Exploring Topological Defects through Differential Geometry and Homotopy Theory

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Abstract

This thesis investigates the mathematical and physical foundations of topological defects. We first introduce the mathematical background, which consists of the theory of Lie groups and their corresponding Lie algebras, and fibre bundles, principal bundles and connections on principal bundles. We also give an introduction to classical field theory, and present the Lagrangian formalism for fields and Yang-Mills theory. We cover spontaneous symmetry breaking, and we explain how this can lead to topological defects using the Kibble mechanism. Finally, we classify topological defects using homotopy groups, for which we develop the underlying framework.

Contents

Chapter 1 | Introduction

Physics is all about describing nature. To do so, physicists formulate models to represent natural phenomena, which often take the form of a set of partial differential equations. By imposing appropriate boundary conditions, the equations help us understand the evolution of physical systems. A key focus is identifying stable solutions, which remain largely unchanged under small perturbations of the initial conditions.

In the early 19^{th} century, the British engineer John Scott Russell observed the phenomenon of solitary waves, stable wavefronts moving through shallow water with a constant velocity, without losing their shape. By the end of the same century, Dutch physicists Korteweg and de Vries discovered a partial differential equation – now called the Korteweg-de Vries equation –, which describes Russell's solitary waves. Their equation allowed for solitary wave solutions. In general, solutions to partial differential equations which are stable localised waves are called solutions.

Midway the 20^{th} century, similar stability phenomena were predicted in solid-state physics and condensed matter physics, in the form of stable vortices in type II superconductors and superfluids. Later, these were also observed. In this case however, stable solutions derived their stability from topological obstructions, preventing the solutions to decay to a lowest energy state. These solutions can be formed during phase transitions and are called *topological defects*. During this time, physicists started to look for topological defects in other places, such as cosmology, particle physics and quantum field theory. The understanding of topological defects provided all kinds of insights about physics, ranging from large-scale structures in the Universe to elementary particle interactions.

In this thesis, we explore topological defects through differential geometry, algebraic topology and classical field theory. Field theory describes physical fields – assignments of physical quantities to every point in time and space – and their evolution. As topological defects are fields themselves, field theory is foundational for their study. However, additional mathematical tools are necessary to understand topological defects.

Topological defects are characterised by their inability to be continuously deformed into a vacuum state – a configuration of the system with the lowest energy. Mathematically, the concept of continuous deformations are captured by homotopies. Related to homotopies are homotopy groups, which are algebraic invariants of topological spaces. Studying the homotopy groups of the vacuum manifold – the set of vacuum states, which can be identified with a smooth manifold under some conditions – gives insight into the variety of possible topological defects.

Beyond algebraic topology, we are also interested in how topological defects are formed in the first place. An important mechanism explaining the formation of topological defects is spontaneous symmetry breaking. For studying symmetry breaking, some Yang-Mills theory is needed, which relies on Lie groups and principal bundles – topics rooted in differential geometry. Thus, topological defects are of great interest to both mathematically inclined physicists and topologists interested in physical applications. The aim of this thesis is to illustrate to both the elegance of topological defects and the underlying theory, from a mathematical perspective.

In order to describe topological defects and symmetry breaking, we establish a mathematical framework for formulating symmetries through Lie groups and Lie algebras. Lie groups are groups which have also a smooth structure; they capture the essence of smooth symmetries. To each Lie group, a Lie algebra is associated, which describe infinitesimal transformations related to the symmetries. Lie groups and their Lie algebras are discussed in chapter 2.

This lays the groundwork for understanding the symmetries of fields. Fields show up quite often in physics, so we must develop some classical field theory. The basis for field theory is the Lagrangian density, which is a functional depending on one or more fields, their derivatives with respect to space and time coordinates, and possibly absolute space and time coordinates. The Euler-Lagrange equations then give differential equations describing the evolution of the fields.

Often a field theory is much easier to describe using some redundancy, by representing a physical state by multiple field configurations. This allows local transformations of the field, called gauge transformations, which keep the Lagrangian density invariant. Yang-Mills theory gives a way to construct such invariant Lagrangian densities. Geometrically, Yang-Mills theory is described using fibre bundles, principal bundles and Ehresmann connections, which are all covered in chapter 3. The Lagrangian formalism and Yang-Mills theory are presented in chapter 4.

With this foundation of Lie groups and field theory, we are ready for symmetry breaking and topological defects. We introduce homotopy groups in chapter 5, which is the main tool for studying and characterising topological defects. These serve as higher-dimensional counterparts to the fundamental group. We also give some theorems and procedures to calculate homotopy groups of topological spaces, such as the long exact sequence of homotopy groups related to fibre bundles. On the way, we meet some intriguing algebraic topology when we relate homotopy groups of the physically relevant compact Lie groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ to the homotopy groups of higher dimensional spheres. These homotopy groups are only partly known – computing these groups is one of the major unsolved problems in algebraic topology.

Afterwards, we are finally ready to delve into the fascinating realm of topological defects in chapter 6. We first look at the spontaneous breaking of symmetries. These symmetries can either be global symmetries or gauge symmetries, which are local. The latter is the basis for the Higgs mechanism, which is a mechanism for mass generation of fields, which we shortly touch upon. We will also relate the vacuum manifold to the symmetry group – describing the symmetry of the system – and the stabiliser of a vacuum state under the symmetry group.

Then the Kibble mechanism explains how symmetry breaking happens in physics, and how it can lead to topological defects. We end the chapter with characterising different types of topological defects using homotopy groups. More exactly, topological defects are possible if the n-th homotopy group of the vacuum manifold is non-trivial, for an $n \in \{0, 1, 2, 3\}$. We also state some implications of their (non)-existence, by means of considering a Grand Unified theory, which tries to unify three of the four fundamental forces: electromagnetism, the weak interaction and the strong interaction.

At last, chapter 7 will be an outlook for further research directions and has some concluding remarks.

Prerequisites

We shall assume the reader is familiar with the basics of algebraic topology (most notably homotopies, fundamental groups and covering maps) and differentiable geometry (smooth manifolds, tangent spaces and the differential of a smooth map, smooth vector and tensor fields, differential forms). While this text occasionally references singular homology and de Rham cohomology, a background in these topics is not required for understanding the main content. The first chapter of *Algebraic Topology* of Hatcher [1] is an excellent reference for an introduction to algebraic topology. For smooth manifolds we recommend Lee's *Introduction to Smooth Manifolds* [2]. The relevant chapters are 1–4 and 8–14.

Extensive background knowledge of physics is not necessarily required, but some understanding of classical mechanics (the Lagrangian formalism, the Legendre transforms), introductory quantum mechanics (wave functions, wave-particle duality, spin), special relativity, and electromagnetism (Maxwell's equations, the potential formulation) could be advantageous for the physical context of chapters 4 and 6. For the keen reader, *Introduction to Electrodynamics* by Griffiths [3] is a recommendation. Chapter 12 of that book gives also a nice preliminary to special relativity, and its relation to electrodynamics. For classical mechanics, Goldstein's *Classical Mechanics* [4] is a classic.

Conventions and notation

Lastly we would like to state some conventions and notation used in this thesis. A smooth manifold M is always assumed to be Hausdorff and second-countable. Smooth in all contexts is understood as $\mathcal{C}^\infty$, i.e. continuous partial derivatives of all orders in the real sense. By $\mathcal{C}^\infty(M)$ we denote the ring of smooth functions $f : M \to \mathbb{R}$. For another smooth manifold N, $\mathcal{C}^\infty(M, N)$ is the set of smooth functions $F : M \to N$. For a point $p \in M$, its tangent space at p is denoted by T_pM, and the tangent bundle by TM. The cotangent space and cotangent bundle are denoted by T_p^*M and T^*M respectively. For a smooth map $F : M \to N$ between smooth manifolds, $\mathrm{d}F : TM \to TN$ is its differential, which at p is symbolised by $\mathrm{d}F_p : T_pM \to T_{F(p)}N$. For legibility, the subscript p in $\mathrm{d}F_p$ is sometimes omitted when it is clear the differential map is considered at p. The $\mathcal{C}^\infty(M)$-module of smooth vector fields $X : M \to TM$ on M is denoted by $\mathfrak{X}(M)$, and X evaluated at a point p is indicated by X_p. At times, X is considered as a derivation $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$. In that case we write Xf or $X(f)$ for the evaluation of a vector field X at the smooth function $f \in \mathcal{C}^\infty(M)$. Likewise, the $\mathcal{C}^\infty(M)$-module of smooth differential k-forms $\omega : M \to \bigwedge^k(T^*M)$ on M is written as $\Omega^k(M)$. ω_p is the evaluation of ω at a point $p \in M$. The pullback of k-forms by F is written as $F^* : \Omega^k(N) \to \Omega^k(M)$.

Throughout this thesis, $\overline{z}$ denotes the complex conjugate of a complex number $z \in \mathbb{C}$, and $A^* = \overline{A}^\top$ the adjoint of a matrix $A \in \mathrm{Mat}(m \times n, \mathbb{C})$. Sesquilinear maps are assumed to be conjugate-linear in the first argument and linear in the second argument. The cyclic group of n elements is denoted $\mathbb{Z}_n$.

Chapter 2 | Lie groups and Lie algebras

Physical systems are often studied by looking at their symmetries. These could be very apparent and geometric, such as translational and rotational symmetry, but also abstract[1], such as invariance under a constant rotation of the wave function $\psi \to e^{i\alpha}\psi$ in quantum mechanics, for $\alpha \in \mathbb{R}$, or the electric field in electrostatics stays the same after its potential transform as $V \to V + V_0$ for $V_0 \in \mathbb{R}$. Mathematically, symmetries are described using groups, but we also want to capture the notion of smoothness. For instance, in the case of circular symmetry, we can smoothly vary a rotation angle. Lie groups combine these two concepts.

Furthermore, we examine vector fields on Lie groups that remain invariant under the group operation of the Lie group. These constitute an interesting algebraic structure, called a Lie algebra, which is intimately linked to the tangent space at the identity of the Lie group. The exponential map allows us to go back from the Lie algebra to the Lie group, encapsulating the notion of infinitesimal transformations. Through the exponential map, we can identify subgroups H of a Lie group G as Lie subgroups, where the Lie algebra of H forms a Lie subalgebra of the Lie algebra of G. As an application we identify frequently encountered Lie groups in mathematics and physics, and we determine their corresponding Lie algebras.

Lastly, we explore some representations of Lie groups and Lie algebras. In particular, we discuss the adjoint representation, which is commonly used in the study of principal bundles and in gauge theory.

This treatise on Lie groups is based on chapter 3 of *Foundations of Differentiable Manifolds and Lie Groups* [5], mostly the first 3 sections and the sections "Exponential Map" and "The Adjoint Representation". In the following chapter, the field $\mathbb{F}$ can either mean $\mathbb{R}$ or $\mathbb{C}$.

2.1 Lie groups

Definition (Lie group). A (real[2]) Lie group is a smooth manifold G with a group structure, such that the map $G \times G \to G$ given by $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ is smooth.

Note that the last condition is equivalent with the maps $(\sigma, \tau) \mapsto \sigma\tau$ and $\sigma \mapsto \sigma^{-1}$ being smooth. Frequently, we shall be using the symbol e for the identity of the Lie group. Before we continue, here are a couple of Lie groups:

[1] These are certain instances of gauge symmetries, which we touch upon in sections 4.3 and 4.4.

[2] There are also complex Lie groups, which are modelled after complex manifolds and the group operations then must be holomorphic maps. In this thesis we only consider real Lie groups, so any reference to Lie groups should be understood as referring to real Lie groups.

- $\mathbb{R}^n$ and $\mathbb{C}^n$ with vector addition;
- $\mathbb{R}^*$, $\mathbb{R}_{>0}$, $\mathbb{C}^*$ and $S^1 \subseteq \mathbb{C}^*$ with multiplication;
- the general linear group $\mathrm{GL}_n(\mathbb{F})$, consisting of the invertible $n \times n$ matrices over $\mathbb{F}$ with matrix multiplication;
- closed subgroups of $\mathrm{GL}_n(\mathbb{F})$, which are called matrix groups, such as $\mathrm{SL}_n(\mathbb{F})$, $\mathrm{O}(n)$ and $\mathrm{U}(n)$; we discuss these in section 2.3;
- the $\mathbb{F}$-linear automorphisms of a finite-dimensional $\mathbb{F}$-vector space V, denoted by $\mathrm{Aut}(V)$, with composition as group operation;
- for G and H Lie groups, their product manifold $G \times H$ can be given a Lie group structure by taking the direct product of groups;
- a group G with countably many elements, with the discrete topology and the structure of a 0-dimensional smooth manifold.

Definition (Lie group homomorphism). A map $\varphi : G \to H$ between Lie groups is called a Lie group homomorphism if it is smooth and a group homomorphism. If φ is also bijective and its inverse is also a Lie group homomorphism, then φ is a Lie group isomorphism.

Note that the composition of smooth maps is again smooth, as is the identity map of a Lie group, so Lie groups with Lie group homomorphisms form a category, denoted by **LieGrp**.

The determinant map $\det : \mathrm{GL}_n(\mathbb{F}) \to \mathbb{F}^*$ is an example of a Lie group homomorphism. The map $\psi : \mathbb{R}_{>0} \times S^1 \to \mathbb{C}^*$, given by $(r, z) \mapsto rz$, is an instance of a Lie group isomorphism.

Definition (Lie subgroup). Let G be a Lie group. A Lie subgroup is a Lie group H alongside with a smooth immersion $i : H \to G$ (a smooth injective map such that $\mathrm{d}i_\sigma : T_\sigma H \to T_{i(\sigma)}G$ is injective for all $\sigma \in H$), that is also a group homomorphism. A Lie subgroup H is called a closed subgroup if $\operatorname{im} i$ is closed in G.

Oftentimes, H is considered as a subset of G, i.e. $i : H \to G$ is the inclusion map. Beware that in general, H does not carry the subspace topology of G. A special type of Lie subgroup is the connected component of the identity (called the identity component) of a Lie group:

Proposition 2.1. *Let G be a Lie group and $H \subseteq G$ the connected component of the identity element e. Then H is a Lie subgroup of G. Moreover, all connected components are diffeomorphic. The set of connected components can be identified with the Lie group G/H.*

Proof. Since G is locally path-connected, every point in H has an open neighbourhood in G that is path-connected (and which is thus also contained in H), hence H is open in G. This makes H into a smooth manifold, which is also path-connected. It then suffices to prove that the map $\psi : G \times G \to G$, given by $(\sigma, \tau) \mapsto \sigma\tau^{-1}$, is closed under H, since then H will be a Lie group. The inclusion map is then automatically a smooth immersion and a group homomorphism, and this implies that H is a Lie subgroup. Let $\sigma, \tau \in H$. H is path-connected, so there exists a path $\gamma : [0, 1] \to H$, such that $\gamma(0) = e$ and $\gamma(1) = \tau$. Then $\tilde{\gamma} : [0, 1] \to H$, defined by $\tilde{\gamma}(t) = \psi(\sigma, \gamma(t))$, is a path from $\tilde{\gamma}(0) = \sigma$ to $\tilde{\gamma}(1) = \sigma\tau^{-1}$. Thus $\sigma\tau^{-1} \in H$ holds indeed.

Let $\tilde{H}$ be a connected component of G, and choose a $\tau \in \tilde{H}$. Then $\varphi : H \to \tilde{H}$, given by $\varphi(\sigma) = \tau\sigma$, is a diffeomorphism. In particular this means that $\tilde{H} = \tau H$, proving the correspondence with the set of path-components and G/H. Since $c_\tau : H \to \tau H \tau^{-1}$ for $\tau \in G$, given by $\sigma \mapsto \tau\sigma\tau^{-1}$, is continuous and surjective, $\tau H \tau^{-1}$ is path-connected. Since $e \in \tau H \tau^{-1}$, we have $\tau H \tau^{-1} = H$ and thus $H \triangleleft G$. G/H therefore has a group structure. Every manifold has countably many path-components, so G/H is countable and can be given a Lie group structure with the discrete topology. ■

2.2 Lie algebras

Call to mind that for a smooth manifold M and smooth vector fields $X, Y \in \mathfrak{X}(M)$, the Lie bracket $[X,Y]$ defines a new smooth vector field on M, defined by $[X,Y](f) = X(Yf) - Y(Xf)$ for all $f \in \mathcal{C}^\infty(M)$. Additionally, the Lie bracket is $\mathbb{R}$-bilinear, anti-symmetric and satisfies the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

It happens that these algebraic properties are interesting in its own right. These rules form an algebraic structure, which is called a Lie algebra:

Definition (Lie algebra). A (real[3]) **Lie algebra** is a real vector space $\mathfrak{g}$ with a bilinear operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (an $\mathbb{R}$-algebra) with the following two additional properties for all $x, y, z \in \mathfrak{g}$:

- $[x,y] = -[y,x]$ (anti-commutativity);
- $[[x,y],z] + [y,z],x] + [z,x],y] = 0$ (Jacobi identity).

$[\cdot,\cdot]$ is called the **Lie bracket** of the Lie algebra. A linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ that is closed under $[\cdot,\cdot]$ is called a **Lie subalgebra**. A map between Lie algebras is a **Lie algebra homomorphism** if it is $\mathbb{R}$-linear and preserves the Lie bracket.

It is convention to denote Lie algebras with fraktur letters, such as $\mathfrak{g}$. Other examples of Lie algebras include:

- for any $\mathbb{R}$-vector space $\mathfrak{g}$, define $[x,y] = 0$ for all $x, y \in \mathfrak{g}$;
- the real vector space $\mathrm{Mat}_n(\mathbb{F})$, with the commutator $[A,B] = AB - BA$;
- for V an $\mathbb{F}$-vector space, the vector space $\mathrm{End}(V)$ of $\mathbb{F}$-linear endomorphisms $f : V \to V$, with $[f,g] = f \circ g - g \circ f$;
- $\mathbb{R}^3$ with the cross product;
- $\mathbb{R}^2$ with a basis (x,y), such that $[x,y] = y$.

Let G be a Lie group. Then we shall see that there is a Lie algebra $\mathfrak{g}$ associated to it, isomorphic as a vector space to T_eG. We use the map $l_\sigma : G \to G$ as shorthand for the left multiplication $\tau \mapsto \sigma\tau$, for a given element $\sigma \in G$. Likewise $r_\sigma : G \to G$ denotes right multiplication.

We consider maps $X : G \to TG$ (which need not even be continuous) such that $\pi \circ X = \mathrm{id}_G$, for $\pi : TG \to G$ the natural projection map. We refer to such maps as rough vector fields, since if X is smooth, then X is a smooth vector field.

[3] Again, it makes sense to consider Lie algebras over any field, but we only consider $\mathbb{R}$ here.

Definition (Left-invariant vector field). Let $X : G \to TG$ be a rough vector field. X is called **left-invariant** if for all $\sigma \in G$, the following condition holds:

$$\mathrm{d}l_\sigma \circ X = X \circ l_\sigma.$$

We denote the set of all left-invariant vector fields by $\mathfrak{g}$.

Note that a left-invariant vector field X is completely determined by X_e, for

$$X_\sigma = (X \circ l_\sigma)_e = \mathrm{d}l_\sigma(X_e), \quad \sigma \in G.$$

Conversely, a rough vector field with this property is left-invariant, since for all $\sigma, \tau \in G$, we have

$$(\mathrm{d}l_\sigma \circ X)_\tau = \mathrm{d}l_\sigma(X_\tau) = \mathrm{d}l_\sigma \circ \mathrm{d}l_\tau(X_e) = \mathrm{d}l_{\sigma\tau}(X_e) = X_{\sigma\tau} = (X \circ l_\sigma)_\tau.$$

This gives an identification between $\mathfrak{g}$ and T_eG. But a lot more is true:

Theorem 2.2. *Let G be a Lie group, and $\mathfrak{g}$ the set of left-invariant vector fields.*

(a) *$\mathfrak{g}$ is an $\mathbb{R}$-vector space of dimension $\dim G$, which is isomorphic to T_eG via $X \mapsto X_e$;*

(b) *every left-invariant rough vector field is automatically smooth;*

(c) *$\mathfrak{g}$ is closed under the Lie-bracket for vector fields and thus forms a Lie algebra.*

Proof. See proposition 3.7 in [5]. ∎

$\mathfrak{g}$ is called the **Lie algebra** of the Lie group G. Per convention, the Lie algebra of a general Lie group is written as the corresponding symbols in lower-case fraktur. Since $\mathfrak{g}$ is a finite dimensional vector space, $\mathfrak{g}$ can be given a smooth manifold structure by identifying it with $\mathbb{R}^n$ by a choice of basis, for $n = \dim \mathfrak{g}$. The particular choice of a particular basis does not matter. So at times, $\mathfrak{g}$ is considered as a smooth manifold.

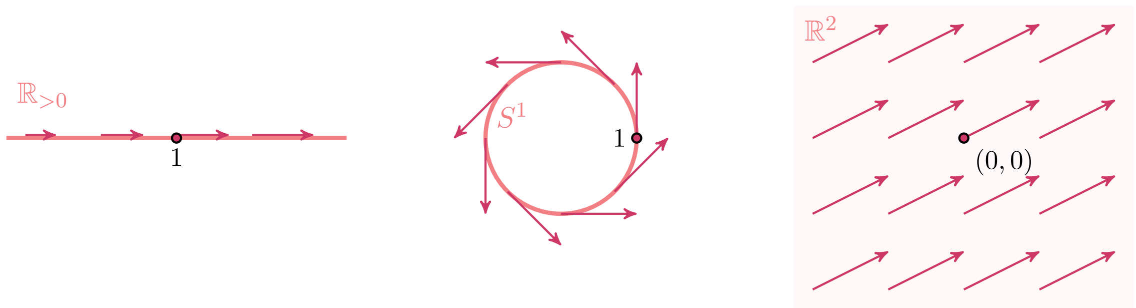


Fig. 2.1: Examples of left-invariant vector fields for the Lie groups $\mathbb{R}_{>0}$, S^1 and $\mathbb{R}^2$. All three are embedded in $\mathbb{R}^2$, with standard coordinates x and y. The sketched left-invariant vector fields are $X = x\frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $Z = 2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ respectively. These vector fields can be found by starting with an $X_e \in T_eG$, and calculating $X_\sigma = \mathrm{d}l_\sigma(X_e)$.

For our study, one of the most important class of Lie algebras are Lie algebras of $\mathrm{Aut}(V)$, for V a finite-dimensional real or complex vector space. Often we identify $\mathrm{Aut}(V)$ with $\mathrm{GL}_n(\mathbb{F})$ for $n = \dim_{\mathbb{F}} V$, by choosing a basis for V.

Proposition 2.3. *The Lie algebra of the Lie group* $\mathrm{GL}_n(\mathbb{F})$ *is isomorphic to* $\mathfrak{gl}_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$ *with* $[A, B] = AB - BA$. *Likewise* $\mathrm{Aut}(V)$ *for* V *a finite-dimensional* $\mathbb{F}$*-vector space has Lie algebra isomorphic to* $\mathrm{End}(V)$ *with* $[f, g] = f \circ g - g \circ f$.

Proof. For $\mathrm{GL}_n(\mathbb{R})$, a left-invariant vector field X can be identified with X_e, which can be written as $X_e = \sum_{ij} a^{ij} \frac{\partial}{\partial x^{ij}}\big|_e$, for x^{ij} the coordinate function for the i, j-entry of a matrix in $\mathrm{GL}_n(\mathbb{R})$. Then $A = (a^{ij})_{ij}$ is the corresponding element in $\mathfrak{gl}_n(\mathbb{F})$. For the calculation of the Lie-bracket, see 3.10.b of [5]. The derivations of the Lie algebras of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{Aut}(V)$ work similar. ■

We have seen so far that to a Lie group G, we can assign a Lie Algebra $\mathfrak{g}$. Let $\varphi : G \to H$ be a Lie group homomorphism. Then the differential induces a map $\mathrm{d}\varphi_e : T_eG \to T_eH$. For $X \in \mathfrak{g}$, by considering the left-invariant vector field corresponding to $\mathrm{d}\varphi_e(X_e)$, we get a map

$$\mathrm{d}\varphi : \mathfrak{g} \to \mathfrak{h}.$$

A natural question to ask is whether φ induces a Lie algebra homomorphism between $\mathfrak{g}$ and $\mathfrak{h}$. This is indeed the case:

Proposition 2.4. *The assignment* $\mathrm{Lie} : \mathbf{LieGrp} \to \mathbf{LieAlg}_{\mathbb{R}}$ *of a Lie group to its Lie algebra and a Lie group homomorphism to its differential at the identity is a functor.*

Proof. Let $\varphi : G \to H$ be a Lie group homomorphism between Lie groups. This induces a linear map $\mathrm{d}\varphi_e : T_eG \to T_eH$. By $\mathfrak{g} \cong T_eG$ and $\mathfrak{h} \cong T_eH$, this gives us a linear map $\mathrm{d}\varphi : \mathfrak{g} \to \mathfrak{h}$. Let $X, Y \in \mathfrak{g}$ and take $\tilde{X} = \mathrm{d}\varphi(X)$, $\tilde{Y} = \mathrm{d}\varphi(Y)$. We want to show that $\mathrm{d}\varphi([X, Y]) = [\tilde{X}, \tilde{Y}]$; then $\mathrm{d}\varphi$ is a Lie algebra homomorphism. What follows is writing out a lot of definitions.

First note that for all $\sigma \in G$,

$$\begin{aligned}\mathrm{d}\varphi(X_\sigma) &= \mathrm{d}\varphi \circ \mathrm{d}l_\sigma(X_e) = \mathrm{d}(\varphi \circ l_\sigma)(X_e) = \mathrm{d}(l_{\varphi(\sigma)} \circ \varphi)(X_e) \\ &= \mathrm{d}l_{\varphi(\sigma)} \circ \mathrm{d}\varphi(X_e) = \mathrm{d}l_{\varphi(\sigma)}(\tilde{X}_e) = \tilde{X}_{\varphi(\sigma)}.\end{aligned}$$

Here we have used that $\varphi \circ l_\sigma(\tau) = \varphi(\sigma\tau) = \varphi(\sigma)\varphi(\tau) = l_{\varphi(\sigma)} \circ \varphi(\tau)$. The same relation holds obviously for Y and $\tilde{Y}$. By using the above statement, we have for all $f \in \mathcal{C}^\infty(H)$ that

$$\begin{aligned}[\tilde{X}, \tilde{Y}]_e(f) &= \mathrm{d}\varphi(X_e)(\tilde{Y} f) - \mathrm{d}\varphi(Y_e)(\tilde{X} f) \\ &= X_e(\tilde{Y}(f) \circ \varphi) - Y_e(\tilde{X}(f) \circ \varphi) \\ &= X_e((\tilde{Y} \circ \varphi)(f)) - Y_e((\tilde{X} \circ \varphi)(f)) \\ &= X_e((\mathrm{d}\varphi \circ Y)(f)) - Y_e((\mathrm{d}\varphi \circ X)(f)) \\ &= X_e(Y(f \circ \varphi)) - Y_e(X(f \circ \varphi)) \\ &= \mathrm{d}\varphi([X, Y]_e)(f),\end{aligned}$$

proving $\mathrm{d}\varphi([X, Y]) = [\tilde{X}, \tilde{Y}]$. By properties of the differential of a smooth map, we have $\mathrm{d}(\mathrm{id}_G) = \mathrm{id}_{\mathfrak{g}}$ and for $\psi : H \to K$ a Lie group homomorphism, $\mathrm{d}(\psi \circ \varphi) = \mathrm{d}\psi \circ \mathrm{d}\varphi$, concluding the statement. ■

2.3 The exponential map

We have ended the previous section with the fact that a Lie group homomorphism $\varphi : G \to H$ induces a Lie algebra homomorphism $\mathrm{d}\varphi : \mathfrak{g} \to \mathfrak{h}$. But is the reverse also true? That is, given a

Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$, does there exists a Lie group homomorphism $\varphi : G \to H$ such that $\psi = \mathrm{d}\varphi$? Theorem 3.27 of [5] (whose proof goes beyond our scope) answers this question affirmatively, in the case of G being simply connected. φ is then even unique with this property. We will use this only for the additive Lie group $G = \mathbb{R}$, which is indeed simply connected.

