Multifield Inflation: Rapid-Turn Attractors and Scaling Solutions

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Abstract

Inflation is a period of accelerated expansion of the universe, taking place a fraction of a second after the Big Bang. The expansion is driven by one or more scalar fields acting on spacetime. The possibility of multiple such scalar fields is motivated by existing results from for example string theory. In multifield inflation, the field-space has a non-trivial Riemannian geometry, generalising single-field inflation. As the evolution of the scalar fields is governed by differential equations, it may be analysed as a dynamical system. In this thesis, two-field models of inflation are studied in the mathematical context of dynamical systems and differential geometry. This is based on the fundamentals of general relativity, which are introduced from both a mathematical and physical point of view. A dynamical system is derived from the background equations of motion in a FLRW spacetime, based on concepts from dynamical systems theory and following examples in existing literature. This in done in two distinct ways. First, the existence of an attractor for rapid-turn inflation is studied. Second, different types of scaling solutions are studied, which are trajectories in field-space along which the rate of change of the inverse Hubble parameter is constant. The two different approaches are then compared.

Contents

Chapter 1

Introduction

The evolution of the universe is the main concern of the field of cosmology. Cosmology encompasses a wide variety of topics; from the formation of the first stars to the coagulation of black holes. One particular concept of interest is inflation: the exponential expansion of the universe, taking place a fraction of a second after the Big Bang. The concept of inflation provides solutions to several problems associated with standard Big Bang cosmology, such as the horizon problem and the flatness problem (see e.g. the discussion in [Baumann, 2012]). In addition, it provides an elegant mechanism for generating density fluctuations in the primordial universe, transforming microscopic quantum fluctuations into the anisotropies observed in the cosmic microwave background (CMB) [Mukhanov and Chibisov, 1981]. Accidentally discovered in 1965, the CMB consists of microwave radiation that fills up the entire observable universe. It is understood to be an afterglow of the Big Bang, and is thus an important source of data for the early universe. The fluctuations in the CMB in turn led to the formation of the large-scale structures observed in the universe today (see e.g. [Durrer, 2008]). At the basis of the theory of cosmology lies General Relativity. First formulated by Albert Einstein, general relativity provides the mathematical framework for the dynamics and structure of spacetime. Concepts such as the spacetime manifold, its metric and differential structure are defined, supported by the rigorous mathematical theory of differentiable geometry.

Inflationary models posit that very shortly after the Big Bang, the universe underwent a period of exponential expansion. In the simplest models for inflation, the expansion is driven by a single hypothetical scalar field ϕ (the *inflaton*) rolling down a potential [Linde, 1982], [Albrecht and Steinhardt, 1982]. To ensure the occurrence of accelerated expansion, the scalar field must slowly roll down the potential, hence these models are called *single-field slow-roll models*. A minimal requirement for this is that the inverse of the Hubble parameter (H^{-1}) must be near-constant. These types of models have been very successful; being consistent with for example observations of the CMB by the *Planck* satellite (see for example [Akrami et al., 2020]).

In light of recent observational and theoretical developments, it has become relevant to consider more complex models of inflation. One way to extend single-field inflation models, is by considering models with not one, but multiple scalar fields driving the inflation. The existence of these multiple scalar fields is suggested by high-energy theories (as discussed by [Baumann and McAllister, 2014]). When considering a model with n scalar fields, these fields are seen as coordinates on an n-dimensional smooth manifold, called *field-space* (see e.g. [Gong, 2017]). In general, the metric on field-space is assumed to be non-flat, as suggested by many of the high-energy theories.

In this work, two-field models of inflation are considered. The mathematical theory of dynamical systems is used to study the models, by writing them as a system of ordinary differential equations. In order to reduce the complexity of the analysis, the existence of two specific types of solutions is studied. The first approach (in Chapter 5) focuses on finding an attractor solution with a large, slowly varying turn rate. The second approach is to study the stability of so-called *scaling solutions*; solutions that are characterised by the time-derivative of the inverse Hubble parameter H^{-1} being constant along the solution.

This thesis is organised as follows. In Chapter 2, the foundations of differential geometry are described. Concepts needed for understanding the mathematical structure of general relativity are introduced, such as smooth manifolds, tensors and the Christoffel connection. In addition, the connection between these abstract mathematical concepts and the formulation of general relativity is made. Chapter 3 gives an introduction in cosmology, and specifically (multifield) inflation. Next, in Chapter 4, a short introduction in the theory of dynamical systems is given, explaining concepts, such as attractors, that are needed for the analysis of the inflationary models. In Chapters 5 and 6, the analysis of two-field models of inflation is approached in two ways, as described above. Finally, in Chapter 7differences between the two approaches are highlighted. In Chapter 8, conclusions and an outlook are offered.

Chapter 2

General Relativity as Geometry

General relativity is a geometric theory of gravity, in which gravity is considered to be a property of four-dimensional spacetime instead of as a force. One of the fundamental ideas of general relativity is that time and space are not seen as separate variables, but rather as part of a four-dimensional spacetime manifold. This means that general relativity has a solid basis in differential geometry. In this chapter the basic geometric concepts underlying general relativity are introduced. We will for example see the definition of a manifold, along with its tangent and cotangent spaces. After that, we will consider several notions defined on a manifold, like tensors, the metric and the covariant derivative. Finally, we will relate all these mathematical concepts to the theory of general relativity. In this chapter, we follows the exposition in [Lee, 2003] (Sections 2.1-2.4) and [Lee, 2019] (Sections 2.5-2.7).

Remark. Before starting with the construction of a manifold, we introduce an important convention, the *Einstein summation convention*. The statement of this convention is that every repeated index in an equation should be summed over. For example, if $i \in \{1, 2, 3\}$, we write

$$y = c_i x^i \quad \text{to mean} \quad y = \sum_{i=1}^{3} c_i x^i. \tag{2.1}$$

2.1 Manifolds

Roughly speaking, a manifold is a topological space that resembles (is diffeomorphic to) Euclidean space at each point. An important subset of manifolds are smooth manifolds, on which calculus can be done in a standard way. The four-dimensional spacetime manifold is an example of this class. We now define the concept of a manifold in a more rigorous way. Starting with the definition of a topological manifold, we impose some additional structure on it, which is then used to define a smooth manifold.

Definition 2.1.1. A topological space M is called a *topological manifold of dimension*

n if it satisfies:

1. M is Hausdorff.
2. M is second-countable, i.e. there exists a collection $\mathcal{U} = \{U_i\}$ of countably many open sets $U_i \subset X$, such that every open set $V \subset X$ can be written as $V = \bigcup_{U_i \subset V} U_i$.
3. M is locally Euclidean of dimension n, i.e. each $p \in M$ has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 2.1.2. Let M be a topological manifold of dimension n. A *coordinate chart* is a pair (U, ϕ), where $U \subseteq M$ is open and $\phi : U \to V$ is a homeomorphism with $V = \phi(U) \subseteq \mathbb{R}^n$ open.

A coordinate chart is thus only a function on a certain open subset of the manifold. In order to impose a useful structure on the entire topological manifold, we need to completely cover it with coordinate charts, such that those charts are compatible in a certain sense. We now formalise these notions.

Definition 2.1.3. Suppose that (U_1, ϕ_1) and (U_2, ϕ_2) are coordinate charts for a topological n-manifold M, such that $U_1 \cap U_2 \neq \emptyset$. Then the homeomorphism given by

$$\phi_2 \circ \phi_1^{-1} : \phi_1 (U_1 \cap U_2) \to \phi_2 (U_1 \cap U_2) \tag{2.2}$$

is called the *transition map* from ϕ_1 to ϕ_2.

Definition 2.1.4. Two coordinate charts (U_1, ϕ_1) and (U_2, ϕ_2) on a topological n-manifold M are *smoothly compatible* if either $U_1 \cap U_2 = \emptyset$, or the transition map $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ is a diffeomorphism.

Remark. Recall the following definitions: If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open, then a function $F : U \to V$ is *smooth* or C^∞ if every component function of F has continuous partial derivatives of every order. Furthermore, F is called a *diffeomorphism* if F is a homeomorphism with smooth inverse.

Definition 2.1.5. A collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ on a topological n-manifold M is called an *atlas* for M if the charts cover M, i.e. $\bigcup_\alpha U_\alpha = M$. An atlas $\mathcal{A}$ is *smooth* if any two charts in $\mathcal{A}$ are smoothly compatible.

Having defined a smooth atlas, we now need to account for the fact that there are different possibilities for defining a smooth atlas on M. This is done by defining a specific atlas that contains 'all' compatible coordinate charts.

Definition 2.1.6. Let M be an n-dimensional topological manifold, $\mathcal{A}$ a smooth atlas on M and (U, ϕ) any chart that is smoothly compatible with all charts in A. Then $\mathcal{A}$ is called *maximal* if $(U, \phi) \in \mathcal{A}$.

Having eliminated the technical issue of having different possible smooth atlases on M, we can finally define a smooth manifold:

Definition 2.1.7 (Smooth Manifold)**.** Let M be an n-dimensional topological manifold and $\mathcal{A}$ a maximal smooth atlas. Then $\mathcal{A}$ is called a *smooth structure on* M and the pair $(M, \mathcal{A})$ is called a *smooth manifold* or *differentiable manifold.*

Remark. Often we will simply refer to M as 'being' the smooth manifold.

2.1.1 Coordinate Representation

Now that the smooth structure on a manifold M has been defined, we can consider a convenient way of expressing points on the manifold. This is done by using a (local) coordinate representation, supplied by the charts in the smooth structure on M:

Definition 2.1.8. Let M be a differentiable manifold and $\mathcal{A}$ the maximal smooth atlas. If $(U, \phi) \in \mathcal{A}$, then it is called a *smooth chart* and the coordinate map ϕ is called a *smooth coordinate map.*

Having chosen a certain smooth chart (U, ϕ) on M, the smooth coordinate map $\phi : U \to V$ gives a certain identification between U and V. Using this identification, a point $p \in U \subseteq M$ can be represented by its coordinates $\phi(p) = \left(x^1, \ldots, x^n\right)$. The n-tuple $\left(x^1, \ldots, x^n\right)$ is then said to be the *(local) coordinate representation for* p.

2.1.2 Smooth Functions and Maps

It is useful to not only define when a manifold is smooth, but also when a mapping on the manifold is smooth.

Definition 2.1.9. Let M be a n-dimensional smooth manifold and $k > 1$ an integer. A function $f : M \to \mathbb{R}^k$ is called *smooth* if for every $p \in M$ there exists some smooth coordinate chart (U, ϕ) such that $p \in U$ and such that the composition $f \circ \phi^{-1}$ is smooth on $\phi(U) \subseteq \mathbb{R}^n$.

Remark. An important class of smooth functions is that where $k = 1$. We denote the set of all such real-valued functions $f : M \to \mathbb{R}$ by $C^\infty(M)$. This is, in fact, a vector space over $\mathbb{R}$ under summation and scalar multiplication.

Just like points $p \in M$, functions $f : M \to \mathbb{R}^k$ can be represented in terms of the local coordinates associated with some coordinate chart for M. Suppose we have a function $f : M \to \mathbb{R}^k$ and a chart (U, ϕ), then the function $\hat{f} : \phi(U) \to \mathbb{R}^k$ given by $\hat{f}(x) = f \circ \phi^{-1}(x)$ is called the *coordinate representation of f*. Note that by definition 2.1.9, f is smooth if and only if its coordinate representation $\hat{f}$ is smooth for some chart around each point $p \in M$.

A generalisation of the notion of a smooth function $f : M \to \mathbb{R}$ on a manifold, is the definition of a smooth map *between* manifolds (as $\mathbb{R}^k$ is a smooth manifold itself).

Definition 2.1.10. Let M, N be differentiable manifolds. A map $F : M \to N$ is *smooth* if for every $p \in M$, there exist smooth charts (U, ϕ) and (V, ψ) with $p \in U$, $F(p) \in V$ and $F(U) \subseteq V$, such that the composition $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

2.2 Tangent Space

We have seen the basic definition of a smooth manifold, and have defined functions on it. However, in order to perform basic calculus, we need to define some more structure. This we will now introduce, in the form of a tangent space T_p, defined at every point p on a manifold. The tangent space is a vector space over $\mathbb{R}$ that is spanned by the tangent vectors to M at p.

Definition 2.2.1 (Tangent Space)**.** Let M be an n-dimensional smooth manifold and $p \in M$. A linear map $D : C^{\infty}(M) \to \mathbb{R}$ is called a *derivation at* p if D obeys the Leibniz rule:

$$\forall f, g \in C^{\infty}(M): \quad D(fg) = D(f) \cdot g(x) + f(x) \cdot D(g). \tag{2.3}$$

The *tangent space* T_pM to M at p is the set of all derivations at p and an element of T_pM is called a *tangent vector at* p.

As the definition of the tangent space is somewhat abstract, it can be helpful to be able to visualise the tangent vectors as arrows that attached to the base point p and are tangent to the manifold M, see Figure 2.1.

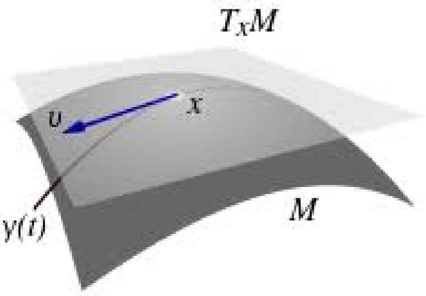

Figure 2.1: A smooth manifold M with the tangent space T_xM to the point $x \in M$. In addition, $v \in T_xM$ is a tangent vector at x.
Credit: Wikipedia Tangent space (consulted 13 Juni 2024).

Settting the operations (for D_1, D_2 derivations, $f, g \in C^{\infty}(M)$ and $\lambda \in \mathbb{R}$)

$$(D_1 + D_2)(f) \coloneqq D_1(f) + D_2(f), \text{ and} \tag{2.4a}$$

$$(\lambda \cdot D_1)(f) \coloneqq \lambda \cdot D(f), \tag{2.4b}$$

the tangent space is turned into a vector space of dimension n. We do not prove this. However, it is worth noting that the proof makes use of the coordinate charts that map open subsets $U \subseteq M$ onto open subsets $V \subseteq \mathbb{R}^n$.

2.2.1 Coordinates on the Tangent Space

As the tangent space is a vector space, it is natural to define a basis for it. The standard way of doing this, is by using local coordinates that correspond to a certain coordinate map:

Definition 2.2.2 (Coordinate Basis)**.** Let M be an n-dimensional smooth manifold. Given a coordinate map $\phi = \left(x^1, x^2, \ldots, x^n\right) : U \to \mathbb{R}^n$ with $U \subseteq M$ open and $p \in U$, the *coordinate basis* for T_pM is given by $\{\partial_1, \ldots, \partial_n\} = \{\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\}$, such that for every smooth $f : M \to \mathbb{R}$ and for all $i \in \{1, \ldots, n\}$ one has

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) \coloneqq \left(\frac{\partial}{\partial x^i}\left(f \circ \phi^{-1}\right)\right)(\phi(p)) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}), \tag{2.5}$$

where $\hat{f}$ and $\hat{p} = \left(p^1, \ldots, p^n\right)$ are the coordinate representations of f and p, respectively.

Using the coordinate basis, any tangent vector $v \in T_pM$ can be uniquely written as a linear combination

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p, \tag{2.6}$$

where the components v^i of v are given by $v^i = v\left(x^i\right) = v^j \frac{\partial x^i}{\partial x^j}(p)$. Note that the Einstein summation convention was used in equation 2.6.

We have seen how the coordinate basis is defined for a choice of coordinate chart (U, ϕ) and how the components of a vector can be determined for this coordinate basis. However, what if we were to choose another coordinate chart (V, ψ), leading to another coordinate basis and other vector components? How can the two different coordinate representations for some tangent vector to a point p on the manifold be related?

