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Abstract

We apply geometrical methods to the quantum brachistochrone problem, whose equation of motion was recently written as a limit case of a well known Lax pair [MC24]. First we discuss the theories of Lie groups, Poisson geometry and Riemann surfaces in that order. This knowledge is applied to the study of Lax equations and their integrability. The final chapter applies these methods to the quantum brachistochrone problem for the smallest case $\mathfrak{su}(2)$.

Contents

Chapter 1

Introduction

In the eighteenth century Euler studied and solved the motion of a spinning rigid body, which was subsequently generalised by Lagrange in the case the body experiences a gravitational force. Near the end of the next century, Kowaleski found another exotic integrable case of the spinning top and showed that, under reasonable assumptions, this is the only other integrable system [Kov88]. Her result still remained unsatisfactory from a practical and theoretical point of view. Unlike many other known integrable systems, it could not be solved by a separation of variables and required an ingenious change of variables to reduce the integrals to complicated expressions involving hyperelliptic functions. Moreover, there was no obvious reason the system had to be integrable: its configurational symmetry combined with Noether's theorem simply does not produce enough integrals.

Seemingly unrelated, a small revolution took place in the 1960's and 1970's, centered around a differential equation introduced by Korteweg and de Vries in 1895 which describes the motion of waves on a shallow surface of water. By rewriting the PDE as a commutator differential equation, now known as a Lax pair, researchers could exploit the seemingly mysterious symmetries of the system and prove the integrability of the KdV-equation. Most significantly, these ideas lead to the development of the inverse scattering method for solving PDE's, which expresses the solution of a Hamiltonian equation of motion in terms of a Riemann-Hilbert problem; the integrals are given by spectral invariants of an auxiliary differential operator.

Following this discovery, Lax pairs have shown up numerously throughout mathematical physics and are almost always accompanied by a complete integrability of the dynamical system in question. The perspective also shed new light on many classical finite dimensional systems, such as the Kowaleski top, whose symmetry could now be understood in terms of a Hamiltonian reduction scheme and solutions were simplified [BRS89]. For many such systems the resulting Riemann Hilbert problem can be expressed and solved explicitly using algebraic geometric methods.

A Lax pair has recently been found by Cheianov and Malikis in their study of the quantum brachistochrone problem [MC24]. The quantum brachistochrone problem is concerned with finding the minimal cost of achieving a given unitary when the driving Hamiltonians are restricted to a subset of the allowed phase space. Such a restriction often arises out the physical conditions of the system in question. They show that the resulting equation of motion can be written as a limit case of the classical spinning top and hence may be integrated completely.

In this thesis we apply these algebraic geometric methods to solve the for quantum brachistochrone problem for small dimensions. Chapters 2-4 discuss the relevant mathematical machinery, Chapter 5 delves into the theory of Lax equations and Chapter 6 studies the quantum brachistochrone problem.

Chapter 2 is to give a brief overview of Lie groups and Lie algebras. We first see how the additional group structure simplifies the smooth structure and introduce subgroups. Then we give a brief tour through Lie group actions and representations, and end with a discussion of the adjoint and coadjoint representation of a Lie group on its Lie algebra.

In Chapter 3 we discuss Poisson manifolds for the purpose of describing Hamiltonian reduction in the fifth chapter. A Poisson structure is a Lie algebra on the set of smooth functions of a phase space which satisfies the Leibniz rule and represents the most general mathematical framework where one can study Hamiltonian dynamics. We discuss their structure theory and Lie group actions on a symplectic manifold.

Chapter 4 gives an introduction in the theory of Riemann surfaces; there is a substantial mathematical literature and we have striven to only introduce as many concepts as we use later. We start with a general description of complex manifolds and their smooth, holomorphic and meromorphic structure. Thereafter we discuss compact Riemann surfaces, divisors and line bundles.

The focus of Chapter 5 is a description of Lax equations that occur in the dual Lie algebra equipped with two Poisson structures. After describing the solution of the equation of motion in terms of a factorisation problem, we discuss Hamiltonian reduction and construct integrable Lax pairs for the classical spinning top. At the end we give a detailed overview of solving the factorisation problem in terms of the Baker-Akhiezer function associated the spectral data of a Lax equation.

Chapter 6 gives a short qualitative description of the history and physical motivation behind the quantum brachistochrone problem. We follow Cheinovs and Malikis' derivation of the equation of motion [MC24] and move on to solve the resulting differential equations for the simplest case.

The reader is assumed to be familiar with approximately a semesters equivalent of differential geometry; concepts such tensor fields and de Rham's theorem should be known. In addition, we will use the notion of a distribution on occasion; a quick introduction is given in the first Appendix. Similarly, we heavily rely on sheaf cohomology in our discussion of Riemann surfaces; the second Appendix contains a refresher for basic definitions of sheaves and Céch cohomology, but by no means should serve as a first introduction to the subject.

The thesis requires a minimal knowledge of physics; where it helps, ideas from physics help to motivate certain concepts, such as Hamiltonian dynamical systems in Chapter 3 or the Schrödinger equation in Chapter 6. At the beginning of Chapter 6 we give a qualitative physical picture of the quantum brachistochrone problem which relies on a basic knowledge of quantum information.

Chapter 2

Lie Groups and and their Lie algebras

A Lie group is a topological group with an additional smooth structure. They were named after Sophus Lie who developed the theory in the late 1890's for local Lie groups, when the notion of a smooth manifold was still mysterious. It turns out that a Lie group is closely associated with a specific finite-dimensional vector space: a Lie algebra.

We give a modern introduction of Lie groups and Lie algebras. First we see how the additional group structure simplifies the smooth structure. The exponential map allows us to answer questions about a Lie group in terms of its Lie algebra. Thereafter we give a short tour through Lie group actions and then discuss the (co)adjoint representations.

2.1 Smooth structure of a Lie group

The additional group structure on G allows us to make more general statements about the differentiable structure than in the case of a smooth manifold.

Definition 2.1. A Lie group G is group that is also a differentiable manifold such that the groups operations of multiplication $m : G \times G \to G$ and inversion $\iota : G \to G$ are smooth.

Alternatively, it suffices to check that the composite map $G \times G \ni (g, h) \mapsto gh^{-1} \in G$ is smooth. This condition is also necessary.

Example 2.1. The following familiar groups are also Lie groups.

(1) The Euclidean space $\mathbb{R}^n$ is an Abelian group, where the map $(x, y) \mapsto x - y$ is clearly smooth.

(2) Denote by $GL_n(\mathbb{C})$ the group of invertible $n \times n$ matrices with matrix multiplication as group operation. Then $GL_n(\mathbb{C})$ is an open subset of $\mathbb{C}^n$ and inherits the structure of a smooth manifold. Indeed, $GL_n(\mathbb{C}) = \det^{-1}(\mathbb{C} \setminus \{0\})$, where $\det : \mathbb{C}^{n^2} \to \mathbb{C}$ is the determinant map. To check that group multiplication and inversion are smooth one may verify that these operations are polynomials of the matrix coefficients in local coordinates.

Any Lie group element $g \in G$ corresponds to a unique diffeomorphism of G given by *left-translation* $\lambda_g : h \mapsto gh$ or *right-translation* $\rho_g : h \mapsto hg^{-1}$. The differentials of these maps yield isomorphisms $d\lambda_g : T_hG \xrightarrow{\sim} T_{gh}G$ and $d\rho_g : T_hG \xrightarrow{\sim} T_{hg^{-1}}G$. In a certain sense, this means that is suffices to study the tangent space at the identity. The following construction makes this precise.

A vector field $X \in \mathscr{X}(G)$ is called *left-invariant* if $\lambda_g^* X = X$ for all $g \in G$.

Theorem 2.1. *Let $\mathfrak{g}$ be the set of left-invariant vector fields G. Then $\mathfrak{g}$ forms a subalgebra of the set $\mathscr{X}(G)$ of vector fields on G, with respect to the commutator bracket, and has the structure of an $\mathbb{R}$-vector space naturally isomorphic to T_eG.*

Proof. Given two left-invariant vector fields $X, Y \in \mathfrak{g}$ it is easy to show that $\lambda^*[X, Y] = [X, Y]$, where $[\cdot, \cdot]$ is the standard commutator operation on smooth vector fields. $\mathfrak{g}$ is thus a Lie algebra as the Jacobi identity holds in $\mathscr{X}(G)$, and an $\mathbb{R}$-vector space with pointwise addition and scalar multiplication.

Consider the map $\phi : \mathfrak{g} \to T_eG$ given by $X \mapsto X_e$. Then ϕ is a homomorphism. If $X_e = 0$ then left-invariance implies $X_g = d\lambda_g(X_e) = 0$ for all $g \in G$ and ϕ is injective. To prove surjectivity, define for $Y \in T_eG$ the vector field Z by $Z_g = d\lambda_g(Y)$. One can easily check that Z is smooth and left-invariant. (See [Lee12, Theorem 8.37] for the details.) □

Suppose $X_1, \ldots, X_n \in T_eG$ is an $\mathbb{R}$-basis and by abuse of notation identify these vectors with left-invariant vector fields $X_1, \ldots, X_n \in \mathfrak{g}$ on G. At any point $g \in G$ it follows that the n-tuple $((X_1)_g, \ldots (X_n)_g)$ forms a basis of the tangent space T_gG. These vector fields thus form a global frame of G. We summarise our discussion.

Corollary 2.1. *Any Lie group G admits a global left-invariant frame. Moreover, $TG \cong G \times \mathfrak{g}$ by left translations.*

The integral curves of left-invariant vector fields are also well behaved. Recall that a vector field X on a smooth manifold M is said to be *complete* if the integral curve γ of X at any point extends to a smooth curve $\gamma : \mathbb{R} \to M$.

Proposition 2.1. *Any left-invariant vector field $X \in \mathfrak{g}$ is complete.*

Proof. See [Lee12, Theorem 9.18]. □

2.1.1 Lie subgroups

We discuss the notion of a Lie subgroup and methods for constructing them.

2.1.2 Subgroups

Suppose G is a Lie group and let $H \subseteq G$ be a subgroup. There exist various notions of subgroups. For example, it is typically not useful to only require that H itself is a Lie group, as H can be equipped with the discrete topology and a trivial smooth structure. Requiring a Lie subgroup to be an immersed submanifold turns out to be the 'correct' setting and loses little generality for applications.

Definition 2.2. A Lie subgroup $H \subseteq G$ is a subgroup which is also a Lie group such that the inclusion $H \hookrightarrow G$ is an immersion.

If $H \subseteq G$ is a Lie subgroup and $K \subseteq H$ is a Lie subgroup, it is easy to see that $K \subseteq G$ is a Lie subgroup with the definition above.

Example 2.2. The following examples are subgroups of the Lie group $G = GL_n(\mathbb{C})$ and are typically called *matrix groups*.

(1) The subgroup $GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$ can be identified with an open subset of $\mathbb{R}^{n^2}$ and is seen to be a Lie subgroup by restricting the inclusion $\mathbb{R}^{n^2} \hookrightarrow \mathbb{C}^{n^2}$.

(2) The *special linear group* $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$ of matrices with determinant 1 is the level set of the smooth determinant map. As 1 is a regular value, $SL_n(\mathbb{R})$ is an embedded submanifold.

(3) Let $O(n)$ be the *orthogonal group* consisting of all $n \times n$ real-entried matrices A such that $AA^T = I_n$ with I_n the n-diagonal identity matrix. Then $\det(A) = \pm 1$ and $O(n) \subset GL_n(\mathbb{R})$. In fact, $O(n)$ is the level set

of the identity I_n of the map $\Psi : GL_n(\mathbb{R}) \ni A \mapsto AA^T \in \mathrm{Mat}_n(\mathbb{R})$. One can verify that the map Ψ has constant rank such that $O(n)$ is an immersed submanifold. (See Example 7.27 in [Lee12].)

(4) The subgroup $SO(n)$ is the intersection $SL_n(\mathbb{R}) \cap O(n)$, or in other words, the connected component of $O(n)$ at the identity. In view of this last statement it is an open submanifold of $O(n)$ and also a Lie subgroup.

(5) In identical fashion one can show that the *unitary group* $U(n)$ of the complex-entried matrices A with $AA^\dagger = I_n$ and the *special unitary group* $SU(n)$ those matrices with determinant 1 are both Lie subgroups of $GL_n(\mathbb{C})$.

2.1.3 Lie group homomorphisms

A Lie group homomorphism between two Lie groups G and H is a group homomorphism $f : G \to H$ which is also smooth.

Theorem 2.2. *Suppose G and H are Lie groups and $f : G \to H$ is Lie group homomorphism. Then $df : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. Suppose $X, Y \in \mathfrak{g}$ and let $X_e, Y_e \in T_eG$ be their values at the identity. It is easy to verify that $d\phi[X_e, Y_e] = [d\phi(X_e), d\phi(Y_e)]$. Thus the vector fields $d\phi[X, Y]$ and $[d\phi(X), d\phi(Y)]$ agree at the identity and in view of Theorem 2.1 these coincide. □

Corollary 2.2. *Let $f : G \to H$ be a diffeomorphism which is also a group homomorphism. Then $df : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism.*

Proposition 2.2. *Let $H \subseteq G$ be a Lie subgroup and $\mathfrak{h}$ (resp. $\mathfrak{g}$) the Lie algebra of H (resp. G). Then there exists a canonical inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.*

Proof. First note that the group inclusion $\iota : H \to G$ is a group homomorphism and an immersion. The corresponding Lie algebra homomorphism $d\iota : \mathfrak{h} \to \mathfrak{g}$ from Theorem 2.2 is then seen to be injective and the claim follows. □

The converse also turns out to be true. One can define an involutive distribution D given by

$$D_g = \{X_p \in T_P G : X \in \mathfrak{h}\} \tag{2.1}$$

and apply the global Frobenius theorem*. The leaf passing through the identity is then seen to be a subgroup; for the full details we refer to [Lee12, Theorem 19.16].

Theorem 2.3. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a subalgebra of $\mathfrak{g}$. Then there exists a unique closed subgroup $H \subseteq G$ such that $T_eH \cong \mathfrak{h}$.*

2.1.4 Exponential map

The exponential map of a Lie group further explores the relationship between a Lie group and its Lie algebra.

By a one-parameter subgroup $H = \{g_t\}_{t\in\mathbb{R}} \subset G$ we mean an $\mathbb{R}$-indexed subgroup such that $g_0 = e$ and $g_t g_s = g_{t+s}$. It is easy to check the following.

Proposition 2.3. *There is a one-to-one correspondence between one-parameter subgroups of G and integral curves of left-invariant vector fields.*

To some degree this result makes a correspondence between group elements, in terms of one-parameter subgroups, and Lie algebra elements in $\mathfrak{g}$, in terms of integral curves. The exponential map makes this idea concrete and allows us to easily translate structure on the Lie algebra $\mathfrak{g}$ to structure on the Lie group G.

Definition 2.3. For $X \in \mathfrak{g}$ let $\gamma_X(t)$ be the integral curve of X passing through the identity at time $t = 0$. The *exponential map* $\exp : \mathfrak{g} \to G$ is given by:

$$\exp(X) := \gamma_X(1). \tag{2.2}$$

Note that $\exp : \mathfrak{g} \to G$ is well-defined in view of Proposition 2.1.

The following proposition describes basic properties of the exponential map.

Proposition 2.4. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The following statements are true.*

(1) For any $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, $\exp tX = \gamma_X(t)$.

(2) The exponential map is smooth.

(3) For any $X \in \mathfrak{g}$ and $t, s \in \mathbb{R}$, $\exp((t+s)X) = \exp(tX)\exp(sX)$.

*for a discussion of distributions and the global Frobenius Theorem see Appendix A. We will use distributions later in Chapter 3 and Chapter 5.

(4) *The exponential map is smooth and restricts to a diffeomorphism of a neighborhood of the origin* 0 *in* $\mathfrak{g}$ *to a neighborhood of the identity* e *in* G.

Proof. See [Lee12, Proposition 20.5] and [Lee12, Proposition 20.8]. □

The exponential map is a natural transformation.

Proposition 2.5. *Suppose* G *and* H *are Lie groups,* $\mathfrak{g}$ *and* $\mathfrak{h}$ *the corresponding Lie algebras and* $f : G \to H$ *a Lie group homomorphism. Then the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "df"] \arrow[d, "\exp_G"] & \mathfrak{h} \arrow[d, "\exp_H"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

Proof. It is clear that f takes one-parameter groups in G to one-parameter groups in H. The claim follows from differentiation and Proposition 2.4. □

The exponential map *in abstracto* can be difficult to work with. Fortunately, for matrix groups $G \subset GL_n(\mathbb{C})$ it coincides with the ordinary exponential defined as follows. Given any matrix $A \in GL_n(\mathbb{C})$ let $\exp_E(A)$ be the formal power series given by:

$$\exp_E(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}, \tag{2.3}$$

where the E stands for 'Euclidean'. To see that this series actually converges, denote by $||\cdot||$ the *Frobenius norm* on $GL_n(\mathbb{C})$ which is the inherited norm from the canonical inclusion $GL_n(\mathbb{C}) \hookrightarrow \mathbb{R}^{2n^2}$. The Frobenius norm will make another appearance in Chapter 6. Concretely, given a matrix $A \in G$ with elements A^i_j it is easy to see that $||A|| = \mathrm{Tr}(AA^\dagger)$. For any $m \in \mathbb{N}$ the partial sum s_m of the series in eq. 2.3 is bound from above by $\sum_{n=0}^{m} ||A||^n (n!)^{-1}$ and the series converges uniformly.

Denote by $\gamma_A : \mathbb{R} \to GL_n(\mathbb{C})$ the one-parameter group given by $\exp_E(tA)$. By the uniform convergence demonstrated above we may differentiate to find

$$\frac{d}{dt} \exp_E(tA) = A \exp_E(tA). \tag{2.4}$$

In particular $d/dt|_{t=0} \exp_E(tA) = A$ and the uniqueness of integral curves guarantees that γ_A is the one-parameter group associated to the vector $A \in \mathfrak{g}$.

We use the exponential map to compute the Lie algebra $\mathfrak{g}$ of the Lie group $U(n)$, which is typically denoted $\mathfrak{u}(n)$. Suppose $X \in \mathfrak{g}$ and t sufficiently small as in Proposition such that the exponential map is a diffeomorphism. As $\exp(tX) \in G$ is a group element is satisfies the identity

$$\exp(tX)\exp(tX)^\dagger = I_n \tag{2.5}$$

$$= (1 + tX + O(t^2))(1 + tX^\dagger + O(t^2)). \tag{2.6}$$

(cf. equation 2.3.) Then differentiating eq. 2.6 on both sides and evaluating at $t = 0$ yields the identity

$$X + X^\dagger = 0, \quad X \in \mathfrak{u}(n). \tag{2.7}$$

In other words, $\mathfrak{u}(n)$ is the Lie algebra of anti-Hermitian matrices.

To find the Lie algebra $\mathfrak{su}(n)$ of the special unitary group $SU(n)$ we need the following simple lemma about the determinant map.

Lemma 2.1. *Let $A \in GL_n(\mathbb{C})$ be a matrix and I_n the identity matrix. Then $\det(I_n + tA) = 1 + t\operatorname{Tr}(A) + O(t^2)$ as $t \to 0$.*

Now suppose $X \in \mathfrak{su}(n)$ and let $\gamma_X(t)$ be the integral curve of X starting at the identity. Then $\det(\gamma_X(t)) = 1$ for all $t \in \mathbb{R}$. Differentiating at $t = 0$ with the above lemma now yields $\operatorname{Tr}(X) = 0$. We summarise this discussion.

Proposition 2.6. *The Lie algebra $\mathfrak{su}(n)$ of the Lie group $SU(n)$ is given by:*

$$\mathfrak{su}(n) = \left\{ X \in \operatorname{Mat}_n(\mathbb{C}) : X + X^\dagger = 0, \ \operatorname{Tr}(X) = 0 \right\}.$$

2.2 Lie group actions

Lie group actions appear often and generalise ordinary group actions familiar from group theory. In the case of Lie groups the discussion becomes more involved as the group G and the acted upon manifold M posses a differentiable structure. We see how a group action transports smooth structure and give general conditions when the quotient space is smooth. Then we discuss Lie group actions on a vectorspace and the adjoint representation.

Definition 2.4. Let G is a Lie group and M a smooth manifold. A *smooth right-action* of G on M is a smooth map $\theta : M \times G \to M$ written as $G \times M \ni (p, g) \mapsto p \cdot g \in M$ that satisfies:

$$\begin{aligned} (p \cdot g_1) \cdot g_2 = p \cdot (g_1 g_2), &\quad g_1, g_2 \in G, \quad p \in M, \\ p \cdot e = p, &\quad p \in M. \end{aligned}$$

We also say that the triple (G, M, θ) is a right-G-action on M or that M is a G-space when the action is clear from the context. One can define a left-action similarly and the theories of left- and right actions are identical up to some signs. When not specified otherwise, we always mean a smooth right-action by the word action.

2.2.1 Orbits and actions

Given a G-space M, fix an element p and let

$$\mathcal{O}_p = \{x \in M : \exists g \in G \text{ with } g \cdot p = x\} \tag{2.8}$$

be the *orbit* of p in M. We call the map $\theta^{(p)} : G \ni g \mapsto p \cdot g$ the *orbit map*. On the other hand, we can also fix an element g to obtain a smooth map $\theta_g : M \to M$ with inverse $\theta_{g^{-1}}$. This leads to a dual map $G \to \mathrm{Diff}(M)$ by sending $g \mapsto \theta_g$.

The *isotropy group* G_p of G at $p \in M$ is the subgroup of all $g \in G$ such that $p \cdot g = p$.
There are various types of actions one might want to study. The following appear frequently. Let (G, M, θ) be a G-action.

- The action is said to be *free* if $p \cdot g_1 = p \cdot g_2$ implies $g_1 = g_2 (\in G)$. For a free action the orbit map $\theta^{(p)} : G \to M$ is then a diffeomorphism.
- We say G acts *properly* if the map $\theta : M \times G \to M$ is a proper map, i.e. preimages of compact sets are compact.
- The action is called *transitive* if the orbit map is surjective for some $p \in M$. Equivalently, for each $p, q \in M$ there exists a $g \in G$ such that $p \cdot g = q$. The manifold M in this case is called a *homogeneous* G-space.

Example 2.3. Suppose G is a Lie group and let $\mathfrak{g}$ be its Lie algebra. There exist natural left actions of G on itself by left and right multiplications:

$$\begin{aligned} \lambda &: G \times G \ni (g, h) \mapsto gh, \\ \rho &: G \times H \ni (g, h) \mapsto hg^{-1}. \end{aligned} \tag{2.9}$$

This action is free and transitive, and proper if the Lie group G is compact.

Given an action (G, M, θ) we can associate to a vector $X \in \mathfrak{g}$, a vector field on M as follows. Suppose $p \in M$ and consider the curve $\gamma(t)$ in M given by $\exp(tX) \cdot p$. The tangent vector of this curve at p given by

$$\gamma'(0) = \frac{d}{dt}_{t=0} (\exp(tX) \cdot p) \in T_p M \tag{2.10}$$

is well-defined. By varying p in M we obtain a rough vector field denoted X_F called the *fundamental vector field* The following proposition, shows, among other things, that this vector field is smooth.

Proposition 2.7. *Let (G, M, θ) be an action and $X \in \mathfrak{g}$. Then the fundamental vector field X_F of X is smooth. Moreover, the* fundamental map $\sigma : \mathfrak{g} \to \mathscr{X}(M)$ *is a Lie algebra homomorphism.*

Proof. See [Lee12, Lemma 20.14] and [Lee12, Theorem 20.15]. □

We can state the main result. The main idea is to apply the global Frobenius theorem to the distribution

$$D_p = T_p(G \cdot p), \tag{2.11}$$

which is involutive by the proposition above. One has to show that the distribution is smooth and construct an atlas for the quotient space M/G; both arguments are worked out in [Lee12, Theorem 21.20].

Theorem 2.4 (Quotient Manifold Theorem). *Let (M, G, θ) be a free and proper action. The quotient space M/G has the structure of a topological manifold of dimension* $\dim M - \dim G$, *and a unique smooth structure such that the quotient map $M \to M/G$ is a smooth submersion.*

This result now quickly gives a classification of homogeneous G-spaces.

Theorem 2.5 (Homogeneous Manifold Theorem). *Let M be a homogeneous G-space, $p \in M$ and let $G_p \subset G$ be the isotropy subgroup of G. There exists a diffeomorphism $G/G_p \cong M$*

2.2.2 Group actions on a vectorspace

In the case M is also a vectorspace, we additionally require that the action acts *linearly* on M. In other words, the action of G on M induces a map $\rho : G \to \mathrm{Aut}\, M$ as before, and this is called a *representation* of G on the vectorspace M.

There is an extensive literature about Lie group representations. We give the definition and then study the adjoint representation of a Lie group on its Lie algebra.

Example 2.4. Let G be a Lie group and $\rho : G \to V$ a representation. We define the dual representation $\rho^* : G \to V^*$ by

$$\rho^*(g)\phi := \phi \circ \rho(g^{-1}), \quad g \in G, \quad \phi \in V^*. \tag{2.12}$$

It is easy to check that ρ^* is a group homomorphism.

The following section describes the canonical group representation of a Lie group G on its Lie algebra $\mathfrak{g}$, known as the *adjoint representation*.