Let G be a Lie group, and choose a left-invariant vector field $X \in \mathfrak{g}$. Then

$$\begin{aligned} \psi : \mathfrak{r} &\to \mathfrak{g}, \\ \lambda \frac{\partial}{\partial x} &\mapsto \lambda X \end{aligned}$$

defines a Lie algebra homomorphism, for $\mathfrak{r}$ the Lie algebra of $\mathbb{R}$. By theorem 3.27 in [5], there exists a unique Lie group homomorphism $\exp_X : \mathbb{R} \to G$, such that $\mathrm{d}(\exp_X)_0(\frac{\partial}{\partial x}) = X$. Generally, a Lie group homomorphism $\mathbb{R} \to G$ is called a **one-parameter subgroup**, so $\exp_X$ can also be described as the unique one-parameter subgroup, such that its tangent vector at 0 is X_e.

Definition (The exponential map). Let G be a Lie group. Then the **exponential map** is defined as the map $\exp : \mathfrak{g} \mapsto G$, given by $X \mapsto \exp_X(1)$.

For $G = \mathbb{R}_{>0}$ (embedded in $\mathbb{R}$), we choose a left-invariant vector field $X = \lambda x \frac{\partial}{\partial x}$, for a $\lambda \in \mathbb{R}$. Then we look for a Lie group homomorphism $\gamma : \mathbb{R} \to \mathbb{R}_{>0}$, such that $\gamma'(0) = \lambda$. Then

$$\gamma'(x) = \lim_{t \to 0} \frac{\gamma(x+t) - \gamma(x)}{t} = \gamma(x) \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} = \lambda \gamma(x).$$

Using $\gamma(0) = 1$, we find the solution to be $\gamma(x) = e^{\lambda x}$. This gives $\exp(X) = \gamma(1) = e^{\lambda}$, which is just the ordinary exponential map, explaining the nomenclature. A lot more properties carry over to the exponential map between the Lie algebra and the Lie group it comes from:

Proposition (Properties of the exponential map). *Let G be a Lie group and $X \in \mathfrak{g}$.*

(a) *for all $t \in \mathbb{R}$, we have $\exp(tX) = \exp_X(t)$;*

(b) *for all $s, t \in \mathbb{R}$, we have $\exp((s+t)X) = \exp(sX)\exp(tX)$ and $(\exp(tX))^{-1} = \exp(-tX)$;*

(c) *$l_\sigma \circ \exp_X : \mathbb{R} \to G$ is the unique integral curve of X starting at $\sigma \in G$;*

(d) *$\exp : \mathfrak{g} \to G$ is smooth and $\mathrm{d}(\exp)_0 : T_0\mathfrak{g} \to T_eG$ is the identity by the identifications $\mathfrak{g} \cong T_0\mathfrak{g} \cong T_eG$;*

(e) *for $\varphi : G \to H$ a Lie group homomorphism, the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{d}\varphi} & \mathfrak{h} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\varphi} & H \end{array}$$

Proof. We refer to the proof of theorems 3.31 and 3.32 in [5]. ■

The exponential map can thus also be understood as generating the trajectory along the left-invariant vector field X, starting at the identity e, in the direction of a chosen tangent vector $X_e \in T_eG$. In particular, this implies that left-invariant vector fields are complete.

In most practical cases, there is an easy way to evaluate the exponential map:

Example (The exponential map for $\mathrm{GL}_n(\mathbb{F})$). From analysis, we know already the matrix exponential, defined for $A \in \mathrm{Mat}_n(\mathbb{F})$ as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Some of its most important properties include:

- $\det e^A = e^{\operatorname{tr} A}$ for $A \in \mathrm{Mat}_n(\mathbb{F})$;
- the matrix exponential is a map $\mathrm{Mat}_n(\mathbb{F}) \to \mathrm{GL}_n(\mathbb{F})$;
- $e^{A+B} = e^A e^B$ for $A, B \in \mathrm{Mat}_n(\mathbb{F})$ such that $[A, B] = 0$.

The first point follows by writing A in its Jordan normal form (over $\mathbb{C}$). $\det e^A = e^{\operatorname{tr} A} \neq 0$ implies that e^A is also invertible, giving the second point. The third follows from rearranging the power series. For $A \in \mathrm{Mat}_n(\mathbb{F})$, we consider the path

$$\gamma_A : \mathbb{R} \to \mathrm{GL}_n(\mathbb{F}),$$
$$t \mapsto e^{tA}.$$

For all $s, t \in \mathbb{R}$, we have $[sA, tA] = 0$, so $\gamma_A(s + t) = \gamma_A(s)\gamma_A(t)$. Furthermore,

$$\gamma_A'(0) = \lim_{t \to 0} \frac{1}{t} \left(e^{tA} - I_n\right) = A + \lim_{t \to 0} t \sum_{k=0}^{\infty} \frac{A^{k+2}}{(k+2)!} t^k = A,$$

since the infinite sum is bounded in the operator norm by $\|A\|^2 e^{\|A\|}$ for $|t| \leq 1$. So γ_A is a one-parameter subgroup of $\mathrm{GL}_n(\mathbb{F})$, with tangent vector A (under identification). This one-parameter subgroup is unique, so $\gamma_A = \exp_A$, thus $\exp(A) = e^A$ holds in this case.

Point (d) of the above proposition implies that $\exp : \mathfrak{g} \to G$ is a diffeomorphism from a open neighbourhood of 0 in $\mathfrak{g}$ to an open neighbourhood of e in G, by the inverse function theorem. This allows us to prove that certain subgroups (in an algebraic sense, such subgroups we call abstract subgroups) of Lie groups are automatically a Lie subgroup, and immediately we know their Lie algebra as well:

Theorem 2.5. *Let G a Lie group, $A \subseteq G$ an abstract subgroup, and let $\mathfrak{a} \subseteq \mathfrak{g}$ a linear subspace. Let $U \subseteq \mathfrak{g}$ and $V \subseteq G$ be open neighbourhoods of 0 and the identity e respectively, such that $\exp|_U : U \to V$ is a diffeomorphism. If*

$$\exp(\mathfrak{a} \cap U) = A \cap V,$$

then A with the subspace topology has a Lie group structure, and A is a Lie subgroup of G. Furthermore, $\mathfrak{a}$ is the Lie algebra of A, which forms a Lie subalgebra of $\mathfrak{g}$.

An important result of theorem 2.5 is the closed-subgroup theorem:

Theorem 2.6. *Let G be a Lie group, and $A \subseteq G$ a closed abstract subgroup. Then A has a unique manifold structure such that A is a Lie subgroup of G.*

The proofs of these statements are quite technical, so we refer to the proof of theorems 3.20, 3.34 and 3.42 in [5].

Theorems 2.5 and 2.6 are really useful to establish common closed subgroups of $\mathrm{GL}_n(\mathbb{F})$ as Lie subgroups of $\mathrm{GL}_n(\mathbb{F})$. These closed subgroups are called matrix groups. Additionally, theorem 2.5 enables us to determine their Lie algebras. We give the results for subgroups preserving a sesquilinear form (which is just a bilinear form for $\mathbb{R}$-vector spaces) and the subgroup of matrix group, consisting of the matrices with determinant 1:

Proposition 2.7.

- *Let* $M \in \mathrm{GL}_n(\mathbb{F})$ *and let* $G = \{A \in \mathrm{GL}_n(\mathbb{F}) : A^*MA = M\}$ *be the set of linear transformations that preserve the non-degenerate sesquilinear form represented by* M*. Then* G *is a Lie subgroup of* $\mathrm{GL}_n(\mathbb{F})$*, with Lie algebra* $\mathfrak{g} = \{X \in \mathfrak{gl}_n(\mathbb{F}) : MX + X^*M = 0\}$.
- *Let* $G \subseteq \mathrm{GL}_n(\mathbb{F})$ *be a closed subgroup, and define* $SG = \{A \in G : \det A = 1\}$*. Then* SG *is a Lie subgroup of* $\mathrm{GL}_n(\mathbb{F})$*, with Lie algebra* $\mathfrak{sg} = \{X \in \mathfrak{g} : \operatorname{tr} X = 0\}$.

Proof. Let G and $\mathfrak{g}$ as in the first point for an $M \in \mathrm{GL}_n(\mathbb{F})$. It is easily checked that G is an abstract subgroup of $\mathrm{GL}_n(\mathbb{F})$, and that $\mathfrak{g}$ is a linear subspace of $\mathfrak{gl}_n(\mathbb{F})$. Let $U \subseteq \mathfrak{g}$, $V \subseteq \mathrm{GL}_n(\mathbb{F})$ be open neighbours of 0 and I_n respectively, such that $\exp|_U : U \to V$ is a diffeomorphism. Define

$$\tilde{U} = U \cap \{-MXM^{-1} : X \in U\} \cap \{X^* : X \in U\}.$$

$X \mapsto -MXM^{-1}$ and $X \mapsto X^*$ for $X \in \mathrm{Mat}_n(\mathbb{F})$ are $\mathbb{R}$-linear isomorphisms between finite dimensional vector spaces, hence homeomorphisms. This means that $\tilde{U}$ is an open neighbourhood of 0. Set $\tilde{V} = \exp \tilde{U}$ and let $X \in \tilde{U} \cap \mathfrak{g}$. Then by $X^*M = -MX$, we have

$$(e^X)^* M e^X = (e^X)^*(Me^XM^{-1})M = e^{X^*}e^{MXM^{-1}}M = e^{X^*}e^{-X^*}M = M,$$

thus we see that $\exp\,(\tilde{U} \cap \mathfrak{g}) \subseteq \tilde{V} \cap G$. Now let $X \in \tilde{U}$, such that $e^X \in \tilde{V} \cap G$. Then $(e^X)^*Me^X = M$, so

$$e^{X^*} = (e^X)^* = Me^{-X}M^{-1} = e^{-MXM^{-1}}.$$

Since $X^*, -MXM^{-1} \in \tilde{U}$, we can take inverses on both sides. This gives $X^* = -MXM^{-1}$, and thus $X \in \mathfrak{g}$, showing $\tilde{V} \cap G \subseteq \exp\,(\tilde{U} \cap \mathfrak{g})$. Applying theorem 2.5 then gives the first statement.

Now we let G, SG and $\mathfrak{sg}$ as in the second point. SG is an abstract subgroup of $\mathrm{GL}_n(\mathbb{F})$, and $\mathfrak{sg}$ a linear subspace of $\mathfrak{gl}_n(\mathbb{F})$. We define

$$U' = U \cap \{X \in \mathfrak{g} : |\operatorname{tr} X| < 2\pi\},$$

which is also an open neighbourhood of 0. Set $V' = \exp U'$. For $X \in U' \cap \mathfrak{sg}$, we have $\det e^X = e^{\operatorname{tr} X} = e^0 = 1$, so $e^X \in V' \cap SG$. Conversely, let $X \in U'$ such that $e^X \in V' \cap SG$. Now $1 = \det e^X = e^{\operatorname{tr} X}$, so $\operatorname{tr} X \in 2\pi i\mathbb{Z}$. By $X \in U'$, we actually have $\operatorname{tr} X = 0$, so $X \in U' \cap \mathfrak{g}$. The statement then follows by applying theorem 2.5 again. ■

Lie subgroups of $\mathrm{GL}_n(\mathbb{F})$ are also ubiquitous in physics. Proposition 2.7 shows that the following regular occurring groups are all in fact Lie groups:

Example 2.8. The special linear group $\mathrm{SL}_n(\mathbb{F})$ are the $n \times n$ matrices over $\mathbb{F}$ with determinant 1. Its Lie algebra $\mathfrak{sl}_n(\mathbb{F})$ consists of the traceless matrices in $\mathfrak{gl}_n(\mathbb{F})$.

Example 2.9. The orthogonal group consists of the linear transformations preserving lengths and angles, i.e. $A \in \mathrm{O}(n) \iff \forall v, w \in \mathbb{R}^n$ we have $\langle Av, Aw\rangle = \langle v, w\rangle$ for the Euclidean inner product. This requirement is equivalent with $A^\top A = I_n$. By proposition 2.7, its Lie algebra is $\mathfrak{o}(n) = \{X \in \mathfrak{gl}_n(\mathbb{R}) : X + X^\top = 0\}$, which are the anti-symmetric matrices. $\mathrm{O}(n)$ consists of rotations and reflections, which have $\det A = \pm 1$. Sometimes we only care about rotations – which have determinant 1 – which constitute the special orthogonal group $\mathrm{SO}(n)$. Because $A + A^\top = 0$ automatically implies $\operatorname{tr} A = 0$, we see that $\mathfrak{so}(n) = \mathfrak{o}(n)$. This is also evident by the fact that $\mathrm{SO}(n)$ is the identity component of $\mathrm{O}(n)$, which we prove in section 5.3.

Example 2.10. The generalised orthogonal group $\mathrm{O}(p, q)$, with $p + q = n$, are the real $n \times n$ matrices which preserve the bilinear form represented by the matrix $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. In special relativity, the group $\mathrm{O}(1, 3)$ is of particular interest. It is called the Lorentz group, and consists of all Lorentz transformations. The Lorentz group has 4 connected components. Most often we only consider the orthochronous Lorentz transformations, i.e. the transformations that preserve the direction of time, denoted by $\mathrm{O}^+(1, 3)$. These are the matrices $\Lambda \in \mathrm{O}(1, 3)$ such that $\Lambda_{00} > 0$. $\mathrm{SO}^+(3, 1)$ preserves additionly the orientation of space, so it has the extra condition $\det \Lambda = 1$. All these Lie groups have Lie algebra

$$\mathfrak{o}(1, 3) = \left\{ \begin{pmatrix} 0 & b^\top \\ b & A \end{pmatrix} : A \in \mathfrak{so}(3),\ b \in \mathbb{R}^3 \right\}.$$

Example 2.11. The unitary group is defined as $\mathrm{U}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A^* A = I_n\}$, and it is the complex analogue of the orthogonal group. Its Lie algebra is $\mathfrak{u}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) : X + X^* = 0\}$. $\mathrm{SU}(n)$ is its determinant 1 counterpart, and $\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}_n(\mathbb{C})$. $\mathrm{SU}(n)$ in particular appears often as symmetry group of gauge theories of the Standard Model; we consider some in sections 6.2 and 6.5.

2.4 The adjoint representation

Often Lie groups are studied by considering their representations. This is most often done in physics, where the representations of the Lie groups in examples 2.8, 2.9, 2.10 and 2.11 are of special interest. For a Lie group G, a **Lie group representation** is a Lie group homomorphism $G \to H$, for $H = \mathrm{GL}_n(\mathbb{F})$ or $H = \mathrm{Aut}(V)$, for an $n \in \mathbb{Z}_{\geq 1}$ or V a finite-dimensional $\mathbb{F}$-vector space. Likewise, a **Lie algebra representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$, for $\mathfrak{h} = \mathfrak{gl}_n(\mathbb{F})$ or $\mathfrak{h} = \mathrm{End}(V)$. By taking the differential of a Lie group representation, we automatically get a Lie algebra representation by proposition 2.4.

In our physical context, we are mostly interested in the adjoint representation. For a Lie group G and an element $\sigma \in G$, let $c_\sigma : G \to G$ be the conjugation map $\tau \mapsto \sigma\tau\sigma^{-1}$. This is a Lie group automorphism, so this induces a Lie algebra automorphism $\mathrm{d}c_\sigma : \mathfrak{g} \to \mathfrak{g}$, which we denote by Ad_σ. This in turn gives a map

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(\mathfrak{g}), \\ \sigma &\mapsto \mathrm{Ad}_\sigma, \end{aligned}$$

where $\mathrm{Aut}(\mathfrak{g})$ is the group of Lie algebra automorphisms of $\mathfrak{g}$ (which is thus a subgroup of the automorphism group $\mathrm{Aut}_{\mathrm{vec}}(\mathfrak{g})$ of $\mathfrak{g}$, where $\mathfrak{g}$ is considered merely as a $\mathbb{R}$-vector space). Since $\mathrm{Aut}(\mathfrak{g})$ is closed in $\mathrm{Aut}_{\mathrm{vec}}(\mathfrak{g})$, $\mathrm{Aut}(\mathfrak{g})$ itself is a Lie group by the closed-subgroup theorem 2.6. By theorem 3.54 of [5], its Lie algebra is the set of derivations

$$\mathrm{Der}(\mathfrak{g}) = \{T \in \mathrm{End}_{\mathrm{vec}}(\mathfrak{g}) : T[X, Y] = [TX, Y] + [X, TY]\ \forall X, Y \in \mathfrak{g}\} \subseteq \mathrm{End}_{\mathrm{vec}}(\mathfrak{g}).$$

Ad is a group homomorphism, since $c_{\sigma\tau} = c_\sigma \circ c_\tau$ for all $\sigma, \tau \in G$, and Ad is smooth by theorem 3.45 in [5]. Thus Ad defines a representation, called the adjoint representation. Its differential descends to a Lie algebra representation

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}),$$

which is also called the **adjoint representation**. For Lie subgroups $G \subseteq \mathrm{GL}_n(\mathbb{F})$, calculating the adjoint representation is particularly straightforward:

Proposition 2.12. *Let $G \subseteq \mathrm{GL}_n(\mathbb{F})$ be a Lie subgroup. For $A \in G$ and $X, Y \in \mathfrak{g}$, the adjoint representations are given by $\mathrm{Ad}_A X = AXA^{-1}$ and $\mathrm{ad}_X Y = [X, Y]$.*

Proof. Let $A \in G$ and $X \in \mathfrak{g}$. We can write $X = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}\right)\big|_{t=0}$ by properties of the exponential map. Then

$$\mathrm{Ad}_A X = \mathrm{d}c_A \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}\right)\bigg|_{t=0} \right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(Ae^{tX}A^{-1}\right)\bigg|_{t=0} = A\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}\right)\bigg|_{t=0} \right)A^{-1} = AXA^{-1}.$$

Now let $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned}\mathrm{ad}_X Y &= \left(\mathrm{d}(\mathrm{Ad})\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}\right)\bigg|_{t=0} \right)\right)(Y) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}\, e^{tX}\right)\bigg|_{t=0} \right)(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}_{e^{tX}} Y\right)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}Ye^{-tX}\right)\bigg|_{t=0} = XY + 0 - YX = [X, Y].\end{aligned}$$

■

For abelian Lie groups, the adjoint representation is trivial:

Lemma 2.13. *Let G be an abelian Lie group. For $A \in G$, $X, Y \in \mathfrak{g}$, we have $\mathrm{Ad}_A X = X$ and $\mathrm{ad}_X Y = 0$.*

Proof. Since G is abelian, $c_A = \mathrm{id}_G$ for all $A \in G$, and thus $\mathrm{Ad}_A X = X$. $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is a constant map, so $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ sends everything to 0, i.e. $\mathrm{ad}_X Y = 0$. ■

Up to this point, we have covered quite some theory for Lie groups. While the structure of Lie groups is a fascinating subject of study, often in mathematics and physics a Lie group acts on other spaces. Key examples are principal homogeneous spaces and principal bundles, which locally resemble the product of a smooth manifold and a principal homogeneous space. These are central to the next chapter, where we also revisit the exponential map and the adjoint representation.

In chapter 4, compact Lie groups emerge as symmetry groups in Yang-Mills theory, which gives rise to specific gauge fields. Physical fields representing particles are acted upon by these Lie groups in a specific Lie group representation, with the adjoint representation frequently occurring in particle physics. In chapter 6, we encounter Lie groups in the context of spontaneous symmetry breaking, which forms the basis for topological defects. These Lie groups are often compact, especially when gauge fields are included. Notable examples of compact Lie groups are finite groups, $\mathrm{U}(1)$, $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$.

In order to study topological defects, we need to understand the homotopy groups of these compact Lie groups. While the topology of finite groups and $\mathrm{U}(1)$ is straightforward, the homotopy groups of the other examples exhibit a rich structure. We link their homotopy groups to those of spheres, using Lie group actions, as illustrated in section 5.3. In this way, Lie groups provide a foundation for studying topological defects in this thesis, and really for many areas of differential geometry and contemporary theoretical physics.

Chapter 3 | Principal G-bundles

Much of physics is done locally, which suffices for many purposes, but it fails to account for global effects due to topological non-trivialities. As topological defects arise from global effects, we need to develop some theory related to fibre bundles. These are smooth manifolds E with a projection map $\pi : E \to M$, such that E is locally diffeomorphic to the product manifold of the base manifold M and the fibre space F. Typically, this local product structure does not extend globally. The fibre space F can be endowed with additional structure, such as that of a vector space, resulting in vector bundles. The tangent bundle is the most well-known example of a vector bundle, and it is used to create all kinds of new vector bundles.

Another often occurring specialisation of a fibre bundle is that of principal G-bundles, for a Lie group G. In this case, the fibre space is given the structure of a principal homogeneous G-space, which is a smooth manifold that is transitively and freely acted upon by G. Important examples include the Hopf fibrations and the projection map $G \to G/H$, for G a Lie group and $H \subseteq G$ a closed subgroup. Principal G-bundles form the basis for describing fields in field theory, which we explore in chapter 4. Additionally, principal G-bundles are useful in calculations for homotopy groups. This technique is used thoroughly in chapters 5 and 6.

Furthermore, in this chapter, we introduce differential k-forms that take values in a finite dimensional $\mathbb{R}$-vector space; we consider particularly finite dimensional Lie algebras. The primary goal of Lie algebra-valued differential k-forms in this thesis is defining Ehresmann connections and their curvature, which play a significant role in Yang-Mills theory, which is addressed in section 4.4.

This chapter is based on parts of chapters 21 ("Vector-Valued Forms"), 27 ("Principal Bundles"), 28 ("Connections on a Principal Bundle") and 30 ("Curvature on a Principal Bundle") of *Differential Geometry* [6] and the first two chapters of [7]; the sections about fibre bundles and principal bundles are also inspired by the lecture notes of Meinrenken [8].

3.1 Fibre bundles

The basis for this chapter are fibre bundles and their morphisms. These readily generalise to vector bundles and principal G-bundles.

Definition (Fibre bundle). Let F, E, B be smooth manifolds. A fibre bundle is a smooth map $\pi : E \to B$ that is locally trivial, which means that there exists an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of B, with diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$ called local trivialisations, such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow[\phi]{\sim} & U_\alpha \times F \\ & \searrow^{\pi} & \downarrow \mathrm{proj}_{U_\alpha} \\ & & U_\alpha \end{array}$$

B is called the **base space**, E the **total space** and F the **fibre space**. If we also want to stress the fibre in a fibre bundle, we use the notation $F \longrightarrow E \xrightarrow{\pi} B$.

Note that a fibre bundle $\pi : E \to B$ must be automatically surjective. For $b \in B$, we denote $E_b = \pi^{-1}(b)$ the fibre of b in E. By local triviality, E_b is diffeomorphic to F, explaining *fibre* in the word 'fibre bundle'. We want a morphism of fibre bundles to preserve these fibres:

Definition (Fibre bundle morphism). Let $\pi : E \to B$, $\pi' : E' \to B'$ be fibre bundles. A **bundle morphism** is a smooth map $\varphi : E \to E'$, such that for all $b \in B$, $\varphi(E_b) \subseteq E'_{\varphi(b)}$. A **bundle isomorphism** is a diffeomorphism $\varphi : E \to E'$, such that for all $b \in B$, $\varphi(E_b) = E'_{\varphi(b)}$.

A bundle morphism $\varphi : E \to E'$ as above induces a map $f : B \to B'$ on the base spaces, such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ \downarrow \pi & & \downarrow \pi' \\ B & \xrightarrow{f} & B' \end{array}$$

By looking at the local trivialisations, we see that f is smooth.

For a chosen base manifold B and fibre space F, $B \times F$ with the projection map to B is a fibre bundle, the **trivial bundle**. A fibre bundle $\pi : E \to B$ bundle isomorphic to a trivial bundle and inducing the identity on B is called **trivialisable**. In figure 3.1, we give an example of a trivialisable and two non-trivialisable fibre bundles.

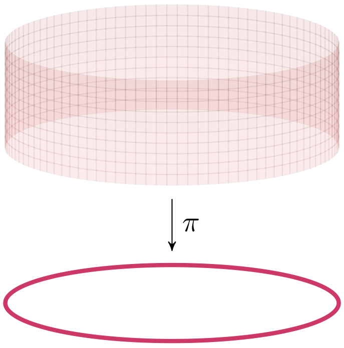
Often for a fibre bundle $\pi : E \to B$, we want to assign to every $b \in B$ and element in its fibre E_b, in a smooth manner. This is exactly the notion of a section of a fibre bundle.

Definition (Section of a fibre bundle). Let $\pi : E \to B$ be a fibre bundle, and $U \subseteq B$ an open subset. A smooth map $s : U \to E$ is called a **local section** if the following diagram commutes:

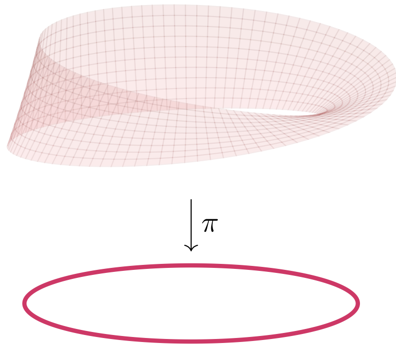
$$\begin{array}{ccc} U & \xrightarrow{s} & E \\ & \searrow^{\mathrm{id}} & \downarrow \pi \\ & & U \end{array}$$

The set of local sections on U of $\pi : E \to B$ is denoted by $\Gamma^\infty(U, E)$. A **(global) section** of $\pi : E \to B$ is a local section with $U = B$.

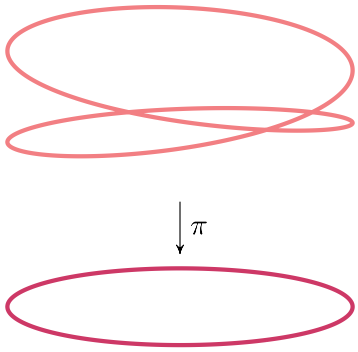
Not every fibre bundle has sections, an example being the one in figure 3.1c.



(a) The cylinder $S^1 \times (-1,1)$, mapping each line segment to its base circle. This is a trivialisable fibre bundle.