Suppose we have the smooth coordinate charts (U, ϕ) and (V, ψ) on M and $p \in U \cap V$. Denote the coordinate functions of ϕ by x^i and those of ψ by $x^{i'}$. The basis vectors for the coordinate bases are then related as follows:

$$\partial_i = \left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial x^{i'}}{\partial x^i}(\hat{p}) \left.\frac{\partial}{\partial x^{i'}}\right|_p = \frac{\partial x^{i'}}{\partial x^i}\partial_{i'}, \tag{2.7}$$

and the components of a vector $v \in T_pM$, given by $v = v^i\partial_i = v^{i'}\partial_{i'}$ transform as

$$v^{i'} = \frac{\partial x^{i'}}{\partial x^i} v^i. \tag{2.8}$$

2.2.2 The Tangent Bundle

Thus far, we have only defined the tangent space per point on a manifold. It would, however, be useful to extend these notions to the entire manifold. This is where the notion of a *tangent bundle* becomes useful.

Definition 2.2.3 (Tangent Bundle)**.** Let M be a smooth manifold. The *tangent bundle* TM of M is the disjoint union of the tangent spaces to all points of M, that is

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM. \tag{2.9}$$

We will not go into much detail about the tangent bundle, but it is worth noting that if M is a smooth n-dimensional manifold, then the tangent bundle TM comes equipped with a natural topology and smooth structure, such that it is again a smooth manifold, although of dimension $2n$ instead of n.

2.2.3 Vector Fields

The concept of a vector field is well-known for Euclidean spaces: a vector field is a continuous map that attaches an 'arrow' to each point. We now need to extend this concepts to smooth manifolds. In this context, a vector field is some continuous map assigning a tangent vector (an element of the tangent space) to each point (see Figure 2.2):

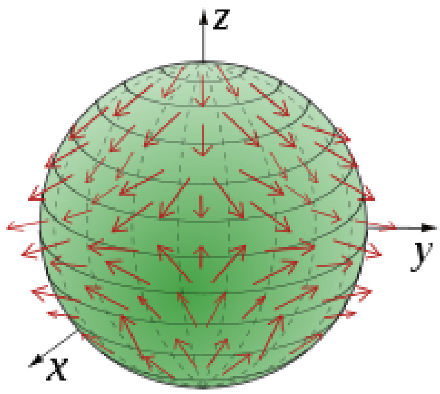

Figure 2.2: A vector field (visualised by the red arrows) defined on a manifold. In this case, the manifold is a simple 2-sphere.
Credit: Wikipedia Tangent space (consulted 13 Juni 2024).

Definition 2.2.4 (Vector Field)**.** Suppose M is a smooth manifold. A *vector field* is a continuous map $X : M \to TM$, where $p \mapsto X_p$, such that $X_p \in T_pM$ for every $p \in M$.

Remark. In addition, we can define a *smooth vector field* as a vector field with a smooth map $X : M \to TM$.

Just like we did before for any vector in a tangent space, we can define the component functions of a vector field X:

Definition 2.2.5. Suppose M is a smooth manifold of dimension n and $X : M \to TM$ a vector field. If (U, ϕ) is a smooth coordinate chart for M, the value X_p can be written in terms of n coordinate basis vectors and *component functions* $X^i : U \to \mathbb{R}$ as

$$X_p = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p = X^i(p)\partial_i, \tag{2.10}$$

where (x^i) are the local coordinates associated to ϕ.

Not only can the vector field be expressed in terms of components, but in fact the smoothness of the field is directly related to the smoothness of its component functions. This fact is expressed in the following Lemma:

Lemma 2.2.6.
Let M be a smooth manifold and $X : M \to TM$ a vector field. Suppose that (U, ϕ) is a smooth coordinate chart on M, then the restriction $X_{|_U} : U \to TM$ is smooth if and only if the component functions $X^i : U \to \mathbb{R}$ with respect to this chart are smooth.

An important example of a vector fields are the coordinate vector fields we previously used to define the coordinate basis for the tangent space:

Example 2.2.7.
Let (U, ϕ) be a smooth coordinate chart on M. The mapping

$$p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p \tag{2.11}$$

defines the i-th coordinate vector field *on U. Obviously, the component functions are constant, as the coordinate vector fields precisely define the coordinate basis. Using this and Lemma 2.2.6, we see that the coordinate vector fields are trivially smooth.*

2.3 Cotangent Space

In Section 2.2, a vector space called the tangent space was defined at each point p on a smooth manifold M. Another space that can be defined at each point p is the *cotangent space*. The cotangent space does not consist of vectors, but of linear functionals, and its definition makes use of that of the tangent space. Before defining the cotangent space, we define the dual vector space and consider a lemma about the bases for a vector space and its dual:

Definition 2.3.1. Let V be a vector space over a field F. Then the *dual space* V^* is the set of all linear functionals $\phi : V \to F$.

Elements of the dual space are called *covariant vectors*, or *dual vectors*. The dual V^* also becomes a vector space over F if it is equipped with the following addition and scalar multiplication:

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x) \tag{2.12a}$$

$$(a\varphi)(x) = a(\varphi(x)), \tag{2.12b}$$

for all $\varphi, \psi \in V^*$, $x \in V$ and $a \in F$.

In addition, if V has some finite dimension n, its dual V^* is of the same dimension (see e.g. [Rynne and Youngson, 2000]). Since the dual consists of linear functionals acting on elements of V, the bases for V and its dual V^* are related to each other in a specific way, as stated by the following Proposition.

Proposition 2.3.2.
Let V be an n-dimensional vector space and denote its dual by V^*. Suppose that $\{\hat{e}_{(1)}, \ldots, \hat{e}_{(n)}\}$ is a basis for V and let $\hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(n)} \in V^*$ such that

$$\hat{\theta}^{(\mu)}\left(\hat{e}_{(\nu)}\right) = \delta^{\mu}_{\nu}. \tag{2.13}$$

Then $\{\hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(n)}\}$ is a basis for V^*, called the dual basis *to $\left(\hat{e}_{(\nu)}\right)$, and therefore $\dim V^* = \dim V = n$.*

Definition 2.3.3 (Cotangent Space)**.** Let M be a smooth manifold. For every $p \in M$, the *cotangent space at p T_p^*M*, is defined to be the dual space of the tangent space T_pM:

$$T_p^*M = (T_pM)^*, \tag{2.14}$$

such that every $f \in T_p^*M$ is a linear map $f : T_pM \to \mathbb{R}$.

Since the cotangent space is defined as the dual of the tangent space, the basis vectors for T_pM give rise to the set of dual basis vectors for T_p^*M as discussed in Proposition 2.3.2.

2.3.1 Coordinates on the Cotangent Space

Following Proposition 2.3.2, we know that the basis of the cotangent space is induced by the basis of the tangent space. Suppose we have smooth coordinates x^i on an open subset $U \subseteq M$. Then at every point $p \in M$, the coordinate basis for T_pM induces a dual basis for T_p^*M. For the moment, we do not yet know what the dual basis vectors induced by the coordinate basis are given by, so we denote them by $\left(\lambda^i|_p\right)$. In Section 2.3.4, we will see that the dual basis vectors are in fact given by the differentials of the coordinate functions x^i. Using the dual basis vectors, any covector $\omega \in T_p^*M$ can be decomposed in terms of its components as $\omega = \omega_i \lambda^i|_p$, which are given by (using Proposition 2.3.2)

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right). \tag{2.15}$$

Now suppose we choose another set of smooth coordinates $(x^{i'})$ on another open subset $U' \subseteq M$ such that $p \in U'$. How are the components $\partial/\partial x^{i'}$ and $\partial/\partial x^i$ of some covector $\omega \in T_p^*M$ then related?

Since the dual basis is induced by the basis for the tangent space, we can use equation 2.7:

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \omega\left(\frac{\partial x^{i'}}{\partial x^i}(p)\left.\frac{\partial}{\partial x^{i'}}\right|_p\right) = \frac{\partial x^{i'}}{\partial x^i}(p)\omega_{i'}. \tag{2.16}$$

2.3.2 The Cotangent Bundle

Just as for the tangent spaces, it is useful to be able to think about cotangent spaces as something defined on the whole manifold. For this, the notion of a *cotangent bundle* is used:

Definition 2.3.4 (Cotangent Bundle)**.** Let M be a smooth manifold. The *cotangent bundle* T^*M of M is the disjoint union of cotangent spaces to all points of M, i.e.

$$T^*M = \bigsqcup_{p \in M} T_p^*M = \bigcup_{p \in M} \{p\} \times T_p^*M. \tag{2.17}$$

As with the tangent bundle, the cotangent bundle comes with a natural topology and smooth structure, turning it in a $2n$-dimensional smooth manifold.

2.3.3 Covector Fields

Using the cotangent bundle, we can now define a covector field, which assigns a covector to each point on the manifold in a continuous way.

Definition 2.3.5 (Covector Field)**.** Suppose M is a smooth manifold. A *covector field* is a continuous map $\omega : M \to T^*M$, where $p \mapsto \omega_p$, such that $\omega_p \in T_p^*M$ for every $p \in M$.

Remark. In the same way as for vector fields, *smooth* covector fields can be defined, depending on the continuity and smoothness of the function ω.

In any smooth coordinate chart, we can define component functions of the covector field:

Definition 2.3.6. Let M be a smooth manifold and suppose we have smooth local coordinates on an open subset $U \subseteq M$. A covector field ω can then be written in terms of coordinate coordinate fields $\left(\lambda^i\right)$ as $\omega = \omega_i \lambda^i$. Here, we have n functions $\omega_i : U \to \mathbb{R}$, called *component functions of* ω, given by

$$\omega_i(p) = \omega_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right). \tag{2.18}$$

In fact, the smoothness of the covector field ω is directly related to the smoothness of its component functions:

Lemma 2.3.7.
Let M be a smooth manifold and $\omega : M \to T^*M$ a covector field. Then ω is smooth if and only if the component functions $\omega_i : U \to \mathbb{R}$ are smooth in every smooth coordinate chart (U, ϕ).

Finally, we will introduce a convenient concept for bases of the (co)tangent space:

Definition 2.3.8 ((Co)frame)**.** Let M be a smooth manifold and $U \subseteq M$ open. A *local frame for M over U* is an ordered tuple of vector fields $(E_1, \dots, E_n)$ on U such that $(E_i|_p)$ is a basis for T_pM at every $p \in U$. Completely analogous, a *local coframe for M over U* is an ordered tuple of covector fields $(\varepsilon^1, \dots, \varepsilon^n)$ on U such that $(\varepsilon^i|_p)$ is a basis for T_p^*M at every $p \in U$. The tuple of (co)vector fields forms a *global (co)frame* if $U = M$.

Example 2.3.9.

Let (U, ϕ) be any smooth chart on M. Then the coordinate vector fields (∂_i) form a local frame over U, called the coordinate frame*, just as the coordinate (co)vector fields $\left(\lambda^i\right)$ form a local coframe over U, called the* coordinate coframe*. Since the component functions of the fields are constants, the coordinate (co)frame is smooth by Lemma 2.3.7.*

As we saw before, the bases of a vector space and its dual are closely related. So, let U be an open subset of a smooth manifold M and $(E_1, \ldots, E_n)$ a local frame for TM over U. Then there is a unique local coframe $(\varepsilon^1, \ldots, \varepsilon^n)$ over U such that $\left(\varepsilon^i|_p\right)$ is the basis dual to $(E_i|_p)$ for every $p \in U$. This coframe is then *dual* to the frame (E_i). For example, the coordinate frame and coordinate coframe are dual to each other.

2.3.4 The Differential of a Function

For a function f on the Euclidean space $\mathbb{R}^n$, the gradient is given by $\left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)$, so by a vector field of which the components consist of the partial derivatives of f. However, this form of the gradient is strongly dependent of the choice of coordinates. To consider partial derivatives in a coordinate-independent way, we may interpret them as components of a covector field, which we defined in Section 2.3.5.

Definition 2.3.10. Let M be a smooth manifold and $f : M \to \mathbb{R}$ a smooth function. The *differential of f* is a covector field $\mathrm{d}f$ defined by

$$\mathrm{d}f_p(v) = vf \quad \text{for } v \in T_pM. \tag{2.19}$$

It can be shown that the covector field $\mathrm{d}f$ is smooth. Having seen the abstract definition, we can investigate what the coordinate representation of the differential looks like: Let $\phi = \left(x^i\right)$ be a smooth coordinate map on an open subset $U \subseteq M$ and (λ^i) the coordinate coframe associated with the coordinates. Now suppose that $\mathrm{d}f$ can be written as $\mathrm{d}f_p = A_i(p)\lambda^i|_p$ for some functions $A_i : U \to \mathbb{R}$. Using the definition of the differential, the functions A_i are then given by

$$A_i(p) = \mathrm{d}f_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p), \tag{2.20}$$

so the coordinate representation is given by

$$\mathrm{d}f_p = \frac{\partial f}{\partial x^i}(p)\lambda^i|_p. \tag{2.21}$$

Thus, we see that the component functions of the differential are the partial derivatives of f with respect to the coordinates of the chosen coordinate chart. Therefore we can indeed interpret the differential as a coordinate-independent generalisation of the Euclidean gradient.

Using the coordinate representation of the differential, we can finally find a better expression for the dual of the coordinate basis. Suppose we take f to be a coordinate function, i.e. $f = x^i : U \to \mathbb{R}$, then

$$\mathrm{d}x^i|_p = \frac{\partial x^i}{\partial x^j}(p)\lambda^j|_p = \delta^i_j \lambda^j|_p = \lambda^i|_p. \tag{2.22}$$

This means that the coordinate covector field λ^i is exactly the differential $\mathrm{d}x^i$ and we may write the coordinate representation of the differential as

$$\mathrm{d}f_p = \frac{\partial f}{\partial x^i}(p)\mathrm{d}x^i|_p. \tag{2.23}$$

Concluding this section, we thus find that the dual of the coordinate basis is given by $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^n\}$.

2.4 Tensors

A tensor is an object that can be seen as a generalisation of vectors and dual vectors. Although tensors can be hard to visualise, they can be thought of as a higher-dimensional matrix. We first introduce the definition of tensors on a vector space, where we define a tensor as a real multilinear function. We will see two special types of tensor: *symmetric* and *antisymmetric* tensors, which possess certain symmetries under permutations of their arguments. Finally, we define tensor fields and tensor bundles, which are a generalisation of the (co)vector fields and bundles we saw before.

2.4.1 Tensors as Multilinear Functions

We define a tensor as a real-valued multilinear function. For this, it is useful to first see the formal definition of such a function:

Definition 2.4.1. Let $V_1, \ldots, V_k, W$ be vector spaces. A map $F : V_1 \times \ldots \times V_k \to W$ is called *multilinear* if it is linear in all its arguments, i.e. when for each i

$$F(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = aF(v_1, \ldots, v_i, \ldots, v_k) + a'F(v_1, \ldots, v_i', \ldots, v_k). \tag{2.24}$$

Remark. We denote the set of all multilinear maps $V_1 \times \ldots \times V_k \to W$ by $L(V_1, \ldots, V_k; W)$. This is a vector space under pointwise addition and scalar multiplication.

An important example of such a multilinear map is the tensor product:

Example 2.4.2.
Let $V_1, \ldots, V_k, W_1, \ldots, W_l$ be vector spaces over $\mathbb{R}$ and suppose $F \in L(V_1, \ldots, V_k; \mathbb{R})$ and $G \in L(W_1, \ldots, W_l; \mathbb{R})$. The tensor product of F and G *is defined as a map*

$$F \otimes G : V_1 \times \ldots V_k \times W_1 \times \ldots \times W_l \to \mathbb{R} \tag{2.25}$$

given by

$$F \otimes G(v_1, \dots, v_k, w_1, \dots, w_l) = F(v_1, \dots, v_k)G(w_1, \dots, w_l). \tag{2.26}$$

Also note that $F \otimes G \in L(V_1, \dots V_k, W_1, \dots, W_l; \mathbb{R})$.