Adjoint representation of a Lie group and Lie algebra

For a group element $g \in G$ let λ_g resp. ρ_g denote the left resp. right translation maps corresponding to g. Write $c_g := \lambda_g \circ \rho_g : G \ni h \mapsto ghg^{-1}$ for their composition. Note that c_g is a group isomorphism with inverse $c_{g^{-1}}$, which fixes the identity e in particular. The differential of this map at the identity is an isomorphism of $T_eG \cong \mathfrak{g}$ to itself and is typically written $\mathrm{Ad}_g : \mathfrak{g} \ni X \mapsto dc_g(X) \in \mathfrak{g}$. For arbitrary $g \in G$, this description gives a group homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ called the *adjoint representation* of G.

The Lie algebra $\mathfrak{g}$ also has a natural action on itself by means of the Lie bracket $[\cdot, \cdot]$. That is to say, given any element $X \in \mathfrak{g}$ there is an endomorphism $\mathrm{ad}_X : Y \mapsto [X, Y]$ of $\mathfrak{g}$, and the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is called the *adjoint representation* of $\mathfrak{g}$.

The following proposition shows that the adjoint representation ad of $\mathfrak{g}$ is the differential of Ad. Its proof is not interesting for our applications; we refer to [Lee12, Theorem 20.27] or [Var13, Theorem 2.3.10].

Theorem 2.6. *For any $X \in \mathfrak{g}$*

$$\mathrm{ad}_X = \frac{d}{dt}_{|t=0} \mathrm{Ad}_{\exp(tX)} \tag{2.13}$$

as an element of End$(\mathfrak{g})$. In addition the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{ad}} & \mathfrak{g} \\ \big\downarrow \exp & & \big\downarrow \exp \\ G & \xrightarrow{\text{Ad}} & G \end{array}$$

The dual $\mathfrak{g}^*$ and Ad-invariant functions

Let $\mathfrak{g}^* := \mathrm{Hom}(V, \mathbb{F})$ be the dual vectorspace of $\mathfrak{g}$. For $\xi \in \mathfrak{g}^*$ denote $\xi(X) = \langle \xi, X \rangle$. There is a natural action of G on $\mathfrak{g}^*$ called the *coadjoint representation* by setting

$$\langle \mathrm{Ad}^*_g(\xi), X \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}}(X) \rangle, \quad \xi \in \mathfrak{g}^*, \quad X \in \mathfrak{g}. \tag{2.14}$$

One can easily verify the identity $(\mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h)(\xi) = \mathrm{Ad}^*_{gh}(\xi)$.

Suppose $f \in C^\infty(\mathfrak{g}^*)$ is a smooth map and denote its differential by df. We can identify df with a linear map denoted by the same symbol $df : \mathfrak{g}^* \to \mathfrak{g}$ as follows. There exists a canonical linear isomorphism $\mathfrak{g} \cong \mathfrak{g}^{**}$ with respect to the vectorspace structure. Moreover, given a finite-dimensional vector space V there is for any $v \in V$ a natural isomorphism $T_v V \cong V$. The differential of f at $\xi \in \mathfrak{g}^*$ written as $df(\xi)$ is thus a linear functional on $\mathfrak{g}^*$ and an element of $\mathfrak{g}$. We write this as a map

$$df : \mathfrak{g}^* \to \mathfrak{g}. \tag{2.15}$$

Definition 2.5. A function $f : \mathfrak{g} \to \mathbb{F}$ (resp. $f : \mathfrak{g}^* \to \mathbb{F}$) is said to be Ad-*invariant* (resp. Ad*-*invariant*) if for all $X \in \mathfrak{g}$ (resp. $X \in \mathfrak{g}^*$) and $g \in G$ we have $f(\mathrm{Ad}_g(X)) = f(X)$ (resp. $f(\mathrm{Ad}^*_g(X)) = f(X)$.

We later need the following lemma and state it only for Ad*-invariant functions.

Lemma 2.2. *Suppose $f \in C^\infty(\mathfrak{g}^*)$ is an Ad* invariant function. Then for all $\xi \in \mathfrak{g}^*$*

$$\mathrm{ad}^*_{df(\xi)}(\xi) = 0. \tag{2.16}$$

Moreover, the following diagram commutes for any $g \in G$:

$$\begin{array}{ccc} \mathfrak{g}^* & \xrightarrow{df} & \mathfrak{g} \\ \big\downarrow \mathrm{Ad}^*_g & & \big\downarrow \mathrm{Ad}^*_g \\ \mathfrak{g}^* & \xrightarrow{df} & \mathfrak{g} \end{array}$$

Proof. For any $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$ using Theorem 2.6 and Ad^*-invariance yields

$$\begin{aligned}\langle \mathrm{ad}^*_{df(\xi)}(\xi), X\rangle &= -\langle (\xi), \mathrm{ad}_{df(\xi)}(X)\rangle = -\langle \mathrm{ad}^*_X(\xi), df(\xi)\rangle \\ &= -\frac{d}{dt}_{|t=0} f(\mathrm{Ad}^*_{\exp tX}(\xi))) = -\frac{d}{dt}_{|t=0} f(\xi) = 0.\end{aligned} \tag{2.17}$$

Commutativity of the diagram is proven in [AVV13, Lemma 2.9]. □

A Lie algebra is called *simple* if it contains no non-trivial subalgebras and *semi-simple* if it can be written as a direct sum of simple subalgebras. Its proof uses far more representation theory than we have at our disposal; a mostly self-contained discussion can be found in [Bum+04, Chapter 10, Proposition 10.5].

Proposition 2.8. *Let $\mathfrak{g}$ be a semi-simple Lie algebra of a Lie group G. The bilinear form*

$$\langle X|Y\rangle := \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y), \quad X, Y \in \mathfrak{g}, \tag{2.18}$$

called the Killing-form *is symmetric and Ad-invariant, and non-degenerate if and only if the Lie algebra $\mathfrak{g}^*$ is semi-simple.*

Nearly all Lie algebras that we work with in this text are semi-simple, such as the Lie algebras of Example 2.2.

A non-degenerate bilinear form $\langle \cdot|\cdot\rangle$ such as the Killing form allows us to identify a Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ as follows. By differentiating the Ad-invariance identity

$$\langle \mathrm{Ad}_g(Y)|\mathrm{Ad}_g(Z)\rangle = \langle Y|Z\rangle, \quad Y, Z \in \mathfrak{g}, \tag{2.19}$$

we obtain

$$\langle \mathrm{ad}_X(Y)|Z\rangle + \langle Y|\mathrm{ad}_X(Z)\rangle = 0. \tag{2.20}$$

Let $\wedge$ denote the induced isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ by the non-degenerate form $\langle \cdot, \cdot\rangle$. Then the following diagram commutes for any $g \in G$:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\wedge"] \arrow[d, "\mathrm{Ad}_g"] & \mathfrak{g}^* \arrow[d, "\mathrm{Ad}^*_g"] \\ \mathfrak{g} \arrow[r, "\wedge"] & \mathfrak{g}^* \end{tikzcd}$$

Indeed, equation (2.19) implies for arbitrary $X, Y \in \mathfrak{g}$:

$$\begin{aligned}\langle \mathrm{Ad}^*_g(\widehat{X}), Y\rangle &= \langle \widehat{X}, \mathrm{Ad}_{g^{-1}}(Y)\rangle = \langle X|\mathrm{Ad}_{g^{-1}}(Y)\rangle \\ &= \langle \mathrm{Ad}_g(X)|Y\rangle = \langle \widehat{\mathrm{Ad}_g(X)}, Y\rangle.\end{aligned} \tag{2.21}$$

The adjoint and coadjoint actions thus naturally become identified. This plays an important role when we study the canonical Poisson structure of the dual $\mathfrak{g}^*$ in the next chapter.

Chapter 3

Poisson structures and integrability

A Poisson structure is the mathematical description of a mechanical system and is a Lie-Bracket on the space of smooth functions of the phase space, allowing for a study of Hamiltonian mechanics. They first appeared in a Poisson's study of conserved variables of mechanical systemsin the following form:

$$\{f,g\} := \sum_{i=1}^{2n} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial x^i} \right), \quad f,g \in C^\infty(\mathbb{R}^{2n}),$$

where $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ are standard coordinates. Poisson found that $\{f, H\} = 0$ and $\{g, H\} = 0$ implies $\{\{f, g\}, H\} = 0$, for an arbitrary smooth function $H \in C^\infty(M)$. Indeed, Jacobi later proved that this bracket operation satisfies the Jacobi identity, and the identity now bears his name.

We develop the structure theory of Poisson manifolds in terms of the distribution of Hamiltonian vector fields and view a symplectic manifold maximally regular Poisson manifold. At the end we discuss group actions on symplectic manifold and the notion of an integrable system.

3.1 Poisson Structures

We introduce Poisson manifolds.

3.1.1 The Poisson bracket

Suppose M is a smooth manifold. A *Poisson structure* on M is a Lie bracket $\{\cdot, \cdot\}$ on the $\mathbb{R}$-vectorspace $C^\infty(M)$ that satisfies the Leibniz rule in each argument:

$$\{f, gh\} = h\{f, g\} + g\{f, h\}, \quad f, g, h \in C^\infty(M). \tag{3.1}$$

The pair $(M, \{\cdot, \cdot\})$ is called a *Poisson manifold* and the bracket $\{\cdot, \cdot\}$ is called the Poisson bracket. Note that the Poisson bracket of any constant function vanishes.

In this text many Poisson structures appear on vectorspaces. Denote by M an n-dimensional vector space. For a given set of coordinate functions $\{x^i\}$ corresponding to some basis of M, the *structure functions* $x^{ij} := \{x^i, x^j\}$ are the components of an anti-symmetric 2-tensor on M that satisfy the following identities:

$$\begin{aligned} &x^{ij} = -x^{ji}, \\ &\sum_{m=1}^{n} \left(\frac{\partial x^{ij}}{\partial x^m} x^{mk} + \frac{\partial x^{jk}}{\partial x^m} x^{mi} + \frac{\partial x^{ki}}{\partial x^m} x^{mj} \right) = 0, \quad i, j, k = 1, \ldots, n. \end{aligned} \tag{3.2}$$

The second equation is an easy consequence of the Jacobi identity.

Conversely, given an indexed set $\{x^{ij}\}_{i,j=1}^n$ of smooth functions satisfying equation (3.2) the vectorspace $\mathbb{R}^n$ may be equipped with a Poisson structure by setting $\{x^i, x^j\} := x^{ij}$, where $\{x^i\}_{i=1}^n$ are the canonical coordinate projection maps.

Example 3.1. Suppose $A \in \mathrm{Mat}_n(\mathbb{C})$ is any constant skew-symmetric $n \times n$ matrix. From linear algebra it is known that there exists a coordinate transformation and a non-negative integer $r \in \mathbb{N}$ such that

$$A = \begin{pmatrix} 0 & I_r & 0 \\ -I_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

The matrix A satisfies eq. (3.2) and hence determines a Poisson structure $\{\cdot, \cdot\}_A$. We call the integer $2r$ the *rank* of $\{\cdot, \cdot\}_A$.

Example 3.2 (Kirillov bracket)**.** Suppose G is a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual. Define the *Kirillov*-bracket on $\mathfrak{g}^*$ by

$$\{f, g\}(\xi) := \langle \xi, [df(\xi), dg(\xi)] \rangle, \quad \xi \in \mathfrak{g}^*, \quad f, g \in C^\infty(\mathfrak{g}^*). \tag{3.4}$$

(Cf. Section 3.2.) The Leibniz property follows from the product rule and Jacobi identity from the same identity for $[\cdot, \cdot]_{\mathfrak{g}}$. The pair $(\mathfrak{g}^*, \{\cdot, \cdot\})$ is called the Lie-Poisson structure and plays an important role in the later chapters of this text.

As the Poisson bracket is a biderivation, we consider the following 2-tensor called the *Poisson tensor*:

$$\begin{aligned} P : \Omega(M) \times \Omega(M) &\longrightarrow \mathbb{R}, \\ (df, dg) &\longmapsto \{f, g\}. \end{aligned} \tag{3.5}$$

Note that P is well-defined as the Poisson bracket annihilates constant functions. Also, $\{\cdot,\cdot\}$ can be reconstructed from the Poisson tensor P: an easy computation shows that the components of the tensor P in local coordinates (x^i) are given by

$$P = x^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}. \tag{3.6}$$

The map P leads by duality to a smooth map $\tilde{P} : \Omega(M) \to \mathscr{X}(M)$ given by

$$\begin{aligned} \tilde{P} : \Omega(M) &\longrightarrow \mathscr{X}(M), \\ df &\longmapsto (dg \mapsto P(df, dg)). \end{aligned} \tag{3.7}$$

We denote the corresponding bundle homomorphism $\tilde{P} : T^*M \to TM$ with the same symbol.

We need the notion of a morphism between Poisson manifolds.

Definition 3.1. Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be Poisson manifolds. A *Poisson morphism* is a smooth mapping $\phi : M \to N$ such that for all $f, g \in C^\infty(N)$:

$$\phi^*\{f, g\}_N = \{\phi^* f, \phi^* g\}_M \tag{3.8}$$

A *Poisson submanifold* $(N, \{\cdot,\cdot\}')$ of $(M, \{\cdot,\cdot\})$ is then a submanifold $\iota : N \hookrightarrow M$ with a Poisson structure $\{\cdot,\cdot\}'$ such that the inclusion ι is a Poisson morphism. If such a structure $\{\cdot,\cdot\}'$ exists, it must be unique.

3.1.2 Hamiltonian vector field

On a Poisson manifold $(M, \{\cdot,\cdot\})$ we can associate a vector field to smooth functions called the Hamiltonian vector field.

Suppose $f \in C^\infty(M)$. The dual map to $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to \mathbb{R}$, denoted X_- is given by

$$\begin{aligned} X_- : C^\infty(M) &\longrightarrow \mathscr{X}(M) \\ f &\longmapsto (g \mapsto \{f, g\}). \end{aligned} \tag{3.9}$$

Note that X_{-} is well-defined as $\{\cdot,\cdot\}$ is a derivation in each argument. We write $X_{-}(f) = X_f$ and X_f is the *Hamiltonian vector field* of f. Observe that $X_f = \tilde{P}(df)$. A Poisson manifold allows us to study Hamiltonian mechanics in more general spaces than $\mathbb{R}^{2n}$.

Let $\mathrm{Ham}(M) = X_{-}(C^\infty(M))$ be the set of all Hamiltonian vector fields. Then $\mathrm{Ham}(M)$ is an $\mathbb{R}$-module with pointwise operations. The following lemma shows that $\mathrm{Ham}(M)$ is also a subalgebra of $\mathscr{X}(M)$.

Lemma 3.1. *Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. For any $f,g \in C^\infty(M)$:*

$$[X_f, X_g] = X_{\{f,g\}} \tag{3.10}$$

The map $X_{-} : C^\infty(M) \to \mathscr{X}(M)$ is therefore is Lie algebra homomorphism.

Proof. This is equivalent to the Jacobi identity. Indeed, choose arbitrary $f,g,h \in C^\infty(M)$ and rewrite the Jacobi identity as

$$\begin{aligned} &\{f,\{g,h\}\} + \{g,\{h,f\}\} = \{\{f,g\},h\}, \\ &X_f(X_g(h)) - X_g(X_f(h)) = X_{\{f,g\}}(h). \end{aligned} \tag{3.11}$$

The claim follows. □

For $p \in M$ let $\mathrm{Ham}_p(M) \subseteq T_pM$ be the linear subspace spanned by the Hamiltonian vector fields. Varying p in M gives a singular distribution called the *Hamiltonian distribution* D_{Ham}. The previous lemma showed that the distribution D_{Ham} is involutive. It is locally described by

$$\mathrm{Ham}_p(M) = \mathrm{span}\{X_{f_1}, \ldots, X_{f_{\dim \mathrm{Ham}_p(M)}}\}, \tag{3.12}$$

where X_i are Hamiltonian vector fields, $i = 1, \ldots, \dim \mathrm{Ham}_p(M)$. Observe that these vector fields may not be linearly independent throughout any open set $U \setminus \{p\}$, with U some neighborhood of p. We give a more explicit characterisation later (see Theorem 3.1).

Definition 3.2. Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold and $p \in M$. The *rank* $\mathrm{rk}_p\{\cdot,\cdot\}$ at p is the natural number $\dim \mathrm{Ham}_p(M)$. Furthermore, a Poisson manifold $(M,\{\cdot,\cdot\})$ is called *regular* if $\mathrm{rk}_p\{\cdot,\cdot\} = \mathrm{const}$ for all $P \in M$.

A quick comparison shows that this notion agrees with Example 3.1.

The following two propositions display simple properties of the rank and we refer for the proofs to [AVV13, Proposition 3.13, Proposition 3.16].

Proposition 3.1. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $s \in \mathbb{N}$. The following subset of M is open:*

$$M_{(s)} := \{p \in M : \ rk_p\{\cdot,\cdot\} \geq 2s\}. \tag{3.13}$$

Proposition 3.2. *Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be Poisson manifolds, $p \in M$ and $\phi : M \to N$ a Poisson morphism. Then $rk_p\{\cdot,\cdot\}_M \geq \ rk_{\phi(p)}\{\cdot,\cdot\}_N$.*

A Poisson structure often arises from a so-called *symplectic structure*. The following section investigates symplectic structures and shows that a symplectic manifold is a regular Poisson manifold with maximal rank.

3.1.3 Symplectic manifolds

A *symplectic* manifold (M, ω) is a smooth manifold M together with a smooth closed non-degenerate 2-form ω. The non-degenerate property means that for all $p \in M$ the tangent space T_pM has a skew-symmetric non-degenerate bilinear form ω_p that induces an isomorphism $T_pM \cong T_p^*M$. Globally this gives a bundle isomorphism denoted by the same symbol:

$$\hat{\omega} : TM \longrightarrow T^*M \tag{3.14}$$

and thus an isomorphism

$$\hat{\omega} : \mathscr{X}(M) \longrightarrow \Omega(M). \tag{3.15}$$

Example 3.3. Let $M = \mathbb{R}^{2n}$ be a smooth manifold and consider the 2-form ω given with respect to the standard global coordinates $(x^i)_{i=1}^{2n}$ by

$$\omega = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}. \tag{3.16}$$

In other words, the components of ω in this basis are given by

$$\omega_{ij} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{3.17}$$

The matrix ω_{ij} is nonsingular and the 2-form ω is nondegenerate. Note the similarity with Example 3.1.

Many symplectic structures arise in the following context.

Proposition 3.3. *Let M be a smooth manifold and T^*M its cotangent bundle. There exists a canonical symplectic closed 2-form ω on T^*M such that (T^*M, ω) is a symplectic manifold.*

Proof. Let $p \in M$ and choose (x^i, ξ^i) standard local coordinates in a neighborhood of p in T^*M. Suppose $\xi \in T_p^*M$ is a covector and denote by $\pi : T^*M \to M$ the canonical projection map that sends $(x^i, \xi^i) \mapsto (x^i)$. The map π is smooth and its differential at (p, ξ) is a linear map $d\pi_{(p,\xi)} : T_{(p,\xi)}(T^*M) \to T_pM$. Define the 1-form $\theta \in \Omega(T^*M)$ in local coordinates by $T_{(p,\xi)} \ni X \mapsto \xi(d\pi_{(p,\xi)}(X))$. It is easy to show that θ is smooth and in standard local coordinates is described by $\theta = \xi^i dx_i$. The 2-form $\omega := d\theta$ is clearly closed and non-degenerate by the same arguments as Example 3.3. $\square$

The 1-form θ above is called the *tautological* 1-form on T^*M. We actually showed that the cotangent bundle has a symplectic structure whose form is given by (3.16). Significantly enough, it turns out that any symplectic manifold can locally be written in this form, known as the *Darboux Theorem*. We will later see a generalisation of this result to Poisson manifolds known as the *Splitting Theorem*.

For $f \in C^\infty(M)$ there corresponds a vector field $X_f = \hat{\omega}^{-1}(df)$ and X_f is called the Hamiltonian vector field of f for obvious reasons. We use the convention $X_f(g) = \omega(X_f, X_g)$. Moreover, we can define a bracket on on $C^\infty(M)$ by setting $\{f, g\} := \omega(X_f, X_g)$. The only non-trivial property to show $\{\cdot, \cdot\}$ is a Poisson bracket, is the Jacobi identity. The following lemma shows that this is equivalent to the requirement $d\omega = 0$.

Lemma 3.2. *Suppose M is a smooth manifold with a non-degenerate two-form ω and define the bracket $\{f, g\} = \omega(X_f, X_g)$ for $f, g \in C^\infty(M)$. Then $(M, \{\cdot, \cdot\})$ is a Poisson manifold if and only if $d\omega = 0$.*

Proof. Recall the following formula for vector fields $X_1, X_2, X_3 \in \mathscr{X}(M)$:

$$
\begin{aligned}
d\omega(X_1, X_2, X_3) = \frac{1}{3}[&X_1\omega(X_2, X_3) + X_2\omega(X_3, X_1) + X_3\omega(X_1, X_2) \\
&- \omega([X_1, X_2], X_3) - \omega([X_2, X_3], X_1) - \omega([X_3, X_1], X_2)].
\end{aligned} \tag{3.18}
$$

One may now easily rewrite $d\omega = 0$ to the Jacobi identity. The converse is the argument in reverse. $\square$

Any symplectic manifold (M, ω) is thus a Poisson manifold. The following proposition shows that the converse can also hold.

Proposition 3.4. *Let $(M^{2n}, \{\cdot, \cdot\})$ be a Poisson manifold such that $rk_p\{\cdot, \cdot\} = 2n$ for all $p \in M$. Then there exists a symplectic structure ω on M such that its induced Poisson structure coincides with $\{\cdot, \cdot\}$.*

Proof. Let $\tilde{P} : T^*M \to TM$ be the bundle homomorphism from before. Then $\tilde{P}$ is stalkwise invertible and there exists an inverse map $\tilde{P}^{-1} : TM \to T^*M$. Note that $\tilde{P}^{-1}$ is smooth as $\tilde{P}$ has constant rank. We define $\hat{\omega} := \tilde{P}^{-1}$ and let $\omega \in \wedge^2\Omega(M)$ be the corresponding 2-form on M. It suffices to show that $d\omega = 0$. For any Hamiltonian vector fields using the identity eq. (3.18) yields

$$\begin{aligned} 3d\omega(X_f, X_g, X_h) &= X_f\omega(X_g, X_h) - \omega([X_g, X_h], X_f) + \text{cyclic terms} \\ &= 2\left(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}\right) \\ &= 0, \end{aligned} \tag{3.19}$$

by the Jacobi identity, where $, f, g, h \in C^\infty(M)$. This finishes the proof. □

The full converse need not be true, i.e. not every Poisson manifold is a symplectic manifold. It turns out that more can be said about this relationship. We sketch the proof below and need a result about the local structure.

3.1.4 Local structure of a Poisson manifold

The following structure result is generalisation of the Darboux theorem for symplectic manifolds. [AVV13, Theorem 3.25]

Theorem 3.1 (Splitting Theorem). *Suppose $(M, \{\cdot, \cdot\})$ is a Poisson manifold of dimension n, let $p \in M$ be arbitrary and denote the rank of $\{\cdot, \cdot\}$ at p by $2r$, $s := n - 2r$. There exists a coordinate neighborhood U of p with coordinates $(q^1, \dots, q^r, p^1, \dots p^r, z^1, \dots, z^s)$ centered at p, such that on U*

$$\{\cdot, \cdot\} = \sum_{i=1}^{r} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i} + \frac{1}{2} \sum_{k,l=1}^{s} \phi_{kl} \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l}, \tag{3.20}$$

where the components ϕ_{kl} are smooth functions that depend only on $z^1, \dots z^s$, and vanish at p.

Sketch: proceed with induction on r. If $r = 1$ there exists a smooth function p_1 such that $X_{p_1}(p) \neq 0$. Moreover, there exist local coordinates such that $X_{p_1} = \partial/\partial q_1$. Note that $\{p_1, q_1\} = 1$. Then X_{p_1} and X_{q_1} form an involutive distribution as $[X_{p_1} X_{q_1}] = X_1 = 0$ and applying the Frobenius theorem gives a decomposition of M into 2-dimensional submanifolds. The local description of these leaves then implies eq. (3.20) and the general case hence follows from induction. □

Let $(M, \{\cdot,\cdot\})$ be a *regular* Poisson manifold. The Hamiltonian distribution D_{Ham} is an involutive distribution and by the Frobenius theorem there exists a foliation of the manifold M into immersed submanifolds of dimension $\operatorname{rk}\{\cdot,\cdot\} = 2r$. Each leaf has by restriction a Poisson structure $\{\cdot,\cdot\}'$ whose rank coincides with the dimension of the leaf, and hence determines a symplectic structure by Proposition 3.4. The leaves are called the *symplectic leaves* of the Poisson manifold $(M, \{\cdot,\cdot\})$ and the foliation a *symplectic foliation*.

In fact, it turns out that the distribution need not be regular. Any Poisson manifold admits a decomposition into symplectic leaves with varying dimensions. This follows from (3.1); details can be found in [AVV13, Theorem 3.26]. To finish this section, we state a general result for determining whether a submanifold is a Poisson submanifold that we will use later. Its proof rather lengthy and can be found in [AVV13, Proposition 3.33].