(b) The Möbius bundle. Each line segment is again mapped to the base circle, but this fibre bundle is not trivialisable[1].



(c) The map $\pi : S^1 \to S^1$, given by $z \mapsto z^2$, which is a double covering of S^1. The fibre space is $\{-1, 1\}$. Since the total space is connected whereas $S^1 \times \{-1, 1\}$ is not, this fibre bundle is non-trivialisable.

Fig. 3.1: Three non-isomorphic examples of fibre bundles with base space S^1.

One can provide additional algebraic structure to the fibre F in a fibre bundle, such as Lie groups, vector spaces or (Lie) algebras, giving rise to group bundles, vector bundles and (Lie) algebra bundles. The local trivialisations must then respect those structures on each fibre. We highlight vector bundles.

Definition (Vector bundle). Let $F = V$ be a finite dimensional $\mathbb{R}$-vector space (with its usual smooth structure). A vector bundle is a fibre bundle $\pi : E \to M$, such that for every $p \in M$, its fibre E_p carries an $\mathbb{R}$-vector space structure, and that the local trivialisations $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times V$ can be chosen to be fibrewise linear, i.e. for $p \in M$, $\phi_\alpha|_{E_p} : E_p \to \{p\} \times V$ is linear. A vector bundle morphism is a fibre bundle morphism between vector bundles which is linear restricted to the fibres.

Every vector bundle $\pi : E \to M$ has a canonical section $s : M \to E$, called the zero section, which assigns the additive identity element in E_p to each element $p \in M$. In fact, for an open subset $U \subseteq M$, $\Gamma^\infty(U, E)$ has the structure of a $\mathcal{C}^\infty(U)$-module, by fibrewise addition and scalar multiplication.

One of the most important vector bundles associated to a smooth manifold M is the tangent bundle $TM \to M$. Sections of the tangent bundle are exactly the smooth vector fields, i.e. $\mathfrak{X}(M) = \Gamma^\infty(M, TM)$. For $E \to M$, $F \to M$ vector bundles, we can apply constructions in linear

[1] Assume the Möbius bundle were trivialisable. In particular, the total space would be homeomorphic to $S^1 \times (-1, 1) \cong \mathbb{C}^*$. By leaving out a circle in the middle of the total space, the resulting space is still connected. However, $\mathbb{C}^*$ without a simple closed curve is disconnected by the Jordan curve theorem, giving a contradiction.

algebra to the individual vector spaces, such as taking direct sums, tensor products, k-th exterior and symmetric powers, the dual vector space or taking the vector space of homomorphisms. These constructions lead to new vector bundles, namely the Whitney sum $E \oplus F \to M$, the tensor bundle $E \otimes F \to M$, the k-th exterior and symmetric bundles $\bigwedge^k E \to M$ and $\sum^k E \to M$, the dual bundle $E^* \to M$ and the Hom-bundle $\mathrm{Hom}(E, F) \to M$. These construction can be applied to the tangent bundle, to get for example the cotangent bundle $T^*M \to M$, the bundle $\bigwedge^k(T^*M) \to M$ and the (k, ℓ)-tensor bundle. Sections of these are smooth covector fields $\Omega^1(M) = \Gamma^\infty(M, T^*M)$, smooth differential k-forms $\Omega^k(M) = \Gamma^\infty(M, \bigwedge^k(T^*M))$, and smooth tensor fields, respectively.

3.2 Principal G-bundles

In physics, we are mostly interested in Lie groups because they *do* something with a space. This is rigorously described using group actions. Let G be a Lie group. Then a **smooth left action** on a smooth manifold X is a smooth map $\varphi : G \times X \to X$ such that it is a left action, i.e.

- $\varphi(e, x) = x$, for all $x \in X$;
- $\varphi(gh, x) = \varphi(g, \varphi(h, x))$, for all $g, h \in G$ and $x \in X$.

Often we simply write $g \cdot x$ or gx for $\varphi(g, x)$. A smooth right action is defined analogously. A smooth manifold X which is smoothly acted upon by a Lie group G, is called a **G-manifold**. A map $f : X \to Y$ between G-manifolds is called **G-equivariant** if for all $x \in X$ and $g \in G$, $f(gx) = gf(x)$, or the appropriate variations if G acts on one or two of the spaces on the right.

We have already seen instances of smooth actions of Lie groups, namely Lie group representations $\rho : G \to \mathrm{Aut}(V)$ for a finite-dimensional $\mathbb{F}$-vector space V induce a smooth left action

$$\begin{aligned} \tilde{\rho} : G \times V &\to V, \\ (g, v) &\mapsto \rho(g)(v). \end{aligned}$$

We shall encounter also other smooth actions.

One of the main theorems in group theory related to group actions is the orbit-stabiliser theorem. In this thesis, we shall be using the following specific form for smooth transitive actions. For the proof, see theorem 3.62 in [5]. The theorem relies on the fact that for G a Lie group and $H \subseteq G$ a closed subgroup, G/H can be given a smooth structure such that the projection map is smooth. For more details, we again refer to [5], theorem 3.58.

Theorem 3.1. *Let $\varphi : G \times X \to X$ be a transitive smooth left action of the Lie group G on the smooth manifold X. Choose a point $x \in X$, and consider its stabiliser subgroup $H = \mathrm{Stab}_x(G) = \{g \in G : gx = x\}$. Then the map $\psi : G/H \xrightarrow{\sim} X$, given by $gH \mapsto gx$, defines a diffeomorphism.*

For a lot of applications, besides transitivity we also demand the smooth action to be free, i.e. if there exist $g \in G$ and $x \in X$ such that $gx = x$, then $g = e$:

Definition (Principal homogeneous G-space). Let G be a Lie group. A **principal homogeneous G-space** is a smooth manifold X with a transitive and free smooth G-action.

An important example is a Lie group G, which acts on itself by left multiplication. In fact, all principal homogeneous G-spaces are isomorphic[2] to this example; let X be a principal homogeneous G-space and $x_0 \in X$ a point, then

$$\psi : G \xrightarrow{\sim} X, \\ g \longmapsto gx_0$$

defines such isomorphism (the fact that ψ^{-1} is smooth is non-trivial, but it immediately follows from theorem 3.1).

But why would we consider principal homogeneous G-spaces for a given Lie group G in the first place, if they turn out to be all isomorphic? They are all isomorphic, but not *canonically* so; we had to choose a point $x_0 \in X$ after all. Due to this aspect principal homogeneous G-spaces often occur in physics. For instance, if one drops a ball from a small height $y = y_0$, then its velocity on a later time at position $y = y_1$ does not depend on the absolute coordinates y_0 and y_1, but on their difference $\Delta y = y_0 - y_1$. Choosing another reference point for our coordinates does not change this fact. Thus the height of the ball can be thought of as an element in a principal homogeneous $\mathbb{R}$-space and the height difference as an element of $\mathbb{R}$; *the* height of the ball makes only sense upon choosing a reference point – often the floor.

Another example is in quantum mechanics, where one cannot speak of an absolute phase of a quantum state, as multiplying with a $z \in \mathbb{C}$ with $|z| = 1$ does not alter the physical state. Phase differences are measurable though, which constitute the group $\mathrm{U}(1)$. The absolute phase then is an element of a principal homogeneous $\mathrm{U}(1)$-space.

Often in physics, we moreover want to choose a reference point locally. Essentially, we want to combine the notion of a principal homogeneous G-space with a fibre bundle:

Definition (Principal G-bundle and its morphisms). Let G be a Lie group. A **principal G-bundle** is a fibre bundle $\pi : \mathcal{P} \to B$ such that each fibre $\mathcal{P}_b$ for $b \in B$ is also a principal homogeneous G-space, and such that the local trivialisations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ can be chosen to be G-equivariant restricted to the fibres. The G-action on $U_\alpha \times G$ is defined by $g(b, h) = (b, gh)$, for $G \in g$ and $(b, h) \in U_\alpha \times G$. $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is called G-equivariant if for all $b \in U_\alpha$, $p \in \mathcal{P}_b$ and $G \in g$,

$$g\phi_\alpha(p) = \phi_\alpha(gp).$$

A **homomorphism of principal G-bundles** between principal G-bundles $\pi : \mathcal{P} \to B$ and $\pi' : \mathcal{P}' \to B'$ is a bundle morphism $\varphi : \mathcal{P} \to \mathcal{P}'$, which is also G-equivariant on the fibres. In other words, for every $b \in B$, $p \in \mathcal{P}_b$ and $g \in G$, we have $g\varphi(p) = \varphi(gp)$, or equivalent expressions if G acts on the right.

A local section of a principal G-bundle $\pi : \mathcal{P} \to B$ then can be interpreted as smoothly choosing a reference point in each principal homogeneous G-space $\mathcal{P}_b$, for b in an open subset of B. Depending on the structure of the principal bundle, this may be possible globally or only locally.

The example in figure 3.1c is a principal $\mathbb{Z}_2$-bundle. Another interesting example (not only mathematically, but also in physics, see for example [9]), is the Hopf fibrations. The following description is based on examples 4.44 up to 4.47 in [1].

Example (Hopf fibrations). The complex numbers with unit length S^1 act upon the unit sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by scalar multiplication. The orbit space of this action can be identified

[2] Two G-manifolds are said to be isomorphic if there exists a G-equivariant diffeomorphism between them.

with $\mathbb{CP}^n$, making the projection map into a principal S^1-bundle (it is easily checked that the action on each fibre is smooth, free and transitive). The same procedure can be done for $\{\pm 1\} \cong S^0$ acting on $S^n \subseteq \mathbb{R}^{n+1}$ and the unit quaternionic numbers of unit length[3] acting on $S^{4n+3} \subseteq \mathbb{H}^{n+1}$, giving principal bundles

$$S^0 \longrightarrow S^n \xrightarrow{\pi} \mathbb{RP}^n, \quad S^1 \longrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n, \quad S^3 \longrightarrow S^{4n+3} \xrightarrow{\pi} \mathbb{HP}^n.$$

The case $n = 1$ are called **Hopf fibrations**. By using stereographic projections, we find that the projective lines $\mathbb{RP}^1$, $\mathbb{CP}^1$ and $\mathbb{HP}^1$ are diffeomorphic to S^1, S^2 and S^4 respectively. Then the principal bundles reduce to the following principal bundles:

$$S^0 \longrightarrow S^1 \xrightarrow{\pi} S^1, \quad S^1 \longrightarrow S^3 \xrightarrow{\pi} S^2, \quad S^3 \longrightarrow S^7 \xrightarrow{\pi} S^4.$$

This example does not generalise to the octonions $\mathbb{O}$, as the octonions with norm 1 do not form a group. However, there is still a related fibre bundle $S^7 \longrightarrow S^{15} \xrightarrow{\pi} S^8$.

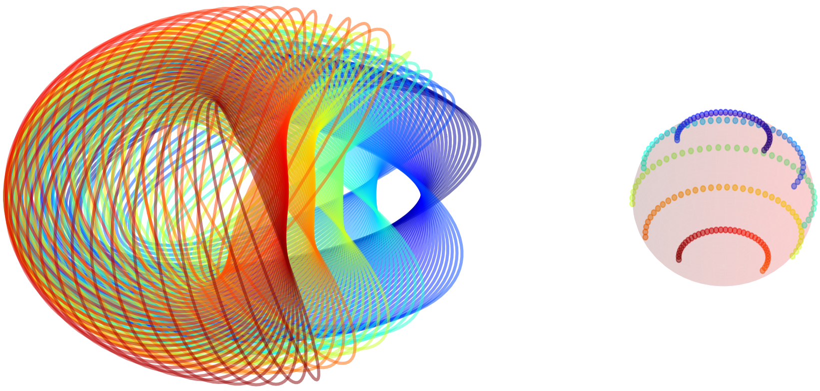


Fig. 3.2: The Hopf fibration $S^1 \longrightarrow S^3 \xrightarrow{\pi} S^2$. On the right, points on S^2 are chosen, the corresponding fibres are illustrated on the left in the same colour. S^3 is visualised by projecting it stereographically onto $\mathbb{R}^3$, and then scaling it to the open unit ball, via $\mathbf{r} \mapsto \frac{\mathbf{r}}{\sqrt{\|\mathbf{r}\|^2+1}}$. The circles in the left picture are disjoint, since each circle corresponds to a different point on S^2.

As a surprising byproduct, the Hopf fibrations provide information of the homotopy groups of spheres. This astonishing fact we discuss in section 5.3.

A last example is one which we shall encounter when studying topological defects in chapter 6:

Example (The induced principal H-bundle for a closed subgroup). Let G be a Lie group, and $H \subseteq G$ be a closed subgroup. We have already seen that the left cosets G/H can be given a smooth manifold structure such that the natural projection map $\pi : G \to G/H$ is a smooth map. It is even a fibre bundle with fibre space H, see theorem 3.58 in [5] for further details. In fact, $\pi : G \to G/H$ has a principal H-bundle structure, where H acts on the right on fibres gH via

$$\begin{aligned} gH \times H &\to gH, \\ (gh, h') &\mapsto ghh'. \end{aligned}$$

[3] The group of quaternions with unit length $\{q \in \mathbb{H} : \|q\| = 1\}$ can be identified with SU(2), which in turn is diffeomorphic to S^3.

3.3 Lie algebra-valued forms and Ehresmann connections

Having established principal G-bundles, we now turn our focus to Ehresmann connections on principal G-bundles. These are crucial in describing Yang-Mills theory geometrically. Before we can do that however, we have to introduce smooth vector-valued differential k-forms. This section is solely a short overview of smooth vector-valued differential k-forms and connections on principal G-bundles. For a thorough overview, we refer to chapters 21, 28 and 30 in [6].

Let M be a general smooth manifold, and $\omega \in \Omega^k(M)$ a smooth k-form. At every point $p \in M$, ω can be considered as an alternating k-linear map

$$\omega_p : T_pM \times \ldots \times T_pM \to \mathbb{R}.$$

For our discussion about connections on principal bundles, we want these alternating k-linear maps to take values in a finite dimensional real vector space V. So we want a vector-valued k-form induce alternating k-linear maps

$$\omega_p : T_pM \times \ldots \times T_pM \to V$$

for each point $p \in M$, in a smooth way. The way to accomplish this is by applying the tensor product with V on each $\bigwedge^k(T_p^*M)$. More precisely:

Definition (Smooth vector-valued differential k-forms). Let M be a smooth manifold, V a finite dimensional $\mathbb{R}$-vector space and $k \geq 0$. Define

$$\Omega^k(M, V) := \Gamma^\infty\left((M \times V) \otimes \bigwedge^k(T^*M)\right),$$

which is the set of sections of the tensor bundle of the trivial bundle $M \times V$ and $\bigwedge^k(T^*M)$. An element of $\Omega^k(M, V)$ is called a **smooth V-valued differential k-form**. We often shorten this to 'V-valued k-form' for conciseness.

Like 'ordinary' k-forms, a V-valued k-form $\omega \in \Omega^k(M, V)$ can indeed be considered as a smooth map $\omega : \bigwedge^k(TM) \to V$. At each point $p \in M$, ω then can be regarded as an alternating k-linear map $\omega_p : T_pM \times \ldots \times T_pM \to V$.

To do calculations with V-valued k-forms, it is often useful to choose a basis $(v_1, \ldots, v_n)$ for V. Then we can write $\omega \in \Omega^k(M, V)$ in terms of the basis $(v_1, \ldots, v_n)$:

$$\omega = \sum_{i=1}^n v_i \otimes \omega^i, \quad \omega^i \in \Omega^k(M).$$

Additionally, by choosing a local coframe $(\varepsilon^1, \ldots, \varepsilon^m)$ on an open subset $U \subseteq M$, we can write ω in terms of coordinate functions on U :

$$\omega = \sum \omega^i_{j_1,\ldots,j_k} v_i \otimes (\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k}), \quad \omega^i_{j_1,\ldots,j_k} \in \mathcal{C}^\infty(U)$$

Using coordinate expressions, we can define some common operations on vector-valued k-forms:

Defining proposition (Operations on V-valued k-forms). Let $\omega \in \Omega^k(M, V)$ be a V-valued k-form, which we write as $\omega = \sum v_i \otimes \omega^i$ after a choice of basis $(v_1, \ldots, v_n)$. Then the following definitions are basis-independent:

- The exterior derivative $\mathrm{d} : \Omega^k(M, V) \to \Omega^{k+1}(M, V)$ is defined as

$$\mathrm{d}\omega := \sum_{i=1}^{n} v_i \otimes \mathrm{d}\omega^i \in \Omega^{k+1}(M, V).$$

- Let $f : N \to M$ be a smooth map between smooth manifolds. Then the pullback $f^* : \Omega^k(M, V) \to \Omega^k(N, V)$ is defined as

$$f^*\omega := \sum_{i=1}^{n} v_i \otimes f^*\omega^i \in \Omega^k(N, V).$$

- For an n-dimensional Lie algebra $V = \mathfrak{g}$, the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g}$ induces a bilinear map $[\cdot, \cdot] : \Omega^k(M, \mathfrak{g}) \times \Omega^\ell(M, \mathfrak{g}) \to \Omega^{k+\ell}(M, \mathfrak{g})$, via

$$[\omega, \eta] := \sum_{1 \leq i,j \leq n} [v_i, v_j] \otimes (\omega^i \wedge \eta^j).$$

Let $\pi : \mathcal{P} \to M$ be a principal G-bundle. In this section, we assume that G acts on the right. We denote the right action for $g \in G$ by $r_g : \mathcal{P} \to \mathcal{P}$. By examining the local trivialisations, we see that $\pi : \mathcal{P} \to M$ is a smooth submersion, meaning that $\mathrm{d}\pi_p : T_p\mathcal{P} \to T_{\pi(p)}M$ is surjective for all $p \in \mathcal{P}$. We define $\mathcal{V}_p \subseteq T_p\mathcal{P}$ as $\mathcal{V}_p = \ker \mathrm{d}\pi_p$, which is called the vertical tangent space at p. Vectors in $\mathcal{V}_p$ are the tangent vectors to paths in $\mathcal{P}$ that map to a single fibre.

However, there is no canonical way to define a notion of moving 'perpendicular' to vertical vectors. This is exactly what a connection accomplishes, by specifying a subspace $\mathcal{H}_p \subseteq T_p\mathcal{P}$ for every point $p \in \mathcal{P}$ in a smooth manner and respecting the G-action, such that $T_p\mathcal{P} = \mathcal{V}_p \oplus \mathcal{H}_p$. Respecting the G-action is called right-invariance, and means that for all $Y_p \in \mathcal{H}_p$, we have $\mathrm{d}r_g(Y_p) \in \mathcal{H}_{pg}$. Then $T_{\pi(p)}M$ can be identified with $\mathcal{H}_p$ via the isomorphism theorem: $T_{\pi(p)}M \cong T_p\mathcal{P}/\ker \mathrm{d}\pi_p \cong \mathcal{H}_p$. $\mathcal{H}_p$ is called the horizontal tangent space at p.

In this way, a connection 'connects' adjacent fibres in $\mathcal{P}$; it is then possible to lift a vector field $X \in \mathfrak{X}(M)$ on M uniquely to a right-invariant horizontal vector field on $\mathcal{P}$. This situation is sketched in figure 3.3.

We can easily define vector fields on $\mathcal{P}$ that are always vertical:

Defining proposition (Fundamental vector field). Let $\pi : \mathcal{P} \to M$ be a principal G-bundle, and $X \in \mathfrak{g}$ a left-invariant vector field on G. Then for all $p \in \mathcal{P}$, we define

$$\underline{A}_p := \frac{\mathrm{d}}{\mathrm{d}t}\left(p \cdot e^{tX}\right)\bigg|_{t=0} \in T_p\mathcal{P}.$$

Then $\underline{A}$ is a smooth vector field on $\mathcal{P}$, called a fundamental vector field.

In fact, the vertical vectors are precisely those vectors that come from fundamental vector fields (corollary 27.19 in [6]). Another important identity is given by proposition 27.13 in [6], which states that for $p \in \mathcal{P}$, $A \in \mathfrak{g}$ and $g \in G$, the following holds:

$$\mathrm{d}r_g(\underline{A}_p) = \left(\underline{\mathrm{Ad}_{g^{-1}} A}\right)_{pg}. \tag{3.1}$$

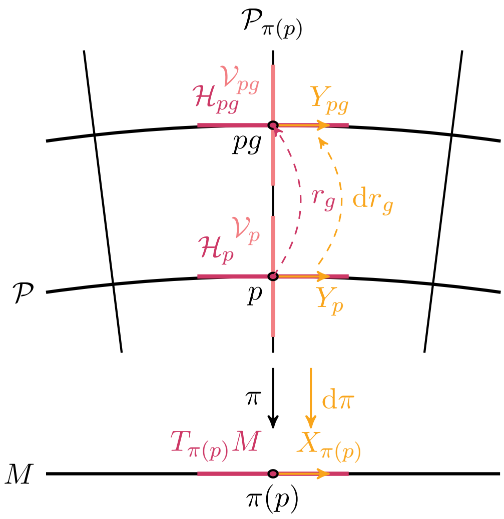


Fig. 3.3: Illustration of a principal bundle $\pi : \mathcal{P} \to M$ and a chosen set of right-invariant horizontal tangent spaces. We choose a point $p \in \mathcal{P}$. Its tangent space $T_p\mathcal{P}$ can be decomposed in a vertical tangent space $\mathcal{V}_p = \ker \mathrm{d}\pi_p$ and a horizontal tangent space $\mathcal{H}_p$, such that $T_p\mathcal{P} = \mathcal{V}_p \oplus \mathcal{H}_p$. G acts on the fibre $\mathcal{P}_{\pi(p)}$ the point p belongs to. We have required that horizontal tangent spaces are mapped to horizontal tangent spaces via the differential of the action. In this way, we can identify a vector $X_{\pi(p)} \in T_{\pi(p)}M$ with right-invariant vectors $Y_{pg} \in \mathcal{H}_{pg}$, for every $g \in G$.

An easy way to describe a connection on a principal G-bundle is through a $\mathfrak{g}$-valued 1-form, called an Ehresmann connection:

Definition (Ehresmann connection). Let $\pi : \mathcal{P} \to M$ be a principal G-bundle. An **Ehresmann connection** is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, such that

1. for all $A \in \mathfrak{g}$ and $p \in \mathcal{P}$, $\omega_p(\underline{A}_p) = A$;
2. for all $g \in G$, $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ as maps $T\mathcal{P} \to \mathfrak{g}$.

For an Ehresmann connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, the horizontal tangent space at $p \in \mathcal{P}$ is defined as $\mathcal{H}_p := \ker \omega_p \subseteq T_p\mathcal{P}$. The first condition states exactly that $\mathcal{V}_p \oplus \mathcal{H}_p = T_p\mathcal{P}$. The second condition ensures that ω_p is G-equivariant, with respect to the following right actions:

$$\overset{g \mapsto \mathrm{d}r_g}{\circlearrowright} \qquad \overset{g \mapsto \mathrm{Ad}_{g^{-1}}}{\circlearrowright}$$
$$T_p\mathcal{P} \xrightarrow{\ \omega_p\ } \mathfrak{g}$$

For vertical vectors, this condition is consistent with equation (3.1); for horizontal vectors $Y_p \in \mathcal{H}_p$, it states the right-invariance $\mathrm{d}r_g(Y_p) \in \mathcal{H}_{pg}$, as

$$\omega_{pg}\left(\mathrm{d}r_g(Y_p)\right) = \omega_{r_g(p)}\left(\mathrm{d}r_g(Y_p)\right) = (r_g^*\omega)_p(Y_p) = \mathrm{Ad}_{g^{-1}}(\omega_p(Y_p)) = \mathrm{Ad}_{g^{-1}}(0) = 0.$$

The curvature associated to a connection measures how the connection varies:

Definition (Curvature of a connection). Let $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ be an Ehresmann connection for a principle G-bundle $\pi : \mathcal{P} \to M$. Then its **curvature** is defined as $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{P}, \mathfrak{g})$.

An important property of the curvature (see theorem 30.4 in [6]) is that $\mathrm{d}\Omega = [\Omega, \omega]$ holds, which is called the **second Bianchi identity**. In the next chapter, we study principal bundles in a physical context, and we give a physical interpretation of connections, their curvature and the second Bianchi identity.

Chapter 4 | Fields and gauge theories

A (physical) field is an assignment of a physical quantity to every point in space and time. Examples are ubiquitous in physics: temperature, acoustic fields describing sound waves, the electric and magnetic field, the gravitational field, fields describing elementary particles such as electrons and quarks, and many more. Topological defects, which we study in chapter 6, are also fields, so it makes sense to study fields. In section 4.1, we first give a quick description of fields and spacetime, here $\mathbb{R}^4$ with the Minkowski metric, on which relativistic fields are modelled.

Next is section 4.2, where we introduce classical field theory, by generalising Lagrangian mechanics and Hamiltonian mechanics to fields. This allows us to construct the equations of motion from a given Lagrangian density by an action principle. We also encounter the energy functional, which will play a vital role in symmetry breaking in chapter 6.

Electromagnetism is one of the most prominent classical field theories. In section 4.3, we present Maxwell's equations in a covariant form using the language of differential geometry, unveiling their full elegance. In this formalism, the potential formulation also immediately follows in a natural way. We argue that this potential formulation is more descriptive, as it is able to combine electrodynamics, special relativity and quantum mechanics, forming the basis for quantum electrodynamics. This is all made possible by gauge-invariance; we can transform the four-potential without altering observable quantities.

By requiring gauge-invariance under local U(1)-gauge transformations, electrodynamics automatically emerges. Yang-Mills theory, which we cover in section 4.4, tries to generalise this principle to invariance under compact Lie groups G. Here we use the theory about principal G-bundles, which we have established in chapter 3. Yang-Mills theory has been incredibly successful in physics, with the Standard Model of particle physics as its quintessence.

4.1 Fields and spacetime

First, we want to represent physical fields mathematically. For our purposes, we model space and time after the smooth manifold $M = \mathbb{R}^4$ (with its usual differential structure). Then fields can be seen as sections of some vector bundle over M, which we assume to be smooth. For instance, a **real scalar field** is a section $\phi \in \Gamma^\infty(M, M \times \mathbb{R})$, which can be identified with a smooth function $\phi : M \to \mathbb{R}$. Replacing $\mathbb{R}$ by $\mathbb{C}$, we get a **complex scalar field**. Combining multiple scalar fields will result in a field $\phi : M \to \mathbb{R}^n$, for some $n \geq 2$. A **vector field** is a section $A \in \Gamma^\infty(M, TM)$, which can be identified with a smooth map $A : M \to TM$, for TM the tangent bundle. Analogously, covector fields and tensor fields are sections of T^*M and the (k, ℓ)-tensor bundle over M.