We have previously stated that the space of all multilinear functions of a certain type is a vector space. The tensor product seen Example 2.4.2 above proves to be useful in defining a basis for this space:

Lemma 2.4.3.
Let $V_1, \dots, V_k$ be real vector space of dimensions $n_1, \dots, n_k$, respectively. Suppose that, for every $j \in 1, \dots, k$, $\left(E_1^{(j)}, \dots, E_{n_j}^{(j)}\right)$ is a basis for V_j and $\left(\varepsilon_{(1)}^1, \dots, \varepsilon_{(j)}^{n_j}\right)$ the associated basis for V_j^*. Then a basis for $L(V_1, \dots, V_k; \mathbb{R})$ is given by

$$\mathcal{B} = \left\{\varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k} : 1 \leq i_1 \leq n_1, \dots, 1 \leq i_k \leq n_k\right\} \tag{2.27}$$

and thus the vector space $L(V_1, \dots, V_k; \mathbb{R})$ has dimension $n_1 \cdots n_k$.

Having seen general multilinear functions, we can define tensors:

Definition 2.4.4 (Tensor)**.** Let V be a finite-dimensional real vector space. A *(mixed) tensor of rank (k,l)* is a multilinear function

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \times \underbrace{V \times \cdots \times V}_{l \text{ times}} \to \mathbb{R}. \tag{2.28}$$

The space of all tensors of rank (k,l) is denoted by $T^{(k,l)}(V)$ and given by

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ times}}. \tag{2.29}$$

Note that a vector can thus be seen as a rank $(0,1)$ tensor, while a dual vector is a rank $(1,0)$ tensor.

Since a tensor is a special case of a multilinear map, a basis for $T^{(k,l)}(V)$ can be derived directly from Lemma 2.4.3.

Corollary 2.4.5.
Let V be a vector space over $\mathbb{R}$ with $\dim V = n$. Suppose the basis for V is given by (E_i) and $\left(\varepsilon^j\right)$ is the dual basis for V^*. A basis for $T^{(k,l)}(V)$ is then given by

$$\left\{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} : 1 \leq i_1, \dots, i_k, j_1, \dots, j_l \leq n\right\}, \tag{2.30}$$

so $\dim T^{(k,l)}(V) = n^{k+l}$.

Using the basis definition, the tensor components of A can be defined by the action of the tensor on basis and dual basis vectors:

$$T^{i_1 \dots i_k}{}_{j_1 \dots j_l} = T\left(\varepsilon^{i_1}, \dots, \varepsilon^{i_k}, E_{j_1}, \dots, E_{j_l}\right), \tag{2.31}$$

such that any arbitrary tensor can be written in terms of its components as

$$T = T^{i_1 \dots i_k}{}_{j_1 \dots j_l} E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}. \tag{2.32}$$

2.4.2 (Anti)symmetric Tensors

In this section we will consider two special classes of tensors: symmetric and antisymmetric ones. A symmetric tensor does not change its value when its arguments are rearranged. A notable example of a symmetric tensor it the dot product. An antisymmetric tensor, such as the determinant, simply changes sign under permutation of its arguments. We will now formalise these notions.

Definition 2.4.6. Let V be a finite-dimensional vector space. A tensor T of rank $(0, l)$ (a covariant l-tensor) is *symmetric* if (for $1 \leq i < j \leq l$):

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_l) = T(v_1, \dots, v_j, \dots, v_i, \dots, v_l). \tag{2.33}$$

Definition 2.4.7. Let V be a finite-dimensional vector space. A covariant l-tensor is *antisymmetric* if for every $i \neq j$ it holds that

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_l) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_l). \tag{2.34}$$

2.4.3 Tensor Bundles and Tensor Fields

Having defined the tangent and cotangent bundles in Sections 2.2.2 and 2.3.2, respectively, we now see the generalisation of both concepts. This generalisation comes in the form of the tensor bundle:

Definition 2.4.8. [Tensor Bundle] Let M be a smooth manifold. The *bundle of tensors of type* (k, l) is given by

$$T^{(k,l)}TM = \bigsqcup_{p \in M} T^{(k,l)}\left(T_pM\right). \tag{2.35}$$

For any choice of $k, l \in \mathbb{N}$, such a bundle is called a *tensor bundle over* M. Note that a tensor bundle is precisely the disjoint union of the spaces of all tensors of rank (k, l) (denoted by $T^{(k,l)}(T_pM)$) defined at every point $p \in M$.

Remark. By choosing $(k, l) = (1, 0)$, we get $T^{(1,0)}TM = TM$, the tangent bundle. In addition, $(k, l) = (0, 1)$ yields $T^{(0,1)}TM = T^*M$, so the cotangent bundle.

Analogous to (co)tangent bundles, a tensor bundle can be used to define a tensor field, which continuously assigns a tensor to each point on the manifold M. This is formally formulated as follows:

Definition 2.4.9. [Tensor Field] Suppose M is a smooth manifold. A *tensor field* (of type (k, l)) is a continuous map $A : M \to T^{(k,l)}TM$ with $p \mapsto A_p$, such that $A_p \in T^{(k,l)}(T_pM)$ for every $p \in M$. We call A a *smooth* tensor field if the map $A : M \to T^{(k,l)}TM$ is smooth.

Using the bases for the tangent and cotangent spaces, we can define component functions of a tensor field with respect to these bases:

Definition 2.4.10. Let M be a smooth manifold of dimension n and suppose we have smooth local coordinates on an open subset $U \subseteq M$. A tensor field $A : M \to T^{(k,l)}TM$ can then be written in terms of coordinate vector fields and coordinate covector fields as

$$A = A^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_l}. \tag{2.36}$$

The functions $A^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} : U \to \mathbb{R}$ are called the *component functions of* A, and are given by (analogous to the tensor components of equation 2.31)

$$A^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}(p) = A_p \left(dx^{i_1}|_p, \ldots, dx^{i_k}|_p, \left.\frac{\partial}{\partial x^{j_1}}\right|_p, \ldots, \left.\frac{\partial}{\partial x^{j_l}}\right|_p \right). \tag{2.37}$$

Finally, we can relate the smoothness of the tensor field to the smoothness of its component functions:

Lemma 2.4.11.
Let M be a smooth manifold and let $A : M \to T^{(k,l)}TM$ be a tensor field. Then A is smooth if and only if the component functions $A^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}$ are smooth in every coordinate chart.

In (physics) literature, the distinction between tensors T (acting on a single point $p \in M$ or, more generally, an arbitrary vector space V) and tensor fields A is often not made very clear. A tensor field is often simply referred to as a 'tensor' and denoted by T. Therefore also the tensor field components $A^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}$ are often referred to as 'being' the tensor components $T^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}$.

2.5 Metrics

In any arbitrary vector space V, the inner product allows us to make sense of geometric quantities such as the lengths of vectors and the distances between them. In order to extend these notions to smooth manifolds, we define a metric on them. We first introduce the *Riemannian metric*, which is the most straightforward generalisation of the inner product. Later, generalise this notion to *pseudo-Riemannian metrics.*

2.5.1 Riemannian Metrics

Definition 2.5.1 (Riemannian Manifold)**.** Let M be a smooth manifold. A *Riemannian metric* on M is a smooth tensor field of type $(0, 2)$, such that it is an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ at each $p \in M$. In addition, a *Riemannian manifold* is a pair (M, g), where M is a smooth manifold and g is some Riemannian metric on M.

By definition, g_p is an inner product, so we use the following notation for $v, w \in T_pM$:

$$g_p(v, w) = \langle v, w \rangle_g. \tag{2.38}$$

The length of a vector $v \in T_pM$ is then given by $|v|_g = \langle v, v \rangle_g^{1/2}$.

Example 2.5.2.

The most standard example of a Riemannian manifold is $(\mathbb{R}^n, \overline{g})$*, where* $\overline{g}$ *is called the* Euclidean metric. *The value of* $\overline{g}$ *at each* $x \in \mathbb{R}^n$ *is simply the dot product: writing* $v, w \in T_x\mathbb{R}^n$ *in terms of the standard coordinates* $\left(x^1, \ldots, x^n\right)$ *as* $v \sum_i v^i \partial_i|_x$ *and* $w = \sum_i w^i \partial_i|_x$*, we get*

$$\langle v, w\rangle_{\overline{g}} = \sum_{i=1}^{n} v^i w^i. \tag{2.39}$$

From now on, when we use $\mathbb{R}^n$ *as a Riemannian manifold, we always assume the metric on it is the Euclidean metric* $\overline{g}$*.*

Later, in Section 2.7 we will introduce a rigorous notion of curvature of manifolds. For now, we start with a definition of flatness. For this, we first define isometries between manifolds.

Definition 2.5.3. Suppose (M, g) and $(\tilde{M}, \tilde{g})$ are Riemannian manifolds. We say that (M, g) and $(\tilde{M}, \tilde{g})$ are *isometric*, if there exists an *isometry* between them, i.e. a smooth bijection $\phi : M \to \tilde{M}$ such that each differential $\mathrm{d}\phi_p : T_pM \to T_{\phi(p)}\tilde{M}$ is a linear isometry (which preserves inner products, i.e. $\langle v, v\rangle = \langle \mathrm{d}\phi_p(v), \mathrm{d}\phi_p(v)\rangle$).

Furthermore, we say that a map $\phi : M \to \tilde{M}$ is a *local isometry* if every point $p \in M$ has a neighbourhood $U \subseteq M$, such that $\phi(U) \subseteq \tilde{M}$ is open and $\phi|_U$ is an isometry.

The notion of flatness makes use of isometries:

Definition 2.5.4. Let M be an n-dimensional Riemannian manifold. We say that M is *flat* if it is locally isometric to $\mathbb{R}^n$ with the Euclidean metric.

Before considering the more general case of pseudo-Riemannian metrics, we first define two properties of a Riemannian manifold that are very useful in cosmology and that we will see in Chapter 3:

Definition 2.5.5 (Isotropy)**.** Let (M, g) be a Riemannian manifold. Let $p \in M$, and let $\mathrm{Iso}_p(M, g)$ denote the set of all isometries $\phi : M \to M$ that fix p, i.e. for which $\phi(p) = p$. For each $\phi \in \mathrm{Iso}_p(M, g)$, the differential is a map $\mathrm{d}\phi_p : T_pM \to T_pM$. In addition the map $I_p : \mathrm{Iso}_p(M, g) \to \mathrm{GL}(T_pM)$, given by $I_p(\phi) = \mathrm{d}\phi_p$, is a representation of $\mathrm{Iso}_p(M, g)$, called the *isotropy representation*. We say that M is *isotropic* at p if the isotropy representation I_p acts transitively on the set of unit vectors in T_pM.

Definition 2.5.6 (Homogeneity)**.** Let (M, g) be a Riemannian manifold. We say that M is *homogeneous* if for every pair $p, q \in M$, there is an isometry $\phi : M \to M$ such that $\phi(p) = q$.

The two properties are related in the following way: if a manifold is isotropic everywhere, then is is homogeneous, and if it is homogeneous and isotropic at (at least) one point, then it is isotropic everywhere. In words, isotropy means that a manifold looks the same in every direction, while homogeneity means that it looks the same at every point on the manifold.

2.5.2 Pseudo-Riemannian Metrics

Although the Riemannian metric is a useful generalisation of the inner product, it needs to be generalised even further to be useful for applications in general relativity. The direct generalisation of the Riemannian metric is the *pseudo-Riemannian metric*. Before defining this, however, some preliminary concepts from linear algebra need to be covered.

Lemma 2.5.7.
Let V be a finite-dimensional vector space and suppose that q is a symmetric bilinear form, i.e. a symmetric tensor of rank $(0,2)$ on V. Then there exists a linear map $\hat{q}: V \to V^*$, given by

$$\hat{q}(v)(w) = q(v,w) \quad \textit{for all} \quad v, w \in V. \tag{2.40}$$

If $\hat{q}$ is an isomorphism, we say that q is nondegenerate.

The nondegeneracy of the linear map q is related to some other concepts, as captured by the following Lemma:

Lemma 2.5.8.
Let V be a finite-dimensional vector space and q a symmetric bilinear form defined on it. Then the following are equivalent:

1. *q is nondegenerate.*
2. *For every nonzero $v \in V$, there is a $w \in W$ such that $q(v,w) \neq 0$.*
3. *If, for some basis $\left(\varepsilon^i\right)$ of V^*, q can be written as $q = q_{ij}\varepsilon^i\varepsilon^j$, then the matrix (q_{ij}) is invertible.*

An inner product is in fact an example of a nondegenerate symmetric bilinear form, a fact which allows us to define the pseudo-Riemannian metric. From now on, we use the term *scalar product* for a general nondegenerate symmetric bilinear form on a finite-dimensional vector space V, and denote it by $\langle \cdot, \cdot \rangle$. A vector space V with a scalar product defined on it is then a *scalar product space*. Just like for the inner product, we say that two vectors $v, w \in V$ are *orthogonal* if $\langle v, w \rangle = 0$.

An important property of the scalar product is that it can be decomposed in a unique way:

Lemma 2.5.9.
Let (V,q) be a scalar product space of dimension n. Then there exists a basis $\left(\beta^i\right)$ for V^*, such that q can be written as

$$q = \left(\beta^1\right)^2 + \cdots + \left(\beta^r\right) - \left(\beta^{r+1}\right)^2 - \cdots - \left(\beta^{r+s}\right)^2, \tag{2.41}$$

with respect to this basis. Here, $r, s \geq 0$ are integers such that $r + s = n$.

It can in fact be shown that the numbers r and s do not depend on the choice of basis. The integer s is called the *index* of q, while the pair (r,s) is called the *signature* of q.

Using the decomposition in Lemma 2.5.9 and the definition of nondegeneracy, we may now define the pseudo-Riemannian metric:

Definition 2.5.10 (Pseudo-Riemannian Manifold)**.** Suppose that M is a smooth manifold. A *pseudo-Riemannian metric* (sometimes called a *semi-Riemannian metric*) is a smooth symmetric tensor field of rank $(0, 2)$ that is nondegenerate at each point $p \in M$ and has the same signature everywhere. The pair (M, g), where M is a smooth manifold and g a pseudo-Riemannian metric, is called a *pseudo-Riemannian manifold.*

Remark. Since the inner product is a special case of the scalar product, every Riemannian metric is also pseudo-Riemannian.

Example 2.5.11.
A generalisation of Euclidean space is pseudo-Euclidean space*, which is a simple example of a pseudo-Riemannian manifold. A pseudo-Euclidean space of signature* (r, s) *(denoted by* $\mathbb{R}^{r,s}$*), is the manifold* $\mathbb{R}^{r+s}$*, with standard coordinates* $\left(\xi^1, \dots, \xi^r, \tau^1, \dots, \tau^s\right)$*, and the pseudo-Riemannian metric* $\overline{q}^{(r,s)}$ *given by*

$$\overline{q}^{(r,s)} = \left(\mathrm{d}\xi^1\right)^2 + \cdots + \left(\mathrm{d}\xi^r\right)^2 - \left(\mathrm{d}\tau^1\right)^2 - \cdots - \left(\mathrm{d}\tau^s\right)^2 . \tag{2.42}$$

For every point p on a (pseudo-)Riemannian manifold M, a local frame can be found that is orthonormal in a neighbourhood of p:

Definition 2.5.12. Let (M, g) be a (pseudo-)Riemannian manifold and $U \subseteq M$ open. A local frame (E_i) on U is called an *orthonormal frame* if the vectors $E_1|_p, \dots, E_n|_p$ form an orthonormal basis for T_pM at every $p \in U$.

