Contact Processes on Random Graphs

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Abstract

We study the contact process on (dynamic) random graphs. The thesis is structured into four chapters, with the first three providing a survey of relevant literature. Chapter 1 covers the classic theory of contact processes on lattices and regular trees, as well as recent research into contact processes on more general graphs. Chapter 2 introduces contact processes on configuration models and other random graphs, including Erdős–Rényi graphs, preferential attachment graphs and dynamic scale-free graphs. Chapter 3 provides an overview of results concerning contact processes in various random dynamic environments, where the recovery rate or the infection rate varies depending on the environment of the vertex or the edge, respectively. Chapter 4 focuses on two special dynamic random environments: one where vertices recover at rate 0 in environment 0, and the other where edges transfer infections at rate 0 in environment 0. Meanwhile, the environment of vertices or edges switches between 0 and 1 in a Markovian way. We establish monotonicity properties as underlying parameters are varied, by investigating the point process of valid infections in the graphical representation. The key idea is to couple two renewal processes via hazard rates such that the random set of epoch times of one renewal process is a subset of that of the other.

Contents

Chapter 1

Contact Processes on Lattices and Trees

This chapter offers an overview of the research into contact processes on lattices and trees, starting from the publication of the pioneer paper [1] in 1974. Section 1.1 serves as a preliminary that introduces notation, definitions, and important concepts, including the graphical representation, the upper invariant measure, and the phase transitions and associated critical values on finite and infinite graphs. Sections 1.2 and 1.3 provide a brief account of classical results about contact processes on lattices and regular trees (both infinite and finite), respectively. We refer the reader to Part I of [19] for more details. Moving forward, Sections 1.4 and 1.5 highlight the latest advances in the area of contact processes on inhomogeneous trees (periodic trees and Galton-Watson trees) and more general finite graphs, respectively.

1.1 Notation and Basic Concepts

In this section we first introduce notation and then turn to the basic concepts of *contact processes*, including the graphical representation, the upper invariant measure, the phase transition of contact processes on infinite graphs and on finite graphs.

We begin with the notation to be used throughout this thesis:

- For $p \in [0,1]$, let $\mathrm{Ber}(p)$ denote the Bernoulli distribution with mass p on 1. For $\lambda > 0$, let $\mathrm{Exp}(\lambda)$ denote the exponential distribution with rate λ, and let Poi_λ denote the Poisson point process on $[0,\infty)$ with intensity λ.
- For a random variable D and a distribution μ, $D \sim \mu$ means that the distribution of D is μ.
- $\xrightarrow{\text{d}}$, $\xrightarrow{\text{p}}$ and $\xrightarrow{\text{a.s.}}$ denote convergence in distribution, in probability and almost surely, respectively. A property holds with high probability when the probability that it holds tends to 1 as $n \to \infty$.

- For functions $f\colon \mathbb{R}\to\mathbb{R}$ and $g\colon \mathbb{R}\to\mathbb{R}$, we write $f(x)=O(g(x))$, $x\to\infty$, if there exist $M>0$ and $x_0>0$ such that $\left|f(x)\right|\le Mg(x)$ for $x\ge x_0$, $f(x)=\Theta(g(x))$, $x\to\infty$, if $f(x)=O(g(x))$ and $g(x)=O(f(x))$ as $x\to\infty$, and $f(x)\sim g(x)$, $x\to\infty$, if $\lim_{x\to\infty} f(x)/g(x)=1$.
- We assume graphs to be locally finite, connected and with countably many vertices. For vertices u and v in a graph, we write $u\sim v$ if u and v are neighbours, and denote by $|u-v|$ the length of the shortest path between u and v.
- Unless otherwise stated, we always use $\mathbb{P}$ to denote the probability measure under which all randomness is defined in a model.

We start to define the contact process $(\xi_t)_{t\ge0}$ with infection rate $\lambda>0$ on a graph $G=(V,E)$. A *configuration* $\eta\in\{0,1\}^V$ is often identified with $A=\left\{v\in V\,\middle|\,\eta(v)=1\right\}$. Define $\left|\eta\right|:=\sum_{v\in V}\eta(v)$, and define η^v by

$$\eta^v(u):=\begin{cases}1-\eta^v(u) & \text{if } u=v,\\ \eta(u) & \text{if } u\neq v,\end{cases}\qquad v\in V.$$

We interpret the value 0 and 1 of the *state* $\eta(v)$ of a vertex v as *healthy* and *infected* , respectively. In particular, [0] and [1] denote the configuration identically equal to 0 and 1, respectively, and $\delta_{[0]}$ denotes the Dirac measure centred at [0].

Definition 1.1.1. The *contact process* with *infection rate* $\lambda>0$ on G is the continuous-time Markov chain $(\xi_t)_{t\ge0}$ with state space $\{0,1\}^V$ and transition rates given by

$$c(v,\eta)=\begin{cases}\lambda\sum\limits_{u\sim v}\eta(u) & \text{if } \eta(v)=0,\\ 1 & \text{if } \eta(v)=1,\end{cases}\tag{1.1.1}$$

where $c(v,\eta)$ denotes the rate for the process to jump from state η to state η^v.

Denote by $(\xi_t^A)_{t\ge0}$ a copy of $(\xi_t)_{t\ge0}$ with $\xi_0=A$. Writing $(\xi_t^{\{v\}})_{t\ge0}$ as $(\xi_t^v)_{t\ge0}$ and writing $(\xi_t^V)_{t\ge0}$ as $(\xi_t^{[1]})_{t\ge0}$ should cause no confusion.

Remark 1.1.2. Harris investigated *contact interactions* in [1], where a vertex is infected at rate λ_k if it has k infected neighbours. We do not confront this generalization of contact processes in our thesis, but we remind the reader that the contact process is a simplified model, although it is already challenging mathematically.

Remark 1.1.3. Possible self-loops in G can be erased since they have no effect on the dynamics. If vertices u and v are connected by k edges in G, then an infection transmits via each edge independently at rate λ, i.e., an infection transmits from u to v (or from v to u) at rate $k\lambda$.

Next, we introduce the *graphical representation* of $(\xi_t)_{t\geq 0}$, which is based on the following equivalent description of the dynamics:

- Infected vertices become healthy at rate 1.
- An infected vertex transmits its infection to a neighbour at rate λ.
- Infection and recovery happen independently from vertex to vertex.

Let $\{\mathscr{R}_v, \mathscr{I}_e \mid v \in V, e \in E\}$ be a set of independent Poisson point processes, where the intensity of $\mathscr{R}_v$ is 1 and the intensity of $\mathscr{I}_e$ is λ. Place a *recovery mark* $\circ$ at $(x,t) \in V \times [0,\infty)$ for $v \in V$ and $t \in \mathscr{R}_v$. Also, place a double-headed *infection arrow* $\leftrightarrow$ between (u,t) and (v,t) for $\{u,v\} \in E$ and $t \in \mathscr{I}_{u,v}$. See Figure 1.1.

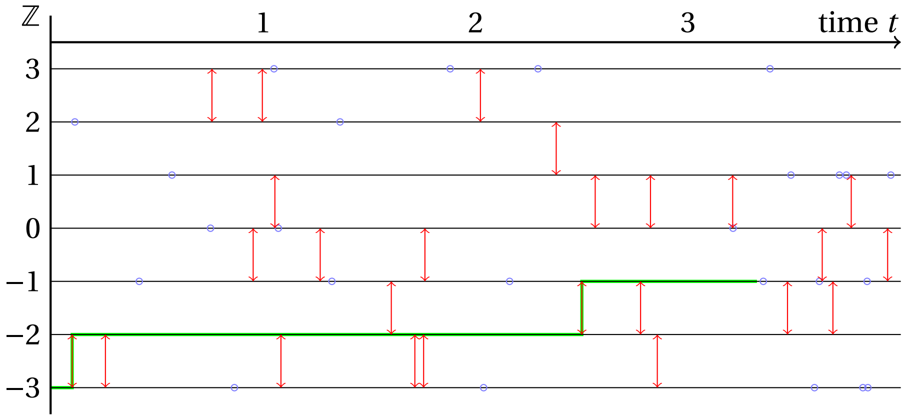

Figure 1.1: *The (partial) graphical representation of* $(\xi_t)_{t\geq 0}$ *in case* $G = \mathbb{Z}$ *and* $\lambda = 1.2$

An *active path* in $V \times [0,\infty)$ is a connected oriented path that moves along the time lines in the increasing direction and along the infection arrows, but without passing any recovery marks. See the green line in Figure 1.1 for an example. Then, for $A \subseteq V$, $v \in \xi_t^A$ if and only if there exists an active path from $(u,0)$ to (v,t) for some $u \in A$.

Our graphical representation is slightly different from the one first introduced in [2], but it does coincide with the dynamics of $(\xi_t)_{t\geq 0}$:

- If u and v are both infected or both healthy at time t, then infection arrows between them do not have any effect until the state of one of them changes.
- If only one of u and v is infected, say u, then u transfers its infection to its neighbour v at rate λ in the graphical representation, since the next infection arrow between u and v appears at rate λ.

Additionally, we introduce some properties that follow from the graphical representation. Let $t \geq 0$, $A, B \subseteq V$ and $(\xi'_t)_{t\geq 0}$ be the contact process on G with infection rate λ'. Then

$$\begin{aligned} \xi_t^A &\preceq \xi_t^B, \quad \text{if } A \subseteq B, && \text{(monotone coupling)}\\ \xi_t^A &\preceq \xi_t'^A, \quad \text{if } \lambda \le \lambda', && \text{(monotonicity)}\\ \xi_t^{A\cup B} &= \xi_t^A \cup \xi_t^B, && \text{(additivity)}\\ \mathbb{P}\left(\xi_t^A \cap B \neq \emptyset\right) &= \mathbb{P}\left(\xi_t^B \cap A \neq \emptyset\right). && \text{(duality)} \end{aligned}$$

Here, the monotone coupling and the additivity follow directly, and the monotonicity follows from coupling of Poisson point processes of infection arrows. As for the duality, we reverse the time direction of $(\xi_t^B)_{t\ge0}$, and the equality follows from the symmetry of the graphical representation.

Let μ_t be the distribution of $\xi_t^{[1]}$. By monotone coupling and the Markov property, $\mu_{t+s} \preceq \mu_t$ for $t,s \ge 0$. By the compactness of the space of all probability measures on $\{0,1\}^V$, the weak limit $\bar{\nu} := \lim_{t\to\infty}\mu_t$ exists. Moreover, $\bar{\nu}$ is the largest invariant measure of $(\xi_t)_{t\ge0}$, and is called the *upper invariant measure*.

Remark 1.1.4. The graphical representation not only facilitates the application of properties of Poisson point processes to obtain coupling, but also provides a method to use ideas and terminology in the theory of percolation. Moreover, the Poisson point processes in the graphical representation can be generalised to renewal processes ([48]) and even hidden Markov chains ([23]).

Finally, we introduce the definitions of *critical values*, *survival* and *extinction*, where Definition 1.1.5 is adapted from Definition 1.1 of [14]. Since G is a connected graph, the following definitions do not depend on v.

Definition 1.1.5. For $A \subseteq V$, the *survival probability* $p_\lambda(A)$ and *extinction time* τ_G^A are defined by

$$p_\lambda(A) := \mathbb{P}\left(\xi_t^A \neq [0] \text{ for } t \ge 0\right), \quad \tau_G^A := \inf\left\{t \ge 0 \,\middle|\, \xi_t^A = [0]\right\},$$

respectively. We write τ^A instead of τ_G^A when G is fixed, and write τ_G^v when $A = \{v\}$. Let $v \in V$. Define

$$\begin{aligned} \lambda_1(G) &:= \inf\left\{\lambda > 0 \,\middle|\, p_\lambda(\{v\}) > 0\right\},\\ \lambda_2(G) &:= \inf\left\{\lambda > 0 \,\middle|\, \mathbb{P}\left(\limsup_{t\to\infty} \xi_t^v(v) = 1\right) > 0\right\},\\ \lambda_2^-(G) &:= \inf\left\{\lambda > 0 \,\middle|\, \limsup_{t\to\infty} \mathbb{P}\left(\xi_t^v(v) = 1\right) > 0\right\},\\ \lambda_2^+(G) &:= \inf\left\{\lambda > 0 \,\middle|\, \liminf_{t\to\infty} \mathbb{P}\left(\xi_t^v(v) = 1\right) > 0\right\}. \end{aligned}$$

Here, $\lambda_1(G)$ and $\lambda_2(G)$ are called the *weak survival critical value* and the *strong survival critical value* of the contact process on G, respectively.

Definition 1.1.6. Let $v \in V$. We say that $(\xi_t)_{t\geq 0}$ *dies out* if $p_\lambda(\{v\}) = 0$, and that $(\xi_t)_{t\geq 0}$ *survives* if $p_\lambda(\{v\}) > 0$. Moreover, we say that $(\xi_t)_{t\geq 0}$ *survives strongly* if $\mathbb{P}\left(\limsup_{t\to\infty} \xi_t^v(v) = 1\right) > 0$, and that $(\xi_t)_{t\geq 0}$ *survives weakly* if it survives but does not survive strongly.

It is immediate that $0 \leq \lambda_1(G) \leq \lambda_2(G) \leq \lambda_2^-(G) \leq \lambda_2^+(G)$. By monotonicity, $(\xi_t)_{t\geq 0}$ dies out if $\lambda < \lambda_1(G)$, survives weakly if $\lambda_1(G) < \lambda < \lambda_2(G)$, and survives strongly if $\lambda > \lambda_2(G)$.

Since contact processes always die out on finite graphs, we must consider a graph sequence $(G_n)_{n\in\mathbb{N}}$ from a model of interest, and study the asymptotic behavior as $n \to \infty$ of the *extinction time* $\tau_{G_n}^A$ of the contact process with fixed infection rate on G_n. It should not be surprising that the critical values for infinite graphs still plays a key role in the phase transition on an n-dependent time scale of contact processes on the corresponding sequence of finite graphs. We refer the reader to later sections for relevant mathematical results, and to Appendix A.1 for an account of results on finite graphs.

1.2 Lattices

In this section we introduce the phase transition of contact processes on infinite lattices and finite lattices briefly.

We begin with infinite lattices. Note that by Theorem 1.2.1 (d), we can and will write $\lambda_1(\mathbb{Z}^d)$ as $\lambda_c(\mathbb{Z}^d)$.

Theorem 1.2.1. *Let $(\xi_t)_{t\geq 0}$ be the contact process on $\mathbb{Z}^d$ with infection rate $\lambda > 0$.*

(a) $1/(2d-1) \leq \lambda_1(\mathbb{Z}^d) \leq 2/d$, $\lim_{d\to\infty} d\lambda_1(\mathbb{Z}^d) = 1/2$.

(b) If $\lambda = \lambda_1(\mathbb{Z}^d)$, then $(\xi_t)_{t\geq 0}$ dies out.

(c) Recall that $p_\lambda(A)$ denotes the survival probability and $\bar{\nu}$ denotes the upper invariant measure. For $A \subseteq \mathbb{Z}^d$,

$$\xi_t^A \xrightarrow{\text{d}} p_\lambda(A)\bar{\nu} + (1 - p_\lambda(A))\delta_{[0]}, \text{ as } t \to \infty.$$

(d) There is no intermediate phase, i.e., $\lambda_1\left(\mathbb{Z}^d\right) = \lambda_2\left(\mathbb{Z}^d\right)$.

Remark 1.2.2. Theorem 1.2.1 (a) is adapted from Theorem 7.1 of [1], Theorem 1.2 of [3] and Corollary 6.3 of [5]. Theorems 1.2.1 (b) and 1.2.1 (c) appear as Theorem 1 and Theorem 4 of [11], respectively. Theorem 1.2.1 (d) is a direct corollary of Theorem 1.2.1 (c).

We also repeat Theorem 2 of [11] as an intermezzo, which introduces “slabs” to connect infinite lattices and finite lattices, and plays a key role in the proof of the phase transition:

Theorem 1.2.3. *For $n \in \mathbb{N}$, let $\lambda_1\left(G_{d,n}\right)$ be the weak survival critical value of the contact process with infection rate λ on $G_{d,n} = [-n,n]^{d-1} \times \mathbb{Z}$. Then,*

$$\lim_{n\to\infty} \lambda_1\left(G_{d,n}\right) = \lambda_c(\mathbb{Z}^d) \quad d \geq 2.$$

Moreover, we combine Theorems 2.30 and 2.54 of [19] to estimate the survival probability in the supercritical case, and repeat Theorem 2.48 of [19] to estimate the extinction time in the subcritical case:

Theorem 1.2.4. *Suppose that $\lambda > \lambda_c(\mathbb{Z}^d)$ and $A \subseteq \mathbb{Z}^d$. Then there are positive constants C and ε, independent of A and t, such that*

$$\mathbb{P}\left(t < \tau^A < \infty\right) \leq C\exp(-\varepsilon t), \quad \mathbb{P}\left(\tau^A < \infty\right) \leq \exp(-\varepsilon|A|).$$

Moreover,

$$\mathbb{P}\left(\tau^{\{0\}} = \infty\right) \geq \frac{\lambda - \lambda_c(\mathbb{Z}^d)}{\lambda(3 + \mathrm{e} + 2d\lambda\mathrm{e})}.$$

Theorem 1.2.5. *If $\lambda \in (0, \lambda_c(\mathbb{Z}^d))$, then there exists an $\varepsilon(\lambda) > 0$ such that*

$$\mathbb{P}\left(\xi_t^0 \neq [0]\right) \leq \exp(-\varepsilon(\lambda)t), \quad t \geq 0.$$

Next, we turn to finite lattices. For $n \in \mathbb{N}$, let $\tau_n^{[1]}$ be the extinction time of the contact process on $\{1,\ldots,n\}^d$ with infection rate λ and initial state $[1]$. The papers [8, 9, 10] study the case $d = 1$, but the result for the subcritical case is extended to $d \geq 2$ by Theorem 3.3 of [19], and the result for the supercritical case is extended by [15, 20]. However, there appear to be no extensions of Theorem 1.7 of [10] to higher dimensions for the critical case. We state the results for the three cases in Theorems 1.2.6 to 1.2.8.

Theorem 1.2.6. *Suppose that $\lambda \in (0, \lambda_c(\mathbb{Z}^d))$. Then the limit*

$$\gamma_-(\lambda) := \lim_{t\to\infty} t^{-1} \log \mathbb{P}\left(\xi_t^0 \neq [0]\right)$$

exists and is positive, where $(\xi_t)_{t\geq 0}$ is the contact process on $\mathbb{Z}^d$ with infection rate λ. Moreover,

$$\frac{\tau_n^{[1]}}{\log n} \xrightarrow{\text{p}} \frac{d}{\gamma_-(\lambda)} \text{ as } n \to \infty.$$

Theorem 1.2.7. *If $d = 1$ and $\lambda = \lambda_c(\mathbb{Z})$, then*

$$\lim_{n\to\infty} \mathbb{P}\left(an \leq \tau_n^{[1]} \leq bn^4\right) = 1, \quad a, b > 0.$$

Theorem 1.2.8. *Suppose that* $\lambda > \lambda_c(\mathbb{Z}^d)$. *Then*

$$\tau_n^{[1]}/\mathbb{E}\left[\tau_n^{[1]}\right] \xrightarrow{\mathrm{d}} \mathrm{Exp}(1) \text{ as } n \to \infty.$$

Moreover, $n^{-d}\log \mathbb{E}[\tau_n^{[1]}]$ *converges to a positive constant as* $n \to \infty$.

Last but not least, the contact process on finite lattices exhibits a property called *metastability* in the supercritical case: the process persists for a long time in a state that resembles the equilibrium on the infinite lattice and afterwards quickly dies out. The papers [6, 7] studied the one-dimensional case, but the result has been extended to any dimension by Theorem 3.9 of [19]:

Theorem 1.2.9. *The limit*

$$\gamma_+(\lambda) := -\lim_{n\to\infty} n^{-d} \log \mathbb{P}\left(\tau_n^{[1]} < \infty\right)$$

exists. Moreover,

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-d}\log \tau_n^{[1]} \geq \gamma\right) = 0, \quad \text{for each } \gamma > \gamma_+(\lambda),$$
$$\lim_{n\to\infty} \mathbb{P}\left(n^{-d}\log \tau_n^{[1]} \leq \delta\right) = 0, \quad \text{for some } \delta > 0.$$

1.3 Regular Trees

In this section we introduce the phase transition of contact processes on infinite regular trees and finite regular trees. Compared to the phase transition on lattices, a intermediate regime exists in the phase transition on regular trees. But first we clarify our notation, since various conventions for regular trees are used in different papers. Let $d \geq 2$.

Definition 1.3.1. (a) A function l from a tree $\mathbb{T}$ to $\mathbb{Z}$ is called a *level function* on $\mathbb{T}$ if, for each $x \in \mathbb{T}$, $l(y) = l(x) - 1$ for exactly one neighbour y of x, and $l(y) = l(x) + 1$ for the other neighbours y of x.

(b) The infinite tree in which every vertex has $d+1$ edges is called the *infinite* $(d+1)$*-regular tree*, and is denoted by $\mathbb{T}^d$. Pick a level function l on $\mathbb{T}^d$, and take $\left\{e_n \in \mathbb{T}^d \mid n \in \mathbb{Z}\right\}$ such that $l(e_n) = n$ and $e_n \sim e_{n+1}$ for $n \in \mathbb{Z}$. This provides an *embedding* of $\mathbb{Z}$ in $\mathbb{T}^d$.

(c) The *finite regular graph* $\mathbb{T}_n^d$ is the subgraph of $\mathbb{T}^d$ with vertex set

$$\left\{x \in \mathbb{T}^d \,\middle|\, 0 \leq l(x) \leq n\right\}.$$

Theorems 2.1 and 2.2 of [14] and Theorem 1.2 of [18] give the following result, which shows that the weak survival critical value $\lambda_1(\mathbb{T}^d)$ and the strong survival critical value $\lambda_2(\mathbb{T}^d)$ do not coincide:

Theorem 1.3.2. *The following are true:*

$$\frac{1}{d} \leq \lambda_1(\mathbb{T}^d) < \frac{1}{d-1}, \quad \lambda_2(\mathbb{T}^d) \geq \frac{1}{2\sqrt{d}}, \quad \lambda_1(\mathbb{T}^d) < \lambda_2(\mathbb{T}^d) < \infty.$$

Next, we list results on each phase of the contact process on $\mathbb{T}^d$. Let $(\xi_t)_{t\geq 0}$ be the contact process with infection rate λ on $\mathbb{T}^d$. Two key ingredients for the proofs in [19] are the functions φ and β defined by:

$$\varphi(\varrho) := \lim_{t\to\infty}\left(\mathbb{E}\left[\sum_{x\in\xi_t}\varrho^{l(x)}\right]\right)^{1/t}, \quad \varrho \geq 0,$$

$$\beta(\lambda) := \lim_{n\to\infty}\left(\mathbb{P}\left(e_n \in \xi_t \text{ for some } t\right)\right)^{1/n}, \quad \lambda > 0. \tag{1.3.1}$$

Remark 1.3.3. We refer the reader to Pages 86 to 91 and Pages 96 to 103 of [19] for the interpretation of φ and β, respectively. Another important tool is the *branching random walk*, which can be regarded as a contact process where we track the multiplicity of infections. We refer the reader to Pages 80 to 85 of [19].

In the critical case $\lambda = \lambda_1(\mathbb{T}^d)$, $(\xi_t^{e_0})_{t\geq 0}$ dies out by Theorem 2, Lemma 1 and Corollary 1 of [16]:

Theorem 1.3.4. *Suppose that $\lambda = \lambda_1(\mathbb{T}^d)$. Then there exists a constant $C(d)$ such that*

$$1 \leq \mathbb{E}\left[\left|\xi_t^{e_0}\right|\right] \leq C(d) \text{ for } t > 0. \tag{1.3.2}$$

Moreover, $\left|\xi_t^{e_0}\right| \xrightarrow{a.s.} \infty$ as $t \to \infty$ on the event that $(\xi_t^{e_0})_{t\geq 0}$ survives. Hence, the contact process $(\xi_t^{e_0})_{t\geq 0}$ dies out.

(1.3.2) originates in the following theorem, which was first proved in [13] for the biased voter model. [16] points out that the proof holds in fact for any translation invariant additive nearest-neighbour interaction.

Theorem 1.3.5. *There exist constants c_λ and $C(d)$ such that*

$$\exp(c_\lambda t) \leq \mathbb{E}\left[\left|\xi_t^{e_0}\right|\right] \leq C(d)\exp(c_\lambda t).$$

Moreover, c_λ is a continuous function of λ.

Remark 1.3.6. There are no other graphs for which the behavior of $\mathbb{E}\left[\left|\xi_t^O\right|\right]$ is known at the critical value, according to [16]. Moreover, Liggett conjectured on Page 93 of [19] that $\sup_{t\geq 0}\mathbb{E}[|\xi_t|] < \infty$ fails for the critical contact process on $\mathbb{Z}^d$.

In the supercritical case $\lambda > \lambda_2(\mathbb{T}^d)$, Theorem 4.70 of [19] gives the following complete convergence theorem:

Theorem 1.3.7. *Recall that $p_\lambda(A)$ is the survival probability and $\bar{\nu}$ is the upper invariant measure. If $\lambda > \lambda_2(\mathbb{T}^d)$, then, for $A \subseteq \mathbb{T}^d$,*

$$\xi_t^A \xrightarrow{\mathrm{d}} p_\lambda(A)\bar{\nu} + (1 - p_\lambda(A))\delta_{[0]} \text{ as } t \to \infty.$$

Before considering the intermediate phase, we mention that Liggett gives the continuity of the survival probability $p_\lambda(A)$ as a function of λ for $A \subseteq \mathbb{R}^d$ in Theorem 4.71 of [19].

In the intermediate regime $\lambda_1(\mathbb{T}^d) < \lambda < \lambda_2(\mathbb{T}^d)$ there are infinitely many extremal invariant measures. Here, we repeat the two constructions given in Theorems 4.107 and 4.121 of [19]. Define the *boundary* $\partial\mathbb{T}^d$ to be the class of semi-infinite self-avoiding paths emanating from e_0. We identify a vertex $x \in \mathbb{T}^d$ with the finite path that leads from e_0 to x. A base for the natural topology on $\partial\mathbb{T}^d$ is given by collections $D(x_0, \ldots, x_n)$ of paths with a common initial segment $\{x_0 = e_0, x_1, \ldots, x_n\}$. Fix a $\vartheta \in (0,1)$. We define a metric distance for this topology: for $x, y \in \partial\mathbb{T}^d \cup \mathbb{T}^d$, let z be the endpoint (other than e_0) of the intersection of the self-avoiding paths from e_0 to x and from e_0 to y, and let

$$\mathrm{dist}(x, y) := \vartheta^{|e_0 - z|}\left(2 - \vartheta^{|x-z|} - \vartheta^{|y-z|}\right).$$

Let γ be the uniform probability measure on $\partial\mathbb{T}^d$, i.e.,

$$\gamma(D(x_0, \ldots, x_n)) = \frac{1}{(d+1)d^{n-1}}, \quad n \in \mathbb{N}.$$

Theorem 1.3.8. *Suppose that $\alpha\colon \mathbb{T}^d \to [0,\infty)$ is uniformly bounded, and that the limit*

$$\alpha(z) := \lim_{x \to z} \alpha(x)$$

exists for $z \in \partial\mathbb{T}^d$ γ-almost everywhere. Then there exists an ivariant measure ν_α for the contact process $(\xi_t)_{t\geq 0}$ such that

$$\lim_{x \to z} \nu_\alpha(\{A \mid x \in A\})\,(d\beta)^{|x-e_0|} = \alpha(z), \quad \gamma\text{-a.e. } z \in \partial\mathbb{T}^d,$$

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{|x-e_0|\geq n, |y-e_0|\geq n, |x-y|\geq k} \nu_\alpha\left(\left\{A \,\middle|\, x, y \in A\right\}\right)(d\beta)^n = 0,$$

where $\beta = \beta(\lambda)$ is given by (1.3.1). Moreover, if α_1 and α_2 are uniformly bounded on $\mathbb{T}^d$ with $\alpha_1 \leq \alpha_2$, then the corresponding invariant measures can be taken to satisfy $\nu_{\alpha_1} \preceq \nu_{\alpha_2}$.

Theorem 1.3.9. *For $x \in \mathbb{T}^d \setminus \{e_0\}$, let*

$$S'(x) := \left\{ y \in \mathbb{T}^d \,\middle|\, \left|y - e_0\right| = \left|y - x\right| + |x - e_0| \right\}.$$

Let $B := \bigcup_{n=1}^{N} S'(x_n)$ *be a disjoint union. Then there exists an invariant measure* ν_B *for* $(\xi_t)_{t\geq 0}$ *such that*

$$\nu_B(\{A \mid x \in A\}) = \lim_{t\to\infty} \mathbb{P}\left(\xi_t^x \cap B \neq \emptyset\right).$$

Furthermore,

$$\begin{aligned}\bar{\nu}(\{A \mid x \in A\}) - \nu_B(\{A \mid x \in A\}) &\leq u(|x - x_n|), \quad x \in S'(x_n),\\ \nu_B(\{A \mid x \in A\}) \leq \sum_{n=1}^{N} u(|x - x_n|) &\leq \sum_{n=1}^{N} \beta^{|x-x_n|}, \quad x \notin B.\end{aligned}$$

Finally, we turn to the extinction time τ_n^A of the contact process on $\mathbb{T}_n^d$ with infection rate λ and initial state $A \subseteq \mathbb{T}^d$. Theorem 1.3.10 focuses on the subcritical case, which appears as Theorem 1.2 of [28], and Theorem 1.3.11 on the supercritical case, which is a combination of Theorem 1.4 of [21] and Theorems 1.5 and 1.6 of [28].

Theorem 1.3.10. *If* $\lambda \in (0, \lambda_2(\mathbb{T}^d))$, *then there exists a* $c > 0$ *such that*

$$n^{-1}\tau_n^{[1]} \xrightarrow{\mathrm{p}} c \text{ as } n \to \infty.$$

Theorem 1.3.11. *Suppose that* $\lambda > \lambda_2(\mathbb{T}^d)$.

(a) Let $\beta \in (0,1)$. *There exist* $c, \varepsilon > 0$ *such that*

$$\mathbb{P}\left(\tau_n^{e_0} \geq c\exp\left(\beta\left|\mathbb{T}_n^d\right|\right)\right) \geq \varepsilon \text{ and } \lim_{n\to\infty} \mathbb{P}\left(\tau_n^{[1]} \geq c\exp\left(\beta\left|\mathbb{T}_n^d\right|\right)\right) = 1.$$

(b) There exists a $c > 0$ *such that* $\left|\mathbb{T}_n^d\right|^{-1} \log \mathbb{E}\left[\tau_n^{[1]}\right] \to c$ *as* $n \to \infty$.

(c) $\tau_n^{[1]}/\mathbb{E}[\tau_n^{[1]}] \xrightarrow{\mathrm{d}} \mathrm{Exp}(1)$ *as* $n \to \infty$.

(d) There exists a $\delta > 0$ *such that, for any* $\alpha > 0$ *and any* n *large enough (depending on* α*),*

$$\inf_{A\subseteq \mathbb{T}_n^d, A\neq\emptyset} \mathbb{P}\left(\tau_n^A > \alpha\mathbb{E}\left[\tau_n^{[1]}\right]\right) > \delta\exp(-\alpha).$$

1.4 Periodic Trees and Galton-Watson Trees

Although Pemantle pioneered the study of contact processes on trees in [14], there has been considerable progress concerning more general trees only in recent papers [41, 42]. In this section we introduce recent research into contact processes on periodic trees and Galton-Watson trees. Recall the definition of the critical values λ_1 and λ_2^+ in Definition 1.1.5.