Proposition 3.5. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $N \subset M$ an immersed submanifold, There exists a Poisson structure $\{\cdot,\cdot\}'$ on N such that N is a Poisson submanifold if and only if the restriction of every Hamiltonian vector field on M to N is tangent to N.*

3.2 The Lie Poisson structure $\mathfrak{g}^*$

The most important Poisson manifold of this text is the canonical Poisson structure on the dual Lie algebra $\mathfrak{g}^*$ belonging to some Lie group G.

Example 3.2 already introduced the Kirillov bracket. We now show how it arises in a canonical manner. The space of linear functions on $\mathfrak{g}^*$, the double dual $\mathfrak{g}^{**}$, can be naturally identified with $\mathfrak{g}$. For the sake of notation, let $X_* \in \mathfrak{g}^{**}$ be the element corresponding to $X \in \mathfrak{g}$. $\mathfrak{g}^{**}$ has the natural structure of a Lie algebra by setting $[X_*, Y_*] = [X, Y]_*$. As such, $\mathfrak{g}^*$ can be given a Poisson structure whose structure functions are precisely the structure constants of $\mathfrak{g}$ by setting

$$\{X_*, Y_*\} := [X_*, Y_*]. \tag{3.21}$$

we claim that

$$\{f, g\}(\xi) = \langle \xi, [df(\xi), dg(\xi)] \rangle, \quad f, g \in C^\infty(\mathfrak{g}^*), \tag{3.22}$$

i.e. this bracket coincides with the Kirillov bracket. Indeed, the equation is valid for linear functions on $\mathfrak{g}^*$ and both sides are a derivation in the smooth functions f and g.

The integral curves of the Hamiltonian vector fields of $(\mathfrak{g}^*, \{\cdot,\cdot\})$ are easily described.

Lemma 3.3. *Let $f \in C^\infty(\mathfrak{g}^*)$ and X_f its Hamiltonian vector field. Its integral curve $\xi(t)$ starting at $\xi \in \mathfrak{g}^*$ is given by:*

$$\frac{d}{dt}\xi(t) = \mathrm{Ad}^*_{df(\xi(t))}(\xi) \tag{3.23}$$

Proof. Suppose $\xi(t)$ is a curve passing through ξ at $t = 0$. For the sake of notation write $\zeta = \xi(t)$. Expanding the right-hand-side of the condition $\xi'(t) = (X_f)_{\xi(t)}$ yields for arbitrary $g \in C^\infty(\mathfrak{g}^*)$

$$\begin{aligned}(X_f)_\zeta(g) &= \{g, f\}(\zeta) = \{\zeta, [dg(\zeta), df(\zeta)]\} \\ &= \{\zeta, -\mathrm{ad}_{df(\zeta)}(dg(\zeta))\} = \{\mathrm{ad}^*_{df(\zeta)}(\zeta), dg(\zeta)\}.\end{aligned} \tag{3.24}$$

On the other hand, the left-hand-side of the same condition can be written using the chain rule as

$$\xi'(t)(g) = \frac{d}{dt}_{|t=0}(g \circ \xi)(t) = \langle \dot{\xi}, dg(\zeta)\rangle. \tag{3.25}$$

Equations (3.24) and (3.25) now imply eq. (3.23). □

If the Lie algebra admits an Ad-invariant non-degenerate bilinear form, then we can also rewrite (3.23) as a so-called *Lax-equation*

$$\frac{d\xi}{dt} = [\xi, -df(\xi)], \tag{3.26}$$

We will study Lax equations extensively in Chapter 5.

We describe the symplectic leaves of the Poisson manifold $(\mathfrak{g}^*, \{\cdot,\cdot\})$. For $\xi \in \mathfrak{g}^*$ let $\mathcal{O}_\xi$ be the orbit of ξ in $\mathfrak{g}^*$ and G_ξ the isotropy subgroup of ξ. The orbit $\mathcal{O}_\xi$ can be given the structure of a smooth manifold by $G/G_\xi \cong \mathcal{O}_\xi$ (cf. Theorem 2.5), as G clearly acts transitively on any orbit. Then $T_\xi\mathcal{O}_\xi = \{\mathrm{ad}^*_X(\xi) : X \in \mathfrak{g}\}$ as the tangent space is spanned by the fundamental vector fields of the group action.

Proposition 3.6. *Let G be a Lie group and $\mathfrak{g}^*$ its dual Lie algebra equipped with the canonical Lie-Poisson structure. The symplectic leaves of $\mathfrak{g}^*$ are precisely the coadjoint orbits in $\mathfrak{g}^*$.*

Proof. It suffices to show for any point $\phi \in \mathcal{O}_\xi$ that

$$T_\phi\mathcal{O}_\xi = \mathrm{Ham}_\phi(\mathfrak{g}^*). \tag{3.27}$$

Indeed, then $\mathcal{O}_\xi$ is the symplectic leaf associated to the Hamiltonian distribution at ξ. By eq. (3.23) and the comment above we have

$$T_\xi\mathcal{O}_\xi = \{\mathrm{ad}^*_{df(\xi)}(\xi) : f \in C^\infty(\mathfrak{g}^*)\} = \mathrm{Ham}_\phi(\mathfrak{g}^*) \tag{3.28}$$

which completes the proof. □

3.3 Hamiltonian group actions

We investigate group actions on symplectic and Poisson manifolds which preserve the respective structure of the space.

Definition 3.3. Let G be a Lie group, $g \in G$ arbitrary and (M, ω) a symplectic manifold. (M, ω) is called a *symplectic G-space* if there exists a right action (G, M, θ) such that $\theta_g^* \omega = \omega$, and the quadruple is written (G, M, θ, ω).

Often we also want the fundamental vector fields $\sigma(\mathfrak{g})$ to be Hamiltonian.

Definition 3.4. Let (M, ω) be a symplectic G-space and let $\mathrm{Ham}(M)$ denote the set of Hamiltonian vector fields on ω. M is called strongly symplectic if $\sigma(\mathfrak{g}) \subset \mathrm{Ham}(M)$, where σ maps elements in $\mathfrak{g}$ to their fundamental vector field.

Definition 3.5. For a strongly symplectic right action (G, M, θ, ω), a lift $\lambda : \mathfrak{g} \to C^\infty(M)$ of σ is a Lie algebra homomorphism such that the following diagram with exact row commutes:

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r, "X_-"] & \mathrm{Ham}(M) \arrow[r] & 0 \\
 & & \mathfrak{g} \arrow[u, "\lambda"] \arrow[ur, "\sigma"] & &
\end{tikzcd}
$$

The triple (M, ω, λ) is called a *Hamiltonian G-space*. One also writes $\lambda(X) = H_X$.

Note that for arbitrary $X, Y \in \mathfrak{g}$ this means:

$$\{H_X, H_Y\} = H_{[X,Y]_{\mathfrak{g}}}, \tag{3.29}$$

where $[\cdot, \cdot]_{\mathfrak{g}}$ denotes the Lie bracket operation in $\mathfrak{g}$.

To give the above concept of a lift λ more clarity, consider the following construction. Let $X_1, \ldots, X_n \in \mathfrak{g}$ be a vectorspace basis and let $\phi_1, \ldots \phi_n$ be smooth such that $\sigma(X_i) = \phi_i$ for $i = 1, \ldots, n$. The existence of a lift is then equivalent to the condition that the map

$$
\begin{aligned}
\psi : \mathfrak{g} &\longrightarrow C^\infty(M) \\
X_i &\longmapsto \phi_i,
\end{aligned}
\tag{3.30}
$$

is a Lie algebra homomorphism, $i = 1, \ldots, n$.

Given any strongly symplectic G-space this may not be possible. However, one can replace the Lie algebra $\mathfrak{g}$ by $\mathfrak{g} \times \mathbb{R}$ to make the action Hamiltonian. For the details see [Wal18, Section 3.2].

Example 3.4. Suppose G is a Lie group and $\xi \in \mathfrak{g}^*$ such that its orbit $\mathcal{O}_\xi$ is non-trivial. Let ω be the symplectic structure on $\mathcal{O}_\xi$. It is explicitly given as follows: for $\phi \in \mathcal{O}_\xi$ and $x, y \in T_\phi \mathcal{O}_\xi$, with $x = \mathrm{ad}_X^* \phi$ resp. $y = \mathrm{ad}_Y^* \phi$ for $X, Y \in \mathfrak{g}$, we have

$$\omega_\phi(x, y) = \langle \phi, [X, Y] \rangle. \tag{3.31}$$

We first show that $(G, \mathcal{O}_\xi, \mathrm{Ad}^*)$ is a symplectic G-space. Indeed, for any $g \in G$ we have

$$\begin{aligned} ((\mathrm{Ad}_g^*)^* \omega)_\phi(x, y) &= \omega_{\mathrm{Ad}_g^* \phi}(x, y) = \langle \mathrm{Ad}_g^* \phi, [\mathrm{Ad}_g X, \mathrm{Ad}_g Y] \rangle \\ &= \omega_\phi(x, y). \end{aligned} \tag{3.32}$$

In fact, we can directly show that the G-action is Hamiltonian. To this end, note that the inclusion $\iota : \mathcal{O}_\xi \to \mathfrak{g}^*$ is a Poisson morphism and let $\lambda : \mathfrak{g} \to C^\infty(\mathcal{O}_\xi)$ be the dual map of ι given by

$$\lambda : X \longmapsto (\phi \longmapsto \langle \phi, X \rangle). \tag{3.33}$$

One can easily verify that λ is the desired lift. In particular, $(\mathcal{O}_\xi, \omega)$ is a strongly symplectic G-space.

We generalise the argument of the previous example. Given any Hamiltonian G-space (M, ω, λ) we can construct a natural map from M to $\mathfrak{g}^*$.

Definition 3.6. Let (M, ω, λ) be a Hamiltonian G-space and $X \in \mathfrak{g}$. The *moment map* $\mu : M \to \mathfrak{g}^*$ is defined by the relationship

$$\langle \mu(p), X \rangle = H_X(p), \quad p \in M. \tag{3.34}$$

One can easily verify from (3.29) that μ is a Poisson map and G-equivariant. Clearly, given any G-equivariant Poisson map $\mu : M \to \mathfrak{g}^*$ on a symplectic G-space M, the action is Hamiltonian. In many instances it is easier to construct a moment map, as we see below.
The following proposition shows that any G-space M can be extend to a Hamiltonian G-space on the cotangent bundle.

Proposition 3.7. *Let (G, M, θ) be an action. Then there exists a natural action denoted $\theta^\#$ on T^*M which is Hamiltonian.*

Proof. Fix a group element $g \in G$ and using θ identify it with a diffeomorphism $\theta_g : M \to M$. The map θ_g lifts to a bundle automorphism

$$\theta_g^\# : T^*M \longrightarrow T^*M \tag{3.35}$$

by setting $\theta_g^\#(p,\xi) = (\theta_g(p), (\theta_g^*)^{-1}\xi)$, where $\theta_g^* : \Omega(M) \to \Omega(M)$ is the pullback of 1-forms. We show that $\theta_g^\#$ preserves the tautological 1-form on T^*M, which we denote ρ. For any $(p,\xi) \in T^*M$ and $v \in T_{(p,\xi)}(T^*M)$ we have

$$\begin{aligned} ((\theta_g^\#)^*\rho)_{(p,\xi)}(v) &= \rho_{(\theta_g(p),(\theta_g^*)^{-1}(\xi))}((d\theta_g^\#)_{(p,\xi)}v) \\ &= (\theta_g^*)^{-1}(\xi)(d\pi_{(\theta_g(p),(\theta_g^*)^{-1}(\xi))}((d\theta_g^\#)_{(p,\xi)}v)) \\ &= \xi(\theta_g^{-1}(d\pi_{(\theta_g(p),(\theta_g^*)^{-1}(\xi))}((d\theta_g^\#)_{(p,\xi)}v))) \\ &= \xi(d\pi_{(p,\xi)}(v)) \\ &= \rho_{(p,\xi)}(v). \end{aligned} \tag{3.36}$$

We next show this action by is Hamiltonian constructing a moment map. Let $\sigma : \mathfrak{g} \to \mathscr{X}(M)$ be the fundamental map and consider its dual given by $\mu_p : T_p^*M \to \mathfrak{g}^*$ given by $\mu_p(\xi)(X) = \xi(\sigma(X)_p)$, where $\xi \in T_p^*M$. Denote the global map by $\mu : T^*M \to \mathfrak{g}^*$. To prove μ is a Poisson mapping, we introduce some notation. Let $H_X \in C^\infty(T^*M)$ be given by $\mu_X(p) = \mu(p)(X)$ and $\tau : \mathfrak{g} \to \mathscr{X}(T^*M)$ the fundamental map of the action $\theta_g^\#$. We need to show

$$(dH_X)_{(p,\xi)}(v) = \omega(v, \tau(X)), \quad v \in T_{(p,\xi)}(T^*M). \tag{3.37}$$

First note that $d\pi_{(p,\xi)}(\tau(X)_{(p,\xi)}) = \sigma(X)_p$. Thus

$$\theta_{(p,\xi)}(\tau(X)_{(p,\xi)}) = \xi(\sigma(X)_p) = H_X((p,\xi)) \tag{3.38}$$

such that with *Cartan's magic formula* we have

$$\iota_{\tau(X)}\omega = \iota_{\tau(X)}d\theta = L_{\tau(X)}\theta - d\iota_{\tau(X)}\theta = -d(\theta(\tau(X))) = -dH_X \tag{3.39}$$

which is the desired equation (3.37), where the Lie derivative term vanishes as G acts by symplectomorphisms and hence preserves θ. In addition, we also need to show that μ is equivariant with respect to the coadjoint action. This easily follows from the naturality of the exponential map. The action is hence Hamiltonian and the proof is complete. □

Example 3.5. In Example 2.3 we discussed natural actions of G on itself. We identify $T^*G \cong G \times \mathfrak{g}^*$ by means of left-translations. One can easily verify that the induced action in the sense above is given by

$$\begin{aligned} l_g &: (h, v) \longmapsto (gh, v), \\ r_g &: (h, v) \longmapsto (hg^{-1}, \mathrm{Ad}_g^*(v)). \end{aligned} \tag{3.40}$$

Hence, the corresponding moment maps are respectively given by

$$\begin{aligned} \mu_l &: (h, v) \longmapsto v, \\ \mu_r &: (h, v) \longmapsto -\mathrm{Ad}_h^*(v). \end{aligned} \tag{3.41}$$

Chapter 4

Riemann Surfaces

This chapter gives a down to earth introduction to the theory of Riemann surfaces. Many references to the literature are provided for the sake of clarity and space. In particular, we rely heavily on [For12] and have adapted many ideas from their presentation.

We start with the definition of a Riemann surface and study elementary properties of a complex structure in a connected space. Sheaves conveniently describe the holomorphic and smooth structure, and play an important role in our discussion. Thereafter we briefly describe the Riemann surface associated to an algebraic equation and go on to investigate divisors and holomorphic line bundles.

4.1 Complex structure

Suppose X is a topological 2-manifold and $x \in X$. Let (U, z) and (V, ζ) be coordinate neighborhoods both containing x, denote $\widehat{U} = z(U) \subseteq \mathbb{R}^2$, $\widehat{V} = \zeta(V) \subseteq \mathbb{R}^2$ and identify them with open subsets in $\mathbb{C}$ via the canonical homeomorphism $(x, y) \mapsto x + iy$. The two charts are called *holomorphically compatible if* the *transition function*

$$\zeta \circ z^{-1} : z(U \cap V) \longrightarrow \zeta(U \cap V)$$

is a biholomorphic map, i.e. the map is holomorphic with a holomorphic inverse. Note that both the domain and image are subsets of $\mathbb{C}$. A *holomorphic atlas* $\mathcal{U}$ is a collection of holomorphically compatible charts that covers X. More precisely, one defines an equivalence class of holomorphic atlases by

$$\mathcal{U} \sim \mathcal{V} \Longleftrightarrow \mathcal{U} \cup \mathcal{V} \quad \text{is an atlas.}$$

Any atlas determines a maximal atlas by applying Zorn's lemma to the set of atlases with partial ordering $\subseteq$. A *complex 1-manifold* is a pair $(X, \mathcal{U})$ consisting of a topological 2-manifold X with a holomorphic atlas $\mathcal{U}$. One can similarly define higher dimensional complex manifolds; see for instance [Sch02, Chapter 1]. Note that a complex 1-manifold is in particular a smooth 2-manifold.

Definition 4.1. A *Riemann surface* is a complex connected 1-manifold.

The additional requirement of connectivity simplifies many of the arguments and, crucially, leads to the Identity Theorem for Riemann surfaces. All complex manifolds that appear in this text are connected. Note that an open subset of a Riemann surface has the canonical structure of a Riemann surface in the usual manner.

A morphism in the category of Riemann surfaces is a holomorphic map. Let X, Y be Riemann surfaces and $f : X \to Y$ a continuous. The mapping f is called *holomorphic* if for every $x \in X$ there exist coordinate neighborhoods (U, z) centered at x and (V, ζ) centered at $f(x) \in Y$ such that the coordinate representation

$$\widehat{f} := \zeta \circ f \circ z^{-1} : z(U) \longrightarrow \zeta(V)$$

is a holomorphic map.

A holomorphic *function* $f : X \to \mathbb{C}$ is a holomorphic mapping from X to $\mathbb{C}$. We denote the set of holomorphic functions on the Riemann surface X by $\mathcal{O}(X)$.

Example 4.1. The following Riemann surfaces appear frequently throughout the text.

(1) The complex plane with the standard atlas $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$ is a Riemann surface.

(2) We describe the *Riemann sphere*, denoted $\mathbb{CP}^1$, which is the one-point compactification of $\mathbb{C}$. Indeed, define $\mathbb{CP}^1 := \mathbb{C} \cup \{\infty\}$, with ∞ a formal symbol, and equip it with the basis consisting of (i) open subsets of $\mathbb{C}$ (ii) sets of the form $(\mathbb{C} \setminus K) \cup \{\infty\}$, with $K \subseteq \mathbb{C}$ a compact set. With spherical projection one can show that $\mathbb{CP}^1$ is homeomorphic to 2-sphere S^2. Consider the atlas given by

$$U_1 = \mathbb{CP}^1 \setminus \{\infty\}, \quad z_1 = \mathrm{id}_{\mathbb{C}},$$
$$U_2 = \mathbb{CP}^1 \setminus \{0\}, \quad z_2 = \begin{cases} z^{-1} & \text{if } z \neq \infty \\ 0 & \text{if } z = \infty \end{cases} .$$

The maps z_1, z_2 are homeomorphisms and the transition function is given by the holomorphic map $z^{-1} : \mathbb{C}^* = \mathbb{C} \setminus \{0\} \to \mathbb{C}^*$. $\mathbb{CP}^1$ with the standard atlas $\{(U_1, z_1), (U_2, z_2)\}$ is called the Riemann sphere.

(3) Suppose $w_1, w_2 \in \mathbb{C}^*$ are linearly independent over $\mathbb{R}$. The lattice spanned by $\{w_1, w_2\}$ is given by

$$\Gamma := \mathbb{Z}w_1 + \mathbb{Z}w_2.$$

Define the *complex-torus* $\mathbb{C}/\Gamma$ as follows. We equip the space C/Γ with the natural quotient topology inherited from the canonical projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$. It is easy to see that π is also open and $\mathbb{C}/\Gamma$ is compact and connected. The complex structure exists and is determined uniquely from the projection π.

We state the removable singularity theorem for holomorphic functions on a Riemann surface; it follows immediately from the same result on the complex plane.

Theorem 4.1 (Removable Singularity Theorem). *Suppose U is an open subset of a Riemann surface and $a \in U$. If the function $f \in \mathcal{O}(U \setminus \{a\})$ is bounded in some neighborhood of a, then f may be holomorphically extended to all of U.*

In addition, the following result generalises the Identity Theorem in the plane to arbitrary Riemann surfaces. The assumption of connectivity is necessary; for the proof we refer to [For12, Theorem 1.11].

Theorem 4.2 (Identity Theorem). *Let $f, g \in \mathcal{O}(X)$ and assume that there exists a subset $A \subseteq X$ with an accumulation point such that $f|_A = g|_A$. Then $f = g$ on all of X.*

The meromorphic functions, almost holomorphic functions with discretely many blowup points, often give great insight into the structure of the Riemann surface.

Definition 4.2. A function f on a Riemann surface X is called *meromorphic* if there exists a closed discrete subset $A \subseteq X$ such that for all $a \in A$ we have $\lim_{x \to a} |f(x)| = \infty$ and $f \in \mathcal{O}(X \setminus A)$. Equivalently, a meromorphic function is a non-constant holomorphic map $X \to \mathbb{CP}^1$.

We write $\mathscr{M}(X)$ for the set of meromorphic functions on X. This space has the natural structure of a $\mathbb{C}$-algebra by pointwise operations and continuing holomorphically across removable singularities if necessary.
The following result forms the foundation for the rest of our discussion; any holomorphic map can locally be expressed in the simple form of eq. (4.1). As a result, holomorphic maps are particularly well-behaved. A direct proof can be found in [For12, Theorem 2.1].

Theorem 4.3. *Let X, Y be Riemann surfaces, $f : X \to Y$ a holomorphic map and $x \in X$. Then there exist coordinate neighborhoods (U, z) centered at x and (V, ζ) centered at $f(x) \in Y$ such that $f(U) \subseteq V$ and the coordinate representation of f is given by*

$$\widehat{f} = z^k. \tag{4.1}$$

Note: the integer k in eq. (4.1) is called the *multiplicity* of f at x, written $v(f, x)$:

$$v(f, x) := k \tag{4.2}$$

with k as in Theorem 4.3.

We state some immediate consequences and then discuss how holomorphic maps can be understood as covering maps.

Corollary 4.1. *Suppose $f : X \to Y$ be a holomorphic map between Riemann surfaces. Then f is open.*

Corollary 4.2. *Suppose $f : X \to Y$ is an injective holomorphic mapping of Riemann surfaces. Then f maps biholomorphically onto its image $f(X)$.*

Proposition 4.1. *Suppose X is a compact Riemann surface, Y a Riemann surface and $f : X \to Y$ a non-constant holomorphic mapping. Then f is surjective and Y is compact.*

Proof. The image is open and compact, hence closed. The claim follows from the assumptions that Y is connected and $f(X)$ non-empty. □

Corollary 4.3 (Liouville)**.** *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic bounded function. Then f is constant.*

Proof. The map f is bounded in a neighborhood of $\infty \in \mathbb{CP}^1$ and hence can be extended to a holomorphic function $f : \mathbb{CP}^1 \to \mathbb{C}$. The result follows from Proposition 4.1. □

4.2 Branching points and covering maps

Following from our description above, holomorphic mappings turn out to be local homeomorphisms at most points of the domain and behave as covering maps; this only fails on a closed discrete subset of the domain, called the branching points of the holomorphic map. To make the description more precise, we turn to the theory of covering spaces.

Definition 4.3. A holomorphic map $f : X \to Y$ between Riemann surfaces X and Y is called *unbranched* at $p \in X$ if f is locally injective at p, i.e. f is a local homeomorphism (cf. Corollary 4.2). On the other hand, the point $q \in X$ is called a *branching point* if f is not unbranched at q. The map is called unbranched if there exist no branching points, and called branched otherwise.

Example 4.2. Suppose $k \in \mathbb{N}_{>1}$ and consider the monic polynomial $f = z^k$ as a meromorphic function on $\mathbb{CP}^1$. The branching points of f are precisely $0, \infty \in \mathbb{CP}^1$ where the polynomial equation $z^k = a \in \mathbb{C}$ fails to have k solutions. One can easily see that the multiplicity of f at these points is precisely k, such that with multiplicities counted any element has k solutions [*].

Recall that a continuous map between topological spaces $f : X \to Y$ is called proper if the preimages of compact sets are compact. The following result generalises the above example. Its proof relies on elementary properties of covering maps. See [For12, Sections 4.21-4.24] for a discussion.

Theorem 4.4. *Let X and Y be Riemann surfaces and $f : X \to Y$ a proper non-constant holomorphic map. There exist a, possible empty, closed discrete set of branching points $A \subset X$ and an integer n, called the number of sheets, such that (i) the restriction $f' : X \setminus f^{-1}(f(A)) \to Y \setminus f(A)$ is a holomorphic n-sheeted covering map (ii) for any element in the image $y \in Y$, the fiber $f^{-1}(y)$ consists of, counting multiplicities, n points. Elements of the subset $f(A)$ are called* critical points *for f.*

Corollary 4.4. *Suppose X is a Riemann surface and $f : X \to \mathbb{CP}^1$ is a meromorphic function. Then, counting multiplicities, the number of poles coincides with the number of zeros.*

4.3 Smooth structures on a Riemann surface

Any Riemann surface also carries a smooth structure. In some instances this smooth structure is easier to work with and gives more direct perspectives.

Suppose X is a Riemann surface and $\mathcal{U}$ a holomorphic atlas for X. Let (U, z) be a chart and identify $z(U) \subseteq \mathbb{C}$ with an open subset of $\mathbb{R}^2$ via the

[*] For the complex surfaces associated to level sets, one can show that the implicit function generalises with the same assumptions as the real case. For a direct proof see for example [Dub09, Lemma 2.1.5]

canonical homeomorphism $x + iy \mapsto (x, y)$. A function f on a Riemann surface is said to be smooth if it is smooth in a coordinate representation at any point, and we write $\mathcal{E}$ for the *sheaf of smooth functions* on a Riemann surface X. Moreover, it is well known that the holomorphic functions f in $\mathbb{C}$ are precisely those for which $\partial_{\bar{z}} f = 0$, where we define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{4.3}$$

In other words, there is an equality $\mathcal{O}(X) = \ker \mathcal{E}(X) \xrightarrow{\partial_{\bar{z}}} \mathcal{E}(X)$. Also, the pair $(z, \bar{z})$ is sometimes called complex coordinates.