Lemma 2.5.13.
Let (M, g) *be a (pseudo-)Riemannian manifold. For every* $p \in M$*, there exists a smooth orthonormal frame on a neighbourhood* $U \subseteq M$ *of* p*.*

As stated before, a Riemannian metric is a special case of a pseudo-Riemannian metric. From a geometric point of view, Riemannian metrics form the most interesting category of pseudo-Riemannian metrics. There is however another category of the pseudo-Riemannian metrics that is of (almost) equal importance, especially in physics. This category is formed by the Lorentz metrics, which are just pseudo-Riemannian metrics of signature $(r, 1)$. In particular, the pseudo-Euclidean metric $\overline{q}^{(r,1)}$ is called the *Minkowski metric*, and the Lorentz manifold $\mathbb{R}^{(r,1)}$ is called $(r+1)-$*dimensional Minkowski space.* In general relativity, the Lorentz metric is allowed to vary from point to point to account for gravitational effects. As it so happens, most results from Riemannian metrics also apply to pseudo-Riemannian metrics and thus to Lorentz metrics.

2.5.3 Local Representation of the Metric

Usually, we the metric is represented as some $n \times n$ matrix. We will now see how this matrix form is derived. Let (M, g) be a (pseudo-)Riemannian manifold and suppose

$\left(x^1, \ldots, x^n\right)$ are smooth local coordinates on an open subset $U \subseteq M$. The metric g can then be written as

$$g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{2.43}$$

Here, we have n^2 smooth components functions g_{ij} (for $i, j \in \{1, \ldots, n\}$). The components form a symmetric $n \times n$ matrix (g_{ij}), with components given by $g_{ij}(p) = \langle \partial_i|_p, \partial_j|_p \rangle$, such that they depend smoothly on p. In addition, the matrix is nonsingular everywhere: if $v \in T_pM$ such that $g_{ij}(p)v^j = 0$, then $\langle v, v \rangle = g_{ij}(p)v^i v^j = 0$, implying $v = 0$.

The expression for the metric in terms of its components can be simplified further by using the *symmetric product* (see for example [Lee, 2003], Chapter 12), yielding

$$g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{2.44}$$

In physics, the metric on a manifold is often denoted in the following way:

$$\mathrm{d}s^2 = g_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{2.45}$$

where $\mathrm{d}s^2$ describes the (square of) the distance between two infinitesimally close points on the manifold.

We can write the metric components even more generally than in equation 2.44. Suppose that (E_i) is a smooth local frame for TM on an open subset $U \subseteq M$, with dual coframe (ε^i). Then a local decomposition of g is given by

$$g = g_{ij} \varepsilon^i \varepsilon^j, \tag{2.46}$$

where the components are given by

$$g_{ij}(p) = \langle E_i|_p, E_j|_p \rangle. \tag{2.47}$$

Again, the matrix (g_{ij}) is nonsingular, smoothly dependent on p, and symmetric.

2.5.4 Raising and Lowering Indices

Although vectors and covectors are defined quite differently, they can be easily transformed into each other by applying a (pseudo-)Riemannian metric. We discuss the construction below.

Let (M, g) be a (pseudo-)Riemannian manifold. Defining a bundle homomorphism (see also Lemma 2.5.7) $\hat{g} : TM \to T^*M$ by

$$\hat{g}(v)(w) = g_p(v, w) \tag{2.48}$$

for all $p \in M$ and $v, w \in T_pM$, it follows that for any two smooth vector fields X, Y on M

$$\hat{g}(X)(Y) = g(X, Y). \tag{2.49}$$

As it turns out, the function $\hat{g}$ is smooth, and $\hat{g}(X)$ is a smooth covector field on M. Given a smooth local frame (E_i) and its dual (ε^i), we use the local expression for g, so $g = g_{ij}\varepsilon^i\varepsilon^j$. In addition, let $X = X^iE_i$ be a smooth vector field. The covector field $\hat{g}(X)$ can then be written in terms of its components as

$$\hat{g}\,(X) = X_j\varepsilon^j = (g_{ij}X^i)\varepsilon^j. \tag{2.50}$$

We thus used the metric g_{ij} to lower the components of the vector field X. Analogously, we say that $\hat{g}\,(X)$ is obtained from X by *lowering an index* and we call $\hat{g}\,(X)$ X *flat*, or $X^\flat$.

Before we continue with raising an index, note that the map $\hat{g}$ is invertible, since the matrix (g_{ij}) is nonsingular everywhere. Thus, the inverse $\hat{g}^{-1}$ corresponds to the inverse matrix (g^{ij}), of which the components satisfy $g^{ij}g_{jk} = g_{kj}g^{ji} = \delta^i_k$.

Now, given a covector field $\omega = \omega_j\varepsilon^j$, we find a vector field given by

$$\hat{g}^{-1}(\omega) = \omega^iE_i = (g^{ij}\omega_j)E_i, \tag{2.51}$$

which we call ω *sharp* and denote by $\omega^\sharp$. It is obtained from ω by *raising an index.* It is worth mentioning that $\flat$ and $\sharp$ are called *musical isomorphisms.*

Not only the indices of (co)vector fields be lowered and raised. In fact, this can be done with tensors of any rank. We shall not go into too much detail about this and just show an example:

Example 2.5.14.
Suppose that A is a tensor (field) of rank $(1,2)$, written is terms of a local frame by

$$A = A_i{}^j{}_k\varepsilon^i \otimes E_j \otimes \varepsilon^k. \tag{2.52}$$

We can then lower the middle index to obtain (the components of) a rank $(0,3)$ tensor:

$$A_{ijk} = g_{jl}A_i{}^j{}_k. \tag{2.53}$$

Besides rasing and lowering indices, the musical isomorphisms can be used to take the trace of $(0,k)$ tensor fields that are defined on (pseudo-)Riemannian manifolds (for $k \geq 2$). By raising an index, a $(1,k-1)$ tensor field is obtained:

Definition 2.5.15. Let (M,g) be a pseudo-Riemannian manifold and h a $(0,k)$ $(k \geq 2)$ tensor field defined on M. The *trace of h with respect to g* is defined as

$$\mathrm{Tr}_g\, h := \mathrm{Tr}(h^\sharp). \tag{2.54}$$

Without going into detail about what the trace of $h^\sharp$ looks like, we consider the simple case $k = 2$, which is the most important for applications. It then holds that

$$\mathrm{Tr}_g\, h = g^{ij}h_{ij} = h^i_i, \tag{2.55}$$

which is equal to the ordinary trace of the matrix (h_{ij}) if the frame is orthonormal.

The final application of the musical isomorphisms we discuss is taking the inner(/scalar) product of covectors:

Definition 2.5.16. Suppose (M, g) is a (pseudo-)Riemannian manifold and let $p \in M$. A scalar product on the cotangent space T_p^*M is defined by

$$\langle \omega, \eta \rangle_g := \langle w^\sharp, \eta^\sharp \rangle_g. \tag{2.56}$$

Remark. Using the definition above (2.5.16) and equation 2.51, we find that

$$\langle \omega, \eta \rangle_g = g_{kl} \left(g^{ki} \omega_i \right) \left(g^{lj} \eta_j \right) = \delta^i_l g^{lj} \omega_i \eta_j = g^{ij} \omega_i \eta_j. \tag{2.57}$$

This can also be written as

$$\langle \omega, \eta \rangle_g = \omega_i \eta^i = \omega^j \eta_j. \tag{2.58}$$

As an aside, note that it is also possible to take inner products of tensors.

2.6 Connections and Covariant Derivatives

When working on an arbitrary (pseudo-)Riemannian manifold, the standard directional derivative loses much of its power (as described in [Lee, 2019], Section 4.1). Therefore, a new kind of derivative must be introduced; the *connection* (and the closely related *covariant derivative*). In this section, we give the definition of a connection and highlight some of its applications. To properly define a connection, we first need to introduce a generalisation of the (co)tangent and tensor bundles we saw in Sections 2.2.2,2.3.4 and 2.4.3; a *smooth vector bundle*:

Definition 2.6.1. A smooth vector bundle consists of two smooth manifolds M and E, together with a smooth surjection $\pi : E \to M$. In addition, for every $p \in M$, there exists an open neighbourhood $U \subseteq M$ of p, a $k \in \mathbb{N}$ and a diffeomorphism $\phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ that satisfies for all $x \in U$:

1. $(\pi \circ \phi)(x, v) = x \quad \forall\, v \in \mathbb{R}^k$,
2. $v \mapsto \phi(x, v)$ is a linear isomorphism between $\mathbb{R}^k$ and $\pi^{-1}(\{x\})$.

The definition of a connection makes use of vector bundles:

Definition 2.6.2 (Connection)**.** Let M be a smooth manifold, and let $\pi : E \to M$ be a smooth vector bundle over M. Denote the space of all smooth maps $\sigma : M \to E$ satisfying $\pi \circ \sigma = \mathrm{Id}_M$ by $\Gamma(E)$ and the space of all smooth vector fields on M by $\mathfrak{X}$. A *connection in* E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \tag{2.59}$$

given by $(X, Y) \mapsto \nabla_X Y$, that satisfies the following properties:

1. For any two functions $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$,
$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y. \tag{2.60}$$
2. For any $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(E)$,
$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2. \tag{2.61}$$
3. ∇ satisfies the product rule: for any smooth function f on M,
$$\nabla_X (fY) = f \nabla_X Y + (Xf) Y. \tag{2.62}$$

The operation $\nabla_X Y$ is called the *covariant derivative of* Y *in the direction of* X.

Although a connection is globally defined, it is actually a *local operator*. In particular, the value of $\nabla_X Y$ at a point follows directly from the values of X and Y at that point. This statement is made precise by the following Lemma:

Lemma 2.6.3.
Let ∇ be a connection in a smooth vector bundle $\pi : E \to M$. For every smooth vector field $X \in \mathfrak{X}(M)$, every smooth map $Y \in \Gamma(E)$, and any $p \in M$, the value $\nabla_X Y|_p$ only depends on the values of Y in a neighbourhood of p and the value of X at p. By writing X in terms of its components with respect to the coordinate vector fields, we obtain
$$\nabla_X Y|_p = \nabla_{X^i \partial_i} Y|_p = X^i(p) \nabla_{\partial_i} Y|_p. \tag{2.63}$$

Remark. Using Lemma 2.6.3, we can introduce some notation. Let X be a vector field defined on a neighbourhood of p, such that $X|_p = v$, i.e. $v \in T_pM$. In addition, suppose that Y is a smooth function $Y : M \to E$ satisfying $\pi \circ Y = \mathrm{Id}_M$ that is defined in a neighbourhood of p. Lemma 2.6.3 shows that the value $\nabla_X Y|_p$ does not depend on whether and how Y is defined on the entire manifold. We set
$$\nabla_v Y = \nabla_X Y|_p. \tag{2.64}$$

2.6.1 Connections in the Tangent Bundle

Thus far, we have seen the general definition of a connection in any smooth manifold E. However, we are interested in (pseudo-)Riemannian geometry, in which the focus lies on connections in the tangent bundle. Often, this is simply called a *connection on* M, or sometimes an *affine connection*. Note that it is only possible to define a connection in TM because it is in fact a smooth manifold (see Section 2.2.2).

A connection in the tangent bundle TM is a map
$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \tag{2.65}$$
which satisfies the properties in Definition 2.6.2. For every smooth manifold M, the tangent bundle TM admits a connection. We now investigate what a connection on the tangent bundle looks like when a local frame is chosen:

Proposition 2.6.4.
Let M be a smooth n-manifold and $U \subseteq M$ an open subset. Suppose that (E_i) is a smooth local frame for TM on U. For any $i, j \in \{1, \ldots, n\}$, the vector field $\nabla_{E_i} E_j$ can be expanded as

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k. \tag{2.66}$$

As $i, j, k \in \{1, \ldots, n\}$, we get n^3 smooth functions $\Gamma^k_{ij} : U \to \mathbb{R}$, which are called connection coefficients *of ∇.*

In fact, the connection can be completely described in terms of its connection coefficients on the neighbourhood U, as stated in the following Proposition:

Proposition 2.6.5.
Let M be a smooth manifold, $U \subseteq M$ open and ∇ a connection in TM. Suppose that (E_i) is a smooth local frame over U, and let $\{\Gamma^k_{ij}\}$ be the connection coefficients of ∇ with respect to it. For any two vector fields X, Y, we have

$$\nabla_X Y = \left(X\left(Y^k\right) + X^i Y^j \Gamma^k_{ij}\right) E_k. \tag{2.67}$$

The connection coefficients of a connection with respect to one local frame can easily be related to the connection coefficients with respect to some other local frame:

Proposition 2.6.6.
Let M be a smooth manifold and ∇ a connection in the tangent bundle TM. Suppose that (E_i) and $\left(\tilde{E}_i\right)$ are two smooth local frames for TM on an open subset $U \subseteq M$. Denote the connection coefficients of ∇ with respect to the two frames as Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$. If the two frames are related by $\tilde{E}_i = A^j_i E_j$ for some matrix (A^j_i), then the connection coefficients are related by

$$\tilde{\Gamma}^k_{ij} = \left(A^{-1}\right)^k_p A^q_i A^r_j \Gamma^p_{qr} + \left(A^{-1}\right)^k_p A^q_i E_q \left(A^p_j\right). \tag{2.68}$$

2.6.2 Connections in Tensor Bundles

Before, in Section 2.6.1, the connection in the tangent bundle was introduced and used to compute the covariant derivative of vector fields. However, it can also be indirectly used to compute the covariant derivatives of any tensor field A of rank (k, l) for some $k, l \geq 0$, as it induces connections in all tensor bundles over a smooth manifold M. The covariant derivative $\nabla_X A$ is linear in X. Therefore, all covariant derivatives can be taken together in a new tensor field of rank $(k, l+1)$, called the *total covariant derivative* of A.

Proposition 2.6.7 (Total Covariant Derivative)**.**
Let M be a smooth manifold with a connection ∇ in the tangent bundle TM. Denoting the space of all smooth covector fields ω by $\mathcal{T}^1(M)$, the total covariant derivative of A

for a smooth tensor field A of rank (k, l), may be defined as a map

$$\nabla A : \underbrace{\mathcal{T}^1(M) \times \cdots \times \mathcal{T}^1(M)}_{ktimes} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l+1times} \to C^\infty(M), \tag{2.69}$$

that is given by

$$(\nabla A)(\omega^1, \ldots, \omega^k, Y_1, \ldots Y_l, X) = (\nabla_X A)(\omega^1, \ldots, \omega^k, Y_1, \ldots Y_l), \tag{2.70}$$

such that ∇A is a smooth tensor field of rank $(k, l+1)$.

The components of the total covariant derivative may be straightforwardly calculated with the following Proposition:

Proposition 2.6.8.
Let M be a smooth manifold and ∇ a connection in TM. Suppose that (E_i) is a smooth local frame for TM, with $\{\Gamma^k_{ij}\}$ the connection coefficients of ∇ with respect to it. If A is a (k, l) tensor field, the components of the total covariant derivative are given by

$$A^{i_1 \cdots i_k}{}_{j_1 \cdots j_l; m} = E_m \left(A^{i_1 \cdots i_k}{}_{j_1 \cdots j_l} \right) + \sum_{s=1}^{k} A^{i_1 \cdots p \cdots i_k}{}_{j_1 \cdots j_l} \Gamma^{i_s}_{mp} - \sum_{s=1}^{l} A^{i_1 \cdots i_k}{}_{j_1 \cdots p \cdots j_l} \Gamma^{p}_{m j_s}. \tag{2.71}$$

Remark. For a covector field ω, equation 2.71 reduces to

$$w_{i;m} = E_m \omega_i - \omega_k \Gamma^k_{ji}. \tag{2.72}$$

In equations 2.71 and 2.72, the semicolon indicates that the indices after the semicolon are the result of differentiation.

2.6.3 The Covariant Directional Derivative

Having seen how to take the 'normal' covariant derivative, we will also define the *covariant directional derivative*; a covariant derivative along a curve. First we define vector and tensor fields along a curve:

Definition 2.6.9. Let M be a smooth manifold and suppose that $\gamma : I \to M$ is a smooth curve.