We begin with periodic trees, which generalise of the concept of regular trees. We say that a tree $\mathbb{T}$ with a level function l is $(a_1,\dots,a_n)$-*periodic* if the vertices in level $mn+r$ have degree a_r for $m \in \mathbb{Z}$ and $1 \le r \le n$. Let $\mathbb{T}[a_1,\dots,a_n]$ denote the subgraph obtained from the $(a_1,\dots,a_n)$-periodic tree by only keeping the vertices whose distance to the root is no greater than n. The authors of [41] give the following upper bound of the survival time of the contact process on the finite tree in Theorem 1.4:

Theorem 1.4.1. *Let $c>0$. Let $\tau_k^{[1]}$ be the extinction time of the contact process with infection rate $\sqrt{c(\log n)/n}$ on $\mathbb{T}[n,a_1,\dots,a_k]$ starting from all vertices infected. If $\max_{1\le i\le k} a_{n,i} \le Cn^{1-\delta}$ for some $C,\delta>0$, then for any $\varepsilon>0$, when n is sufficiently large,*

$$\mathbb{E}\left[\tau_k^{[1]}\right] \le C_0 n^{c(1+\varepsilon)} \log n,$$

where C_0 is some positive constant depending on k but not on C,δ.

Theorems 1.2 and 1.3 of [41] give the following result on the critical values of contact processes on periodic trees, where both the maximum degree and smaller degrees play a role:

Theorem 1.4.2. *Let $k \in \mathbb{N}$ be fixed. Consider the critical values $\lambda_1(n)$ and $\lambda_2^+(n)$ of the contact process on the $(n,a_{n,1},\dots,a_{n,k})$-periodic tree. Assume that the limit*

$$b := \lim_{n\to\infty} \log(a_{n,1}\cdots a_{n,k})/\log n$$

exists and that $\max_{1\le i\le k} a_{n,i} \le Cn^{1-\delta}$ for some $C,\delta>0$. Then

$$\lambda_2^+(n) \sim \sqrt{(k-b)\log n/(2n)}, \quad n\to\infty.$$

Moreover, if $k-2b-1>0$, then

$$\lambda_1(n) \sim \sqrt{(k-2b-1)\log n/(2n)}, \quad n\to\infty.$$

If $k-2b-1<0$, then

$$\lim_{n\to\infty} \log\lambda_1(n)/\log n = -(b+1)/(k+1), \quad n\to\infty.$$

Next, we turn to Galton-Watson trees. Let D be an $\mathbb{N}_0$-valued random variable with distribution $\boldsymbol{p}=(p_k)_{k\in\mathbb{N}_0}$. The Galton-Watson tree with offspring distribution $\boldsymbol{p}$ is constructed as follows: starting with the root, each individual independently has k children with probability p_k. We first define three kinds of tails:

Definition 1.4.3. We say that $\boldsymbol{p}$ has an *exponential tail* if $\mathbb{E}\left[\exp(cD)\right] < \infty$ for some $c > 0$, a *subexponential tail* if $\limsup_{k\to\infty} k^{-1}\log p_k = 0$, and a *heavy tail* if $\mathbb{E}\left[\exp(cD)\right] = \infty$ for each $c > 0$.

We summarise the results about the critical values $\lambda_1(\boldsymbol{p})$ and $\lambda_2^+(\boldsymbol{p})$ of the contact process on the Galton-Watson tree with offspring distribution $\boldsymbol{p}$ in the following two theorems, respectively, where the first one appears as Theorem 1.2 of [42] and Theorem 1 of [44], and the second one is a combination of Theorem 3.2 of [14], Theorem 1.3 and Theorem 1.4 of [42]. Note that Theorem 1.4.5 (b) is a direct corollary of Theorem 1.4.5 (a).

Theorem 1.4.4. *(a) If $p_k = (1-p)^{k-1}p$, $k \in \mathbb{N}$, for some $p \in (0,1)$, then $\lambda_1(\boldsymbol{p}) \le p/(1-p)$.*

(b) If $\boldsymbol{p}$ has an exponential tail, then $\lambda_1(\boldsymbol{p}) > 0$.

Theorem 1.4.5. *(a) There are constants c_2 and c_3 such that*

$$\lambda_2^+(\boldsymbol{p}) \le c_3\sqrt{k^{-1}r_k\log r_k\log k}, \quad k \ge 2,$$

where r_k is the maximum of 2 and $-c_2\log(kp_k)/\log\mathbb{E}[D]$.

(b) If $p_k = c_\gamma\exp(-k^\gamma)$, $k \in \mathbb{N}_0$, with $0 < \gamma < 1$, where c_γ is the normalisation constant, then $\lambda_2^+(\boldsymbol{p}) = 0$.

(c) If $p_k = 2^{-k}$, $k \in \mathbb{N}$, then $\lambda_2^+(\boldsymbol{p}) \le 2.5$.

(d) If $\mathbb{E}[D] > 1$ and $\boldsymbol{p}$ has a subexponential tail, then $\lambda_2^+(\boldsymbol{p}) = 0$.

1.5 General Finite Graphs

In this section we collect results about contact processes on a sequence $(G_n)_{n\in\mathbb{N}}$ of general graphs. We start from trees with bounded degrees and graphs with a spanning tree, and end with graphs with bounded degrees and even general connected graphs. Recall that $\tau_G^{[1]}$ denotes the extinction time of the contact process on a graph G with initial state [1]. The following four theorems appear as Theorem 1.1 of [31], Theorem 1.2 of [31], a combination of Theorems 1.3 and 1.4 of [32], and Theorem 1.2 of [36], respectively.

Theorem 1.5.1. *Let $\Lambda(n,d)$ be the set of trees with n vertices and with degrees bounded by $d \ge 2$. If $\lambda > \lambda_c(\mathbb{Z})$, then there exists a $c > 0$ such that*

$$\lim_{n\to\infty}\inf_{G_n\in\Lambda(n,d)}\mathbb{P}\left(\tau_{G_n}^{[1]} \ge \mathrm{e}^{cn}\right) = 1, \quad \lim_{n\to\infty}\inf_{G_n\in\Lambda(n,d)} n^{-1}\log\mathbb{E}\left[\tau_{G_n}^{[1]}\right] \ge c.$$

Theorem 1.5.2. *Let $d \ge 2$, and let $\mathscr{T}(n,d)$ be the set of graphs with a spanning tree in $\Lambda(n,d)$. If $\lambda > \lambda_c(\mathbb{Z})$, then, as $n \to \infty$,*

$$\tau_{G_n}^{[1]}/\mathbb{E}\left[\tau_{G_n}^{[1]}\right] \xrightarrow{\mathrm{d}} \mathrm{Exp}(1), \quad G_n \in \mathscr{T}(n,d).$$

Theorem 1.5.3. *Let $d \geq 2$, and let $\mathscr{G}(n,d)$ be the set of connected graphs with n vertices and with degrees bounded by $d+1$.*

(a) If $\lambda < \lambda_1(\mathbb{T}^d)$, then there exists a $C < \infty$ such that

$$\lim_{n\to\infty} \inf_{G_n \in \mathscr{G}(n,d)} \mathbb{P}\left(\tau_{G_n}^{[1]} < C \log n\right) = 1.$$

(b) If $\lambda > \lambda_c(\mathbb{Z})$, then there exists a $c > 0$ such that

$$\lim_{n\to\infty} \inf_{G_n \in \mathscr{G}(n,d)} \mathbb{P}\left(\tau_{G_n}^{[1]} > ce^{cn}\right) = 1.$$

(c) The contact process with infection rate $\lambda \leq \lambda_1(\mathbb{T}^d)$ on any graph with degrees bounded by $d+1$ dies out.

Remark 1.5.4. Theorem 1.5.3 is based on Theorems 1.5.1, 1.5.2 and 2.1.2, where the last one is about the extinction time of contact processes on random regular graphs.

Theorem 1.5.5. *Suppose that $\lambda > \lambda_c(\mathbb{Z})$ and $\varepsilon > 0$. Then there exists a $c(\varepsilon) > 0$ such that, for any connected graph G with $n \geq 2$ vertices,*

$$\mathbb{E}\left[\tau_G^{[1]}\right] \geq \exp\left(c(\varepsilon) n (\log n)^{-1-\varepsilon}\right).$$

Moreover, for any nonempty subset A of the vertex set of G,

$$\mathbb{P}\left(\tau_G^A > c(\varepsilon) n (\log n)^{-1-\varepsilon}\right) > c(\varepsilon).$$

Chapter 2

Contact Processes on Configuration Models

This chapter provides an overview of results obtained for contact proccesses on various configuration models and other random graphs, including Erdős-Rényi graphs, preferential attachment graphs and dynamic scale-free graphs. Configuration models are considered to be more realistic models of real-world networks, and a vast body of research has been devoted to them. In Section 2.1 we begin with contact processes on random regular graphs, both static and dynamic. In Section 2.2 we introduce the configuration model with i.i.d. degrees and present results about contact processes on configuration models with different degree distributions, excluding the Poisson degree distribution and the power-law distribution with exponent greater than 2. We group the former with Erdős-Rényi graphs in Section 2.3 due to their close connection, while the latter is discussed in Section 2.4 alongside preferential attachment graphs because they share the scale-free property. In Section 2.5 we introduce contact processes on dynamic scale-free graphs, which share the scale-free property and similar strategies of proof with static scale-free graphs (see Remarks 2.4.3 and 2.5.3).

2.1 Random Regular Graphs

Random regular graphs are configuration models specifed by constant degree sequences. In this section we first introduce the construction and then list results about contact processes on static random regular graphs and on switching random regular graphs.

The sequence $(\deg(v))_{v\in V}$ is called the *degree sequence* of a graph $G = (V, E)$. We begin with the definition of the configuration model specified by a prescribed degree sequence:

Definition 2.1.1. Fix $n \in \mathbb{N}$. Let $V = \{1, \dots, n\}$ be the vertex set, and let $\boldsymbol{d} = (d_i)_{i=1}^n$ be a prescribed degree sequence with the number of edges $m = \frac{1}{2}\sum_{i=1}^n d_i$ being an integer. Assign d_i half-edges to vertex i for $i \in V$. Let $E_0 := \emptyset$, H_0 the set of $2m$ half-edges, and $v(h)$ the vertex to which the half edge h is attached for $h \in H_0$. Generate recursively for k from 1 to m:

- Let a_k be the half-edge in H_{k-1} with the smallest subscript, and let b_k be the uniform random variable chosen from $H_{k-1} \setminus \{a_k\}$.
- Set $E_k := E_{k-1} \cup \{\{v(a_k), v(b_k)\}\}$ and $H_k := H_{k-1} \setminus \{a_k, b_k\}$.

We call the graph (V, E_m) the *configuration model* with degree sequence $\boldsymbol{d}$.

In other words, (V, E_m) is the random multigraph constructed by assigning d_i half-edges to vertex i for $i \in V$, and pairing all half-edges uniformly at random to become edges. Note that pairing half-edges uniformly at random is a choice. It is possible that other choices lead to the same distribution for the random graph.

We continue to define the random regular graphs. Let $d, n \in \mathbb{N}$ with dn even. The random d-regular graph G_n on the vertex set $V_n := \{1, \dots, n\}$ is the configuration model with degree sequence $(d_i)_{i=1}^n$ such that $d_i = d$ for all i. Thanks to Theorem 7.12, Proposition 7.15 and Corollary 7.17 of [37], we are able to pick G_n according to $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}\left(\cdot \mid G_n \text{ is simple}\right)$, since the probablity of simple graphs is positive and the distribution of G_n under $\tilde{\mathbb{P}}$ is uniform over the collection of all undirected d-regular graphs on V_n. For convenience, we pick G_n according to $\tilde{\mathbb{P}}$ and fix G_n once it is chosen.

Next, we introduce results about the contact process $\left({}^n\xi_t\right)_{t\geq 0}$ with infection rate λ on G_n. The first result is about the phase transition, which is given by Theorem 1.2 of [32]:

Theorem 2.1.2. *Let $\tau_n^{[1]}$ be the extinction time of $({}^n\xi_t^{[1]})_{t\geq 0}$ for $n \in \mathbb{N}$.*

(a) For every $\lambda \in (0, \lambda_1(\mathbb{T}^{d-1}))$ there exists a constant $C < \infty$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\tau_n^{[1]} < C \log n\right) = 1.$$

(b) For every $\lambda \in (\lambda_1(\mathbb{T}^{d-1}), \infty)$ there exists a constant $c > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\tau_n^{[1]} > ce^{cn}\right) = 1.$$

Another interesting result is the cutoff phenomenon revealed by Theorem 1.1 of [35] for the time when $({}^n\xi_t^u)_{t\geq 0}$ infects a vertex distinct from u. Let c_λ and p_λ denote the exponential growth rate (see Theorem 1.3.5) and the survival probability of the contact process on $\mathbb{T}^{d-1}$ with infection rate λ, respectively.

Theorem 2.1.3. *Let $\lambda > \lambda_1(\mathbb{T}^{d-1})$. Fix vertices $u, v \in V_n$. For any $\varepsilon \in (0, 1/32)$, there exists a sequence $(g_n(\varepsilon))_{n\in\mathbb{N}}$ of constants converging to 0 as $n \to \infty$, such that for asymptotically almost every G_n,*

$$\mathbb{P}\left(v \in {}^n\xi_t^u \text{ for some } t \leq (1-\varepsilon)c_\lambda^{-1} \log n\right) \leq g_n(\varepsilon),$$

$$\mathbb{P}\left(v \in {}^n\xi^u_{(1+\varepsilon)c_\lambda^{-1}\log n}\right) \geq (1-g_n(\varepsilon))p_\lambda^2.$$

Here, a property is said to hold asymptotically almost surely *if the probability of the set of graphs satisfying this property tends to* 1 *as* $n \to \infty$.

Theorem 2.1.3 suggests that if ${}^n\xi^u_t$ does not die out quickly, then it will enter a "quasi-stationary" state around time $c_\lambda^{-1}\log n$ with the fraction of infected vertices approximately p_λ, which is asserted by Theorem 1.2 in [35]:

Theorem 2.1.4. *Let $\lambda > \lambda_1(\mathbb{T}^{d-1})$ and $\varepsilon \in (0,1/32)$. For any $\delta > 0$, there exists a sequence $(f_n(\delta))_{n\in\mathbb{N}}$ of constants converging to* 0 *as $n \to \infty$ such that, for asymptotically almost every G_n,*

$$\mathbb{P}\left((1-\delta)np_\lambda \leq \left|{}^n\xi^{[1]}_{(1+\varepsilon)c_\lambda^{-1}\log n}\right| \leq (1+\delta)np_\lambda\right) \geq 1-f_n(\delta).$$

Moreover, for any sequence $(t_n)_{n\in\mathbb{N}}$ of times with $t_n > (1+\varepsilon)c_\lambda^{-1}\log n$,

$$\mathbb{P}_{G_n}\left((1-\delta)np_\lambda \leq \left|{}^n\xi^{[1]}_{t_n}\right| \leq (1+\delta)np_\lambda \;\middle|\; {}^n\xi^{[1]}_{t_n} \neq [0]\right) \geq 1-f_n(\delta).$$

Finally, we introduce the result given in [46] that the swiching dynamics can aid the spread of the infection on random regular graphs. Let $H_{n,d} := V_n \times \{1,\dots,d\}$ be the set of half-edges. Sample φ uniformly at random from the set

$$\left\{\varphi\colon H_{n,d} \to H_{n,d} \;\middle|\; \varphi \text{ is a bijection}, \varphi = \varphi^{-1}, \varphi(h) \neq h \text{ for all } h \in H_{n,d}\right\}$$

and regard $E_{n,d} := \left\{\{(x,a),\varphi((x,a))\} \;\middle|\; (x,a) \in H_{n,d}\right\}$ as the set of edges. The local-rewiring dynamics are as follows: Let $e = \{(x,a),(y,b)\}$ and $e' = \{(x',a'),(y',b')\}$ be edges in $E_{n,d}$ with $(x,a) < (y,b)$ and $(x',a') < (y',b')$ in lexicographic order of $H_{n,d}$. The positive switch $\Gamma^{\mathtt{m}}$ with mark $\mathtt{m} = (\{e,e'\},+)$ is the transformation that removes the edges e,e' and adds two new edges $\{(x,a),(x',a')\}$ and $\{(y,b),(y',b')\}$. Similarly, the negative switch $\Gamma^{\mathtt{n}}$ with mark $\mathtt{n} = (\{e,e'\},-)$ is the transformation that removes the edges e,e' and adds two new edges $\{(x,a),(y',b')\}$ and $\{(x',a'),(y,b)\}$.

Let $({}^nG_t)_{t\geq 0}$ be the continuous-time Markov chain on the space of d-regular graphs on V_n, where ${}^nG_0$ is drawn uniformly at random from the d-regular graphs on V_n. Given the state ${}^nG_t$ at time t, for each positive switch mark $\mathtt{m}$, the process jumps to $\Gamma^{\mathtt{m}}({}^nG_t)$ at rate $v/(nd)$, where $v > 0$ is called the switch rate. It is readily seen that the uniform distribution on random d-regular graphs on V_n is stationary. Theorem 2 of [46] proves that the extinction time of the contact process on the switching random regular graph is at least exponential in the supercritical case, but the subcritical case and the monotonicity of $v \mapsto \bar{\lambda}_d(v)$ are still open:

Theorem 2.1.5. *Let $d \geq 3$. For every $v > 0$, there exists a $\bar{\lambda}_d(v) \in (0, \lambda_c(\mathbb{T}^d))$ such that the following holds. For any $\lambda > \bar{\lambda}_d(v)$, there exists a $c > 0$ such that the extinction time $\tau_n^{[1]}$ of the contact process with infection rate λ on the switching random d-regular graph $({}^nG_t)_{t\geq 0}$ with switch rate v satisfies*

$$\lim_{n\to\infty} \mathbb{P}\left(\tau_{G_t}^{[1]} > \exp(cn)\right) = 1.$$

2.2 Configuration Models with i.i.d. Degrees

Compared with random regular graphs, the configuration model with i.i.d. degrees is more widely used in the research of real-world networks. In this section we first list notation, simplicity, connectedness and common degree distributions. After that we present contact processes on configuration models with i.i.d. degrees sampled from a distribution with exponential tail, a distribution with heavy tail and a power-law distribution with exponent $\alpha \leq 2$, respectively.

We begin with the definitions. Let D be a random variable with probability distribution $\boldsymbol{p} = (p_k)_{k\in\mathbb{N}_0}$ on $\mathbb{N}_0$. The *configuration model G_n with n vertices and degree distribution $\boldsymbol{p}$* is constructed as follows.

(a) Let $(d_i')_{i=1}^n$ be a sequence of i.i.d. random variables with probability distribution $\boldsymbol{p}$.
(b) Let $d_i := d_i'$ for $i \in \{1,\dots,n-1\}$, and $d_n := d_n + \mathbf{1}_{\{\sum_{i=1}^n d_i' \text{ is odd}\}}$.
(c) Let G_n be the configuration model with degree sequence $(d_i)_{i=1}^n$.

We may also define $(d_i)_{i=1}^n$ by conditioning on the event that $\sum_{i=1}^n d_i'$ is even in step (b), i.e.,

$$\mathbb{P}\left((d_i)_{i=1}^n = \cdot\right) := \mathbb{P}\left((d_i')_{i=1}^n = \cdot \,\middle|\, \sum_{i=1}^n d_i' \text{ is even}\right), \tag{2.2.1}$$

which makes no difference as $n \to \infty$.

Definition 2.2.1. We say that $(G_n)_{n\in\mathbb{N}}$ is *scale-free with exponent α* if

$$\lim_{k\to\infty} \log\left(1 - \sum_{i=0}^k p_i\right) \Big/ \log k = -\alpha + 1, \tag{2.2.2}$$

and has a *power-law degree distribution with exponent α* if

$$\lim_{k\to\infty} \log p_k / \log k = -\alpha. \tag{2.2.3}$$

Note that (2.2.3) implies (2.2.2), but (2.2.3) is still too strict when the function $k \mapsto p_k$ is not smooth.

[17] and Theorem 7.21 in [37] give the following results on the connected component and the probability of simplicity of the model, respectively:

Theorem 2.2.2. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of configuration models with degree distribution $\boldsymbol{p}$.*

(a) G_n contains a unique connected component of size linear in n if and only if $\mathbb{E}[D(D-2)]>0$.

(b) Let $\nu := \mathbb{E}[D(D-1)]/\mathbb{E}[D]$. If $\mathbb{E}[D]^2<\infty$, then

$$\lim_{n\to\infty}\mathbb{P}\left(G_n \text{ is simple}\right)=\exp(-\nu/2-\nu^2/4).$$

Next, we look at the phase transition of contact processes on $(G_n)_{n\in\mathbb{N}}$ when $\boldsymbol{p}$ has an exponential tail or a heavy tail, which appears as Theorems 3 and 4 of [44], respectively:

Theorem 2.2.3. *Suppose that $\mathbb{E}\left[D^2\right]<\infty$ and $\mathbb{E}[D(D-2)]>0$. Pick G_n according to $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}\left(\cdot \,\middle|\, G_n \text{ is simple}\right)$ instead. Let $({}^n\xi_t)_{t\ge 0}$ be the contact process with infection rate λ on G_n, and let $\tau_n^{[1]}$ be the extinction time of $({}^n\xi_t^{[1]})_{t\ge 0}$.*

(a) If $\boldsymbol{p}$ has an exponential tail, then there exist constants $0<\underline{\lambda}(\boldsymbol{p})\le\overline{\lambda}(\boldsymbol{p})<\infty$ such that, for $\lambda\in(0,\underline{\lambda}(\boldsymbol{p}))$,

$$\lim_{n\to\infty}\mathbb{P}\left(\tau_n^{[1]}\le n^{1+\varepsilon}\right)=1,\quad \varepsilon>0,$$

while, for $\lambda>\overline{\lambda}(\boldsymbol{p})$, there exist positive constants c and C such that

$$\lim_{n\to\infty}\mathbb{P}\left(\exp(cn)\le\tau_n^{[1]}\le\exp(Cn)\right)=1. \tag{2.2.4}$$

(b) If $\boldsymbol{p}$ has a heavy tail, then for $\lambda>0$ there exist positive constants c and C such that (2.2.4) holds.

Finally, we list Theorems 1.1 and 1.2 of [30], which investigate contact processes on configuration models specified by power-law degree distributions with exponent $\alpha\le 2$. In this case, the degree distribution has infinite mean and thus is not typical in scale-free networks.

Theorem 2.2.4. *Suppose that $\alpha\in(1,2]$ and $p_k=c_\alpha k^{-\alpha}$ for $k\in\mathbb{N}$, where c_α is the normalising constant. Consider the contact process $({}^n\xi_t)_{t\ge 0}$ with infection rate $\lambda>0$ on G_n.*

(a) There is a constant $c(\lambda)$ such that

$$n^{-1}\left|{}^n\xi_{t_n}^{[1]}\right|\xrightarrow{\text{p}}\sum_{k=1}^{\infty}\frac{k\lambda}{k\lambda+1}p_k \text{ as } n\to\infty$$

for any sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}t_n=\infty$ and $t_n\le\exp(c(\lambda)n)$ for $n\in\mathbb{N}$.

(b) Let $\tau_n^{[1]}$ *be the extinction time of* $({}^n\xi_t^{[1]})_{t\geq 0}$*. Then*

$$\tau_n^{[1]}/\mathbb{E}\left[\tau_n^{[1]}\right] \xrightarrow{\text{d}} \text{Exp}(1) \text{ as } n \to \infty,$$

and there exists a $C > 0$ *such that* $\mathbb{E}\left[\tau_n^{[1]}\right] \leq \exp(Cn)$ *for* $n \in \mathbb{N}$*.*

Remark 2.2.5. Theorems 1.1 and 1.2 of [30] also hold for $(\tilde{G}_n)_{n\in\mathbb{N}}$, where $\tilde{G}_n$ is the configuration model on $\{1,\ldots,n\}$ specified by the degree distribution $(p_{n,k})_{k=1}^n$ with $p_{n,k} = c_{n,\alpha}k^{-\alpha}$. Here, $c_{n,\alpha}$ is the normalising constant.

2.3 Poisson Degree Distribution and Erdős–Rényi Graphs

Configuration models with Poisson degree distribution and Erdős-Rényi graphs have a close relation, since the degree distribution of Erdős-Rényi graphs converges to a Poisson distribution when the product of the number of vertices and the edge probability converges to a constant. In [45] the authors establish a new method to prove metastability of contact processes on these two models, which relies on bounding the total infection rate from below uniformly, over all sets with a fixed number of nodes. This method is different from the method in Remark 2.4.3 for contact processes on scale-free graphs.

We begin with Proposition 4.2 (1) of [45], which is on configuration models with a Poisson degree distribution:

Theorem 2.3.1. *For* $n \in \mathbb{N}$*, let* G_n *be the configuration model specified by the Poisson degree distribution with rate* μ *on* $\{1,\ldots,n\}$*, and let* $\tau_n^{[1]}$ *be the extinction time of the contact process with infection rate* λ *on* G_n *starting from all vertices infected. If* $\exp(\mu) > 2\exp(\mu/\sqrt{2}) - 1$*, then there exist* $\lambda, c > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{E}\left[\tau_n^{[1]} \,\middle|\, G_n\right] > \exp(cn)\right) = 1.$$

We refer to Proposition 4.2 (2) of [45] for an estimate of the constant c. In fact, this theorem is an application of Theorem 4.1 of [45]:

Theorem 2.3.2. *For* $n \in \mathbb{N}$*, let* G_n *be the configuration model with degree distribution* $\boldsymbol{p}$ *on* $\{1,\ldots,n\}$*, and let* $\tau_n^{[1]}$ *be the extinction time of the contact process with infection rate* λ *on* G_n *starting from all vertices infected. If* $\mathbb{E}_{\boldsymbol{p}}\left[2^{-D/2}\right] < 1/2$ *and* $\lambda > \mu_0$*, then there exists a* $c > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{E}\left[\tau_n^{[1]} \,\middle|\, G_n\right] > \exp(cn)\right) = 1.$$

Here, μ_0 is defined as follows. Put

$$\begin{aligned}
\mathrm{nlg}(0) &:= 0, \quad \mathrm{nlg}(x) := x\log x, \quad x>0,\\
\varphi(a_1,a_2;\varrho) &:= \frac{1}{2}\mathrm{nlg}(a_1+a_2)+\frac{1}{2}\mathrm{nlg}(a_1-\varrho)+\frac{1}{2}\mathrm{nlg}(a_2-\varrho)\\
&\quad -\mathrm{nlg}(a_1)-\mathrm{nlg}(a_2)+\mathrm{nlg}(\varrho),\\
R(x) &:= \sup_{a\in\mathbb{R}}\left(ax-\log\mathbb{E}\left[\exp(aD)\right]\right),\\
\Psi(\gamma,\varrho) &:= \inf_{a_1,a_2>0}\left(\varphi(a_1,a_2;\varrho)+\gamma R(a_1/\gamma)+(1-\gamma)R(a_2/(1-\gamma))\right).\\
H(\gamma) &:= -\gamma\log\gamma-(1-\gamma)\log(1-\gamma),\\
\Gamma &:= \left\{\gamma\in(0,1/2)\;\middle|\;\Psi(\gamma,0)>H(\gamma)\right\}.
\end{aligned}$$

Then,

$$\mu_0 := \inf\left\{\gamma/\varrho\;\middle|\;\gamma\in\Gamma,\varrho\in(0,\gamma(1-\gamma)\mathbb{E}[D]),\Psi(\gamma,\varrho)>H(\gamma)\right\}.$$

Next, we introduce results about contact processes on Erdős-Rényi graphs. Let G_n be the Erdős–Rényi graph with vertex set $V_n=\{1,\dots,n\}$ and edge probability p_n. Namely, each edge $\{i,j\}$ is in G_n with probability p_n independently for $i,j\in V_n$ with $i\neq j$. Let $({}^n\xi_t)_{t\geq 0}$ be the contact process with infection rate λ_n on G_n. Theorems 3.1 and 3.2 of [45] give lower bounds on the extinction time $\tau_n^{[1]}$ of $({}^n\xi_t^{[1]})_{t\geq 0}$ when $np_n\to\infty$ as $n\to\infty$ and when np_n is constant, respectively:

Theorem 2.3.3. *Suppose that $np_n\to\infty$ as $n\to\infty$.*

(a) If $np_n\lambda_n\to\infty$ as $n\to\infty$, then for $\varepsilon\in(0,1)$,

$$\mathbb{E}\left[\tau_n^{[1]}\;\middle|\;G_n\right]\geq\exp\left((1-\varepsilon)n\log(np_n\lambda_n)\right)\text{ with high probability.}$$

(b) Let $\gamma>1$. If $np_n\lambda_n=\gamma$ for $n\in\mathbb{N}$, then for $\varepsilon\in(0,1)$,

$$\mathbb{E}\left[\tau_n^{[1]}\;\middle|\;G_n\right]\geq\exp\left(n((1-\varepsilon)\log\gamma+\gamma^{-1}-1)\right)\text{ with high probability.}$$

Theorem 2.3.4. *Suppose that $\sigma>4\log 2$ and $p_n=\sigma/n$ for $n\in\mathbb{N}$. Then there exist functions*

$$\lambda_0(\sigma)=\sigma^{-1}(1+o(1)),\quad \varepsilon(\sigma)=o(1),\quad \text{as }\sigma\to\infty,$$

such that, for each $\lambda>\lambda_0(\sigma)$, there exists a $\eta>0$ such that

$$\mathbb{E}\left[\tau_n^{[1]}\;\middle|\;G_n\right]>\exp(\eta n)\text{ with high probability.}$$

Moreover, if $\lambda>\lambda_0(\sigma)$, then with high probability

$$n^{-1}\log\left(\mathbb{E}\left[\tau_n^{[1]}\;\middle|\;G_n\right]\right)\geq(1-\varepsilon(\sigma))(\log(\lambda\sigma)+\lambda^{-1}\sigma^{-1})-1.$$

2.4 Scale-Free Graphs

In this section we look at contact processes on configuration models with power-law distributions with exponent larger than 2 and preferential attachment graphs. Both of them are models for scale-free networks, and proofs rely on the presence of a small number of vertices with high degrees (see Remark 2.4.3).

In the first half of this section, we focus on configuration models with a power-law degree distribution $\boldsymbol{p} = (p_k)_{k\in\mathbb{N}_0}$ with exponent $\alpha > 2$. Note that in Theorems 2.4.1 and 2.4.4 we assume $\alpha > 3$, and (see Theorem 2.2.2) we can and will condition on the event that the graph is simple and $\{d_1 + \cdots + d_n \text{ is even}\}$. In other theorems G_n may have self-loops and multi-edges.

Theorem 1 of [26] shows that the critical value is 0 when $\alpha > 3$, while Theorem 1.3 in [31] generalises the result to the case $\alpha > 2$:

Theorem 2.4.1. *Suppose that* $\alpha > 3$ *and* $p_0 + p_1 + p_2 = 0$. *For any* $\lambda > 0$ *there is a* $p(\lambda) > 0$ *such that, for any* $\delta > 0$,

$$\lim_{n\to\infty} \inf_{0\le t\le \exp(n^{1-\delta})} \mathbb{P}\left(n^{-1}\left|{}^n\xi_t^{[1]}\right| \ge p(\lambda)\right) = 1.$$

Theorem 2.4.2. *Suppose that* $\alpha > 2$ *and* $p_0 + p_1 + p_2 = 0$. *For any* $\lambda > 0$ *there exists a* $c > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\tau_n^{[1]} \ge \mathrm{e}^{cn}\right) = 1.$$

Remark 2.4.3. The proofs of the above two theorems use different ideas. The proof of Theorem 2.4.1 utilizes the "hubs", i.e., a small number of vertices with degree $d \gg \lambda^{-2}$. Indeed, a vertex v with degree $d \gg \lambda^{-2}$ typically has order λd infected neighbours. Once v gets healthy, the probability that none of its infected neighbours infects v within a unit time is $\exp(-d\lambda^2)$, which is rather small. Thus, the contact process can survive for a long time starting from a sufficient number of infected vertices. However, Theorem 2.4.2 follows from Theorem 1.5.1. The authors of [31] conjectured that the theorem also holds when $\alpha > 1$.