Coordinates on $M = \mathbb{R}^4$ are denoted by $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. t is the temporal coordinate,

x, y and z the spatial coordinates. Note that in reality, $x^0 = ct$, where c represents the speed of light. For convenience, we adopt natural units so that $c = 1$. While this choice is purely administrative, it is crucial to bear in mind during calculations. A vector field $X \in \mathfrak{X}(M)$ can be decomposed in terms of the standard basis[1]:

$$X = X^0 \frac{\partial}{\partial t} + X^1 \frac{\partial}{\partial x} + X^2 \frac{\partial}{\partial y} + X^3 \frac{\partial}{\partial z}.$$

To shorten such expressions, we introduce the notation $\partial_\mu := \frac{\partial}{\partial x^\mu}$. Furthermore, we shall be using the Einstein summation convention, which states that if an undefined index appears both below and above in a term, then it is implicitly summed over. For example, we would write $X = X^\mu \partial_\mu$ in this case, or $\omega = \omega_\mu \mathrm{d}x^\mu$ for a covector field $\omega \in \Omega^1(M)$. When Greek indices appear, the summation is over the temporal and spatial indices, whereas for Latin indices the summation is only over space components.

We endow M with the **Minkowski metric** $\eta : M \to T^*M \otimes T^*M$, defined by

$$\eta = \eta_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \quad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

Then for smooth vector fields $X, Y \in \mathfrak{X}(M)$, we see that

$$\eta(X,Y) = \eta_{\mu\nu} \mathrm{d}x^\mu(X) \mathrm{d}x^\nu(Y) = \eta_{\mu\nu} X^\mu X^\nu = X^0 Y^0 - X^1 Y^1 - X^2 Y^2 - X^3 Y^3.$$

$\eta(X,Y)$ is also commonly denoted as $\langle X, Y \rangle$. M with the Minkowski metric is known as **Minkowski space**.

At each point in M, η is a non-degenerate symmetric bilinear form on T_pM, thereby endowing M with the structure of a pseudo-Riemannian manifold. The diffeomorphisms of T_pM preserving this bilinear form and keeping the origin fixed, are precisely the Lorentz transformations in $\mathrm{O}(1,3)$, which we encountered in example 2.10. Physically, η allows measurements of 'distances' in spacetime, such as the proper time interval along a curve in spacetime, which corresponds to the time measured by a clock moving along that curve. The Minkowski metric also divides a vector $v \in T_pM$ into one of three categories:

- $\boldsymbol{\langle v, v \rangle > 0}$ **(timelike)**. Curves in spacetime where the tangent vector to the curve (the four-velocity) at every point is timelike represent matter, which moves slower than the speed of light.
- $\boldsymbol{\langle v, v \rangle = 0}$ **(lightlike)**. Photons and other massless particles move at the speed of light, with their four-velocity being everywhere lightlike.
- $\boldsymbol{\langle v, v \rangle < 0}$ **(spacelike).** Particles with spacelike four-velocity move faster than the speed of light, although there is currently no evidence for their existence.

For a more comprehensive discussion of special relativity, see chapter 12 of *Introduction to Electrodynamics* [3].

The non-degeneracy of η allows for raising and lowering indices:

$$X_\mu = \eta_{\mu\nu} X^\nu, \quad X^\mu = \eta^{\mu\nu} X_\nu,$$

[1] For general smooth manifolds M, this can only be done locally. Smooth manifolds that allow a set of smooth vector fields that form a basis on each tangent space T_pM for $p \in M$, are called *parallelisable*. Examples include Lie groups G such as $\mathbb{R}^4$ (take a basis for T_eG, and identify these vectors with the smooth vector fields in $\mathfrak{g}$), S^7 and the product of parallelisable manifolds.

for $\eta^{\mu\nu}$ such that $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho$. Here we are considering the Minkowski metric, for which $\eta^{\mu\nu} = \eta_{\mu\nu}$ holds in particular. This equivalence means that a vector $X \in T_pM$ with components X^μ can be interpreted as a covector in T^*_pM with indices X_μ, and vice versa. These transformations are called the **musical isomorphisms** $\flat : T_pM \xrightarrow{\sim} T^*_pM$ and $\sharp : T^*_pM \xrightarrow{\sim} T_pM$.

In chapter 13 of [2], it is proven that these isomorphisms extend to vector bundle isomorphisms between the tangent bundle TM and cotangent bundle T^*M. These isomorphisms then can be interpreted as $\mathcal{C}^\infty(M)$-linear isomorphisms

$$\begin{aligned} \flat : \mathfrak{X}(M) &\xrightarrow{\sim} \Omega^1(M), \\ X &\longmapsto X_\mu \mathrm{d}x^\mu, \end{aligned} \qquad\qquad \begin{aligned} \sharp : \Omega^1(M) &\xrightarrow{\sim} \mathfrak{X}(M), \\ \omega &\longmapsto \omega^\mu \partial_\mu. \end{aligned}$$

Thus, smooth vector fields can naturally be interpreted as smooth covector fields, and vice versa. All of this is possible whenever M has a pseudo-metric η. In general, the indices of tensor fields can also be raised and lowered in an analogous manner.

Raising and lowering indices also allow for the following common inner product expression in physics

$$\langle X, Y\rangle = \eta_{\mu\nu}X^\mu Y^\nu = X_\mu Y^\mu = X^\mu Y_\mu, \quad X, Y \in \mathfrak{X}(M).$$

By the above discussion, it makes sense to define η also for covector fields $\zeta, \omega \in \Omega^1(M)$, via

$$\langle \zeta, \omega\rangle := \langle \zeta^\sharp, \omega^\sharp\rangle,$$

which is a non-degenerate symmetric bilinear form on each cotangent space T^*_pM. η can even be extended to a non-degenerate symmetric bilinear form on each $\bigwedge^k(T^*_pM)$, via

$$\left\langle \zeta^1 \wedge \ldots \wedge \zeta^k, \omega^1 \wedge \ldots \wedge \omega^k \right\rangle := \det\left(\langle \zeta^i, \omega^j\rangle\right)_{ij}, \quad \zeta^1, \omega^1, \ldots, \zeta^k, \omega^k \in \Omega^1(M),$$

and extending this bilinearly. This allows for a natural isomorphism $\star : \Omega^k(M) \xrightarrow{\sim} \Omega^{n-k}(M)$, which is uniquely defined by the relation

$$\zeta \wedge \star\omega = \langle \zeta, \omega\rangle \, \mathrm{d}^4x, \quad \zeta, \omega \in \Omega^k(M).$$

$\star$ is called the **Hodge star operator**, and it has some interesting properties, notably $\star\star = (-1)^{k+1}\,\mathrm{id}_{\Omega^k(M)}$. This operator can also be used to define differential operators in a coordinate-free manner. For example, the four-divergence $\operatorname{div} X = \star\mathrm{d}(\star X^\flat)$, for a smooth vector field $X \in \mathfrak{X}(M)$. We encounter the Hodge star operator again when we talk about electromagnetism and gauge fields.

It is important to note that the constructions presented in this section generalise to general smooth pseudo-Riemannian manifolds. For a more comprehensive treatment of working with tensor fields, manipulating indices and the Hodge star operator, we refer to chapters 12 and 13 in [10], which serve as the foundation of this section.

4.2 The Lagrangian formalism

In Lagrangian mechanics, a mechanical system is described using a **Lagrangian** $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$, which is a smooth function depending on independent generalised position coordinates $\mathbf{q} = (q_1, \ldots, q_n)$, their time derivatives $\dot{\mathbf{q}} = (\dot{q}_1, \ldots, \dot{q}_n)$, and time t itself. In a lot of situations, the Lagrangian is given by $L = T - V$, which is the difference of the kinetic energy T and the potential energy V. The equations of motion are determined by **Hamilton's principle** – also known as the **principle of stationary action** –, which states that the evolution of the system is such that the action integral

$$\mathcal{S}[\mathbf{q}(t)] = \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \mathrm{d}t$$

is stationary. This gives the following equations, called the **Euler-Lagrange equations**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{4.1}$$

The quantity $p_i := \frac{\partial L}{\partial \dot{q}_i}$ is called the **generalised momentum**, and rightly so. For instance, for a particle with mass m and kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ moving in a velocity-independent potential $V(\mathbf{r}, t)$, the generalised momenta are simply the linear momenta $p_x = mx$, etc. If one describes the movement in cylindrical coordinates (r, φ, z) instead, then $p_\varphi = mr^2\dot{\varphi}$, which is just the angular momentum around the z-axis. Sometimes it is more useful to express the equations of motion in terms of $\mathbf{q}$, $\mathbf{p}$ and t instead of $\mathbf{q}$, $\dot{\mathbf{q}}$ and t. This is accomplished by the Legendre transform

$$H(\mathbf{q}, \mathbf{p}, t) = p_i\dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t).$$

H is called the **Hamiltonian**. **Hamiltonian mechanics** gives a very useful framework for quantum mechanics and statistical physics, amongst others.

By calculating dH and dL, we get the first order partial differential equations, which are called **Hamilton's equations**:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{4.2}$$

Physically, H often represents the total energy (such as the aforestated example where $L = T - V$, $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$ and $V = V(x, y, z, t)$; then $H = T + V$), but in general, this need not always be the case. Nonetheless, we can deduce that when the Lagrangian does not depend on time explicitly, then $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$, hence the quantity H is conserved. This makes H an interesting quantity in its own right.

The Lagrangian formalism generalises to fields. Most notably, we want to generalise equation (4.1). Throughout this section, we follow chapter 12 of *Classical Mechanics* [4]. Let ϕ_ρ be a collection of fields, where ρ is a general index for field components (for instance, it can be an index for a vector field, or double indices for a rank 2 tensor field, etc.). Then the **Lagrangian density** $\mathcal{L}$ is a function of the field components, their spacetime derivatives and absolute spacetime coordinates on $M = \mathbb{R}^4$:

$$\mathcal{L} = \mathcal{L}\left(\phi_\rho, \partial_\mu\phi_\rho, x^\mu\right).$$

The Lagrangian density $\mathcal{L}$ can often be expressed as the difference between the kinetic energy density $\mathcal{T}$ and the potential energy density $\mathcal{V}$. We use Lagrangian densities ("Lagrangian per volume") rather than Lagrangians themselves, because the fields have uncountable many degrees of freedom, so it would not make any sense to attribute kinetic and potential energy to a single degree of freedom. It is the same reason why in a material with a certain mass, we use the mass density rather than the individual masses of the constituents. Nonetheless, colloquially the word 'Lagrangian' is also applied to $\mathcal{L}$.

For $\Omega \subseteq M$ a compact region in spacetime, the **action** is defined as

$$\mathcal{S}[\phi(x^\mu)] = \int_\Omega \mathcal{L}\left(\phi_\rho(x^\mu), \partial_\mu\phi_\rho(x^\mu), x^\mu\right) \mathrm{d}^4x. \tag{4.3}$$

Here $\mathrm{d}^4x = \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the volume form of M. **Hamilton's principle** then states that the variation $\delta\mathcal{S}$ must be 0, which means that the evolution of the fields $\phi_\rho^{(0)}$ is such that $\mathcal{S}$ does not change up to first order. That is to say, for a variation $\phi_\rho = \phi_\rho^{(0)} + \alpha\zeta_\rho$ for smooth functions ζ_ρ which vanish at $\partial\Omega$, and α a real parameter, that

$$\left.\frac{\mathrm{d}(\mathcal{S}[\phi])}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0.$$

By solving this (for the lengthy calculation we refer to section 12.2 in [4]), one finds the **Euler-Lagrange equations** for fields, which determine the evolution for the field components ϕ_ρ:

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\rho)}\right] - \frac{\partial\mathcal{L}}{\partial\phi_\rho} = 0. \tag{4.4}$$

Note the apparent similarities between equations (4.1) and (4.4). To describe Lagrangian mechanics geometrically, all the possible configurations the system can be in forms an n-dimensional manifold, the **configuration manifold** Q. The generalised coordinates $\mathbf{q}$ parameterise this manifold (locally). A particular instance of a pair $(\mathbf{q}, \dot{\mathbf{q}})$ of generalised position and velocity is then exactly an element of the tangent bundle TQ. The Lagrangian $L(\mathbf{q}, \mathbf{p}, t)$ can then be interpreted as a smooth function $L : TQ \times \mathbb{R} \to \mathbb{R}$.

The pair $(\mathbf{q}, \mathbf{p})$ of generalised position and momentum is an element of the cotangent bundle T^*Q, as can readily be seen from the definition $p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$. The cotangent bundle T^*Q can be given a symplectic structure via specifying a non-degenerate closed differential 2-form $\omega \in \Omega^2(T^*Q)_{\text{closed}}$, which allows an isomorphism between smooth vector fields and smooth covector fields on T^*Q. The differential $\mathrm{d}H$ of the Hamiltonian $H : T^*Q \to \mathbb{R}$ (which for simplicity we assume to be time-independent) can then be interpreted as a vector field, whose flow describes the evolution of the system. This is a geometric interpretation of Hamilton's equations (4.2). The Legendre transform transforms a function from the tangent bundle to a function from the cotangent bundle. Chapter 4 and 8 of [11] give a short introduction on the geometric description of classical mechanics.

However, in contrast to Lagrangian mechanics, a field has uncountably many degrees of freedom, so this manifold description does not work. Still classical field theory can be described using jet manifolds and jet bundles, which allow to describe partial differential equations (such as the Euler-Lagrange equations) on fibre bundles. This reaches far beyond the scope of this thesis, but for the very inclined reader, we refer to [12].

Like in the discrete case, we can define the **generalised momentum** $\pi_\rho = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_\rho}$. Again, the **Hamiltonian density** $\mathcal{H}$ is obtained by doing a Legendre transform

$$\mathcal{H}(\phi_\rho, \partial_i\phi_\rho, \pi_\rho, x^\mu) = \pi_\rho\dot{\phi}_\rho - \mathcal{L}(\phi_\rho, \partial_\mu\phi_\rho, x^\mu), \tag{4.5}$$

where summation over ρ is implicit. The quantities $\dot{\phi}_\rho$ in the Lagrangian density are replaced by the generalised momenta π_ρ in the Hamiltonian density. The equations of motion are given by

$$\dot{\phi}_\rho = \frac{\partial\mathcal{H}}{\partial\pi_\rho}, \quad \dot{\pi}_\rho = -\frac{\partial\mathcal{H}}{\partial\phi_\rho} + \frac{\mathrm{d}}{\mathrm{d}x^i}\left(\frac{\partial\mathcal{H}}{\partial(\partial_i\phi_\rho)}\right), \quad \frac{\partial\mathcal{H}}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t},$$

which is a field analogue of equations (4.2). In general, we call the expression

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\rho)}\partial_\nu\phi_\rho - \mathcal{L}\delta^\mu_\nu,$$

the **stress-energy tensor**, for δ^μ_ν the Kronecker delta. By a quick calculation, we find

$$\frac{\mathrm{d}T^\mu_\nu}{\mathrm{d}x^\mu} = -\frac{\partial\mathcal{L}}{\partial x^\nu},$$

where implicit summation is over μ. If $\mathcal{L}$ does not depend explicitly on x^μ and the fields are confined to a compact volume $K \subseteq \mathbb{R}^3$ (i.e. they are 0 outside K at all times), then the quantity

$$R_\nu = \int_K T^0_\nu \mathrm{d}^3x$$

is conserved, i.e. $\frac{\mathrm{d}R_\nu}{\mathrm{d}t} = 0$, which follows immediately from $\frac{\mathrm{d}T^\mu_\nu}{\mathrm{d}x^\mu} = 0$ and the divergence theorem. In particular, we are interested in the $\nu = 0$ case, since T^0_0 can be identified with the energy density (c.f. equation (4.5) and the expression for T^0_0). Then integrating over all space gives the total energy, also called the energy functional:

Definition (Energy functional and vacuum states). For a Lagrangian density $\mathcal{L} = \mathcal{L}(\phi_\rho(x^\mu), \partial_\mu\phi_\rho(x^\mu), x^\mu)$, its energy functional is defined by

$$E[\phi] = \int_{\mathbb{R}^3} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\rho} \dot{\phi}_\rho - \mathcal{L} \right) \mathrm{d}^3x. \tag{4.6}$$

A field solution ϕ (a solution to the Euler-Lagrange equations (4.4) for the Lagrangian density $\mathcal{L}$) that minimises E, is called a vacuum state. If all the vacuum states are constant in spacetime, i.e. $\partial_\mu\phi = 0$ for all μ, then we can identify the vacuum states with the constant values they map to, which constitute the vacuum manifold $\mathcal{M}$.

Note that by the above discussion, $E[\phi]$ is independent of time if $\mathcal{L}$ does not explicitly depend on time. Under some general form of the Lagrangian, the vacuum manifold $\mathcal{M}$ is indeed a smooth manifold. This is later specified in proposition 6.1.

Since space and time have an equal footing in this Lagrangian formalism (in contrast to Lagrangian mechanics with a countable degrees of freedom, and the Hamiltonian formalism), it is much easier to make the field theory compatible with special relativity. If we assume the fields are Lorentz covariant and the Lagrangian density $\mathcal{L}$ is a Lorentz scalar[2], then equation (4.4) is clearly Lorentz covariant. We end this section with an example of a relativistic field.

Example (The Klein-Gordon equations). Let ψ be a complex scalar field. Its real and imaginary part are independent real scalar fields, so ψ and $\overline{\psi}$ are also independent. We consider the following Lagrangian:

$$\mathcal{L} = \partial_\mu\overline{\psi}\partial^\mu\psi - m^2\overline{\psi}\psi, \tag{4.7}$$

for m a real constant. It is clearly Lorentz covariant. Using the Euler-Lagrange equations (4.4), we find

$$\partial_\mu\partial^\mu\psi + m^2\psi = 0, \quad \partial_\mu\partial^\mu\overline{\psi} + m^2\overline{\psi} = 0. \tag{4.8}$$

These equations are called the **Klein-Gordon equations**, which describe a free spin-0 particle with mass m (using natural units $c = \hbar = 1$). Solutions are superpositions of travelling waves

$$\psi = Ae^{ik_\mu x^\mu}, \quad -k_\mu k^\mu + m^2 = 0,$$

with A a constant amplitude and k^μ the four-wave vector. By $k^\mu = (\omega, \mathbf{k})$, the waves have dispersion relation $\omega^2 = k^2 + m^2$. For $m = 0$, the Klein-Gordon equations reduce to the wave equation, with waves travelling at the speed of light.

[2] Let $\Lambda \in \mathrm{O}(1,3)$ be a Lorentz transformation, and set $\tilde{\Lambda} = \Lambda^{-1}$. A rank (r,s)-tensor $T \in V^{\otimes r} \otimes (V^*)^{\otimes s}$ for $V = \mathbb{R}^4$ transforms (by choosing a basis for V) via $T^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_s} \to \Lambda^{\mu_1}_{\rho_1} \cdots \Lambda^{\mu_r}_{\rho_r} \tilde{\Lambda}^{\sigma_1}_{\nu_1} \cdots \tilde{\Lambda}^{\sigma_s}_{\nu_s} T^{\rho_1,\dots,\rho_r}_{\sigma_1,\dots,\sigma_s}$. An object transforming in this way under Lorentz transformations, is called Lorentz covariant. For example, the four-velocity u^μ is Lorentz covariant and called a Lorentz vector, as it transforms as a vector: $u^\mu \to \Lambda^\mu_\nu u^\nu$. In contrast, $u_\mu u^\mu$ transforms as a scalar, since $u_\mu u^\mu \to \tilde{\Lambda}^\nu_\mu \Lambda^\mu_\rho u_\nu u^\rho = \delta^\nu_\rho u_\nu u^\rho = u_\mu u^\mu$. $u_\mu u^\mu$ is therefore invariant under Lorentz transformations, and called a Lorentz scalar.

4.3 Electromagnetism as a gauge theory

In this and the following section, we mostly follow chapters 1, 2 and 4 from [13], chapter 5 of [14] and section 2.1 of [15]. These all cover gauge theories. In this section, we present the most well-known gauge theory, namely electromagnetism, which exhibits U(1) symmetry. In the next section, we generalise this to general compact Lie groups G, called Yang-Mills theory.

Amongst the most well-established classical field theories is electromagnetism, which describes the interactions between charges and the electric fields $\mathbf{E}$ and magnetic fields $\mathbf{B}$ in space and time. The epitome of electromagnetism are the Maxwell equations, which are given by

$$\begin{aligned} &\nabla \cdot \mathbf{B} = 0 && \text{(Gauß's magnetic law)}, && \nabla \cdot \mathbf{E} = \rho && \text{(Gauß's law)}, \\ &\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law)}, && \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} && \text{(Maxwell-Ampère law)}. \end{aligned}$$

Here again we have used natural units such that the permittivity of vacuum ε_0 and the permeability of vacuum μ_0 are both 1. Here ρ is the electric charge density ("the amount of charge per volume"), and $\mathbf{J}$ is the current density ("the amount of charge per unit time flowing through a unit area, pointing in the direction the charge is flowing"). By taking the divergence of the Maxwell-Ampère law, and substituting Gauß's law, we derive the following

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

This expression is a local version of the conservation of electric charge. Integrating over a volume $K \subseteq \mathbb{R}^3$ and applying the divergence theorem yields the following result:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\oint_{\partial K} \mathbf{J} \cdot \mathrm{d}\mathbf{a},$$

meaning that the total charge Q in the volume K can only change by moving charges across the boundary ∂K.

Note that Maxwell's equations can be written in a Lorentz covariant manner (for more details than described here, we recommend chapter 12 of [3] for further reading), and even coordinate-free. In fact, it was this observation that led to the formulation of special relativity. However, in this form, it is not immediately apparent that Maxwell's equations are Lorentz covariant, nor do $\mathbf{E}$ and $\mathbf{B}$ transform in a 'nice' way under Lorentz transformations. The solution is describing $\mathbf{E}$ and $\mathbf{B}$ using an alternating rank 2 tensor field, i.e. a smooth 2-form, called the **electromagnetic field tensor**:

$$\begin{aligned} F &= \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= \mathrm{d}t \wedge (E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z) - (B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y) \in \Omega^2(M). \end{aligned}$$

One can check that the components $F_{\mu\nu}$ indeed transform under Lorentz transformations $\Lambda \in \mathrm{O}(3,1)$ as $F_{\mu\nu} \to \tilde{\Lambda}^\rho_\mu \tilde{\Lambda}^\sigma_\nu F_{\rho\sigma}$, for $\tilde{\Lambda} = \Lambda^{-1}$. Thus F is indeed the quantity we are looking for. The Hodge star operator gives another natural smooth 2-form:

$$\begin{aligned} \star F &= \frac{1}{2} \tilde{F}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\mu \\ &= -(E_x \mathrm{d}y \wedge \mathrm{d}z + E_y \mathrm{d}z \wedge \mathrm{d}x + E_z \mathrm{d}x \wedge \mathrm{d}y) - \mathrm{d}t \wedge (B_x \mathrm{d}x + B_y \mathrm{d}y + B_z \mathrm{d}z) \in \Omega^2(M). \end{aligned}$$

Lastly, the quantity $J^\mu = (\rho, \mathbf{J})$ is a Lorentz vector, called the **four-current density**. The corresponding covector is given by $J_\mu = (\rho, -\mathbf{J})$. Since the Maxwell equations are given by derivatives of quantities in F, for our presentation it is more natural to consider the four-current density as a smooth 3-form, which we still denote with J:

$$\begin{aligned} J &= \star(-J_\mu \mathrm{d}x^\mu) = \star(-\rho \mathrm{d}t + J_x \mathrm{d}x + J_y \mathrm{d}y + J_z \mathrm{d}z) \\ &= -\rho \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}t \wedge (J_x \mathrm{d}y \wedge \mathrm{d}z + J_y \mathrm{d}z \wedge \mathrm{d}x + J_z \mathrm{d}x \wedge \mathrm{d}y) \in \Omega^3(M). \end{aligned}$$

Then let us calculate the exterior derivatives of F and $\star F$:

$$\mathrm{d}F = 0 \qquad \text{(Gauß-Faraday law)}, \qquad \mathrm{d}(\star F) = J \qquad \text{(Gauß-Ampère law)}.$$

We thus have reduced the four Maxwell equations to two elegant equations which are Lorentz covariant. The local conservation of electric charge then immediately follows, by $\mathrm{d}J = \mathrm{d}^2(\star F) = 0$. In coordinates, these equations are given by $\partial_\mu F^{\mu\nu} = J^\nu$, $\partial_\mu \tilde{F}^{\mu\nu} = 0$ and $\partial_\mu J^\mu = 0$.

And yet, we can refine these two equations even further. The condition $\mathrm{d}F = 0$ means exactly that F must be a closed 2-form. Recall that we are still working on the smooth manifold $M = \mathbb{R}^4$, which is contractible. By the Poincaré lemma, the second de Rham cohomology group $H^2_{\mathrm{dR}}(M)$ is trivial, meaning that there exists[3] a covector field $A \in \Omega^1(M)$, such that $F = \mathrm{d}A$, or $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in coordinates. Then $\mathrm{d}F = 0$ holds automatically. A is called the **four-potential**. By writing $A^\mu = (V, \mathbf{A})$, the four-potential can be identified with the ordinary scalar and vector potential in electrodynamics, which give the electric and magnetic fields:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Using the four-potential, all of the Maxwell equations can be stated in the following neat form:

$$\mathrm{d}(\star \mathrm{d}A) = J.$$

Note however, that F does define A only up to a closed 1-form $\omega \in \Omega^1(M)_{\text{closed}}$. Since $M = \mathbb{R}^4$ is simply connected, ω is exact, so this means that the equations of motions are invariant under

$$A \mapsto A + \mathrm{d}f, \quad f \in \mathcal{C}^\infty(M).$$

This phenomenon where multiple field configurations correspond to a single physical configuration is called **gauge invariance**, and $f \in \mathcal{C}^\infty(M)$ is said to define a **gauge transformation** of A.

At first glance, the four-potential and gauge transformations might appear to be merely theoretical constructs that provide an elegant formulation of Maxwell's equations. However, their significance extends far beyond formalism. For one, in order to describe electrodynamics in the Lagrangian or Hamiltonian formalism, we have to use the four-potential. For instance, the action integral in equation (4.3) is given by

$$\mathcal{S} = \int_\Omega \left(-\frac{1}{2} \mathrm{d}A \wedge \star \mathrm{d}A + A \wedge J \right) = \int_\Omega \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \right) \mathrm{d}^4 x.$$

The resulting Euler-Lagrange equations are then exactly the Maxwell equations, when applying Hamilton's principle on the field components A_μ.

Additionally, gauge equivalence classes are physically measurable. Two potentials A and A' defined on an open subset $U \subseteq M$ are equivalent if $A' = A + \mathrm{d}f$, for some $f \in \mathcal{C}^\infty(U)$. Since M is simply-connected, every closed smooth 1-form is exact, so each F corresponds with exactly one gauge equivalence class. This is not the case however when $H^1_{\mathrm{dR}}(U)$ is non-trivial, which can be seen in the solenoidal Aharonov-Bohm effect:

[3] We tacitly assume that F is defined everywhere. When a solution F only exists on an open subset $U \subseteq M$ and represents a non-trivial element in $H^2_{\mathrm{dR}}(U)$, then no $A \in \Omega^1(U)$ can exist such that $F = \mathrm{d}A$. An example is $F = -\frac{q}{4\pi} \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$ for a constant $q \in \mathbb{R}_{\neq 0}$, which is only defined on $M \setminus \ell$, for ℓ the line $\ell = \mathbb{R} \times \{(0,0,0)\}$. $[F]$ is a non-trivial element in $H^2_{\mathrm{dR}}(M \setminus \ell)$, which can be seen by noting that $M \setminus \ell$ is homotopy equivalent with S^2, and $H^2_{\mathrm{dR}}(S^2) \cong \mathbb{R}$ is generated by the volume form $\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$. F corresponds with $\mathbf{E} = 0$, $\mathbf{B} = \frac{q}{4\pi r^2} \hat{\mathbf{r}}$, and thus represents a magnetic monopole field. For the curious reader, we refer to chapter 1 of [15].