- A *(smooth) vector field along* γ is a continuous (smooth) map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}$ for every $t \in I$. It is said to be *extendible* if there is a smooth vector field $\tilde{V}$ on a neighbourhood of $\gamma(I)$ such that $V = \tilde{V} \circ \gamma$.
- More generally, a *(smooth) tensor field along* γ is a continuous (smooth) map $\sigma : I \to T^{(k,l)}TM$ such that $\sigma(t) \in T^{(k,l)} \left(T_{\gamma(t)} M \right)$ for each $t \in I$. Extensibility is defined in the same way as for vector fields.

The set of all smooth vector fields along the curve γ is denoted by $\mathfrak{X}(\gamma)$, and is a real vector space under pointwise vector addition and scalar multiplication.

Now we are in a position to define the covariant directional derivative:

Theorem 2.6.10 (Covariant Directional Derivative)**.**
Let M be a smooth manifold and ∇ a connection in TM. Suppose that V, W are smooth vector fields along γ. For every smooth curve $\gamma : I \to M$, ∇ determines a unique operator

$$\mathcal{D}_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma), \tag{2.73}$$

called the covariant directional derivative *along γ. It satisfies the following properties:*

1. *It is linear over $\mathbb{R}$: for every $a, b \in \mathbb{R}$*

$$\mathcal{D}_t(aV + bW) = a\mathcal{D}_t(V) + b\mathcal{D}_t(W). \tag{2.74}$$

2. *Product rule: for every smooth function f on I*

$$\mathcal{D}_t\,(fV) = \dot{f}V + f\mathcal{D}_tV. \tag{2.75}$$

3. *If V is extendible, then for every extension $\tilde{V}$ of V,*

$$\mathcal{D}_tV(t) = \nabla_{\dot{\gamma}(t)}\tilde{V}. \tag{2.76}$$

A (nearly) equivalent definition of the covariant directional derivative is captured in the following formula:

$$\mathcal{D}_tV(t) = \dot{V}^j(t)\ \partial_j|_{\gamma(t)} + V^j(t)\nabla_{\dot{\gamma}(t)}\ \partial_j|_{\gamma(t)} = \left(\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\Gamma^k_{ij}(\gamma(t))\right)\partial_k|_{\gamma(t)}\,. \tag{2.77}$$

Here, the $\dot{V}^j(t)$ terms are the derivatives of the components of V, which are given by $V(t) = V^j(t)\ \partial_j|_{\gamma(t)}$. More generally, for any smooth tensor field there is an analogous covariant directional derivative along γ. In particular, for covector fields $\omega = \omega_i \mathrm{d}x^i$, the covariant directional derivative is given by

$$\mathcal{D}_t\omega(t) = \left(\dot{\omega}_k(t) - \dot{\gamma}^i(t)\omega_j(t)\Gamma^k_{ij}(\gamma(t))\right)\mathrm{d}x^k. \tag{2.78}$$

2.6.4 The Levi-Civita Connection

In Section 2.2.3, it was established that every tangent bundle TM admits a connection. In fact, many different connections can be defined. We shall now describe two properties, *metric compatibility* and *symmetry*, that determine a unique connection on every (pseudo-)Riemannian manifold; the *Levi-Civita connection.*

The first property, metric compatibility, essentially means that a connection satisfies the product rule:

Definition 2.6.11. Let (M, g) be a (pseudo-)Riemannian metric. A connection ∇ on TM is said to be *compatible with* g if it satisfies the product rule for all $X, Y, Z \in \mathfrak{X}(M)$:

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle. \tag{2.79}$$

If a connection satisfies 2.79, it is also said to be a *metric connection.* Metric compatibility can be defined in a number of equivalent ways, as stated by the following Proposition:

Lemma 2.6.12.
Let (M, g) be a (pseudo-)Riemannian manifold and suppose that ∇ is a connection in TM. Then the following are equivalent:

1. *∇ is a metric connection.*
2. *The total covariant derivative of the metric vanishes identically: $\nabla g \equiv 0$.*
3. *If (E_i) is any smooth local frame, the connection coefficients $\{\Gamma^k_{ij}\}$ satisfy*

$$\Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} = E_k \left(g_{ij} \right). \tag{2.80}$$

4. *If V, W are smooth vector fields along some smooth curve γ, it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle V, W \rangle = \langle \mathcal{D}_t V, W \rangle + \langle V, \mathcal{D}_t W \rangle. \tag{2.81}$$

As it turns out, many different metric connections can be defined for all (pseudo-)Riemannian manifolds. Therefore, this property is now sufficient for defining the unique Levi-Civita connection. The other property required for this is *symmetry*:

Definition 2.6.13. Let (M, g) be a smooth manifold. A connection ∇ on the tangent bundle is *symmetric* if it satisfies

$$\nabla_X Y - \nabla_Y X \equiv [X, Y] \quad \text{for all} \quad X, Y \in \mathfrak{X}(M). \tag{2.82}$$

Remark. In Definition 2.6.13, the object $[X, Y]$ is a *Lie bracket*, of which the coordinate expression is given by

$$[X, Y] = X \left(Y^i \right) \frac{\partial}{\partial x^i} - Y \left(X^i \right) \frac{\partial}{\partial x^i}. \tag{2.83}$$

We are now ready to define the Levi-Civita connection. The Theorem that guarantees the existence of this connection is called the *Fundamental Theorem of Riemannian Geometry*:

Theorem 2.6.14 (The Levi-Civita Connection)**.**
Let (M, g) be a (pseudo-)Riemannian manifold. Then there exists a unique connection ∇ on TM that is compatible with g and symmetric. This connection is called the .

The next Proposition gives us some useful formulas for computing the Levi-Civita connection:

Proposition 2.6.15.
Let (M, g) e a (pseudo-)Riemannian manifold and ∇ the Levi-Civita connection on it. The following then holds:

1. *If $X, Y, Z \in \mathfrak{X}(M)$, then*
$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left(X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \right). \tag{2.84}$$

2. *In any smooth coordinate chart for M, the connection coefficients are given by*
$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right). \tag{2.85}$$
These coefficients are called Christoffel symbols*. As the Levi-Civita connection is symmetric, the Christoffel symbols are symmetric in the lower two indices.*

3. *Suppose that (E_i) is some smooth local frame on an open subset $U \subseteq M$. Writing c^k_{ij} for the n^3 smooth functions $U \to \mathbb{R}$ defined by*
$$[E_i, E_j] = c^k_{ij} E_k. \tag{2.86}$$
In this frame, the coefficients of the Levi-Civita connection are then given by
$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(E_i g_{jl} + E_j g_{il} - E_l g_{ij} - g_{jm} c^m_{il} - g_{lm} c^m_{ji} + g_{im} c^m_{lj} \right). \tag{2.87}$$

From now on, we shall always use the Levi-Civita connection, without explicitly stating so. A final interesting results about this connection is that its total covariant derivative commutes with the musical isomorphisms:

Lemma 2.6.16.
Suppose that A is some smooth tensor field of rank (k, l) on a (pseudo-)Riemannian manifold M. If $k \geq 1$, then
$$\nabla \left(A^\flat \right) = (\nabla A)^\flat. \tag{2.88}$$
Similarly, if $l \geq 1$, then
$$\nabla \left(A^\sharp \right) = (\nabla A)^\sharp. \tag{2.89}$$

2.7 Curvature

In this Section, we will define what it means for a manifold to have curvature. Recall that in Section 2.5.1, we defined that a Riemannian manifold is *flat* if it is locally isometric to some Euclidean space (see Example 6.14). Similarly, a pseudo-Riemannian manifold is said to be flat if it is locally isometric to some pseudo-Euclidean space (see 2.5.11). We will see that there is a more rigorous way of checking whether a manifold is flat, namely by using the Riemann curvature tensor. Starting with the flatness criterion, we will show the steps that motivate the definition of this tensor.

Definition 2.7.1. If ∇ is any connection on a smooth manifold M, it is said to satisfy the *flatness criterion*, if for any smooth vector fields X, Y, Z on an open subset $U \subseteq M$, it holds that

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z, \tag{2.90}$$

where the expression on the right hand side involves a Lie bracket and simply means $\nabla_{[X,Y]} Z = \nabla_{XY-YX} Z$.

Remark. As we saw in Section 2.6.14, the Levi-Civita connection is a unique connection that is especially 'well-behaved'. It also satisfies the flatness criterion.

2.7.1 The Riemann Curvature Tensor

We are not only interested in flatness, but also in curvature. To this end, we make the following definition:

Definition 2.7.2. Let (M, g) be a (pseudo-)Riemannian manifold. We define the *Riemann curvature endomorphism* as the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.91}$$

Remark. This map R defines a $(1, 3)$ tensor field on M. For any $X, Y \in \mathfrak{X}(M)$, the map $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $Z \mapsto R(X, Y)Z$ is a smooth bundle endomorphism, called the *curvature endomorphism determined by X and Y*. Therefore, the name Riemann curvature endomorphism is justified.

Since it is a $(1, 3)$ tensor field, the Riemann curvature endomorphism can be written in terms of any local frame as

$$R = R^l{}_{ijk} E_l \otimes \varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k. \tag{2.92}$$

Its coefficients are then defined by

$$R(E_i, E_j) E_k = R^l{}_{ijk} E_l. \tag{2.93}$$

In equations 2.92 and 2.93 we adopted the convention that the upper indices come first. We will continue to use this convention from now on. As stated by the following Proposition, there is a straightforward way of computing the coefficients $R^l{}_{ijk}$:

Proposition 2.7.3.
Let (M, g) be a (pseudo-)Riemannian manifold. For any set of smooth local coordinates, the components of the Riemann curvature endomorphism are given by

$$R^l{}_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}. \tag{2.94}$$

A more useful way to work with the components ${R^l}_{ijk}$ of the Riemann curvature endomorphism, is by encoding them in a tensor of rank $(0, 4)$, which is derived from R as follows:

Definition 2.7.4 (Riemann Curvature Tensor)**.** Let (M, g) be a (pseudo-)Riemannian manifold and R the Riemann curvature endomorphism. Then the *Riemann (curvature) tensor* is the rank $(0, 4)$ tensor field $Rm = R^\flat$.

In any choice of smooth local coordinates, the coefficients $R_{ijkl} = g_{lm}{R^m}_{ijk}$ are given by

$$R_{lijk} = g_{lm}\left(\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}\right), \tag{2.95}$$

which follows immediately from equation 2.94.

As stated before, the Riemann tensor can be used to rigorously check whether a manifold is flat:

Theorem 2.7.5.
A (pseudo-)Riemannian manifold is flat if and only if its curvature tensor vanishes identically.

The Riemann tensor has a number of symmetries, making computations of its components significantly easier. The following Proposition summarises all symmetries.

Proposition 2.7.6.
Let (M, g) be a (pseudo-)Riemannian manifold and suppose that W, X, Y, Z are vector fields on M. Then the Riemann tensor satisfies the following statements:

1. *Skew-symmetry in the first two arguments: $Rm\,(W, X, Y, Z) = -Rm\,(X, W, Y, Z)$. In terms of the tensor components, this means $R_{lijk} = -R_{iljk}$*
2. *Skew-symmetry in the last two arguments: $Rm\,(W, X, Y, Z) = -Rm\,(W, X, Z, Y)$, or $R_{lijk} = -R_{likj}$ in terms of the tensor components.*
3. *Symmetry in permutation of the two pairs of arguments:*
   $Rm\,(W, X, Y, Z) = Rm\,(Y, Z, W, X)$, or $R_{lijk} = R_{jkli}$
4. *$Rm\,(W, X, Y, Z) + Rm\,(X, Y, W, Z) + Rm\,(Y, W, X, Z) = 0$, or*
   $R_{lijk} + R_{ijlk} + R_{jlik} = 0$.

2.7.2 The Ricci Tensor

As the Riemann curvature tensor is of rank $(0, 4)$, it can be quite complicated to work with. Thus, it is useful to construct a simpler tensor based on it, while retaining some of the information carries by the Riemann tensor. To this end, the Ricci tensor is constructed.

Definition 2.7.7 (The Ricci Tensor)**.** Let (M, g) be a (pseudo-)Riemannian manifold and X, Y vector fields on M. Then the *Ricci tensor* is a map

$$Rc\,(X, Y) = \mathrm{Tr}\,(Z \mapsto R\,(Z, X)\,Y)\,, \tag{2.96}$$

of which the components R_{ij} are given by

$$R_{ij} = R^{k}{}_{kij} = g^{km} R_{mkij}. \tag{2.97}$$

Derived from the Ricci tensor is the Ricci scalar:

Definition 2.7.8. Let (M, g) be a (pseudo-)Riemannian manifold and X, Y vector fields on M. Then the *Ricci scalar*, sometimes called the *scalar curvature*, is the trace of the Ricci tensor. It is denoted by R or S and given by

$$R = \mathrm{Tr}_g\, Rc = R_i^{\ i} = g^{ij} R_{ij}. \tag{2.98}$$

2.8 Connecting Differential Geometry to General Relativity

Thus far, we have seen a great number of concepts that are of importance in differential geometry, such as tensors and covariant derivatives. In order to apply these concepts to general relativity, it is necessary to understand the geometrical structure of spacetime.

In short, in general relativity, spacetime is modeled as a four-dimensional smooth manifold, equipped with a Lorentz metric (as defined in Section 2.5.2). Unless explicitly stated otherwise, we assume that the (co)tangent bundle is equipped with the coordinate (co)frame (see example 2.3.9).

The connection on the Lorentz metric that is used in general relativity is always the Levi-Civita connection (defined in Section 2.6.14). Since the standard (co)frame on the manifold M is taken to be the coordinate (co)frame, the connection coefficients of the Levi-Civita connection are always the Christoffel symbols, as follows from Proposition 2.6.15.

In addition, an extra condition is imposed on the Lorentzian metric g: its Ricci curvature R has to satisfy Einstein's equation (explained in [Lee, 2019], [Choquet-Bruhat, 2009]), given by

$$Rc - \frac{1}{2} Rg = \frac{8\pi G}{c^4} T, \tag{2.99}$$

where g is as usual the metric, Rc the Ricci tensor and R the Ricci scalar. The constants in this equation are the speed of light in vacuum, which has the value [Mohr et al., 2022]

$$c \approx 3.00 \cdot 10^{8}\,\mathrm{m\,s^{-1}}, \tag{2.100}$$

and the gravitational constant G, which has the value

$$G \approx 6.67 \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}. \tag{2.101}$$

In addition, T is a symmetric 2-tensor field called the *stress-energy tensor*, which describes the density and flux of energy and momentum at each point in spacetime. We shall see this tensor field in more detail in Section 3. Furthermore, Rc in this equation denotes the Ricci curvature tensor, and R denotes the Ricci scalar, which is the trace of the Ricci tensor.

In addition, we shall from now on follow the convention that the signature of the Lorentz metric g is given by $(3, 1)$, or, in the language of physics, the signature $(- + ++)$.

Later, in Section 3.6, we will see the definition of another smooth manifold; the field-space manifold. We leave the explanation of this concept until then, but it is worth noting that this is a Riemannian manifold with metric tensor $\mathcal{G}$.

Besides establishing the geometric structure of spacetime, it is useful to introduce the most important conventions that are used in general relativity. The first convention is the Einstein summation convention, which we have already been using in the entirety of this Section. Recall that the statement of the Einstein summation convention is that every repeated index should be summed over, see equation 2.1. Another standard practice is to refer to (co)vectors or tensors in terms of their components. As the basis on the (co)tangent space is always understood to be the coordinate basis (or its dual), it is not necessary to explicitly mention the basis vectors. Thus, a tangent vector $v \in T_pM$, written in terms of the coordinate basis as

$$v = v^i \left.\frac{\mathrm{d}}{\mathrm{d}x^i}\right|_p, \tag{2.102}$$

is simply written as v^i. Anaogously, a tensor (field)

$$T = T^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \, E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}, \tag{2.103}$$

where (E_i) and (ε^j) are the coordinate frame and coframe respectively, is referred to as $T^{i_1 \ldots i_k}{}_{j_1 \ldots j_l}$.

