 020 [arXiv:1408.6522 [hep-th]].

[52] M.B. Green and J.H.Schwarz, âCovariant Description of Superstrings,â Phys.Lett. 136B(1984)367

[53] N. Berkovits and W. Siegel, âSuperspace Effective Actions For 4âD Compactifications of Heterotic and Type II Superstrings,â Nucl.Phys.B462(1996)213.

[54] J. de Boer and K.Skenderis, âCovariant Computation of the Low Energy Effective Action of the Heterotic Superstringâ, Nucl.Phys.B481(1996)129.

[55] D. Friedan, E. Martinec, and S. Shenker, âConformal Invariance, Supersymmetry, and String Theory,â Nucl.Phys.B271(1986)93.

[56] M.R. Gaberdiel and S. Gerigk, âThe massless string spectrum on $AdS_3 \times S^3$ from the supergroup,â JHEP1110(2011)045 [arXiv:1107.2660[hep-th]]

[57] J. Troost, â Massless particles on supergroups and $AdS_3 S^3$ supergravity,â JHEP 1107(2011)042[arXiv:1102.0153[hep-th]].

[58] S. Gerigk, âString States on $AdS_3 S^3$ from the Supergroup,â JHEP1210(2012) 084[arXiv:1208.0345[hep-th]].

[59] L. Eberhardt, "AdS_3/CFT_2 at higher genus," (2020) arXiv:2002.11729v1 [hep-th]

[60] L. Eberhardt, M. Gaberdiel, "String theory on AdS_3 and the symmetric orbifold of liouville theory," (2019) arXiv:1903.00421v1 [hep-th]

[61] L. Eberhardt, M. Gaberdiel, "Strings on $AdS_3 \times S^3 \times S^3 \times S^1$," (2019) arXiv:1904.01585v1 [hep-th]

[62] A. Dei, L. Eberhardt, M. Gaberdiel, "Three-point functions in AdS_3/CFT_2 holography," (2019) arXiv:1907.13144v1 [hep-th]

[63] A. Dei, L. Eberhardt, "String correlators on AdS_3: Three-point functions," (2020) arXiv:2105.12130v2 [hep-th]

[64] A. Dei, L. Eberhardt, "String correlators on AdS_3: Four-point functions," (2021) arXiv:2107.01481v2 [hep-th]

[65] A. Dei, L. Eberhardt, "String correlators on AdS_3: Analytic structure and dual CFT" (2022) arXiv:2203.13264v2 [hep-th]

[66] A. Dei, M. Gaberdiel, R. Gopakumar, B. Knighton, "Free field worldsheet correlators for AdS_3" (2020) arXiv:2009.11306v1 [hep-th]

[67] H. Bertle, A. Dei, M. Gaberdiel, "Stress-energy tensor correlators from the world-sheet," (2020) arXiv:2012.08486v1 [hep-th]

[68] R. Gopakumar, âFrom free fields to AdS space. I,â Phys. Rev. D 70 (2004) 025009 [hep-th/0308184]

[69] R. Gopakumar, âFrom free fields to AdS space. II,â Phys. Rev. D 70 (2004) 025010 [hep-th/0402063].

[70] R. Gopakumar, âFrom free fields to AdS space. III,â Phys. Rev. D 72 (2005) 066008 [hep-th/0504229].

[71] M. Gaberdiel, R. Gopakumar, B. Knighton, P. Maity, "From Symmetric Product CFTs to AdS_3,"(2021) arXiv:2011.10038v2 [hep-th]

[72] M. Gaberdiel, R. Gopakumar, "The String Dual to Free N=4 Super Yang-Mills," (2021) arXiv:2104.08263v2 [hep-th]

[73] M. Gaberdiel, R. Gopakumar, "The Worldsheet Dual of Free Super Yang-Mills in 4D," (2021) http://arxiv.org/abs/2105.10496v1

[74] F. Galvagno, M. Gaberdiel, "Worldsheet dual of free N=2 quiver gauge theories," (2022) arXiv:2206.08795v2 [hep-th]

[75] L. Eberhardt, "A perturbative CFT dual to string AdS_3 with pure NS-NS flux," (2021) arXiv:2110.07535v1 [hep-th]

[76] V. Dotsenko, V. Fateev, "Conformal Algebra and Multipoint Correlation Functions in Two-Dimensional Statistical Models," Nucl. Phys. B240 (1984) 312.

[77] M. Fiset, M. Gaberdiel, K. Naderi, V. Sriprachyakul, "Perturbing the symmetric orbifold from the worldsheet," (2022) arXiv:2212.12342v1 [hep-th]

[78] M. Gaberdiel, B. Knighton, J. Vosmera, "D-branes in $AdS_3 S^3 \mathbb{T}^4$ at $k = 1$ and their holographic duals" (2011) arXiv:2110.05509v1 [hep-th]

[79] M. Gaberdiel, B. Nairz, "BPS Correlators for AdS_3/CFT_2 (2022) arXiv:2207.03956v1 [hep-th]

[80] O. Lunin, S.D. Mathur,"Three point functions for M^N/S_N orbifolds with $\mathcal{N} = 4$ supersymmetry,â Commun. Math. Phys. 227 (2002) 385 [hep-th/0103169].

[81] A. Dei, L. Eberhardt, "Correlators fo the symmetric product orbifold," (2020) arXiv:1911.08485v2 [hep-th]

[82] L. Eberhardt, "Partition functions of the Tensionless string," (2020) arXiv:2008.07533v2 [hep-th]

[83] S. Hawking, D. Page,“Thermodynamics of black holes in Anti-De Sitter space," Comm. Math. Phys. 87, 577-588 (1983)

[84] L. Susskind, “Some speculations about black hole entropy in string theory," (1993) arXiv:hep-th/9309145v2

[85] G. Horowitz, J. Polchinski, “A correspondence principle for black holes and strings," (1997) arXiv:hep-th/9612146v2

[86] L. Eberhardt, “Summing over Geometries in String Theory," (2021) arXiv:2102.12355v1 [hep-th]

[87] L. Eberhardt, “The Disk Partition Function in String Theory," (2021) arXiv:2105.08726v1 [hep-th]