Another interesting result given by Theorem 2 of [26] is on a quasi-stationary distribution in which a randomly chosen vertex is occupied with a positive probability:

Theorem 2.4.4. *Suppose that* $\alpha > 3$. *Let* X_n *be the uniform random variable on* $\{1,\dots,n\}$. *Define a measure* ${}^n\xi_\infty^1$ *on* $\{1,\dots,n\}$ *by*

$$\mathbb{P}\left({}^n\xi_\infty^1 \cap A \ne \emptyset\right) := \mathbb{P}\left({}^n\xi^A_{\exp(\sqrt{n})} \ne \emptyset\right), \quad A \subseteq \{1,\dots,n\}.$$

There is a $\lambda_0 > 0$ such that, if $0 < \lambda < \lambda_0$ and $0 < \delta < 1$, then there exist two constants $c(\delta,\alpha)$ and $C(\delta,\alpha)$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(c\lambda^{1+(\alpha-2)(2+\delta)} \le \mathbb{P}\left(X_n \in {}^n\xi^1_\infty\right) \le C\lambda^{1+(\alpha-2)(1-\delta)}\right) = 1.$$

Finally, we introduce the following result on the typical density of infected vertices, which appears as Theorem 1.1, Theorem 1.3 and Proposition 1.4 in [27]:

Theorem 2.4.5. *Suppose that $\alpha > 2$ and $p_0 + p_1 + p_2 = 0$. Define $\varrho_\alpha(\lambda)$ by*

$$\varrho_\alpha(\lambda) := \begin{cases} \lambda^{1/(3-\alpha)} & \text{if } 2 < \alpha \le \frac{5}{2}, \\ \lambda^{2\alpha-3}\log^{2-\alpha}(\lambda^{-1}) & \text{if } \frac{5}{2} < \alpha \le 3, \\ \lambda^{2\alpha-3}\log^{4-2\alpha}(\lambda^{-1}) & \text{if } \alpha > 3. \end{cases}$$

(a) There exist $c, C > 0$ such that, for $\lambda > 0$ small enough and a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers satisfying $t_n \to \infty$ and $\log t_n = o(n)$ as $n \to \infty$,

$$\lim_{n\to\infty} \mathbb{P}\left(c\varrho_\alpha(\lambda) \le n^{-1}\left|{}^n\xi^{[1]}_{t_n}\right| \le C\varrho_\alpha(\lambda)\right) = 1.$$

(b) Let T be the Galton-Watson tree with root O for which the degree distribution of O is $\boldsymbol{p}$, and all other vertices have degree distribution $\boldsymbol{q} = (q_k)_{k\in\mathbb{N}_0}$ given by

$$q_k := kp_k \Big/ \sum_{i=0}^{\infty} ip_i\,, \quad k \in \mathbb{N}_0.$$

Let $\gamma_{\boldsymbol{p}}(\lambda)$ be the survival probability of the contact process on T with infection rate λ and initial state $\{O\}$. Then, for any $\lambda, \varepsilon > 0$ and any sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers satisfying $t_n \to \infty$ and $\log t_n = o(n)$ as $n \to \infty$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|n^{-1}\left|{}^n\xi^{[1]}_{t_n}\right| - \gamma_{\boldsymbol{p}}(\lambda)\right| > \varepsilon\right) = 0.$$

Moreover, for λ small enough there exist $c, C > 0$ such that

$$c\varrho_\alpha(\lambda) \le \gamma_{\boldsymbol{p}}(\lambda) \le C\varrho_\alpha(\lambda).$$

In the second half of this section, we turn to the preferential attachment graph, where the same strategy in Remark 2.4.3 can be used to sustain infection. Here, we use the sequential model of preferential attachment graphs, which seems less natural, but is easier to analyze because it is exchangeable. Let $m \in \mathbb{N}$, $\alpha \in [0, 1)$ and $V_n := \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

Definition 2.4.6. Let G_1 contain one vertex 1 and no edges, and let G_2 contain two vertices 1 and 2, and m edges connecting them. For $n \geq 3$, create G_n by adding vertex n and edges $\{v, w_{n,1}\}, \dots, \{v, w_{n,m}\}$ to G_{n-1}. Here, $w_{n,1}, \dots, w_{n,m}$ is drawn inductively from V_{n-1}: conditional on G_{n-1}, $w_{n,1}, \dots, w_{n,i-1}$, set $w_{n,i}$ to be $k \in V_{n-1}$ with probability

$$\alpha_{n-1}^{(i)} \frac{1}{n-1} + (1 - \alpha_{n-1}^{(i)}) \frac{\deg_{n-1}^{(i)}(k)}{Z_{n-1}^{(i)}},$$

where $\deg_{n-1}(k)$ denotes the degree of vertex k in the graph G_{n-1},

$$\deg_{n-1}^{(i)}(k) := \deg_{n-1}(k) + \sum_{j=1}^{i-1} \mathbf{1}_{\{w_j = k\}},$$

$$Z_{n-1}^{(i)} := \sum_{k=1}^{n-1} \deg_{n-1}^{(i)}(k) = 2m(n-2) + i - 1,$$

$$\alpha_{n-1}^{(i)} := \alpha \frac{2m(n-1)}{2m(n-2) + 2m\alpha + (1-\alpha)(i-1)}.$$

Then $(G_n)_{n \in \mathbb{N}}$ is called the sequence of *preferential attachment graphs* with parameters m and α in the sequential model.

We continue with $(G_n)_{n \in \mathbb{N}}$ constructed above. Let $({}^n\xi_t)_{t \geq 0}$ be the contact process with infection rate $\lambda > 0$ on G_n for $n \in \mathbb{N}$. We introduce Theorem 1.1 and Proposition 1.2 of [33], which give sharp bounds for the density of infected vertices at an almost exponential time and the asymptotic behaviour of the extinction time for $({}^n\xi_t)_{t \geq 0}$, respectively:

Theorem 2.4.7. *There exist $c, C > 0$ such that, for λ small enough,*

$$\lim_{n \to \infty} \mathbb{P}\left(c\lambda^{1+2\psi} \left|\log \lambda\right|^{-\psi} \leq n^{-1} \left| {}^n\xi_{t_n}^{[1]} \right| \leq C\lambda^{1+2\psi} \left|\log \lambda\right|^{-\psi} \right) = 1,$$

where $\psi := (1+\alpha)/(1-\alpha)$, and $(t_n)_{n \in \mathbb{N}}$ is any sequence satisfying

$$\lim_{n \to \infty} t_n = \infty, \quad t_n \leq \exp\left(c\lambda^2 n (\log n)^{-\psi} \right), \ n \in \mathbb{N}.$$

Theorem 2.4.8. *Let $\tau_n^{[1]}$ be the extinction time of $({}^n\xi_t^{[1]})_{t \geq 0}$. Then*

$$\tau_n^{[1]} / \mathbb{E}\left[\tau_n^{[1]} \right] \xrightarrow{\mathrm{d}} \mathrm{Exp}(1) \text{ as } n \to \infty.$$

Remark 2.4.9. The proof of Proposition 1.2 of [33] also shows the metastability of $({}^n\xi_t)_{t \geq 0}$, namely, after time $n^{\log n}$, either $({}^n\xi_t)_{t \geq 0}$ dies out or it equals the contact process starting from [1].

2.5 Dynamic Scale-Free Graphs

In this section we describe the recent research in [34, 39, 49] regarding contact processes on scale-free random graphs evolving according to three different stationary dynamics. They show that there is a phase transition when the power-law exponent is greater than 4, which is different from contact processes on corresponding static scale-free graphs (see Theorems 2.4.2 and 2.4.8).

We begin with the construction of dynamic scale-free graphs. Let $\beta > 0$, $\gamma \in (0,1)$, $\eta \in \mathbb{R}$, $\varkappa > 0$ and $\alpha := 1 + \gamma^{-1} \in (2, +\infty)$. Set

$$\varkappa_i := \varkappa\left(\frac{n}{i}\right)^{\gamma\eta}, \quad p_{i,j} := \min\left(\frac{1}{n}p\left(\frac{i}{n}, \frac{j}{n}\right), 1\right), \quad n \in \mathbb{N}, i, j \in \{1, \dots, n\},$$

where $p(x, y)$ is one of the following four kernels:

- the *factor kernel* $p(x, y) := \beta x^{-\gamma} y^{-\gamma}$;
- the *preferential attachment kernel* $p(x, y) := \beta \min(x, y)^{-\gamma} \max(x, y)^{\gamma-1}$;
- the *strong kernel* $p(x, y) := \beta \min(x, y)^{-\gamma}$;
- the *weak kernel* $p(x, y) := \beta \max(x, y)^{-\gamma-1}$.

For $n \in \mathbb{N}$, let $G^{(n)}$ be the random graph with vertex set $V_n := \{1, \dots, n\}$ that contains each edge $\{i, j\}$ independently with probability $p_{i,j}$. It is easy to check that $(G^{(n)})_{n\in\mathbb{N}}$ is a scale-free graph sequence with power-law exponent α for all four kernels. We construct the dynamic graph $(G_t^{(n)})_{t\geq 0}$ with $G_0^{(n)} := G^{(n)}$ according to one of the following three dynamics:

- In Dynamics I and II, every vertex i is updated independently (at rate $\varkappa$ in Dynamics I and at rate $\varkappa_i$ in Dynamics II), and upon updating receives a new set of adjacent edges according to the kernel p, independently of the previous state.
- In Dynamics III, every unordered pair $\{i, j\}$ of distinct vertices is updated independently at rate $\varkappa_i + \varkappa_j$, and upon updating, vertices i and j are connected by an edge with probability $p_{i,j}$, independently of the previous state.

We run the infection process $({}^n\xi_t)_{t\geq 0}$ with infection rate $\lambda > 0$ on $(G_t^{(n)})_{t\geq 0}$. Let $\tau_n^{[1]}$ be the extinction time of $({}^n\xi_t^{[1]})_{t\geq 0}$. In [34, 39, 49] the terms *slow extinction*, *metastability* and *metastability exponent* are vital and hence we list them in the following two definitions. Since the term "fast extinction" is defined differently in the three papers, we will not use it to avoid possible confusion.

Definition 2.5.1. We say that there is *slow extinction* if for all $\lambda > 0$ there exists a $c > 0$ such that $\mathbb{P}\left(\tau_n^{[1]} \leq \exp(cn)\right) \leq \exp(-cn)$ for $n \in \mathbb{N}$.

Definition 2.5.2. We call the infection process $({}^n\xi_t)_{t\geq 0}$ *metastable* if there exists an $\varepsilon > 0$ such that, for any $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \in \mathscr{S}(\varepsilon)$,

$$\liminf_{n\to\infty} I_n(t_n) > 0, \quad \liminf_{n\to\infty} I_n(s_n) > 0, \quad \lim_{n\to\infty} (I_n(t_n) - I_n(s_n)) = 0,$$

where $\mathscr{S}(\varepsilon)$ is the set of sequences $(s_n)_{n\in\mathbb{N}}$ of positive real numbers satisfying $\lim_{n\to\infty} t_n = \infty$ and $t_n < \exp(\varepsilon n)$ for $n \in \mathbb{N}$, and

$$I_n(t) := \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} {}^n\xi_t^{[1]}(i)\right], \quad n \in \mathbb{N}, t > 0.$$

Moreover, we call χ the *metastable exponent* of the infection process if

$$\chi = \lim_{\lambda\downarrow 0} \frac{\log(\liminf_{n\to\infty} I_n(t_n))}{\log\lambda} = \lim_{\lambda\downarrow 0} \frac{\log\left(\limsup_{n\to\infty} I_n(t_n)\right)}{\log\lambda}.$$

Before listing results, we expain how the authors of [34, 39, 49] adapt the idea of utilizing “hubs” in Remark 2.4.3 for the static case to the dynamic case.

Remark 2.5.3. For convenience, we use the word *star* to denote a subgraph consisting of a “hub” and its neighbours. In the static case we rely on a small number of “hubs” to sustain the infection. However, dynamic scale-free graphs are more complicated, e.g., a star may disappear if the “hub” has only few neighbours after an updating. The authors of [39] consider the infection between stars and use the following four *relevant survival strategies* to sustain survival. We refer to Theorem 1 of [39] for formal definitions.

- *Quick direct spreading*: stars directly infect sufficiently many other stars before their recoveries.
- *Quick indirect spreading*: stars infect sufficiently many neighbours before their recoveries, and these neighbours subsequently infect other stars.
- *Delayed direct spreading*: if the degree of a star is of larger order than λ^{-2}, then it remains infected for a long time and infect other stars directly.
- *Delayed indirect spreading*: the mechanism is similar as the delayed direct spreading, but the star infects other stars via its neighbours.

Next, we list the results concerning Dynamics I as Theorems 2.5.4 and 2.5.5, which appear as Theorem 2.1 of [34], and Propositions 1 and 2 of [39], respectively.

Theorem 2.5.4. *Suppose that p is the factor kernel and $(G_t^{(n)})_{t\geq 0}$ evolves according to Dynamics I.*
(a) If $\gamma < 1/3$, then there exist $\lambda, C > 0$ such that $\mathbb{E}[\tau_n^{[1]}] \leq Cn^{\gamma}\log n$ for $n \in \mathbb{N}$.

(b) *If* $1/3 < \gamma < 1$*, then there is slow extinction and metastability, and the metastability exponent is given by*

$$\chi = \begin{cases} 2/(3\gamma - 1) & \text{if } 1/3 < \gamma < 2/3, \\ \gamma/(2\gamma - 1) & \text{if } 2/3 < \gamma < 1. \end{cases}$$

Theorem 2.5.5. *If* p *is the preferential attachment kernel and* $(G_t^{(n)})_{t\geq 0}$ *evolves according to Dynamics I, then there is slow extinction and metastability, and the metastability exponent is given by*

$$\chi = \begin{cases} (3 - 2\gamma)/\gamma & \text{if } 0 < \gamma < 3/5, \\ (3 - \gamma)/(3\gamma - 1) & \text{if } 3/5 < \gamma < 1. \end{cases}$$

We move to Theorem 3 of [39] concerning Dynamics II:

Theorem 2.5.6. *Suppose that* p *is the factor kernel and* $(G_t^{(n)})_{t\geq 0}$ *evolves according to Dynamics II.*

(a) *If* $\eta < 1/2$ *and* $\gamma < (3 - 2\eta)^{-1}$*, or if* $\eta \geq 1/2$ *and* $\gamma < 1/2$*, then there exist* $\lambda, c > 0$ *such that* $\mathbb{E}[\tau_n^{[1]}] \leq cn^c$ *for* $n \in \mathbb{N}$*.*

(b) *If* $\eta < 1/2$ *and* $\gamma > (3 - 2\eta)^{-1}$*, or if* $\eta \geq 1/2$ *and* $\gamma > 1/2$*, then there is slow extinction and metastability, and the metastability exponent* χ *is given by*

$$\chi = \begin{cases} \frac{2-2\gamma\eta}{3\gamma-2\gamma\eta-1} & \text{if } \gamma < \frac{2}{3+2\eta}, \\ \frac{\gamma}{2\gamma-1} & \text{if } \gamma > \frac{2}{3+2\eta}. \end{cases}$$

Theorem 2.5.7. *Assume that* p *is the preferential attachment kernel and* $(G_t^{(n)})_{t\geq 0}$ *evolves according to Dynamics II.*

(a) *If* $\eta \geq 1/2$ *and* $\gamma < 1/2$*, then there exist* $\lambda, c > 0$ *such that* $\mathbb{E}[\tau_n^{[1]}] \leq cn^c$ *for* $n \in \mathbb{N}$*.*

(b) *If* $\eta < 1/2$*, or if* $\eta \geq 1/2$ *and* $\gamma > 1/2$*, then there is slow extinction and metastability, and the metastability exponent* χ *is given by*

$$\chi = \begin{cases} \frac{3-2\gamma-2\gamma\eta}{\gamma-2\gamma\eta} & \text{if } \eta < 1/2 \text{ and } \gamma < \frac{3}{5+2\eta}, \\ \frac{3-\gamma-2\gamma\eta}{3\gamma-2\gamma\eta-1} & \text{if } \eta < 1/2 \text{ and } \frac{3}{5+2\eta} < \gamma < \frac{1}{1+2\eta}, \\ \frac{1}{2\gamma-1} & \text{if } \gamma > \frac{1}{1+2\eta}. \end{cases}$$

Remark 2.5.8. According to Remark 1 of [39], when $\eta > 1/2$ and $\gamma < 1/2$, $\mathbb{E}[\tau_n^{[1]}]$ is even subpolynomial in n in both kernels.

Finally, we list Theorem 1 of [49] on Dynamics III as the following three theorems:

Theorem 2.5.9. *Suppose that p is the factor kernel and $(G_t^{(n)})_{t\geq 0}$ evolves according to Dynamics III.*

(a) If $\eta \geq 1/2$ and $\alpha > 3$, then there exist $\lambda, c > 0$ such that $\mathbb{E}[\tau_n^{[1]}] \leq c(\log n)^c$ for $n \in \mathbb{N}$.
(b) If $\eta < 1/2$ or $\alpha < 3$, then there is slow extinction and metastability. Moreover, the metastability exponent χ is given by

$$\chi = \begin{cases} \frac{1}{3-\alpha} & \text{if } \begin{cases} & \eta \leq 0 \text{ and } \alpha \leq 5/2, \\ \text{or} & 0 \leq \eta \leq 1/2 \text{ and } \alpha \leq 5/2+\eta, \\ \text{or} & \eta \geq 1/2 \text{ and } \alpha < 3, \end{cases} \\ 2\alpha - 3 & \text{if } \eta \leq 0 \text{ and } \alpha \geq 5/2, \\ \frac{2\alpha-3-2\eta}{1-2\eta} & \text{if } 0 \leq \eta < 1/2 \text{ and } \alpha > 5/2+\eta. \end{cases}$$

Theorem 2.5.10. *Suppose that p is the preferential attachment kernel or the strong kernel, and $(G_t^{(n)})_{t\geq 0}$ evolves according to Dynamics III.*

(a) If $\eta \geq 1/2$ and $\alpha > 3$, then there exist $\lambda, c > 0$ such that $\mathbb{E}[\tau_n^{[1]}] \leq c(\log n)^c$ for $n \in \mathbb{N}$.
(b) If $\eta < 1/2$, or if $\eta \geq 1/2$ and $\alpha < 3$, then there is slow extinction and metastability. Moreover, the metastability exponent χ is given by

$$\chi = \begin{cases} 2\alpha - 3 & \text{if } \eta \leq 0, \\ \frac{2\alpha-3-2\eta}{1-2\eta} & \text{if } 0 < \eta < 1/2 \text{ and } \alpha \geq 2+2\eta, \\ \frac{\alpha-1}{3-\alpha} & \text{if } \alpha < 3 \text{ and } \alpha < 2\eta+2. \end{cases}$$

Theorem 2.5.11. *Suppose that p is the weak kernel and $(G_t^{(n)})_{t\geq 0}$ evolves according to Dynamics III. Then there is slow extinction and metastability, and the metastability exponent is $\alpha - 1$.*

Chapter 3

Contact Processes in Random Environments I

In this chapter we provide an overview of recent research on contact processes in random environments, where the environment is either for vertices (Sections 3.1 and 3.2) or for edges (Sections 3.3 to 3.6). In Section 3.1 we look at Broman's model, where each vertex recovers at rate δ_i in environment i for $i \in \{0,1\}$ and the environment of each vertex swithes between 0 and 1 independently. Remenik's model in Section 3.2 is different: vertices, whether healthy or infected, are *blocked* at rate α and then unblocked at rate $\alpha\delta$, becoming healthy once unblocked. Sections 3.3 to 3.5 delve into contact processes on dynamic bond percolation where edges open and close dynamically. In Section 3.3 we construct the graphical representation and discuss some basic properties. Section 3.4 introduces results for the homogeneous setting where the dynamics of all edges are the same and the graph is vertex-transitive, while Section 3.5 introduces results for dynamic bond percolation on complete graphs, involving long range edges. Finally, in Section 3.6, we introduce contact processes on long range percolation on $\mathbb{Z}$ and lattices with dynamic range, which are two variants of dynamic bond percolation.

3.1 Broman's Randomly Evolving Environment

In this section we introduce the contact process in Broman's randomly evolving environment, where infected vertices recover at rate δ_0 in environment 0 and recover at rate δ_1 in environment 1. Assume that $0 \le \delta_0 \le \delta_1$, $p \in [0,1]$ and $\gamma > 0$.

We begin with the one-dimensional static version investigated in [12]. Let $(B_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. random variables with distribution $\mathrm{Ber}(p)$. The transition rates of the contact process $(\xi_t)_{t\ge 0}$ on $\mathbb{Z}$ in the environment $(B_n)_{n\in\mathbb{Z}}$

is given by

$$c(n,\eta) = \begin{cases} \left|\eta_t \cap \{n-1, n+1\}\right| & \text{if } \eta(n) = 0, \\ \delta_{B_n} & \text{if } \eta(n) = 1. \end{cases}$$

Theorems 1 and 2 of [12] provide upper bounds on the rate of growth and show that $(\xi_t)_{t\geq 0}$ has an intermediate phase, respectively:

Theorem 3.1.1. *Set* $\Omega_\infty := \{\xi_t^0 \neq \emptyset \text{ for } t \geq 0\}$, $\varrho_n := \inf\left\{t \geq 0 \,\middle|\, n \in \xi_t^0\right\}$ *and* $\delta_c := \sup\{\delta > 0 \mid \mathbb{P}_\delta(\Omega_\infty) > 0\}$, *where* $\mathbb{P}_\delta$ *is the law of the process when* $\delta_1 = \delta_0 = \delta$. *Suppose that* $\delta_1 > \delta_c$ *and* $\delta_0 = 0$. *Then the limit*

$$\gamma_\perp(\delta_1) := \lim_{n\to\infty} -n^{-1} \log \mathbb{P}\left(\sup_{t\geq 0} \sup \xi_t^0 \geq n\right)$$

exists. Moreover,

(a) If $\gamma_\perp(\delta_1) < -\log p$, *then* ϱ_n/n *converges almost surely on* Ω_∞ *to a positive number as* $n \to \infty$.

(b) If $\gamma_\perp(\delta_1) \geq -\log p$, *then for* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\log \varrho_n / \log n + \gamma_\perp(\delta_1)/\log p\right| > \varepsilon, \Omega_\infty\right) = 0.$$

Theorem 3.1.2. *Suppose that* $p < 1$. *There is a* $\delta_c(\delta_1, p) > 0$ *such that, if* $\delta_0 < \delta_c(\delta_1, p)$, *then for almost surely every environment* $e \in \{0,1\}^{\mathbb{Z}}$,

$$\mathbb{P}^e\left(\xi_t^0 \neq [0] \text{ for } t \geq 0\right) > 0,$$

where $\mathbb{P}^e$ *is the distribution of the contact process in the fixed environment* e.

Next, we turn to the contact process in the randomly evolving environment on a graph $G = (V, E)$ investigated in [23] by Broman, which is a pair of processes $(B_t, \xi_t)_{t\geq 0}$ with state space $(\{0,1\} \times \{0,1\})^V$. For $v \in V$, $B_t(v)$ denotes the environment that vertex v sees and $\xi_t(v)$ denotes the state of v at time t. The transition rates of vertex v at time t are given by Table 3.1. Denote by π_q the product measure of $\text{Ber}(q)$ on V for $q \in [0,1]$.

Broman's main tool (Theorem 1.4 of [23]) is to couple the point process of recovery marks in the graphical representation of the contact process in a randomly evolving environment, which is embedded in a hidden Markov chain, with the Poisson point process in the graphical representation of ordinary contact processes:

Theorem 3.1.3. *Let* $(B_t, X_t)_{t\geq 0}$ *be a Markov process on* $\{0,1\} \times \mathbb{N}_0$ *with initial distribution* $B_0 \sim \text{Ber}(p)$ *and* $X_0 = 0$, *and with transition rates given by Table 3.2. Let* $\tau_k := \inf\{t \geq 0 \mid X_t = k\}$ *for* $k \in \mathbb{N}$, *and let* $X := \{\tau_k \mid k \in \mathbb{N}\}$. *Define*

$$\lambda_{\max}(\delta_0, \delta_1, \gamma, p) := \max\{\lambda \geq 0 \mid \text{Poi}_\lambda \preceq X\},$$

from	to	at rate
$(0,0)$	$(1,0)$	γp
$(0,1)$	$(1,1)$	γp
$(1,0)$	$(0,0)$	$\gamma(1-p)$
$(1,1)$	$(0,1)$	$\gamma(1-p)$
$(0,0)$	$(0,1)$	$\sum_{u\sim v}\xi_t(u)$
$(1,0)$	$(1,1)$	$\sum_{u\sim v}\xi_t(u)$
$(0,1)$	$(0,0)$	δ_0
$(1,1)$	$(1,0)$	δ_1

Table 3.1: *Transition rates of* $(B_t,\xi_t)_{t\geq 0}$

from	to	at rate
$(0,k)$	$(1,k)$	γp
$(1,k)$	$(0,k)$	$\gamma(1-p)$
$(0,k)$	$(0,k+1)$	δ_0
$(1,k)$	$(1,k+1)$	δ_1

Table 3.2: *Transition rates of* $(B_t,X_t)_{t\geq 0}$

$$\lambda_{\min}(\delta_0,\delta_1,\gamma,p) := \min\{\lambda\geq 0 \mid X \preceq \mathrm{Poi}_\lambda\}.$$

Then $\lambda_{\min}(\delta_0,\delta_1,\gamma,p) = \delta_1$ *for* $p>0$*, and*

$$\lambda_{\max}(\delta_0,\delta_1,\gamma,p) = \frac{1}{2}\left(\delta_0+\delta_1+\gamma-\sqrt{(\delta_1-\delta_0-\gamma)^2+4\gamma(1-p)(\delta_1-\delta_0)}\right).$$

In case $\delta_0=\delta_1=\delta$, $(\xi_t^v)_{t\geq 0}$ is an ordinary contact process, and we denote by δ_{c1} and δ_{c2} the critical values of $(\xi_t^v)_{t\geq 0}$, i.e., $(\xi_t^v)_{t\geq 0}$ dies out when $\delta>\delta_{c2}$, $(\xi_t^v)_{t\geq 0}$ survives weakly when $\delta_{c1}<\delta<\delta_{c2}$, and $(\xi_t^v)_{t\geq 0}$ survives strongly when $\delta<\delta_{c1}$. We repeat the phase transition obtained in Theorem 1.8 and Proposition 1.9 of [23] as follows:

Theorem 3.1.4. *Suppose that* $v\in V$, $\delta_1<\infty$, $B_0\sim\pi_p$ *and* $D\in\mathbb{N}$ *is the maximum degree of* G.

(a) Assume that $\delta_{c1}<\delta_0<\delta_{c2}<\delta_1$. *There exists a* $p_{c2}=p_{c2}(\delta_0,\delta_1,\gamma)\in[0,1]$ *such that* $(\xi_t^v)_{t\geq 0}$ *dies out if* $p>p_{c2}$ *and survives weakly if* $p<p_{c2}$. *Moreover,* $p_{c2}<1$ *if* $\gamma>\delta_{c2}-\delta_0$, *and* $p_{c2}>0$ *if* $\gamma\geq D$.

(b) Assume that $\delta_0<\delta_{c1}\leq\delta_{c2}<\delta_1$. *There exist* $p_{c2}=p_{c2}(\delta_0,\delta_1,\gamma)\in[0,1]$ *and* $p_{c1}=p_{c1}(\delta_0,\delta_1,\gamma)\in[0,1]$ *such that* $p_{c1}\leq p_{c2}$, $(\xi_t^v)_{t\geq 0}$ *dies out if* $p>p_{c2}$, *survives weakly if* $p_{c1}<p<p_{c2}$, *and survives strongly if* $p<p_{c1}$. *Moreover,* $p_{c2}<1$ *if* $\gamma>\delta_{c2}-\delta_0$, $p_{c1}<1$ *if* $\gamma>\delta_{c1}-\delta_0$, *and* $p_{c1},p_{c2}>0$ *if* $\gamma\geq D$.

(c) Assume that $\delta_0<\delta_{c1}<\delta_1<\delta_{c2}$. *There exists a* $p_{c1}=p_{c1}(\delta_0,\delta_1,\gamma)\in[0,1]$ *such that* $(\xi_t^v)_{t\geq 0}$ *survives weakly if* $p>p_{c1}$ *and survives strongly if* $p<p_{c1}$. *Moreover,* $p_{c1}<1$ *if* $\gamma>\delta_{c1}-\delta_0$, *and* $p_{c1}>0$ *if* $\gamma\geq D$.

(d) Fix $i\in\{1,2\}$ *and assume that* $\delta_0<\delta_{ci}$. *Then*

$$\limsup_{\gamma\to\infty} p_{ci}(\delta_0,\delta_1,\gamma)\leq\frac{\delta_{ci}-\delta_0}{\delta_1-\delta_0}.$$

Remark 3.1.5. Broman conjectured that $0<p_{c1},p_{c2}<1$ holds under a weaker condition, and the limit of $p_{ci}(\delta_0,\delta_1,\gamma)$ as $\gamma\to\infty$ exists.

Finally, Theorems 1.2 and 1.4 of [25] show that the survival probability is independent of the initial distribution and that the critical process dies out, respectively:

Theorem 3.1.6. *Assume that $G = \mathbb{Z}^d$, $0 < \delta_0 \le \delta_1$, $q \in [0,1]$ and $A \subseteq \mathbb{Z}^d$ with $|A| < \infty$. Denote by $(\xi_t^{\pi_q,A})_{t\ge0}$ the infection process with $\xi_0 = A$ and $B_0 \sim \pi_q$. Set*

$$p_c(q,A) := \inf\left\{ p > 0 \,\middle|\, \mathbb{P}\left(\xi_t^{\pi_q,A} \neq \emptyset \text{ for } t > 0\right) > 0 \right\},$$

where $\inf\emptyset := 1$. Then $p_c(q,A)$ is independent of both A and q. Moreover, if $p_c \in (0,1]$ and $p = p_c$, then $(\xi_t^{\pi_q,A})_{t\ge0}$ dies out.

3.2 Remenik's Dynamic Random Environment

In this section we introduce the contact process in Remenik's dynamic random environment, which was first investigated in [24]. Different from Broman's model in Section 3.1, Remenik's model adds an additional *blocked* state -1 to each vertex. Once a vertex is blocked, it must first be unblocked and return to the healthy state 0. Remenik claims that his model is natural: if a vertex becomes uninhabitable, then the infection disappears.