Chapter 3

General Relativity, Cosmology and Inflation

Having introduced all mathematical concepts that form the framework of general relativity, we turn to the physics behind inflation. We start with a Section on general relativity, in which Einstein's equation and the stress-energy tensor are introduced, amongst other things. Then, in Section 3.2, a general introduction of the field of cosmology is given. After starting with a discussion about homogeneity and isotropy, the Friedmann-Lemaître-Robertson-Walker is introduced and the Friedmann equations are derived. Unless stated otherwise, [Carroll, 2019] is used as a source for these two sections. In Section 3.3, the need for a period of inflation in the very early universe is motivated by two problems with standard Big Bang cosmology: the horizon and flatness problems. Next, in Section 3.4, it is explained how inflation solves those problems and how it helps to generate cosmological perturbations. Finally, in Sections 3.5 and 3.6, single-field and multifield inflation are introduced, respectively.

Remark. Unless specifically stated otherwise, we from now on use natural units, meaning that the speed of light c and the the reduced Planck constant $\hbar$ are set to 1. Furthermore, the reduced Planck mass $M_{\text{pl}} = 1/(8\pi G)$ is also set to 1.

Remark. When referring to the components of some object (the metric, a tensor, etc.), we use Greek indices (e.g. μ, ν) for quantities related to spacetime, Latin indices such as i, j for quantities related to the spatial part of spacetime, and Latin indices (e.g. a, b) for quantities related to field-space (which is introduced in Section 3.5).

3.1 Physics of General Relativity

The physical idea behind the concept of a curved spacetime manifold is that curvature *is* gravity. Gravity influences the behaviour of matter, and in turn matter determines the gravitational field. The larger the effects of gravity are in a certain region of space, the more strongly curved spacetime is in that region. In the vicinity of a black hole, for example, spacetime is strongly curved.

More specifically, spacetime (or the metric on it) responds to the presence of energy and momentum (both carried by matter). The field equation that the spacetime metric must satisfy is ultimately a postulate, motivated by the known physics of Newtonian mechanics and special relativity. It turns out that this postulate agrees extremely well with experimental tests (as explained in [Will, 2014], for example). In terms of the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R (see 2.7.2) and the stress-energy tensor $T_{\mu\nu}$, the *Einstein field equation*, or just *Einstein's equation*, is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{3.1}$$

where we have used $8\pi G = 1$ (see also equation 2.99). Note that we have used the convention of referring to tensors in terms of their components (as introduced in Section 2.8) in equation 3.1. An important feature of the Einstein field equation, and of general relativity as a whole, is that the energy-momentum tensor is the source of the gravitational field (and thus of the curvature of spacetime). In words, Einstein's equation can be stated as follows [Baez and Bunn, 2005]: "The expansion of the volume of any set of particles initially at rest is proportional to (minus) the sum of the energy density and the three components of pressure."

To understand how the stress-energy tensor is defined, we consider the derivation of Einstein's equation through the principle of least action (see [Rindler, 2018], [Lee et al., 2018] for a more detailed explanation of this method). We start by defining an action that is suitable for general relativity. As proposed by Hilbert, the simplest choice for a Lagrangian is the Ricci scalar R. Assuming that we our working on a 4-manifold with a Lorentzian metric, the action corresponding to this Lagrangian is

$$S_H = \int \sqrt{-g}R\mathrm{d}^4x, \tag{3.2}$$

which is called the *Einstein-Hilbert action*. Here, g denotes the trace of the metric $g_{\mu\nu}$ (and not the metric *function*, as it often was in Chapter 2). In fact, the choice for the Lagrangian is unique in some way (see [Carroll, 2019] for a more in-depth discussion). The Einstein-Hilbert action encodes for the gravitational part of the action on spacetime. To account for matter, we must consider an action of the form

$$S = \frac{1}{2}S_H + S_M, \tag{3.3}$$

where S_M is the action corresponding to matter. Varying this action with respect to the inverse metric (as the metric is the dynamical variable on the spacetime manifold) yields

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \tag{3.4}$$

Rearranging the terms and defining the stress-energy tensor as

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{3.5}$$

we recover the Einstein field equation (see equation 3.1).

If we consider matter in the form of a scalar field ϕ acting on spacetime (see also Section 3.5), it follows that the associated stress-energy tensor is given by

$$\begin{aligned} T^{(\phi)}_{\mu\nu} &= -2\frac{1}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} \\ &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\sigma\rho}\nabla_\rho\nabla_\sigma\phi - g_{\mu\nu}V(\phi). \end{aligned} \tag{3.6}$$

Additionally, for a perfect fluid the stress-energy tensor reduces to (given as a tensor of rank $(2,0)$)

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}, \tag{3.7}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric and U^μ is the four-velocity (vector field) of the fluid.

3.2 Basic Ideas of Cosmology

Cosmological models are based on the assumption that the universe looks the same everywhere, at least on the largest scales. This is usually expressed by saying that the universe is *homogeneous* and *isotropic*, and the combination of these properties is known as the *cosmological principle*. Isotropy means that, on the largest scales, space looks the same in every direction, while homogeneity refers to space looking geometrically the same at every point. Although these are very intuitive concepts, they are actually also properties that can be formally assigned to a manifold M, as defined in Chapter 2.

3.2.1 The FLRW Metric

A very useful property of homogeneity and isotropy is that together they imply that a manifold is maximally symmetric. Observational evidence shows that space is homogeneous and isotropic. Since space is a part of the four-dimensional spacetime-manifold, one might assume that all of spacetime is in fact maximally symmetric. This turns out to not be the case, as this results in solutions to the Einstein field equation that are only valid if there is no matter present in the universe (see e.g. [Carroll, 2019], Section 8.1). Therefore, it is necessary to restrict the analysis to that for which we have observational evidence, namely that the universe is *spatially* homogeneous and isotropic. Spacetime is therefore modelled as $\mathbb{R} \times \Sigma$, with $\mathbb{R}$ representing the time coordinates and Σ being a maximally symmetric three-dimensional Riemannian manifold (see also [O'Neill, 1983] for the decomposition of spacetime in this way). Note that the definitions of isotropy

(Definition2.5.5) and homogeneity (Definition 2.5.6) are thus sufficient for the present discussion. The metric on spacetime is therefore of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left(\gamma_{ij}(u)\mathrm{d}u^i\mathrm{d}u^j\right), \tag{3.8}$$

where the three-dimensional metric γ_{ij} has to satisfy

$$^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \tag{3.9}$$

and defining $k = {}^{(3)}R/6$. The coordinates used in equation 3.8 are called *comoving coordinates*, meaning that there are no cross terms $\mathrm{d}t\mathrm{d}u^i$ and that the $\mathrm{d}t^2$ coefficient does not depend on the u^i.

Any two dimensional metric of the form 3.8 can be put in the form (see for example [Carroll, 2019], Chapter 5)

$$\mathrm{d}\sigma^2 = e^{2\beta(\bar{r})}\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{3.10}$$

By computing the components of the Ricci tensor, it can be deduced that $\beta = -1/2\ln(1-k\bar{r}^2)$. In addition, we define a dimensionless scale factor (with $R(t_0) = R_0$)

$$a(t) = \frac{R(t)}{R_0}, \tag{3.11}$$

and a curvature parameter

$$\kappa = \frac{k}{R_0^2}, \tag{3.12}$$

which have dimensions of distance and 1/length^2, respectively. Also note that the definition of the scale factor implies that in the present $a(t_0) = 1$. This way, we obtain

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right), \tag{3.13}$$

as the metric for a spatially homogeneous and isotropic universe. This is know as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The scale-factor $a(t)$ governs the size of the spatial slice Σ. Its evolution will be discussed in the next section. Before moving on, we note that the nonzero components of the Ricci tensor for the FLRW metric are given by

$$R_{00} = \frac{-3\ddot{a}}{a}, \tag{3.14a}$$

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1-\kappa r^2}, \tag{3.14b}$$

$$R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa), \tag{3.14c}$$

$$R_{33} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\sin^2\theta. \tag{3.14d}$$

3.2.2 The Friedmann Equations

In order to determine anything about the evolution of the universe, one needs to know the behaviour of the scale factor $a(t)$. We will derive this behaviour by considering Einstein's equation.

We model matter and energy as a perfect fluid, which is at rest in comoving coordinates, and therefore has four-velocity

$$U^\mu = (1, 0, 0, 0). \tag{3.15}$$

We offer some more insight into perfect fluids in Section 3.2.3. Recalling the expression for the stress-energy tensor for a perfect fluid (equation 3.7), we see that it straightforwardly reduces to $T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p)$ for the given four-velocity, so the trace is

$$T = T^\mu_\mu = -\rho + 3p. \tag{3.16}$$

To deduce the Friedmann equations, recall that Einstein's equation can be written as

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \tag{3.17}$$

Using equation 3.14a and the components of the FLRW metric, $\mu = \nu = 0$ yields the (first) Friedmann equation

$$\left(\frac{\ddot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a^2}, \tag{3.18}$$

while $\mu, \nu \in \{1, 2, 3\}$ leads to the second Friedmann equation, given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p). \tag{3.19}$$

All choices for $\mu, \nu \neq 0$ lead to the same equation, due to the isotropy of the spatial slices of the spacetime manifold.

The rate of expansion is typically characterised by the *Hubble parameter*, defined as

$$H := \frac{\dot{a}(t)}{a(t)}. \tag{3.20}$$

At the present time, the value of the Hubble parameter is known as the *Hubble constant* H_0, which has been measured to be $H_0 = 70 \pm 10$ km/sec/Mpc.

3.2.3 Perfect Fluids

In the previous section, we modelled all matter and radiation in the universe as a perfect fluid. A perfect fluid is an idealised material that can be completely characterised by the mass density and isotropic pressure (pressure that is uniform in all directions). But what sorts of perfect fluids are relevant in cosmology and how do they contribute to the energy density in the universe?

To obtain a relation between the energy density in the universe and the scale factor, consider the zeroth component of the equation for conservation of energy. It can be derived using the Christoffel symbols associated with the FLRW metric:

$$\begin{aligned} 0 &= \nabla_\mu T^\mu{}_0 \\ &= \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda \\ &= -\dot{\rho} - 3(\rho + p)\frac{\dot{a}}{a}. \end{aligned} \tag{3.21}$$

To solve this equation, we need to introduce an *equation of state*, i.e. a relation between ρ and p. For perfect fluids relevant in cosmology, the equation of state is given by

$$p = w\rho, \tag{3.22}$$

where w is constant. Sometimes w itself is said to be 'the equation of state'. Substituting this in equation 3.21, we obtain

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a}. \tag{3.23}$$

Integrating this gives $\rho \propto a^{-3(1+w)}$. Three kinds of important cosmological fluids are *matter*, *radiation*, and *vacuum energy*, which we shall now discuss separately.

- **Matter** consists of non-colliding, non-relativistic particles with negligible pressure. Ordinary stars and galaxies are examples of matter. Thus, it holds that $w = 0$, and the energy density is given by $\rho_M \propto a^{-3}$. This can be interpreted as the decrease of particle density as the universe expands. A universe in which matter makes the greatest contribution to the energy density is said to be *matter dominated.*

- **Radiation** is a term used to describe for instance electromagnetic radiation, but also massive particles that move at near-relativistic speeds. The equation of state for radiation is $p_R = 1/3\rho_R$, so the energy density decreases as $\rho_R \propto a^{-4}$. This can be explained in the following way: the number density of the particles of which radiation consists falls of as a^{-3}, just as for matter. In addition, the particles lose energy as a^{-1} due to redshifting.

- **Vacuum energy** is the energy density associated with empty space, or vacuum. Due to quantum fluctuations, space always has a nonzero energy density, even in the absence of matter or radiation. Unlike for matter and radiation, the vacuum energy density is constant: $\rho_\Lambda \propto 1$.

It is thought that at early times, the universe was radiation-dominated, as the universe was much smaller then. At future times, the universe might become vacuum-dominated, since the matter and radiation densities decrease as the universe expands, but the vacuum energy density remains constant.

3.3 Motivation for Inflation

An important challenge in cosmology is specifying the 'initial conditions' for the universe that could have led to the universe as it can be observed today. This is called the *Cauchy problem* in general relativity (see e.g. the discussion in [Ringström, 2009]). As we will see in this section, ordinary Big Bang theory requires a very specific set of initial conditions that would have allowed the universe to evolve to its state today. We consider two specific cases (the Big Bang puzzles) that lead to the requirement of such specific initial conditions. The standard Big Bang model neither explains nor predicts the need for such specific initial conditions. But how does specifying the initial condition of the universe even work? To do this, a spatial slice Σ of the four-dimensional spacetime manifold must be considered. On Σ, the positions and velocities of all particles must then be defined, after which the known laws of gravity and fluid dynamics can be used to study the time evolution of the system defined on Σ.

3.3.1 The Horizon Problem

For the Cauchy problem in general relativity, the initial distribution of matter may be described by using functions $\rho(\boldsymbol{x})$ and $p(\boldsymbol{x})$ to define the matter density and pressure, respectively. Observations of the cosmic microwave background imply that the inhomogeneities in the matter distribution were very small in the past (see e.g. [Akrami et al., 2020]). Since inhomogeneities grow in time (see [Bretón et al., 2010], for example), it reasonable to assume that they were even smaller at the earliest times. Thus, the early universe must have had a high degree of homogeneity. However, in the conventional Big Bang picture, this early universe consisted of many patches of space that were causally disconnected, and so there is no explanation for why all these patches evolved so similarly. The problem of explaining the large-scale homogeneity of the observed universe is thus called the *horizon problem.*

To make these statements a bit more precise, we define the *comoving (particle) horizon* τ as the causal horizon, or the maximum distance travelled by a light ray in a time t:

$$\tau := \int_0^t \frac{1}{a(t')}\mathrm{d}t' = \int_0^t \frac{1}{a(t')}\frac{1}{\dot{a}(t')}\frac{\mathrm{d}a(t')}{\mathrm{d}t'}\mathrm{d}t' = \int_0^t \frac{1}{a^2(t')}\frac{1}{H(t')}\frac{\mathrm{d}a(t')}{\mathrm{d}t'}\mathrm{d}t' = \int_0^a \frac{\mathrm{d}\ln a}{aH}. \tag{3.24}$$

In this equation, we also see the term $(aH)^{-1}$, which is called the *comoving Hubble radius.* From equations 3.21 and 3.23, it follows that the comoving Hubble radius $(aH)^{-1}$ evolves as

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)}, \tag{3.25}$$

where w is the equation of state of the fluid dominating the universe. Thus, using equation 3.24, it follows that

$$\tau \propto a^{\frac{1}{2}(1+3w)}. \tag{3.26}$$

In the standard Big Bang picture, it holds that $w > 0$, as described in Section 3.2.3, and thus both the comoving Hubble radius $(aH)^{-1}$ and the comoving horizon τ monotonically increase with time. This means that, as the time increases, the fraction of the universe that is in causal contact increases as well. Therefore, patches of the universe that are currently coming into causal contact with each other, were not causally connected when the CMB was formed, even though the CMB shows that there was very little inhomogeneity in the universe at that time.

3.3.2 The Flatness Problem

Another part of the Cauchy problem of the universe is the specification of the fluid velocities at all points in space. In order for the universe to evolve homogeneously, those velocities need to have had very precise initial conditions. If they had been slightly too small, the universe would have collapsed within moments. On the other hand, it would have expanded far too rapidly if the initial velocities had been slightly too large (see the discussion in e.g. [Ryden, 2017]).