One key difference between holomorphic and smooth functions is the existence of the partition of unity. The failure of existence of a 'holomorphic partition of unity' suggests that the extension of holomorphic functions is a more subtle question. This plays an important role when we discuss meromorphic functions on and sheaf cohomology. The following result makes this more concrete: the first cohomology group of $\mathcal{E}$ vanishes.

Theorem 4.5. *For any Riemann surface X: $H^1(X, \mathcal{E}) = 0$.*

Sketch: for any open cover $\mathcal{U} = (U_i)_{i \in I}$ and 1-cocycle $f = (f_{ij})_{i,j \in I} \in Z^1(\mathcal{U}, \mathcal{E})$ define the cochain $g_i = \sum_j \psi_j f_{ij}$ on U_{ij} that then satisfies $\delta g = f$, where $(\psi_i)_{i \in I}$ is a partition of unity subordinate to $\mathcal{U}$. The claim then follows. □

The tangent and cotangent space are easily expressed in terms of a smooth structure. Let X be a Riemann surface, $p \in X$ and denote by $T_p X$ (resp. $T_p^* X$) the ordinary tangent space (resp. cotangent space) when X is considered as a smooth manifold. Given any smooth function $f \in \mathcal{E}(X)$, we may define the differential df in the usual way. The following proposition shows that the $T_p^* X$ can also be understood in terms of the complex coordinates.

Proposition 4.2. *Suppose X is a Riemann surface, $p \in X$ and $(U, z = x + iy)$ is a coordinate neighborhood centered containing p. Then both $\{d_p x, d_p y\}$ and $\{d_p z, d_p \bar{z}\}$ form a basis of $T_p^* X$. Moreover, for any $f \in \mathcal{E}(X)$ we have the following equality:*

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\ &= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}. \end{aligned} \tag{4.4}$$

Suppose (U, z) and (V, z') are coordinate neighborhoods centered at $p \in X$, then we have:

$$\frac{\partial z}{\partial z'} = c, \qquad \frac{\partial \overline{z}}{\partial \overline{z'}} = \overline{c},$$
$$\frac{\partial z}{\partial \overline{z'}} = 0, \qquad \frac{\partial \overline{z}}{\partial z'} = 0,$$

where $c \in \mathbb{C}^*$. Thus, the linear spaces $T_p^{1,0}X := \mathbb{C}d_p z \subset T_p^* X$ and $T_p^{0,1}X := \mathbb{C}d_p\overline{z}$ remain invariant under a change of coordinates.

Let $\mathcal{E}^{(1)}$ denote sheaf of smooth 1-forms on a Riemann surface X. We define the subsheaf $\mathcal{E}^{1,0}$ (resp. $\mathcal{E}^{0,1}$) as the subset of smooth 1-forms such that on any open neighborhood $U \subseteq X$, $\mathcal{E}^{1,0}(U)$ (resp. $\mathcal{E}^{0,1}(U)$) consists of $\omega \in \mathcal{E}^{(1)}(U)$ with $\omega = f dz$ in any local coordinates (resp. $\omega = f d\overline{z}$), with $f \in \mathcal{E}(U)$. $\mathcal{E}^{1,0}$ and $\mathcal{E}^{0,1}$ have the standard restriction maps, and there exists a natural sheaf isomorphism $\mathcal{E}^{(1)} \cong \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$. Moreover, let Ω be the sheaf of holomorphic 1-forms ω of the form $\omega = f dz$ in local coordinates on an open set U, where $f \in \mathcal{O}(U)$.

Definition 4.4. Suppose U is an open subset of a Riemann surface, $p \in U$ and $f \in \mathcal{E}(U)$. The *holomorphic differential d'* (resp. *antiholomorphic differential d''*) is the homomorphism $\mathcal{E}(U) \to \mathcal{E}^{(1,0)}$ (resp. $\mathcal{E}(U) \to \mathcal{E}^{(0,1)}(U)$) given by $f \mapsto \partial_z f dz$ (resp. $f \mapsto \partial_{\overline{z}} f d\overline{z}$).

Note that $d = d' + d''$ with the canonical isomorphism $\mathcal{E}^{(1)} \cong \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$.

Denote by $\mathcal{E}^{(2)}$ the sheaf of smooth 2-forms on a Riemann surface X and let $d : \mathcal{E}^{(1)} \to \mathcal{E}^{(2)}$ be the standard differential that sends smooth k-forms to smooth $k+1$-forms. It is easy to check that for any $p \in X$ we have $d_p z \wedge d_p \overline{z} = -2i d_p x \wedge d_p y$. One can similarly define the (anti)holomorphic operators d', d'' on smooth 1-forms in terms of local coordinates.

We may interpret holomorphic 1-forms as the kernel of a differential operator.

Lemma 4.1. *Suppose U is an open subset of a Riemann surface and $\omega \in \mathcal{E}^{1,0}(U)$. Then $\omega \in \Omega(U)$ if and only if $d''\omega = d\omega = 0$.*

Suppose (U, z) is a coordinate neighborhood on a Riemann surface centered at $p \in U$ and $\omega \in \Omega(U \setminus p)$ a holomorphic 1-form. We may assume that $z(U) \subset \mathbb{C}$ is the open disk of radius 1. In local coordinates we can write $\omega = f dz$, where f can be identified with a holomorphic function on $z(U \setminus \{0\})$ which has a Laurent expansion $f = \sum_{n=\nu}^{\infty} c_n z^n$, $\nu \in \mathbb{Z}$. The *residue* of ω at p is defined

$$\mathrm{Res}_p(\omega) := c_{-1}. \tag{4.5}$$

For the proof of the following lemma see [For12, Section 9.9].

Lemma 4.2. *The residue of a holomorphic* 1*-form is independent on the choice of coordinates.*

Definition 4.5. A 1-form ω on a Riemann surface X is called *meromorphic* if there exists a closed discrete subset A such that $\omega \in \Omega(X \setminus A)$ and ω has a pole at every $a \in A$. The vectorspace sheaf of meromorphic 1-forms is written $\mathscr{M}^{(1)}$. Elements ω of $\mathscr{M}^{(1)}(X)$ are also called *Abelian differentials*, that are (i) of the first kind if ω is holomorphic everywhere, i.e. $A = \varnothing$ (ii) of the second kind if the residue of ω is zero at every pole (iii) of the third kind otherwise.

Note: The theory of complex integration can be constructed analogously to that on smooth manifolds and we omit a discussion here. Generalising integration of complex functions on the plane, holomorphic and meromorphic functions are well-behaved. We give a short discussion at the end of Section 4.4.

4.4 Analytic continuation and more sheaves

We have seen above that the problem of extending holomorphic functions presents many subtleties. Analytic continuation studies this problem by investigating such extensions as multi-valued functions. We later need the notion of analytic continuation along a curve.

Given a Riemann surface X, suppose $p, q \in X$ and recall that the stalk of $\mathcal{O}$ at p or q can be identified with the ring of series of series expansions: $\mathcal{O}_p \cong \mathbb{C}\{z - p\}$. Let $\phi \in \mathcal{O}_p$ be a function germ and $\gamma : [0,1] \to X$ a piecewise smooth curve, $\gamma(0) = p$, $\gamma(1) = q \in X$. $\psi \in \mathcal{O}_q$ is said to be the result of *analytic continuation* along γ if there exists a one parameter group of function germs $\phi_t \in \mathcal{O}_{\gamma(t)}$, $0 \le t \le 1$, such that:

(i) $\phi_0 = \phi$ and $\phi_1 = \psi$;

(ii) for each $t \in [0,1]$ there exists an open interval $T \ni t$, an open neighborhood U of $\gamma(T)$ and a holomorphic function $f \in \mathcal{O}(U)$ such that for all $t \in T$

$$f_{\gamma(t)} = \phi_t, \tag{4.6}$$

where f_p denotes the function germ of f at $p \in X$.

We will see an example of analytic continuation in Example 4.3.

A classical 'continuation' is that of closed differential forms. For any closed 1-form ω the function $p \mapsto \int^p \omega$ is in general multi valued. We have the following more precise statement.

Proposition 4.3. *Suppose X is a Riemann surface and suppose ω is a smooth closed 1-form on X. There exist a connected topological space $\widehat{X}$, a covering map $p : \widehat{X} \to X$ and a smooth function $f \in \mathcal{E}(\widehat{X})$ such that $df = p^*\omega$.*

If the space M is simply connected then any closed form has a primitive. The following construction gives a less restrictive condition.

Definition 4.6. Suppose ω is a a closed differential form on a Riemann surface X. The integrals $\int_a \omega$, where a runs through the fundamental group of X, are called the *periods* of ω and denoted p_σ.

We have the following useful condition.

Proposition 4.4. *A closed smooth 1-form ω on a Riemann surface has a primitive if and only if all periods p_σ vanish.*

Proof. It is well-known that any exact 1-form has vanishing periods. Conversely, suppose ω is exact and let f' be a primitive on the universal cover $\pi : \tilde{X} \to X$ of $\pi^*\omega$. As all periods vanish, f' is constant on the fibers of π and hence determines a smooth primitive f on X which is the desired function. This completes the proof. □

Note: These statements are also clearly true on arbitrary smooth manifolds.

Let $\mathcal{O}$ be the sheaf of holomorphic functions on a Riemann surface X. We have seen above that $H^1(X, \mathcal{E}) = 0$ and there exists an injective sheaf homomorphism $\mathcal{O} \hookrightarrow \mathcal{E}$. We can say somewhat more. To this end we need the following important lemma. [For12, Theorem 13.2]

Lemma 4.3 (Dolbeoult's Lemma). *Suppose $g \in \mathcal{E}(X)$, where $X = D_R(0)$ is the open disk centered at 0 of radius $R \in (0, \infty]$. There exists a smooth function $f \in \mathcal{E}(X)$ such that*

$$\frac{\partial f}{\partial \bar{z}} = g.$$

It is now easy to prove the following.

Proposition 4.5. *Suppose X is a Riemann surface. There exist short exact sequences of sheaves*

$$\begin{aligned} 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \xrightarrow{d''} \mathcal{E}^{1,0} \longrightarrow 0, \\ 0 \longrightarrow \Omega \longrightarrow \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \longrightarrow 0. \end{aligned} \tag{4.7}$$

As a consequence, we have isomorphisms

$$\begin{aligned} H^1(X,\mathcal{O}) &\cong \mathcal{E}^{1,0}(X)/d''\mathcal{E}(X), \\ H^1(X,\Omega) &\cong \mathcal{E}^{(2)}(X)/d\mathcal{E}^{1,0}(x). \end{aligned} \tag{4.8}$$

The dimension $g := \dim H^1(X,\mathcal{O})$ is called the *genus* of a Riemann surface X and plays an important role in the rest of our discussion. For compact Riemann surfaces one can proof with some analytical machinery that $g < \infty$; a discussion can be found in [For12, Chapter 14]. We now describe its relation to the familiar genus of compact topological manifolds, the latter being understood as the number of 'holes' of a topological space.

Let X be a compact Riemann surface and let $H_1(X,\mathbb{Z})$ be the first smooth *homology group* of X. Recall that

$$\overline{a} \sim \overline{b} \in H_1(X,\mathbb{Z}) \Longleftrightarrow \int_a \omega = \int_b \omega \quad \forall \omega \in Z^1(X,\mathcal{E}), \quad a,b \in Z_1(X,\mathbb{Z}), \tag{4.9}$$

i.e. de Rham's theorem applied to the first smooth homology group of X, where $Z^1(X,\mathcal{E})$ denotes all closed smooth 1-forms on X. (Note: we identify the de Rham cohmology and Céch cohomology groups.) Indeed, de Rham's theorem gives an equality $\dim H_1(X,\mathbb{Z}) = \dim H^1(X,\mathcal{E})$. Consider the following short exact sequence of sheaves:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow{d''} \Omega \longrightarrow 0. \tag{4.10}$$

The resulting long exact sequence is given by

$$\begin{aligned} 0 \longrightarrow H^0(X,\mathbb{C}) &\longrightarrow H^0(X,\mathcal{O}) \longrightarrow H^0(X,\Omega) \\ \longrightarrow H^1(X,\mathbb{C}) &\longrightarrow H^1(X,\mathcal{O}) \longrightarrow H^1(X,\Omega) \\ \longrightarrow H^2(X,\mathbb{C}) &\longrightarrow H^2(X,\mathcal{O}). \end{aligned} \tag{4.11}$$

It turns out that $H^2(X,\mathcal{O}) = 0$ and $\dim H^0(X,\Omega) = \dim H^1(X,\mathcal{O})$ by a remarkable result; in other words, there exist precisely g linearly independent holomorphic differentials on a compact Riemann surface of genus

g. These are consequences of the so-called *Serre duality theorem*. We do not have the means and time to discuss this; for a neat introduction see [AVV13, Section 5.1].

These dimensions imply $\dim H^1(X,\mathbb{C}) = 2g$. With some work one can show that there exists an isomorphism $H^1(X,\mathbb{C}) \cong H^1(X,\mathcal{E})$; see for example [For12, Chapter 19, Theorem 19.14]. We have thus recovered the familiar equality $\dim H_1(X,\mathbb{Z}) = 2g$.

$H_1(X,\mathbb{Z})$ can be given the structure of symplectic space, with a two form given by the *intersection number* of two curves, denoted by $a \cdot b$, $a, b \in H_1(X,\mathbb{Z})$. For a more detailed discussion see for example [FM11, Section 6.1]. This intersection number does not depend on the choice of representatives. Most importantly, we will use later that there exists a basis $a_1, \dots, a_g, b_1, \dots b_g \in H_1(X,\mathbb{Z})$ such that $a_i \cdot a_j = 0$, $a_i \cdot b_j = \delta_{ij}$ and $b_i \cdot b_j = 0$ called a *canonical basis of cycles*. Given any basis $\omega'_1, \dots \omega'_g \in H^1(X,\Omega)$ there exist matrices

$$2\pi i A_{ij} := \int_{a_i} \omega'_j, \quad B'_{ij} := \int_{b_i} \omega'_j, \tag{4.12}$$

where the factor $2\pi i$ is placed for convenience. One can show that the matrix A_{ij} is invertible and after changing to a basis $\omega_1, \dots, \omega_g$ where $A_{ij} = \delta_{ij}$ we define the *period matrix* B by

$$B_{ij} = \int_{b_i} \omega_j. \tag{4.13}$$

One can show that B_{ij} is a symmetric matrix. For a more detailed discussion see [AVV13, Section 5.2]. The subset Λ in $\mathbb{C}^g$ of γ running through $\int_\gamma \overrightarrow{\omega}$ is called the *period-lattice*. From our description above, it is clear that Λ is actually a lattice. One can similarly show as above that a different basis leads to an isomorphic lattice.

Definition 4.7. Suppose X is a compact Riemann surface of genus g and let $\Lambda \subset \mathbb{C}^g$ be the period lattice of X. The quotient space $\mathbb{C}^g/\Lambda$ is called the *Jacobian* of X.

We discuss the Jacobian again at the end of this chapter.

Integration of meromorphic 1-forms

Suppose X is a Riemann surface and $\omega \in \mathscr{M}^{(1)}(X)$ a meromorphic 1-form. Denote its set of poles by A. It is well known that for any $\tilde{\omega} \in \mathcal{E}^{(1)}(X)$ with compact support we have

$$\iint_X d\tilde{\omega} = 0. \tag{4.14}$$

In view of the second isomorphism of equation (4.8), let $\xi \in \mathcal{E}^{(2)}(X \setminus A)$ be a representative of the holomorphic 1-form $\omega \in \Omega(X \setminus A)$ and define

$$\operatorname{Res}(\overline{\xi}) = \frac{1}{2\pi i} \iint_X \xi. \tag{4.15}$$

We show that (4.15) can be computed in terms of the residues of ω. A *Mittag-Leffler distribution* for an open cover $\mathcal{U} = (U_i)_{i\in I}$ is a 0-cohain $\mu = (\omega_i)_{i\in I} \in C^0(\mathcal{U}, \mathscr{M}^{(1)})$ such that the difference $\omega_i - \omega_j$ is holomorphic. In other words, $\delta\mu \in Z^1(X, \Omega)$. The residue of μ at $a\imath U_i$ is defined

$$\operatorname{Res}_a(\mu) := \operatorname{Res}_a(\omega_i). \tag{4.16}$$

Note that this is well-defined as the difference is holomorphic. On a compact Riemann surface, we define $\operatorname{Res}(\mu) = \sum_{x\in X} \operatorname{Res}_x(\mu)$. The following generalises the residue theorem from complex analysis.

Theorem 4.6. *Suppose X is a compact Riemann surface and $\mu \in C^0(X, \mathscr{M}^{(1)})$ is a Mittag-Leffler distribution. Let $\overline{\delta\mu}$ denote the cohomology class of $\delta\mu$. The residues of equations* (4.15) *and* (4.16) *agree:*

$$\operatorname{Res}(\overline{\delta\mu}) = \operatorname{Res}(\mu). \tag{4.17}$$

A proof can be found in [For12, Theorem 17.3].

4.5 Riemann surface of an algebraic function

We state how the solution sets of algebraic equations over $\mathbb{C}$ can be interpreted as a Riemann surface.

The following result describes how *singular* Riemann surfaces may be completed. A direct proof can be found in [For12, Theorem 8.4].

Lemma 4.4. *Let X be a Riemann surface, $A \subseteq X$ a closed discrete subset and let $X' = X \setminus A$. Suppose Y' is another Riemann surface and $\pi' : Y' \to X'$ a proper unbranched holomorphic covering. Then there exist a Riemann surface Y, a proper branched holomorphic covering $\pi : Y \to X$ and a fiber-preserving biholomorphic map $\phi : Y \setminus \pi^{-1}(A) \to Y'$ such that $\pi' = \pi \circ \phi^{-1}$.*

The following is the main result of this section. One can show that for regular values of a polynomial, i.e. its coefficients do not blow up and the polynomial factors linearly, there is a local Riemann surface associated to such an algebraic equation. The proof of the following glues together a surface on $|\mathcal{O}|$, where the critical points of the above lemma are the blowup loci of the meromorphic functions and the points where the discriminant vanishes; $|\mathcal{O}|$ denotes the discrete sheaf of $\mathcal{O}$ (see Appendix B). Details are given in the reference [For12, Theorem 8.9].

Theorem 4.7. *Suppose X is a Riemann surface and*

$$P(T) = T^n + c_1 T^{n-1} + \cdots + c_n \in \mathscr{M}(X)[T] \tag{4.18}$$

is an irreducible polynomial of degree n. Then there exist a Riemann-surface, a branched holomorphic n-sheeted covering $\pi : Y \to X$ and a meromorphic function $F \in \mathscr{M}(Y)$ such that $(\pi^* P)(F) = 0$. The triple (Y, π, F) is uniquely characterised by the following property. Given any other triple (Z, τ, G) such that $(\tau^* P)(G) = 0$, there exists a unique fiber-preserving map $\sigma : Z \to Y$ such that $G = \sigma^* F$.

Note: the triple (Y, π, F) is called the *algebraic function* associated to the polynomial $P(T)$.

Example 4.3. [For12, Example 8.10] Suppose $a_1, \ldots, a_k \in \mathbb{C} \setminus 0$ are distinct and consider the Laurent polynomial

$$P(z) = \lambda^{-m}(z - a_1) \cdots (z - a_k), \quad m \in \mathbb{N}_{<k}. \tag{4.19}$$

We interpret $P(z)$ as a meromorphic on the Riemann sphere and describe the Riemann surface constructed from the algebraic equation

$$T^2 - P(z) \in \mathscr{M}(\mathbb{CP}^1)[T]; \tag{4.20}$$

its algebraic function is commonly denoted $\sqrt{P(z)}$. We describe the behaviour near the critical points $A = \{0, \infty, a_1, \ldots, a_k\}$.
(i) choose $r > 0$ such that the open disk $U_j = D_r(a_j)$ contains no other singular points, $r = 1, \ldots, k$. As U_j is simply connected and the function $g(z) = \lambda^{-m} \Pi_{i \neq j}(z - a_i)$ does not vanish, there exists a function germ h such that $h^2 = g$. We can write

$$f = (z - a_j) h^2(z). \tag{4.21}$$

Let $0 < \rho < r$, $\theta \in \mathbb{R}$ and let $\zeta = a_j + \rho e^{i\theta}$. As the function f is regular on U_j, there exists a function germ $\phi_z \zeta \in \mathcal{O}_\zeta$ such that

$$\phi(z) = \sqrt{\rho} e^{i\theta/2} h. \tag{4.22}$$

Hence, analytically continuation along the curve $a_j + \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$, results in the negative of the original function germ. The map $\pi : V_j \to U_j$ is a *connected* two-sheeted covering and a_j has one original; otherwise analytic continuation along the curve $a_j + \rho e^{i\theta}$ would produce the same germ.
(ii) We describe a neighborhood of π near ∞. Let $U_\infty \ni \infty$ be a sufficiently small simply connected neighborhood containing no other critical points. Then

(a) there exists a function h such that $f = h^2$ if $k - m$ is even;

(b) there exists a function h such that $f = zh^2$ if $k - m$ is odd.

By our description above, π is unbranched over $z = \infty$ in the first case as π is locally a disconnected two-sheeted cover; in the second case π is branched over $z = \infty$ with multiplicity 2.
(iii) The same argument for $z = \infty$ works for $z = 0$, with two different cases: m is odd or m is even.

The Riemann surface associated to an algebraic equation can also be compactified. Although we do not have the space and algebraic-geometric means to discuss the details of *normalisation*, the following proposition summarises the results we need for later.

Proposition 4.6. *Suppose $P(T) \in \mathscr{M}(\mathbb{C})[T]$ is an irreducible polynomial and let X' be the Riemann surface associated to it. There exists a projective embedding of X' into $\mathbb{CP}^2$. Let $\overline{X'}$ denote its closure. Then the normalisation X of $\overline{X'}$ is a compact Riemann surface.*

Note that normalisation and nonsingularity are equivalent in complex dimension one.

4.6 Divisors

Divisors generalise the order of a meromorphic function at a point. Many invariants of a Riemann surface are neatly described by its set of meromorphic functions and divisors describe the meromorphic functions.

Definition 4.8. A *divisor D* on a Riemann surface X is a mapping

$$D : X \to \mathbb{Z}$$

such that for any compact subset $K \subseteq X$, $D(x) = 0$ for all but finitely many $x \in K$. It is typical to write $D_x = D(x)$.

Suppose U is an open subset of a Riemann surface and $f \in \mathscr{M}(U) \setminus \{0\}$. The following generalises the order of meromorphic functions on the plane.

$$\operatorname{ord}_x(f) := \begin{cases} 0 & \text{if } f \text{ is holomorphic in a neighborhood of } x \\ k & \text{if zero of order } k \text{ at } x \\ \text{-k} & \text{if pole of order } k \text{ at } x \\ \infty & \text{if } f \text{ is zero in a neighborhood of } x \end{cases} \tag{4.23}$$

Define the map $(f)(x) := \operatorname{ord}_x(f)$. By the Identity Theorem (f) is a divisor. A divisor D is said to be *principal* if there exists a meromorphic function $f \in \mathscr{M}(X)$ with $(f) = D$. Moreover, two divisors D, D' are said to be equivalent if their difference is principal. Clearly, any two meromorphic 1-forms are equivalent.

Given a divisor D on a compact Riemann surface X there exist only finitely many points $x \in X$ where $D_x \neq 0$. The following notion is therefore well-defined:

$$\deg D := \sum_{x \in X} D_x \tag{4.24}$$

Any principal divisor therefore has degree 0. The converse need not be true; this question is investigated later in section 4.7. We now describe in some detail how divisors can describe the meromorphic structure of a Riemann surface. Define the *divisor sheaf* $\mathcal{O}_D$ of D as follows: for any open subset U of a Riemann surface let

$$\mathcal{O}_D(U) := \{f \in \mathscr{M}(U) : (f)_x \geq -D_x \quad \forall x \in U\}. \tag{4.25}$$

With the standard restriction maps $\mathcal{O}_D$ is a sheaf. The minus-sign is placed for convenience and makes the resulting expressions easier to read. Note that for the zero divisor $D = 0$ we have $\mathcal{O}_D = \mathcal{O}$ as an equality of sheaves. Given two equivalent divisors D, D' such that $D - D' = (f)$, the map $\mathcal{O}_D(U) \ni g \mapsto fg \in \mathcal{O}_{D'}(U)$ is an isomorphism for any subset U and hence determines a sheaf isomorphism $\mathcal{O}_D \cong \mathcal{O}_{D'}$.

Example 4.4. Let U be some open subset of a Riemann surface and $a \in U$. We define the *skyscraper divisor* P_a at a to be following divisor:

$$P_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases} .$$

Then $\mathcal{O}_{P_a}$ is the sheaf of meromorphic functions with a simple pole at a.