First, we follow the way in [24] to construct the model and its graphical representation. The contact process $(\xi_t)_{t\ge0}$ in Remenik's dynamic random environment on $\mathbb{Z}^d$ is a Markov process with state space $\{-1,0,1\}^{\mathbb{Z}^d}$ and with transition rates of vertex x at time t given by

$$\begin{array}{cccll} 0 & \longrightarrow & 1 & \text{at rate} & \lambda \sum_{y\sim x} \xi_t(y) \\ 1 & \longrightarrow & 0 & \text{at rate} & 1 \\ 0,1 & \longrightarrow & -1 & \text{at rate} & \alpha \\ -1 & \longrightarrow & 0 & \text{at rate} & \alpha\delta \end{array}$$

The graphical representation of $(\xi_t)_{t\ge0}$ is constructed by placing symbols on $\mathbb{Z}^d \times [0,\infty)$ according to the set

$$\left\{ \mathscr{R}^x, \mathscr{I}^{x,y}, \mathscr{B}^x, \mathscr{U}^x \,\middle|\, x, y \in \mathbb{Z}^d, |x-y| = 1 \right\}$$

of independent Poisson point processes on $[0,\infty)$. Here, for $x, y \in \mathbb{Z}^d$ with $|x-y| = 1$,

- the intensity of $\mathscr{R}^x$ is 1, and we put a *recovery mark* at (x,t) for $t \in \mathscr{R}^x$.
- the intensity of $\mathscr{I}^{x,y}$ is λ, and we put an *infection arrow* from (x,t) to (y,t) for $t \in \mathscr{I}^{y,z}$.
- the intensity of $\mathscr{B}^x$ is α, and we put a *block mark* at (x,t) for $t \in \mathscr{B}^x$.

- the intensity of $\mathscr{U}^x$ is $\alpha\delta$, and we put an *unblock mark* at (x,t) for $t \in \mathscr{U}^x$.

Consider an initial configuration ξ_0, for $x \in \mathbb{Z}^d$ and $t \geq 0$, set $\xi_t(x) := -1$ if $\max\left\{s \leq t \mid s \in \mathscr{B}^x \cup \mathscr{U}^x\right\} \in \mathscr{B}^x$, or if $\xi_0(x) = -1$ and $[0,t] \cap \mathscr{U}^x = \emptyset$. Set

$$B_t := \left\{ x \in \mathbb{Z}^d \,\middle|\, \xi_t(x) = -1 \right\}.$$

We say that there is an *active path* from (x,s) to (y,t) if there is a connected oriented path, moving along the time lines in the increasing direction and passing along infection arrows, but without passing recovery marks and space-time points that are set to -1. Set $A_0 := \left\{x \in \mathbb{Z}^d \mid \xi_0(x) = 1\right\}$ and

$$A_t := \left\{ y \in \mathbb{Z}^d \,\middle|\, \text{there exists an active path from } (x,0) \text{ to } (y,t) \text{ for some } x \in A_0 \right\}.$$

Then $A_t = \left\{x \in \mathbb{Z}^d \mid \xi_t(x) = 1\right\}$. We say that $(\xi_t)_{t\geq 0}$ *survives* if

$$\mathbb{P}\left(A_t \neq \emptyset \,\middle|\, A_0 = \{0\}, B_0 = \mathbb{Z}^d \setminus \{0\} \right) > 0, \quad t \geq 0.$$

Otherwise, we say that $(\xi_t)_{t\geq 0}$ *dies out.*

The attractiveness and the monoticity properties follow from the graphical representation (see Section 1 and Propostion 2.1 of [24]):

Theorem 3.2.1. *Let μ_1 and μ_2 be two probability measures on $\{-1,0,1\}^{\mathbb{Z}^d}$ with $\mu_1 \preceq \mu_2$. Then $\xi_t^{\mu_1} \preceq \xi_t^{\mu_2}$ for $t \geq 0$. Consequently, the following two weak limits*

$$\bar{v} := \lim_{t\to\infty} \xi_t^{[1]}, \quad \underline{v} := \lim_{t\to\infty} \xi_t^{[-1]}$$

exist, where $[-1]$ is the configuration with all vertices blocked. Moreover, they are invariant, and any invariant measure v satisfies $\underline{v} \preceq v \preceq \bar{v}$.

Theorem 3.2.2. *Consider another contact process $(\xi'_t)_{t\geq 0}$ in Remenik's dynamic random environment on $\mathbb{Z}^d$ with parameters λ', α' and δ'. Assume that $(\xi_t)_{t\geq 0}$ survives. If $\lambda < \lambda'$, $\alpha = \alpha'$ and $\delta = \delta'$, or if $\lambda = \lambda'$, $\alpha = \alpha'$ and $\delta < \delta'$, then $(\xi'_t)_{t\geq 0}$ also survives.*

Moreover, Remenik's model satisfies the self-duality relation (Propostion 2.2 of [24]). To present the result, we define the probability measure ν_A on $\{-1,0,1\}^{\mathbb{Z}^d}$ for $A \subseteq \mathbb{Z}^d$ as follows: -1's are chosen first according to their equilibrium measure μ_δ and then 1's are placed at every site in A that is not blocked by a -1. Here, the equilibrium measure μ_δ is given by the product measure

$$\mu_\delta\left(\{\xi \mid \xi(x) = -1\}\right) = 1 - \mu_\delta\left(\{\xi \mid \xi(x) \neq -1\}\right) = \frac{1}{1+\delta}, \quad x \in \mathbb{Z}^d.$$

Theorem 3.2.3. *Assume that* $U, V, W \subseteq \mathbb{Z}$ *with* U *or* V *finite. Then*

$$\mathbb{P}^{\nu_U}\left(A_t \cap V \neq \emptyset, B_t \cap W \neq \emptyset\right) = \mathbb{P}^{\nu_V}\left(A_t \cap U \neq \emptyset, B_0 \cap W \neq \emptyset\right).$$

Finally, we repeat the results on phase transition and complete convergence that appear as Theorems 1 and 2 of [24]:

Theorem 3.2.4. *(a) If* $\lambda < (\alpha+1)\lambda_c(\mathbb{Z}^d)$*, then* $(\xi_t)_{t\geq 0}$ *dies out.*
(b) There exists a $\delta_p > 0$ *such that* $(\xi_t)_{t\geq 0}$ *dies out for any* λ*,* α *and* $\delta < \delta_p$*.*
(c) If $\bar{\lambda}(\lambda,\alpha,\delta) > (\alpha+1)2d\lambda_c(\mathbb{Z}^d)$*, then* $(\xi_t)_{t\geq 0}$ *survives. Here,*

$$\bar{\lambda}(\lambda,\alpha,\delta) := \frac{1}{2}\left(2d\lambda + \alpha(1+\delta) - \sqrt{(2d\lambda - \alpha(1+\delta))^2 + 8d\alpha\lambda}\right).$$

Theorem 3.2.5. *Denote by* $\tau := \inf\{t \geq 0 \mid A_t = \emptyset\}$ *the extinction time of* $(\xi_t)_{t\geq 0}$*. Then, for any initial distribution* μ*,*

$$\xi_t^\mu \xrightarrow{\mathrm{d}} \mathbb{P}^\mu(\tau < \infty)\,\underline{\nu} + \mathbb{P}^\mu(\tau = \infty)\,\bar{\nu},$$

where $\bar{\nu}$ *and* $\underline{\nu}$ *are the upper and lower invariant measures of* $(\xi_t)_{t\geq 0}$ *defined in Theorem 3.2.1, respectively.*

3.3 Dynamic Bond Percolation (Setup)

Instead of putting vertices in a dynamic random environment, edges can also open and close randomly, and infections can only be transmitted via open edges. In this section we give the graphical representation of the contact process on dynamic bond percolation and introduce the research in [43] in the homogeneous setting.

We begin with the construction. Let $G = (V,E)$ be a connected graph with bounded degree. Let $\boldsymbol{\lambda} := (\lambda_x)_{x\in V}$, $\boldsymbol{p} := (p_e)_{e\in E}$ and $\boldsymbol{v} := (v_e)_{e\in E}$ be sequences of numbers in $(0,\infty)$, $[0,1]$ and $(0,\infty)$, respectively. For an *environment* $g \in \{0,1\}^E$, we interpret the value 1 and 0 of the *state* $g(e)$ of an edge $e \in E$ as *open* and *closed*, respectively. Let

$$\mathrm{GR} := \left\{\mathscr{C}^e, \mathscr{O}^e, \mathscr{I}^e, \mathscr{R}^x \;\middle|\; e \in E, x \in V\right\} \tag{3.3.1}$$

be a set of independent Poisson point processes on $[0,\infty)$, where $\mathscr{C}^e$, $\mathscr{O}^e$, $\mathscr{I}^e$ and $\mathscr{R}^x$ have intensity $v_e(1-p_e)$, $v_e p_e$, λ_e and 1, respectively. Set

$$G_t^e := \begin{cases} G_0^e, & \text{if } [0,t] \cap (\mathscr{C}^e \cup \mathscr{O}^e) = \emptyset, \\ 1, & \text{if } \max\left([0,t] \cap (\mathscr{C}^e \cup \mathscr{O}^e)\right) \in \mathscr{O}^e, \\ 0, & \text{if } \max\left([0,t] \cap (\mathscr{C}^e \cup \mathscr{O}^e)\right) \in \mathscr{C}^e, \end{cases} \qquad t \geq 0, e \in E.$$

Here, $\max\emptyset := \infty$. See Figure 3.1, where we place a purple line segment at $t \in \mathscr{C}^e$ and an olive line segment at $t \in \mathscr{O}^e$ for $e \in E$. We fill the area where the edge is closed with gray, and place recovery marks and infection arrows in the same way as in Section 1.1.

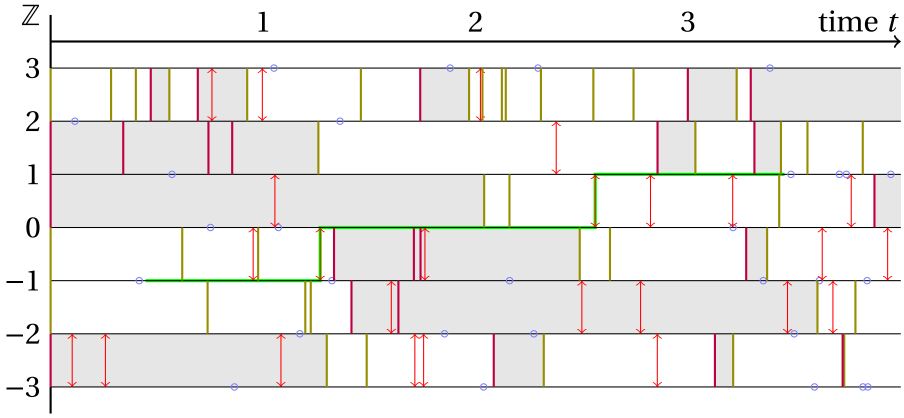

Figure 3.1: *The (partial) graphical representation of* $(\xi_t)_{t\ge0}$ *in case* $G = \mathbb{Z}$, $\lambda_e = 1.2$, $p_e = 0.6$ *and* $v_e = 2.5$ *for* $e \in E$.

Different from ordinary contact processes, only infections via open edges are *valid*. An *active path* in $V \times [0,\infty)$ is a connected oriented path that moves along the time lines in the increasing direction and along the **valid** infection arrows, but without passing any recovery marks. See the green line in Figure 3.1 for an example. Define the *infection process* $(\xi_t)_{t\ge0}$ on $\{0,1\}^V$ by setting $\xi_t(x) := 1$ if and only if there exists an active path from $(y,0)$ to (x,t) for some $y \in \xi_0$, for $t > 0$ and $x \in V$. Namely, there exists a non-decreasing sequence $(t_i)_{i=0}^n$ of times and a sequence $(x_i)_{i=0}^n$ of vertices satisfying:

- $t_0 = 0$, $\xi_0(x_0) = 1$, $t_n = t$ and $x_n = x$.
- For $0 \le i \le \lfloor (n-1)/2 \rfloor$, $x_{2i} = x_{2i+1}$ and $[t_{2i}, t_{2i+1}] \cap \mathscr{R}^{x_{2i}} = \emptyset$.
- For $0 \le i \le \lfloor n/2 \rfloor - 1$, $t_{2i+1} = t_{2i+2}$ and $t_{2i+1} \in \mathrm{VI}^{\{x_{2i+1}, x_{2i+2}\}}$, where

$$\mathrm{VI}^e := \left\{ t \in \mathscr{I}^e \,\middle|\, G_t^e = 1 \right\}. \tag{3.3.2}$$

Clearly, $\left\{ (G_t^e)_{t\ge0} \,\middle|\, e \in E \right\}$ is a set of independent continuous-time Markov chains on $\{0,1\}$, where each $(G_t^e)_{t\ge0}$ evolves as follows:

$$0 \to 1 \quad \text{at rate } v_e p_e, \qquad 1 \to 0 \quad \text{at rate } v_e(1-p_e).$$

The continuous-time Markov chain $(G_t)_{t\ge0}$ with state space $\{0,1\}^E$ given by $G_t(e) := G_t^e$ for $t \ge 0$ and $e \in E$ is called *dynamic bond percolation* on G with *density* $\boldsymbol{p}$ and *speed* $\boldsymbol{v}$. Writing $G_t(\{x,y\})$ as $G_t(x,y)$ should cause no confusion. Then the process $(G_t, \xi_t)_{t\geq 0}$ is called the *contact process on dynamic bond percolation* with infection rate $\boldsymbol{\lambda}$, density $\boldsymbol{p}$ and speed $\boldsymbol{v}$ on G. We call $(\xi_t)_{t\geq 0}$ the *ordinary contact process* with infection rate $\boldsymbol{\lambda}$ on G if $p_e = 1$ for $e \in E$. For convenience, we introduce the following notation:

- For two sequences $\boldsymbol{a} := (a_e)_{e\in E}$ and $\boldsymbol{a}' := (a'_e)_{e\in E}$, we write $\boldsymbol{a} \leq \boldsymbol{a}'$ if $a_e \leq a'_e$ for $e \in E$.
- Let $\pi_{\boldsymbol{p}}$ denote the product measure $\bigotimes_{e\in E} \mathrm{Ber}(p_e)$ on $\{0,1\}^E$.
- For $A \subseteq G$ and a probability measure μ on $\{0,1\}^E$, we denote by $(G^\mu_t)_{t\geq 0}$ a copy of $(G_t)_{t\geq 0}$ with $G_0 \sim \mu$, and denote by $(\xi^{\mu,A}_t)_{t\geq 0}$ a copy of the infection process starting with $G_0 \sim \mu$ and $\xi_0 = A$. We write $(\xi^{\pi_{\boldsymbol{p}},A}_t)_{t\geq 0}$ as $(\xi^A_t)_{t\geq 0}$.
- If $\boldsymbol{\lambda}$ is a sequence of with value λ, then "with infection rate λ" should cause no confusion. The same holds for "with density p", "with speed v", and the product measure π_p.
- We write $\mathbb{P}$ to denote the joint probabililty measure for dynamic bond percolation and the infection process, which may be emphasized by writing their parameters as subscripts of $\mathbb{P}$.

We list three basic properties as follows, where the first two can be checked in a standard way, and the proof of Theorem 3.3.1 (c) will be given in Section 4.2:

Theorem 3.3.1. *Let $A, B \subseteq V$, and let μ and ν be probability measures on $\{0,1\}^E$.*

(a) If $G_0 \sim \pi_{\boldsymbol{p}}$, then $G_t \sim \pi_{\boldsymbol{p}}$ for $t \geq 0$.
(b) If $A \subseteq B$, then $\xi^{\mu,A}_t \preceq \xi^{\mu,B}_t$ for $t \geq 0$.
(c) If $\mu \preceq \nu$, then $G^\mu_t \preceq G^\nu_t$ and $\xi^{\mu,A}_t \preceq \xi^{\nu,A}_t$ for $t \geq 0$.

Finally, we introduce the following three monoticity properites:

Theorem 3.3.2. *Let $A \subseteq V$, and let μ be a probability measure on $\{0,1\}^E$. Let $(G'_t, \xi'_t)_{t\geq 0}$ be the contact process on dynamic bond percolation with infection rate $\boldsymbol{\lambda}'$, density $\boldsymbol{p}'$ and speed $\boldsymbol{v}'$ on G.*

(a) If $\boldsymbol{\lambda} \leq \boldsymbol{\lambda}'$, $\boldsymbol{p} = \boldsymbol{p}'$ and $\boldsymbol{v} = \boldsymbol{v}'$, then $\xi^{\mu,A}_t \preceq \xi'^{\mu,A}_t$.
(b) If $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, $\boldsymbol{p} \leq \boldsymbol{p}'$ and $\boldsymbol{v} = \boldsymbol{v}'$, then $\xi^{\mu,A}_t \preceq \xi'^{\mu,A}_t$ and $\xi^A_t \preceq \xi'^A_t$.
(c) $(\xi^{\pi_{\boldsymbol{p}},A}_t)_{t\geq 0}$ is stochastically dominated from above by the ordinary contact process on G with infection rate $\boldsymbol{\lambda}$, and from below by the ordinary contact process on G with infection rate $\tilde{\boldsymbol{\lambda}}$ given by

$$\tilde{\lambda}_e := \frac{1}{2}\left(\lambda_e + v_e - \sqrt{(\lambda_e + v_e)^2 - 4\lambda_e v_e p_e}\right), \quad e \in E. \tag{3.3.3}$$

Remark 3.3.3. Theorem 3.3.2 follows from a coupling of the point processes of valid infections in the graphical representation. We will give the proof of Theorems 3.3.2 (a) and 3.3.2 (b) to make the thesis self-contained, while Theorem 3.3.2 (c) follows from Theorem 3.1.3, and a simpler version appears as Proposition 2.2 of [43], which can be generalised without revising the proof.

3.4 Dynamic Bond Percolation (Homogeneous)

In Section 3.3 we have encountered the graphical representation and some basic properties of the contact process on dynamic bond percolation. In this section we introduce the research in [43] under a homogeneity condition, i.e., the graph is vertex-transitive and the parameters are constant.

Let $G = (V, E)$ be a vertex-transitive graph with infinitely many vertices and finite constant degree, and let $(G_t, \xi_t)_{t\geq 0}$ be the contact process on dynamic bond percolation on G with infection rate $\lambda > 0$, density $p \in (0,1)$ and speed $v > 0$. Recall that $(\xi_t^A)_{t\geq 0}$ starts with $G_0 \sim \pi_p$ and $\xi_0 = A$. The authors of [43] define the critical value and immunity region as follows:

Definition 3.4.1. The *critical value* $\lambda_c(G, v, p)$ of $(G_t, \xi_t)_{t\geq 0}$ is defined by

$$\lambda_c(G, v, p) := \inf\left\{\lambda > 0 \,\middle|\, \mathbb{P}_{\lambda,v,p}\left(\xi_t^x \neq [0] \text{ for } t > 0\right)\right\}, \quad x \in V.$$

The *immunity region* $\mathfrak{I}(G)$ is defined by

$$\mathfrak{I}(G) := \left\{(v, p) \,\middle|\, v \in (0,\infty), p \in (0,1), \lambda_c(G, v, p) = \infty\right\}.$$

The choice of x does not affect the critical value. Section 2.1 of [43] points out that $\lambda_c(G, v, p)$ is the same for any initial condition that contains a positive but finite number of infected vertices. The word *immunity* is appropriate since $(\xi_t)_{t\geq 0}$ dies out for any infection rate $\lambda > 0$ if (v, p) is in the immunity region.

Besides the monotonicity properties in Theorem 3.3.2, the contact process on dynamic bond percolation in the homogeneous setting satisfies the following extra monotonicity property:

Theorem 3.4.2. *Let $(G'_t, \xi'_t)_{t\geq 0}$ be the contact process on dynamic bond percolation on G with infection rate $\lambda' > 0$, density $p' \in (0,1)$ and speed $v' > 0$. If $p = p'$, $v \leq v'$ and $\lambda v' \leq \lambda' v$, then $\xi_{t/v}^A \preceq \xi'^A_{t/v'}$ for $t \geq 0$ and $A \subseteq V$.*

The idea behind the property is explained in Section 3.2 of [43], although the property itself is not explicitly stated. This result is natural since the ratios of the recovery rate, the infection rate and the edge switching rate matters, not their actual values. We will not prove this theorem, since we will provide a generalised version Theorem 4.1.4 in Section 4.1.

With these monoticity properties, results on the critical value can be derived. Here, Theorem 3.4.3 (a) follows from Theorem 3.3.2 (a), Theorem 3.4.3 (b) appears as Proposition 2.1 of [43] and follows from Theorem 4.1.4, and Theorem 3.4.3 (c) appears as Theorem 2.9 of [43] and follows from Theorem 3.1.3.

Theorem 3.4.3. *(a) The function $p \mapsto \lambda_c(G, v, p)$ is nonincreasing for $v > 0$.*

(b) *The function* $v \mapsto v^{-1}\lambda_c(G,v,p)$ *is nonincreasing for* $p \in (0,1)$.
(c) $\lambda_1(G) \le \lambda_c(G,v,p) \le \hat{\lambda}(G,v,p)$, *where* $\lambda_1(G)$ *is the weak survival critical value of the contact process on* G, *and*

$$\hat{\lambda}(G,v,p) = \begin{cases} \lambda_1(G)(v-\lambda_1(G))/(vp-\lambda_1(G)) & \text{if } vp > \lambda_1(G), \\ \infty & \text{otherwise.} \end{cases}$$

However, Theorem 3.4.3 is not enough to describe the shape of the immunity region, which is useful for practical application. Theorem 2.3, Theorem 2.6, Corollary 2.8 and Theorem 2.9 of [43] give the expected shape of the immunity region shown in Figure 1 of [43].

Theorem 3.4.4. *(a) For any* $p \in (0,1]$, $p\lambda_c(G,v,p)$ *converges to the weak survival critical value of the contact process on* G *as* $v \to \infty$.
(b) *For all* $v > 0$ *there is a* $p_0(G,v) \in (0,1)$ *such that* $\lambda_c(G,v,p) = \infty$ *for all* $p < p_0(G,v)$.
(c) *There exists a* $p_1(G) \in (0,1)$ *such that* $\lambda_c(G,v,p) < \infty$ *for every* $p > p_1(G)$ *and* $v > 0$, *while for every* $p < p_1(G)$ *there exists a* $v > 0$ *with* $\lambda_c(G,v,p) = \infty$.
(d) *Fix* $v \in (0,1)$. *For any* $p < p_0(G,v)$ *there are constants* $\beta_0, \beta_1 > 0$ *(which may depend on* p *and* v *but not on* λ*), such that, for any* $A \subseteq V$,

$$\mathbb{E}\left[\inf\left\{ t \ge 0 \,\middle|\, \xi_t^A \ne [0] \right\}\right] \le \beta_0 \log|A| + \beta_1.$$

Finally, we introduce the results for $G = \mathbb{Z}^d$ or $\mathbb{Z}$. Theorem 2.4 of [43] gives a slightly different version for the case $G = \mathbb{Z}$:

Theorem 3.4.5. *(a) For all* $p \in [0,1)$, $\lim_{v\to 0} \lambda_c(\mathbb{Z},v,p) = \infty$.
(b) *Suppose that* $G = \mathbb{Z}$ *and fix* $p \in (0,1)$. *For* v *small enough there are constants* $\beta_0, \beta_1 > 0$ *(which may depend on* p, v *and* λ*) such that, for any* $A \subseteq \mathbb{Z}$,

$$\mathbb{E}\left[\inf\left\{ t \ge 0 \,\middle|\, \xi_t^A \ne [0] \right\}\right] \le \beta_0 \log|A| + \beta_1.$$

Remark 3.4.6. We may guess that larger values of v promote the infection, since they make it easier for infected vertices to infect previously unreachable vertices. However, according to Remark 2.5 of [43], the effect of v is necessarily subtler. By Theorem 1.2.8 the extinction time of the supercritical contact process on $\{1,\dots,n\}$ is exponential in n. Note that, with a large probability, at least one vertex in A is contained in a connected interval in G_0 of logarithmic length in A. Hence, the extinction time $\inf\left\{ t \ge 0 \,\middle|\, \xi_t^A \ne [0] \right\}$ is polynomial in $|A|$ in the static case. When compared to the logarithm extinction time in Theorem 3.4.5 (b), the authors of [43] find that the infection process in the static environment is more resilient than the same infection process on a slightly dynamic environment.

Theorem 1.1 of [48] extends Theorem 3.4.5 (a) to $\mathbb{Z}^d$:

Theorem 3.4.7. *Let $d \geq 2$, and let $p_c(\mathbb{Z}^d)$ be the critical density for Bernoulli bond percolation on $\mathbb{Z}^d$. Then*

$$\lim_{v \downarrow 0} \lambda_c(\mathbb{Z}^d, v, p) = \infty \text{ if } p < p_c(\mathbb{Z}^d),$$
$$\sup_{v \geq 0} \lambda_c(\mathbb{Z}^d, v, p) < \infty \text{ if } p > p_c(\mathbb{Z}^d).$$

Recall $p_1(G)$ defined in Theorem 3.4.4 (c). This theorem shows that $p_1(\mathbb{Z}^d) \leq p_c(\mathbb{Z}^d)$. We know that $p_1(\mathbb{Z}) < 1 = p_c(\mathbb{Z})$, but it remains an open question whether strict inequality holds for $d \geq 2$.

3.5 Dynamic Bond Percolation (Long Range)

In Section 3.4 we have introduced the contact process on dynamic bond percolation in a homogeneous setting. In this section, we turn to the latest research [50] for the contact process on dynamic bond percolation in a more general setting.

We begin with the setting. Let $G = (V, E)$ be a complete graph, and let $\boldsymbol{p} = (p_e)_{e \in E}$ and $\boldsymbol{v} = (v_e)_{e \in E}$ be sequences of numbers in $[0,1]$ and $(0, \infty)$, respectively. Let $\gamma > 0$ and $q \in (0,1]$. Let $(G_t, \xi_t)_{t \geq 0}$ be the contact process with dynamic edges on G with infection rate $\lambda > 0$, density $\tilde{\boldsymbol{p}} = (\tilde{p}_e)_{e \in E}$ and speed $\tilde{\boldsymbol{v}} = (\tilde{v}_e)_{e \in E}$, where

$$\tilde{p}_e := q p_e, \quad \tilde{v}_e := \gamma v_e, \quad e \in E.$$

Assume that $G_0 \sim \pi_{\tilde{\boldsymbol{p}}}$. To ensure that the transition rates of $(\xi_t)_{t \geq 0}$ are finite, we assume that

$$\sum_{y \in V \setminus \{x\}} v_{\{x,y\}} p_{\{x,y\}} < \infty, \qquad \sum_{y \in V \setminus \{x\}} v_{\{x,y\}}^{-1} < \infty, \qquad x \in V.$$

Remark 3.5.1. We repeat the reason to assume the two inequalities in Section 1 of [50], since this also explains the words “long range” in the title of this section. Let $(V, \tilde{E})$ be a vertex-transitive subgraph of (V, E) with finite constant degree, and equip (V, E) with the distance between edges defined in $(V, \tilde{E})$. The two inequalities imply that $v_{\{x,y\}} p_{\{x,y\}} \to 0$ and $v_{\{x,y\}} \to \infty$ as $|x - y| \to \infty$. Namely, the proability that an edge connecting two vertices with a long distance is open is very small and, thus, infection over long distance becomes more unlikely as the distance increases. Moreover, the second inequality assumes that all egdes attached to a vertex are updated in a finite time.

It is non-trivial to see whether or not ξ_t stays finite starting with ξ_0 finite, and whether or not a vertex will connect to infinitely many neighbours via open edges at some time. Thanks to Propositions 5.6 and 5.7 of [50], we need not worry too much:

Theorem 3.5.2. *(a) If ξ_0 is finite, then $\mathbb{E}[\xi_t] < \infty$ for $t \geq 0$.*
(b) If $A \subseteq V$ is finite, then $(G_t, \xi_t^A)_{t\geq 0}$ is a well-defined Feller process on $\{0,1\}^{V\cup E}$.

Next, we turn to the weak survival critical value $\lambda_c(\gamma, q)$ defined by

$$\lambda_c(\gamma,q) := \inf\left\{\lambda \geq 0 \,\middle|\, \mathbb{P}_{\lambda,\gamma,q}\left(\xi_t^A \neq \emptyset \text{ for } t \geq 0\right) > 0\right\}, \quad A \subseteq V, 0 < |A| < \infty.$$

Note that $\lambda_c(\gamma, q)$ does not depend on A. Theorem 2.1 of [50] shows that $\lambda_c(\gamma, q)$ coincides with the critical value defined by weak convergence:

Theorem 3.5.3. *$\lambda_c(\gamma, q)$ equals the infimum of $\lambda > 0$ such that $(G_t, \xi_t^{[1]})$ does not converge weakly to $\pi_{\tilde{p}} \otimes \delta_{[0]}$ as $t \to \infty$.*

Theorem 2.2 and Corollary 2.3 of [50] utilize coupling in Theorem 3.3.2 (c) to obtain the following result for fast speed:

Theorem 3.5.4. *Let $\bar{\lambda}_c(\gamma, q)$ and $\lambda_c^\infty(q)$ be the weak survival critical values of the ordinary contact process on G, starting with one infected vertex, with infection rates $(\bar{a}_e(\lambda,\gamma,q))_{e\in E}$ and $(\lambda q p_e)_{e\in E}$, respectively, where*

$$\bar{a}_e(\lambda,\gamma,q) := \frac{1}{2}\left(\lambda + \gamma v_e - \sqrt{(\lambda+\gamma v_e)^2 - 4 v_e p_e \lambda\gamma q}\right), \quad e \in E.$$

Then $\lambda_c(\gamma, q) \leq \bar{\lambda}_c(\gamma, q)$ and

$$\limsup_{\gamma\to\infty} \lambda_c(\gamma,q) \leq \lim_{\gamma\to\infty} \bar{\lambda}_c(\gamma,q) = \lambda_c^\infty(q) < \infty.$$

Similar to Theorem 3.4.4, [50] also provides results on the immunity region (see Definition 3.4.1) in Theorem 2.4, Corollary 2.5 and Theorem 2.6:

Theorem 3.5.5. *(a) For $\gamma > 0$, there exists a $q_0(\gamma) \in (0,1]$ such that $\lambda_c(\gamma, q) = \infty$ for $q < q_0(\gamma)$. Moreover, the function $\gamma \mapsto q_0(\gamma)$ is non-increasing on $(0,\infty)$.*
(b) For $q \in (0,1]$, there exists a $\gamma_0(q) \in [0,\infty)$ such that $\lambda_c(\gamma, q) = \infty$ for $\gamma < \gamma_0(q)$ and $\lambda_c(\gamma, q) < \infty$ for $\gamma > \gamma_0(q)$.
(c) There exists a $q_1 \in (0,1]$ such that $\lim_{\gamma\to 0} \lambda_c(\gamma, q) = \infty$ for $q < q_1$. Moreover, $q_1 = 1$ if G is the complete graph on $\mathbb{Z}$, and

$$\sum_{y\in\mathbb{N}} y v_{\{0,y\}} p_{\{0,y\}} < \infty, \quad \sum_{y\in\mathbb{N}} y v_{\{0,y\}}^{-1} < \infty.$$

3.6 Other Variants

In this section we present contact processes on long range percolation on $\mathbb{Z}$ in [29] and on lattices with dynamic range in [47], which are two variants of dynamic bond percolation. The former is a static model, while in the latter, each vertex updates the range within which it can transfer infection, independently.