Example (The solenoidal Aharonov-Bohm effect). Imagine a long impermeable solenoid. When no current is running through the solenoid, there are no electric and magnetic fields. With a current, there is a uniform magnetic field in the solenoid, yet there is still no electric and magnetic field outside the solenoid. So in both cases, $F = 0$ outside the solenoid.
The quantity $\oint A_\mu \mathrm{d}x^\mu$, integrated over a loop in spacetime, is gauge-invariant, i.e. invariant under $A \to A + \mathrm{d}f$, which follows immediately from the gradient theorem. Let γ be a circular loop in space with radius r around the solenoid. Let $\mathcal{S}$ be the disk with radius r, which has γ as boundary. It turns out that in quantum mechanics, this expression (up to a scaling constant) describes the phase difference of the wave function of a particle with charge q passing around the solenoid:

$$\Delta\varphi = \frac{q}{\hbar}\oint_\gamma \mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell}.$$

$\Delta\varphi$ is observable, as is seen in electron scattering experiments. $\Delta\varphi$ is proportional to the magnetic flux through the surface $\mathcal{S}$ using Stokes theorem:

$$\Phi = \int_\mathcal{S} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = \int_\mathcal{S} (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = \oint_\gamma \mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell}.$$

The potential $\mathbf{A} = \frac{\Phi}{2\pi s}\hat{\boldsymbol{\theta}}$ outside the solenoid described in cylindrical coordinates satisfies the above equation and $\mathbf{B} = \nabla \times \mathbf{A} = 0$ there. When no current flows through the solenoid, $\mathbf{A}$ reduces to 0, as there is no flux and thus $\Delta\varphi = 0$, as one would expect. But when current is flowing, there is a magnetic flux Φ, and so $\Delta\varphi = \frac{q\Phi}{\hbar} \neq 0$, which can only be attributed to the potential $\mathbf{A}$, and not to the electric and magnetic fields alone, as in both situations $\mathbf{E} = 0$ and $\mathbf{B} = 0$. Geometrically, the corresponding covector fields are $A = 0$ and $A = -\frac{\Phi}{2\pi}\mathrm{d}\theta$, and their difference is closed but not exact. This means that while these describe the same electromagnetic field tensor F, they belong to different gauge equivalence classes.

Lastly, gauge transformations are essential in coupling the field of a charged particle to that of the electromagnetic field, and by requiring invariance under local phase shifts, this coupling arises naturally. We give an example of this, called scalar electrodynamics, which describes the interaction between the electromagnetic field and a charged spin 0-particle, such as a charged pion $\pi^\pm$ particle:

Example (Scalar electrodynamics). We consider a complex scalar field ψ, and a Lagrangian density of the form

$$\mathcal{L} = \partial_\mu\overline{\psi}\partial^\mu\psi - V(|\psi|), \tag{4.9}$$

for some potential V only depending on the amplitude of ψ. We have already seen such an example, namely the Klein-Gordon Lagrangian (4.7), where $V(|\psi|) = m^2|\psi|^2$. We see that $\mathcal{L}$ is invariant under global phase shifts $\psi \to e^{i\alpha}\psi$, and so are the equations of motions

$$\partial_\mu\partial^\mu\psi + V'(|\psi|)\psi = 0, \quad \partial_\mu\partial^\mu\overline{\psi} + V'(|\psi|)\overline{\psi} = 0.$$

But say we want $\mathcal{L}$ and the equations of motion to be invariant under local phase shifts $\psi \mapsto e^{i\alpha(x^\mu)}\psi$, for $\alpha \in \mathcal{C}^\infty(M)$ a smooth function from spacetime M to $\mathbb{R}$. For convenience, we write $\alpha(x) := \alpha(x^\mu)$, but we still assume that α depends on *all* spacetime coordinates. The potential $V(|\psi|)$ is invariant under this transformation $\psi \mapsto e^{i\alpha(x)}\psi$, but the term $\partial_\mu\overline{\psi}\partial^\mu\psi$ is clearly not, as it will contain partial derivatives of α. We denote a gauge transformation with a subscript 'gt'. Since we can add a four-gradient of a scalar function to A_μ without changing any physics, we choose the following transformations:

$$\psi_{\text{gt}} = e^{i\alpha(x)}\psi, \qquad \overline{\psi}_{\text{gt}} = e^{-i\alpha(x)}\overline{\psi}, \qquad (A_\mu)_{\text{gt}} = A_\mu + \frac{1}{e}\partial_\mu\alpha, \tag{4.10}$$

for e the electric charge. This factor is mathematically quite arbitrary, but it is used to give physical meaning. By defining the smooth function $g = e^{i\alpha} : M \to \mathrm{U}(1)$, and letting g^{-1} the composition of g with the group inverse of U(1), equation (4.10) has the following form, which is illuminating for the next section

$$\psi_{\mathrm{gt}} = g\psi, \qquad \overline{\psi}_{\mathrm{gt}} = \overline{g}\overline{\psi}, \qquad (A_\mu)_{\mathrm{gt}} = A_\mu + \frac{i}{e} g \partial_\mu (g^{-1}). \tag{4.11}$$

We define the expression $D_\mu := \partial_\mu - ieA_\mu$. Then by looking at the transformation rule in equation (4.10), we see that $(D_\mu \psi)_{\mathrm{gt}} = e^{i\alpha(x)} D_\mu \psi$. This is really interesting, since this means that $\overline{D}_\mu \overline{\psi} D^\mu \psi$ is invariant under gauge transformations, exactly what we wanted. So $\mathcal{L}$ is gauge-invariant, when we replace the 'ordinary' derivatives ∂_μ with the operator D_μ. The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{D}_\mu \overline{\psi} D^\mu \psi - V(|\psi|) \tag{4.12}$$

is invariant under local phase shifts, and so are the equations of motion (which are obtained by varying the fields ψ, $\overline{\psi}$ and A_μ in the action integral):

$$D_\mu D^\mu \psi + V'(|\psi|)\psi = 0, \quad \overline{D}_\mu \overline{D}^\mu \overline{\psi} + V'(|\psi|)\overline{\psi} = 0, \quad \partial_\mu F^{\mu\nu} = J^\nu, \tag{4.13}$$

where the four-current density J^μ is given by $J^\mu = -ie\left(\overline{\psi} D^\mu \psi - \psi \overline{D}^\mu \overline{\psi}\right)$, which is also gauge-invariant. Asking for invariance for local phase shifts automatically led to an electromagnetic-like interaction!

The above example can straightforwardly be generalised to describe spin-1/2 particles, such as electrons and positrons. However, in this case, the Lagrangian (4.9) has to be modelled after the Dirac equation, which makes this adaptation more complex due to the bispinor nature of the field in the Dirac equation. The resulting Lagrangian density is crucial however, as it forms the foundation of quantum electrodynamics, a theory that unifies the principles of electromagnetism, special relativity and quantum mechanics. It gives a complete description of the interaction between matter and light.

4.4 Yang-Mills theory

In the previous section, we saw that demanding local invariance under transformations $\psi \to g(x)\psi$, for $g : M \to \mathrm{U}(1)$ a smooth function, leads automatically to Maxwell's equations. Yang-Mills theory seeks to generalise this gauge invariance to compact Lie groups G. To achieve this, we have to provide a geometric interpretation of the generalisation of the four-potential A, the electromagnetic field tensor F and gauge transformations. This section will heavily rely on the theory of principal bundles discussed in chapter 3, especially section 3.3. In this section, $\mathbb{F}$ is the field $\mathbb{R}$ or $\mathbb{C}$.

General gauge theory

In general, one would start with a compact[4] Lie group G and a principal G-bundle $\pi : \mathcal{P} \to M$, for M a general smooth manifold representing spacetime. This setup describes the global structure of the G-action on the fields. The $\mathfrak{g}$-valued 1-form $A \in \Omega^1(\mathcal{P}, \mathfrak{g})$ will be an Ehresmann connection on this principle bundle. A is also referred to as a *gauge field* in the physical literature. By choosing a local gauge, which is a local section $s \in \Gamma^\infty(U, \mathcal{P})$, we can identify A with a $\mathfrak{g}$-valued 1-form $A_s \in \Omega^1(U, \mathfrak{g})$ on an open subset $U \subseteq M$, via pulling back A via s: $A_s = s^*A$. This is exactly the four-potential in electromagnetism (up to a factor i) when $G = \mathrm{U}(1)$, since $\mathfrak{g} = i\mathbb{R}$. The curvature of A is given by $F = \mathrm{d}A + \frac{1}{2}[A, A] \in \Omega^2(\mathcal{P}, \mathfrak{g})$. Since pullbacks commute with the bilinear form $[\cdot, \cdot]$, $F_s = s^*F \in \Omega^2(U, \mathfrak{g})$ is given by $F_s = \mathrm{d}A_s + \frac{1}{2}[A_s, A_s]$. It can be interpreted as a generalisation of the electromagnetic field tensor on an open subset $U \subseteq M$.

We also want to describe the interactions of fields (describing particles) with the gauge field A. These fields can be each of a different type and are acted upon by different Lie group representations of G, so we expect that in general, each field corresponds with a section ϕ of a corresponding vector bundle, which depends on the principal G-bundle $\pi : \mathcal{P} \to M$. This is exactly the notion of an associated vector bundle. Then the connection A induces a so-called covariant derivative on each associated vector bundle, which is the D_μ operator we have encountered earlier. A gauge transformation is a smooth map $g : U \to G$, which can be identified with a bundle automorphism of the principal G-bundle $\pi|_{\mathcal{P}_U} : \mathcal{P}_U \to U$, where $\mathcal{P}_U = \pi^{-1}(U)$. Note that some mathematicians call this bundle automorphism a gauge transformation, and a gauge transformation is called a *physical* gauge transformation). Gauge transformations act on A_s, F_s and ϕ according to some transformation rule.

This framework allows us to construct a gauge-invariant Lagrangian, akin to equation (4.12), which forms the foundation of Yang-Mills theory. By adding all types of interactions, Yang-Mills theory can even describe the Standard Model of particle physics, but some additional quantum field theory is needed to make it work. For a detailed mathematical treatment of Yang-Mills theory, we refer to chapters 4 and 5 of [14].

Some simplifications

For our purposes however, this presentation is a bit excessive and unnecessary, since we only consider $M = \mathbb{R}^4$. Since M is contractible, all principal G-bundles $\pi : \mathcal{P} \to M$ are in fact trivialisable (corollary 4.2.9 in [14]), so there always exists a global section $s \in \Gamma^\infty(M, \mathcal{P})$. We can thus work on M instead of open subsets $U \subseteq M$ (for instance, this is not possible in the monopole example given in footnote [3]). From now on, we will be only working with $A_s \in \Omega^1(M, \mathfrak{g})$ and $F_s \in \Omega^2(M, \mathfrak{g})$, and we suppress the subscript s.

We also want to introduce a field ϕ, which is acted upon by G. Let $\rho : G \to \mathrm{Aut}(V)$ be a Lie group representation of G, for V a finite dimensional $\mathbb{F}$-vector space. In general, this is done by letting ϕ be a section of an associated vector bundle derived from the principal G-bundle and the representation ρ. For our purposes, fields can be simplified to smooth maps $\phi : M \to V$, where G acts on ϕ via

$$g\phi := \rho(g) \circ \phi.$$

When the representation ρ is clear, it is often left out. A covariant derivative is a generalisation

[4] Physicists prefer to work with compact Lie groups G in Yang-Mills theory, because the Killing-form, which is a symmetric bilinear form on $\mathfrak{g}$ given by $\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$, is negative-definite. Furthermore, the representation theory of compact Lie groups is much nicer than that of general Lie groups. Oftentimes it is presumed that $\mathfrak{g}$ is semi-simple as well, since then κ is non-degenerate. This has certain implications for the Lagrangian constructed in Yang-Mills theory. We do not have to worry about these details, however.

of the directional derivative in Euclidean geometry, and in this case it is given by

$$\begin{aligned}\nabla^A : \mathfrak{X}(M) \times \mathcal{C}^\infty(M, V) &\to \mathcal{C}^\infty(M, V),\\ (X, \phi) &\mapsto \mathrm{d}\phi(X) + \mathrm{d}\rho(A(X))\phi.\end{aligned} \tag{4.14}$$

Since $\mathrm{d}\rho$ is Lie algebra homomorphism $\mathrm{d}\rho : \mathfrak{g} \to \mathrm{End}(V)$, $\mathrm{d}\rho$ also acts on ϕ in a natural way. The covariant derivative satisfies the following properties:

- $\nabla^A_X \phi$ is $\mathcal{C}^\infty(M)$-linear in $X \in \mathfrak{X}(M)$ and $\mathbb{R}$-linear in $\phi \in \mathcal{C}^\infty(M, V)$;
- the Leibniz rule is satisfied: for $f \in \mathcal{C}^\infty(M)$, we have $\nabla^A_X(f\phi) = (\mathrm{d}f)(X)\phi + f\nabla^A_X\phi$.

Using these simplifications, there is no need to introduce associated bundles or the general theory of connections and covariant derivatives. Once more, the interested reader is encouraged to look at chapters 4 and 5 of [14], and the relevant chapters in [6].

We can proceed to Yang-Mills theory:

Definition (Ingredients for Yang-Mills theory on $\mathbb{R}^4$). We require the following data:

- A compact Lie group G, which we assume to be a matrix group, with Lie algebra $\mathfrak{g}$.
- A Lie group representation $\rho : G \to \mathrm{Aut}(V)$, for V a finite dimensional $\mathbb{F}$-vector space;
- $A \in \Omega^1(M, \mathfrak{g})$, which is called the gauge field.
- The field strength F of A, which is given by $F = \mathrm{d}A + \frac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g})$.
- $\phi \in \mathcal{C}^\infty(M, V)$ a smooth function representing a field;
- Smooth maps $g : M \to G$, called gauge transformations; they act on A and ϕ via

$$A_{\mathrm{gt}}(x) = g(x)Ag(x)^{-1} + g(x) \cdot \mathrm{d}\left(g(x)^{-1}\right), \quad \phi_{\mathrm{gt}} = g(x)\phi := \rho(g(x)) \circ \phi. \tag{4.15}$$

The transformation rule in equation in (4.15) needs some explanation. The first part of the transformation rule of A can be recognised as the adjoint representation, and indeed represents a $\mathfrak{g}$-valued 1-form (c.f. proposition 2.12). In the second part, $g \cdot \mathrm{d}g^{-1}$ is understood to be matrix multiplication of g with the componentwise exterior derivative of g^{-1}. It is non-trivial that $g \cdot \mathrm{d}g^{-1} \in \Omega^1(M, \mathfrak{g})$, but it can easily be verified for $g = \exp \circ \alpha$, for α a smooth function $\alpha : M \to \mathfrak{g}$. Observe that the transformation rule (4.15) is a generalisation of equation (4.11), which we met when considering electrodynamics.

We want to consider these transformation rules in coordinates. Let $n = \dim \mathfrak{g}$ be the dimension of the Lie algebra $\mathfrak{g}$ of G, and choose a basis $(T^1, \ldots, T^n)$ for $\mathfrak{g}$. Then we can write $A \in \Omega^1(M, \mathfrak{g})$ and $F \in \Omega^2(M, \mathfrak{g})$ as follows in local coordinates:

$$A(x) = A^a_\mu(x) T^a \otimes \mathrm{d}x^\mu, \quad F(x) = \frac{1}{2} F^a_{\mu\nu}(x) T^a \otimes (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu), \quad A^a_\mu, F^a_{\mu\nu} \in \mathcal{C}^\infty(M).$$

Here the summation over a is implicit. For conciseness, we define $A_\mu := A^a_\mu T^a$ and $F_{\mu\nu} := F^a_{\mu\nu} T^a$; these components can be considered as $n \times n$ matrix functions. The components of A and F are of course related as well, which follows immediately for the component expression for the exterior derivative for $\mathfrak{g}$-valued forms:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{4.16}$$

Note that equation (4.16) reduces to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for abelian Lie groups G, such as U(1) in electromagnetism. For a gauge transformation $g : M \to G$, A_μ and $F_{\mu\nu}$ transform as

$$(A_\mu)_{\mathrm{gt}} = gA_\mu g^{-1} + g\partial_\mu g^{-1}, \quad (F_{\mu\nu})_{\mathrm{gt}} = gF_{\mu\nu}g^{-1}. \tag{4.17}$$

Again, we recognise the adjoint representation. So when G is abelian, $(F_{\mu\nu})_{\mathrm{gt}} = F_{\mu\nu}$, which is what we have seen for the electromagnetic field tensor, but for a general compact Lie group G, this does not need to hold.

Looking at the transformation rule in (4.17), we see that the expression $\mathcal{L}_{\mathrm{YM}} = \frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu})$ is gauge invariant and Lorentz covariant, so it is ideal for a term in a Lagrangian density. The resulting equations by applying the action principle are called the **Yang-Mills equations**[5]. These are a generalisation of the Gauß-Ampère law in vacuum in electromagnetism. Note that the second Bianchi identity (see section 3.3) already states that $\mathrm{d}F = [F, A]$, which is a generalisation to the Gauß-Faraday law. In the case of the abelian group U(1), this identity reduces to $\mathrm{d}F = 0$. Often in a physics context, the basis $(T^a)_a$ is normalised, such that $\operatorname{tr}\big(T^aT^b\big) = -\frac{1}{2}\delta_{ab}$. This condition makes calculations a lot easier. In that case, $\mathcal{L}_{\mathrm{YM}}$ can be written as $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a}$.

Now we introduce the field $\phi \in \mathcal{C}^\infty(M, V)$. In coordinates, the covariant derivative (c.f. equation (4.14)) is given by

$$\nabla^A_\mu \phi = \partial_\mu \phi + \mathrm{d}\rho(A_\mu)\phi. \tag{4.18}$$

For the matrix group $G \subseteq \mathrm{GL}_n(\mathbb{F})$, two representations are often used in physics, namely the **fundamental representation** $\rho_{\mathrm{fund}} : G \to \mathrm{GL}_n(\mathbb{F})$, with $\rho_{\mathrm{fund}}(X) = X$, and the adjoint representation $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$, which we covered in section 2.4. We can also consider the trivial representation $\rho_{\mathrm{triv}} : G \to \mathrm{Aut}(V)$, by sending every element to the identity (then there is no coupling between A and ϕ). In those three cases, equation (4.18) reduces to

$$\nabla^A_{\mu,\mathrm{fund}}\phi = \partial_\mu\phi + A_\mu\phi, \quad \nabla^A_{\mu,\mathrm{Ad}}\phi = \partial_\mu\phi + [A_\mu, \phi], \quad \nabla^A_{\mu,\mathrm{triv}}\phi = \partial_\mu\phi. \tag{4.19}$$

In these cases, it is easily verified that $(\nabla^A_\mu\phi)_{\mathrm{gt}} = g\nabla^A_\mu\phi$, but this holds in general.

Lastly, we want a G-equivariant symmetric bilinear form $\langle\cdot,\cdot\rangle_V : V \times V \to \mathbb{F}$ on V, i.e. for all $v, w \in V$ and $g \in G$, $\langle gv, gw\rangle_V = \langle v, w\rangle_V$. These can be the usual Euclidean inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$ for the fundamental representations of SO(n) and SU(n), respectively. We then are able to construct the gauge-invariant term $\mathcal{L}_{\mathrm{kin}} = \frac{1}{2}\left\langle\nabla^A_\mu\phi, \nabla^{\mu,A}\phi\right\rangle_V$. Putting everything together, and adding a potential function $U(\phi)$ which is invariant under gauge transformations, we get the following gauge-invariant Lagrangian density

$$\mathcal{L} = \frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\left\langle\nabla^A_\mu\phi, \nabla^{\mu,A}\phi\right\rangle_V - U(\phi), \tag{4.20}$$

which is the generalisation to equation (4.12).

In chapter 6, we will be only considering Lagrangian densities of the form (4.20), which is called a **Yang-Mills-Higgs Lagrangian density**. The focus will be on the compact groups $G = \mathrm{U}(1)$, $G = \mathrm{SO}(n)$ and $G = \mathrm{SU}(n)$.

[5]As an unrelated side node, the Yang-Mills equations are a wonderful example of a topic in physics that later turned out to be extremely useful in mathematics. By studying moduli spaces of Yang–Mills connections, Donaldson and Freedman proved the existence of certain topological 4-manifolds, which cannot be given a smooth structure, as opposed to topological n-manifolds for $n \leq 3$, which can always be given a smooth structure. Even more remarkably, their theorems allowed to construct *exotic* $\mathbb{R}^4$'s, smooth 4-manifolds homeomorphic to the Euclidean $\mathbb{R}^4$, but not diffeomorphic. By a theorem of Taubes, there are even uncountably many of them! [16]

Chapter 5 | Higher homotopy groups

Homotopy groups will be the final mathematical ingredient we need to understand and classify topological defects. Homotopy groups $\pi_n(X, x_0)$ are higher-dimensional generalisations of the fundamental group $\pi_1(X, x_0)$ of a pointed topological space (X, x_0). They can detect 'holes' the fundamental group can simply not see, but are known for their intricacy; even the homotopy groups of such a 'simple' space as S^2 are hard to compute. And yet homotopy groups retain some desirable properties of fundamental groups, such as the functoriality, being invariant under homotopies and behaving well under products. Homotopy groups are even abelian for $n \geq 2$.

In chapter 3, we considered fibre bundles $F \longrightarrow E \xrightarrow{\pi} B$. It turns out that the homotopy groups of the fibre F, the total space E and base space B are related in a long exact sequence. This gives a tool for calculating homotopy groups by considering certain fibre bundles. In particular, for G a Lie group and $H \subseteq G$ a closed subgroup, the fibre bundle $H \longrightarrow G \xrightarrow{\pi} G/H$ is considered. As a consequence, there is a relationship between the homotopy groups of G, H and G/H. This particular instance is worked out for the compact Lie groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, and we show that their homotopy groups can be linked to the homotopy groups of k-spheres S^k.

5.1 Introduction to homotopy groups

The following section is based on section 4.1 of Hatcher's *Algebraic Topology* [1].

Let (X, x_0) be a pointed topological space, i.e. a topological space X with a distinguished base point $x_0 \in X$. The fundamental group $\pi_1(X, x_0)$ can either be considered as the homotopy classes of loops $\gamma : [0, 1] \to X$ starting and ending at the base point x_0, or homotopy classes of continuous maps $f : (S^1, s_0) \to (X, x_0)$. Both viewpoints generalise to higher homotopy groups, and each can be fruitful at times.

We start of with the first perspective. For the natural number $n \geq 1$, we let $I^n = [0, 1]^n$ be the n-cube with its usual Euclidean topology. Its boundary ∂I^n consists of all the points in I^n with at least one coordinate equal to 0 or 1. For completeness, we allow $n = 0$ as well and define $I^0 = \{0\}$, $\partial I^0 = \varnothing$.

We need a higher-dimensional analogue of the homotopy of loops used in the fundamental group, in order to generalise the fundamental group. We give a more general definition, which we need later.

Definition (Homotopy of continuous maps between paired spaces). Let (X, A) and (Y, B) be paired spaces, i.e. topological spaces X and Y with $A \subseteq X$ and $B \subseteq Y$ subspaces with the subspace topology. A continuous map $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$, such that $f(A) \subseteq B$. Let $f, g : (X, A) \to (Y, B)$ be continuous maps. A **homotopy** of f and g is a continuous map $H : [0,1] \times X \to Y$ such that for all $x \in X$ and $t \in [0,1]$, we have

$$H(0,x) = f(x), \quad H(1,x) = g(x), \quad H(t,A) \subseteq B.$$

If there exists such a homotopy, we call f and g **homotopic**, denoted as $f \simeq g$.

It is not hard to see that $\simeq$ is an equivalence relation. In particular, $\pi_n(X, x_0)$ is defined as the equivalence classes of continuous maps from I^n to X, which map ∂I^n to x_0:

$$\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \to (X, x_0) \text{ continuous}\} /_{\simeq}.$$

For $n = 0$, by identifying $f : (I^0, \partial I^0) \to (X, x_0)$ with $f(0)$, $\pi_0(X, x_0)$ consists of the classes of points that can be connected by a path, thus $\pi_0(X, x_0)$ can be considered as the path-components of X. For $n = 1$, $\pi_1(X, x_0)$ consists of path-homotopy classes of paths which are x_0 at 0 and 1, and as a consequence, $\pi_1(X, x_0)$ is the fundamental group.

Like loops, we can concatenate continuous maps $f, g : (I^n, \partial I^n) \to (X, x_0)$:

$$\begin{aligned} f + g : (I^n, \partial I^n) &\to (X, x_0) \\ (s_1, \dots, s_n) &\mapsto \begin{cases} f(2s_1, s_2, \dots, s_n), & 0 \leq s_1 \leq 1/2, \\ g(2s_1 - 1, s_2, \dots, s_n), & 1/2 \leq s_1 \leq 1. \end{cases} \end{aligned}$$

Likewise, inversion is defined as

$$\begin{aligned} -f : (I^n, \partial I^n) &\to (X, x_0), \\ (s_1, \dots, s_n) &\mapsto f(1 - s_1, s_2, \dots, s_n). \end{aligned}$$

Theorem 5.1. *Let (X, x_0) be a pointed topological space. For $n \geq 1$, $\pi_n(X, x_0)$ with the aforementioned operations on the homotopy classes and identity element $[x \mapsto x_0]$, forms a group, called the **n-th homotopy group**. For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.*

Proof. The proof that $\pi_1(X, x_0)$ is a group can be adjusted with little effort to show that in general the addition and inversion are well-defined for homotopy classes and that these make $\pi_n(X, x_0)$ into a group. This is because the addition and inversion formulae only involve the first coordinate.

Let $n \geq 2$. The fact that $\pi_n(X, x_0)$ is abelian can be wonderfully illustrated, see also figure 5.1. Let $f, g : (I^n, \partial I^n) \to (X, x_0)$ be arbitrarily given continuous maps. Then $f + g : (I^n, \partial I^n) \to (X, x_0)$ can be regarded as the map where the left part of I^n is mapped as f and the right part as g. The interior of these halves can be shrunk in a continuous manner within the n-cube to two smaller n-cubes; everything outside these smaller n-cubes is mapped to x_0. But these smaller n-cubes can be moved to each other's initial place without intersecting each other along the way. This is done by first moving down the n-cube mapping accordingly to f, and simultaneously moving up the n-cube mapping accordingly to g. This can be repeated in other directions as well, such that at the end, the two n-cubes have swapped. Then by making these two bigger again, we have constructed a homotopy of $f + g$ to $g + f$, such that ∂I_n is sent to x_0 at every time. Thus $[f] + [g] = [f + g] = [g + f] = [g] + [f]$, proving that $\pi_n(X, x_0)$ is abelian. ■

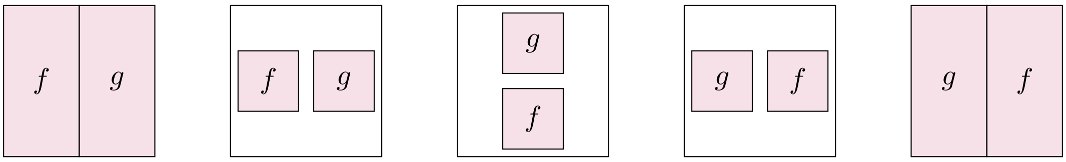


Fig. 5.1: Illustration why $f + g \simeq g + f$. The parts of I^n given by the image of f and g are shrunk, are exchanged and then expanded. The white parts and the black borders are mapped to x_0.