The Riemann-Roch theorem gives an extremely useful relation between the cohomology groups of $\mathcal{O}_D$ and can often be rewritten to find either the degree of a divisor or the genus of a Riemann surface (cf. Theorem 4.9). One proof relies induction, starting with $D = 0$ and adding skyscraper divisors in each induction step. Details as usual are found in [For12, Theorem 16.9].

Theorem 4.8 (Riemann-Roch). *Let D be a divisor on a compact Riemann surface X and . Then $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite dimensional vector spaces and*

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D. \tag{4.26}$$

Closely related is the *Riemann-Hurwitz formula*, which relates the genera† of two compact Riemann surfaces if there exists a holomorphic map between them. A proof which uses Serre's duality theorem can be found in [For12, Theorem 17.14].

Recall that $v(f, x)$ denotes the multiplicity of f at x.

Theorem 4.9 (Riemann-Hurwitz). *Let X and Y be compact Riemann surfaces of genus g and g' respectively and $f : X \to Y$ a branched holomorphic n-sheeted covering map. There is an equality*

$$g = \frac{1}{2} \sum_{x \in X} (v(f, x) - 1) + n(g' - 1) + 1. \tag{4.27}$$

We will use this formula to calculate the genera of some algebraic curves later in Chapter 6. The following elaborates on Example 4.3.

Example 4.5. Let X be the Riemann surface determined from the algebraic equation (4.20) and $\pi : X \to \mathbb{CP}^1$ the associated 2-sheeted cover. We have proven in Example 4.3 that π is branched (i) over a_j with multiplicity 2 (ii) over $\lambda = \infty$ and $\lambda = 0$ precisely when $k - m$ is odd respectively m is odd, with multiplicity 2. The Riemann-Hurwitz formula gives

$$g = \frac{1}{2} \sum_{x \in X} (v(f, x) - 1) - 1, \tag{4.28}$$

as π is a two-sheeted cover, where g is the genus of X. By our discussion above we have

$$g = \begin{cases} \frac{k-2}{2} & \text{if } m \text{ is even and } k \text{ is even,} \\ \frac{k-1}{2} & \text{if } m \text{ is even and } k \text{ is odd,} \\ \frac{k-1}{2} & \text{if } m \text{ is odd and } k \text{ is odd,} \\ \frac{k}{2} & \text{if } m \text{ is odd and } k \text{ is even.} \end{cases} \tag{4.29}$$

†'genera' is the plural of genus.

4.7 Abel's theorem, line bundles and the Picard group

Any principal divisor has degree zero, but the converse may not be true. Abel's theorem gives a necessary and sufficient condition for the converse to hold. This discussion of principal divisors naturally leads to the introduction of the Jacobian of a compact Riemann surface.

Let $C_1(X, \mathbb{Z})$ denote the free Abelian group of piece-wise smooth curves and denote the boundary operator by ∂. It is well known that $\partial : C_1(X, \mathbb{Z}) \to C_0(X, \mathbb{Z})$ is surjective. We identify $C_0(X, \mathbb{Z})$ with the set of divisors $\mathrm{Div}(X)$ in the following natural manner: given any piece-wise smooth curve γ, define the divisor $D := \partial\gamma$ which takes the value 1 at $\gamma(1)$ and -1 at $\gamma(0)$.

Theorem 4.10. *Suppose D is a divisor on a compact Riemann surface X with* $\deg D = 0$*. Let γ be a curve such that $\partial\gamma = D$. Then D is principal if and only if for all closed* 1*-forms ω we have*

$$\int_\gamma \omega = 0. \tag{4.30}$$

The proof is rather technical, see [For12, Theorem 20.7].

We now discuss holomorphic line bundles and their relation to divisors. By a complex n-vector bundle we mean a vector bundle where the fibers consist of $\mathbb{C}^n$.

Definition 4.9. Let E be a complex n-vector bundle and $\mathcal{U} = (U_i, \Psi_i)_{i \in I}$ a collection of local trivialisations, i.e., an atlas for E. The atlas is called holomorphic and E a *holomorphic vector bundle* of dimension n if the transition functions are holomorphic:

$$g_{ij} = \Psi_i \circ \Psi_j^{-1} \in GL_n(\mathcal{O}(U_{ij})). \tag{4.31}$$

A *holomorphic line bundle* is a complex vector bundle of dimension 1.

Given a holomorphic vector bundle E it is easy to verify the cocycle relation $g_{ij}g_{jk} = g_{ik}$ on U_{ijk}. The converse also turns out t be true. To this end, let $GL_n(\mathcal{O})$ be the sheaf of invertible holomorphic matrices such that for any open set $U \subseteq X$, $GL_n(\mathcal{O}(U))$ consists of $n \times n$ invertible matrices with holomorphic functions on U as coefficients. For the proof of the lemma below, one can define an appropriate equivalence relation on the trivial bundle $X \times \mathbb{C}^n$; see [For12, Theorem 29.7].

Lemma 4.5. *For every 1-cocycle $g = (g_{ij})_{i,j\in I} \in Z^1(X, GL_n(\mathcal{O}))$ there exists a holomorphic n-vector bundle E such that the transition function of E is given by g.*

Definition 4.10 (Sheaf of holomorphic sections)**.** Suppose E is a holomorphic n-vector bundle on a Riemann surface X and let $p \in X$. A section $f : X \to E$ is called *holomorphic* if there exists a local trivialisation (U_i, Ψ_i) at p such that the map

$$\Psi_i \circ f : p \longmapsto (p, f_i(p))$$

is holomorphic, i.e. $f_i : U \to \mathbb{C}^n$ is a holomorphic map. Note that f_i satisfies the relationship

$$f_i = g_{ij} f_j, \tag{4.32}$$

where g_{ij} is the transition function corresponding to E. We denote the *sheaf of holomorphic sections* by $\mathcal{O}_E$.

One can similarly construct the sheaf of meromorphic sections of a line bundle and we leave this to the reader.

We now describe the relation between holomorphic line bundles and divisors. To this end we cite a result about the existence of meromorphic functions on a compact Riemann surface. See [For12, Corollary 14.13].

Proposition 4.7. *Let X be a compact Riemann surface and $p_1, \dots, p_n \in X$ distinct. For any n-tuple $c_1, \dots, c_n \in \mathbb{C}$, there exists a meromorphic function f such that $f(a_i) = c_i$, $i = 1, \dots, n$.*

Let D be a divisor on a Riemann surface X and $\mathcal{U} = (U_i)_{i\in I}$ an open cover of coordinate neighborhoods of X for which there exist meromorphic functions $\psi = (\psi_i)_{i\in I}$ with $\psi_i \in (M)(U_i)$, such that $(\psi_i) = D$ on U_i. Consider the 2-cochain

$$g_{ij} = \psi_i / \psi_j \in \mathcal{O}(U_{ij}). \tag{4.33}$$

Indeed, at any point $p \in X$ with $D_p \neq 0$ the singularity at g_{ij} at p may be removed. Clearly $(g_{ij}) \in Z^1(\mathcal{U}, \mathcal{O})$ and hence determines a holomorphic line bundle L. We claim that $\mathcal{O}_D \cong \mathcal{O}_L$ as sheaves. Indeed, given a function $f \in \mathcal{O}_D(U)$, $U \subseteq X$ open, there exist holomorphic functions f_i on $U_i \cap U$ such that $f = f_i/\psi_i$. Conversely, given any local section f_i one may show with eq. (4.32) that f_i/ψ_i is independent of the choice of index and hence determines an element $f \in \mathcal{O}(U)$.

On the other hand, suppose we are given a line bundle L on a compact Riemann surface. We want to construct a divisor D with $\mathcal{O}_D \cong \mathcal{O}_L$ We first the following result about the existence of meromorphic sections.

Theorem 4.11. *Suppose E is a holomorphic vector bundle on a Riemann surface X and $Y \subset X$ a relatively compact open subset of X. Given any $a \in Y$ there exists a meromorphic section which has a pole at a.*

Proof. See [For12, Theorem 29.16]. □

Now, there exists a global meromorphic section ψ of L that is nontrivial. For a local chart on L define the divisor $D = (\psi_i)$, where ψ_i is the coordinate representation of ψ. It is easy to see that D is independent of the choice of coordinates. We now show $\mathcal{O}_L \cong \mathcal{O}_D$. For every holomorphic section $f \in \mathcal{O}_E(U)$ we have $(f/\psi) \geq -D$ on $U_i \cap U$. Conversely, given any holomorphic function with$(f) \geq -D$ the function $f\psi$ is a holomorphic section of E. This proves the claim.
We summarise our discussion with the following Theorem.

Theorem 4.12. *Suppose X is a compact Riemann surface. For every divisor D there exists a holomorphic line bundle L such that $\mathcal{O}_D \cong \mathcal{O}_L$. The converse is also true.*

Note : in view of the correspondence above, we may define the degree of a holomorphic line bundle as the degree of the associated divisor.

Definition 4.11. Suppose X is a compact Riemann surface of genus g. The *Picard group* is the quotient group $\mathrm{Pic}(X) := \mathrm{Div}(X)/\mathrm{Div}_P(X)$, where $\mathrm{Div}_P(X)$ is the set of principal divisors on a Riemann surface. We also define $\mathrm{Pic}_0(X) = \mathrm{Div}_0(X)/\mathrm{Div}_P(X)$, where $\mathrm{Div}_0(X)$ consists of the divisors of degree 0. Equivalently, $\mathrm{Pic}(X)$ can be characterised as the sets of isomorphism classes of holomorphic line bundles.

Consider the map $\Psi : \mathrm{Pic}_0(X) \to \mathrm{Jac}(X)$ given as follows. For a divisor D let γ be a smooth curve such that $\partial\gamma = D$. Consider the map

$$D \longmapsto \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \right). \tag{4.34}$$

Abel's theorem has already shown that the kernel of this map is precisely given by $\mathrm{Pic}_P(X)$. It turns out the map Ψ is also surjective, known as *Jacobi's inversion problem.*

Proposition 4.8. *For any compact Riemann surface X the map Ψ constructed above is a group isomorphism $Pic_0(X) \cong \mathrm{Jac}(X)$.*

Chapter 5

Integration of Lax equations

A Lax equation is a commutator differential equation

$$\dot{L} = [L, M],$$

for some appropriate objects L and M. The pair (L, M) is called a *Lax pair*. Besides linear differential equations, these often turn out to be completely integrable dynamical systems and appear frequently throughout mathematical physics, such as in the KdV-equation and in motion of a classical spinning top. The present chapter is devoted to studying Lax equations in the dual Lie algebra $\mathfrak{g}^*$, which we use in the next chapter to study the quantum Brachistochrone problem.

We first discuss Lie algebras with an additional Lie bracket, an R-bracket, and see how Lax equations naturally appear as the integral curve of the Hamiltonian vector field of a Casimir. Finding a solution to such a Lax equation turns out to be equivalent to solving a matrix Riemann Hilbert problem. To this end, we introduce twisted loop algebras that are later used to attack this problem with algebraic geometric methods. Before doing so, we present a large class of integrable dynamical systems, among which is a Lax pair that generalises the equation of motion of a spinning top. We closely follow Chapters 2-6 and Chapter 8-10 of [RS94].

5.1 The R-bracket and loop algebras

The R-bracket is a simple scheme for constructing an additional Lie bracket.

Definition 5.1. Let $\mathfrak{g}$ be a Lie algebra and fix an endomorphism $R \in \mathrm{End}(\mathfrak{g})$.

The *R-bracket* $[\cdot,\cdot]_R$ is the bilinear map given by

$$[X,Y]_R := \frac{1}{2}\left([X,RY]+[RX,Y]\right). \tag{5.1}$$

It is easy to verify that $[\cdot,\cdot]_R$ is skew-symmetric. In general the bracket in eq. (5.1) may not satisfy the Jacobi identity. There exist several easy to check sufficient conditions for this to hold, known as the condition.

The simplest and most important examples occur when there exists a Lie algebra decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_\pm \subset \mathfrak{g}$ are nontrivial subalgebras of $\mathfrak{g}$. Let $P_\pm$ be the projection operators on the subspaces $\mathfrak{g}_\pm$ and define the endomorphism $R = P_+ - P_-$. Then it is easy to verify that $[\cdot,\cdot]_R$ satisfies the Jacobi identity, and we denote the double Lie algebra by

$$\mathfrak{g}_R = \mathfrak{g}_+ \ominus \mathfrak{g}_-. \tag{5.2}$$

This R-bracket will make another appearance in Section 5.3.

Given a double Lie algebra $\mathfrak{g}_R$ as above, we can equip the dual $\mathfrak{g}_R^*$ with two Poisson brackets. Let $I(\mathfrak{g}^*)$ be the ring of Casimirs in $\mathfrak{g}^*$. I.e. $I(\mathfrak{g}^*) \subset C^\infty(\mathfrak{g}^*)$ is the ideal consisting the smooth functions such that

$$\{\phi, f\} = 0, \quad \phi \in I(\mathfrak{g}^*), \quad f \in C^\infty(M), \tag{5.3}$$

with $\{\cdot,\cdot\}$ the canonical Poisson structure in $\mathfrak{g}^*$.

The following theorem describes the integral curves of Casimirs in $\mathfrak{g}^*$.

Theorem 5.1. *Let $\phi \in I(\mathfrak{g}^*)$. The integral curves of the Hamiltonian vector field X_ϕ in $\mathfrak{g}_R^*$ starting at $L \in \mathfrak{g}^*$ are given by*

$$\frac{dL}{dt} = \mathrm{ad}^*_{\mathfrak{g}}(M)(L), \quad M = \frac{1}{2}R(d\phi(L)). \tag{5.4}$$

Note: The pair (L, M) is called a *Lax pair* for the Hamiltonian ϕ. If $\mathfrak{g}$ admits an Ad-invariant non-degenerate bilinear form, such as the Killing form, equation (5.4) can be written as the Lax equation

$$\frac{dL}{dt} = [L, M]. \tag{5.5}$$

In the case of the projection decomposition $R = P_+ - P_-$, it is more convenient to write (5.4) as a set of two equations

$$\frac{dL}{dt} = [L, M_\pm], \quad M_\pm = \frac{1}{2}P_\pm d\phi(L). \tag{5.6}$$

Proof. The proof follows Lemma 3.3 and uses Lemma 2.2. We can write the R-bracket of ϕ and $\psi \in C^\infty(\mathfrak{g}^*)$ as

$$\begin{aligned}\{\phi, \psi\}_R(L) &= \frac{1}{2}\langle L, [Rd\phi(L), d\psi(L)] + [d\phi(L), Rd\psi(L)]\rangle \\ &= -\frac{1}{2}\langle \mathrm{ad}^*_{\mathfrak{g}}(Rd\phi(L))(L), d\psi(L)\rangle,\end{aligned} \tag{5.7}$$

in view of Lemma 2.2. The claim now follows as in Proposition 3.3. □

One can write down an explicit solution of the equation of motion (5.4) in the case that the R-bracket is given by the simple form (5.2). This amounts to a factorisation in the Lie group G. Let $G_\pm \subseteq$ be the unique subgroups corresponding to the subalgebras $\mathfrak{g}_\pm$ (cf. Theorem 2.3).

Theorem 5.2. *Let $\phi \in I(\mathfrak{g}^*)$ and write $X(t) = d\phi(L(t))$. Let $g_\pm(t)$ be smooth curves in $G_\pm$ that solve the Riemann problem*

$$\exp tX = g_+(t)^{-1} g_-(t), \tag{5.8}$$

with $g_\pm(0) = e$. The integral curve of eq. (5.4) *with $L(0) = L$ is explicitly given on $G_\pm$ by*

$$L(t) = \mathrm{Ad}^*_G(g_\pm(t))(L). \tag{5.9}$$

Proof. After differentiating eq. (5.9) at time t we get

$$\frac{dL}{dt}(t) = \mathrm{ad}^*_{\mathfrak{g}}(\partial_t g_+(t) g_+(t)^{-1})(L) = \mathrm{ad}^*_{\mathfrak{g}}(\partial_t g_-(t) g_-(t)^{-1})(L). \tag{5.10}$$

It suffices to show $\partial_t g_\pm(t) g_\pm(t)^{-1} = -M_\pm(t)$, where $M_\pm(t) = P_\pm X$. Rewriting (5.8) as $\partial_t g_+(t) \exp tX = g_-(t)$ and differentiating gives us

$$\frac{d}{dt} g_+(t) g_+(t) + \mathrm{ad}^*_{\mathfrak{g}}(g_-(t))(X(t)) = \frac{d}{dt} g_-(t) g_-(t)^{-1}. \tag{5.11}$$

The Ad^*_G-invariance of the Hamiltonian ϕ implies $X(t) = \mathrm{Ad}^*_G(g_\pm(t))(X(t))$ and the identities $\partial_t g_+ g_+^{-1} \in \mathfrak{g}_\pm$ with the uniqueness of integral curves prove the claim. □

5.1.1 $\mathbb{Z}$-grading and loop algebras

The classical spinning top systems admit a Lax representation in terms of an associated twisted loop algebra. We introduce the notion of a $\mathbb{Z}$-grading.

Definition 5.2. A $\mathbb{Z}$-*grading* of a Lie algebra $\mathfrak{g}$ is a $\mathbb{Z}$-indexed collection of subalgebras $(g_i)_{i\in\mathbb{Z}}$ such that

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i, \tag{5.12}$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

Any $\mathbb{Z}$-graded Lie algebra admits a natural decomposition into subalgebras

$$\mathfrak{g}_+ := \bigoplus_{i\geq 0} \mathfrak{g}_i, \quad g_- := \bigoplus_{i<0} \mathfrak{g}_i, \tag{5.13}$$

following the decomposition in eq. (5.2). Denote the corresponding Lie brackets by $[\cdot,\cdot]_\pm$. For any integers $n, m \in \mathbb{N}$ we can easily describe the dual of $\mathfrak{g}' = \oplus_{i=-m}^n \mathfrak{g}_i$ by the identification $(\mathfrak{g}')^* \cong \oplus_{i=-m}^n \mathfrak{g}_i^*$, where $\mathfrak{g}_i^*$ is the dual to $\mathfrak{g}_i$.

Formally, the dual $\mathfrak{g}^*$ of $\mathfrak{g}_+ \ominus \mathfrak{g}_-$ is the projective limit of the subspaces $\bigoplus_{i=0}^m \mathfrak{g}_i^*$ and $\bigoplus_{i=-n}^{-1} \mathfrak{g}_i^*$. For simplicity, we restrict this limit to always mean finite dimensional subspaces and any element $\xi \in \mathfrak{g}^*$ can be written as a finite sum of elements in $\mathfrak{g}_i^*$.

Let $\mathfrak{g} = \mathfrak{g}_+ \ominus \mathfrak{g}_-$ be a decomposition as in eq. (5.2) and fix an element $\xi = \xi_+ + \xi_- \in \mathfrak{g}^*$ with $\xi_\pm \in \mathfrak{g}_\pm$ such that $\xi_+ \in \oplus_{i=0}^n \mathfrak{g}_i^*$ for some $n \in \mathbb{N}$ (resp. $\xi_- \in \oplus_{i=-1}^{-n-1} \mathfrak{g}_i$ for some $m \in \mathbb{N}$). Let $P_\pm$ be the projection operator onto the subalgebras $\mathfrak{g}_\pm$. One can easily verify the following equation:

$$\mathrm{ad}_{\mathfrak{g}}^*(X)(\xi) = \mathrm{ad}_{\mathfrak{g}_+}^*(X_+)(\xi_+) - \mathrm{ad}_{\mathfrak{g}_-}^*(X_-)(\xi_-), \quad X_\pm = P_\pm X. \tag{5.14}$$

Lemma 5.1. *Fix $n \geq m \geq 0$ and $\xi \in g_{m-1}^*$. The subspace $f + \bigoplus_{i=-m}^{-n-1} \mathfrak{g}_i^*$ is a Poisson subspace.*

Proof. In view of Proposition 3.5 we need to show that the integral curve of any Hamiltonian vector field remains this subspace. According to Theorem 5.4 this amounts to showing that it is closed under the coadjoint action. The result now follows from equation (5.14). □

5.1.2 Twisted loop algebras

The additional algebraic structure of the twisted loop algebra later plays the role of a spectral parameter when tackling the associated factorisation problem to a Lax equation.

Let $\mathfrak{g}$ be a complex Lie algebra. The *loop algebra* $\mathfrak{L}(\mathfrak{g})$ consists of all Laurent polynomials with coefficients in $\mathfrak{g}$:

$$\mathfrak{L}(\mathfrak{g}) := \mathfrak{g}[\lambda, \lambda^{-1}] := \left\{ \sum_i x_i \lambda^i,\ x_i \in \mathfrak{g} \right\}, \quad \lambda \in \mathbb{C}. \tag{5.15}$$

For some fixed endomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ with $\sigma^m = \mathrm{id}_{\mathfrak{g}}$, we define the *twisted loop algebra* as

$$\mathfrak{L}(\mathfrak{g}, \sigma) := \{ p(\lambda) \in \mathfrak{L}(\mathfrak{g}) : \sigma p(\lambda) = p(\exp(2\pi i/m)\lambda) \}. \tag{5.16}$$

Clearly both $\mathfrak{L}(\mathfrak{g})$ and $\mathfrak{L}(\mathfrak{g}, \sigma)$ admit a $\mathbb{Z}$-grading with respect to powers of λ. Let P_i the projection operator on the i-subspace and denote the corresponding decomposition of the twisted loop algebra as $\mathfrak{L}_{\pm}(\mathfrak{g}, \sigma)$ (cf. (5.13)). Given a non-degenerate Ad_G-invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, the Lie algebra $\mathfrak{L}(\mathfrak{g}, \sigma)$ can also be equipped with a non-degenerate bilinear form as the next simple lemma shows.

Lemma 5.2. *Let $\langle \cdot | \cdot \rangle$ be a bilinear mapping on $\mathfrak{L}(\mathfrak{g}, \sigma)$ given by*

$$\langle X(\lambda) | Y(\lambda) \rangle := \mathrm{Res}_{\lambda=0} \lambda^{-1} \langle X(\lambda), Y(\lambda) \rangle d\lambda. \tag{5.17}$$

Then $\langle \cdot | \cdot \rangle$ is non-degenerate.

5.2 Hamiltonian reduction and spinning top systems

In many applications of interest the phase space of a dynamical systems can be obtained via a so-called Hamiltonian reduction. A Hamiltonian reduction consists of a Hamiltonian group action on a smooth manifold such that the resulting quotient is the phase space under study. Typically the larger space is more symmetric and easier to work with. We construct a class of integrable systems which describe Lax pairs for spinning top systems.

The difficult part often is to describe the symplectic leaves of the resulting phase space. The notion of a dual pair helps us with this.

Definition 5.3 (Dual pair)**.** Let M be symplectic manifold and U, V two Poisson manifolds. Two Poisson maps $f : M \to U$ and $g : M \to V$ are said to be a dual pair if the Lie algebras $f^* C^\infty(U)$ and $g^* C^\infty(U)$ are centralisers of each other in $C^\infty(M)$;

$$\{\xi, \zeta\}_M = 0, \quad \xi \in f^* C^\infty(U), \quad \zeta \in g^* C^\infty(V). \tag{5.18}$$

Without loss of generality, the maps f and g may be assumed to be surjective. A surjective Poisson map $f : M \to U$ is also called a *symplectic realisation*.

We have the following general result.

Theorem 5.3. *Let $U \xleftarrow{f} M \xrightarrow{g} V$ be a dual pair. The connected components of the sets $f(g^{-1}(v))$, $v \in V$ a regular value, and $g(f^{-1}(u))$, $u \in U$ a regular value, are symplectic leaves in U and V respectively.*

Proof. We only prove the first claim as the second is identical. Suppose $v \in V$ and let $\mathcal{F}_U$ be the foliation in M determined from the involutive distribution

$$D_p = \{X_h : h \in f^*C^\infty(U)\}, \quad p \in M. \tag{5.19}$$

Consider now $g^{-1}(v)$ and recall that $\{X_{h_1}, X_{h_2}\}_M = X_{h_1}h_2$. By assumption $g^{-1}(v)$ is a properly embedded submanifold of M. As f and g form a dual pair we have $\{h_1, h_2\} = 0$ for all $h_1 \in f^*C^\infty(M)$ and $h_2 \in g^*C^\infty(M)$. This means that all h_2 are constant on any leaf of the foliation $\mathcal{F}_U$ and hence that g is constant on every leaf. This implies that $g^{-1}(v) \supset \mathcal{F}_{U,p}$ for some $p \in g^{-1}(v)$, where $\mathcal{F}_{U,p}$ is the leaf starting at p. Hence

$$g^{-1}(v) = \bigcup_{p \in g^{-1}(v)} \mathcal{F}_{U,p}. \tag{5.20}$$

Applying the surjective map f and passing to the connected components completes the proof. □

We specialise the above situation to Hamiltonian group actions. Given a symplectic G-space M such that the quotient manifold is smooth, we may identify smooth functions on M/G with G-constant functions on M. As G acts by symplectomorphisms, the set of G-constant smooth functions on M is a subalgebra of $C^\infty(M)$, such that $C^\infty(M/G)$ inherits a Poisson structure from M. We now have the following proposition.