We begin with long range percolation G_s on $\mathbb{Z}$ with exponent $s > 1$. Let $E := \{\{i, j\} \mid i, j \in \mathbb{Z}, i \neq j\}$, and let G_s be the random element on $\{0,1\}^E$ such that, independently,

$$\mathbb{P}\left(G_s(i,j) = 1\right) = \left|i - j\right|^{-s}.$$

We identify G_s with the edge set $\{(i,j) \in E \mid G_s(i,j) = 1\}$. Note that G_s is locally finite if and only if $s > 1$. Let $\lambda_1(G_s)$ denote the weak survival critical value of the contact process on $(\mathbb{Z}, G_s)$. By the ergodicity of G_s, there is an $\lambda_1(s) \geq 0$ such that $\lambda_1(G_s) = \lambda_1(s)$ almost surely. Theorem 1.1 of [29] is the following:

Theorem 3.6.1. $\inf\{s > 1 \mid \lambda_1(s) > 0\} \leq 102$.

Next, we construct the graphical representation of the contact process with dynamic range on $\mathbb{Z}^d$ and list the main results of [47]. Let $|x - y|$ denote the distance between vertices x and y in $\mathbb{Z}^d$. Let $E := \{(x, y) \mid x, y \in \mathbb{Z}^d, x \neq y\}$, and let N be a random variable on $\mathbb{N}_0$. Let

$$\mathrm{GR} := \left\{\mathscr{R}^x, \mathscr{I}^e, \mathscr{T}^e, N_{x,n} \,\middle|\, x \in \mathbb{Z}^d, e \in E, n \in \mathbb{N}\right\} \tag{3.6.1}$$

be a set of independent random elements, where $\mathscr{R}^x$, $\mathscr{I}^e$ and $\mathscr{T}^e$ are Poisson point processes on $[0,\infty)$ with intensity 1, λ and 1, respectively, and $N_{x,n}$ is a random variable with the same distribution as N. Define the nth time $T_{x,n}$ when vertex $x \in \mathbb{Z}^d$ updates its range as

$$T_{x,0} := 0, \quad T_{x,n} := \min\left\{t \in \mathscr{T}^x \,\middle|\, t > T_{x,n-1}\right\}, \quad n \geq 1.$$

The *range* $r_x(t)$ of $x \in \mathbb{Z}^d$ at time t is given by

$$r_x(t) := N_{x,n}, \quad t \in [T_{x,n-1}, T_{x,n}).$$

Note that, for a pair (x, y) of distinct vertices, the event that x is in the range of y and the event that y is in the range of x are independent. Hence, unlike for the graphical representation in Sections 1.1 and 3.3, we place a recovery mark $\circ$ at $(x, t) \in \mathbb{Z}^d \times [0,\infty)$ for $x \in \mathbb{Z}^d$ and $t \in \mathscr{R}^x$, and we place an infection arrow $\rightarrow$ from (x, t) and (y, t) for $(x, y) \in E$ and $t \in \mathscr{I}^{(x,y)}$. Define the point process $\mathrm{VI}^{(x,y)}$ of *valid infections* on $(x, y) \in E$ by

$$\mathrm{VI}^{(x,y)} := \left\{t \in \mathscr{I}^{(x,y)} \,\middle|\, |x - y| \leq r_x(t)\right\}, \quad (x, y) \in E.$$

An *active path* in $\mathbb{Z}^d \times [0,\infty)$ is a connected oriented path that moves along the time lines in the increasing direction and along the valid infection arrows, but without passing any recovery marks. Define the *contact process with dynamic* *range* $(\xi_t)_{t\geq 0}$ on $\mathbb{Z}^d$ as follows. For $t > 0$ and $x \in \mathbb{Z}^d$, set $\xi_t(x) := 1$ if and only if there exists an active path from $(y, 0)$ to (x, t) for some $y \in \xi_0$. We omit the formal definition, since it is similar to the one in Section 3.3.

Theorems 1.1 and 1.2 in [47] are about the phase transition:

Theorem 3.6.2. *Let* $\vartheta(\lambda) := \mathbb{P}\left(\xi_t^0 \neq \emptyset \text{ for } t \geq 0\right)$.
(a) If $\mathbb{E}\left[N^d\right] < \infty$*, then there exists a* $\lambda > 0$ *such that* $\vartheta(\lambda) = 0$.
(b) If $\limsup_{n\to\infty} n\mathbb{P}\left(N^d \geq n\right) > 0$*, then* $\vartheta(\lambda) > 0$ *for all* $\lambda > 0$.

Chapter 4

Contact Processes in Random Environments II

This chapter investigates the contact process in Broman's randomly evolving environment with $\delta_0 = 0$ (see Section 3.1) and on dynamic bond percolation (see Section 3.3). We aim to obtain non-trivial conditions for stochastic ordering of infection processes with different parameters. In Section 4.1 we introduce our idea and present some results on the contact process on dynamic bond percolation, whose proofs are covered in Sections 4.2 to 4.5. In Section 4.2 we prove several monotonicity properties by coupling the Poisson point processes in the graphical representation. Section 4.3 delves into coupling of two renewal processes with different lifetime distributions via hazard rates, which serves as a crucial ingredient. In Section 4.4 we calculate and analyse the hazard rates of the lifetime distribution and delay distribution of the renewal process in which the point process of valid infections is embedded in. Combining the results of Sections 4.3 and 4.4, we derive a non-trivial monotonicity property in Section 4.5. In Section 4.6 we apply the same method to the contact process in Broman's randomly evolving environment with $\delta_0 = 0$. Finally, in Section 4.7 we discuss interesting open questions.

4.1 Introduction

In this section we introduce our ideas and present the results for the contact process on dynamic bond percolation. The vital ingredient is Theorem 4.1.5, and the main results are Theorems 4.1.12 to 4.1.14.

We begin with preliminary definitions and notation. For a probability distribution L, we denote by $L(x)$, $f_L(x)$ and $\mathscr{L}_L(z)$ its cumulative distribution function, probability density function and Laplace transform, respectively.

Definition 4.1.1. Let L be a probability distribution with probability density function f_L. The *hazard rate* r_L of L is defined by

$$r_L(x) := \frac{f_L(x)}{1 - L(x)}, \quad x \in \mathbb{R}, L(x) \neq 1.$$

Definition 4.1.2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative independent random variables on $[0,\infty)$, where the distribution of X_n is L for $n \geq 2$, and the distribution of X_1 is D. Define $Y_t := \sup_{n\in\mathbb{N}_0}\{\sum_{i=1}^n X_i \leq t\}$ for $t \geq 0$.

(a) We call $(Y_t)_{t\geq 0}$ the *D-delayed L-renewal process* if $D \neq L$. Otherwise, we call $(Y_t)_{t\geq 0}$ the *zero-delayed L-renewal process.*

(b) We call $(Y_t)_{t\geq 0}$ the *stationary L-renewal process* if the mean μ of L is finite and $D(x) = \mu^{-1}\int_0^x (1 - L(y))\,\mathrm{d}y$.

We call L the *lifetime distribution*, D the *delay distribution*, $S_n := \sum_{i=0}^n X_i$ the *nth (renewal) epoch*, and the set $\{S_n \mid n \in \mathbb{N}\}$ the *epoch set* of $(Y_t)_{t\geq 0}$. Moreover, we denote by $\mathrm{RP}(D,L)$ and $\mathrm{RP}(L)$ the epoch set of the D-delayed L-renewal process and the epoch set of the zero-delayed L-renewal process, respectively.

For the literal meaning of stationarity and its equivalence with the definition in Definition 4.1.2 (b), we refer the reader to Section 4.2 of [22].

Let $G = (V,E)$ be a graph. Let $\boldsymbol{\lambda} = (\lambda_e)_{e\in E}$, $\boldsymbol{\lambda}' = (\lambda'_e)_{e\in E}$, $\boldsymbol{p} = (p_e)_{e\in E}$, $\boldsymbol{p}' = (p'_e)_{e\in E}$, $\boldsymbol{v} = (v_e)_{e\in E}$ and $\boldsymbol{v}' = (v'_e)_{e\in E}$ be sequences of numbers in $(0,\infty)$, $(0,\infty)$, $[0,1]$, $[0,1]$, $(0,\infty)$ and $(0,\infty)$, respectively. Let $(G_t,\xi_t)_{t\geq 0}$ be the contact process on dynamic bond percolation with infection rate $\boldsymbol{\lambda}$, density $\boldsymbol{p}$ and speed $\boldsymbol{v}$ on G. We stick to the notation introduced in Section 3.3, including the graphical represention given by (3.3.1), the point process of valid infections given by (3.3.2), the product measure $\pi_{\boldsymbol{p}}$ on $\{0,1\}^E$, etc. Let μ be a probability distribution on $\{0,1\}^E$, and let π_1 denote $\pi_{\boldsymbol{p}}$ with $p_e = 1$ for $e \in E$.

A symbol with $'$ means that it is defined in the same way as the symbol without $'$ but for the contact process $(G'_t,\xi'_t)_{t\geq 0}$ on dynamic bond percolation on G with infection rate $\boldsymbol{\lambda}'$, density $\boldsymbol{p}'$ and speed $\boldsymbol{v}'$ rather than $\boldsymbol{\lambda}$, $\boldsymbol{p}$ and $\boldsymbol{v}$. For example,

$$\mathrm{GR}' := \left\{\mathscr{C}'^e, \mathscr{O}'^e, \mathscr{I}'^e, \mathscr{R}'^x \,\middle|\, e \in E, x \in V\right\}$$

is the set of independent Poisson point processes for the graphical representation of $(G'_t,\xi'_t)_{t\geq 0}$, VI'^e is the point process of valid infections via edge e of $(G'_t,\xi'_t)_{t\geq 0}$, etc.

Next, we introduce our ideas and results in this chapter. We have already seen Theorems 3.3.2 (a) and 3.3.2 (b), and we aim to find additional sufficient conditions for the following coupling:

$$\xi_t^{\pi_1,A} \preceq \xi_t'^{\pi_1,A}, \quad t \geq 0, A \subseteq V, \tag{4.1.1}$$

$$\xi_t^{\mu,A} \preceq \xi_t'^{\mu,A}, \quad t \geq 0, A \subseteq V, \tag{4.1.2}$$

$$\xi_t^{\pi_{\boldsymbol{p}},A} \preceq \xi_t'^{\pi_{\boldsymbol{p}'},A}, \quad t \geq 0, A \subseteq V. \tag{4.1.3}$$

Conditions involving $\boldsymbol{v}$ are especially interesting, since Remark 3.4.6 reveals the subtlety of the effect of the speed $\boldsymbol{v}$. We first exploit coupling of the Poisson point processes in the graphical representation to obtain the following two theorems, which show that the effect of $\boldsymbol{v}$ on the contact process can be dominated in some sense by the effect of $\boldsymbol{p}$ or the effect of $\boldsymbol{\lambda}$. Here, Theorem 4.1.3 is a new result and Theorem 4.1.4 is a generalization of Theorem 3.4.2.

Theorem 4.1.3. *If $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ and $\frac{1-p_e'}{1-p_e} \leq \frac{v_e}{v_e'} \leq \frac{p_e'}{p_e}$ for $e \in E$, then (4.1.2) and (4.1.3) hold.*

Theorem 4.1.4. *Let $\alpha \in (0,1]$. Assume that $\lambda_e \leq \alpha\lambda_e'$, $p_e = p_e'$ and $v_e = \alpha v_e'$ for $e \in E$. Then $\xi_t^{\mu,A} \preceq \xi_{\alpha t}'^{\mu,A}$ for $t \geq 0$ and $A \subseteq V$.*

Next, we consider the coupling of ξ_t and ξ_t' at a deeper level. Instead of coupling the Poisson point processes in the graphical representation, we couple VI^e and VI'^e directly, just like we did for ordinary contact processes. Define

$$\mathrm{VI}_t^e := \left|\left\{ s \leq t \,\middle|\, s \in \mathrm{VI}^e \right\}\right|, \quad t \geq 0, e \in E.$$

We will prove in Theorem 4.1.8 that $(\mathrm{VI}_t^e)_{t\geq 0}$ is a renewal process. Then the question is: When can two renewal processes be coupled such that the epoch set of one renewal process is a subset of the epoch set of the other? We made a survey of the literature on coupling of renewal processes. Most papers focus on coupling of two renewal processes with the same lifetime distribution and different delays. However, the idea of separating the hazard rate of the lifetime distribution in Brown's paper [4] inspires us to construct a coupling to prove the following theorem and corollaries.

Theorem 4.1.5. *Let L, L', D and D' be probability distributions on $[0,\infty)$ with $L(x)$, $L'(x)$, $D(x)$, $D'(x) < 1$ for $x \in \mathbb{R}$. If*

$$r_D(x) \leq r_{D'}(x), \quad r_D(u+x) \leq r_{L'}(x), \quad r_L(u+x) \leq r_{L'}(x), \quad u,x \geq 0, \tag{4.1.4}$$

then $\mathrm{RP}(D,L) \preceq \mathrm{RP}(D',L')$. Moreover, $\mathrm{Poi}_c \preceq \mathrm{RP}(D,L) \preceq \mathrm{Poi}_C$, where

$$c := \inf\{r_D(x), r_L(x) \mid x \geq 0\}, \quad C := \sup\{r_D(x), r_L(x) \mid x \geq 0\}.$$

The condition for two stationary renewal process is simpler.

Corollary 4.1.6. *Let L and L' be probability distributions on $[0,\infty)$ with finite mean μ and μ', respectively. Assume that $L(x)$, $L'(x) < 1$ for $x \in \mathbb{R}$. Let D and D' be probability distributions on $[0,\infty)$ given by $D(x) = \mu^{-1}\int_0^x (1-L(y))\,\mathrm{d}y$ and $D'(x) = \mu'^{-1}\int_0^x (1-L'(y))\,\mathrm{d}y$ for $x \geq 0$.*

(a) *If* $r_D(u+x) \le r_{L'}(x)$ *and* $r_L(u+x) \le r_{L'}(x)$ *for* $u, x \ge 0$, *then* $\mathrm{RP}(D, L) \preceq \mathrm{RP}(D', L')$.
(b) *If* r_L *is non-increasing and* $r_L \le r_{L'}$, *then* $\mathrm{RP}(D, L) \preceq \mathrm{RP}(D', L')$.

Since $\mathrm{RP}(L)$ and $\mathrm{RP}(L, L)$ are the same, the following corollary for zero-delayed renewal processes is immediate.

Corollary 4.1.7. *Let* L *and* L' *be two probability distributions on* $[0, \infty)$ *with* $L(x), L'(x) < 1$ *for* $x \in \mathbb{R}$.
(a) *If* $r_L(u+x) \le r_{L'}(x)$ *for* $u, x \ge 0$, *then* $\mathrm{RP}(L) \preceq \mathrm{RP}(L')$.
(b) *If* $0 < c \le r_L(x) \le C$ *for* $x \in [0, \infty)$, *then* $\mathrm{Poi}_c \preceq \mathrm{RP}(L) \preceq \mathrm{Poi}_C$.
(c) *If* r_L *or* $r_{L'}$ *is non-increasing and* $r_L(x) \le r_{L'}(x)$ *for* $x \ge 0$, *then* $\mathrm{RP}(L) \preceq \mathrm{RP}(L')$.

To apply Theorem 4.1.5, we need to compute the hazard rate of the lifetime distribution of $(\mathrm{VI}^e_t)_{t\ge 0}$. The first step is to calculate the Laplace transform of its lifetime distribution. For convenience, we view λ, p and v as functions of e, and we omit e when focusing on a single edge. Define two probability distributions L and D by their Laplace transforms

$$\mathscr{L}_L(z) = \frac{\lambda(z+pv)}{z^2 + z(\lambda+v) + \lambda p v}, \quad \mathscr{L}_D(z) = \frac{\lambda p v}{z^2 + z(\lambda+v) + \lambda p v}. \tag{4.1.5}$$

Theorem 4.1.8. (a) *If* $G^e_0 = 1$, *then* $(\mathrm{VI}^e_t)_{t\ge 0}$ *is the zero-delayed* L*-renewal process.*
(b) *If* $G^e_0 = 0$, *then* $(\mathrm{VI}^e_t)_{t\ge 0}$ *is the* D*-delayed* L*-renewal process.*
(c) *If* $G^e_0 \sim \mathrm{Ber}(p_e)$, *then* $(\mathrm{VI}^e_t)_{t\ge 0}$ *is the stationary* L*-renewal process.*

We proceed to calculate and analyse the hazard rates r_L and r_D of L and D, respectively. Define

$$z_1 := \frac{\lambda + v - \sqrt{(\lambda+v)^2 - 4\lambda p v}}{2}, \quad z_2 := \frac{\lambda + v + \sqrt{(\lambda+v)^2 - 4\lambda p v}}{2}, \tag{4.1.6}$$

$$a_1 := \lambda \frac{pv - z_1}{z_2 - z_1}, \quad a_2 := \lambda \frac{pv - z_2}{z_1 - z_2}, \quad b := \frac{z_1 z_2}{z_2 - z_1}. \tag{4.1.7}$$

Theorem 4.1.9. *Suppose that* $p \in (0,1)$. *The hazard rates* r_L *and* r_D *are given by*

$$r_L(x) = \frac{a_1 \exp(-z_1 x) + a_2 \exp(-z_2 x)}{\frac{a_1}{z_1}\exp(-z_1 x) + \frac{a_2}{z_2}\exp(-z_2 x)}, \quad x \ge 0, \tag{4.1.8}$$

$$r_D(x) = \frac{\exp(-z_1 x) - \exp(-z_2 x)}{\frac{1}{z_1}\exp(-z_1 x) - \frac{1}{z_2}\exp(-z_2 x)}, \quad x \ge 0. \tag{4.1.9}$$

Moreover, $r_L(0) = \lambda$, $r_D(0) = 0$, $\lim_{x\to\infty} r_L(x) = \lim_{x\to\infty} r_D(x) = z_1$, *and*

$$\frac{\mathrm{d}r_L}{\mathrm{d}x}(x) = -\frac{\lambda(1-p)v}{(1-L(x))^2}\exp(-(\lambda+v)x) < 0, \quad x \ge 0, \tag{4.1.10}$$

$$\frac{\mathrm{d}r_D}{\mathrm{d}x}(x) = \frac{\lambda p v}{(1-D(x))^2}\exp(-(\lambda+v)x) > 0, \quad x \ge 0. \tag{4.1.11}$$

The reader may have noticed that z_1 equals $\tilde{\lambda}_e$ defined in (3.3.3), which we will explain in Remark 4.4.2. With Corollary 4.1.6 (b), Corollary 4.1.7 (b) and Theorem 4.1.9, we get a generalised version of Theorem 3.3.2 (c):

Theorem 4.1.10. *If $\lambda_e \le z_1'(e)$ for $e \in E$, then (4.1.1) and (4.1.3) hold.*

By analysing z_1, we obtain the following corollary of Theorem 4.1.10:

Theorem 4.1.11. *(a) Fix $\boldsymbol{\lambda}$, $\boldsymbol{p}$, $\boldsymbol{v}$, $\boldsymbol{p}'$ and $\boldsymbol{v}'$. If $\lambda_e < p_e' v_e'$ for $e \in E$, then there exists a $\boldsymbol{\lambda}_0' = (\lambda_{e,0}')_{e\in E}$ such that (4.1.1) and (4.1.3) hold for any $\boldsymbol{\lambda}' \ge \boldsymbol{\lambda}_0'$.*

(b) Fix $\boldsymbol{\lambda}$, $\boldsymbol{p}$, $\boldsymbol{v}$, $\boldsymbol{\lambda}'$ and $\boldsymbol{v}'$. If $\lambda_e < \min(\lambda_e', v_e')$ for $e \in E$, then there exists a $\boldsymbol{p}_0' = (p_{e,0}')_{e\in E}$ such that $p_{0,e}' < 1$ for $e \in E$ and (4.1.1) and (4.1.3) hold for any $\boldsymbol{p}' \ge \boldsymbol{p}_0'$.

(c) Fix $\boldsymbol{\lambda}$, $\boldsymbol{p}$, $\boldsymbol{v}$, $\boldsymbol{\lambda}'$ and $\boldsymbol{p}'$. If $\lambda_e < \lambda_e' p_e'$ for $e \in E$, then there exists a $\boldsymbol{v}_0' = (v_{e,0}')_{e\in E}$ such that (4.1.1) and (4.1.3) hold for any $\boldsymbol{v}' \ge \boldsymbol{v}_0'$.

The advantage of Theorem 4.1.5 over Theorem 3.1.3 lies in the possibility to couple two renewal processes without an intermediate Poisson point process. We prepare for this by proving the following theorem.

Theorem 4.1.12. *Consider the following two conditions:*

$$\lambda \le \lambda', \quad v + 2z_1' \ge \lambda + \lambda' + v', \quad \lambda(1-p)v \le \lambda'(1-p')v', \quad z_1 \le z_1'. \tag{4.1.12}$$

$$\begin{gathered} \lambda \le \lambda', \quad v + 2z_1' \ge \lambda + \lambda' + v', \quad \lambda(1-p)v \le \lambda'(1-p')v', \\ \lambda p v < \lambda' p' v', \quad \lambda + v \ge \lambda' + v'. \end{gathered} \tag{4.1.13}$$

If (4.1.12) holds, then $r_L \le r_{L'}$. If (4.1.13) holds, then $r_L \le r_{L'}$ and $r_D \le r_{D'}$.

Almost immediately we have the following two theorems:

Theorem 4.1.13. *Consider all parameters as functions of $e \in E$. If (4.1.12) holds for $e \in E$, then (4.1.1) and (4.1.3) hold. If (4.1.13) holds for $e \in E$, then both (4.1.2) and (4.1.3) hold.*

Theorem 4.1.14. *Fix $\boldsymbol{\lambda}$, $\boldsymbol{p}$, $\boldsymbol{v}$, $\boldsymbol{p}'$ and $\boldsymbol{v}'$. If $z_1(e) < v_e' p_e'$ for $e \in E$, then there exists a $\boldsymbol{\lambda}_0' = (\lambda_{e,0}')_{e\in E}$ such that (4.1.1) and (4.1.3) hold for any $\boldsymbol{\lambda}' \ge \boldsymbol{\lambda}_0'$.*

Since $z_1(e) < \lambda_e$, Theorem 4.1.14 is an extension of Theorem 4.1.11 (a). This is because we directly compare r_L and $r_{L'}$ instead of comparing them with the constant hazard rate of an intermediate Poisson point process. In fact, we will see in Remark 4.5.4 that the estimate for $\lambda_{e,0}'$ in Theorem 4.1.13 is fairly good.

- In Section 4.2, we prove Theorem 3.3.1 (c) as promised, and prove Theorems 3.3.2 (a), 3.3.2 (b), 4.1.3 and 4.1.4 by coupling the Poisson point processes in the graphical representation.
- In Section 4.3, we construct a coupling via hazard rates to prove Theorem 4.1.5 and Corollary 4.1.6.
- In Section 4.4, we prove Theorems 4.1.8 and 4.1.9.
- In Section 4.5, we prove Theorems 4.1.10 to 4.1.14.

4.2 Coupling of Graphical Representation

In this section we first prove Theorem 3.3.1 (c), Theorem 3.3.2 (a) and Theorem 3.3.2 (b) as promised in Section 3.3, and then prove Theorem 4.1.3 and Theorem 4.1.4. The proofs rely only on coupling, although the ideas behind the last two theorems are not as straightforward as those behind the first three.

We begin with the notation for coupling. Recall the graphical representations

$$\mathrm{GR} := \left\{ \mathscr{C}^e, \mathscr{O}^e, \mathscr{I}^e, \mathscr{R}^x \,\middle|\, e \in E, x \in V \right\}, \quad \mathrm{GR}' := \left\{ \mathscr{C}'^e, \mathscr{O}'^e, \mathscr{I}'^e, \mathscr{R}'^x \,\middle|\, e \in E, x \in V \right\}$$

of $(\xi_t)_{t\geq 0}$ and $(\xi'_t)_{t\geq 0}$. A symbol with $\hat{}$ means that it is a copy in the coupling. For example, $\hat{\mathscr{C}}^e$, $\hat{\mathscr{O}}^e$, $\hat{\mathscr{I}}^e$ and $\hat{\mathscr{R}}^x$ are the corresponding point processes in $\hat{\mathrm{GR}}$ to $\mathscr{C}^e$, $\mathscr{O}^e$, $\mathscr{I}^e$ and $\mathscr{R}^x$, respectively.

Next, we give the proofs of Theorem 3.3.1 (c), Theorem 3.3.2 (a) and Theorem 3.3.2 (b).

Proof of Theorem 3.3.1 (c). Assume that $\mu \preceq \nu$. Then there exists a coupling $((\hat{G}^e_0)_{e\in E}, (\hat{G}'^e_0)_{e\in E})$ of $(G^e_0)_{e\in E}$ and $(G'^e_0)_{e\in E}$ such that $\hat{G}^e_0 \leq \hat{G}'^e_0$ for $e \in E$. Moreover, set $\hat{\mathrm{GR}} := \mathrm{GR}$ and $\hat{\mathrm{GR}}' := \mathrm{GR}$. Fix $e \in E$. Let $T^e := \min(\mathscr{C}^e \cup \mathscr{O}^e)$. When $t \geq T^e$, $\hat{G}^e_t = \hat{G}'^e_t = 1$ if $\max\{s \leq t \mid \mathscr{C}^e \cup \mathscr{O}^e\}$ is in $\mathscr{O}^e$, and $\hat{G}^e_t = \hat{G}'^e_t = 0$ otherwise. When $t < T^e$, $\hat{G}^e_t \leq \hat{G}'^e_t$, since $\hat{G}^e_t = \hat{G}^e_0$ and $\hat{G}'^e_t = \hat{G}'^e_0$. Then $\hat{G}^e_t \leq \hat{G}'^e_t$ for $t \geq 0$, and hence $\hat{\mathrm{VI}}^e \preceq \hat{\mathrm{VI}}'^e$. Therefore, $\hat{\xi}^{\mu,A}_t \preceq \hat{\xi}^{\mu',A}_t$ for $t \geq 0$. □

Proof of Theorem 3.3.2 (a). Assume that $\boldsymbol{\lambda} \leq \boldsymbol{\lambda}'$, $\boldsymbol{p} = \boldsymbol{p}'$ and $\boldsymbol{\nu} = \boldsymbol{\nu}'$. Let

$$\hat{\mathrm{GR}}' := \mathrm{GR}', \quad \hat{\mathrm{GR}} := \left\{ \mathscr{C}'^e, \mathscr{O}'^e, \hat{\mathscr{I}}^e, \mathscr{R}^x \,\middle|\, e \in E, x \in V \right\},$$

where $\hat{\mathscr{I}}^e$ is obtained from $\mathscr{I}'^e$ by keeping each point independently with probability λ_e/λ'_e. Then $(\hat{\mathrm{GR}}, \hat{\mathrm{GR}}')$ is a coupling of GR and GR′. Note that the only difference between $\hat{\mathrm{GR}}$ and $\hat{\mathrm{GR}}'$ is that $\hat{\mathscr{I}}^e \preceq \hat{\mathscr{I}}'^e$ for $e \in E$. Therefore $\hat{\mathrm{VI}}^e \preceq \hat{\mathrm{VI}}'^e$ for $e \in E$. By the definition of the infection process, we have $\hat{\xi}^{\mu,A}_t \preceq \hat{\xi}'^{\mu,A}_t$ for $t \geq 0$, i.e., $\xi^{\mu,A}_t \preceq \xi'^{\mu,A}_t$ for $t \geq 0$. □

Proof of Theorem 3.3.2 (b). Assume that $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, $\boldsymbol{p} \leq \boldsymbol{p}'$ and $\boldsymbol{\nu} = \boldsymbol{\nu}'$. For $e \in E$, let $(x^e_n)_{n\in\mathbb{N}}$ be a Poisson point process on $[0,\infty)$ with intensity ν_e, and let $(M^e_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with

$$\mathbb{P}\left(M^e_1 = 0\right) = p_e, \quad \mathbb{P}\left(M^e_1 = 1\right) = p'_e - p_e, \quad \mathbb{P}\left(M^e_1 = 2\right) = 1 - p'_e.$$

Define

$$\hat{\mathscr{C}}^e := \left\{ x_n \,\middle|\, n \in \mathbb{N}, M^e_n = 1 \text{ or } 2 \right\}, \quad \hat{\mathscr{O}}^e := \left\{ x_n \,\middle|\, n \in \mathbb{N}, M^e_n = 0 \right\},$$

$$\hat{\mathscr{C}}'^e := \{x_n \mid n \in \mathbb{N}, M_n^e = 2\}, \quad \hat{\mathscr{O}}^e := \{x_n \mid n \in \mathbb{N}, M_n^e = 0 \text{ or } 1\},$$

and

$$\hat{\mathrm{GR}} := \{\hat{\mathscr{C}}^e, \hat{\mathscr{O}}^e, \mathscr{I}^e, \mathscr{R}^x \mid e \in E, x \in V\},$$

$$\hat{\mathrm{GR}}' := \{\hat{\mathscr{C}}'^e, \hat{\mathscr{O}}'^e, \mathscr{I}^e, \mathscr{R}^x \mid e \in E, x \in V\}.$$

Then $(\hat{\mathrm{GR}}, \hat{\mathrm{GR}}')$ is a coupling of GR and GR$'$, and the only difference between $\hat{\mathrm{GR}}$ and $\hat{\mathrm{GR}}'$ is that

$$\hat{\mathscr{C}}^e \supseteq \hat{\mathscr{C}}'^e, \quad \hat{\mathscr{O}}^e \subseteq \hat{\mathscr{O}}'^e. \tag{4.2.1}$$

Note that, for $n \in \mathbb{N}$ and $t \in [x_n, x_{n+1})$, $\hat{G}_t^e = 1$ if $x_n \in \hat{\mathscr{O}}^e$ and $\hat{G}_t^e = 0$ if $x_n \in \hat{\mathscr{C}}^e$. Hence, $\hat{\mathrm{VI}}^e \cap [x_1, \infty) \subseteq \hat{\mathrm{VI}}'^e \cap [x_1, \infty)$. Since $\mathbb{P}\left(G_0^e = 1\right) = \mathbb{P}\left(G_0'^e = 1\right)$, we can couple G_0^e and $G_0'^e$ such that $\hat{G}_0^e = \hat{G}_0'^e$. Then $\hat{\mathrm{VI}}^e \cap [0, x_1) = \hat{\mathrm{VI}}'^e \cap [0, x_1)$. Thus, $\hat{\xi}_t^{\mu,A} \preceq \hat{\xi}_t'^{\mu,A}$ for $t \geq 0$, i.e., $\xi_t^{\mu,A} \preceq \xi_t'^{\mu,A}$ for $t \geq 0$. Together with Theorem 3.3.1 (c), we obtain that $\xi_t^{\pi_{\boldsymbol{p}},A} \preceq \xi_t'^{\pi_{\boldsymbol{p}'},A}$ for $t \geq 0$. □

Moving on to Theorem 4.1.3, we first connect the effects of $\boldsymbol{p}$ and $\boldsymbol{v}$. The motivation behind the result is to know more about the effect of $\boldsymbol{v}$. Although [43] points out the subtlety of the role of v, and [43, 48] focus on the asymptotic behaviour as the speed $\boldsymbol{v}$ tends to 0 or ∞, they do not investigate the effect of varying v_e between 0 and ∞.