There is also another way to look at homotopy groups. For $n \geq 1$, a continuous map $f : (I^n, \partial I^n) \to (X, x_0)$ can also be regarded as a continuous map $g : (S^n, s_0) \to (X, x_0)$ for a chosen point $s_0 \in S^n$. The reason is that S^n is homeomorphic to the quotient space under the following equivalence relation on I^n:

$$x \sim y \iff x = y \ \vee \ x, y \in \partial I^n.$$

So we are really identifying the boundary ∂I^n with a single point s_0. We reobtain f (up to homotopy) from g by 'opening up' S^n at s_0 and sending the boundary ∂I^n to x_0. In this way, we can regard $\pi_n(X, x_0)$ also as homotopy classes of continuous functions from (S^n, s_0) to (X, x_0):

$$\pi_n(X, x_0) = \{f : (S^n, s_0) \to (X, x_0) \text{ continuous}\} /_{\simeq}.$$

This is also an equivalent definition for $n = 0$.

For $f, g : (S^n, s_0) \to (X, x_0)$ continuous, $f + g$ can be understood as $(f \vee g) \circ \psi : (S^n, S_0) \to (X, x_0)$. $\psi : S^n \to S^n \vee S^n$ takes an equator of S^n passing through s_0 (which is an S^{n-1}) and squeezes it to the point s_0, resulting in two copies of S^n, kissing each other at the point s_0. The resulting space is the wedge sum $S^n \vee S^n$. Then f and g are combined into one map $f \vee g$, which is possible since $f(s_0) = g(s_0) = x_0$:

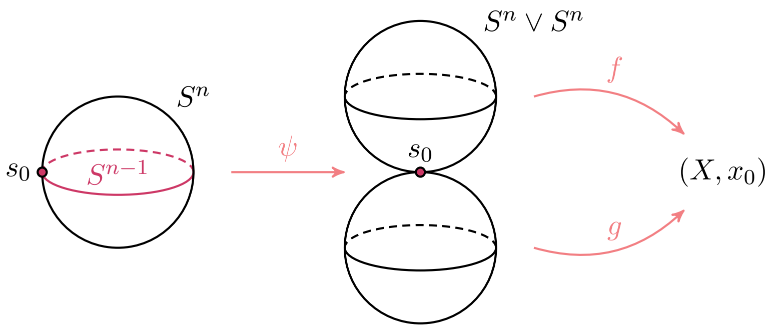


Fig. 5.2: Sketch of $f + g$. S^n is split into two n-spheres by shrinking the equator to a point. Then the upper n-sphere maps to X according to f, the lower n-sphere according to g.

As for the fundamental group, if X is path-connected, then for all $n \geq 1$ we have that $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic for two base points $x_0, x_1 \in X$. In most situations we then simply speak of $\pi_n(X)$. For the proof we refer to section 4.1 of *Algebraic Topology* [1]. Functoriality also carries over:

Proposition 5.2. *Let $\varphi : (X, x_0) \to (Y, y_0)$ be a continuous map. Then for all $n \geq 0$, this induces a well-defined map*

$$\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0),$$
$$[f] \mapsto [\varphi \circ f].$$

Moreover, $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_n(X,x_0)}$ *and for* $\psi : (Y, y_0) \to (Z, z_0)$ *continuous, one has* $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$*. For* $n \geq 1$*,* φ_* *is a group homomorphism.*

Proof. For $n \geq 0$, let $f, g : (S^n, s_0) \to (X, x_0)$ be continuous, such that $[f] = [g]$ in $\pi_n(X, x_0)$. Then there exists a homotopy $H : [0, 1] \times S^n \to X$ from f to g, which sends s_0 to x_0 at all times. Then $\varphi \circ H$ is a homotopy from $\phi \circ f$ to $\phi \circ g$, sending s_0 to y_0 at all times. So $\varphi_*[f] = \varphi_*[g]$, and hence it is well-defined. $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_n(X,x_0)}$ and $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ follow immediately from the definition.

Now let $f, g : (S^n, s_0) \to (X, x_0)$ be continuous. Then by definition of the addition, $\varphi \circ (f + g) = (\varphi \circ f) + (\varphi \circ g)$, so

$$\varphi_*([f] + [g]) = \varphi_*([f + g]) = [(\varphi \circ f) + (\varphi \circ g)] = \varphi_*[f] + \varphi_*[g],$$

concluding the proof. ■

Proposition 5.2 tells us exactly that the following assignments are functors

$$\pi_0 : \mathbf{Top_\bullet} \to \mathbf{Set_\bullet}, \quad \pi_1 : \mathbf{Top_\bullet} \to \mathbf{Grp}, \quad \pi_n : \mathbf{Top_\bullet} \to \mathbf{Ab}, n \geq 2.$$

This immediately implies that homeomorphic topological spaces have isomorphic homotopy groups. But we can do better:

Proposition 5.3. *Let* $\varphi, \psi : (X, x_0) \to (Y, y_0)$ *be continuous such that* $\varphi \simeq \psi$*. Then* $\varphi_* = \psi_*$*.*

Proof. Let $H : [0, 1] \times X \to Y$ such a homotopy, i.e. H is continuous such that for all $t \in [0, 1]$ and $x \in X$, we have $H(0, x) = \varphi(x)$, $H(1, x) = \psi(x)$ and $H(t, x_0) = y_0$. Let $f : (S^n, s_0) \to (X, x_0)$ be continuous, and define $\tilde{H} : [0, 1] \times S^n \to Y$ via $(t, s) \mapsto H(t, f(s))$. Per construction, $\tilde{H}$ is continuous and $\tilde{H}(0, s) = \varphi \circ f(s)$, $\tilde{H}(1, s) = \psi \circ f(s)$ and $\tilde{H}(t, s_0) = y_0$. Thus $\tilde{H}$ is a homotopy from $\varphi \circ f$ to $\psi \circ f$. In particular, $\varphi_*[f] = \psi_*[f]$, hence $\varphi_* = \psi_*$. ■

Corollary 5.4. *Let* $\varphi : (X, x_0) \to (Y, y_0)$ *be a homotopy equivalence fixing the base points. Then* $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ *is an isomorphism for all* $n \geq 1$*.*

This corollary states amongst other things that all contractible spaces (X, x_0) (where the homotopy keeps the base point x_0 fixed) have the same homotopy groups as the space consisting of a single point, an important example being $\mathbb{R}^n$ for $n \geq 1$. Since there is only a single pointed map from (S^k, s_0) to the singleton space, $\pi_k(X, x_0)$ is trivial for all $k \geq 1$.

By looking at the fundamental group, another expected property of homotopy groups is that it respects products. This is indeed the case:

Theorem 5.5. *Let* $\{(X_\alpha, x_\alpha)\}_\alpha$ *be a collection of pointed spaces and* (X, x) *the pointed product space. Then* $\pi_n(X, x) \cong \prod_\alpha \pi_n(X_\alpha, x_\alpha)$ *for all* $n \geq 0$*.*

Proof. Let $p_\alpha : (X, x) \to (X_\alpha, x_\alpha)$ and $q_\alpha : \prod_\beta \pi_n(X_\beta, x_\beta) \to \pi_n(X_\alpha, x_\alpha)$ be the projection maps. By the universal property of the product of pointed sets / groups / abelian groups, there exists a unique morphism $q : \pi_n(X, x) \to \prod_\beta \pi_n(X_\beta, x_\beta)$ such that $q_\alpha \circ q = (p_\alpha)_*$. It thus only remains to show that q is a bijection.

Let $f, g : (S^n, s_0) \to (X, x)$ be continuous maps, such that $q([f]) = q([g])$. Then $(p_\alpha)_*[f] = (p_\alpha)_*[g]$ for all α, so there exist homotopies $H_\alpha : [0,1] \times S^n \to X_\alpha$ from $p_\alpha \circ f$ to $p_\beta \circ g$, such that $H_\alpha(t, s_0) = x_\alpha$. The universal property of products gives a continuous map $H : [0,1] \times S^n \to X$, such that $H(0,s) = f(s)$, $H(1,s) = g(s)$ and $H(t, s_0) = x$. This follows from the uniqueness of the map $H(0, \_) : S^n \to X$, such that $p_\alpha \circ H(0, \_) = p_\alpha \circ f$; f clearly satisfies this condition. $H(1,s) = g(s)$ and $H(t, s_0) = x$ follow similarly. We conclude that $[f] = [g]$ holds, proving injectivity.

Now let $f_\alpha : (S^n, s_0) \to (X_\alpha, x_\alpha)$ be arbitrarily given continuous maps. The universal property of products gives use a continuous map $f : (S^n, s_0) \to (X, x_0)$, such that $p_\alpha \circ f = f_\alpha$. Then $q_\alpha \circ q([f]) = (p_\alpha)_*[f] = [f_\alpha]$, concluding surjectivity. ■

Covering maps give also an easy way to calculate homotopy groups:

Proposition 5.6. *Let $p : (Y, y_0) \to (X, x_0)$ be a covering map. Then $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective and $p_* : \pi_n(Y, y_0) \to \pi_n(X, x_0)$ is an isomorphism for $n \geq 2$.*

Proof. For $n \geq 1$, let $f : (S^n, s_0) \to (Y, y_0)$ such that $[f] \in \ker p_*$. Then there exists a homotopy $H : [0,1] \times S^n \to X$ between $p \circ f$ and the constant map $s \mapsto x_0$. By the homotopy lifting property of covering maps, there exists a homotopy $\tilde{H} : [0,1] \times S^n \to Y$ such that $H = p \circ \tilde{H}$ from f to the constant map $s \mapsto y_0$, hence p_* is injective. For $n \geq 2$, a continuous map $f : (S^n, s_0) \to (X, x_0)$ can be lifted to a continuous map $\tilde{f} : (S^n, s_0) \to (Y, y_0)$ such that $p \circ \tilde{f} = f$ proving surjectivity (such a lift exists because S^n is simply connected and locally path-connected for $n \geq 2$). ■

For instance, this proposition gives isomorphisms of homotopy groups for $k \geq 2$ for several standard spaces, such as the circle S^1, the n-torus $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$ for $n \geq 1$ and the real projective space $\mathbb{RP}^n$ for $n \geq 1$:

$$\pi_k(S^1) \cong \pi_k(\mathbb{R}) \cong 0, \quad \pi_k(T^n) \cong \pi_k(\mathbb{R}^n) \cong 0, \quad \pi_k(\mathbb{RP}^n) \cong \pi_k(S^n).$$

Moreover, the homotopy groups of connected compact surfaces Σ can also easily be calculated, considering these are all classified, alongside with their universal covers[1]. If Σ is not homeomorphic to S^2 or $\mathbb{RP}^2$, the universal covering space of Σ is contractible, giving $\pi_k(\Sigma) \cong 0$ for $k \geq 2$.

5.2 The long exact sequence for fibre bundles

This section takes inspiration from the *fibre bundle* section in Hatcher's *Algebraic Topology* [1].

In this section, we study the relation between the homotopy groups of the fibre space F, the total space E and the base space B of a fibre bundle, as defined in section 3.1. Let $F \longrightarrow E \xrightarrow{\pi} B$ be a fibre bundle[2], and choose a base point $b_0 \in B$ in the base manifold and a point $x_0 \in F = E_{b_0}$ in the corresponding fibre of b_0. Then we can relate the homotopy groups $\pi_n(F, x_0)$, $\pi_n(E, x_0)$

[1] A compact surface is either homeomorphic to S^2, a connected sum of tori T^2 or a connected sum of real projective planes $\mathbb{RP}^2$ (see theorems 6.15 and 10.22 in [17]). S^2 is the universal covering space of S^2 and $\mathbb{RP}^2$, $\mathbb{R}^2$ of T^2 and $\mathbb{RP}^2 \# \mathbb{RP}^2$ (the Klein bottle) and the hyperbolic disk B^2 of the remaining compact surfaces (theorem 12.29 in [17]).

[2] The construction in this section works in general for so-called *Serre fibrations*, but these go beyond our scope; a fibre bundle is a special kind of Serre fibration, see proposition 4.48 in [1].

and $\pi_n(B, b_0)$ for $n \geq 1$. Note that the inclusion map $i : F \to E$ already induces a group homomorphism $i_* : \pi_n(F, x_0) \to \pi_n(E, x_0)$, and the projection $\pi : E \to B$ induces a group homomorphism $\pi_* : \pi_n(E, x_0) \to \pi_n(B, b_0)$.

We would like to link $\pi_n(B, b_0)$ and $\pi_{n-1}(F, x_0)$ as well. This is a lot more intricate, however. First, we consider I^{n-1} as a face of ∂I^n, namely the side with $s_n = 0$. Then we define

$$J^{n-1} = \overline{\partial I^n \setminus I^{n-1}} \subseteq \partial I^n,$$

which is just the union of all sides but I^{n-1}. We define the map $\partial : \pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$ as follows. For a continuous map $f : (I^n, \partial I^n) \to (B, b_0)$, we lift it to a continuous map $\tilde{f} : (I^n, \partial I^n) \to (E, F)$, such that $\tilde{f}(J^{n-1}) = \{x_0\}$ and $\pi \circ \tilde{f} = f$. Then by restricting $\tilde{f}$ to $I^{n-1} \subseteq I^n$, the result is a continuous map $g := \tilde{f}\big|_{I^{n-1}} : (I^{n-1}, \partial I^{n-1}) \to (F, x_0)$. We set $\partial[f] = [g]$.

The existence of a lift $\tilde{f}$ that satisfies the above conditions and the fact that $\partial{:}\pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$ is well-defined and a group homomorphism for $n \geq 2$ is non-trivial. For details on this and the proof of the forthcoming profound theorem, we refer to [1].

Theorem 5.7. *Let $F \longrightarrow E \xrightarrow{\pi} B$ be a fibre bundle. For base points $b_0 \in B$ and $x_0 \in F = E_{b_0}$, the following long sequence is exact:*

$$\cdots \longrightarrow \pi_{n+1}(E, x_0) \xrightarrow{\pi_*} \pi_{n+1}(B, b_0) \xrightarrow{\partial_*} \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{\pi_*} \pi_n(B, b_0) \xrightarrow{\partial_*} \pi_{n-1}(F, x_0) \xrightarrow{i_*} \pi_{n-1}(E, x_0) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(E, x_0) \xrightarrow{\pi_*} \pi_1(B, b_0) \xrightarrow{\partial_*} \pi_0(F, x_0) \xrightarrow{i_*} \pi_0(E, x_0) \xrightarrow{\pi_*} \pi_0(B, b_0) \longrightarrow 0$$

Note that $\pi_0(F, x_0)$ and $\pi_0(E, x_0)$ are merely pointed sets, and not groups. Exactness at $\pi_0(F, x_0)$, $\pi_0(E, x_0)$ and $\pi_0(B, b_0)$ is understood as the image of the incoming arrow is precisely those elements that map under the outgoing arrow to the homotopy class of the constant map.

Theorem 5.7 allows us to relate homotopy groups of the fibre, total space and base space in a fibre bundle. For instance, for the fibre bundle $S^1 \longrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n$ for $n \geq 0$, which we have seen in section 3.2, we get $\pi_k(\mathbb{CP}^n) \cong \pi_k(S^{2n+1})$ for $k \geq 3$, since $\pi_k(S^1) \cong 0$ for $k \geq 2$. In particular, for the the complex Hopf fibration, we find $\pi_k(S^2) \cong \pi_k(S^3)$ for $k \geq 3$.

Recall that for a Lie group G and a closed subgroup H, there is a natural principal-H bundle $H \longrightarrow G \xrightarrow{\pi} G/H$. Applying theorem 5.7, we get the following:

Corollary 5.8. *Let G be a Lie group and $H \subseteq G$ be a closed subgroup. Then the following sequence is exact:*

$$\cdots \longrightarrow \pi_n(H, e) \longrightarrow \pi_n(G, e) \longrightarrow \pi_n(G/H, H) \longrightarrow \pi_{n-1}(H, e) \longrightarrow \cdots \longrightarrow \pi_0(G/H, H) \longrightarrow 0$$

5.3 Homotopy groups of spheres and O(n), SO(n), U(n) and SU(n)

A few of the most used Lie groups in physics are O(n), SO(n), U(n) and SU(n), for $n \geq 1$. These are all compact Lie groups, and thus have an elegant representation theory, making them especially well-suited for physical applications such as describing particles in particle physics. A natural question to ask – notably in the context of topological defects and symmetry breaking, which are covered in the next chapter – is what are the homotopy groups of these Lie groups?

We start with SO(n), which is the group of rotations in $\mathbb{R}^n$. We choose a point $v \in \mathbb{R}^n$, say $v = (0, \dots, 0, 1)$. The orbit of v under the action of SO(n) is precisely S^{n-1}. Thus by restricting the SO(n)-action to S^{n-1}, we see that SO(n) acts transitively on S^{n-1}. Matrices $A \in \mathrm{SO}(n)$ that stabilise v are the matrices of the form $A = \left(\begin{smallmatrix} B & 0 \\ w & 1 \end{smallmatrix}\right)$, for $B \in \mathrm{Mat}_{n-1}(\mathbb{R})$ and $w \in \mathbb{R}^{n-1}$. $A^\top A = I_n$ implies $B \in \mathrm{SO}(n-1)$ and $w = 0$. Thus the stabiliser of v can be associated with $\mathrm{SO}(n-1)$. By theorem 3.1, we have that $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ is diffeomorphic to S^{n-1}. Then corollary 5.8 relates the homotopy groups of $\mathrm{SO}(n-1)$, SO(n) and S^{n-1}. Particularly, since S^{n-1} is path-connected for $n \geq 2$ and $\mathrm{SO}(1) \cong \{e\}$, we see that SO($n$) is path-connected as well for $n \geq 1$.

O(n) is not connected, which can be seen by looking at the determinant map $\det : \mathrm{O}(n) \to \mathbb{R}$, which has image $\{\pm 1\}$. By taking only the matrices with determinant 1, we get SO(n), which is path-connected. SO(n) then must be the identity component of O(n). $\mathrm{O}(n)/\mathrm{SO}(n) \cong \mathbb{Z}_2$ by proposition 2.1, so corollary 5.8 gives $\pi_k(\mathrm{O}(n), e) \cong \pi_k(\mathrm{SO}(n), e)$ for $k \geq 1$.

Now we turn our focus to U(n) and SU(n). Note that U(n) is diffeomorphic to $\mathrm{SU}(n) \times S^1$ via the map (for $n \geq 2$ this is not a Lie group homomorphism!)

$$\begin{aligned} \psi : \mathrm{SU}(n) \times S^1 &\to \mathrm{U}(n), \\ (A, z) &\mapsto \mathrm{diag}(z, 1, \dots, 1)A. \end{aligned}$$

Then by theorem 5.5, $\pi_k(\mathrm{U}(n), e) \cong \pi_k(\mathrm{SU}(n), e)$, for $k \geq 0$, $k \neq 1$, and $\pi_1(\mathrm{U}(n), e) \cong \pi_1(\mathrm{SU}(n), e) \times \mathbb{Z}$. As a consequence, we only have to focus on the homotopy groups of SU(n). In this instance, SU(n) acts on $\mathbb{C}^n$. As in the SO(n) case, we now get in a comparable manner $\mathrm{SU}(n)/\mathrm{SU}(n-1) \cong S^{2n-1}$. Therefore the homotopy groups of SU(n), $\mathrm{SU}(n-1)$ and S^{2n-1} are related in an exact sequence.

The previous examples illustrate the necessity of the homotopy groups of n-spheres, so what are they? This turns out to be a really complicated problem in algebraic topology. The first few are given in table 5.1:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{12}$	${\mathbb{Z}_2}^2$	${\mathbb{Z}_2}^2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$

Table 5.1: The first 9 homotopy groups of S^n, for $1 \leq n \leq 9$, from section 4.1 in [1].

A few intriguing patterns are visible in table 5.1:

- $\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$ and below that diagonal, all entries are 0. Spaces (X, x_0) such that $\pi_k(X, x_0) \cong 0$ for all $0 \leq k \leq n$ for a given n, are called **n-connected**. The fact that S^{n+1} is n-connected is a consequence of the Hurewicz theorem, which states that for an n-connected topological space (X, x_0) for $n \geq 1$, $H_k(X) \cong 0$ for $1 \leq k \leq n$, and $H_{n+1}(X) \cong \pi_{n+1}(X, x_0)$, where $H_k(X)$ is the k-th singular homology group of X.
- Along the coloured diagonals, the homotopy groups are the same. These are called the stable homotopy groups, and are due to the Freudenthal suspension theorem.
- The only non-trivial homotopy group of S^1 is the fundamental group; this is a consequence of proposition 5.6, since the universal covering space of S^1 is contractible.
- $\pi_n(S^2) \cong \pi_n(S^3)$ for $n \geq 3$, which is a result of the complex Hopf fibration $S^1 \longrightarrow S^3 \xrightarrow{\pi} S^2$. The quaternionic Hopf fibration $S^3 \longrightarrow S^7 \xrightarrow{\pi} S^4$ also turns up in the table. Note that the inclusion $i : S^3 \to S^7$ is homotopic to the constant map, by virtue of $\pi_3(S^7) \cong 0$. Then proposition 5.3 implies that i_* maps everything to the identity, so the long exact sequence in theorem 5.7 gives short exact sequences

$$0 \longrightarrow \pi_{n+1}(S^7) \longrightarrow \pi_{n+1}(S^4) \longrightarrow \pi_n(S^3) \longrightarrow 0$$

This short exact sequence splits, resulting in $\pi_{n+1}(S^4) \cong \pi_n(S^3) \oplus \pi_{n+1}(S^7)$ for $n \geq 0$.

Calculating non-trivial homotopy groups of spheres goes far beyond the scope of this thesis. However, Hatcher provides an extensive coverage of theorems related to the calculation of homotopy groups of spheres in [1] and [18]. We end this chapter with stating some homotopy groups of SO(n) and SU(n) – we shall need them later in chapter 6. Using table 5.1 and the facts that $\mathrm{SO}(n)/\mathrm{SO}(n-1) \cong S^{n-1}$ and $\mathrm{SU}(n)/\mathrm{SU}(n-1) \cong S^{2n-1}$, one can calculate *some* of the homotopy groups in the following two tables. For instance, the homotopy groups $\pi_k(\mathrm{SO}(n))$ and $\pi_k(\mathrm{SU}(n))$ for a fixed k stabilise for n big enough, because S^{n+1} is n-connected. To calculate all homotopy groups of SO(n) and SU(n) however, one has to resort to advanced algebraic topology machinery. For further details, we refer to section 3 of [19].

	π_1	π_2	π_3	π_4	π_5	π_6
SO(1)	0	0	0	0	0	0
SO(2)	$\mathbb{Z}$	0	0	0	0	0
SO(3)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
SO(4)	$\mathbb{Z}_2$	0	$\mathbb{Z}^2$	${\mathbb{Z}_2}^2$	${\mathbb{Z}_2}^2$	${\mathbb{Z}_{12}}^2$
SO(5)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
SO(6)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
SO(7)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
SO(8)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

Table 5.2: The first 6 homotopy groups of SO(n). The coloured cells indicate where the homotopy group stabilises.

	π_1	π_2	π_3	π_4	π_5	π_6
SU(1)	0	0	0	0	0	0
SU(2)	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
SU(3)	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$
SU(4)	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
SU(5)	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
SU(6)	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
SU(7)	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
SU(8)	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Table 5.3: The first 6 homotopy groups of SU(n). The coloured cells indicate where the homotopy group stabilises.

Chapter 6 | Topological defects

We are finally ready to tackle the main topic of this thesis: *topological defects*. All previous chapters come together here. Before we can consider topological defects, we first must dive into spontaneous symmetry breaking using some field theory. While a given Lagrangian remains invariant under some action of a Lie group G, the vacuum states need not to. In general a vacuum state is invariant under a Lie subgroup $H \subseteq G$. We demonstrate that the vacuum manifold $\mathcal{M}$ can be identified with the homogeneous space G/H.

We can also break local symmetries. Unlike the breaking of global symmetries, which has physical implications, breaking local symmetries is purely a theoretical construct. This construction is crucial, however, as massless fields can acquire mass; this is the essence of the Higgs mechanism.

The Kibble mechanism explains how spontaneous symmetry breaking happens in nature, and that it can lead to topological defects. These are field solutions that cannot be continuously deformed into a trivial vacuum solution. We will give an exact definition in section 6.4. We list a couple of topological defects: domain walls, strings, monopoles and textures. Homotopy groups then come up naturally, as they describe whether these topological defects can form or not. Finally, to bring these concepts to life, we present a detailed example of a Grand Unified theory using the developed homotopy theory. We prove that it allows for monopoles, and give some profound physical implications.

6.1 Breaking of global symmetries

Spontaneous symmetry breaking is the phenomenon in which a Lagrangian density is invariant under a certain action of a Lie group (called the **symmetry group** in this context), but the vacuum states are not. This happens quite often in physics, for instance:

- **Ferromagnets**. Above the Curie temperature, the spins in a ferromagnetic material are essentially randomly oriented. But below the Curie temperature, neighbouring spins align in the same direction, breaking the rotational symmetry, i.e. invariance under SO(3).
- **Crystallisation**. When crystals form, continuous translational symmetry is broken to discrete translational symmetry.
- **The Higgs mechanism**. At very high energy, the force carriers of the electroweak interaction, the photon and the $W^{\pm}$- and Z^0-bosons, are all believed to be massless. Below a critical temperature, the $W^{\pm}$- and Z^0-bosons gain mass whereas the photon stays massless. This process is described by the Higgs mechanism. In this case, the Lagrangian is invariant under $SU(2) \times U(1)$, but a vacuum state under a subgroup isomorphic to $U(1)$ (this is a

different U(1) than the right factor in the product Lie group). A more detailed description is given in section 6.2.

As can be seen in these examples, there is some phase transition going on. We explain this phenomenon in section 6.3. In this section, we illustrate three examples of spontaneous symmetry breaking, adapted from chapter 5 of [13], where the Lagrangian is invariant under certain *global* transformations (in contrast to local gauge transformations, which we cover in section 6.2).