Proposition 5.1. *Let $M \times G \to M$ be a Hamiltonian group action with lift λ, $\pi : M \to M/G$ the canonical projection map and $\mu : M \to \mathfrak{g}^*$ the moment map. Then $M/G \xleftarrow{\pi} M \xrightarrow{\mu} \mathfrak{g}^*$ is a dual pair.*

Proof. Let $p \in M$ and recall that $T_p\mathcal{O}_p = \{X_{\lambda(X)} : X \in \mathfrak{g}\}$. Then for any $g \in \mathbb{C}^\infty(M/G)$ and $f \in C^\infty(M)$ we have by G-invariance $\{\pi^*g, f\} = X_f\pi^*g = 0$ and the claim follows. □

Corollary 5.1. *Symplectic leaves in M/G are the connected components of the sets $\pi(\mu^{-1}(\xi))$, $\xi \in \mathfrak{g}^*$.*

Let $\rho : G \to \mathrm{Aut}(V)$ be a Lie group representation and let $S = G \times_\rho V$ be the semi-direct product of G and V with a group operation given by

$$(g, v) \cdot (h, w) = (gh, v + \rho(g)w).$$

It is easy to check that S is a Lie group of dimension $\dim G + \dim V$ by componentwise differentiation.

Lemma 5.3. *There exists a natural Lie algebra isomorphism $\mathfrak{s} \cong \mathfrak{g} \oplus_\rho V$, where the Lie bracket of the latter in given by*

$$[(X_1, v_1), (X_2, v_2)] = ([X_1, X_2], \rho(X_1)v_2 - \rho(X_2)v_1). \tag{5.21}$$

Below we describe the Poisson structure of the dual $\mathfrak{s}^*$ in terms of Poisson mappings from cotangent bundles to $\mathfrak{s}^*$.

Suppose now G acts on a manifold M and denote the induced Hamiltonian action on T^*M by (cf. Proposition 3.29). Let $\mu : T^*M \to \mathfrak{g}$ be the corresponding momentum map. Suppose additionally that there is map $j : M \to V^*$ which is equivariant with respect to the dual action ρ^* of G on V^* . I.e. for . Let $\pi : T^*M \to M$ be the natural projection.

Proposition 5.2. *The map $\phi : T^*M \to \mathfrak{s}^*$ given by*

$$\phi(p) = \mu(p) \oplus j(\pi(p)) \tag{5.22}$$

is a Poisson morphism.

Proof. Suppose $X, Y \in \int$ are two linear function on $\int^*$. We need to show:

$$\{X \circ \phi, Y \circ \phi\}_{T^*M} = [X, Y] \circ \phi. \tag{5.23}$$

In view of the linearity and skew-symmetry we can consider three separate cases. If both X and Y are in $\mathfrak{g}$ the claim follows from the fact that μ is a Poisson mapping. When X and Y are in V the right side of (5.23) vanishes trivially. Finally, when $X \in \mathfrak{g}$ and $Y \in V$ the claim follows from the equivariance of j. $\square$

We now specify Proposition 5.2 in the case $M = G$. Indeed, let G act on itself by left-multiplication. By Example 3.5 the moment map $T^*G \to \mathfrak{g}^*$ in local coordinates is given by $(g, \xi) \mapsto \xi$. Denote the dual representation of ρ by ρ^* and for a fixed element $a \in V^*$ let $j(g) = \rho^*(g)a$. The map j is G-equivariant by construction. Let $G_a \subseteq G$ be the isotropy subgroup of a and $\mathcal{O}_a$ the G-orbit of a in V^*.

Theorem 5.4. *Let $\mu_a : T^*G \to \mathfrak{s}^*$ be the map*

$$\mu_a(g, \xi) = \xi + \rho^*(g^{-1})a. \tag{5.24}$$

Then μ_a is a Poisson mapping by Proposition 5.2.

5.2.1 Integrable top systems

We finally begin a construction of a Lax pair for the spinning top, using the twisted loop algebra from before.

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra and σ a Cartan involution of $\mathfrak{g}$. Let $\mathfrak{k} \subset \mathfrak{g}$ be the subalgebra of fixed points of σ and $K \subset G$ the corresponding Lie group such that $T_eK \cong \mathfrak{k}$. Write the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\sigma|_{\mathfrak{p}} = -\operatorname{id}_{\mathfrak{p}}$. The dual decomposition can be written as $\mathfrak{g} = \mathfrak{k}^* \oplus \mathfrak{p}^*$.

Let $\mathfrak{L}(\mathfrak{g}, \sigma)$ be the twisted loop algebra of $\mathfrak{g}$ and σ and fix an element $a \in \mathfrak{p}^*$. Consider the subspace of $\mathfrak{L}(\mathfrak{g}, \sigma)^*$ generated by Lax matrices

$$L(\lambda) = a\lambda + l + s\lambda^{-1}, \quad l \in \mathfrak{k}^*, \quad s \in \mathfrak{p}^*. \tag{5.25}$$

Then the Lax matrices (5.25) form a Poisson subspace of $\mathfrak{L}^*(\mathfrak{g}, \sigma)$ by Lemma 5.1. We specify the construction of Section 5.1. Let K act on itself by right multiplication, fix an element $f \in \mathfrak{p}^*$ and define $j(k) = \operatorname{Ad}^*_{k^{-1}}(f)$. Then

$$(l, \xi) \mapsto L(\lambda) = a\lambda + l + \operatorname{Ad}^*_{k^{-1}}(f)\lambda^{-1} \tag{5.26}$$

is a Poisson mapping from T^*K to $\mathfrak{L}^*(\mathfrak{g}, \sigma)$.

From now on we assume that the group K is compact and $\mathfrak{g}$ is semisimple, so that we can identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. Suppose ϕ is a polynomial on $\mathfrak{g}$ which is Ad-invariant and define the function

$$H_\phi(L(\lambda)) := \operatorname{Res}_{\lambda=0} \lambda\phi(\lambda^{-1}L(\lambda))d\lambda. \tag{5.27}$$

Then one can easily check that H_ϕ is also Ad-invariant. We want to find the corresponding Lax pair associated to the Casimir H_ϕ and introduce some notation. Let $\phi_a(l) = \phi(a + \xi)$ where $a \in \mathfrak{p}^*$ is some fixed element as above and define $b := d\phi(a)$. Let $\mathfrak{k}_a$ be the centraliser of a and $\mathfrak{k}^\perp$ the orthogonal subspace with respect to the Killing form. Furthermore, denote the second differential of ϕ_a at $g = 0$ by $d^2\phi_a : \mathfrak{k}_a \to \mathfrak{k}_a$.

$$\omega(l) := \begin{cases} \phi''_a(l) & \text{if } l \in \mathfrak{k}_a \\ \operatorname{ad}_b(\operatorname{ad}_a)^{-1}(l) & \text{if } l \in \mathfrak{k}^\perp \end{cases}. \tag{5.28}$$

Note that $\ker \operatorname{ad}_a \subset \ker \operatorname{ad}_b$.

Theorem 5.5. *The Hamiltonian $H_\phi(L)$ is given by*

$$H_\phi(l, \xi) = \frac{1}{2}\langle l|\omega(l)\rangle + \langle b|\xi\rangle \tag{5.29}$$

The corresponding Lax pair in $\mathfrak{L}_+(\mathfrak{g},\sigma)$ is then

$$\begin{aligned} L(\lambda) &= a\lambda + l + \xi\lambda^{-1}. \\ M_+(\lambda) &= b\lambda + \omega(l). \end{aligned} \tag{5.30}$$

Proof. We want to compute the coefficient of λ^{-2} in the term $\phi(a + l\lambda^{-1} + \xi\lambda^{-2})$. A Taylor expansion shows that $H_\phi = (b,\xi) + \frac{1}{2}d^2\phi_a(l)$. To compute the second differential we split $l = l_0 + l_1$, with $l_0 \in \mathfrak{k}_a$, $l_1 \in \mathfrak{k}^\perp$ and observe that $[a,\mathfrak{p}] = \mathfrak{k}^\perp$. Indeed, $[a,\mathfrak{p}] \subset \mathfrak{k}$ and a simple argument with the Jacobi identity shows $[a,\mathfrak{p}] \cap \mathfrak{k}_a = \{0\}$. Choose $X \in \mathfrak{p}$ such that $[a,X] = l_1$ and define $g = \exp X\lambda^{-1}$. We find:

$$\mathrm{ad}_g(a + \lambda^{-1}l) = a + l_0\lambda^{-1} + \frac{1}{2}[X, l + l_0]\lambda^{-2} + \ldots \tag{5.31}$$

where the other terms are of order λ^{-3}. By the Ad-invariance of ϕ we have $\phi(a + l\lambda^{-1}) = \phi(\mathrm{ad}_g(a + l\lambda^{-1}))$. By Lemma 2.2, we have $\langle b|[X,l_0]\rangle = 0$. Taylor expanding the RHS and calculating the λ^{-2} coefficient and comparing gives

$$\begin{aligned} \phi_a''(l) &= \langle d^2\phi_a(l_0)|l_0\rangle + \langle b|[X,l]\rangle \\ &= \langle d^2\phi_a(l_0)|l_0\rangle + \langle [b,X]|l\rangle = \langle \omega(l)|l\rangle \end{aligned} \tag{5.32}$$

by the invariance of $\langle\cdot|\cdot\rangle$. To determine M_+ we need to find the positive powers of λ of the derivative of H_ϕ. We have $dH_\phi = \lambda d\phi(\lambda^{-1}L(\lambda)) = b\lambda + \omega(l) + \ldots$ from our calculation above. The claim then follows. □

The Lax equation $\dot{L} = [L,M]$ corresponding to (5.30) is given by

$$\begin{aligned} \dot{l} &= [l,\omega(l)] + [s,b], \\ \dot{s} &= [s,\omega(l)]. \end{aligned} \tag{5.33}$$

We now discuss the factorisation problem of (5.8).

5.3 Factorisation in $\mathfrak{L}(\mathfrak{g},\sigma)$

It turns out that the factorisation problem of eq. (5.8) can be answered completely when the Lie algebra $\mathfrak{g}$ is a twisted loop algebra. The following section develops this program.

Let $\mathfrak{L}(\mathfrak{g},\sigma)$ be the twisted loop algebra determined by the complex Lie algebra $\mathfrak{g}$ of the complex group G and the automorphism σ of order n. We have seen that $\mathfrak{L}(\mathfrak{g},\sigma)$ is a Lie algebra. Let $\mathfrak{L}(G,\sigma)$ be the set of functions $g : \mathbb{CP}^1 \to G$ holomorphic in $\mathbb{CP}^1 \setminus \{0,\infty\}$ such that

$$g(\exp(2\pi i/n)\lambda) = \sigma g(\lambda), \tag{5.34}$$

where σ here denotes the induced automorphism $\sigma : G \to G$. The Lie algebra of $\mathfrak{L}(G,\sigma)$ can clearly be identified with $\mathfrak{L}(\mathfrak{g},\sigma)$. The subalgebras $\mathfrak{L}_\pm(\mathfrak{g},\sigma)$ thus correspond to the subgroups $\mathfrak{L}_\pm(G,\sigma)$ of G-valued functions that are holomorphic in $\mathbb{CP}^1 \setminus \{0\}$ and $\mathbb{CP}^1 \setminus \{\infty\}$ respectively.

Consider the Lax equation (5.4) in $\mathfrak{L}(\mathfrak{g},\sigma)$ and let (L,M) be the Lax pair associated to a Hamiltonian ϕ. Let $X(\lambda) = d\phi(L(\lambda))$ and $g_\pm(\cdot,t) \in \mathfrak{L}_\pm(G,\sigma)$ be smooth curves that solve the factorisation problem

$$\exp tX(\lambda) = g_+^{-1}(\lambda,t)g_-(\lambda,t) \tag{5.35}$$

such that $g_\pm(\lambda,0) = e$. We have proven that the solution of the Lax equation in $\mathfrak{L}_\pm(\mathfrak{g},\sigma)$ is given by

$$L(\lambda,t) = \mathrm{Ad}^*_{g_\pm(\lambda,t)}L(\lambda). \tag{5.36}$$

From a geometric point of view, equation (5.35) can be interpreted as a transition function of a holomorphic G-bundle over $\mathbb{CP}^1$ *. This perspective allows to show the existence of solutions to the factorisation problem; a short discussion is given in [RS94, Proposition 8.1 and 8.2].

Equation (5.36) linearises if we translate the problem to the Jacobian of a spectral curve, allowing us to write down an explicit solution.

Let $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{C})$ be an m-dimensional complex matrix Lie subalgebra and $L(\lambda) \in \mathfrak{L}(\mathfrak{g},\sigma)$, with σ an automorphism of order k. The Lax matrix determines an algebraic curve Γ_0 given by

$$\begin{aligned} 0 &= \det(L(\lambda) - \nu I_n) \\ &= \nu^m + p_1(\lambda)\nu^{m-1} + \cdots + p_m(\lambda), \end{aligned} \tag{5.37}$$

where $p_i(\lambda) \in \mathbb{C}[\lambda,\lambda^{-1}]$, $i = 1,\ldots,m$. Let $\lambda,\nu : \Gamma_0 \to \mathbb{CP}^1$ be the coordinate projection maps. If $p \in \Gamma_0$ is not a branching point of λ and the spectrum of $L(\lambda(p))$ is simple, there exists a one-dimensional eigenspace $E(p) \subset \mathbb{C}^m$. This gives a holomorphic line bundle E_L on the subset $\tilde{\Gamma}_0 \subseteq \Gamma_0$ for which these conditions hold. E_L is called the *eigenbundle* of L.

*A discussion of fiber bundles falls outside of the scope of this text. See for example [Hus66].

Let Γ be the nonsingular compactification of Γ_0 as described in Proposition 4.6. Then E_L extends to a line bundle on Γ also denoted E_L. Indeed, the map $p \mapsto E(p)$ is a meromorphic map on Γ_0 and hence a holomorphic map $\Gamma \to \mathbb{CP}^{n-1}$. Moreover, λ and μ extend to meromorphic functions on Γ in the same manner.

We introduce some more terminology. The 'set' consisting of the spectral curve Γ, the eigenbundle E_L and projection maps λ, ν is called the *spectral data* for $L(\lambda)$. We now discuss how eq. (5.36) linearises on the Jacobian $\mathrm{Jac}(\Gamma)$ and the matrix $L(\lambda)$ can be recovered from its spectral data, up to some freedom. In turn, this later allows us to rephrase the factorisation problem (5.35) in terms of this spectral data.

5.3.1 Reconstruction from spectral data

Let $X(\lambda) = d\phi(\lambda)$ as before. Then $[L(\lambda), X(\lambda)] = 0$ by Proposition 2.2. The following simple lemma shows that the spectral curve (5.37) remains invariant under the time-evolution of the Lax equation. Its proof is delayed until the end of this section.

Lemma 5.4. *Suppose $dL/dt = [L, M]$ is a Lax equation where the matrix M possibly depends on time. Then for every $\nu \in \mathbb{C}$ and t*

$$\frac{d}{dt} \det(L(t) - \nu) = 0. \tag{5.38}$$

In view of the identity $[L(\lambda), X(\lambda)] = 0$, the eigenvectors of $L(\lambda)$ are also eigenvectors of $X(\lambda)$:

$$X(\lambda)v = \mu(p)v, \quad , p \in \Gamma, \quad v \in E(p), \tag{5.39}$$

where $\mu : \Gamma \to \mathbb{CP}^1$ is a meromorphic function; let $U_\pm = \{p \in \Gamma : \lambda^\pm(p) \neq \infty\}$, then $U_+ \cup U_- = \Gamma$ and μ is holomorphic in the intersection $U_+ \cap U_-$.

We describe the time evolution of the eigenbundle on Γ. Denote by $E_{L(t)}$ the line bundle at time t and let F_t be the one-parameter family of line bundles associated to the transition function $\exp t\mu$ with respect to the open cover $\{U_+, U_-\}$. From section 4.7 there exists a canonical identification of the line bundles with elements of $\mathrm{Jac}\Gamma$. Let $\mathrm{Jac}_d\Gamma$ denote the shifted Jacobi variety of divisors of degree $d = \deg E$.

Theorem 5.6. *Let $E_{L(t)}$ be the line bundle, viewed as an element of* $\mathrm{Jac}_d\Gamma$*. Then $E_{\lambda(t)} = E_L \otimes F_t$.*

Proof. As $\mathfrak{g}$ is a matrix algebra, $L(\lambda, t) = g_\pm(\lambda, t)L(\lambda)g_\pm^{-1}(\lambda, t)$ from (5.36). Hence on $U_\pm$ there exist bundle isomorphism $E_{L(t)}(p) = g_+(\lambda(p), t)E_L(p)$ resp. $E_{L(t)}(p) = g_-(\lambda(p), t)E_L(p)$. The transition function is then precisely given by

$$g_+^{-1}(\lambda(p), t)g_-(\lambda(p), t)|_{E(p)} = \exp tM(\lambda(p))|_{E(p)} = \exp t\mu(p). \tag{5.40}$$

This completes the proof. $\square$

Let V be the velocity vector of the Lax equation as described above. In view of the Serre duality theorem, the tangent space to $\mathrm{Pic}_d\Gamma$ may be identified with the dual of the space of meromorphic 1-forms on Γ. Let ω be such a differential; then the ω-component of the velocity vector V is given by

$$\omega(V) = \sum_{p:\lambda(p)=0} \mathrm{Res}_p(\mu\omega) \tag{5.41}$$

For a proof we refer to [RS94, Section 10.2, eq.]; some of its ideas appear later in this text.

We can now discuss the reconstruction $L(\lambda)$ from its spectral data, which requires regularity.

Definition 5.4. The Lax matrix $L(\lambda) \in \mathfrak{L}(\mathfrak{g}, \sigma)$ is said to be *strongly regular* if the following conditions hold:

(i) The spectral curve (5.37) is irreducible;

(ii) The spectral curve Γ_0 is irreducible;

(iii) The coefficients of the highest and lowest powers of λ are matrices with a simple spectrum.

Condition (iii) ensures that the bundle is well-behaved near the critical points where $\lambda = 0, \infty$ as we shall see below.

We need some more notation. For a line bundle F on Γ let $\mathcal{O}_F(\Gamma)$ denote the space of global holomorphic sections of Γ. In general the eigenbundle E_L may not have any global holomorphic sections. On the other hand, as a subbundle $E_L \subset \Gamma \times \mathbb{C}^n$ the linear coordinates on $\mathbb{C}^n$ give n linearly independent sections of the dual bundle E_L^*. As such, it is more convenient to work with the dual eigenbundle E_L^*.

Let D be a divisor on Γ and let $\mathcal{O}_F(\Gamma, D)$ be the space of global meromorphic sections ψ of Γ such that $(\psi) \geq -D$. For $a \in \mathbb{CP}^1$ let P_a be the divisor that takes the value 1 at all points $p \in \Gamma$ such that $\lambda(p) = a$. These points p where $P_a(p) = 1$ are said to lie over a.

Definition 5.5. Let F be a line bundle on the compact Riemann surface Γ. F is called *λ-regular* if $\dim \mathcal{O}_F(\Gamma) = n$ and for all $a \in \mathbb{CP}^1 \setminus \{0, \infty\}$ we have $\mathcal{O}_F(\Gamma, P_a) = \{0\}$.

The following proposition shows that the dual eigenbundle E_L^* is λ-regular if the matrix $L(\lambda)$ is regular.

Proposition 5.3. *Suppose $L(\lambda)$ is regular. Then E_L^* is λ-regular.*

Proof. Suppose $\psi \in \mathcal{O}_{E_L^*}(\Gamma)$ is a global holomorphic section. We will show that ψ can be expressed as a linear combination of linear coordinates on $\mathbb{C}^n$. Fix a value $a \in \mathbb{CP}^1 \setminus \{0, \infty\}$ which is not a ramification point of λ and let $p_1, \ldots, p_n \in \Gamma$ be points that lie over a. The linear spaces $E(p_1), \ldots E(p_n)$ span $\mathbb{C}^n$ as the spectrum is simple and the linear functionals $\psi(p_1), \ldots \psi(p_n)$ now determine a linear function $\psi(a)$ on $\mathbb{C}^n$. Varying a gives a $(\mathbb{C}^n)^*$-valued function $\psi(\lambda)$. As ψ is holomorphic the function $\psi(\lambda)$ remains bounded near the ramification points . The regularity condition (iii) of $L(\lambda)$ ensures that $\psi(\lambda)$ is also bounded near $a = 0, \infty$. Hence by Liouville's theorem, the function $\psi(\lambda) =$ const and linear coordinates span the space of holomorphic section on E_L^*.

To show $\mathcal{O}_F(\Gamma, P_a) = \{0\}$, if $\psi(\lambda)$ vanishes for a single value of λ then $\psi(\lambda) = 0$. This completes the proof. □

We introduce more notation. Let $\Gamma' = \Gamma \setminus \lambda^{-1}(\{0, \infty\})$ and $\mathcal{O}_{E_L^*}(\Gamma')$ the space of meromorphic sections that are holomorphic in Γ'. Moreover, let $R = \mathbb{C}[\lambda, \lambda^{-1}]$. Multiplication by Laurent polynomials defines a mapping $r : \mathcal{O}_{E_L^*}(\Gamma) \otimes R \to \mathcal{O}_{E_L^*}(\Gamma')$.

Proposition 5.4. *The mapping $r : \mathcal{O}_{E_L^*}(\Gamma) \otimes R \to \mathcal{O}_{E_L^*}(\Gamma')$ is an isomorphism of R-modules.*

Proof. We first prove injectivity. Suppose $\sum_i \psi_i \lambda^i = 0$, where $\psi_i \in \mathcal{O}_{E_L^*}(\Gamma)$. After multiplying with λ^{-k-1} we obtain $\psi_k \lambda^{-1} = \sum_i \psi_i \lambda^{i-k-1}$ and thus $\psi_k \in \mathcal{O}_{E_L^*}(P_0)$ means $\psi_k = 0$. The same argument shows $\psi_i = 0$ for the other i.

To prove surjectivity, suppose $\psi \in \mathcal{O}_{E_L^*}(\Gamma')$. By the regularity there exists natural numbers n and m and holomorphic functions ψ_1, ψ_2 such that $\psi - \psi_1 \lambda^n - \psi_2 \lambda^{-m}$ has poles at $\lambda = 0$ and $\lambda = \infty$ of at most order $n - 1$ respectively $m - 1$. The claim follows by induction. □

Note: the same claim follows by letting $\Gamma' = \Gamma \setminus \lambda^{-1}(\{\infty\})$ (resp.$\Gamma' = \Gamma \setminus \lambda^{-1}(\{\infty\})$) and instead choosing $R = \mathbb{C}[\lambda]$ (resp. $R = \mathbb{C}[\lambda^{-1}]$).

We can finally discuss the reconstruction of $L(\lambda)$ from the spectral data. Recall the spectral data consists of a compact Riemann surface Γ, an n-sheeted covering $\lambda : \Gamma \to \mathbb{CP}^1$, a dual eigenbundle E_L^* and meromorphic function $\nu : \Gamma \to \mathbb{CP}^1$ which is regular in Γ'. Indeed, this follows from the regularity of $L(\lambda)$ and multiplication by ν therefore gives an R-linear operator in $\mathcal{O}_{E_L^*}(\Gamma')$. By Proposition 5.4 there exists an R-linear operator in $\in \mathcal{O}_{E_L^*}(\Gamma) \otimes R$. Let $\psi = (\psi_1, \ldots, \psi_n)$ be a basis for $\mathcal{O}_{E_L^*}(\Gamma)$. ψ is called a *Baker-Akhiezer function* for the spectral curve Γ. With respect to this basis there exist matrix coefficients $c_{ij} \in R$ such that $\nu\psi_j = \sum_i c_{ij}\psi_i$. Since ν is the eigenvalue of $L(\lambda)$ by eq. (5.37), we have $L(\lambda)\psi = \nu\psi$ and the coefficients of $L(\lambda)$ are therefore given by c_{ij}.

The following theorem summarises this result.

Theorem 5.7. *Let Γ be a compactified nonsingular algebraic curve, $\lambda : \Gamma \to \mathbb{CP}^1$ an n-sheeted covering, E^* a λ-regular line bundle on Γ and $\psi = (\psi_1, \ldots \psi_n)$ a basis for $\mathcal{O}_{E^*}(\Gamma)$. For each meromorphic function ν that is regular on $\Gamma' = \Gamma \setminus \lambda^{-1}(\{0, \infty\})$ there exists a Lax matrix $L(\lambda)$ such that $L\psi = \nu\psi$.*

We now use reconstruction to express the factorisation $g_\pm(\lambda, t)$ in terms of a Baker-Akhiezer function ψ for E_L^*.

5.3.2 Baker-Akhiezer functions

Let $L(\lambda)$ be a Lax matrix with its associated spectral data as before. Moreover, let $(\psi_1, \ldots, \psi_n)$ be a Baker-Akhiezer function for E^*. We have seen that

$$L(\lambda(p))\psi(p) = \nu(p)\psi(p). \tag{5.42}$$

The evolution equation (5.36) implies the identity

$$\psi_\pm(p,t) = g_\pm(\lambda(p), t)\psi(p.0), \tag{5.43}$$

where $\psi_\pm(p,t)$ is the Baker-Akhiezer vector for $L(\lambda, t)$ in the domain $U_\pm = \{p \in \Gamma : \lambda^\pm(p) \neq \infty\}$. This implies

$$\psi_+(p,t) = e^{t\mu(p)}\psi_-(p,t). \tag{5.44}$$

The Baker-Akhiezer function is seen to evolve in the following manner with time. Let $M_\pm = P_\pm d\phi(L)$, where ϕ is the Hamiltonian and $P_\pm$ are the projection operators. Using the identity $\partial_t g_\pm g_\pm^{-1} = -M_\pm$ from the proof of Theorem :

$$\frac{d}{dt}\psi_\pm(p,t) = -M_\pm(\lambda(p))\psi_\pm(p,t). \tag{5.45}$$

Thus in order to solve the Riemann-Hilbert problem, it suffices to compute the Baker-Akhiezer function. This whole approach is justified by the fact that one can do so purely in terms of the associated spectral data.