Proof of Theorem 4.1.3. The assumption implies that

$$p_e \leq p'_e, \quad p_e v_e \leq p'_e v'_e, \quad (1 - p_e) v_e \geq (1 - p'_e) v'_e.$$

Let $\hat{\mathscr{O}}^e$ be the Poisson point process obtained from $\mathscr{O}'^e$ by keeping each point independently with probability $p_e v_e / p'_e v'_e$, and let $\hat{\mathscr{C}}'^e$ be the Poisson point process obtained from $\mathscr{C}^e$ by keeping each point independently with probability $v'_e(1 - p'_e)/v_e(1 - p_e)$. Define

$$\hat{\mathrm{GR}} := \{\mathscr{C}^e, \hat{\mathscr{O}}^e, \mathscr{I}^e, \mathscr{R}^x \mid e \in E, x \in V\}, \quad \hat{\mathrm{GR}}' := \{\hat{\mathscr{C}}'^e, \mathscr{O}'^e, \mathscr{I}^e, \mathscr{R}^x \mid e \in E, x \in V\}.$$

Then $(\hat{\mathrm{GR}}, \hat{\mathrm{GR}}')$ is a coupling of GR and GR$'$ with (4.2.1). With arguments similar to those in the last part of the proof of Theorem 3.3.2 (b), we obtain (4.1.2) and (4.1.3). □

Finally, we prove Theorem 4.1.4 by scaling the time of all the Poisson point processes in the graphical representation of $(\xi'_t)_{t \geq 0}$.

Proof of Theorem 4.1.4. For a point process N on $[0, \infty)$, let $\alpha N := \{\alpha x \mid x \in N\}$. Compare

$$\mathrm{GR} = \{\mathscr{C}^e, \mathscr{O}^e, \mathscr{I}^e, \mathscr{R}^x \mid e \in E, x \in V\},$$

$$\alpha\mathrm{GR}' := \left\{\alpha\mathscr{C}'^e, \alpha\mathscr{O}'^e, \alpha\mathscr{I}'^e, \alpha\mathscr{R}'^x \,\middle|\, e \in E, x \in V\right\}.$$

By assumption, for $e \in E$ and $x \in V$,

$$\mathscr{C}^e \overset{\mathrm{d}}{=} \alpha\mathscr{C}'^e, \quad \mathscr{O}^e \overset{\mathrm{d}}{=} \alpha\mathscr{O}'^e, \quad \mathscr{I}^e \preceq \alpha\mathscr{I}'^e, \quad \mathscr{R}^x \succeq \alpha\mathscr{R}'^x.$$

Hence, GR and $\alpha\mathrm{GR}'$ can be coupled such that $(\xi_t)_{t\geq 0}$ has fewer valid infections and more recoveries than $(\xi'_{\alpha t})_{t\geq 0}$ in the graphical representation, which implies the result. □

4.3 Coupling of Renewal Processes via Hazard Rates

As mentioned in Section 4.1, VI^e is the epoch set of a renewal process, and we seek conditions for $\mathrm{VI}^e \preceq \mathrm{VI}'^e$. Hence we make a digression into renewal theory. With the help of hazard rates, we couple two renewal processes such that the epoch set of one renewal process is a subset of the epoch set of the other.

We begin with notation and a preliminary property of the hazard rate.

Definition 4.3.1. For a cumulative distribution function $F(x)$, the corresponding *survival function* is given by $\bar{F}(x) := 1 - F(x)$, $x \in \mathbb{R}$. Provided that $F(u) \neq 1$, the *survival function* $\bar{F}_u$ of F *at age* $u \geq 0$ is defined by

$$\bar{F}_u(x) := \mathbf{1}_{\{x<0\}} + \mathbf{1}_{\{x\geq 0\}}\bar{F}(u+x)/\bar{F}(u), \quad x \in \mathbb{R}.$$

Given a survival function $\bar{G}$, we denote by G the corresponding cumulative distribution function, which is given by $G(x) = 1 - \bar{G}(x)$.

It is known that for a distribution F on $[0,\infty)$ with $F \neq 1$,

$$\bar{F}(x) = \exp\left(-\int_0^x r_F(s)\,\mathrm{d}s\right), \quad \bar{F}_u(x) = \exp\left(-\int_u^{u+x} r_F(s)\,\mathrm{d}s\right), \quad x, u \geq 0. \quad (4.3.1)$$

Next, we turn to coupling of renewal processes with different lifetime distributions. Let L, L', D and D' be probability distributions on $[0,\infty)$ with $L(x)$, $L'(x)$, $D(x)$, $D'(x) < 1$ for $x \in \mathbb{R}$. Note that $'$ does not represent the derivative. Recall that we denote by $L(x)$, $f_L(x)$ and $r_L(x)$ the cumulative distribution function, probability density function and hazard rate of L, respectively. We need the *defective distribution functions* K_0 and H_u for our coupling.

Definition 4.3.2. A non-decreasing and right-continuous function $G\colon \mathbb{R} \to [0,1]$ is called a *defective distribution function* if

$$\lim_{x\to-\infty} G(x) > 0 \text{ or } \lim_{x\to\infty} G(x) < 1.$$

Lemma 4.3.3. *Assume that $r_D(x) \le r_{D'}(x)$ and $r_D(u+x) \le r_{L'}(x)$ for $u, x \ge 0$. Set*

$$\bar{K}_0(x) := \mathbf{1}_{\{x<0\}} + \mathbf{1}_{\{x\ge 0\}} \frac{\bar{D}'(x)}{\bar{D}(x)}, \quad \bar{H}_u(x) := \mathbf{1}_{\{x<0\}} + \mathbf{1}_{\{x\ge 0\}} \frac{\bar{L}'(x)}{\bar{D}_u(x)}, \quad x \in \mathbb{R}, u \ge 0.$$

Then K_0 and H_u are possibly defective distribution functions.

Proof. By the definition of $\bar{K}_0$ and $\bar{H}_u$, we have $\bar{K}_0(x) = \bar{H}_u(x) = 1$ for $x \le 0$. The right-continuity of $\bar{K}_0$ is ensured by $\bar{D}(x) > 0$ and the right-continuity of $\bar{D}'(x)$ and $\bar{D}(x)$. Similarly, $\bar{H}_u(x)$ is also right-continuous. Moreover, $\bar{K}_0(x)$ and $\bar{H}_u(x)$ are non-increasing in x, since by (4.3.1), for $x_2 \ge x_1 \ge 0$,

$$\begin{aligned} \frac{\bar{K}_0(x_2)}{\bar{K}_0(x_1)} &= \frac{\bar{D}'(x_2)\bar{D}(x_1)}{\bar{D}'(x_1)\bar{D}(x_2)} = \exp\left(\int_{x_1}^{x_2} r_D(s) - r_{D'}(s)\,\mathrm{d}s\right) \le 1, \\ \frac{\bar{H}_u(x_2)}{\bar{H}_u(x_1)} &= \frac{\bar{L}'(x_2)\bar{D}(u+x_1)}{\bar{L}'(x_1)\bar{D}(u+x_2)} = \exp\left(\int_{x_1}^{x_2} r_D(u+s) - r_{L'}(s)\,\mathrm{d}s\right) \le 1. \end{aligned}$$

Hence, $K_0(x)$ and $H_u(x)$ are possibly defective distribution functions. □

The following theorem illustrates the coupling we are after.

Theorem 4.3.4. *Assume that $r_D(x) \le r_{D'}(x)$ and $r_D(u+x) \le r_{L'}(x)$ for $u, x \ge 0$. Let K_0 and H_u be the possibly defective distributions given in Lemma 4.3.3. Construct $X_1', X_2', \dots$ and X as follows:*

- *In Step 1, let Z_1 and W_1 be two independent random variables such that*

$$Z_1 \sim D, \quad W_1 \sim K_0.$$

  Set $X_1' := \min(Z_1, W_1)$ and go to Step 2.
- *In Step m with $m \ge 2$, let Z_m and W_m be two random variables that are conditionally independent of each other and of $(Z_1, W_1), \dots, (Z_{m-1}, W_{m-1})$ given $\sum_{i=1}^{m-1} W_i$ such that*

$$\left(Z_m \,\middle|\, \sum_{i=1}^{m-1} W_i = u\right) \sim D_u, \quad \left(W_m \,\middle|\, \sum_{i=1}^{m-1} W_i = u\right) \sim H_u.$$

  Set $X_m' := \min(Z_m, W_m)$ and go to Step $m+1$.

Then $(X_n')_{n\in\mathbb{N}}$ is a sequence of independent random variables with

$$X_1' \sim D', \quad X_n' \sim L' \text{ for } n \ge 2, \quad X := \sum_{i=1}^{N} X_i' \sim D, \tag{4.3.2}$$

where $N := \inf\{n \in \mathbb{N} \mid Z_n \le W_n\}$. Moreover, $\mathbb{P}(N = \infty) = 0$.

Proof. First, we prove that $(X'_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables with $X'_1 \sim D'$ and $X'_n \sim L'$ for $n \geq 2$. Since X'_1 is the minimum of independent random variables Z_1 and W_1, we have

$$\mathbb{P}\left(X'_1 > x\right) = \mathbb{P}(Z_1 > x)\,\mathbb{P}(W_1 > x) = \bar{D}(x)\bar{K}_0(x) = \bar{D}'(x), \quad x \geq 0.$$

For $u \geq 0$, let Z^*_u and W^*_u be independent random variables with distributions D_u and H_u, respectively. Then, for $m \geq 2$, X'_m is independent of $X'_1, \ldots, X'_{m-1}$ and has distribution L' because of the following formulas:

$$\left(X'_m \,\middle|\, (W_i, Z_i) = (w_i, z_i),\ i = 1, \ldots, m-1\right) \sim \min\left(Z^*_{\sum_{i=1}^{m-1} w_i}, W^*_{\sum_{i=1}^{m-1} w_i}\right),$$

$$\mathbb{P}\left(\min(Z^*_u, W^*_u) > x\right) = \mathbb{P}\left(Z^*_u > x\right)\mathbb{P}\left(W^*_u > x\right) = \bar{D}_u(x)\bar{H}_u(x) = \bar{L}'(x), \quad x \geq 0.$$

Second, we prove that $X \sim D$. On the event $\{N < \infty\}$, $X'_N = Z_N$ and $X'_i = W_i$ for $1 \leq i \leq N-1$. Set $Z_\infty = 0$ and $X = \sum_{i=1}^{N-1} W_i + Z_N$. Note that $\mathbb{P}\left(\sum_{i=2}^{\infty} W_i = \infty\right) = 1$, since, for $x_0 > 0$ and $i \geq 2$,

$$\mathbb{P}(W_i > x_0) \geq \inf_{u\geq 0} \bar{H}_u(x_0) = \inf_{u\geq 0}(\bar{L}'(x_0)/\bar{D}_u(x_0)) \geq \bar{L}'(x_0) > 0.$$

Thus, for $x > 0$, there exists an $m \geq 2$ such that $\sum_{i=1}^{m-1} W_i < x \leq \sum_{i=1}^{m} W_i$, and

$$\begin{aligned}
&\mathbb{P}(X > x \mid W_1 = w_1, \ldots, W_{m-1} = w_{m-1}, W_m = w_m, \ldots)\\
&= \mathbb{P}(Z_1 > w_1) \prod_{i=2}^{m-1} \mathbb{P}\left(Z_i > w_i \,\middle|\, \sum_{j=1}^{i-1} W_j = \sum_{j=1}^{i-1} w_j\right) \mathbb{P}\left(\sum_{i=1}^{m-1} w_i + Z_m > x\right)\\
&= \bar{D}(w_1)\left(\prod_{i=2}^{m-1} \bar{D}_{\sum_{j=1}^{i-1} w_j}(w_i)\right) \bar{D}_{\sum_{i=1}^{m-1} w_i}\left(x - \sum_{i=1}^{m-1} w_i\right)\\
&= \bar{D}(w_1)\frac{\bar{D}(w_1 + w_2)}{\bar{D}(w_1)} \cdots \frac{\bar{D}\left(\sum_{i=1}^{m-1} w_i + x - \sum_{i=1}^{m-1} w_i\right)}{\bar{D}\left(\sum_{i=1}^{m-1} w_i\right)}\\
&= \bar{D}(x), \quad w_1, \ldots, w_{m-1}, w_m, \ldots \geq 0.
\end{aligned}$$

Therefore $X \sim D$. Finally, since $X = \sum_{i=1}^{N-1} W_i + Z_N$ and $\mathbb{P}\left(\sum_{i=1}^{\infty} W_i = \infty\right) = 1$, we have $\mathbb{P}(N = \infty) \leq \mathbb{P}(X = \infty) = 0$. □

The following corollary is the special case of Theorem 4.3.4 where $D = L$ and $D' = L'$:

Corollary 4.3.5. *Assume that $r_L(u + x) \leq r_{L'}(x)$ for $u, x \geq 0$. Then there exist a random variable N with $\mathbb{P}(N < \infty) = 1$, and a sequence of independent random variables $(X'_n)_{n\in\mathbb{N}}$, such that*

$$X'_n \sim L' \text{ for } n \in \mathbb{N}, \quad X := \sum_{i=1}^{N} X'_i \sim L. \tag{4.3.3}$$

Before we apply the coupling, we explain the construction in Theorem 4.3.4. As we have mentioned earlier, we want to couple the epoch sets of two renewal processes $(\mathrm{VI}^e_t)_{t\geq 0}$ and $(\mathrm{VI}'^e_t)_{t\geq 0}$. The only relevant result we were able to find appears in [4], where Brown constructs a random variable N and a sequence of independent random variables $(X'_n)_{n\in\mathbb{N}}$ satisfying (4.3.3) under either of the following conditions:

- L' has an increasing mean residual life (IMRL). Namely, $\mathbb{E}[X] < \infty$ and $\mathbb{E}[X - t \mid X > t]$ is non-decreasing in $t \geq 0$ for the random variable X with distribution L'. Moreover,
$$L(x) = \int_0^x \bar{L}'(y)\,\mathrm{d}y \Big/ \int_0^\infty \bar{L}'(y)\,\mathrm{d}y, \tag{4.3.4}$$
- $r_{L'}$ is non-increasing, and there exists a probability distribution H on $[0,\infty)$ such that
$$\bar{L}(x) = \int_0^\infty \frac{\bar{L}'(x+y)}{\bar{L}'(y)}\,\mathrm{d}H(y), \tag{4.3.5}$$

Note that (4.3.4) and the IMRL condition imply that $r_{L'}$ is non-increasing. Indeed, Brown's result constructs a coupling of epoch sets of the zero-delayed L'-renewal process and the L-delay L'-renewal process. The idea is to decompose the hazard rate of L' into two parts, where one part provides epochs for both processes and the other provides epochs for the renewal process with more epochs. Inspired by this idea, we conjecture that the coupling only depends on r_L and $r_{L'}$, and we manage to extend Brown's result to Theorem 4.3.4 and Corollary 4.3.5. Furthermore, we give an intuitive reason for the idea in Remark 4.3.6.

Remark 4.3.6. We explain the relation between the decomposition of the hazard rate into two parts and the assumption in Corollary 4.3.5. The aim is to construct a random variable N satisfying (4.3.3). Define

$$q(u,x) := \mathbb{P}\left(N = m \,\middle|\, \sum_{i=1}^{m-1} X'_i = u, N > m-1, X'_m = x\right), \qquad u, x \geq 0.$$

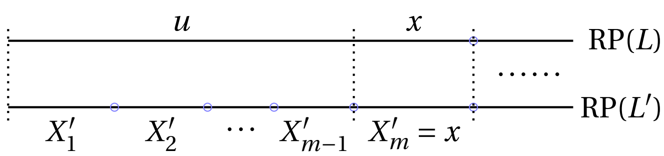

Figure 4.1: *Coupling of* RP(L) *and* RP(L')

See Figure 4.1. Given $N > m-1$, $N = m$ is equivalent to $Z_m \le W_m$. Together with $X'_i = W_i$ for $i < N$ and $X'_m = \min(Z_m, W_m)$, we get

$$q(u,x) = \mathbb{P}\left(Z_m \le W_m \,\middle|\, \sum_{i=1}^{m-1} W_i = u, N > m-1, \min(Z_m, W_m) = x\right).$$

We condition on $\sum_{i=1}^{m-1} W_i = u$ and $N > m-1$. By the proof of Theorem 4.3.4, $\min(Z_m, W_m) \sim L'$. Moreover, $Z_m \sim L_u$ and $W_m \sim H_u$. Thus, the probability density of the event $\min(Z_m, W_m) = x$ and $Z_m \le W_m$, i.e., $x = Z_m \le W_m$, is

$$f_{L_u}(x)\bar{H}_u(x),$$

where $f_{L_u}(x)$ is the density function of L_u. Then we get

$$\begin{aligned} q(u,x) = \frac{f_{L_u}(x)\bar{H}_u(x)}{f_{L'}(x)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - \frac{\bar{L}(u+x)}{\bar{L}(u)}\right)\frac{\bar{L}'(x)}{\bar{L}_u(x)}\frac{1}{f_{L'}(x)} \\ &= \frac{1}{\bar{L}(u)}\frac{\mathrm{d}}{\mathrm{d}x}L(u+x)\frac{\bar{L}(u)}{\bar{L}(u+x)}\frac{\bar{L}'(x)}{f_{L'}(x)} \\ &= \frac{f_L(u+x)}{\bar{L}(u+x)}\frac{\bar{L}'(x)}{f_{L'}(x)} = \frac{r_L(u+x)}{r_{L'}(x)}. \end{aligned}$$

Hence, we need the condition $r_L(u+x) \le r_{L'}(x)$ for $u, x \ge 0$ to ensure $q(u,x) \le 1$. Conditional on $u = \sum_{i=1}^{m-1} X'_i$, $X > u$ and $X'_m = x$, we set $X = \sum_{i=1}^{m} X'_i$ with probability $q(u,x) = r(u+x)/r'(x)$, which amounts to decomposing the hazard rate into two parts.

With the coupling in Theorem 4.3.4 and Corollary 4.3.5, we are ready to prove Theorem 4.1.5.

Proof of Theorem 4.1.5. Assume that (4.1.4) holds. Let

$$\big((X_n'^{(1)})_{n\in\mathbb{N}}, N^{(1)}, X^{(1)}\big) \text{ and } \big(X_n'^{(m)})_{n\in\mathbb{N}}, N^{(m)}, X^{(m)}\big)_{m\ge 2}$$

be a copy of $\big((X'_n)_{n\in\mathbb{N}}, N, X\big)$ in Theorem 4.3.4 satisfying (4.3.2) and a sequence of independent copies of $\big((X'_n)_{n\in\mathbb{N}}, N, X\big)$ in Corollary 4.3.5 satisfying (4.3.3), respectively. For $n \in \mathbb{N}$, let $\alpha(n)$ be the non-negative integer such that $\sum_{i=1}^{\alpha(n)} N^{(i)} \le n < \sum_{i=1}^{\alpha(n)+1} N^{(i)}$, and let $\beta(n) := n - \sum_{i=1}^{\alpha(n)} N^{(i)}$. Set

$$S'_n := \sum_{i=1}^{\alpha(n)} \sum_{j=1}^{N^{(i)}} X_j'^{(i)} + \sum_{i=1}^{\beta(n)} X_i'^{(\alpha(n)+1)}, \quad S_n := \sum_{i=1}^{n} X^{(i)}, \quad n \in \mathbb{N}.$$

It can be easily checked that $\big((S_n)_{n\in\mathbb{N}}, (S'_n)_{n\in\mathbb{N}}\big)$ is a coupling of $\mathrm{RP}(D, L)$ and $\mathrm{RP}(D', L')$ by the construction of $X^{(i)}$ and $X_j'^{(i)}$. Moreover, since $S_n = S'_{\sum_{i=1}^{n} N^{(i)}}$ for $n \in \mathbb{N}$, we have $\{S_n \mid n \in \mathbb{N}\} \subseteq \big\{S'_n \,\big|\, n \in \mathbb{N}\big\}$, and thus $\mathrm{RP}(D, L) \preceq \mathrm{RP}(D', L')$.

Recall that, for $\alpha > 0$, Poi_α is the epoch set of the $\mathrm{Exp}(\alpha)$-delayed $\mathrm{Exp}(\alpha)$-renewal process, and the hazard rate of $\mathrm{Exp}(\alpha)$ is the constant α on $[0,\infty)$. By what we have just proved and the definition of c and C, we have $\mathrm{Poi}_c \preceq \mathrm{RP}(D,L) \preceq \mathrm{Poi}_C$. □

Finally, we prove Corollary 4.1.6 by Theorem 4.1.5 and Lemma 4.3.7.

Lemma 4.3.7. *Let f and g be measurable functions on a Borel subset D of $\mathbb{R}$ with $f(x) \geq 0$ and $g(x) > 0$ for $x \in D$. Assume that $\int_D g(x)\,\mathrm{d}x < \infty$. Then*

$$\inf_{x\in D} \frac{f(x)}{g(x)} \leq \frac{\int_D f(x)\,\mathrm{d}x}{\int_D g(x)\,\mathrm{d}x} \leq \sup_{x\in D} \frac{f(x)}{g(x)}.$$

Proof. Let $\alpha := \inf_{x\in D}(f(x)/g(x))$ and $\beta := \sup_{x\in D}(f(x)/g(x))$. Then $\alpha g(x) \leq f(x) \leq \beta g(x)$ for $x \in D$, and the lemma follows from

$$\alpha \int_D g(x)\,\mathrm{d}x \leq \int_D f(x)\,\mathrm{d}x \leq \beta \int_D g(x)\,\mathrm{d}x. \qquad \square$$

Proof of Corollary 4.1.6. We prove that $r_L \leq r_{L'}$ implies $r_D \leq r_{D'}$, and that the non-increasingness of r_L implies $r_D \leq r_L$. With these two claims, Corollary 4.1.6 is a direct corollary of Theorem 4.1.5.

First, we assume $r_L \leq r_{L'}$. Note that

$$r_D(x) = \frac{f_D(x)}{1-D(x)} = \frac{\mu^{-1}(1-L(x))}{1-\mu^{-1}\int_0^x 1-L(y)\,\mathrm{d}y} = \frac{\bar{L}(x)}{\int_x^\infty \bar{L}(y)\,\mathrm{d}y}, \quad x \geq 0. \tag{4.3.6}$$

A similar formula holds for $r_{D'}(x)$ and, thus, in order to prove $r_D \leq r_{D'}$, it suffices to prove that

$$\frac{\bar{L}(x)}{\bar{L}'(x)} \leq \frac{\int_x^\infty \bar{L}(y)\,\mathrm{d}y}{\int_x^\infty \bar{L}'(y)\,\mathrm{d}y}. \tag{4.3.7}$$

Note that $\bar{L}(x)/\bar{L}'(x)$ is non-decreasing in x, since $r_L \leq r_{L'}$ and

$$\frac{\bar{L}(x)}{\bar{L}'(x)} = \frac{\exp\left(-\int_0^x r_L(t)\,\mathrm{d}t\right)}{\exp\left(-\int_0^x r_{L'}(t)\,\mathrm{d}t\right)} = \exp\left(\int_0^x r_{L'}(t) - r_L(t)\,\mathrm{d}t\right).$$

Then (4.3.7) follows from Lemma 4.3.7.

Next, we assume that r_L is non-increasing. By (4.3.6), we have

$$\begin{aligned} r_D(x) = \frac{\bar{L}(x)}{\int_x^\infty \bar{L}(y)\,\mathrm{d}y} &= \frac{\exp\left(-\int_0^x r_L(t)\,\mathrm{d}t\right)}{\int_x^\infty \exp\left(-\int_0^y r_L(t)\,\mathrm{d}t\right)\mathrm{d}y} \\ &= \frac{1}{\int_x^\infty \exp\left(-\int_x^y r_L(t)\,\mathrm{d}t\right)\mathrm{d}y} \\ &\leq \frac{1}{\int_x^\infty \exp(-(y-x)r_L(x))} = r_L(x), \quad x \geq 0. \end{aligned} \qquad \square$$

After identifying sufficient conditions for coupling of epoch sets of renewal processes, we investigate necessary conditions as a supplement.

Theorem 4.3.8. *Assume that the epoch set of an F-renewal process is stochastically dominated by the epoch set of an F'-renewal process (there is no condition on the first epoch of both renewal processes). Then $F \le F'$.*

Proof. Let $\{S_n \mid n \in \mathbb{N}\}$ and $\{S'_n \mid n \in \mathbb{N}\}$ be the epoch sets of an F-renewal process and an F'-renewal process, respectively, such that

$$\{S_n \mid n \in \mathbb{N}\} \subseteq \{S'_n \mid n \in \mathbb{N}\}.$$

Then there exists an $n \in \mathbb{N}$ such that $\mathbb{P}(S_1 = S'_n) > 0$, since $\mathbb{P}\left(\bigcup_{i=1}^{\infty}\{S_1 = S'_i\}\right) = 1$. Let $A := \{S_1 = S'_n\}$. Then $\mathbb{P}(A, S_2 - S_1 \ge S'_{n+1} - S'_n) = \mathbb{P}(A)$, i.e.,

$$\mathbb{P}\left(S_2 - S_1 \ge S'_{n+1} - S'_n \mid A\right) = 1. \tag{4.3.8}$$

Otherwise, $S'_n < S_2 < S'_{n+1}$ with a positive proability, which contradicts that S_2 coincides with an epoch in $\{S'_n \mid n \in \mathbb{N}\}$.

Note that $S_2 - S_1$ is independent of S_1, and $S'_{n+1} - S'_n$ is independent of S'_n. Therefore $S_2 - S_1$ and $S'_{n+1} - S'_n$ are independent of A. Hence

$$\mathbb{P}(S_2 - S_1 \le x \mid A) = F(x), \quad \mathbb{P}\left(S'_{n+1} - S'_n \mid A\right) = F'(x), \quad x \ge 0.$$

Together with (4.3.8), we get $F \le F'$. □

4.4 The Point Process of Valid Infections

In this section we prepare to apply the coupling of renewal processes in Section 4.3 to obtain conditions for $\mathrm{VI}^e \preceq \mathrm{VI}'^e$. After introducing the *hypoexponential distribution*, we first prove that VI^e is an L-renewal process (Theorem 4.1.8), and then we calculate and analyse the hazard rates of the lifetime distribution and delay distribution of $(\mathrm{VI}^e_t)_{t\ge 0}$ (Theorem 4.1.9). Throughout this section, fix $e \in E$, and for convenience write λ_e, p_e and υ_e as λ, p and υ, respectively.

To simplify the notation, we denote by $\mathrm{HypoExp}_n(\lambda_1,\dots,\lambda_n)$ the *hypoexponential distribution* with parameters $\lambda_1, \dots, \lambda_n$, which is defined as the distribution of the sum of n independent random variables $X_1, \dots, X_n$ with $X_i \sim \mathrm{Exp}(\lambda_i)$ for $1 \le i \le n$. By the formula for the Laplace transform of the sum of independent exponential ly distributed random variables, the Laplace transform of $\mathrm{HypoExp}(\lambda_1,\dots,\lambda_n)$ is $\prod_{i=1}^n \lambda_i/(\lambda_i + z)$.

We start by proving Theorem 4.1.8. We first prove that $(\mathrm{VI}^e_t)_{t\ge 0}$ is a renewal process, and then calculate the Laplace transforms of the lifetime distribution and delay distribution, and finally prove the stationarity of $(\mathrm{VI}^e_t)_{t\ge 0}$ when $G^e_0 \sim \mathrm{Ber}(p_e)$.

Proof of Theorem 4.1.8. Recall that $(G_t^e)_{t\geq 0}$ jumps from 0 to 1 at rate vp and jumps from 1 to 0 at rate $v(1-p)$. Moreover, when the edge e is open (at state 1), infections occur via e at rate λ. Figure 4.2 provides an intuitive description of the dynamics of $(G_t^e, \mathrm{VI}_t^e)_{t\geq 0}$, where the edge is open in olive states and closed in purple states. It is easily checked that $(G_t^e, \mathrm{VI}_t^e)_{t\geq 0}$ is a continuous-time Markov chain on $\{0,1\}\times\mathbb{N}_0$ with transition rates given by Table 4.1 ($k\in\mathbb{N}_0$).

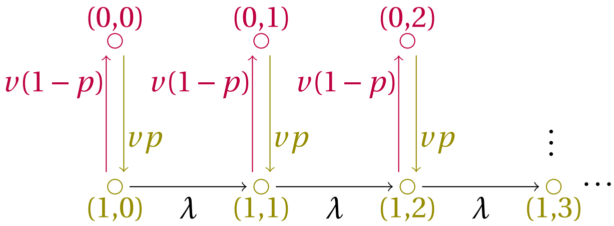

Figure 4.2: *Rates of* $(G_t^e, \mathrm{VI}_t^e)_{t\geq 0}$.

from	to	at rate
$(0,k)$	$(1,k)$	vp
$(1,k)$	$(0,k)$	$v(1-p)$
$(1,k)$	$(1,k+1)$	λ

Table 4.1: *Rates of* $(G_t^e, \mathrm{VI}_t^e)_{t\geq 0}$.

We first prove that $(\mathrm{VI}_t^e)_{t\geq 0}$ is a renewal process in a standard way. For $k\in\mathbb{N}_0$, set $\tau_k := \inf\{t\geq 0 \mid \mathrm{VI}_t^e = k\}$. Note that $G_{\tau_k}^e = 1$ and $\mathrm{VI}_{\tau_k}^e = k$ for $k\in\mathbb{N}$. By the strong Markov property, $(\tau_k - \tau_{k-1})_{k\in\mathbb{N}}$ is a sequence of independent random variables. Moreover, for $k\in\mathbb{N}$, $(G_{t+\tau_k}^e, \mathrm{VI}_{t+\tau_k}^e - k)_{t\geq 0}$ is a Markov chain with initial state $(1,0)$ and dynamics given by Table 4.1. For $k\in\mathbb{N}$, since

$$\tau_{k+1}-\tau_k = \inf\{t>0 \mid \mathrm{VI}_{t+\tau_k}^e - k = 1\},$$

we have $\tau_{k+1}-\tau_k \overset{\mathrm{d}}{=} \sigma_1$, where the distributions of σ_1 and σ_0 (σ_0 will be used later) are given by

$$\mathbb{P}(\sigma_b\in\cdot\,) := \mathbb{P}\left(\inf\{t\geq 0 \mid (G_t^e, \mathrm{VI}_t^e) = (1,1)\}\in\cdot \;\middle|\; (G_0^e, \mathrm{VI}_t^e) = (b,0)\right),\quad b\in\{0,1\}.$$

Moreover, $\tau_1 \overset{\mathrm{d}}{=} \sigma_1$ if $G_0^e = 1$, and $\tau_1 \overset{\mathrm{d}}{=} \sigma_0$ if $G_0^e = 0$. Hence, to prove Theorem 4.1.8, it suffices to prove $\mathscr{L}_{\sigma_1} = \mathscr{L}_L$, $\mathscr{L}_{\sigma_0} = \mathscr{L}_D$ and to show that τ_1 makes the renewal process $(\mathrm{VI}_t^e)_{t\geq 0}$ stationary in the case $G_0^e \sim \mathrm{Ber}(p)$.