Breaking a discrete symmetry

One of the simplest examples where symmetry breaking occurs is the following Lagrangian density for a real scalar field φ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi), \quad V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \eta^2\right)^2, \tag{6.1}$$

for positive constants λ and η. $\mathcal{L}$ is invariant under $\varphi \mapsto \pm\varphi$, thus invariant under an action of $\mathbb{Z}_2$. The Euler-Lagrange equations (4.4) give the following equation of motion:

$$\partial_\mu\partial^\mu\varphi + \lambda\left(\varphi^2 - \eta^2\right)\varphi = 0. \tag{6.2}$$

The potential V is bounded from below, so it makes sense to look at the vacuum states – the field solutions with minimal energy. The energy functional (c.f. equation (4.6)) is given by

$$E[\varphi] = \int_{\mathbb{R}^3} \left((\partial_t\varphi)^2 + (\partial_i\varphi)^2 + V(\varphi)\right) \mathrm{d}^3x,$$

where $(\partial_i\varphi)^2$ is shorthand for $(\partial_x\varphi)^2 + (\partial_y\varphi)^2 + (\partial_z\varphi)^2$. Since $(\partial_t\varphi)^2$, $(\partial_i\varphi)^2$ and $V(\varphi)$ are all non-negative, minimising $E[\varphi]$ is equivalent to minimising each term individually. This gives that $\partial_\mu\varphi = 0$ and that $V(\varphi)$ must be minimal, so $\varphi^2 = \eta^2$ (since we assume φ to be smooth, in particular continuous, these equations must hold everywhere). Equation (6.2) is automatically satisfied with these conditions. Thus the vacuum states are the constant fields

$$\varphi(x^\mu) = \pm\eta.$$

These fields can be identified with the smooth manifold $\mathcal{M} = \{\pm\eta\}$, which we recall from chapter 4, is called the vacuum manifold. A chosen vacuum state, say $\varphi^{(0)}(x^\mu) = \eta$, is *not* invariant under $\mathbb{Z}_2$, as $-\varphi^{(0)}$ is another state than $\varphi^{(0)}$.

In quantum field theory, particles correspond with small perturbations χ around the vacuum state $\varphi^{(0)}$. This means that we write the field as $\varphi(x^\mu) = \varphi^{(0)} + \chi(x^\mu)$, for χ a small scalar field. The Euler-Lagrange equations for this field are

$$\mathcal{L}_\chi = \mathcal{L}[\varphi^{(0)} + \chi] = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V_\chi(\chi), \quad V_\chi(\chi) = \lambda\eta^2\chi^2 + \lambda\eta\chi^3 + \frac{\lambda}{4}\chi^4.$$

Thus the Lagrangian for perturbations is not invariant under $\mathbb{Z}_2$ either, which was to be expected. When χ is *really* small, the Lagrangian $\mathcal{L}_\chi$ reduces to

$$\mathcal{L}_\chi = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m^2\chi^2, \quad m = \sqrt{2\lambda}\eta.$$

This is exactly the Klein-Gordon Lagrangian for a real scalar field (c.f. (4.7) for the Lagrangian of a complex scalar field). In this sense, m can be interpreted as the mass of the particle described by the field χ.

Breaking a U(1) symmetry

We make the previous example a little bit complicated, by considering a complex scalar field ψ, and the following Lagrangian density

$$\mathcal{L} = \partial_\mu \overline{\psi} \partial^\mu \psi - V(|\psi|), \quad V(|\psi|) = \frac{\lambda}{4}\left(|\psi|^2 - \eta^2\right)^2. \tag{6.3}$$

Here λ and η are again positive constants. Once more, ψ and $\overline{\psi}$ are independent fields, as in the Klein-Gordon equations (4.8), yielding the following equation of motion:

$$\partial_\mu \partial^\mu \psi + \frac{\lambda}{2}(|\psi|^2 - \eta^2)\psi = 0, \tag{6.4}$$

and the same equation with ψ replaced by $\overline{\psi}$. This Lagrangian density is invariant under global phase rotations $\psi \to e^{i\alpha}\psi$, for $\alpha \in \mathbb{R}$, so invariant under the Lie group U(1). Minimising the energy functional, taking into account that equation (6.4) must hold, gives the vacuum states

$$\psi(x^\mu) = \eta e^{i\beta}, \quad \beta \in \mathbb{R}.$$

Thus the vacuum manifold is $\mathcal{M} = S^1_\eta$. Again, a vacuum state $\psi^{(0)}$ is not invariant under U(1) anymore. We shortly consider perturbations again around a vacuum state; we choose $\psi^{(0)} = \eta$. The perturbations can be written as

$$\psi(x^\mu) = \left(\psi^{(0)} + \frac{1}{\sqrt{2}}\rho(x^\mu)\right) e^{i\alpha(x^\mu)},$$

The new Lagrangian in terms of the fields ρ and α has a term $\frac{1}{2}\lambda\eta^2\rho^2$ and no α^2 term. This means that ρ has a mass of $\sqrt{\lambda}\eta$, whereas α is massless. A theorem by Goldstone states that massless fields always occur in spontaneous symmetry breaking of a continuous symmetry. To be exact, the number of massless fields occurring in spontaneous symmetry breaking is at least the dimension of the vacuum manifold $\mathcal{M}$.

Partial breaking of a symmetry

In the previous two examples, the Lagrangian is invariant under a Lie group G, whereas a vacuum state is only invariant under the trivial group. Symmetry can also be *partly* broken, thus a vacuum state is still invariant under a non-trivial closed subgroup H of G. We now consider such an example.

Let ϕ be an n-tuple field, so a field $\phi : M \to \mathbb{R}^n$ for $n \geq 2$, and a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\langle \partial_\mu \phi, \partial^\mu \phi \rangle - V(\|\phi\|), \quad V(\|\phi\|) = \frac{\lambda}{4}\left(\|\phi\|^2 - \eta^2\right)^2. \tag{6.5}$$

Here we have used the standard Euclidean norm $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ and the induced norm $\|\cdot\|$. This Lagrangian is invariant under the transformations $\phi \mapsto A\phi$ for $A \in \mathrm{SO}(n)$, as

$$\langle \partial_\mu (A\phi), \partial^\mu (A\phi) \rangle = (A\partial_\mu \phi)^\top A \partial^\mu \phi = \partial_\mu \phi^\top A^\top A \partial^\mu \phi = \partial_\mu \phi^\top \partial^\mu \phi = \langle \partial_\mu \phi, \partial^\mu \phi \rangle.$$

An analogous derivation gives $\|A\phi\| = \|\phi\|$. The vacuum states are

$$\phi(x^\mu) = v, \quad v \in S^{n-1}_\eta,$$

where S^{n-1}_η is the $(n-1)$-sphere with radius η. We identify these fields with the vacuum manifold $\mathcal{M} = S^{n-1}_\eta$. We now choose a vacuum state, say $\phi^{(0)} = v$, $v = (0, \ldots, 0, \eta) \in S^{n-1}_\eta$.

Which $A \in \mathrm{SO}(n)$ keep $\phi^{(0)}$ invariant? These form exactly the stabiliser of $\phi^{(0)}$ of the action of $\mathrm{SO}(n)$ on the vacuum manifold S_η^{n-1}. A brief computation provides

$$\mathrm{Stab}_v(\mathrm{SO}(n)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(n-1) \right\},$$

which we identify with $\mathrm{SO}(n-1)$. Thus we see that a vacuum state is only invariant under $\mathrm{SO}(n-1)$; we say $\mathrm{SO}(n)$ is spontaneously broken to $\mathrm{SO}(n-1)$. This can easily be visualised for $n = 3$, see figure 6.1.

Note that $\mathrm{SO}(n)$ acts transitively on the vacuum manifold S_η^{n-1}, so by theorem 3.1,

$$S_\eta^{n-1} \cong \mathrm{SO}(n)/\,\mathrm{SO}(n-1),$$

i.e. the vacuum manifold is diffeomorphic to the quotient of the whole symmetry group $\mathrm{SO}(n)$ by the stabiliser $\mathrm{SO}(n-1)$. By the transitiveness of the action, all stabilisers are Lie group isomorphic by conjugation, so by the above diffeomorphism, it does not matter which vacuum state we chose. Furthermore, by knowing that the symmetry group $\mathrm{SO}(n)$ acts transitively on the vacuum manifold, and the stabiliser is $\mathrm{SO}(n-1)$, we find the vacuum manifold $\mathcal{M} \cong \mathrm{SO}(n)/\,\mathrm{SO}(n-1)$.

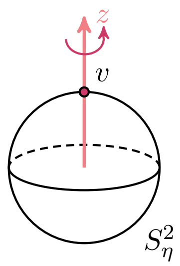


Fig. 6.1: Sketch for $n = 3$: SO(3) is spontaneously broken to SO(2). The vacuum manifold is S_η^2. We can see that the vacuum state $\phi^{(0)}(x^\mu) = v$, for $v = (0, 0, \eta)$, remains invariant under rotations around the z-axis, corresponding to the group of 2D rotations SO(2). However, v is not invariant under rotations around any other rotation axis.

The general case

The above results generalise to the ordinary case where the Lagrangian density is invariant under a Lie group G, where we assume that G acts transitively on the vacuum manifold $\mathcal{M}$. Even gauge fields can be added (but this is not necessary):

Proposition 6.1. *Let ϕ be a field, which we consider as a smooth map $\phi : M \to \mathbb{R}^n$. We assume that G acts smoothly on $\mathbb{R}^n$, via some Lie group representation. Let $\mathcal{L}$ be a Yang-Mills-Higgs Lagrangian density of the form (4.20), which is invariant under the Lie group G. We assume that U takes on a global minimum value $x_{min} \in \mathbb{R}$ somewhere. Assume furthermore that G acts transitively on the vacuum manifold $\mathcal{M}$, and that G spontaneously breaks down to a subgroup $H \subseteq G$. Then the vacuum manifold $\mathcal{M}$ is indeed a smooth manifold, and $\mathcal{M} \cong G/H$.*

Proof. The energy functional associated to the Lagrangian density (4.20) is at a minimum when $F_{\mu\nu} = 0$ (we then choose $A_\mu = 0$) and $\partial_\mu \phi = 0$. ϕ must thus be a constant function, hence we can identify the set of vacuum states with the set

$$\mathcal{M} = \{\phi(0) : \phi \text{ is a vacuum state}\} = U^{-1}(x_{\min}) \subseteq \mathbb{R}^n.$$

We choose a vacuum state $v \in \mathcal{M}$, and let $H = \mathrm{Stab}_v(G)$ be the stabiliser under the smooth G-action on $\mathbb{R}^n$. It is easily shown that H is a closed subgroup of G. When we restrict the G-action to $\mathcal{M}$ (this action need not be smooth, as $\mathcal{M}$ has no manifold structure yet), the stabiliser of v is still H. Since G acts transitively on $\mathcal{M}$, by theorem 21.20 in [2], $\mathcal{M}$ has a unique smooth structure, such that the G-action on $\mathcal{M}$ is smooth. By transitiveness, we can use theorem 3.1, which states that $\mathcal{M} \cong G/H$. ■

Proposition 6.1 is most apparent in the $\mathrm{SO}(n)$ example, but it also holds in the $\mathbb{Z}_2$ and $\mathrm{U}(1)$ example; there H is trivial.

6.2 Breaking of local symmetries: the Higgs mechanism

In the same manner, we can 'break' local symmetry like a global symmetry. This breaking has – opposed to the breaking of global symmetry, which we shall see in the next section – no physical manifestation; it is purely a theoretical device. This mechanism, called the Higgs mechanism, is really important however, as it allows describing gauge fields with mass, such as the W$^{\pm}$- and Z-bosons in the weak interaction. This section is based on chapter 6 in [13].

Breaking a local U(1) symmetry

We consider an instance of the Lagrangian (4.12), which is the Lagrangian for electromagnetism with a complex scalar field ψ:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{D}_\mu\overline{\psi}D^\mu\psi - V(|\psi|), \quad V(|\psi|) = \frac{\lambda}{4}\left(|\psi|^2 - \eta^2\right)^2, \quad D_\mu\psi = \partial_\mu\psi - ieA_\mu\psi. \tag{6.6}$$

This is essentially the same Lagrangian density as in equation (6.3), but a U(1) gauge field is added. $\mathcal{L}$ is invariant under local gauge transformations $(A_\mu)_{\text{gt}} = A_\mu + \frac{1}{e}\partial_\mu\alpha(x)$, $\psi_{\text{gt}} = e^{i\alpha(x)}\psi$, for a smooth function $\alpha : M \to \mathbb{R}$. We can ask ourselves the same question as we did in the previous section: what are the vacuum states, and are they invariant under these gauge transformations? The energy functional this time depends both on the fields A and ψ:

$$E[A_\mu, \psi] = \int_{\mathbb{R}^3}\left(\frac{1}{4}(F_{ij})^2 + \frac{1}{2}(F_{0i})^2 + \overline{D}_0\overline{\psi}D_0\psi + \overline{D}_i\overline{\psi}D_i\psi + V(|\psi|)\right)\mathrm{d}^3x. \tag{6.7}$$

Equation (6.7) is obtained by using the definition of the energy functional in (4.6). This integral is further simplified by using the divergence theorem and the equations of motion, given by equation (4.13).

Again, we can minimise each term individually, which gives $F_{\mu\nu} = 0$, $D_\mu\psi = 0$ and $|\psi| = \eta$. Note that the gauge invariance implies that when $(A_\mu^{(0)}, \phi^{(0)})$ is a vacuum state, that $(A_\mu^{(0)} + \frac{1}{e}\partial_\mu\alpha, e^{i\alpha}\phi^{(0)})$ is also a vacuum state, for $\alpha \in \mathcal{C}^\infty(M)$ a smooth function. Since $F_{\mu\nu} = 0$, we can write $A_\mu = \frac{1}{e}\partial_\mu\beta$, for a $\beta \in \mathcal{C}^\infty(M)$. This implies that $\partial_\mu(e^{-i\beta}\psi) = 0$, so the vacuum solutions are of the form

$$A_\mu(x^\nu) = \frac{1}{e}\partial_\mu\beta, \quad \psi(x^\nu) = \eta e^{i\beta(x^\nu)}, \quad \beta \in \mathcal{C}^\infty(M).$$

Like in previous section, we consider perturbations around the a vacuum state. We choose $A_\mu^{(0)} = 0$ and $\psi^{(0)} = \eta$. Needless to say, these are not invariant under U(1) anymore. Perturbations around $A_\mu^{(0)}$ are described by A_μ itself, perturbations around $\psi^{(0)}$ are described by

$\psi(x^\mu) = \left(\psi^{(0)} + \frac{1}{\sqrt{2}}\rho(x^\mu)\right) e^{i\alpha(x^\mu)}$. Filling in these expressions in (6.6), we find the following long Lagrangian:

$$\mathcal{L}_{\rho,\alpha} = \left(\frac{1}{\sqrt{2}}\partial_\mu\rho + ie\left(A_\mu - \frac{1}{e}\partial_\mu\alpha\right)\left(\eta + \frac{1}{\sqrt{2}}\rho\right)\right)\left(\frac{1}{\sqrt{2}}\partial^\mu\rho - ie\left(A^\mu - \frac{1}{e}\partial^\mu\alpha\right)\left(\eta + \frac{1}{\sqrt{2}}\rho\right)\right)$$
$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\eta^2\rho^2 - \frac{\lambda}{2\sqrt{2}}\eta\rho^3 - \frac{\lambda}{16}\rho^4.$$

In order to simplify this expression, we define new fields $B_\mu := A_\mu - \frac{1}{e}\partial_\mu\alpha$, $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Note that both B_μ and $G_{\mu\nu}$ are gauge-invariant. The Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}M^2 B_\mu B^\mu - \frac{1}{2}m^2\rho^2 + \mathcal{L}_{\text{int}}, \quad M = \sqrt{2}e\eta, \quad m = \sqrt{\lambda}\eta,$$

where $\mathcal{L}_{\text{int}}$ contains higher order interaction terms. Two things are important to notice: the field α is not present in the Lagrangian anymore, and the vector field B^μ has a mass M. Note that terms of the form $m^2 A_\mu A^\mu$ are in general not gauge-invariant, so these do not appear in Lagrangian. However in this way, we can construct massive vector fields.

The above mechanism, called the **Higgs mechanism**, describes how a massless field A_μ 'absorbs' the massless field α, and becomes massive. The number of degrees of freedom stay the same however. Before symmetry breaking, the massless A_μ field has two degrees of freedom (compare this with the two transverse polarisation states of a photon), and α has a single degree of freedom, so in total there are three. When α gets 'eaten up', the massive vector field B_μ has three degrees of freedom; the mass allows for an extra longitudinal polarisation mode.

The Higgs mechanism is possible due to the particular potential for the field φ; if $\varphi = 0$ were the only vacuum state, perturbations around the vacuum state would result in the same Lagrangian. In this context, the field φ is called the **Higgs field**.

The electroweak force

The Higgs mechanism is really significant in the Standard Model. During the 1950's and 1960's, physicists tried to combine electromagnetism and the weak interaction (the force attributable to radiactive decay) into an electroweak interaction, in order to explain parity non-conservation in weak interactions. The Lagrangian describing the electroweak interaction is invariant under $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, where the subscripts tell something about how the representations of these groups act on the corresponding fields. The 'L' means that SU(2) acts only on so-called 'left-handed' particles, whereas the 'Y' is the weak hypercharge, which is a certain property of elementary particles. These details do not matter for the discussion, however. Initially, at very high temperatures before $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is broken, there are four massless fields (a single one for each generator of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, so $3 + 1 = 4$ in total). The symmetry group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ breaks down to a the subgroup $\mathrm{U}(1)_{\text{em}}$ via the Higgs mechanism. Then linear combinations of the original four fields give rise to three massive vector fields – describing the $\mathrm{W}^\pm$- and Z-bosons, mediating the weak interaction –, and a single massless vector field – corresponding with the photon in electrodynamics.

For a thorough discussion on this unification of electromagnetism and the weak interaction, called **Weinberg–Salam theory**, chapter 6 of [13] is certainly recommended. Most notably, the electroweak interaction is indeed observed, as well as the masses of $\mathrm{W}^\pm$- and Z-bosons. In 2012, CERN announced the groundbreaking discovery of the Higgs boson – the particle associated to the Higgs field – based on experiments in the Large Hadron Collider [20].

6.3 The Kibble mechanism

The Lagrangian densities discussed so far are simple models, but describing fields representing true physical particles and their interactions in nature is of course far more intricate and complex. In reality, different kind of corrections must be added to the classical potential V, in order to describe the real interactions. This new potential is called the effective potential V_{eff}. For instance, quantum effects have to be taken into account, such as quantisation of fields. For example, quantum mechanically the electromagnetic field consists of packets of discrete energy, called photons.

Furthermore, thermal interactions are also important, especially in cosmology, where the universe was much hotter in the past. These interactions depend on the number of different types of fermions and bosons, as identical fermions cannot occupy the same quantum state by the Pauli exclusion principle, whereas bosons can. In [21], a simplified correction model, leaving out most of quantum corrections, is calculated for the U(1)-invariant Lagrangian in equation (6.3) (without gauge field) and (6.6) (with gauge field). For brevity we only consider the U(1)-model without a gauge field, but this example generalises to other models, also including gauge fields. The effective potential is given by

$$V_{\text{eff}}(|\psi|) = V(|\psi|) + \left(\alpha|\psi|^2 - \beta\right) T^2 - \gamma T^4, \quad V(|\psi|) = \frac{\lambda}{4}\left(|\psi|^2 - \eta^2\right)^2,$$

for α, β, γ some positive constants, and T the temperature. The nature of α, β and γ is not really important for our discussion (for the interested reader, see [21]). What is important though, is the different temperature régimes:

$$V_{\text{eff}}(|\psi|) \approx \frac{\lambda}{4}\left(|\psi|^2 - \eta^2\right)^2, \quad T \ll \eta, \qquad V_{\text{eff}}(|\psi|) \approx \frac{\lambda}{4}|\psi|^4 + \left(\alpha|\psi|^2 - \beta\right) T^2 - \gamma T^4, \quad T \gg \eta.$$

In the first régime, we have already seen that the vacuum manifold is $\mathcal{M} = S^1_\eta$. This form of the potential is to be expected, as at low temperature there is not much thermal interactions. At high temperatures however, the temperature contributions dominate, and the vacuum manifold is $\mathcal{M} = \{0\}$, so the field takes on values near 0 everywhere. So there is a phase transition from $\psi = 0$ being a stable minimum of the effective potential, to $\psi = 0$ being an unstable local maximum and $|\psi| = \eta$ being stable for constant complex scalar fields ψ. This is illustrated in figure 6.2. The critical temperature in this model is $T_c = \sqrt{6}\eta$.

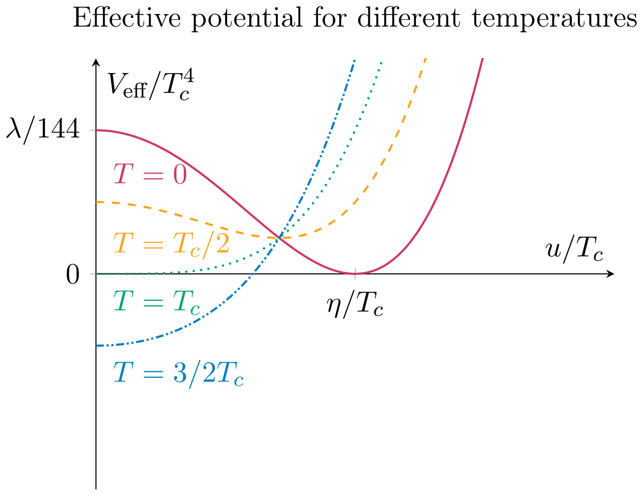


Fig. 6.2: Sketch of effective potential $V_{\text{eff}}(u)$ for the different temperature régimes, for $u \in \mathbb{R}_{\geq 0}$. For the Lagrangian $\mathcal{L} = \partial_\mu \overline{\psi} \partial^\mu - V_{\text{eff}}(|\psi|)$, minimising the energy gives that ψ is a constant field with magnitude u, such that $V_{\text{eff}}(|\psi|)$ is minimal. It thus suffices to look at the minima of $V_{\text{eff}}(u)$. $u = 0$ is stable for $T > T_c$ and unstable for $T < T_c$. In this range, there is a non-zero vacuum expectation value.

When cooling down from $T > T_c$ to $T < T_c$, the field takes on values near the new vacuum manifold $\mathcal{M}$. There is no physical reason to believe that this field takes on the same value everywhere in space; as in general the field is uncorrelated at large distances. This is especially true in cosmology, where the speed of light determines a bound on causality. So we expect local regions where the field takes on nearby values in the vacuum manifold. However, this need not be globally so, it would be very well possible that the field cannot relax to a vacuum state due to a topological obstruction. This field is called a **topological defect**, and the formation process of topological defects by means of spontaneous symmetry breaking from a Lie group G to H, where the field takes on values in the vacuum manifold $\mathcal{M} \cong G/H$, is called the **Kibble mechanism** [21, 22].

Lastly, note that by heating up, we can resolve the topological defects by adding enough energy, so that the field takes on values near the vacuum manifold everywhere. This is nonviable in cosmological contexts, but it explains amongst others why ferromagnets lose their magnetic properties above the Curie temperature in the absence of an external magnetic field.

6.4 Different types of topological defects

We first give a mathematical definition of topological defects:

Definition (Topological defect). Let $\mathcal{L}$ be a Lagrangian density with vacuum manifold $\mathcal{M}$. A **topological defect** is a static field solution ϕ (in some Lorentz frame), such that there exists a suitable non-empty subset $A \subseteq \mathbb{R}^3$ with $\phi(A) \subseteq \mathcal{M}$, such that $\phi|_A : A \to \mathcal{M}$ is not homotopic to a constant map.

This definition captures the essence of a topological obstruction preventing a field to decay to a vacuum solution. However, in practice this definition is quite impractical and too precise to work with in a physical context, as fields are never static in reality, and they never *exactly* take on values in the vacuum manifold $\mathcal{M}$. Yet for theoretical idealisations, this definition suffices.

We now shall look at some common topological defects. In our examples, $A \subseteq \mathbb{R}^3$ will be a closed embedded submanifold of $\mathbb{R}^3$. This section is based on [21].

Domain walls

We first consider the Lagrangian in equation (6.1) again with a real field $\varphi : M \to \mathbb{R}$, which has the vacuum manifold $\mathcal{M} = \{\pm\eta\}$. By the Kibble mechanism, we would expect that there are regions where φ takes on the values near $+\eta$, and regions near $-\eta$, as illustrated in figure 6.3.

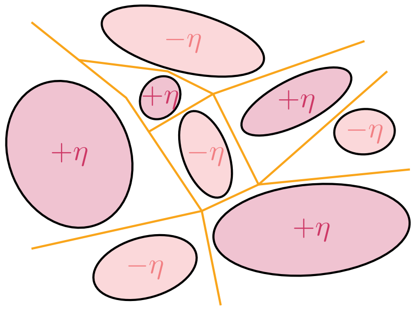


Fig. 6.3: There are clear regions where φ assumes the vacuum value $+\eta$, and regions where φ assumes $-\eta$. Between them the field φ must leave the vacuum manifold by continuity. At these places, the field has a higher energy density, called domain walls.

We choose two different points $a, b \in \mathbb{R}^3$, which together form a 0-sphere $S^0 = \{a, b\}$, such that $\varphi(S^0) \subseteq \mathcal{M}$. Then it could be that $\varphi|_{S^0} : S^0 \to \mathcal{M}$ is not null-homotopic, and thus represents a topological defect. This is exactly the case when φ maps to two different path-components, i.e. the one with $+\eta$ and $-\eta$. This is possible because $\pi_0(\mathcal{M}, x_0)$ is non-trivial, for a chosen base point $x_0 \in \mathcal{M}$. In general, a domain wall is a topological defect ϕ, such that $\phi|_{S^0} : S^0 \to \mathcal{M}$ represents a non-trivial element in $\pi_0(\mathcal{M}, x_0)$. The possibility of domain walls is thus reduced to the question whether $\pi_0(\mathcal{M}, x_0)$ is trivial.

In the particular case of a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \varphi - V(\varphi) \tag{6.8}$$

for φ a real field and the potential V bounded from below, there are exact solutions for domain walls. For simplicity, we assume that φ (which is already static per definition) does not depend on the y- and z-coordinates. Then φ is a solution[1] to

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 = 2V(\varphi)$$

For the 'kink' potential $V_{\text{kink}}(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \eta^2\right)^2$ and the the so-called sine-Gordon potential $V_{\text{sG}}(\varphi) = \lambda\,(1 - \cos(2\pi\varphi/\eta))$, with vacuum manifolds $\mathcal{M}_{\text{kink}} = \{\pm\eta\}$ and $\mathcal{M}_{\text{sG}} = \eta\mathbb{Z}$ respectively, exact solutions can be calculated using separation of variables.

[1] The equation of motion is $\frac{\partial^2\varphi}{\partial x^2} = V'(\varphi)$. By multiplying by $\frac{\partial \varphi}{\partial x}$, using the chain rule and integrating we find $\left(\frac{\partial\varphi}{\partial x}\right)^2 = 2V(\varphi) + c$, for a constant c. By assuming that $\lim_{x\to\pm\infty} \varphi(x) \in \mathcal{M}$, $\lim_{x\to\pm\infty} \frac{\partial\varphi}{\partial x} = 0$, we find $c = 0$.