Let $(\psi^i_\pm)$ be a Baker-Akhiezer function for the spectral data that satisfy the regularity condition

$$\psi^i_-(p,t) = \text{const} \tag{5.46}$$

for any p that lies over $\lambda = \infty$. Suppose $a \in \mathbb{CP}^1 \setminus \{0,\infty\}$ is not a ramification point λ and let $p_1,\ldots,p_n \in \Gamma$ be points that lie over a. We define the $n\times n$ matrices:

$$\Psi_\pm(a,t)_{ij} = \psi^i_\pm(p_j,t) \tag{5.47}$$

and now *define*:

$$g_\pm(a,t) := \Psi_\pm(a,t)\Psi^{-1}_\pm(a,0). \tag{5.48}$$

Note that (5.48) does not depend on the ordering of points $p_1,\ldots,p_n$ as there is a sum over this index.

The following is the main result of this chapter.

Theorem 5.8. *The functions $g_\pm(\lambda,t)$ in equation* (5.48) *are entire functions of $\lambda^{\pm1}$, analytic for all t and solve the factorisation problem* (5.35).

Proof. Let $L(\lambda,t)$ and $X(\lambda,t)$ be the matrices recovered from the Baker-Akhiezer function ψ and the function ν resp μ. Denote $X_\pm = P_\pm X$. We show the validity equation (5.45) for these matrices. Equation (5.44) implies $\partial_t\psi_\pm = e^{\pm t\mu}(\partial_t\psi_\mp \pm \mu\psi_\mp)$, so that $\partial_t\psi_\pm$ are regular in $U_\pm$. By Proposition 5.4 there exist polynomials $-X'_\pm \in \mathfrak{g}\otimes\mathbb{C}[\lambda^\pm]$ such that $\partial_t\psi_\pm = -X'\psi_\pm$. The same equation can be rewritten as $-X'_\pm\psi = e^{\pm t\mu}(-X_\mp \pm \mu)\psi_\mp$, which imply $(X'_\pm - X'_\mp)\psi_\pm = \mu\psi_\pm = X\psi_\pm$ and hence $X'_\pm = X_\pm$.

The proven equation (5.45) now shows that the chosen $g_\pm$ satisfy the differential equation $\partial_t g_\pm = -X_\pm g_\pm$ with the initial condition $g_\pm(\lambda,0) = I_n$. □

We finally give the proof Lemma 5.4.

Proof. For simplicity we assume $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ equipped with the commutator bracket. We first consider the differential equation $\dot{L} = [L,M]$ where M is a constant matrix. The Ansatz

$$L(t) = e^{-tM}Le^{tM} \tag{5.49}$$

is easily seen to solve the differential equation. This implies that $\det(L(t) - \nu I) = \det(L-\nu I)$. For the general case with a time-dependent matrix $M(t)$, by Taylor expansion we have $L(t) = \exp(-tM)L\exp(t) + O(t^2)$ as $t\to 0$ and the claim hence follows. □

Explicit formulae may be obtained for the Baker-Akhiezer functions; Reyman *et. al.* outline a description in [RS94, Section 10.2], and explicit calculations are given in [BRS89]. We incorporate their method directly in the next chapter when we calculate the Baker-Akhiezer function for a specific set of spectral data.

Chapter 6

The quantum Brachistochrone problem

Quantum computers are at the forefront of scientific development. Still, controlling quantum states represents many difficulties given the unique behaviour of quantum mechanics at the smallest scale. From a theoretical point of view, quantum control is concerned with finding a unitary transformation that drives the given state to the desired state. One of the most successful descriptions is in terms of representations of Lie groups and Lie algebras [DP10], which is the focus of this chapter.

In this language, the system evolves on the Lie group $SU(n)$ where n the dimension of the system, being acted upon by some time-dependent Hamiltonian operator $\hat{H}(t) \in -i\mathfrak{su}(n)$. For many applications the Hamiltonian is restricted a linear subspace of the Lie algebra $\mathfrak{g}$, by conditions such as energy constraints or external magnetic fields. The quantum brachistochrone (QB) problem attempts to minimise the cost of the evolution process and corresponds to finding the most efficient evolution algorithm for such restrained systems.

Its first formulation as a variational problem with flexible boundary conditions was recently given in [YC22]; Yang *et. al.* find solutions when the orthogonal subspace to the allowed subset of Hamiltonians is a subalgebra. Most recently, Cheianov and Malikis extended the set of integrable solutions when this orthogonal subspace is not a subalgebra, and were able to write the equation of motion as a limit case of integrable spinning top systems [MC24]. Such integrable solutions turn out to be much more numerically contractible.

6.1 Overview

We discuss the derivation of the equation of motion of the QB problem and realise it as a limiting case of a Lax pair for the spinning tops seen in Chapter 5 following [MC24].

6.1.1 Introduction

We introduce the QB problem. Throughout this discussion let $G = SU(n)$ and $\mathfrak{g} = \mathfrak{su}(n)$ unless stated otherwise. We equip $\mathfrak{g}$ with the Frobenius norm which coincides with the Killing form (cf. Proposition 2.8); recall that $||A|| = \sqrt{\mathrm{Tr}(AA^\dagger)}$. Moreover, identify $\mathfrak{g}$ hereby with its dual $\mathfrak{g}^*$ and elements of $\mathfrak{g}$ with the defining representation.

Definition 6.1. A *quantum evolution* consists of a smooth curve $\hat{H}(t)$, called the *driving Hamiltonian*, and an initial unitary operator $U_0 \in G$. The *unitary state* of the system is the solution of the Schrödinger equation in natural units with an initial condition:

$$i\partial_t \hat{U}(t) = \hat{H}(t)\hat{U}(t), \quad \hat{U}(0) = \hat{U}_0, \quad t \geq 0. \tag{6.1}$$

We assume that the time domain I of (6.1) is a closed interval $I = [0, t_f]$, where t_f is the final time. $\hat{U}_f := \hat{U}(t_f)$ is called the *final unitary*.

As outlined above, we want to realise the final unitary in an optimal manner. To this end, we are interested in the choice of Hamiltonians that minimise the functional

$$S[\hat{H}] = \int_0^{t_f} ||\hat{H}(t)||dt, \tag{6.2}$$

i.e. we want to solve this variational problem with the boundary conditions $\hat{U}(0) = \hat{U}_0$, $\hat{U}(t_f) = \hat{U}_f$. With (6.1) one can check that a different choice of time parameterisation leads to the same functional equation (6.2). We hence assume without loss of generality $t_f = 1$ and can also assume $\hat{U}_0 = I_n \in G$.

Under specific physical conditions such as energy constraints are the presence of external forces, the Hamiltonian may not be able to access the whole space $\mathfrak{g}$. Let $\mathfrak{a}$ be a linear subspace consisting of the accessible Hamiltonians and $\mathfrak{b} = \mathfrak{a}^\perp$ the orthogonal subspace. We require that $\mathfrak{a}$ generates the whole Lie algebra $\mathfrak{g}$, i.e. any unitary should be 'achievable' from a physical point of view*.

*It is unclear whether this assumption guarantees that the evolution equation (6.1) achieves all $U = U_f \in G$; it is well-known that the exponential map of a compact connected manifold is surjective, but this situation requires a different perspective.

To find the minima of the functional (6.2) we can apply the Euler-Lagrange formalism with Lagrange multipliers †. More specifically, choose a basis $\hat{A}_1, \ldots, \hat{A}_r$ of $\mathfrak{a}$ and $\hat{B}_1, \ldots \hat{B}_s$ of $\mathfrak{b}$, and let $(\lambda^i(t))_{i=1}^s$ be Lagrange multipliers. The orthogonality constraint may be written as

$$\mathrm{Tr}(\hat{H}(t)\hat{B}_i) = 0, \quad i = 1, \ldots, s. \tag{6.3}$$

The length functional can now be rewritten using (6.1) as

$$S = \int_0^1 \left[\sqrt{\mathrm{Tr}(\partial_t \hat{U} \partial_t \hat{U}^\dagger)} + \sum_i^m \lambda^i \, \mathrm{Tr}(\hat{B}_i \partial_t \hat{U} \hat{U}^\dagger) \right] dt. \tag{6.4}$$

Define the smooth curve $\hat{D} = \sum_i \lambda^i(t)\hat{B}_i$. The EL-equations given the following differential equation; a derivation is given in Appendix A of [MC24].

$$\frac{d}{dt}(\hat{H} + \hat{D}) + i[\hat{H}, \hat{D}] = 0. \tag{6.5}$$

Note that in the absence of constraints, $\hat{D} = 0$, we recover a constant Hamiltonian $\dot{\hat{H}} = 0$ and the solutions of (6.1) is the familiar exponential.

In fact, (6.5) can be written as a Lax equation. Let $\hat{\mathfrak{t}} := -i(\hat{H} + \hat{D})$ and $\mathscr{P}_{\mathfrak{b}} : \mathfrak{g} \to \mathfrak{b}$ the projection onto the $\mathfrak{b}$ subspace. Then

$$\frac{d}{dt}\hat{\mathfrak{t}} = [\hat{\mathfrak{t}}, \mathscr{P}_{\mathfrak{b}}\hat{\mathfrak{t}}]. \tag{6.6}$$

The symmetry of equation (6.6) can be exploited to to a great extend; a discussion can be found in [MC24, Page 4]. In particular, when $\mathfrak{b}$ is a subalgebra, one can write down an explicit solution in terms of the initial conditions. See for example equation (22) of [MC24].

Remarkably, (6.6) can be written as a limiting case of the Lax pair for spinning top systems we constructed in Section 5.2. This allows us to apply the machinery of Section 5.3 to solve attack this differential equation.

To this end, let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be a non-trivial pseudo-Cartan decomposition of $\mathfrak{g}$. That means:

$$[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{l}, \quad [\mathfrak{p}, \mathfrak{l}] \subseteq \mathfrak{p}. \tag{6.7}$$

In particular, $\mathfrak{l}$ is a subalgebra.

The following is the main result of this section [MC24, Theorem 1]. We use the same notation as in Theorem 5.5.

† A more formal treatment would require Sobolev spaces and falls outside the scope of this text.

Theorem 6.1. *Suppose $\hat{p} \in \mathfrak{p}$ arbitrary, let $\hat{a} = \epsilon\hat{p}$ with $\epsilon \in \mathbb{R}$. Let*

$$\mathfrak{l}_a = \mathfrak{l}_a^{(A)} \oplus \mathfrak{l}_a^{(B)} \tag{6.8}$$

be some, possibly trivial, decomposition of the centraliser of $\hat{a}$ into subalgebras of $\mathfrak{l}_a$. Then there exists a invariant polynomial $\phi(z)$ such that eq. (6.5) *is the limit $\epsilon \to 0$ of the Lax system eq.* (5.33) *where $\mathfrak{g}$ is decomposed as:*

$$\mathfrak{a} = \mathfrak{p} + \mathfrak{l}_a^{(A)}, \quad \mathfrak{b} = \mathfrak{l}^{\perp} + \mathfrak{l}_a^{(B)}. \tag{6.9}$$

Proof. The goal of the proof is to construct a polynomial such that $d\phi(a) = b$. To this end, let $(a_1, \ldots, a_Q)$ denote the spectrum of $\hat{a}$ as acting canonically on the vectorspace $V = \mathbb{C}^n$ and $V_i \subseteq V$ the eigenspace of a_i. Let $\mathfrak{l}_a^i \subseteq \mathfrak{l}_a$ be the largest subalgebra preserving V_i. Then there exists a decomposition

$$\mathfrak{l}_a = \bigoplus_i^Q \mathfrak{l}_a^i \tag{6.10}$$

of the centraliser $\mathfrak{l}_a$ of $\hat{a}$ into the subalgebras $\mathfrak{l}_a^i$. Without loss of generality we can rearrange indices such that

$$\mathfrak{l}_a^{(A)} = \bigoplus_{i=1}^q \mathfrak{l}_a^i, \quad \mathfrak{l}_a^{(B)} = \bigoplus_{i=q+1}^Q \mathfrak{l}_a^i. \tag{6.11}$$

Consider now the following polynomial:

$$\phi(z) := \operatorname{Tr}\left(\frac{z^2}{2} + \frac{1}{2}\psi(z)\prod_{i=1}^Q (z - a_i)^2\right), \tag{6.12}$$

where

$$\psi(z) = -\sum_{k=q+1}^Q \prod_{s=1, s\neq k}^Q \frac{(z - a_s)^2}{(a_k - a_s)^3}. \tag{6.13}$$

An easy calculation shows that the eigenvalues of $\hat{b}$ coincide with $\hat{a}$. Hence $\hat{b} = \hat{a}$. Thus $\omega(\hat{l})$ acts as the identity on $\mathfrak{l}^{\perp}$ (cf. eq. (5.28)). It follows from the definition that

$$\omega = \mathscr{P}^{\perp} + \mathscr{P}^{(B)} = \mathscr{P}_{\mathfrak{b}}, \tag{6.14}$$

where $\mathscr{P}^{(B)} = \mathscr{P}^{(q+1)} = \cdots + \mathscr{P}^{(Q)}$. Substituting these identities in (5.33) and comparing powers of λ yields:

$$\begin{aligned} &[\hat{a}, \hat{b}] = 0, \quad [\hat{a}, \omega(\hat{l})] = [\hat{b}, \hat{l}], \\ &\frac{d}{dt}\hat{l} = [\hat{l}, \omega(\hat{l})] + [\hat{s}, \hat{b}], \quad \frac{d}{dt}\hat{s} = [\hat{s}, \omega(\hat{l})]. \end{aligned} \tag{6.15}$$

The first two equations are satisfied by construction. Equation (6.14) implies

$$\frac{d}{dt}\hat{t} = [\hat{t}, \mathscr{P}_B\hat{t}] + \epsilon[\hat{s}, \hat{p}]. \tag{6.16}$$

Taking $\epsilon \to 0$ completes the proof. $\square$

We now solve an example case $\mathfrak{su}(2)$ and take the limit $\epsilon \to 0$ to recover a solution of the original problem. For later use, recall that we proved in chapter 5 that $dH_\phi = \lambda\phi(\lambda^{-1}L(\lambda))$. With the proof above we thus have $dH_\phi = L(\lambda)$, using notation from Chapter 5, we thus have $\mu(p) = \nu(p)$ for the problem below.

6.2 An analysis of $\mathfrak{su}(2)$

We solve eq. (6.16) and take the limit $\epsilon \to 0$, utilising the techniques from chapter 5. This calculation is inspired by the approach of [BRS89].

Let $G = SU(2)$ and $\mathfrak{g} = \mathfrak{su}(2)$ be the corresponding Lie algebra with the standard basis given by the Pauli matrices:

$$\sigma_x = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.17}$$

Consider the pseudo-Cartan decomposition $\mathfrak{p} = \text{ sp}\{\sigma_x, \sigma_y\}$ and $\mathfrak{l} = \text{ sp}\{\sigma_z\}$, such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$. We fix an arbitrary element $\hat{p} \in \mathfrak{p}$:

$$\hat{p} = \begin{pmatrix} ia_1 & ia_2 \\ ia_2 & ia_1 \end{pmatrix}, \tag{6.18}$$

for some arbitrary $a_1, a_2 \in \mathbb{R}$ and let $\hat{a} = \epsilon\hat{p}$. Note that $\{0\} = \mathfrak{l}_a \subset \mathfrak{l}$. for any choice of constants $a_1, a_2 \in \mathbb{R}$. This leads to a decomposition $\mathfrak{a} = \mathfrak{p}$ and $\mathfrak{b} = \mathfrak{l}$. As the choice of $\hat{p}$ does not influence the resulting decomposition or equation of motion (eq. (6.16)), we can simplify our calculation by setting $a_2 = 0$ and normalise $a_1 = 1$. We want to solve the Riemann-Hilbert problem for the following Lax matrix:

$$\begin{aligned} L(\lambda, t) &= \hat{a}\lambda + \hat{l}(t) + \hat{s}(t)\lambda^{-1} && (6.19) \\ &= \hat{a}\lambda + \alpha(t)\sigma_z + (\beta(t)\sigma_x + \gamma(t)\sigma_y)\lambda^{-1}. && (6.20) \end{aligned}$$

Denote $\alpha = \alpha(0)$, $\beta = \beta(0)$ and $\gamma = \gamma(0)$. We can also write:

$$L(\lambda,0) = \begin{pmatrix} \epsilon\lambda i + \beta i\lambda^{-1} & \alpha + i\gamma\lambda^{-1} \\ -\alpha + i\gamma\lambda^{-1} & -\epsilon\lambda i - \beta i\lambda^{-1} \end{pmatrix}. \tag{6.21}$$

Similarly, we have:

$$M(\lambda,0) = \begin{pmatrix} \epsilon\lambda i & \alpha \\ -\alpha & -\epsilon\lambda i \end{pmatrix}. \tag{6.22}$$

Note that the terms of highest and lowest order in λ have a simple spectrum. The spectral curve of $L = L(\lambda,0)$ is given by:

$$\begin{aligned} 0 = \det(L - \nu I_2) &= \nu^2 + (\epsilon\lambda a_1 + \beta\lambda^{-1})^2 - (-(\gamma\lambda^{-1})^2 - \alpha^2) && (6.23) \\ &=: \nu^2 + p(\lambda), && (6.24) \end{aligned}$$

where:

$$\begin{aligned} p(\lambda) &= \epsilon^2\lambda^2 + \lambda^0(\alpha^2 + 2\epsilon a_1\beta) + \lambda^{-2}(\beta^2 + \gamma^2) \\ &= \epsilon^2\lambda^{-2}(\lambda - a_1)\cdots(\lambda - a_4), \end{aligned} \tag{6.25}$$

where

$$a_i = \pm\sqrt{\delta_\epsilon \pm \sqrt{\delta_\epsilon^2 - \frac{\beta^2+\gamma^2}{\epsilon^2}}}, \quad i = 1,\ldots,4, \quad \delta_\epsilon := -\frac{\alpha^2 + 2\epsilon\beta}{2\epsilon^2}. \tag{6.26}$$

Let $\Gamma_0 \subseteq \mathbb{C}^* \times \mathbb{C}$ be the set of all pairs (λ,ν) satisfying this identity.

6.2.1 Algebraic surfaces for $\mathfrak{su}(2)$

The spectral curve Γ_0 has the symmetries $(\lambda,\nu) \xrightarrow{\tau_1} (-\lambda,\nu)$ and $(\lambda,\nu) \xrightarrow{\tau_2} (\lambda,-\nu)$ that determine involutions of Γ_0. Denote by Γ the nonsingular compactification of Γ_0 and let $C = \Gamma/(\tau_1)$, $E = C/(\tau_2)$ be reduced curves. The two coordinate substitutions $z = \lambda^2$ and $y = \nu^2$ give two two-sheeted covering maps $\Gamma \to C$ resp. $C \to E$.

We can interpret Γ as the Riemann surface determined by the algebraic function $\sqrt{z}$ on C, and similarly C as determined by the algebraic function $\sqrt{y}$ on E. By our arguments in Example 4.5, the surface Γ and C both have genus 1 and E has genus 0, under the assumption that either β or γ is nonzero.

We follow the argument given in [BRS89, Lemma 5.1] and apply it to this situation. Let $\pi : \Gamma \to C$ denote the projection map induced by τ_1. It follows from Riemann-Hurwitz that the two-sheeted cover $\pi : \Gamma \to C$ is unramified: hence, there exists a cycle Z such that any loop on C lifts to Γ if and only if $\langle \gamma, Z\rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the intersection number. The same argument applied to the projection $\kappa' : C \to E$ implies that there are 4 branching points of ρ of multiplicity 2 which we give below.

From Example 4.3, we know that two points lie over $\lambda = \infty$, denote them ∞_+ and ∞_-, and moreover two points lie over $\lambda = 0$, denoted $0_+, 0_-$. It is easy to see with eq. (6.25) that both poles are simple and both zeros are therefore simple as well. Furthermore, (6.25) implies that ν has simple poles at $\infty_\pm, 0_\pm$ and four simple zeros at $a_1, \dots, a_4$.

Similarly, on C there is one simple pole $\overline{\infty}$ that lies over $z = \infty$ and another simple zero $\overline{0}$. Furthermore, ν has two simple poles at $\overline{\infty}$, $\overline{0}$ and two simple zeros at two points $p_\pm$ given by

$$p_\pm = \delta_\epsilon \pm \sqrt{\delta_\epsilon^2 - \frac{\beta^2 + \gamma^2}{\epsilon^2}}. \tag{6.27}$$

Repeating the argument on E yields a simple poles $\overline{\infty}$ and $\overline{0}$ for μ; likewise simple poles of $y = \sqrt{\nu}$ at $\overline{\infty}$ and $\overline{0}$ and simple zeros at $p_\pm$. Hence, the branching points of $\kappa : C \to E$ are given by these four points. Thus C can be obtained from gluing two copies of E along suitable cuts $[\overline{\infty}, \overline{0}]$ and $[p_+, p_-]$.

Note that the function $z = \sqrt{\lambda}$ now considered with domain $E \setminus [\overline{\infty}, \overline{0}]$ becomes unramified, as $\overline{\infty}, \overline{0}$ are the ramification points of z.

Let $a, b \in H_1(C, \mathbb{Z})$ be a canonical basis of cycles on C. We claim that a and b can be chosen such that $\kappa a = -a$ $\kappa' b = -b$ and $a = Z \bmod 2$. Indeed, E is genus 0 and can topologically be identified with a sphere $\mathbb{CP}^1$; the location of the ramification points, (6.27), then shows that cuts can be chosen in the following manner: the first $[\overline{\infty}, \overline{0}]$ along, say, the $x = 0$ axis and $[p_+, p_-]$ counterclockwise in the $z = \text{const}$ plane. Geometrically, a simple analytic continuation argument shows z changes sign along both a and b. Hence setting $a = Z$ proves the claim.

We also need to describe the behaviour of the covering $\kappa : \Gamma \to E$ which has 4 ramification points by the same Riemann-Hurwitz argument. It is clear that these are given by $\infty_\pm$ and $p_\pm$. Let the cut $[\infty_+, \infty_-]$ pass through both points $0_\pm$. We can choose a canonical basis as before.

The following lemma summarises our discussion above.

Lemma 6.1. *There exists a choice of canonical basis $a, b \in H_1(\Gamma, \mathbb{Z})$ such that $\kappa a = -a$, $\kappa b = -b$.*

Note that eq. (6.21) has the symmetry

$$L(\lambda) = I L(-\lambda) I, \tag{6.28}$$

where $I = i\sigma_z$. For the purpose of later computations, we transform to a basis where I is diagonal by conjugating with the matrix $C = \sqrt{2}^{-1}\begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix}$; in this basis $\mathrm{diag}(1,-1) = C^{-1}IC$. For the sake of notation write $L(\lambda)$ instead of $C^{-1}L(\lambda)C$. Then:

$$\begin{aligned} L(\lambda) &= \begin{pmatrix} -i\alpha & -\epsilon\lambda - (\beta+\gamma)\lambda^{-1} \\ -\epsilon\lambda + (\beta+\gamma)\lambda^{-1} & i\alpha \end{pmatrix} \\ &=: L_1\lambda + L_0 + L_{-1}\lambda, \\ M(\lambda) &= \begin{pmatrix} -i\alpha & -\epsilon\lambda \\ -\epsilon\lambda & i\alpha \end{pmatrix} \\ &=: M_1\lambda + M_0. \end{aligned} \tag{6.29}$$

We obtain the new symmetry:

$$L(-\lambda) = \eta L(\lambda)\eta \tag{6.30}$$

The involution τ_1 can be extended to the bundle E_L^* by setting:

$$\psi(\tau_1 p) = \eta\psi(p), \quad p \in \Gamma, \quad \psi(p) \in E_L^*(p). \tag{6.31}$$

in view of the equation above.

6.2.2 Analytical expressions for $\mathfrak{su}(2)$

We begin the program of computing the Baker-Akhiezer function of $E^*_{L(t)}$. This requires the singular behaviour of the meromorphic projection $\nu(\lambda)$ with respect to λ on the spectral curve Γ. Let $\lambda^{\pm}$ be local coordinates on the λ sphere $\mathbb{CP}^1$ near 0 resp. ∞. It follows from eq. 6.24 that near $\lambda = 0$:

$$\nu(\lambda) \sim \pm\lambda^{-1}(\sqrt{\beta^2+\gamma^2}, -\sqrt{\beta^2+\gamma^2}) + o(1), \tag{6.32}$$

where the sign can be chosen later for convenience. Near $\lambda = \infty$ we have:

$$\nu(\lambda) \sim \pm i\epsilon\lambda(1,-1) + O(1). \tag{6.33}$$

Fix a basis $\{(\psi^i_\pm)\}$ of sections of $E^*(L(t))$. The Baker-Akhiezer function ψ satisfies the following analytical properties:
(i) ψ is meromorphic on Γ except at the points that lie over $\lambda = \infty$;
(ii) $\psi\exp(-t\nu)$ is meromorphic on Γ except at $\lambda = 0$ by (5.44);
(iii) the divisor $\widehat{\mathscr{D}}$ of poles of ψ has degree 4 and is time independent;

(iv) ψ is subject to the symmetry condition of eq. 6.31.