Next, we prove that $\mathscr{L}_{\sigma_1} = \mathscr{L}_L$ and $\mathscr{L}_{\sigma_0} = \mathscr{L}_D$. Set

$$q := \frac{v(1-p)}{v(1-p)+\lambda},\quad l_0(z) := \frac{vp}{vp+z},\quad l_1(z) := \frac{v(1-p)+\lambda}{v(1-p)+\lambda+z},\quad z\in\mathbb{R}. \tag{4.4.1}$$

For $n\in\mathbb{N}_0$ and $b\in\{0,1\}$, let $A_n^{(b)}$ be the event that, starting from $(b,0)$, $(G_t^e, \mathrm{VI}_t^e)_{t\geq 0}$ visits $(1-b,0)$ n times before jumping to $(1,1)$. From Figure 4.2, it can be checked that $\mathbb{P}(A_n^{(1)}) = \mathbb{P}(A_n^{(0)}) = q^n(1-q)$ for $n\in\mathbb{N}_0$. Moreover, for $n\in\mathbb{N}_0$,

$$\sigma_1 | A_n^{(1)} \sim \mathrm{HypoExp}_{2n+1}\left(v(1-p)+\lambda, vp, \ldots, v(1-p)+\lambda, vp, v(1-p)+\lambda\right),$$

$$\sigma_0|A_n^{(0)} \sim \text{HypoExp}_{2n+2}\big(vp, v(1-p)+\lambda, \dots, vp, v(1-p)+\lambda\big).$$

By the formula of Laplace transform of convolutions and mixtures, we have

$$\begin{aligned}
\mathscr{L}_L(z) &= \sum_{n=0}^{\infty} q^n(1-q)\mathscr{L}_{\sigma_1|A_n^{(1)}}(z) \\
&= \sum_{n=0}^{\infty} q^n(1-q)(l_1(z)l_0(z))^n l_1(z) = \frac{(1-q)l_1(z)}{1-ql_0(z)l_1(z)}, \qquad z \in \mathbb{R}, \\
\mathscr{L}_D(z) &= \sum_{n=0}^{\infty} q^n(1-q)\mathscr{L}_{\sigma_0|A_n^{(0)}}(z) \\
&= \sum_{n=0}^{\infty} q^n(1-q)(l_0(z)l_1(z))^{n+1} = \frac{(1-q)l_0(z)l_1(z)}{1-ql_0(z)l_1(z)}, \qquad z \in \mathbb{R}.
\end{aligned}$$

Together with (4.4.1) we obtain $\mathscr{L}_{\sigma_1} = \mathscr{L}_L$ and $\mathscr{L}_{\sigma_0} = \mathscr{L}_D$.

Finally, we turn to the distribution of the delay τ_1 of $(\mathrm{VI}_t^e)_{t\geq 0}$ in the case $G_0^e \sim \mathrm{Ber}(p_e)$. Note that $\tau_1|\{G_0^e = 1\} \overset{\mathrm{d}}{=} \sigma_1 \sim L$ and $\tau_1|\{G_0^e = 0\} \overset{\mathrm{d}}{=} \sigma_0 \sim D$. Hence

$$\mathscr{L}_S(z) = p\mathscr{L}_L(z) + (1-p)\mathscr{L}_D(z) = \frac{\lambda p(z+v)}{z^2 + z(v+\lambda) + \lambda v p}, \quad z \in \mathbb{R}.$$

It can be checked that $\mathscr{L}_L(z) = 1 - (\lambda p)^{-1} z\mathscr{L}_S(z)$ for $z \in \mathbb{R}$. Moreover,

$$\begin{aligned}
\mathscr{L}_L(z) &= \int_0^\infty \exp(-zx)\,\mathrm{d}L(x) = -\int_0^\infty \exp(-zx)\,\mathrm{d}\bar{L}(x) \\
&= -\exp(-zx)\bar{L}(x)\big|_0^\infty + \int_0^\infty \bar{L}(x)\,\mathrm{d}(\exp(-zx)) \\
&= 1 - z\int_0^\infty \exp(-zx)\bar{L}(x)\,\mathrm{d}x, \quad z \in \mathbb{R}. \qquad (4.4.2)
\end{aligned}$$

By the bijectivity of the Laplace transform, the probability density function of S is $f_S(x) = \lambda p\bar{L}(x)$, and thus $S(x) = \lambda p \int_0^x \bar{L}(y)\,\mathrm{d}y$. Since S is a probability distribution, we have $\lambda p = \big(\int_0^\infty \bar{L}(x)\,\mathrm{d}x\big)^{-1}$, and hence the S-delayed L-renewal process is stationary. □

However, the Laplace transforms of L and D given in (4.1.5) are not enough. To apply couping of renewal processes, we need to calculate and analyse the hazard rates of L and D. Before doing so, we derive some properties of z_1, z_2 defined in (4.1.6) and a_1, a_2, b defined in (4.1.7).

Theorem 4.4.1. *Suppose that $p \in (0,1)$. Then*

$$z_1, z_2 \in \mathbb{R}, \quad 0 < z_1 < pv < z_2, \quad a_1, a_2 > 0,$$

$$a_1 + a_2 = \lambda, \quad \frac{a_1}{z_1} + \frac{a_2}{z_2} = 1, \quad \frac{b}{z_1} - \frac{b}{z_2} = 1, \tag{4.4.3}$$

$$a_1 a_2 \left(2 - \frac{z_1}{z_2} - \frac{z_2}{z_1}\right) = \lambda(1-p)v, \quad b^2\left(2 - \frac{z_1}{z_2} - \frac{z_2}{z_1}\right) = -\lambda p v. \tag{4.4.4}$$

Proof. First, $z_1, z_2 \in \mathbb{R}$ and $z_1 > 0$, since $(\lambda+v)^2 - 4\lambda p v = (\lambda - v)^2 + 4\lambda(1-p)v > 0$ and $\lambda + v > \sqrt{(\lambda+v)^2 - 4\lambda p v}$. Note that $z_1 < pv$ is equivalent with

$$\lambda + v - 2pv < \sqrt{(\lambda+v)^2 - 4\lambda p v},$$

which follows from $(\lambda + v - 2pv)^2 < (\lambda+v)^2 - 4\lambda p v$, i.e., $4pv^2(p-1) < 0$. Similarly, we have $vp < z_2$. Then $a_1, a_2 > 0$ follows from $z_1 < vp < z_2$ directly.

Second, $a_1 + a_2 = \lambda$ and $\frac{b}{z_1} - \frac{b}{z_2} = 1$ can be checked directly. Moreover,

$$\frac{a_1}{z_1} + \frac{a_2}{z_2} = \frac{\lambda(vp - z_1)z_2 - \lambda(vp - z_2)z_1}{(z_2 - z_1)z_1 z_2} = \frac{\lambda v p(z_2 - z_1)}{(z_2 - z_1)z_1 z_2} = 1.$$

Finally, (4.4.4) follows from the following calculation:

$$\begin{aligned} 2 - \frac{z_1}{z_2} - \frac{z_2}{z_1} &= \frac{2z_1 z_2 - z_1^2 - z_2^2}{z_1 z_2} = -\frac{(z_2 - z_1)^2}{\lambda p v}, \\ a_1 a_2 &= \frac{\lambda^2}{(z_2 - z_1)^2}(vp - z_1)(z_2 - vp) = \frac{\lambda^2}{(z_2 - z_1)^2} v^2 p(1-p), \\ b^2 &= \frac{z_1^2 z_2^2}{(z_2 - z_1)^2} = \frac{\lambda^2 p^2 v^2}{(z_2 - z_1)^2}. \end{aligned}$$

□

With Theorem 4.4.1 we are ready to prove the formulas and properties of the hazard rates in Theorem 4.1.9.

Proof of Theorem 4.1.9. First, we prove (4.1.8) and (4.1.9). Note that

$$\begin{aligned} \mathcal{L}_L(z) &= \lambda \frac{z + pv}{(z+z_1)(z+z_2)} = \frac{pv - z_1}{z_2 - z_1}\frac{\lambda}{z + z_1} + \frac{pv - z_2}{z_1 - z_2}\frac{\lambda}{z+z_2} = \frac{a_1}{z+z_1} + \frac{a_2}{z+z_2}, \\ \mathcal{L}_D(z) &= \frac{\lambda p v}{(z+z_1)(z+z_2)} = \frac{z_1 z_2}{z_2 - z_1}\left(\frac{1}{z+z_1} - \frac{1}{z+z_2}\right) = \frac{b}{z+z_1} - \frac{b}{z+z_2}. \end{aligned}$$

Hence, the probability density functions f_L and f_D of L and D, respectively, are given by

$$f_L(x) = a_1 \exp(-z_1 x) + a_2 \exp(-z_2 x), \quad x \geq 0, \tag{4.4.5}$$

$$f_D(x) = b\exp(-z_1 x) - b\exp(-z_2 x), \quad x \geq 0. \tag{4.4.6}$$

By integrating the probability density functions and using (4.4.3), we obtain

$$L(x) = 1 - \frac{a_1}{z_1}\exp(-z_1 x) - \frac{a_2}{z_2}\exp(-z_2 x), \quad x \geq 0, \tag{4.4.7}$$

$$D(x) = 1 - \frac{b}{z_1}\exp(-z_1 x) + \frac{b}{z_2}\exp(-z_2 x), \quad x \geq 0. \tag{4.4.8}$$

With (4.4.5) to (4.4.8), we obtain (4.1.8) and (4.1.9).

Next, we analyse the hazard rates r_L and r_D. By (4.4.3),

$$r_L(0) = \frac{a_1 + a_2}{\frac{a_1}{z_1} + \frac{a_2}{z_2}} = \lambda, \quad r_D(0) = \frac{1-1}{\frac{1}{z_1} - \frac{1}{z_2}} = 0.$$

Moreover, since $0 < z_1 < z_2$ and $a_1, a_2 > 0$, we have

$$\lim_{x\to\infty} r_L(x) = \lim_{x\to\infty} \frac{a_1 \exp(-z_1 x)}{\frac{a_1}{z_1}\exp(-z_1 x)} = z_1, \quad \lim_{x\to\infty} r_D(x) = \lim_{x\to\infty} \frac{\exp(-z_1 x)}{\frac{1}{z_1}\exp(-z_1 x)} = z_1.$$

Recall $r_L(x) = f_L(x)/(1 - L(x)$, (4.4.5), (4.4.7) and (4.4.4). We have

$$\begin{aligned}
\frac{\mathrm{d}r_L}{\mathrm{d}x}(x) &= \frac{1}{(1-L(x))^2}\left((1-L(x))\frac{\mathrm{d}f_L}{\mathrm{d}x}(x) + (f_L(x))^2\right) \\
&= \frac{1}{(1-L(x))^2} a_1 a_2 \left(2 - \frac{z_1}{z_2} - \frac{z_2}{z_1}\right)\exp(-(z_1+z_2)x) \\
&= -\frac{\lambda(1-p)v}{(1-L(x))^2}\exp(-(\lambda+v)x) < 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{\mathrm{d}r_D}{\mathrm{d}x}(x) &= \frac{1}{(1-D(x))^2}\left((1-D(x))\frac{\mathrm{d}f_D}{\mathrm{d}x}(x) + (f_D(x))^2\right) \\
&= \frac{c^2}{(1-D(x))^2}\left(\frac{z_1}{z_2} + \frac{z_2}{z_1} - 2\right)\exp(-(z_1+z_2)x) \\
&= \frac{\lambda p v}{(1-D(x))^2}\exp(-(\lambda+v)x) > 0.
\end{aligned}$$

□

Remark 4.4.2. It is not a coincidence that the lower bound z_1 of r_L is exactly the same as $\tilde{\lambda}_e$ defined in (3.3.3). Assume that $G_0^e \sim \mathrm{Ber}(p)$. By Theorem 3.1.3, we have

$$\mathrm{Poi}_{\lambda_{\max}(0,\lambda,v,p)} \preceq \mathrm{VI}^e \preceq \mathrm{Poi}_{\lambda_{\min}(0,\lambda,v,p)}, \tag{4.4.9}$$

where $\lambda_{\max}(0,\lambda,v,p) = \frac{1}{2}(\lambda + v - \sqrt{(\lambda+v)^2 - 4\lambda p v}) = z_1$ and $\lambda_{\min}(0,\lambda,v,p) = \lambda$. Meanwhile, we obtain (4.4.9) independently from Corollary 4.1.6, since r_L is non-increasing, has an upper bound λ, and has a lower bound z_1. Moreover, with Corollary 4.1.7 (b), we know that (4.4.9) also holds in the case $G_0^e = 1$.

Our method replies on coupling of renewal processes, and hence cannot be generalised to the case where $\delta_0 \neq 0$ in Theorem 3.1.3. However, our method enables coupling of epoch sets of two renewal processes without an intermediate Poisson point process, which gives more precise results (see Example 4.5.2). With the help of the hazard rates obtained in this section, in Section 4.5 we will apply coupling of renewal processes to prove monotonicity properties of contact processes on dynamic bond percolation.

4.5 Coupling of Point Processes of Valid Infections

Theorem 3.3.2 (c) couples the infection process $(\xi_t)_{t\geq 0}$ with an ordinary contact process on G by coupling VI^e with a Poisson point process via Theorem 3.1.3. In this section, with the help of results in Sections 4.3 and 4.4, we first prove a generalised version Theorem 4.1.10 of Theorem 3.3.2 (c) and its corollary Theorem 4.1.11 independently. Moreover, we are inspired to couple two infection processes $(\xi_t)_{t\geq 0}$ and $(\xi'_t)_{t\geq 0}$ under different parameters by coupling VI^e with VI'^e directly. In the second part of this section, we obtain the sufficient condition for $\xi_t \preceq \xi'_t$ for $t \geq 0$ in Theorem 4.1.13 and its corollary Theorem 4.1.14 by verifying the sufficient condition for $\mathrm{VI}^e \preceq \mathrm{VI}'^e$ in Theorem 4.1.12 .

We begin with Theorem 4.1.10, a simple corollary of Corollary 4.1.6 (b), Corollary 4.1.7 (b), Theorem 4.1.8 and Theorem 4.1.9.

Proof of Theorem 4.1.10. Assume that $\lambda_e \leq z'_1(e)$ and $G^e_0 = 1$ for $e \in E$. By Theorem 4.1.9, λ_e is an upper bound of the hazard rate of the lifetime distribution L of VI^e, and $z'_1(e)$ is a lower bound of the hazard rate of the lifetime distribution L' of VI'^e. If $G^e_0 = 1$, then by Corollary 4.1.7 (b) we have

$$\mathrm{VI}^e \preceq \mathrm{Poi}_{\lambda_e} \preceq \mathrm{Poi}_{z'_1(e)} \preceq \mathrm{VI}'^e, \quad e \in E. \tag{4.5.1}$$

Note that the hazard rate of L is non-increasing. By Theorem 4.1.8 (c) and Corollary 4.1.6 (b), (4.5.1) also holds when $G^e_0 \sim \mathrm{Ber}(p)$. Then (4.1.1) and (4.1.3) hold. □

The proof of Theorem 4.1.11 needs the following theorem on how z_1 varies as λ, p and v change:

Theorem 4.5.1. *Consider z_1 as a function with arguments $\lambda > 0$, $p \in (0,1)$ and $v > 0$. Then*

(a) $\partial z_1/\partial\lambda > 0$ *and* $\lim_{\lambda\to\infty} z_1(\lambda, p, v) = pv$.

(b) $\partial z_1/\partial p > 0$ *and* $\lim_{p\to 1} z_1(\lambda, p, v) = \min(\lambda, v)$.

(c) $\partial z_1/\partial v > 0$ *and* $\lim_{v\to\infty} z_1(\lambda, p, v) = \lambda p$.

Proof. (a) Note that

$$\frac{\partial z_1}{\partial \lambda}(\lambda, p, v) = \frac{1}{2}\left(1 - \frac{\lambda + v - 2pv}{\sqrt{(\lambda+v)^2 - 4\lambda p v}}\right) > 0$$

if and only if $\lambda + v - 2pv < \sqrt{(\lambda+v)^2 - 4\lambda p v}$, which follows from $(\lambda + v - 2pv)^2 < (\lambda+v)^2 - 4\lambda p v$, i.e., $v(p-1) < 0$. Moreover,

$$\begin{aligned}\lim_{\lambda\to\infty} z_1(\lambda, p, v) &= \lim_{\lambda\to\infty} \frac{1}{2}\frac{4\lambda p v}{\lambda + v + \sqrt{(\lambda+v)^2 - 4\lambda p v}} \\ &= \lim_{\lambda\to\infty} \frac{2pv}{1 + \frac{v}{\lambda} + \sqrt{\left(1+\frac{v}{\lambda}\right)^2 - 4\frac{v}{\lambda}p}} = pv.\end{aligned}$$

(b) The claim follows from

$$\frac{\partial z_1}{\partial p}(\lambda, p, v) = \frac{\lambda v}{\sqrt{(\lambda+v)^2 - 4\lambda p v}} > 0,$$

$$\lim_{p\to 1} z_1(\lambda, p, v) = \frac{1}{2}\left(\lambda + v - \sqrt{(\lambda+v)^2 - 4\lambda v}\right) = \min(\lambda, v).$$

(c) Note that λ and v are exchangeable in $z_1(\lambda, p, v)$. Hence the result follows in the same way as Theorem 4.5.1 (a). □

With Theorem 4.5.1, we obtain Theorem 4.1.11 directly from Theorem 4.1.10.

Proof of Theorem 4.1.11. (a) Assume that $\lambda_e < p'_e v'_e$ for each $e \in E$. By Theorem 4.5.1 (a), for $e \in E$ there exists a $\lambda'_{0,e}$ such that $z_1(\lambda'_e, p'_e, v'_e) > \lambda_e$ for $\lambda'_e \geq \lambda'_{0,e}$. Then, by Theorem 4.1.10, (4.1.1) and (4.1.3) hold for any $\boldsymbol{\lambda}' \geq (\lambda'_{0,e})_{e\in E}$.

(b) Assume that $\lambda_e < \min(\lambda'_e, v'_e)$ for $e \in E$. By Theorem 4.5.1 (b), for $e \in E$ there exists a $p'_{0,e} < 1$ such that $z_1(\lambda'_e, p'_e, v'_e) > \lambda_e$ for $p'_e > p'_{0,e}$. Then, by Theorem 4.1.10, (4.1.1) and (4.1.3) hold for any $\boldsymbol{p}' \geq (p'_{0,e})_{e\in E}$.

(c) Assume that $\lambda_e < \lambda'_e p'_e$ for $e \in E$. By Theorem 4.5.1 (c), for $e \in E$ there exists a $v'_{0,e}$ such that $z_1(\lambda'_e, p'_e, v'_e) > \lambda_e$ for $v'_e > v'_{0,e}$. Then, by Theorem 4.1.10, (4.1.1) and (4.1.3) hold for any $\boldsymbol{v}' \geq (v'_{0,e})_{e\in E}$. □

Next, we prove Theorems 4.1.12 to 4.1.14, among which Theorem 4.1.12 is the key. With $r_L \leq r_{L'}$ and $r_D \leq r_{D'}$, coupling of VI^e and VI'^e is immediate from Theorem 4.1.5 and its corollaries, and Theorem 4.1.13 follows. Since the expressions for r_L and r_D are too complicated to be handled directly, we turn to the derivatives of r_L and r_D.

Proof of Theorem 4.1.12. Assume that (4.1.12) holds. By Theorem 4.1.9, $r_{L'}(x) > z_1'$ and $r_L(x) \le \lambda$ for $x \ge 0$, and thus $r_{L'}(x) - r_L(x) > z_1' - \lambda$ for $x \ge 0$. Hence for $x \ge 0$,

$$\frac{\mathrm{d}r_{L'}}{\mathrm{d}x}(x) \Big/ \frac{\mathrm{d}r_L}{\mathrm{d}x}(x) = \frac{(1-L(x))^2}{(1-L'(x))^2} \frac{\lambda'(1-p')v'}{\lambda(1-p)v} \frac{\exp(-(\lambda'+v')x)}{\exp(-(\lambda+v)x)} \tag{4.5.2}$$

$$\ge \exp\left(2\int_0^x r_{L'}(s) - r_L(s)\,\mathrm{d}s\right)\exp((\lambda+v-\lambda'-v')x), \tag{4.5.3}$$

$$> \exp(2(z_1'-\lambda)x)\exp((\lambda+v-\lambda'-v')x)$$

$$= \exp((v+2z_1'-\lambda-\lambda'-v')x) \ge 1, \tag{4.5.4}$$

where (4.5.2) follows from (4.1.10), (4.5.3) follows from $\bar{L}(x) = \exp\left(-\int_0^x r_L(s)\,\mathrm{d}s\right)$ and the assumption $\lambda(1-p)v \le \lambda'(1-p')v'$ in (4.1.12), and (4.5.4) follows from the assumption $v + 2z_1' \ge \lambda + \lambda' + v'$ in (4.1.12). By (4.1.10) we have $\mathrm{d}r_L/\mathrm{d}x < 0$, and thus

$$\frac{\mathrm{d}r_{L'}}{\mathrm{d}x} < \frac{\mathrm{d}r_L}{\mathrm{d}x} < 0.$$

If there exists an $x_0 \ge 0$ such that $r_L(x_0) > r_{L'}(x_0)$, then

$$r_L(x) \ge r_{L'}(x) + r_L(x_0) - r_{L'}(x_0), \quad x \ge x_0.$$

However, by Theorem 4.1.9 and the assumption $z_1 \le z_1'$ in (4.1.12),

$$\lim_{x\to\infty} r_L(x) = z_1 \le z_1' = \lim_{x\to\infty} r_{L'}(x),$$

which gives a contradiction. Thus $r_{L'} \ge r_L$.

Next, instead of (4.1.12), we assume that $\lambda p v < \lambda' p' v'$ and $\lambda + v \ge \lambda' + v'$ and prove that $r_D(x) < r_{D'}(x)$ for $x > 0$ and $z_1 \le z_1'$. Together with (4.1.11) and $\bar{D}(x) = \exp\left(-\int_0^x r_D(s)\,\mathrm{d}s\right)$, we have

$$\frac{\mathrm{d}r_{D'}}{\mathrm{d}x}(x) \Big/ \frac{\mathrm{d}r_D}{\mathrm{d}x}(x) = \frac{(1-D(x))^2}{(1-D'(x))^2} \frac{\lambda' p' v'}{\lambda p v} \frac{\exp(-(\lambda'+v')x)}{\exp(-(\lambda+v)x)}$$

$$\ge \exp\left(2\int_0^x r_{D'}(s) - r_D(s)\,\mathrm{d}s\right). \tag{4.5.5}$$

Note that $\frac{\mathrm{d}r_D}{\mathrm{d}x}(0) = \lambda p v < \lambda' p' v' = \frac{\mathrm{d}r_{D'}}{\mathrm{d}x}(0)$. Moreover, $\frac{\mathrm{d}r_D}{\mathrm{d}x}$ and $\frac{\mathrm{d}r_{D'}}{\mathrm{d}x}$ are continuous. Hence there exists an $x_1 > 0$ such that $\frac{\mathrm{d}r_D}{\mathrm{d}x} < \frac{\mathrm{d}r_{D'}}{\mathrm{d}x}$, and so $r_D < r_{D'}$ on $(0, x_1]$. If there exists an $x_2 \in (x_1, \infty)$ such that $r_D(x_2) = r_{D'}(x_2)$ and $r_D(x) < r_{D'}(x)$ for $x \in (0, x_2)$, then, by (4.5.5),

$$\frac{\mathrm{d}r_{D'}}{\mathrm{d}x}(x) \ge \exp\left(2\int_0^x r_{D'}(s) - r_D(s)\,\mathrm{d}s\right)\frac{\mathrm{d}r_D}{\mathrm{d}x}(x) \ge \frac{\mathrm{d}r_D}{\mathrm{d}x}(x), \quad x \in (0, x_2),$$

which contradicts with $r_D(x_1) < r_{D'}(x_1)$ and $r_D(x_2) = r_{D'}(x_2)$. Therefore $r_D(x) < r_{D'}(x)$ for $x > 0$. Together with Theorem 4.1.9, we have

$$z_1 = \lim_{x\to\infty} r_D(x) \le \lim_{x\to\infty} r_{D'}(x) = z_1'.$$

Hence, if the condition in (4.1.13) holds, then $r_D \le r_{D'}$, (4.1.12) holds, and hence $r_L \le r_{L'}$. □

With Theorem 4.1.12, Theorem 4.1.13 is only a few steps away.

Proof of Theorem 4.1.13. Fix $A \subseteq V$. First, assume that (4.1.12) and $G_0^e = G_0'^e = 1$ hold for $e \in E$. Then, by Theorem 4.1.12, $r_{L_e} \le r_{L_e'}$, and, by Theorem 4.1.8 (a), $(\mathrm{VI}_t^e)_{t\ge 0}$ and $(\mathrm{VI}_t'^e)_{t\ge 0}$ are the zero-delayed L_e-renewal process and L_e'-renewal process, respectively. Note that r_{L_e} is non-increasing by Theorem 4.1.9. Hence we have $\mathrm{VI}^e \preceq \mathrm{VI}'^e$ by Corollary 4.1.7 (c), which implies (4.1.1).

Second, assume that (4.1.12), $G_0^e \sim \mathrm{Ber}(p)$ and $G_0'^e \sim \mathrm{Ber}(p')$ hold for $e \in E$. Then (4.1.3) follows similarly via Theorem 4.1.8 (c) and Corollary 4.1.6 (b).

Finally, assume that (4.1.13) holds for $e \in E$. Then, by Theorem 4.1.12, $r_{L_e} \le r_{L_e'}$ and $r_{D_e} \le r_{D_e'}$ for $e \in E$. Without loss of generality, we assume that $(G_0^e)_{e\in E} = (G_0'^e)_{e\in E} \sim \mu$. Fix $e \in E$. On the event $\{G_0^e = G_0'^e = 1\}$, $\mathrm{VI}^e \preceq \mathrm{VI}'^e$ follows from $r_{L_e} \le r_{L_e'}$ and Corollary 4.1.7 (c). We turn to the event $\{G_0^e = G_0'^e = 0\}$, on which $(\mathrm{VI}_t^e)_{t\ge 0}$ and $(\mathrm{VI}_t'^e)_{t\ge 0}$ are the D_e-delayed L_e-renewal process and D_e'-delayed L_e'-renewal process, respectively. Note that $r_{D_e}(u+x) \le z_1(e) \le z_1'(e) \le r_{L_e'}(x)$ for $u, x \ge 0$, and $r_{L_e}(u+x) \le r_{L_e}(x) \le r_{L_e'}(x)$. Then $\mathrm{VI}^e \preceq \mathrm{VI}'^e$ follows from $r_{D_e} \le r_{D_e'}$ and Theorem 4.1.5. With $\mathrm{VI}^e \preceq \mathrm{VI}'^e$ for $e \in E$, we obtain (4.1.2). Note that (4.1.13) implies (4.1.12) (see the proof of Theorem 4.1.12). Hence, by the second part of this proof, (4.1.3) also holds. □

We provide the following examples to show the usefulness of Theorem 4.1.13. For the Mathematica code that we used to plot Figures 4.3 and 4.4, we refer the reader to Appendix A.2.

Example 4.5.2. For $e \in E$, set λ_e, p_e, v_e, λ_e', p_e' and v_e' to be 1, 0.6, 3.65, 1.7, 0.5, and 2, respectively. Then the last three conditions in (4.1.12) can be checked via the following calculation:

$$\lambda + \lambda' + v' - v - 2z_1' \approx -0.0251191,$$
$$\lambda v(1-p) - \lambda' v'(1-p') = -0.24, \quad z_1 - z_1' \approx -0.00577591.$$

Thus (4.1.1) and (4.1.3) hold. See Figure 4.3 for the hazard rates. However, we cannot apply Theorems 3.3.2 (a) to 3.3.2 (c), 4.1.3 and 4.1.4 in this setting, since Theorem 3.3.2 (c) requires $\lambda_{\min}(0,\lambda,v,p) = \lambda \le z_1' = \lambda_{\max}(0,\lambda',v',p')$, Theorem 3.3.2 (a) requires $p = p'$, Theorem 3.3.2 (b) requires $v = v'$, Theorem 4.1.3 requires $p \le p'$, and Theorem 4.1.4 requires $v \le v'$.

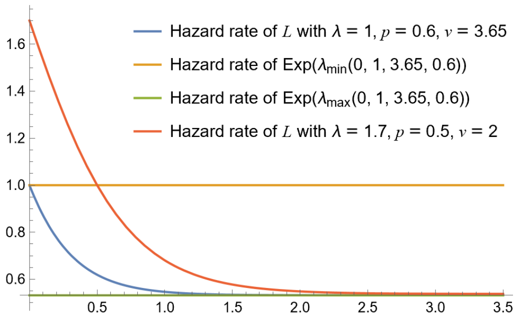

Figure 4.3: *Hazard rates in Example 4.5.2*

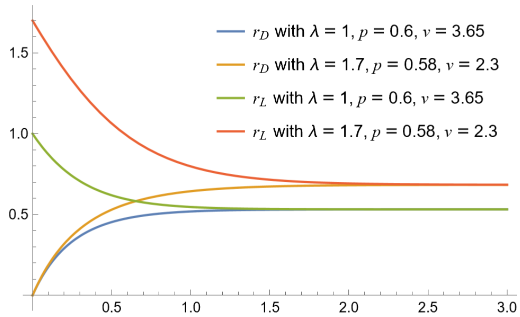

Figure 4.4: *Hazard rates in Example 4.5.3*

Example 4.5.3. For $e \in E$, set λ_e, p_e, v_e, λ'_e, p'_e and v'_e to be 1, 0.6, 3.65, 1.7, 0.58, and 2.3, respectively. Then the last four conditions in (4.1.13) can be checked via the following calculation:

$$\lambda + \lambda' + v' - v - 2z'_1 \approx -0.0177386, \quad \lambda v(1-p) - \lambda' v'(1-p') = -0.1822,$$
$$\lambda p v - \lambda' p' v' \approx -0.0778, \quad \lambda' + v' - \lambda - v \approx -0.65.$$

Thus (4.1.2) and (4.1.3) hold. See Figure 4.4 for the hazard rates. Similar with Example 4.5.2, Theorems 3.3.2 (a) to 3.3.2 (c), 4.1.3 and 4.1.4 does not work here.

Finally, we prove Theorem 4.1.14, which is a corollary of Theorem 4.1.12 and extends Theorem 4.1.11 (a).

Proof of Theorem 4.1.14. Assume that $z_1 < v'p'$. Let $f(\lambda') := \lambda' - 2z_1(\lambda', p', v')$. Then

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda'}(\lambda') = \frac{\lambda' + v' - 2v'p'}{(\lambda' + v')^2 - 4\lambda' p' v'}.$$

Note that $\frac{\mathrm{d}f}{\mathrm{d}\lambda'} \geq 0$ when $\lambda' \geq (2p'-1)v'$. Moreover, $\lim_{\lambda' \to \infty} f(\lambda') = \infty$. Hence there exists a $\lambda'_1 > 0$ such that

$$f(\lambda'_1) \geq \lambda + v' - v, \quad \lambda'_1 \geq (2p'-1)v'.$$

By Theorem 4.5.1 (a), there exists a $\lambda'_2 > 0$ such that

$$z_1(\lambda', p', v') \geq z_1 \text{ for } \lambda' > \lambda'_2.$$

Hence, (4.1.12) holds for $\lambda' \geq \lambda'_0$, where

$$\lambda'_0 := \max\left(\lambda, \lambda'_1, (2p'-1)v', \frac{\lambda v(1-p)}{v'(1-p')}, \lambda'_2\right).$$

Hence (4.1.1) and (4.1.3) follow from Theorem 4.1.13. □

Remark 4.5.4. Theorem 4.1.14 applies (4.1.12), via which we ensure that $r_{L'}$ decreases everywhere faster than r_L, but $r_{L'}$ remains larger than r_L eventually. This seems to be a very strong sufficient condition. But, if we set

$$(\lambda, p, v) := (1, 0.6, 3.65), \quad (p', v') := (0.5, 2)$$

in Theorem 4.1.14, then, by Example 4.5.2 and the proof of Theorem 4.1.14, we can set $\lambda_0' := 1.7$. We see from Figure 4.3 that 1.7 is a fairly good estimate of λ_0'.

As a supplement, we introduce the other condition in (4.5.6) for $r_L \le r_{L'}$ in the following remark, which is already covered in Theorem 4.1.3, but shows that Theorem 4.1.3 is non-trivial.