As the presence of energy and momentum determines the local curvature of space, and thus the precise specification of the initial velocities is referred to as the *flatness problem*. The problem can also be formulated in the following way: per general relativity, spacetime curves in the presence of matter. Why then is the universe observed to be almost Euclidean? Let us now quantify this problem. Consider the first Friedmann equation, rewritten in the convenient form

$$1 - \Omega(a) = \frac{-k}{a^2H^2} \quad \text{where} \quad \Omega(a) := \frac{\rho(a)}{\rho_{crit}(a)} = \frac{\rho(a)}{3Ha^2}. \tag{3.27}$$

As before, the comoving Hubble radius is monotonically increasing in standard cosmology, causing the quantity $|\Omega(a) - 1|$ to also increase with time. The near-flatness of the universe today corresponds to $\Omega(a_0) \approx 1$, meaning that Ω must have been extremely close to 1 in the early universe. But why would this have been the case?

3.4 The Basics of Inflation

As we will see, the concept of *inflation* solves both the horizon and the flatness problem in an elegant way. We saw before that the monotonic increase of the Hubble radius $(aH)^{-1}$ leads to problems in standard Big Bang cosmology. The idea behind inflation is therefore: what if the Hubble radius simply did decrease at some time in the early universe?

Before investigating inflation further, it is important to first stress the relation between the comoving horizon τ and the comoving Hubble radius $(aH)^{-1}$: Since the comoving horizon is an integral of the comoving Hubble radius, a separation greater than τ means that objects have never been in causal contact, while a separation greater than $(aH)^{-1}$ only means that they are not causally connected at *this* time. So it may happen that objects were in causal contact in the early universe but not now, if $(aH)^{-1}$ was much larger in the early universe, after which it decreased for a certain period of time. As an aside, we note that inflation not only solves the Big Bang puzzles, but also explains the fluctuations seen in the CMB, as cosmological perturbations are generated during inflation.

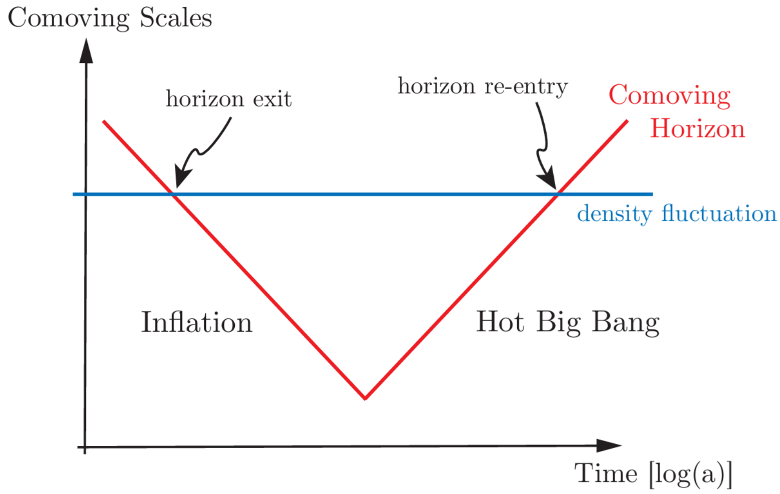

Figure 3.1: Schematic of the decrease of the comoving horizon τ with respect to time. The blue line represents the (quantum) density fluctuations that are 'frozen' when they exit the horizon during inflation (see Section 3.4.2), i.e. when they come to lie above the comoving horzion in the figure.
Credit: [Baumann, 2012].

3.4.1 The Big Bang Puzzles Solved

The decreasing of the comoving Hubble radius during inflation gives an elegant solution for the Big Bang puzzles. For the horizon problem, consider the comoving horizon. Since it decreased during inflation, patches of space that were causally connected before inflation, 'lost' connection during inflation due to the decreasing of τ, and are just now becoming connected again as τ is again increasing. So the homogeneity that is seen in for instance the CMB was in fact established before inflation. For the flatness problem, recall the Friedmann equation, as written in equation 3.27. As the comoving Hubble radius $(aH)^{-1}$ decreases during inflation, the quantity $|\Omega(a) - 1|$ does the same (see equation 3.27. This means that Ω need not necessarily have been so close to 1 in the very early universe!

3.4.2 Quantum Generation of Cosmological Perturbations

As we have seen, inflation is a mechanism that helps solve the Big Bang puzzles. However, there is a second way in which inflation helps to form the universe as we know it. The decrease of the comoving horizon during inflation is namely exactly the mechanism by which quantum fluctuations generated during inflation lead to macroscopic density fluctuations in the universe. Before and during inflation, quantum fluctuations are created on all length scales, i.e. with a spectrum that includes all comoving wavenumbers k_{co}. Here, the comoving wavenumber is related to the standard wavenumber k as follows:

$$k_{co} = a(t)k = a(t) \cdot \frac{2\pi}{\lambda}, \tag{3.28}$$

and the comoving wavelength λ is related to the comoving wavelength λ_{co} by $\lambda = a(t)\lambda_{co}$. Thus, the comoving wavenumber is defined in terms of the scale factor $a(t)$. As the physical wavelength changes along with the scale factor, the comoving wavelength and wavenumber remain constant during the expansion of the universe, and specifically also during inflation.

Quantum fluctuations that are relevant for cosmology are generated inside the horizon, so when they are generated it holds that $k \gg aH$. However, the comoving Hubble radius $(aH)^{-1}$ shrinks during inflation, and so aH increases. Therefore, as k_{co} remains constant, at some point the fluctuations must *exit* the horizons, i.e. $k < aH$. When outside the horizon, the fluctuations are not influenced by causality, are thus 'frozen' until they re-enter the horizon at late times, i.e. as the comoving horizon increases again. This freezing of the perturbations results in the anisotropies that are observed in the CMB. It also explains the large-scale homogeneity of the anisotropies, as the fluctuations were in causal contact when they were formed.

The computation of the spectrum of the quantum fluctuations is beyond the scope of this work. For a fully detailed calculation, the reader might consult *TASI Lectures on Inflation* [Baumann, 2012], especially Section 12. Important to note is that, for any model of inflation (be it single-field of multifield), the spectrum of quantum fluctuations generated during inflation must be consistent with observations of the CMB. This means that any model that is not consistent with the CMB observations must be ruled out.

3.4.3 Conditions for Inflation

Before, we defined inflation as a period of time in which the comoving Hubble radius decreases, so $\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0$. We shall now derive two equivalent conditions:

1. **Accelerated expansion** Differentiating the comoving Hubble radius $(aH)^{-1}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = -(aH)^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\left(a \cdot \frac{\dot{a}}{a}\right) = -\frac{\ddot{a}}{a^2H^2} < 0, \quad \text{so} \quad \frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0. \tag{3.29}$$

from which it follows that $\mathrm{d}/\mathrm{d}t(aH)^{-1} < 0$ if and only if $\ddot{a} > 0$. Therefore, inflation can also be defined as a period of accelerated expansion of space. In addition, the second derivative of the scale factor a can be related to the derivative of the Hubble parameter. Using equation 3.29, we obtain

$$\frac{\ddot{a}}{a} = -aH^2 \cdot -\frac{1}{a^2H^2} \cdot \left(H\dot{a} + a\dot{H}\right) = H^2\left(\frac{1}{aH^2} \cdot \dot{a}H + \frac{1}{aH^2}\dot{H}\right)$$

$$= H^2\left(\frac{\dot{a}}{\dot{a}} - \epsilon\right) = H^2(1-\epsilon), \quad \text{with} \quad \epsilon := -\frac{\dot{H}}{H^2}. \tag{3.30}$$

Accelerated expansion therefore corresponds to the condition

$$\epsilon := -\frac{\dot{H}}{H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}N} < 1. \tag{3.31}$$

Here we use a new derivative, namely the derivative with respect to e-folds N of the expansion during inflation. The e-folds are a measure for time in the sense that if expansion has been going on for an e-fold, then the size of the universe has increased by a factor e. It holds that $\mathrm{d}/\mathrm{d}N = \mathrm{d}/(H\mathrm{d}t) = \mathrm{d}/(\mathrm{d}\ln a)$.

2. Using the second Friedmann equation (equation 3.19), it immediately follows that $\ddot{a}/a > 0$ implies that $\rho < -3p$, meaning that the pressure should be negative.

3.5 Single-field Inflation

In some of the simplest models of inflation, a scalar field ϕ drives the inflation. In the more general case of multifield inflation, which we will discuss later, the inflation is driven by n scalar fields ϕ^a, that are the (smooth) components of a smooth function $\phi : \mathcal{S} \to \mathcal{F}$, where $\mathcal{S}$ denotes the spacetime manifold and $\mathcal{F}$ an n-dimensional field-space manifold. In the case of a single scalar field, there is just one component and the field-space manifold is simply the Euclidean space $\mathbb{R}$. The single scalar field is called the *inflaton*. The nature of the field is not specified, but it is used to parametrise the evolution of the energy density in the universe during inflation. We assume that the field is minimally coupled to gravity. In this case, the action of the field is given by

$$S = \int \mathrm{d}^4x \left(\underbrace{\frac{1}{2}\sqrt{-g}R}_{SH} + \underbrace{(\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]}_{S_\phi} \right), \tag{3.32}$$

such that it is the sum of the Einstein-Hilbert action S_H and the action S_ϕ of the scalar field, which has a canonical kinetic term.

Using the definition of the stress-energy tensor T, recall (see Section 3.1) that the tensor components for a single scalar field ϕ are

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right). \tag{3.33}$$

Note that g denotes the metric on spacetime.

In addition, the evolution of the field is described by

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) + V_{,\phi} = 0. \tag{3.34}$$

These equations originate from classical field theory, which is beyond the scope of this thesis. For a more in-depth discussion of this topic, the reader is advised to consult *Introduction to Classical Field Theory* [Torre, 2022], for example.

We now assume the metric g to be the FLRW metric (see equation 3.13) and take the field to be spatially homogeneous, so $\phi(t, \mathbf{x}) \equiv \phi(t)$. The zeroth component of the stress-energy tensor is given by

$$\begin{aligned} T^0_{\ 0} = g^{0a} T_{a0} &= g^{0a} \partial_a \phi \partial_0 \phi - g^{0a} g_{a0} \left(\frac{1}{2} g^{\sigma b} \partial_\sigma \phi \partial_b \phi + V(\phi) \right) \\ &= g^{00} \partial_0 \phi \partial_0 \phi - \delta^0_0 \left(\frac{1}{2} g^{00} \partial_0 \phi \partial_0 \phi + V(\phi) \right) \\ &\overset{(g^{00}=-1)}{=} -\dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2 - V(\phi) = -\frac{1}{2}\dot{\phi}^2 - V(\phi), \end{aligned} \tag{3.35}$$

and for $i \in \{1, 2, 3\}$ we have

$$\begin{aligned} T^i_{\ i} &= g^{ia} \partial_a \phi \partial_i \phi - g^{ia} g_{ai} \left(\frac{1}{2} g^{\sigma b} \partial_\sigma \phi \partial_b \phi + V(\phi) \right) \\ &= g^{ii} \partial_i \phi \partial_i \phi - \delta^i_i \left(\frac{1}{2} g^{\sigma 0} \partial_0 \phi \partial_0 \phi + V(\phi) \right) \\ &= 0 + \frac{1}{2}\dot{\phi}^2 - V(\phi) = \frac{1}{2}\dot{\phi}^2 - V(\phi). \end{aligned} \tag{3.36}$$

Note that the calculation is significantly simplified due to the diagonality of the FLRW metric tensor. We thus see that the stress-energy tensor takes the form of that for a perfect fluid, with $\rho_\phi = -T^0_{\ 0} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and $p_\phi = T^0_{\ 0} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. The resulting equation of state for the scalar field is therefore

$$w_\phi := \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{3.37}$$

Therefore, if the potential energy V dominates over the kinetic energy term $\frac{1}{2}\dot{\phi}$, the pressure becomes negative (if $w_\phi < 0$) and the expansion of the universe indeed accelerates ($w_\phi < -1/3$, see equation 3.19).

Before continuing our study of single-field inflation, it is necessary to know by which equations it is governed. Two equations are particularly important. The equation of motion for the scalar field follows from equation 3.34, and is given by

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{3.38}$$

The second is the Friedmann constraint, which follows by combining the second Friedmann equation (equation 3.19) with the expressions for the matter density and pressure corresponding with the scalar field (see equation 3.37):

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{3.39}$$

Worth noting is that for large values of the potential, the 'rolling' of the field down the potential is slowed down by Hubble friction, which comes from the term $3H\dot{\phi}$.

3.5.1 The Slow-roll Parameters

The rate of the expansion of space can be captured by the so-called *slow-roll parameters* ϵ and η. The name of these parameters is inspired by the fact that the parameters must be small if slow-roll inflation is to occur. The *first* slow-roll parameter is defined as

$$\epsilon := -\frac{\mathrm{d}\ln H}{\mathrm{d}N} = -\frac{\dot{H}}{H^2}, \tag{3.40}$$

and the *second* slow-roll parameter as (see e.g. [Christodoulidis and Gong, 2024])

$$\eta := \frac{\mathrm{d}\ln \epsilon}{\mathrm{d}N}, \tag{3.41}$$

where $\mathrm{d}/\mathrm{d}N = \mathrm{d}/(H\mathrm{d}t)$ is the derivative with-respect to e-folds N. Note that the definition of η is not universal; it may be defined in a number of different, but related ways (in [Baumann, 2012], for example, it is define as $\eta = -\ddot{\phi}/(H\dot{\phi})$). The slow-roll regime is defined by the slow-roll conditions, which state that

$$\epsilon \ll 1 \quad \text{and} \quad |\eta| \ll 1, \tag{3.42}$$

and ensure that accelerated expansion is sustained for a sufficiently long period of time. The slow-roll parameters will be crucial to the analysis in Chapters 5 and 6.

When the slow-roll parameter ϵ becomes equal to 1 ($\epsilon(\phi_{end}) = 1$), the slow-roll conditions are violated and inflation ends. As an e-fold is defined as the time in which space expands by a factor e, the number of e-folds that inflation lasts for can be calculated from the scale factor:

$$N(\phi) := \ln\frac{a_{end}}{a} = \int_t^{t_{end}} H\mathrm{d}t = \int_\phi^{\phi_{end}} \frac{H}{\dot{\phi}}\mathrm{d}\phi = \int_{\phi_{end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon}}. \tag{3.43}$$

In order to solve the horizon and flatness problems, inflation needs to last for at least 60 e-folds. Furthermore, the density fluctuations observed in the CMB were generated approximately 40 to 60 e-folds before inflation ended (as described by e.g. [Baumann, 2012]).

3.6 Multifield Inflation Models

As was described in the previous section, in single-field inflation a single scalar field, called the *inflaton*, drives inflation. For this inflaton, multiple candidates exist, for example *superpartners* in supersymmetric theories (see for example [Antoniadis et al., 2017], [Aldabergenov et al., 2024]) and *moduli fields* in string theory (see e.g. [Cicoli and Quevedo, 2011], [Abe et al., 2023]). As such, multiple fields have been identified as possible contributors to inflation. It is therefore logical to consider models in which more than 1 field contributes to inflation (as discussed in e.g. [Gong, 2017]), which we do in this section. Such models give rise to a great number of new inflationary mechanisms, such as *hyperinflation* [Brown, 2018], *sidetracked inflation* [Garcia-Saenz et al., 2018] and *angular inflation* [Christodoulidis et al., 2019a]. Observations of for example the CMB may be used to impose constraints on possible models of inflation or to confirm certain properties of multifield inflation models. Another motivation for multifield inflation is that analysis has shown that for single-field inflation models, quite specific parameter values are needed to make the models compatible with observations. Thus, multifield inflation offers the possibility of constructing models that are compatible with observation, without the need for such fine-tuning of parameters [Mukhanov and Steinhardt, 1998].