The divisor $\widehat{\mathscr{D}}$ is not completely determined from this condition, as given a meromorphic function on Γ with $f(\tau_1 p) = f(p)$ and $(f) \leq \widehat{\mathscr{D}}$ we can replace $f\psi$ by $f\psi$. With this freedom we can find a divisor $\mathscr{D}$ on C such that $\widehat{\mathscr{D}}$ is the pullback of the divisor $P_\infty + \mathscr{D}$.

Let $\Psi_-(\lambda, t)$ be the matrix of equation (5.47). It follows from the analycity properties (i) and (ii) that

$$\Psi(\lambda, t) = \pm \left(\Phi(t) + S(t)\lambda^{-1} + \dots\right) \mathrm{diag}(\epsilon\lambda e^{-i\epsilon\lambda t}, -\epsilon\lambda e^{i\epsilon\lambda t}) \tag{6.34}$$

where $\Phi(t)$ and $S(t)$ do not depend on λ. In view of the equations

$$L(\lambda(p), t)\psi(p, t) = \nu(p)\psi(p, t), \quad \frac{\partial}{\partial t}\psi(p, t) = -M(\lambda(p), t)\psi(p, t), \tag{6.35}$$

we obtain the following conditions with (6.33):

$$L_1 = i\epsilon\Phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi^{-1}, \tag{6.36}$$

$$L_0 = [S\Phi^{-1}, L_1], \tag{6.37}$$

$$M_1 = -\Phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi^{-1}, \tag{6.38}$$

$$M_0 = -\partial_t\Phi\Phi^{-1} + [S\Phi^{-1}, M_1]. \tag{6.39}$$

Additionally, the symmetry condition (iv) takes the following form. Recall that the involution τ_1 flips sheets:

$$\Psi(-\lambda, t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi(\lambda, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.40}$$

A simple calculation then shows that Ψ and S are subject to the following symmetry conditions:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Phi, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -S. \tag{6.41}$$

Combining this with (6.36) gives

$$\Phi(t) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{6.42}$$

Furthermore, the symmetry equation combined with (6.37) gives $s_{11}(t) = -s_{12}(t)$, $s_{21}(t) = s_{22}(t)$ and

$$\alpha(t) = 2\epsilon s_{11}(t) + 2\epsilon i s_{22}(t). \tag{6.43}$$

The symmetry condition (6.40) also yields at $\lambda = 0$

$$\Psi(0,t) = \begin{pmatrix} \Psi_{11}(t) & \Psi_{11}(t) \\ -\Psi_{22}(t) & \Psi_{22}(t). \end{pmatrix} \tag{6.44}$$

Our next goal is to find explicit formulae for the Baker-Akhiezer functions. This can miraculously be done in terms of the so-called *Riemann-theta* functions.

6.2.3 Baker-Akhiezer function for $\mathfrak{su}(2)$

The 1-dimensional Riemann theta function with characteristics $\xi, \chi \in \mathbb{C}$ has the following properties:

$$\begin{aligned} \theta[(\xi,\chi)](z|B) &= \sum_{m\in\mathbb{Z}} \exp\left[\frac{1}{2}B(m+\xi)^2 + (z+2\pi i\chi)(m+\xi)\right], \\ \theta(z+2\pi i n) &= \theta(z), \\ \theta(z+Bn) &= \exp\left[-\frac{1}{2}Bn^2 - nz\right]\theta(z), \quad n \in \mathbb{Z}, \end{aligned} \tag{6.45}$$

where B is the period matrix. We introduce functions that have the same meromorphic behaviour as the Baker-Akhiezer functions. Let σ be an Abelian differential of the second kind on C with a double pole at ∞_+, such that $\sigma + dv$ is regular at ∞_+ and we have the normalisation condition

$$\int_a \sigma = 0. \tag{6.46}$$

Define the function

$$\Omega(p) = \int^p \sigma \tag{6.47}$$

where the constant of integration is fixed by the condition

$$\Omega(p) = \epsilon\lambda + O(\lambda^{-1}) \quad \text{as } p \to \infty_+. \tag{6.48}$$

We recall some results from above: λ has a well-defined branch on $\Gamma \setminus a$, specified by its sign $\nu \sim i\epsilon\lambda$; κ acts on C by $(z,\nu) \mapsto (z,-\nu)$. The following is an analogue of [BRS89, Lemma 7.1] and the proof is identical.

Lemma 6.2. *We have the the following identity:*

$$\kappa^*\sigma = -\sigma. \tag{6.49}$$

Moreover, let

$$V := \int_b \sigma. \tag{6.50}$$

Then V coincides with the velocity vector of on JacC *(cf. equation* (5.41)*).*

Let $d\rho$ be an Abelian differential of the third kind which has simple poles at ∞_+ and ∞_- with residues 1 resp. -1. Choose l a path from ∞_+ to ∞_- which does not intersect a and such that

$$\int_a d\rho = 0. \tag{6.51}$$

We soon specify the curve l further. Note that $d\rho$ can be written as the pullback to Γ of a holomorphic differential on E given by

$$d\rho = \frac{z dz}{y - \sqrt{(z-p_+)(z-p_-)}}, \tag{6.52}$$

Define

$$\rho(p) = \int^p d\rho \tag{6.53}$$

with the constant of integration fixed by the condition

$$e^{\rho(p)} = \epsilon\lambda + O(1) \quad \text{as } p \to \infty_+. \tag{6.54}$$

Finally, define the function

$$\omega(p) = \int_{\infty_+}^p \omega, \tag{6.55}$$

where ω is a holomorphic 1-form on C.

The functions $\Omega(p), \rho(p)$ and $\omega(p)$ are multi-valued. We need their behaviour near the points ∞_-, $0_\pm$ and use the path l to do so. To this end we can require that:

(a) l passes through 0_+;

(b) the cycle $l - \kappa l$ is homologous to a.

By hypothesis the periods of σ and $d\rho$ over a are 0 and hence the functions $\Omega(p)$ and $\rho(p)$ have single-valued branches in a neighborhood of $l \cup \kappa' l$, given by the conditions (6.48) and (6.54). Moreover, we assumed $\omega(p)$ has a period $2\pi i$ along a.

We have the following analogue of [BRS89, Lemma 7.2].

Lemma 6.3.

$$\omega(\infty_-) = \pi i, \quad \int_b d\rho = \pi i. \tag{6.56}$$

Proof. We have $l - \kappa' l = a$, $\kappa'^* \omega = -\omega$ and hence

$$\int_l \omega = \int_{l-a} \kappa'^* \omega = -\int_l \omega + 2\pi i.. \tag{6.57}$$

By (4.6) again we have $\int_b d\omega = \int_l \omega$ and the claim follows. $\square$

We describe the other values near the critical points and then give expressions for the Baker-Akhiezer functions. [BRS89, Lemma 7.3]

Lemma 6.4. *The following hold:*

$$\Omega(\infty_-) = 0; \tag{6.58}$$

$$e^{\rho(p_+)} = e^{\rho(p_-)}; \tag{6.59}$$

$$e^{\rho(p)} = -\frac{q^2}{\lambda} + O(1) \quad \textit{as } p \to \infty_-, \tag{6.60}$$

where $q = e^{\rho(0_+)}$.

Proof. For the first identity note that $\Omega(\kappa' p) = \Omega(p)$ in a neighborhood of $l \cup \kappa' l$ in view of (6.48); the claim then follows from the observation $\kappa' \infty_- = \infty_-$. Secondly, note that $e^{\rho(\kappa' p)} = -e^{\rho(p)}$ by the previous lemma and the identity is clear. The third claim takes a bit more work.

Consider $d\rho$ as a differential on E. From the description at the start of the section, we can choose l such that it is symmetric with respect to the involution κ'. We have $\kappa'^* d\rho = -d\rho$, such that $\int_0^{\kappa' p} d\rho = -\int_0^p d\rho$, $\lambda(\kappa' p) = -\lambda(p)$ and $\kappa' \infty_+ = \infty_-$. This completes the proof. $\square$

Let $c \in \mathbb{C}$ such that the divisor of $\theta(\omega(p) + c)$ agrees with that of $\widehat{\mathscr{D}}$ which we defined above. We obtain the following expressions for the Baker-Akhiezer functions; the proof relies on the analycity properties we have described above and the behaviour of the θ functions. More details are given in [RS94, Section 10.2] and[BRS89, Proposition 7.4]

Proposition 6.1. *The functions $\phi_i(p, t)$ given by*

$$\begin{aligned} \phi_1(p,t) &= \frac{\theta[\mathbf{v}](\omega(p) + Vt + c)\theta(c)}{\theta(\omega(p) + c)\theta[\mathbf{v}](Vt + c)} e^{\rho(p) + \Omega(p)t}, \\ \phi_2(p,t) &= -\frac{\theta(\omega(p) + Vt + c)\theta(c)}{\theta(\omega(p) + c)\theta(Vt + c)} e^{\rho(p) + \Omega(p)t}, \end{aligned} \tag{6.61}$$

where $\mathbf{v} = (0, 1/2)$*, have the following analycity properties:*

(i) ϕ_2 *is double-valued and acquires a factor* $(-1)^{\langle \gamma, a \rangle}$ *when continued analytically along a closed curve* γ.

(ii) We have

$$\begin{aligned}
&(\phi_1, \phi_2)(\infty_-) = (0, 0), \\
&(\phi_1, \phi_2)(p) = ((i\epsilon\lambda, -i\epsilon\lambda) + O(1))e^{i\epsilon\lambda t} \quad \text{as } p \to \infty_+.
\end{aligned} \tag{6.62}$$

(iii) The divisor of poles of (ϕ_1, ϕ_2) *coincides with* $\widehat{\mathscr{D}}$.

We now give an expression for $\alpha(t)$ and follow the proof of [BRS89, Lemma 7.5]. We get from (6.43) the identity

$$\begin{aligned}
\alpha(t) &= \lim_{p \to \infty_+} \lambda(2\epsilon\phi_1(p,t) + 2\epsilon i\phi_2(p,t)) \\
&= 2\epsilon \frac{\frac{\partial}{\partial k}\theta[\mathbf{v}](\omega(p) + Vt + c)}{\theta[\mathbf{v}](Vt + c)} - 2\epsilon i \frac{\frac{\partial}{\partial k}\theta(\omega(p) + Vt + c))}{\theta(Vt + c)},
\end{aligned} \tag{6.63}$$

where $k = \lambda^{-1}$ is a local parameter at $p = \infty_+$. Write $\omega = f(k)dk$ near $p = \infty_+$. Then $V = f(0)$ by (5.41). Moreover, the function $\omega(p)$ is given near $p = \infty_+$ by

$$\omega(p) = f(0)k + O(k^2). \tag{6.64}$$

As such, the derivative in eq. (6.63) may be replaced by a time derivative. We have proven the following.

Proposition 6.2. *The time evolution of the function* $\alpha(t)$ *is given by*

$$\alpha(t) = 2\epsilon \frac{\partial}{\partial t} \log \theta[(0, 1/2)](Vt + c) - 2\epsilon i \frac{\partial}{\partial t} \log \theta(Vt + c). \tag{6.65}$$

We can now combine equation (6.44) and (6.61) to calculate the matrix $\Psi(0, t)$.

$$\Psi(0,t) = qe^{\Omega(0_+)t} \frac{\theta(c)}{\theta(\omega(0_+) + c)} \begin{pmatrix} \frac{\theta[\mathbf{v}](\omega(0_+)+Vt+c)}{\theta[\mathbf{v}](Vt+c)} & \frac{\theta[\mathbf{v}](\omega(0_+)+Vt+c)}{\theta[\mathbf{v}](Vt+c)} \\ -\frac{\theta(\omega(0_+)+Vt+c)}{\theta(Vt+c)} & \frac{\theta(\omega(0_+)+Vt+c)}{\theta(Vt+c)}. \end{pmatrix} \tag{6.66}$$

We use the symmetry condition (6.44) to determine the evolution of $\beta(t)$ and $\gamma(t)$. A discussion identical to Section 6.2.2 yields

$$L_{-1} = \sqrt{\beta^2 + \gamma^2}\Psi(0,t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi(0,t)^{-1} \tag{6.67}$$

and we now obtain

$$\cdot \begin{pmatrix} \beta(t) \\ \gamma(t) \end{pmatrix} = \sqrt{\beta^2 + \gamma^2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \Psi_{11}^2(0,t) \\ \Psi_{22}^2(0,t) \end{pmatrix} \tag{6.68}$$

where $\Psi_{11}(0,t)$ and $\Psi_{22}(0,t)$ are given by (6.66).
This rounds off our calculation, with results (6.65) and (6.68). We summarise our discussion.

Theorem 6.2. *The solutions of the equations of motion are given by* (6.65) *and* (6.68)*, where*

1. $\theta[\xi,\chi](z)$ *denotes the* 1*-dimensional theta function with characteristics* $\xi, \chi \in \mathbb{C}$ (6.45)*, and period matrix B given by the integral*

$$B = \int_b \omega; \tag{6.69}$$

2. *V is the velocity vector determined from the meromorphic function v* (6.24) *on* Γ *and equation* (5.41)*;*

3. $\omega(0_+)$ *is the Abel transform*

$$\int_{\infty_+}^{0_+} \omega \tag{6.70}$$

   cut along the canonical cycles;

4. *c is a constant such that the divisor of* $\theta(\omega(p)+c)$ *agrees with the divisor* $\widehat{\mathscr{D}}$ *of poles of the Baker-Akhiezer function.*

Chapter 7

Conclusion and outlook

To come to a conclusion, we have studied the simplest version of the quantum brachistochrone for $\mathfrak{su}(2)$, by writing the equation of motion as the limit case of a spinning top system [MC24].

Chapter 2 introduced Lie groups and their Lie algebras, at the end discussing the canonical representation of a Lie group on either its Lie algebra or dual thereof. In Chapter 3 we gave an account of Poisson manifolds that represent the most general context for studying Hamiltonian dynamics. We developed structure theory of Poisson manifolds and applied that to the canonical Poisson structure on the dual Lie algebra and Hamiltonian group actions. In Chapter 4 we presented an introduction to Riemann surfaces. First we discussed various properties of a complex structure and integration. Using the language of sheaves then simplified our discussion and helped us study divisors and line bundles on a compact Riemann surface.

Lax equations were studied in Chapter 5, particularly in the dual Lie algebra equipped with the canonical Poisson structure. First we discussed how the integration of the equation of motion can be rephrased in terms of a factorisation problem, analogous to the inverse scattering method. We looked at Hamiltonian reduction and used those ideas to construct integrable spinning top systems. Thereafter we extensively studied the factorisation problem in terms of the spectral data of a Lax equation, yielding an explicit solution in terms of algebraic geometric means. Chapter 6 focused on the quantum brachistochrone (QB) problem and solved the resulting equation of motion for the simplest case of $\mathfrak{su}(2)$.

This project was initially aimed at solving the QB problem for a non-trivial decomposition of the Lie algebra $\mathfrak{su}(3)$. That calculation was ultimately scrapped in the final version of this project in view of many difficulties: the resulting expressions quickly become convoluted and the higher dimensionality introduces too much complexity to handle. Moreover, it seems highly unlikely that an algorithm can be produced for finding higher dimensional solutions in this manner, as suggested in [MC24]; the algebraic geometry seems too specific and difficult to find a general solution, as suggested by the lengthy computation in Section 6.2.

Furthermore, it remains unclear to what extend the limit $\epsilon \to 0$ can actually be taken in the resulting solutions; one would need to describe the asymptotic behaviour of the relevant integrals and θ functions, which seems analytically intractable and goes beyond the scope of this text.

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Appendix **A**

Appendix

A Distributions and the Frobenius theorem

A distribution generalises the notion of a frame of vector fields. Whereas any vector field on a smooth manifold leads to a decomposition into integral curves, the global Frobenius theorem states that any smooth smooth distribution admits a decomposition into integral manifolds, as long as it is *involutive*.

We follow [Var13, Section 1.3]; all smooth structures in that text are replaced by smooth structures.

Definition A.1. A *singular distribution* D on an open subset $U \subseteq M$ assigns to every point $p \in U$ a subspace $D_p \subseteq T_pM$. The distribution is called nonsingular if $\dim D_p = \dim D_q$ for all $p, q \in U$ and in this case $\dim D_p$ is called the *dimension* of D.

All distributions are assumed to be nonsingular unless stated otherwise.

A distribution D of dimension m is said to be *smooth* if for every $p \in U$ there exists a coordinate neighborhood containing p and a set of vector fields $\{X_1, \ldots, X_m\} \subset \mathscr{U}$ such that for all $q \in U$

$$T_qM = \operatorname{span}\{(X_1)_q, \ldots, (X_m)_q\}.$$

The vector fields $X_1, \ldots, X_m$ is called a local frame for the distribution.

Example A.1. Any nonvanishing vector field on a smooth manifold determines a smooth distribution of dimension 1.

An *integral manifold* of D is an immersed submanifold $N \subseteq M$ such that $T_pN = D_p$ for all $p \in N$. The distribution is then said to be *integrable* if each point is contained in an integral manifold.

Definition A.2. A distribution D is said to be *involutive* at $p \in M$ if there exists a neighborhood U and a local frame of vector vector which is involutive.

Any integrable is then clearly involutive. The *Frobenius Theorem* shows that the converse is also true; in fact, it shows that any involutive distribution is *completely integrable*: any point $p \in M$ admits a local frame of vector fields $X_1, \ldots, X_{\dim D_p}$ such that $D_p = \mathrm{span}((X_1)_p, \ldots (X_{\dim D_p})_p)$. A simple coordinate argument shows that completely integrable implies integrable.

The straightening theorem gives a simple local form for vector fields; the following result generalises this to involutive frames. A direct proof with induction can be found in [Var13, Lemma 1.3.2].

Proposition A.1. *Suppose M is a smooth manifold, $p \in M$ and U a neighborhood of p. Given a set of linearly independent vector fields $X_1, \ldots, X_k \in \mathscr{X}(U)$ such that $[X_i, X_j] = 0$, $1 \le i, j \le k$, then there exist coordinates (x^i) around p such that*

$$X_i = \frac{\partial}{\partial x^i} + \sum_{1 \le j < i} a_j \frac{\partial}{\partial x^j},$$

where the a_j are smooth functions in a coordinate neighborhood of p.

The reader is now invited to use Proposition A.1 to proof a local version of the Frobenius theorem. Alternatively, a proof of the following is given in [Var13, p. 1.3.3].

Theorem A.1 (Local Frobenius Theorem). *Let D be an involutive distribution on a smooth manifold M. For any $p \in M$ there exists a coordinate neighborhood U such that the restriction $D|_U$ is a completely integrable distribution.*

The remarkable statement of the global Frobenius theorem is that these integral can be extended maximal immersed submanifolds, called the *leaves* of the distribution; the smooth manifold then admits a decomposition into these leaves and is called a D-foliation, or simply foliation. More precisely, this amounts to the following construction: (i) define a system of open sets of unions of integral manifolds (ii) show that this induces a finer topology and that this space satisfies the second axiom of countability (iii) the leaves are then the connected components of the resulting manifold (iv) specify a smooth structure on these integral manifolds. Details can be found in [Var13, Lemma 1.3.4, Lemma 1.3.5 and Theorem 1.3.6]. We only state the final result.

Theorem A.2 (Frobenius Theorem). *Let M be a smooth manifold and D an involutive distribution. Then there exists a collection of disjoint maximal integral* *submanifolds, which are immersed submanifolds, called a* foliation $\mathcal{F} = (\mathcal{F}_i)_{i\in I}$, *such that*

$$M = \bigcup_{i\in I} F_i.$$

Moreover, integral manifolds of D are open submanifolds of exactly one leaf $\mathcal{F}_i$.

B Elementary sheaves and Céch cohomology

Sheaves conveniently describe the local algebraic structure of a topological space. We introduce these objects and their natural cohomology. Céch cohomology allows us to resolve the topological space. We discuss [For12, Chapter 6, Chapter 15]. This serves as a refresher for the relevant definitions we encounter in the text.

Definition B.1. A *presheaf* of Abelian groups $\mathcal{F}$ on a topological space X is for any collection of open sets $\mathcal{U} = (U_i)_{i\in I}$

(i) An Abelian group $\mathcal{F}(U_i)$ for each $I \in I$;

(ii) A collection of *restriction maps* $(\rho_{ij})_{i,j\in I}$ such that

$$\rho_{ii} = \text{ id}_{F(U_i)} \quad \text{for all } i \in I,$$
$$\rho_{ij} \circ \rho_{jk} = \rho_{ik} \text{for every } i, j, k \text{ with } UK \subset U_j \subset U_i.$$

A sheaf has additional regularity conditions which are essential in all applications. We use the notation $U_{ij} = U_i \cap U_j$.

Definition B.2. A *sheaf* $\mathcal{F}$ of Abelian groups on a topological space X is a presheaf of Abelian groups that satisfies the following properties: for any open collection $\mathcal{U} = (U_i)_{i\in I}$ and open set $U = \bigcup_i U_i$ we have

(i) if $f, g \in \mathcal{F}(U)$ and $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$;

(ii) given any elements $f_i \in \mathcal{F}(U_i)$ such that

$$f_i|_{U_{ij}} = f_j|_{U_{ij}}$$

for all $i, j \in I$, then there exists an element $f \in \mathcal{F}(U)$ such that $f|$

Definition B.3 (coboundary operator)**.** Suppose X is a topological space, $\mathcal{F}$ a sheaf of Abelian groups and $\mathcal{U} = (U_i)_{i\in I}$ an open cover of X. Denote

$$U_{i_0,\dots,i_k} = U_{i_0} \cap \cdots \cap U_{i_k}, \quad k \in \mathbb{N}, \quad i_0, \dots, i_k \in I.$$

Let $C^k(\mathcal{U}, \mathcal{F})$ denote the free Abelian group given as follows: $C^k(\mathcal{U}, \mathcal{F})$ is generated by the elements f called k-cochains:

$$f = (f_{i_0 \dots i_k}) \in \Pi_{j=i_0,\dots,i_k} \mathcal{F}(U_j).$$

The *coboundary operator* $\delta : C^k(\mathcal{U},\mathcal{F}) \to C^{k+1}(\mathcal{U},\mathcal{F})$ sends any k-cochain () to the $k+1$-cochain

$$(\delta f)_{i_0 \ldots i_{k+1}} = \sum_{j=1}^{k} \sum_{i_j \le i_m \le i_{k+1}} (-1)^m (f_{i_0 \ldots \widehat{i_m} \ldots i_{k+1}}),$$

where $\widehat{\cdot}$ means that the index is forgotten.

In particular we have

$$\begin{aligned} \delta : C^0(\mathcal{U},\mathcal{F}) &\longrightarrow C^1(\mathcal{U},\mathcal{F}), \\ (f_i)_i &\longmapsto (f_i - f_j)_{ij}, \end{aligned}$$

and

$$\begin{aligned} \delta : C^0(\mathcal{U},\mathcal{F}) &\longrightarrow C^1(\mathcal{U},\mathcal{F}), \\ (f_{ij})_i &\longmapsto (f_{ij} - f_{ik} + f_{jk})_{ijk}. \end{aligned}$$

One can show that $\delta^2 = 0$. It is therefore natural to define

$$\begin{aligned} Z^k(\mathcal{U},\mathcal{F}) &:= \ker\left(C^k(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} C^{k+1}(\mathcal{U},\mathcal{F})\right), \\ B^k(\mathcal{U},\mathcal{F}) &:= \operatorname{im}\left(C^k(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} C^{k+1}(\mathcal{U},\mathcal{F})\right), \end{aligned}$$

and consider the k-th Céch cohomology groups

$$H^k(\mathcal{U},\mathcal{F}) := \frac{Z^k(\mathcal{U},\mathcal{F})}{B^k(\mathcal{U},\mathcal{F})}$$

Elements of $Z^k(\mathcal{U},\mathcal{F})$ resp. $B^k(\mathcal{U},\mathcal{F})$ are called k-cocycles resp. k-coboundaries.

Definition B.4. The k'th *Céch cohomology group* of the sheaf $\mathcal{F}$ on the space X is the projective limit of Abelian groups

$$H^k(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} H^k(\mathcal{U},\mathcal{F}),$$

with respect to a refinement of open covers

Definition B.5. Let $\mathcal{F}$ be a sheaf of Abelian groups on the topological space X and U a neighborhood of $x \in X$. The *stalk* of $\mathcal{F}$ at x is the projective limit of Abelian groups

$$\mathcal{F}_x = \varinjlim V \rho_{UV}(\mathcal{F}(U))$$

with respect to inclusion of neighborhoods $V \ni x$, where ρ_{UV} denotes the restriction map.

Definition B.6. Let X be a topological space. A *short exact sequence* of sheaves $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ on X, written

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

is for every $x \in X$ a stalkwise short exact sequence

$$0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0.$$

A short exact sequence of sheaves gives rise to a long exact sequence of their cohomology groups via connecting homomorphisms; we omit a discussion here, some details can be found in [For12, Theorem 15.12] for $k = 1$ and $k = 2$.