Remark 4.5.5. Assume $\lambda \le \lambda'$. If $\frac{\mathrm{d}r_L}{\mathrm{d}x} < \frac{\mathrm{d}r_{L'}}{\mathrm{d}x}$, then $r_L \le r_{L'}$ and thus $\xi_t \preceq \xi_t'$ for $t \ge 0$. We can easily find a sufficient condition similar to (4.1.12):

$$\lambda < \lambda', \quad \lambda + v + \lambda' - v' - 2z_1 < 0, \quad \lambda' v'(1-p') \le \lambda v(1-p). \tag{4.5.6}$$

Assume that (4.5.6) holds. Since $z_1 < \lambda \le \lambda'$, we have $v < v'$. Since $\lambda < \lambda'$, we have $\frac{1-p'}{1-p} \le \frac{v}{v'}$. Therefore $p < p'$ and hence

$$\frac{1-p'}{1-p} \le \frac{v}{v'} < 1 < \frac{p'}{p}.$$

Hence, by Theorem 3.3.2 (a) and Theorem 4.1.3,

$$\xi_t(\lambda, p, v) \preceq \xi_t(\lambda', p, v) \preceq \xi_t(\lambda', p', v').$$

Therefore the condition in (4.5.6) is not new.

4.6 Application to Broman's Randomly Evolving Environment

In this section we apply the method presented in Sections 4.2 to 4.5 to the contact process in Broman's randomly evolving environment with $\delta_0 = 0$. Instead of coupling the point processes of valid infections, we couple the point processes of *valid recoveries.*

Before presenting our results, we generalise the model to an inhomogeneous setting by assigning each vertex $v \in V$ its own parameters $\delta_v > 0$, $p_v \in [0, 1]$ and $\gamma_v > 0$. Similar to the graphical representation of the contact process on dynamic bond percolation, let

$$\mathrm{GR} := \left\{ \mathscr{B}^v, \mathscr{G}^v, \mathscr{I}^e, \mathscr{R}^v \;\middle|\; e \in E, v \in V \right\} \tag{4.6.1}$$

be a set of independent Poisson point processes on $[0,\infty)$, where the intensities of $\mathscr{B}^v$, $\mathscr{G}^v$, $\mathscr{I}^e$ and $\mathscr{R}^v$ are $(1-p_v)\gamma_v$, $p_v\gamma_v$, 1 and δ_v, respectively. Set

$$B_t^v := \begin{cases} B_0^v, & \text{if } [0,t]\cap(\mathscr{B}^v\cup\mathscr{G}^v)=\emptyset, \\ 1, & \text{if } \max([0,t]\cap(\mathscr{B}^v\cup\mathscr{G}^v))\in\mathscr{G}^v, \\ 0, & \text{if } \max([0,t]\cap(\mathscr{B}^v\cup\mathscr{G}^v))\in\mathscr{B}^v, \end{cases} \qquad t\geq 0, v\in V.$$

Here, $\max\emptyset := \infty$. See Figure 4.5, where, for $v\in V$, we place a purple line segment at $t\in\mathscr{B}^v$ to denote that the environment of v turns *bad* (state 0), and place an olive line segment at $t\in\mathscr{G}^v$ to denote that the environment of v turns *good* (state 1). We fill the area where the vertex is in the bad environment with gray, and place recovery marks and infection arrows in the same way as in Section 1.1.

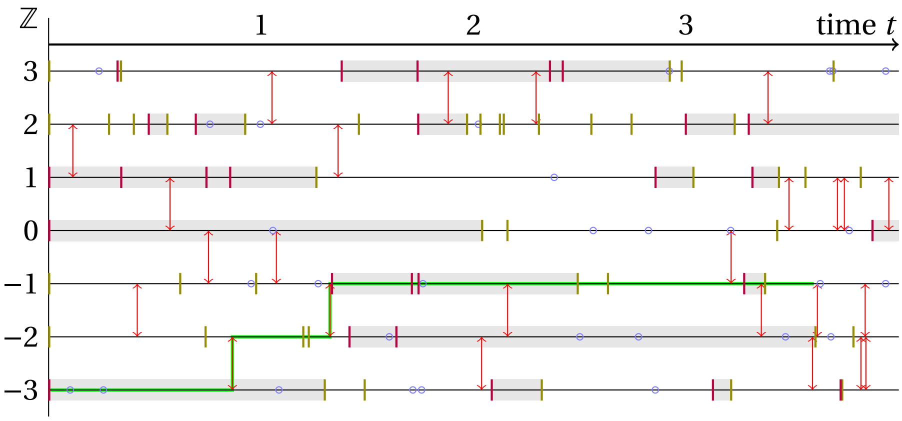

Figure 4.5: *The (partial) graphical representation of $(\xi_t)_{t\geq 0}$ in case $G=\mathbb{Z}$, $\delta_v=1.2$, $p_v=0.6$ and $\gamma_v=2.5$ for $v\in V$.*

Different from ordinary contact processes, only recovery marks in the good environment are *valid*. An *active path* in $V\times[0,\infty)$ is a connected oriented path that moves along the time lines in the increasing direction and along the infection arrows, but without passing any *valid* recovery marks. For example, the green line in Figure 3.1 specifies an active path. Define the *infection process* $(\xi_t)_{t\geq 0}$ on $\{0,1\}^V$ by setting $\xi_t(v) := 1$ if and only if there exists an active path from $(u,0)$ to (v,t) for some $u\in\xi_0$, for $t>0$ and $v\in V$. It can be checked easily that $(B_t^v,\xi_t^v)_{t\geq 0}$ has the following transition rates at time t:

from	to	at rate	from	to	at rate
(0,0)	(1,0)	$\gamma_v p_v$	(0,0)	(0,1)	$\sum_{u\sim v}\xi_t(u)$
(0,1)	(1,1)	$\gamma_v p_v$	(1,0)	(1,1)	$\sum_{u\sim v}\xi_t(u)$
(1,0)	(0,0)	$\gamma_v(1-p_v)$	(1,1)	(1,0)	δ_v
(1,1)	(0,1)	$\gamma_v(1-p_v)$	(0,1)	(0,0)	0

We finish the preparation with some notation.

- Denote by VR^v the point processes of valid recoveries for vertex $v \in V$, i.e., $\mathrm{VR}^v := \{t \in \mathscr{R}^v \mid B_t^v = 1\}$.
- Denote $(\delta_v)_{v\in V}$, $(p_v)_{v\in V}$ and $(\gamma_v)_{v\in V}$ by $\boldsymbol{\delta}$, $\boldsymbol{p}$ and $\boldsymbol{\gamma}$.
- Let μ be a probability measure on $\{0,1\}^V$, and let π_1 denote the Dirac measure centred at the environment in which all vertices are good. Let $\pi_{\boldsymbol{p}}$ denote the product measure $\bigotimes_{v\in V} \mathrm{Ber}(p_v)$.
- A symbol with $'$ means that it is defined in the same way as the symbol without $'$ but for the contact process $(B'_t, \xi'_t)_{t\geq 0}$ in Broman's randomly evolving environment on G with parameters $\boldsymbol{\delta}'$, $\boldsymbol{p}'$ and $\boldsymbol{\gamma}'$, rather than $\boldsymbol{\delta}$, $\boldsymbol{p}$ and $\boldsymbol{\gamma}$.

We aim to find sufficient conditions for the following couplings:

$$\xi_t^{\pi_1,A} \succeq \xi_t'^{\pi_1,A}, \quad t \geq 0, A \subseteq V, \tag{4.6.2}$$

$$\xi_t^{\mu,A} \succeq \xi_t'^{\mu,A}, \quad t \geq 0, A \subseteq V, \tag{4.6.3}$$

$$\xi_t^{\pi_{\boldsymbol{p}},A} \succeq \xi_t'^{\pi_{\boldsymbol{p}'},A}, \quad t \geq 0, A \subseteq V. \tag{4.6.4}$$

Note that we replace $\preceq$ in (4.1.1) to (4.1.3) with $\succeq$ here, since recoveries in the graphical representation of the contact process in Broman's randomly evolving environment hinder the spread.

By the same coupling idea used to prove Theorems 4.1.3 and 4.1.4, we obtain the following two theorems.

Theorem 4.6.1. *If $\delta_v = \delta'_v$ and $\frac{1-p'_v}{1-p_v} \leq \frac{\gamma_v}{\gamma'_v} \leq \frac{p'_v}{p_v}$ for $v \in V$, then (4.6.3) and (4.6.4) hold.*

Proof. The assumption implies that

$$p_v \leq p'_v, \quad \gamma_v(1-p_v) \geq \gamma'_v(1-p'_v), \quad \gamma_v p_v \leq \gamma'_v p'_v.$$

Let $\hat{\mathscr{B}}'^v$ be the Poisson point process obtained from $\mathscr{B}^v$ by keeping each point independently with probability $\gamma'_v(1-p'_v)/\gamma_v(1-p_v)$, and let $\hat{\mathscr{G}}^v$ be the Poisson point process obtained from $\mathscr{G}'^v$ by keeping each point independently with probability $\gamma_v p_v/\gamma'_v p'_v$. Define

$$\hat{\mathrm{GR}} := \{\mathscr{B}^v, \hat{\mathscr{G}}^v, \mathscr{I}^e, \mathscr{R}^v \mid e \in E, v \in V\}, \quad \hat{\mathrm{GR}}' := \{\hat{\mathscr{B}}'^v, \mathscr{G}'^v, \mathscr{I}^e, \mathscr{R}^v \mid e \in E, v \in V\}.$$

Then $(\hat{\mathrm{GR}}, \hat{\mathrm{GR}}')$ is a coupling of GR and GR$'$ with $\mathscr{B}^v \supseteq \hat{\mathscr{B}}'^v$ and $\hat{\mathscr{G}}^v \subseteq \mathscr{G}'^v$. With arguments similar to those in the last part of the proof of Theorem 3.3.2 (b) in Section 4.2, we obtain (4.6.3) and (4.6.4). □

Theorem 4.6.2. *Let $\alpha \in (0,1]$. Assume that $\delta_v \le \alpha\delta'_v$, $p_v = p'_v$ and $\gamma_v = \alpha\gamma'_v$ for $v \in V$. Then $\xi_t^{\mu,A} \ge \xi'^{\mu,A}_{\alpha t}$ for $t \ge 0$ and $A \subseteq V$.*

Proof. For a point process N on $[0,\infty)$, let $\alpha N := \{\alpha x \mid x \in N\}$. Compare

$$\mathrm{GR} = \left\{\mathscr{B}^v, \mathscr{G}^v, \mathscr{I}^e, \mathscr{R}^v \;\middle|\; e \in E, v \in V\right\},$$
$$\alpha\mathrm{GR}' := \left\{\alpha\mathscr{B}'^v, \alpha\mathscr{G}'^v, \alpha\mathscr{I}'^e, \alpha\mathscr{R}'^v \;\middle|\; e \in E, v \in V\right\}.$$

By assumption, for $e \in E$ and $v \in V$,

$$\mathscr{B}^v \overset{\mathrm{d}}{=} \alpha\mathscr{B}'^v, \quad \mathscr{G}^v \overset{\mathrm{d}}{=} \alpha\mathscr{G}'^v, \quad \mathscr{I}^e \ge \alpha\mathscr{I}'^e, \quad \mathscr{R}^v \le \alpha\mathscr{R}'^v.$$

Hence, GR and $\alpha\mathrm{GR}'$ can be coupled such that $(\xi_t)_{t\ge0}$ has more infections and less valid recoveries than $(\xi'_{\alpha t})_{t\ge0}$ in the graphical representation, which implies the result. □

Next, we compare VR^v with VI^e defined in (3.3.2). Define

$$\mathrm{VR}_t^v := \left|\left\{s \le t \;\middle|\; s \in \mathrm{VR}^v\right\}\right|, \quad t \ge 0, v \in V.$$

The dynamics of $(B_t^v, \mathrm{VR}_t^v)_{t\ge0}$ is shown in Figure 4.6, where the vertex is good in olive states and is bad in purple states. Hence $(B_t^v, \mathrm{VR}_t^v)_{t\ge0}$ is a continuous-time Markov chain on $\{0,1\} \times \mathbb{N}_0$ with transition rates given by Table 4.2 ($k \in \mathbb{N}_0$). Note that, compared with the transition rates $(G_t^e, \mathrm{VI}_t^e)_{t\ge0}$ given in Table 4.1, the only difference is that λ_e and v_e are replaced by δ_v and γ_v, respectively, and we obtain results for $(\mathrm{VR}_t^v)_{t\ge0}$ that are similar to Theorems 4.1.8 and 4.1.9. We only write the formula of $\tilde{z}_1(v)$ that corresponds to z_1 in (4.1.6):

$$\tilde{z}_1(v) := \frac{\delta_v + \gamma_v - \sqrt{(\delta_v+\gamma_v)^2 - 4\delta_v p_v \gamma_v}}{2}.$$

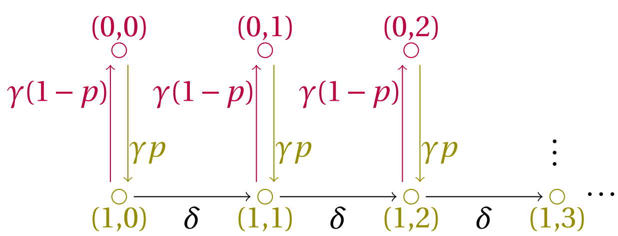

Figure 4.6: *Rates of* $(B_t^v, \mathrm{VR}_t^v)_{t\ge0}$.

from	to	at rate
$(0,k)$	$(1,k)$	γp
$(1,k)$	$(0,k)$	$\gamma(1-p)$
$(1,k)$	$(1,k+1)$	δ

Table 4.2: *Rates of* $(B_t^v, \mathrm{VR}_t^v)_{t\ge0}$.

We present the following results that correspond, successively, with Theorems 4.1.10, 4.1.11, 4.1.13 and 4.1.14.

Theorem 4.6.3. *If $\delta_v \le z_1'(v)$ for $v \in V$, then* (4.6.2) *and* (4.6.4) *hold.*

Theorem 4.6.4. *(a) Fix $\boldsymbol{\delta}$, $\boldsymbol{p}$, $\boldsymbol{\gamma}$, $\boldsymbol{p}'$ and $\boldsymbol{\gamma}'$. If $\delta_v < p_v'\gamma_v'$ for $v \in V$, then there exists a $\boldsymbol{\delta}_0' = (\delta_{v,0}')_{v\in V}$ such that* (4.6.2) *and* (4.6.4) *hold for any $\boldsymbol{\delta}' \ge \boldsymbol{\delta}_0'$.*

(b) Fix $\boldsymbol{\delta}$, $\boldsymbol{p}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}'$ and $\boldsymbol{\gamma}'$. If $\delta_v < \min(\delta_v', \gamma_v')$ for $v \in V$, then there exists a $\boldsymbol{p}_0' = (p_{v,0}')_{v\in V}$ with $p_{0,v}' < 1$ for $v \in V$ such that (4.6.2) *and* (4.6.4) *hold for any $\boldsymbol{p}' \ge \boldsymbol{p}_0'$.*

(c) Fix $\boldsymbol{\delta}$, $\boldsymbol{p}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}'$ and $\boldsymbol{p}'$. If $\delta_v < \delta_v' p_v'$ for $v \in V$, then there exists a $\boldsymbol{\gamma}_0' = (\gamma_{v,0}')_{v\in V}$ such that (4.6.2) *and* (4.6.4) *hold for any $\boldsymbol{\gamma}' \ge \boldsymbol{\gamma}_0'$.*

Theorem 4.6.5. *Assume that, for $v \in V$,*

$$\delta_v \le \delta_v', \quad \gamma_v + 2\tilde{z}_1'(v) \ge \delta_v + \delta_v' + \gamma_v', \quad \delta_v(1-p_v)\gamma_v \le \delta_v'(1-p_v')\gamma_v'.$$

If $\tilde{z}_1(v) \le \tilde{z}_1'(v)$ for $v \in V$, then (4.6.2) *and* (4.6.4) *hold. If $\delta_v p_v \gamma_v < \delta_v' p_v' \gamma_v'$ and $\delta_v + \gamma_v \ge \delta_v' + \gamma_v'$ for $v \in V$, then both* (4.6.3) *and* (4.6.4) *hold.*

Theorem 4.6.6. *Fix $\boldsymbol{\delta}$, $\boldsymbol{p}$, $\boldsymbol{\gamma}$, $\boldsymbol{p}'$ and $\boldsymbol{\gamma}'$. If $\tilde{z}_1(v) < p_v'\gamma_v'$ for $v \in V$, then there exists a $\boldsymbol{\delta}_0' = (\delta_{v,0}')_{v\in V}$ such that* (4.6.2) *and* (4.6.4) *hold for any $\boldsymbol{\delta}' \ge \boldsymbol{\delta}_0'$.*

4.7 Discussions

In this final section we discuss three interesting open questions.

First, we are still far from comprehending the effect of $\boldsymbol{v}$ on the contact process on dynamic bond percolation. With Theorems 4.1.3 and 4.1.12 we understand better how the infection rate $\boldsymbol{\lambda}$, the edge density $\boldsymbol{p}$ and the edge switching speed $\boldsymbol{v}$ affect the infection process. However, we want to obtain results for $\boldsymbol{v}$ that are similar to Theorems 3.3.2 (a) and 3.3.2 (b). As mentioned in Remark 3.4.6, [43] points out that the infection process in the static environment is more resilient than the same infection process in a slightly dynamic environment, so the condition $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, $\boldsymbol{p} = \boldsymbol{p}'$ and $\boldsymbol{v} \le \boldsymbol{v}'$ cannot guarantee the monotonicity property, especially for small $\boldsymbol{v}$ and $\boldsymbol{v}'$. Is there a $\boldsymbol{v}_0$ such that (4.1.1) holds for

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}', \quad \boldsymbol{p} = \boldsymbol{p}', \quad \boldsymbol{v}_0 \le \boldsymbol{v} \le \boldsymbol{v}'? \tag{4.7.1}$$

Unfortunately, our method does not answer this question. The subtlety is that the hazard rate $r_L(x)$ decreases with v for small x and increases with v for large x. See Figure 4.7. Hence it is not possible to guarantee $r_L \le r_{L'}$ with (4.7.1). Further research may identify the $\boldsymbol{v}_0$ or prove that such $\boldsymbol{v}_0$ does not exist for general graphs.

Second, we still do not know whether the condition $r_L(u+x) \le r_{L'}(x)$, $u, x \ge 0$ in Corollary 4.1.7 (a) is necessary for $\mathrm{RP}(L) \preceq \mathrm{RP}(L')$. In Section 4.3 we stop

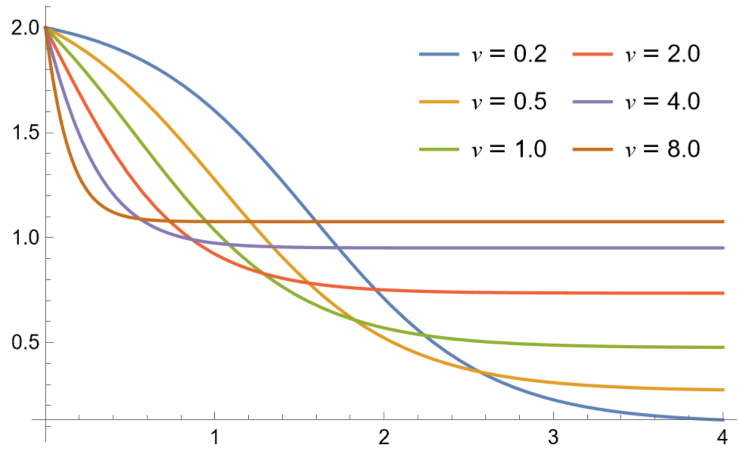

Figure 4.7: *Hazard rates r_L with $\lambda = 2$, $p = 0.6$ and different ν.*

our investigation with the necessary condition $L \succeq L'$ in Theorem 4.3.8. Future research could bridge the gap between the sufficient condition and the necessary condition. This would not only be an addition to renewal theory, but also would benefit the research on *contact processes under renewals* [38, 40, 48], whose point processes of infections and/or recoveries in the graphical representation are replaced with epoch sets of independent renewal processes. Although we only apply the coupling of renewal processes constructed in Section 4.3 to contact processes on dynamic bond percolation and contact processes in Broman's randomly evolving environment, it is clear that our method can be applied to contact processes under renewals, and any progress could prompt research on contact processes under renewals.

Finally, we are interested in coupling point processes embedded in hidden Markov chains. In Section 4.6 we assume that vertices never recover in the bad environment. However, in Broman's model, vertices recover at rate δ_0 in the bad environment, where δ_0 could be positive. The reason for our additional assumption is that, when $\delta_0 > 0$, the point processes of valid recoveries in the graphical representation is embedded in a general hidden Markov chain, which is beyond the range of Theorem 4.1.5. Similarly, the contact process on dynamic bond percolation can be generalised by setting the infection rate as λ_1 when the edge is in the good environment (state 1) and as λ_0 when the edge is in the bad environment (state 0). Then the point processes of valid infections are also embedded in hidden Markov chains. Hence, successful coupling of point processes embedded in hidden Markov chains will help us understand contact processes in these two random environments, and possibly other variants of contact processes.

Index

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Appendix A

Appendix

A.1 The Extinction Time of Contact Processes on Finite Graphs

In Table A.1, we list the results on the extinction time τ_n^A of contact processes with infection rate $\lambda > 0$ and initial configuration A on various finite graphs G_n; c and C denote constants. For the meaning of other symbols, please refer to the related section.

Table A.1: Asymptotic Behavior of the Extinction Time

Graph G_n	Parameters	Extinction Time as $n \to \infty$
$\{1,\dots,n\}^d$ (1.2)	$\lambda < \lambda_c(\mathbb{Z}^d)$	$\tau_n^{[1]}/\log n \xrightarrow{\text{p}} d/\gamma_-(\lambda)$.
	$d=1, \lambda = \lambda_c(\mathbb{Z})$	$\mathbb{P}\left(an \le \tau_n^{[1]} \le bn^4\right) \to 1,\ a, b > 0$
	$\lambda > \lambda_c(\mathbb{Z}^d)$	$\mathbb{P}\left(n^{-d}\log \tau_n^{[1]} \ge \gamma\right) \to 0,$ $\tau_n^{[1]} \Big/ \mathbb{E}\left[\tau_n^{[1]}\right] \xrightarrow{\text{d}} \text{Exp}(1),$ $n^{-d}\log \mathbb{E}\left[\tau_n^{[1]}\right] \to c.$
$\mathbb{T}_n^d$ (1.3)	$\lambda < \lambda_2(\mathbb{T}^d)$	$n^{-1}\tau_n^{[1]} \xrightarrow{\text{p}} c.$
	$\lambda > \lambda_2(\mathbb{T}^d)$	$\mathbb{P}\left(\tau_n^{[1]} \ge c\exp\left(\beta\left\lvert\mathbb{T}_n^d\right\rvert\right)\right) \to 1,$ $\left\lvert\mathbb{T}_n^d\right\rvert^{-1}\log \mathbb{E}\left[\tau_n^{[1]}\right] \to c,$ $\tau_n^{[1]}/\mathbb{E}\left[\tau_n^{[1]}\right] \xrightarrow{\text{d}} \text{Exp}(1).$
$G_n \in \Lambda(n,d)$ [a] (1.5)	$d \ge 2, \lambda > \lambda_c(\mathbb{Z})$	$\inf_{G_n \in \Lambda(n,d)} \mathbb{P}\left(\tau_n^{[1]} \ge \mathrm{e}^{cn}\right) \to 1.$

Continued on next page

Table A.1: Asymptotic Behavior of the Extinction Time (Continued)

$G_n \in \mathscr{T}(n,d)$ [b] (1.5)	$d \geq 2, \lambda > \lambda_c(\mathbb{Z})$	$\tau_n^{[1]}/\mathbb{E}\left[\tau_n^{[1]}\right] \xrightarrow{d} \text{Exp}(1).$
$G_n \in \mathscr{G}(n,d)$ [c] (1.5)	$\lambda < \lambda_1(\mathbb{T}^d)$	$\inf_{G_n \in \mathscr{G}(n,d)} \mathbb{P}\left(\tau_n^{[1]} < C\log n\right) \to 1.$
	$\lambda > \lambda_c(\mathbb{Z})$	$\inf_{G_n \in \mathscr{G}(n,d)} \mathbb{P}\left(\tau_n^{[1]} > ce^{cn}\right) \to 1.$
$\text{RR}(n,d)$ [d] (2.1)	$\lambda < \lambda_1(\mathbb{T}^{d-1})$	$\mathbb{P}\left(\tau_n^{[1]} < C\log n\right) \to 1.$
	$\lambda > \lambda_1(\mathbb{T}^{d-1})$	$\mathbb{P}\left(\tau_n^{[1]} > ce^{cn}\right) \to 1.$
$\text{CMe}(n,\boldsymbol{p})$ [e] (2.2)	$\lambda < \underline{\lambda}(\boldsymbol{p})$	$\mathbb{P}\left(\tau_n^{[1]} \leq n^{1+\varepsilon}\right) \to 1.$
	$\lambda > \overline{\lambda}(\boldsymbol{p})$	$\mathbb{P}\left(\exp(cn) \leq \tau_n^{[1]} \leq \exp(Cn)\right) \to 1.$
$\text{CMs}(n,\boldsymbol{p})$ [f] (2.2)	$\lambda > 0$	$\mathbb{P}\left(\exp(cn) \leq \tau_n^{[1]} \leq \exp(Cn)\right) \to 1.$
$\text{CMpl}(n,\boldsymbol{p})$ [g] (2.2)	$p_0 + p_1 + p_2 = 0$	$\mathbb{P}\left(\tau_n^{[1]} \geq e^{cn}\right) \to 1.$
$\text{PA}(n;m,\alpha)$ [h] (2.4)	$m \in \mathbb{N}_1, \alpha \in [0,1)$	$\tau_n^{[1]}/\mathbb{E}\left[\tau_n^{[1]}\right] \xrightarrow{d} \text{Exp}(1).$

[a] $\Lambda(n,d)$ is the set of trees with n vertices and degree bounded by d.
[b] $\mathscr{T}(n,d)$ is the set of graphs with a spanning tree in $\Lambda(n,d)$.
[c] $\mathscr{G}(n,d)$ is the set of connected graphs with n vertices and degree bounded by $d+1$.
[d] $\text{RR}(n,d)$ is the random d-regular graph with n vertices.
[e] $\text{CMe}(n,\boldsymbol{p})$ is the configuration model with n vertices and degree distribution $\boldsymbol{p}$, where $\boldsymbol{p}$ has an exponenial tail, $\mathbb{E}_{\boldsymbol{p}}\left[D^2\right] < +\infty$ and $\mathbb{E}_{\boldsymbol{p}}[D(D-2)] > 0$.
[f] $\text{CMs}(n,\boldsymbol{p})$ is the configuration model with n vertices and degree distribution $\boldsymbol{p}$, where $\boldsymbol{p}$ does not have an exponenial tail, $\mathbb{E}_{\boldsymbol{p}}\left[D^2\right] < +\infty$ and $\mathbb{E}_{\boldsymbol{p}}[D(D-2)] > 0$.
[g] $\text{CMpl}(n,\boldsymbol{p})$ is the configuration model with n vertices and power-law degree distribution $\boldsymbol{p}$, whose exponent is greater than 2.
[h] $\text{PA}(n;m,\alpha)$ is the preferential attachment graph with n vertices and parameters m and α.

A.2 Mathematica Code for Plotting Hazard Rates

In this section we provide the Wolfram Mathematica (version 13.1.0) code that we use to generate Figures 4.3, 4.4 and 4.7. The first part provides the functions given in (4.1.6) to (4.1.9). The second part checks the condition (4.1.12) and plots Figure 4.3 in Example 4.5.2, while the third part checks the condition (4.1.13) and plots Figure 4.4 in Example 4.5.3. The last part plots Figure 4.7.

```
z1[lambda_,p_,v_]:=(lambda+v-Sqrt[(lambda+v)^2-4*
   lambda*p*v])/2;
z2[lambda_,p_,v_]:=(lambda+v+Sqrt[(lambda+v)^2-4*
   lambda*p*v])/2;
a1[lambda_,p_,v_]:=lambda*(p*v-z1[lambda,p,v])/(z2[
   lambda,p,v]-z1[lambda,p,v]);
a2[lambda_,p_,v_]:=lambda*(p*v-z2[lambda,p,v])/(z1[
   lambda,p,v]-z2[lambda,p,v]);
rL[x_,lambda_,p_,v_]:=(a1[lambda,p,v]*Exp[-z1[lambda
   ,p,v]*x]+a2[lambda,p,v]*Exp[-z2[lambda,p,v]*x])/(
   a1[lambda,p,v]*Exp[-z1[lambda,p,v]*x]/z1[lambda,p
   ,v]+a2[lambda,p,v]*Exp[-z2[lambda,p,v]*x]/z2[
   lambda,p,v]);
rD[x_,lambda_,p_,v_]:=(Exp[-z1[lambda,p,v]*x]-Exp[-
   z2[lambda,p,v]*x])/(Exp[-z1[lambda,p,v]*x]/z1[
   lambda,p,v]-Exp[-z2[lambda,p,v]*x]/z2[lambda,p,v
   ]);

lambda1=1;p1=0.6;v1=3.65;
lambda2=1.7;p2=0.5;v2=2;
lambda1+lambda2+v2-v1-2*z1[0,lambda2,p2,v2]
lambda1*v1*(1-p1)-lambda2*v2*(1-p2)
z1[0,lambda1,p1,v1]-z1[0,lambda2,p2,v2]
Plot[{rL[x,lambda1,p1,v1],lambda1,z1[lambda1,p1,v1],
   rL[x,lambda2,p2,v2]},{x,0,3.5},PlotLegends->
   Placed[{"Hazard rate of L with \[Lambda]==1,p
   ==0.6,v==3.65","Hazard rate of Exp(Subscript[\[
   Lambda], min](0,1,3.65,0.6))","Hazard rate of Exp
   (Subscript[\[Lambda], max](0,1,3.65,0.6))","
   Hazard rate of L with \[Lambda]==1.7,p==0.5,v
   ==2"},Scaled[{0.61, 0.75}]], PlotRange->All]

lambda3=1.7;p3=0.58;v3=2.3;
```

```
lambda1+lambda3+v3-v1-2*z1[lambda3,p3,v3]
lambda1*v1*(1-p1)-lambda3*v3*(1-p3)
lambda1*v1*p1-lambda3*v3*p3
lambda3+v3-lambda1-v1
Plot[{rD[x,lambda1,p1,v1],rD[x,lambda3,p3,v3],rL[x,
   lambda1,p1,v1],rL[x,lambda3,p3,v3]},{x,0,3},
   PlotLegends->Placed[LineLegend[{"Subscript[r, D]
   with \[Lambda]==1,p==0.6,v==3.65","Subscript[r, D
   ] with \[Lambda]==1.7,p==0.58,v==2.3","Subscript[
   r, L] with \[Lambda]==1,p==0.6,v==3.65","
   Subscript[r, L] with \[Lambda]==1.7,p==0.58,v
   ==2.3"}], Scaled[{0.68,0.75}]],PlotRange->All]

lambda4=2;p4=0.6;
v4=0.2;v5=0.5;v6=1.0;v7=2.0;v8=4.0;v9=8.0;
Plot[{rL[x,lambda4,p4,v4],rL[x,lambda4,p4,v5],rL[x,
   lambda4,p4,v6],rL[x,lambda4,p4,v7],rL[x,lambda4,
   p4,v8],rL[x,lambda4,p4,v9]},{x,0,4},PlotLegends->
   Placed[LineLegend[{"v==0.2","v==0.5","v==1.0","v
   ==2.0","v==4.0","v==8.0"},LegendLayout->{"Column
   ",2}],Scaled[{0.75,0.78}]],PlotRange->All]
```