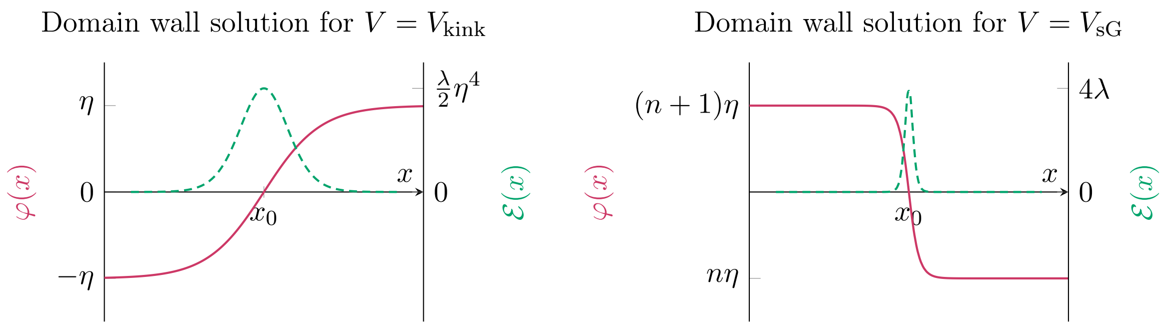


Fig. 6.4: Sketch of topological defect solutions $\varphi(x) = \eta \tanh\left(\eta\sqrt{\frac{\lambda}{2}}(x - x_0)\right)$ to the kink potential V_{kink} and $\varphi(x) = n\eta + \frac{2\eta}{\pi}\arctan\left(e^{-2\pi\sqrt{\lambda}(x-x_0)/\eta}\right)$, for $n \in \mathbb{Z}$, to the sine-Gordon potential V_{sG}. Both solutions cannot be continuously deformed to a constant solution. In both models, the energy density $\mathcal{E}$ is concentrated near x_0, forming the domain wall.

Strings

When the vacuum manifold $\mathcal{M}$ is connected, no domain walls can form. However, there can be higher dimensional analogues. For instance, we consider the U(1)-model as described by the Lagrangian in equation (6.3) for a complex scalar field ψ. In that case, $\mathcal{M} = S^1_\eta$. We embed a large circle S^1_R with radius R in $\mathbb{R}^3$. Then we expect that $\psi(S^1_R) \subseteq \mathcal{M}$ for a static field solution ψ and large R (recall that nature prefers minimising energy, so far away ψ must take on values near $\mathcal{M}$). It could be that the field 'winds around' at large distances and that $\psi|_{S^1_R} : S^1_R \to \mathcal{M}$ is not homotopic to a constant map. This is exactly the case if and only if $\psi|_{S^1_R}$ represents a non-trivial element in $\pi_1(\mathcal{M}, x_0)$. Such topological defects are called strings or vortices. Fields describing strings must necessarily attain values outside the vacuum manifold in the 'middle' of the circle S^1_R; otherwise, we could shrink R to 0, resulting in a homotopy to a constant map. Thus in the middle, there is an accumulation of energy which extends as a string in 3-space, explaining the nomenclature for strings (3 spatial dimensions) and vortices (2 spatial dimensions). A string cannot have endpoints because if it did, the string would tend to relax and attain values within the vacuum manifold near its ends. This relaxation would progressively pull the entire string into the vacuum manifold, eventually causing the string to vanish. Thus strings are either infinite or a closed loop.

The U(1)-models allows strings, since $\pi_1(\mathcal{M}, x_0) \cong \mathbb{Z}$, and an example is given by figure 6.5. However, in comparison to the domain walls examples, it is impossible to come up with exact solutions for strings. For example, consider an ansatz such as $\psi(x^i) = \eta f(s) e^{in\theta}$ for $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ a smooth function. This ansatz describes a string around the z-axis in cylindrical coordinates, winding around $n \in \mathbb{Z}$ times. Substituting this into equation (6.4) describing the equation of motion, we find

$$f''(s) + \frac{1}{s}f'(s) - \frac{n^2}{s^2}f(s) - \frac{\lambda\eta^2}{2}\left(f(s)^2 - 1\right)f(s) = 0,$$

which is a highly non-linear ordinary differential equation. Such solutions have to be found using numerical methods with a computer.

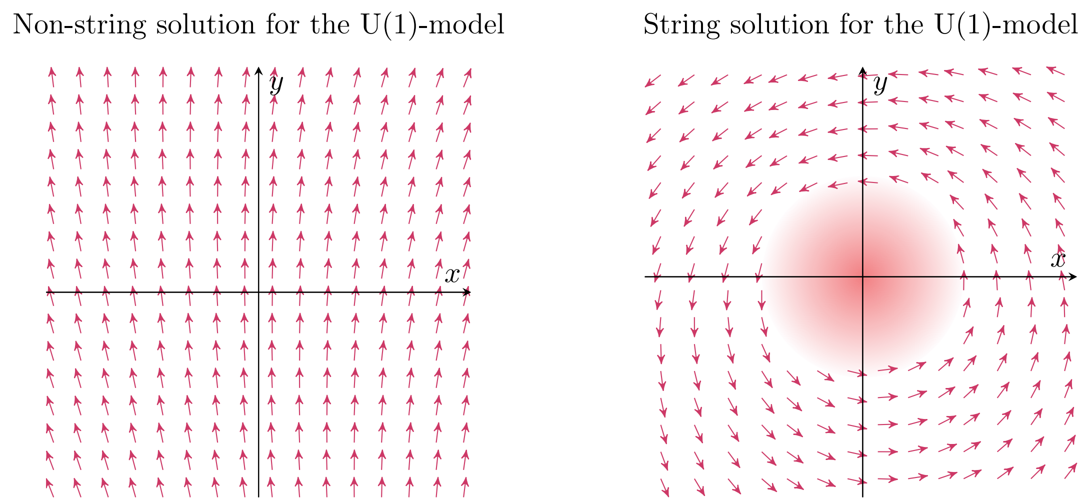


Fig. 6.5: Sketches of what a field solution for the U(1)-model might look like. On the left, there is a 'trivial' solution, i.e. it is homotopic to a vacuum solution, and to the right there is a string solution. Only the field in the xy-plane is drawn. All arrows are of unit length, and represent an element in $\mathcal{M} = S^1_\eta$. Note that the string solution must leave the vacuum manifold $\mathcal{M}$ at the core, where the phase is not defined at certain locations.

Monopoles

The natural next step is considering a Lagrangian $\mathcal{L}$ with a field ϕ and a vacuum manifold $\mathcal{M}$, and a large embedded sphere $S^2_R \subseteq \mathbb{R}^3$, such that $\phi(S^2_R) \subseteq \mathcal{M}$. Then $\psi|_{S^2_R} : S^2_R \to \mathcal{M}$ could 'wrap around' in $\mathcal{M}$, i.e. it represents a non-trivial element in $\pi_2(\mathcal{M}, x_0)$. Such a topological defect is called a monopole. Again, energy is concentrated in the middle of S^2_R. An example model where these monopoles are allowed is the Lagrangian density in equation (6.5) for $n = 3$, since then $\mathcal{M} = S^2_\eta$, and $\pi_2(\mathcal{M}, x_0) \cong \mathbb{Z}$.

Textures

Can we also use $\pi_3(\mathcal{M}, x_0)$ to identify topological defects for a Lagrangian density $\mathcal{L}$ with field ϕ? After all, S^3 cannot be embedded in $\mathbb{R}^3$ [2]. A solution is when we assume that the field ϕ satisfies $\operatorname{im} \phi \subseteq \mathcal{M}$ and that it is asymptotically constant, which means that the following limit exists:

$$L = \lim_{r \to \infty} \phi(r).$$

Then we can extend ϕ to a continuous function $\phi : S^3 \to \mathcal{M}$, and then we can ask whether it is homotopic to a constant map. If $[\phi]$ is non-trivial in $\pi_3(\mathcal{M}, x_0)$, we call ϕ a texture. Note that technically, a texture is not a topological defect according to our definition.

When a symmetry group G is spontaneously broken to a subgroup H, the vacuum manifold $\mathcal{M}$ is diffeomorphic to G/H, as shown in proposition 6.1. Then the homotopy groups of $\mathcal{M}$ determine what kinds of topological defects are possible during breaking this symmetry:

- if $\pi_0(\mathcal{M}, x_0)$ is non-trivial, then domain walls can form; different regions in space can attain different values of the vacuum manifold;
- if $\pi_1(\mathcal{M}, x_0)$ is non-trivial, then the field can loop around in the vacuum manifold, such that it cannot be contracted to a trivial solution; strings can arise;

[2] Assume that $f : S^3 \hookrightarrow \mathbb{R}^3$ is an embedding. By compactness of S^3, a corollary of the invariance of domain theorem (see corollary 2B.4 in [1]) states that f is a homeomorphism. This is a contradiction as $\mathbb{R}^3$ is not compact.

- if $\pi_2(\mathcal{M}, x_0)$ is non-trivial, then the field can wrap around in the vacuum manifold, and monopoles can form;
- if $\pi_3(\mathcal{M}, x_0)$ is non-trivial, textures can form.

Corollary 5.8 is really useful to determine $\pi_n(\mathcal{M}, x_0)$, as these homotopy groups are linked with the homotopy groups of G and H in an exact sequence.

However, a significant caveat hinders the interpretation of these topological defect models as representations of nature. Specifically, all the models discussed in this section so far possess infinite total energy. While this can be demonstrated for each model individually by calculating the energy density and integrating over all space, we will instead use a cute scaling argument, attributed to Derrick:

Proposition (Derrick's theorem). *Let $\phi = (\phi^1, \ldots, \phi^k)$ be a real field with k components, in $(n, 1)$- spacetime. Consider the general form Lagrangian*

$$\mathcal{L} = \frac{1}{2} F_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b - V(\phi),$$

for F_{ab} smooth functions depending on the field components ϕ^a, such that the matrix F is positive-definite for every field ϕ. We also assume that V is bounded from below and takes on this minimum value somewhere. Then when $n > 2$, the only static field solutions with finite energy are the vacuum solutions.

Proof. We consider static field solutions ϕ, so these satisfy $\dot{\phi}^a = 0$. In that case, the energy density is given by

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a} \dot{\phi}^a - \mathcal{L} = -\mathcal{L}.$$

Since ϕ is a solution to the Euler-Lagrange equations, this means that $E[\phi]$ is also stationary. In our case, the energy functional is given by

$$E[\phi] = \int_{\mathbb{R}^n} \left(\frac{1}{2} F_{ab}(\phi) \partial_i \phi^a \partial_i \phi^b + V(\phi) \right) \mathrm{d}^n x = \underbrace{\int_{\mathbb{R}^n} \frac{1}{2} F_{ab}(\phi) \partial_i \phi^a \partial_i \phi^b \mathrm{d}^n x}_{I[\phi]} + \underbrace{\int_{\mathbb{R}^n} V(\phi) \mathrm{d}^n x}_{J[\phi]}.$$

By positive-definiteness of F, $F_{ab}(\phi) \partial_i \phi^a \partial_i \phi^b$ is always non-negative. Then $I[\phi] \geq 0$. Without loss of generality, we can assume that the minimal value V takes on is 0; else we could add a constant to the Lagrangian, which does not change the equations of motion. This then gives that $J[\phi] \geq 0$.

Now, we consider a static field solution ϕ with finite energy, and set $I = I[\phi]$ and $J = J[\phi]$. For $\lambda > 0$, we consider the scaled fields ϕ_λ, given by $\phi^a_\lambda(x^\mu) = \phi^a(\lambda x^\mu)$. Then by doing a linear change of variables in the integrals, we see that

$$E[\phi_\lambda] = I[\phi_\lambda] + J[\phi_\lambda] = \lambda^{2-n} I + \lambda^{-n} J.$$

ϕ is a stationary solution, so

$$\left. \frac{\partial E[\phi_\lambda]}{\partial \lambda} \right|_{\lambda=1} = (2-n) I - n J = 0.$$

Note again that $I, J \geq 0$. The above equation can only hold for $n \geq 3$, when $I = J = 0$. Then the total energy is minimal, and thus ϕ is a vacuum state. ∎

Since we often work in $(3,1)$-spacetime, Derrick's theorem effectively states that topological defects have infinite energy. Even in $(2,1)$-spacetime, there are huge restrictions, as can be seen in the proof. Evidently, Derreck's theorem does not hold for $n = 1$, as there are topological defects in $(1,1)$-spacetime with finite energy. For instance, the kink model and sine-Gordon model in $(1,1)$-spacetime have finite energy, as can be seen in figure 6.4.

There are, however, ways to circumvent Derrick's theorem. For instance, we could allow for time-depend fields, model spacetime after different manifolds or by allowing infinite energies as in the previous examples. Then the energy of strings and monopoles is cut off at a certain radius for physical applications. Yet an even more natural bypass is using gauge fields, of which we give an example:

Local monopoles

We consider a real 3-tuple field ϕ and we add a SO(3) gauge field to the Lagrangian. We assume that SO(3) acts on ϕ by the adjoint action, so ϕ takes values in $\mathfrak{so}(3)$. By example 2.9, the Lie algebra $\mathfrak{so}(3)$ is generated by the matrices

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\operatorname{tr}\big(T^i T^j\big) = -\frac{1}{2}\delta^{ij}$, and $[T^i, T^j] = \frac{1}{2}\varepsilon^{ijk}T^k$, for ε^{ijk} the Levi-Civita symbol. We write $\phi = \phi^a T^a$. Then the Lagrangian is given by (c.f. equations (4.16), (4.19) and (4.20))

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi^a + \frac{1}{2}\varepsilon^{abc}A^b_\mu\phi^c\right)\left(\partial^\mu\phi^a + \frac{1}{2}\varepsilon^{abc}A^{b,\mu}\phi^c\right) - \frac{\lambda}{4}\left(\phi^a\phi^a - \eta^2\right)^2,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \frac{1}{2}\varepsilon^{abc}A^b_\mu A^c_\nu.$$

Here the summation over a, b and c is also implicit. In this case, $\mathcal{M} \cong S^2_\eta$ holds as well and thus $\pi_2(\mathcal{M}, x_0) \cong \mathbb{Z}$. We thus expect monopoles, called **'t Hooft-Polyakov monopoles** in this model. In [21], a topological defect solution is described for ϕ^a and A^a_μ, such that the fields ϕ^a and A^a_μ decay exponentially to 0 at large distances. This means that these solutions have finite energy.

6.5 Applying topological defects: Grand Unified Theories

In section 6.2, we have seen that at high temperatures, electromagnetism and the weak force unite into the electroweak force. The symmetry breaking pattern is $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{em}}$. Note that $\mathcal{M} \cong (\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y) / \mathrm{U}(1)_{\mathrm{em}} \cong S^3$ [23], so no domain walls, strings or monopoles can form (see table 5.1 for the homotopy groups of S^3).

Another fundamental force is the strong interaction, which plays a crucial role in binding quarks within hadrons, such as protons and neutrons, and in holding hadrons together. This interaction explains the stability of most atomic nuclei. Despite the electromagnetic repulsion between the positively charged protons and the neutrality of neutrons, the strong interaction ensures that these nuclei remain stable. Mathematically, the theory of the strong interaction is described by quantum chromodynamics and the gauge symmetry group of the strong interaction is SU(3), which we denote $\mathrm{SU}(3)_C$ (the 'C' stands for colour charge, the electric charge equivalent in quantum chromodynamics). It would be very appealing if electrodynamics, the weak and the strong force can be combined into one force at an even higher temperature than the electroweak unification temperature. This is exactly what Grand Unified Theories seek to accomplish.

More specifically, we want some Lagrangian density with symmetry group G, which breaks down in one or more steps to $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Then we can look at what topological defects can occur. For reasons related to the representation theory of the Standard Model, we take G to be compact, connected and simple (the only proper closed normal subgroups of G are of dimension 0, and G is non-abelian). Candidates include $G = \mathrm{SU}(5)$, $G = \mathrm{SO}(10)$ and the exceptional compact Lie group $G = \mathrm{E}_6$ of dimension 78 [24]. Here we shall be considering $G = \mathrm{SO}(5)$ (called the Georgi–Glashow model), but the following discussion generalises mostly to other compact, connected and simple Lie groups G, since by theorems of Cartan and Bott [19], $\pi_2(G) \cong 0$ and $\pi_3(G) \cong \mathbb{Z}$.

In what sense is $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, which we denote by G_{SM}, a subgroup of $\mathrm{SU}(5)$? We can construct the following faithfull Lie algebra representation

$$\sigma : \mathfrak{su}(3)_C \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_Y \to \mathfrak{su}(5),$$
$$(P, Q, w) \mapsto \begin{pmatrix} P - 2wI_3 & 0 \\ 0 & Q + 3wI_2 \end{pmatrix}.$$

Then σ is the induced map of the following Lie group representation

$$\rho : \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{SU}(5),$$
$$(A, B, z) \mapsto \begin{pmatrix} z^{-2}A & 0 \\ 0 & z^3 B \end{pmatrix}.$$

The only problem is that this representation is not faithful, but $\ker \rho = \langle (\zeta_3 I_3, -I_2, \zeta_6) \rangle \cong \mathbb{Z}_6$. So it is really $\mathrm{SU}(5)$ breaking down to $G_{\mathrm{SM}}/\mathbb{Z}_6$. This is no problem, as long as we use only representations of G_{SM} which factor through $G_{\mathrm{SM}}/\mathbb{Z}_6$.

We now determine the homotopy groups of the vacuum manifold $\mathcal{M} \cong \mathrm{SU}(5)/\left(G_{\mathrm{SM}}/\mathbb{Z}_6\right)$. The homotopy groups of G_{SM} can be easily found, using table 5.3 and theorem 5.5. Applying the long exact sequence in corollary 5.8 to $\pi : G_{\mathrm{SM}} \to G_{\mathrm{SM}}/\mathbb{Z}_6$, we find that π induces an isomorphism between $\pi_n(G_{\mathrm{SM}})$ and $\pi_n(G_{\mathrm{SM}}/\mathbb{Z}_6)$ for $n \geq 2$. The tail looks like

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(G_{\mathrm{SM}}/\mathbb{Z}_6) \longrightarrow \mathbb{Z}_6 \longrightarrow 0$$

We cannot calculate $\pi_1(G_{\mathrm{SM}}/\mathbb{Z}_6)$ from this short exact sequence without looking at the maps, since $\mathbb{Z}_6$ is merely a set here and not a group. But at least we see that $\pi_1(G_{\mathrm{SM}}/\mathbb{Z}_6)$ is non-trivial – it has at least an element with infinite order – which is enough information for our discussion. Then the long exact sequence of the homotopy groups for $\mathcal{M}$ looks like

$$\begin{array}{ccccccc}
 & & \cdots & \longrightarrow & \pi_4(\mathcal{M}) & & \\
\hookrightarrow \mathbb{Z}^2 & \xrightarrow{\;i_*\;} & \mathbb{Z} & \longrightarrow & \pi_3(\mathcal{M}) & & \\
\hookrightarrow 0 & \longrightarrow & 0 & \longrightarrow & \pi_2(\mathcal{M}) & & \\
\hookrightarrow \pi_1(G_{\mathrm{SM}}/\mathbb{Z}_6) & \longrightarrow & 0 & \longrightarrow & \pi_1(\mathcal{M}) & & \\
\hookrightarrow 0 & \longrightarrow & 0 & \longrightarrow & \pi_0(\mathcal{M}) & \longrightarrow & 0
\end{array}$$

We thus see that $\pi_0(\mathcal{M}) \cong 0$, $\pi_1(\mathcal{M}) \cong 0$ and $\pi_2(\mathcal{M}) \cong \pi_1(G_{\mathrm{SM}}/\mathbb{Z}_6) \neq 0$. So this model allows for monopoles, but not for domain walls or strings. In fact, textures do not form either, since $\pi_3(\mathcal{M}) \cong 0$. This can be seen by looking at the following commuting diagram:

$$\begin{array}{ccccc} & & \overset{\rho}{\frown} & & \\ G_{\mathrm{SM}} & \xrightarrow{\pi} & G_{\mathrm{SM}}/\mathbb{Z}_6 & \xhookrightarrow{i} & \mathrm{SU}(5) \\ j \big\uparrow & & & & \big\uparrow i_4 \\ \mathrm{SU}(3) & & \xhookrightarrow{\quad i_3 \quad} & & \mathrm{SU}(4) \end{array}$$

Here i_3, i_4 and j are the natural inclusion maps. By applying the π_3 functor, we note that $(i_3)_*$ and $(i_4)_*$ are isomorphisms – which is exactly the stabilising of homotopy groups of $\mathrm{SU}(n)$ which we have seen in section 5.3. Then $(i_4)_* \circ (i_3)_*$ is an isomorphism and thus i_* must be surjective. Looking back at the long exact sequence, this implies that $\pi_3(\mathcal{M}) \cong 0$.

As of 2022, no monopoles from this symmetry breaking pattern are observed. The expected abundance of monopoles exceeds the observed bounds by far, a discrepancy known as the *monopole problem* in cosmology [24, 25]. This raises intriguing questions: Are Grand Unified Theories incorrect, suggesting that monopoles never existed in the first place? Have monopoles dissipated over time, or is there another mechanism at play that explains their absence?

The existence of Grand Unified Theories and monopoles is still an ongoing debate in present-day particle physics and cosmology. In 1980, Guth proposed the theory of *cosmic inflation*, the idea of exponential expansion of the Universe immediately after the Big Bang, in order to solve the monopole problem [26]. Simultaneously, it solved several other cosmological problems, and this theory continues to be a cornerstone of contemporary cosmology. So whilst the search for topological defects such as monopoles remains unfulfilled, their theoretical exploration has led to significant advancements in our understanding of the Universe.

Chapter 7 | Concluding remarks

In this thesis, we have developed the theory of topological defects from the ground up. First we began by introducing fundamental concepts such as Lie groups and Lie algebras, which turned up in every forthcoming chapter. The exponential map, which we primarily used as a way to calculate Lie algebras of common matrix groups in chapter 2, turned out to be really useful to describe infinitesimal Lie group actions in chapter 3, particularly for fundamental vector fields. The adjoint representation emerged prominently in Yang-Mills theory, illustrating how gauge fields transform under gauge transformations, and in the action of the Lie group SU(2) on fields describing 't Hooft-Polyakov monopoles.

Chapter 3 provided mostly definitions and examples of fibre bundles, principal G-bundles and Ehressmann connections. Their significance became evident, such as in Yang-Mills theory or the role of the principal H-bundle $H \longrightarrow G \overset{\pi}{\longrightarrow} G/H$ in determining the homotopy groups of the the vacuum manifold in chapter 6.

In chapter 4, we introduced spacetime as $\mathbb{R}^4$ with the Minkowski metric, alongside with the Lagrangian formalism. It set the stage for studying spontaneous symmetry breaking and topological defects in chapter 6 and a simplified version of Yang-Mills theory, with electromagnetism as a nice preview.

In chapter 5, we developed a lot of theory related to homotopy groups. The long exact sequence of homotopy groups of a fibre bundle was of particular importance in this thesis, but meanwhile we met the richness of homotopy groups when considering the homotopy groups of n-spheres and compact Lie groups.

Lastly, we explored spontaneous symmetry breaking from a Lie group G to a closed subgroup $H \subseteq G$. When G acts transitively on the vacuum manifold $\mathcal{M}$, we concluded that $\mathcal{M} \cong G/H$. We also covered the Higgs mechanism, and we have seen how spontaneous symmetry breaking can lead to topological defects, by the Kibble mechanism. Sections 6.4 and 6.5 finally brought topological defects and homotopy groups together; topological defects are possible if $\pi_n(\mathcal{M}, x_0)$ is non-trivial, for an $n \in \{0, 1, 2, 3\}$. We also explored some examples of topological defects, and calculated the homotopy groups of the vacuum manifold $\mathcal{M}$. Here, the long exact sequence of homotopy groups for the fibre bundle $H \longrightarrow G \overset{\pi}{\longrightarrow} \mathcal{M}$ came into good use.

This thesis really represents an exploration rather than a straight path towards topological defects. One could introduce the topic with just some field theory and basic theory about homotopy groups alone, without going into details about Lie groups and Lie algebras, principal bundles and connections, the geometry of spacetime and the deep theorems of algebraic topology. They would do fine most of the time, and this would suffice for most practical purposes. However, one walking this path misses the opportunity to fully understand what is really going on mathematically and to appreciate the beauty of the theory.

Still, the exploration is far from complete. In chapter 4, we already concluded that describing the Lagrangian formalism geometrically would surpass the scope of this thesis. This approach is followed in [12] however, where jet manifolds, jet bundles and variational bicomplices are introduced to define the calculus of variations and the Euler-Lagrange equations on fibre bundles. This method also easily allows for gauge fields, so Yang-Mills theory integrates naturally into the theory. Although this may be a feast to geometrists, for physical applications this has no additional benefits.

In section 4.4, we made some simplifications to Yang-Mills theory. For instance, we skipped over associated bundles, and assumed the principal bundles was trivialisable, since the spacetime manifold $M = \mathbb{R}^4$ is contractible. In contrast to a geometric Lagrangian formalism, considering Yang-Mills theory in its full form would have physical advantages. For instance, the Dirac monopole [13] can only be defined on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, which is homotopically equivalent to S^2, which is not contractible. Principal bundles over $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$ in general do not allow for global sections. In this case of the Dirac monopole, the electromagnetic four-potential is defined on two different patches of $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, which together cover $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$. Here the full machinery of principal bundles has to be utilised. In addition, a complete description of the principal bundle theory allows for a lot more different interactions possible in the Lagrangian; such as fermion interactions with gauge bosons and Yukawa interactions between fermions and the Higgs boson, which describe the mass generation mechanism for fermions. For this, we additionally would have to introduce spinor bundles. This is described in chapters 6, 7 and 8 in [14].

Although this thesis explores topological defects, their physical (non)-appearance is somewhat lacking in our discussion. This would be an excellent start for future research. In [21], topological defects are considered in a cosmological context. For instance, it is shown that domain walls cannot form above a critical temperature of about 0.1MeV, as it would imply gravitational effects, measurable in the cosmic microwave background. This suggests that no discrete symmetries in particle physics are broken above this temperature. Another phenomenon that would be observable if topological defects exist, is gravitational lensing: light is bend due to presence of topological defects. In future research, such phenomena could be explored. Topological defects in other physical disciplines, such as condensed matter physics and solid-state physics, could also be an interesting topic of study. As opposed to cosmology, topological defects are indeed observed in these disciplines [27].

Lastly, we used homotopy groups to study topological defects, but we did not fully exploit their information; basically we only considered whether the homotopy groups $\pi_n(\mathcal{M}, x_0)$ were trivial or not. What does the group $\pi_n(\mathcal{M}, x_0)$ tell us in general? Is there a physical interpretation of the group addition in the homotopy group? When $\pi_1(\mathcal{M}, x_0)$ is non-abelian, does this complicate the interactions? These are interesting research questions for follow-up research.

Almost 130 years after the Korteweg-de Vries equation, which marked the beginning of the study of solitons and topological defects, topology has finally made its way into mainstream physics. Topological insulators, topological phase transitions and the quantum hall effect are state of the art topics in modern theoretical physics, which all rely on topology. In 2016, Thouless, Haldane and Kosterlitz were awarded the Nobel Prize in Physics for their groundbreaking work on topological phase transitions, which are also related to topological defects. We expect many interesting physical discoveries to be made on the edge of theoretical physics and topology.

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