We consider a system of n scalar fields. The scalar fields $\phi = (\phi^a)$ are described as coordinates on an n-dimensional smooth manifold $\mathcal{F}$ on which a general Riemannian metric tensor $\mathcal{G}_{ab}$ is defined (see [Nibbelink and van Tent, 2000], [Christodoulidis et al., 2019b], for example). As for single-field inflation, we assume that the scalar fields are minimally coupled to gravity. The action is then given by (as in e.g. [Bjorkmo and Marsh, 2019])

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(R - \frac{1}{2}\mathcal{G}_{ab}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - V(\phi^1,\ldots,\phi^n)\right). \tag{3.44}$$

As described in Section 3.1, the stress-energy tensor corresponding to the fields can be obtained by varying their action with respect to the inverse spacetime metric $g^{\mu\nu}$, resulting in [Abedi and Abbassi, 2017]

$$T_{\mu\nu} = \mathcal{G}_{ab}\partial_\mu\phi^a\partial_\nu\phi^b - g_{\mu\nu}\left(\frac{1}{2}\mathcal{G}_{ab}g^{\rho\sigma}\partial_\rho\phi^a\partial_\sigma\phi^b + V(\phi^1,\ldots\phi^n)\right). \tag{3.45}$$

As usual, the 00 and ij (for $i, j \in \{1, 2, 3\}$) components of this stress-energy tensor then give the Friedmann equations. For multifield inflation, these are given by (as described by [Christodoulidis et al., 2019b])

$$3H^2 = \frac{1}{2}\mathcal{G}_{ab}\dot{\phi}^a\dot{\phi}^b + V, \tag{3.46a}$$

$$\dot{H} = -\frac{1}{2}\mathcal{G}_{ab}\dot{\phi}^a\dot{\phi}^b. \tag{3.46b}$$

Finally, the background equations of motion for the scalar fields ϕ^a can be obtained by varying the action with respect to ϕ^a, i.e. $\delta S_\phi/\delta\phi^a$. This results in

$$\mathcal{D}_t\dot{\phi}^a + 3H\dot{\phi}^a + \mathcal{G}_{ab}V_{,b} = 0, \tag{3.47}$$

where D_t is the covariant directional derivative (see Section 2.6.3) with respect to a smooth curve ϕ^a in field-space, i.e.

$$\mathcal{D}_t\dot{\phi}^a = \ddot{\phi}^a + \Gamma^a_{bc}\dot{\phi}^b\dot{\phi}^c. \tag{3.48}$$

Another useful covariant directional derivative is the one with respect to e-folds; $\mathcal{D}_N$. It is related to $\mathcal{D}_t$ as

$$ \tag{3.49} $$

As an aside, note that these (3.47) are called the *background* equations of motion because they correspond to a homogeneous and isotropic spacetime with the FLRW metric. In practice, the FLRW spacetime is subject to perturbations, but these are not taken into account in the derivation of 3.47. Thus, they are in a way the zeroth order of equations of motion for the scalar fields and are called the *background* equations of motion.

The slow-roll parameters ϵ and η are defined in exactly the same way as for single-field inflation (see equations 3.40 and 3.41). For slow-roll inflation to occur, the conditions $\epsilon, |\eta| \ll 1$ should be satisfied.

3.6.1 The Kinematic Basis

As stated before, the fields ϕ^a are seen as coordinates on the field-space manifold $\mathcal{F}$. As described in Chapter 2, a suitable frame for the (co)tangent bundle would therefore be the coordinate (co)frame. However, since each (co)tangent space is just a vector space, we are free to introduce some other frame. A possible choice is a local orthogonal frame, meaning that the basis vectors of each tangent space are orthogonal with respect to the metric. Such a frame is sometimes called a vielbein basis (as described in [Carroll, 2019]). While we shall not discuss the vielbein formalism in detail, we do highlight an important example; the *kinematic basis.* The kinematic basis is a local orthonormal frame, that is defined tangent and perpendicular to the inflationary trajectory (as in [Gong, 2017], for example). The unit tangent vector is defined as

$$T^a := \frac{\dot{\phi}^a}{\mathcal{G}_{ab}\dot{\phi}^a\dot{\phi}^b}, \tag{3.50}$$

and the unit normal as the unit vector orthogonal to the tangent, i.e. as the vector N^a that satisfies

$$\mathcal{G}_{ab}T^aN^b = 0. \tag{3.51}$$

From this, it follows that the derivative of T^a is proportional to N^a, so $\mathcal{D}_tT^a \propto N^a$. In Chapter 5, we will see another example of such a local orthogonal basis, the *gradient basis.*

Chapter 4

Dynamical Systems Background

In Chapter 3, we have seen the equations governing multifield inflation (3.46a, 3.46b, 3.47). In Chapters 5 and 6, these equations are used to define a system of ordinary differential equations, i.e. a dynamical system. The analysis focuses on the stability of solutions, and in particular on attractor solutions. Therefore, we first need to introduce these concepts as defined by the mathematical theory of dynamical systems. First, in Section 4.1, we introduce the flow of a dynamical system. Then, in Section 4.2, the stability of sets and specifically of orbits is studied. Finally, attractors are defined in Section 4.3 and in Section 4.4, the theory of this chapter is related to the analysis in Chapter 6. Throughout this chapter, [Meiss, 2007] is used as a reference.

4.1 Flow and Orbits

Generally speaking, a *dynamical system* is a rule that defines a trajectory on a set of states (the *phase space*) as a function of one parameter (which we call *time*). The phase space is some manifold M, which is often taken to be $\mathbb{R}^n$. The time t may be either discrete or continuous. In the case that t is discrete, the dynamical system is called a *mapping* (see e.g. [Alligood et al., 2000]). We, however, focus on systems in which the time variable is continuous, i.e. $t \in \mathbb{R}$.

The flow of a dynamical system is essentially the collection of all solutions to the system. More formally, this is defined as:

Definition 4.1.1. Suppose we have a dynamical system for which the phase space is some manifold M. A *complete flow* $\varphi_t(x)$ is a differnetiable mapping $\varphi : \mathbb{R} \times M \to M$ that satisfies the following properties:

1. $\varphi_0(x) = x$ for all $x \in \mathbb{R}$,
2. $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $t, s \in \mathbb{R}$.

Here, the composition $\circ$ is given by $\varphi_t \circ \varphi_s(x) \coloneqq \varphi_t(\varphi_s(x))$.

Remark. The second property ($\varphi_t \circ \varphi_s = \varphi_{t+s}$), is called the *group property.* If this property is not satisfied (but the first property is), the flow is not complete. From the group property, the useful result can be derived that two trajectories cannot cross if the flow is complete. Furthermore, it follows that for every $t \in \mathbb{R}$

$$\varphi_{-t} \circ \varphi_t = \varphi_{-t+t} = \varphi_0 = \mathrm{Id}, \tag{4.1}$$

so $(\varphi_t)^{-1} = \varphi_{-t}$ is also differentiable (and in particular continuous).

A vector field $f : M \to \mathbb{R}^n$, defined by

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\bigg|_{t=0} \tag{4.2}$$

is associated with the flow φ. The following Lemma explains the exact connection:

Lemma 4.1.2.
If $\varphi_t(x)$ is a flow on a manifold M, then it is a solution of the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x_0) = f(\varphi_t(x_0)), \quad \varphi_0(x_0) = x_0, \tag{4.3}$$

for the vector field defined by 4.2.

Intrinsically connected with dynamical systems is the concept of an *orbit.* An orbit is the collection of all states that correspond to a specific initial condition:

Definition 4.1.3. Let M be a manifold and let $\varphi : \mathbb{R} \times M \to M$ be the flow of some dynamical system. Suppose that $x_0 \in M$ is some initial condition. The *orbit of* x_0 is then defined as

$$\Gamma(x_0) := \{\varphi_t(x_0) \, : \, t \in \mathbb{R}\}, \tag{4.4}$$

and the *forward orbit* is defined as

$$\Gamma^+(x_0) := \{\varphi_t(x_0) \, : \, t \geq 0\}. \tag{4.5}$$

Related to the forward orbit is the concept of an omega-limit set, which captures the behaviour of the flow φ_t as $t \to \infty$.

Definition 4.1.4. Suppose that M is a manifold and let $\varphi : \mathbb{R} \times M \to M$ be a flow on M. We say that $y \in M$ is a *limit point* of $\Gamma^+(x)$ if there exists a strictly increasing sequence $\{t_j\}_{j=1}^{\infty}$ such that $\varphi_{t_j}(x) \to y$ as $j \to \infty$.

Definition 4.1.5. Under the assumptions of the previous definition, the *omega-limit set* of $\Gamma^+(x)$ is defined as the set of all limit points of $\Gamma^+(x)$ and denoted by $\omega(x)$.

Especially interesting types of orbits are equilibria and periodic orbits. An *equilibrium* or *fixed point* is a point $\tilde{x} \in M$ such that $\varphi_t(\tilde{x}) = \tilde{x}$ for all $t \in \mathbb{R}$. It follows that its orbit is given by $\Gamma(\tilde{x}) = \{\tilde{x}\}$. A *periodic orbit* γ is essentially a closed loop in phase space.

Each point x_γ on a periodic orbit has the property that there is a time $T \geq 0$ such that the obit of x_γ returns to itself:

$$\varphi_T(x_\gamma) = x_\gamma. \tag{4.6}$$

An orbit is in fact a special case of an invariant set, which is defined as follows.

Definition 4.1.6. Let M be a manifold with a flow $\varphi : \mathbb{R} \times M \to M$ defined on it. A set $\Lambda \subseteq M$ is called *invariant* under the flow φ if $\varphi_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$, or if for all $x \in \Lambda$ it holds that $\varphi_t(x) \in \Lambda$ for all $t \in \mathbb{R}$. Analogously, the set Λ is *forward invariant* if $\varphi_t(\Lambda) \subseteq \Lambda$ for all $t > 0$.

4.2 Stability and Lyapunov Exponents

Equilibrium points are of specific interest when studying a dynamical system. Making the connection between flows and their associated vector fields, we see that is x^* is an equilibrium of the flow, then it holds that $f(x^*) = 0$. The following important Theorem about (the stability) of equilibrium points is well-known:

Theorem 4.2.1 (Hartman-Grobman).
Let x^* be a hyperbolic equilibrium point of a C^1 vector field $f : M \to \mathbb{R}$, where M ia an arbitrary manifold. Then, there exists a neighbourhood N of x^* such that the flow of the system $\dot{x} = f(x)$ is topologically conjugate to the flow of its linearisation $\dot{\xi} = Df(x^*)\xi$.

This Theorem allows us to classify the stability of a hyperbolic equilibrium x^* by calculating the eigenvalues of the matrix $Df(x^*)$; if the real parts of all eigenvalues are negative, x^* is stable, while x^* is unstable if one of the eigenvalues has positive real part.

In case we are interested in the stability of an orbit, we need a more general notion of linear stability, which allows us to compute some quantity that serves as the analogue of the eigenvalues corresponding to an equilibrium. Suppose we are interested in the stability of a particular trajectory $\varphi_t(x^*)$, which we call the *fiducial trajectory* (and we call x^* the *fiducial point*). For the stability of the fiducial trrajectoy, consider a trajectory $\varphi_t(x^* + \varepsilon v_0)$ starting near the fiducial point. Assuming that the initial deviation v_0 evolves as $v(t)$, its evolution is described by

$$\dot{v} = Df(\varphi_t(x^*))v := A(t)v, \tag{4.7}$$

where Df is the Jacobian as usual. The matrix $A(t) = Df(\varphi_t(x^*))$ may be seen as a linear operator, that acts on a vector $v(t) \in T_{\varphi_t(x)}M$, to give the velocity at the point $y(t) = \varphi_t(x) + \varepsilon v(t)$ for ε sufficiently small. Here, $T_{\varphi_t(x)}M$ is the tangent space to the point $\varphi_t(x) \in M$, as defined in Section 2. The fundamental matrix solution of the differential equation 4.7 is given by $\Phi(t;x) = D_x\varphi_t(x)$ and is a linear operator $\Phi(t;x) : T_xM \to T_{\varphi_t(x)}M$.

Informally, the Lyapunov exponents of an orbit are defined as the asymptotic growth rate of the length of the tangent vectors $v(t)$:

$$|\Phi(t;x)v| \sim e^{\mu t}|v|. \tag{4.8}$$

However, it is not a priori clear that the length of the tangent vector grows approximately exponentially. This is however ascertained by the following Lemma:

Lemma 4.2.2.
Let M be a manifold with a flow φ on it. Let $\varphi_t(x^*)$ be the fiducial trajectory and $\varphi_t(x^* + \varepsilon v_0)$ a trajectory starting near the fiducial point. Suppose that $\Phi(t; x)$ is the fundamental matrix solution of

$$\dot{\Phi} = A(t)\Phi = Df(\phi_t(x^*))v, \quad \Phi(0; x) = I, \tag{4.9}$$

where I is the identity matrix. In addition, suppose that $||A(t)|| \leq k$ for all $t \geq 0$. Then for any tangent vector v there exists constants $c, \tilde{c} \geq 0$ such that for all $t \geq 0$

$$\tilde{c}e^{-Kt} \leq |\Phi(t; x)| \leq ce^{Kt}. \tag{4.10}$$

From this, it follows that the function $\ln |\Phi v|/t$ is bounded for every $t \geq 0$. Since any bounded function has limit points, this allows us to make the following definition:

Definition 4.2.3. Under the assumptions of Lemma 4.10, the set of limit points of $\ln |\Phi v|/t$ is called the *Lyapunov spectrum*:

$$Sp(x, v) := \{\lambda = \lim_{j \to \infty} \frac{1}{t_j} \ln |\Phi(t_j; x)v| \,:\, \text{sequences } \{t_j\} \text{ with } t_j \to \infty\}. \tag{4.11}$$

Two limits of special interest are the lim inf and lim sup. Using these, it can be shown that the Lyapunov spectrum is a closed interval. In particular, when the limits coincide the spectrum is reduced to a single point. Now recall that a fixed point is asymptotically stable if the (real part of the) largest eigenvalue corresponding to it is negative. In analogy to this, when considering Lyapunov exponents we are interested in the value of the largest growth rate. The *Lyapunov exponents* are therefore defined as follows:

Definition 4.2.4 (Lyapunov exponents)**.** Let M be a manifold with a flow φ on it. Let $\varphi_t(x^*)$ be the fiducial trajectory and $\varphi_t(x^* + \varepsilon v_0)$ a trajectory starting near the fiducial point. Suppose that $\Phi(t; x)$ is the fundamental matrix solution and $v \in T_{x^*}M$. Then the *Lyapunov exponent* is defined as the supremum limit of $\ln |\Phi v|/t$, i.e.

$$\mu(x, v) := \limsup_{t \to \infty} \frac{1}{t} \ln |\Phi(t; x)v|. \tag{4.12}$$

Two useful results are that an orbit has a maximum and that almost every orbit has one zero Lyapunov exponent, as captured by the following Lemma:

Lemma 4.2.5.
Let M be an n-manifold with a flow φ defined on it. Suppose that $\varphi_t(x)$ is a bounded trajectory of the flow. Then the following results hold:

1. *The trajectory has at most n distinct Lyapunov exponents.*

2. *If, in addition, $\varphi_t(x)$ does not converge to an equilibrium of the flow, then the trajectory has at least one zero Lyapunov exponent.*

In fact, this zero Lyapunov exponent corresponds to $v = f(\varphi_t(x))$, so it is in the direction of the flow.

4.3 Attractors

The analysis in Section 5 is mostly concerned with finding attractor solutions for the dynamical system that describes multifield inflation. Roughly speaking, an attractor is an invariant set towards which all trajectories in the vicinity move. We now discuss the more formal construction of an attractor, starting with the definition of stability of an invariant set:

Definition 4.3.1. Let M be a manifold and let φ be a flow on M. An invariant set Λ is *stable* if for any neighbourhood N of Λ there exists another neighbourhood $\tilde{N} \subseteq N$of Λ such that if $x \in \tilde{N}$, then $\varphi_t(x) \in N$ for all $t \geq 0$.
In particular, Λ is *asymptotically stable* if it is stable and there is a neighbourhood U of Λ such that for every $x \in U$, the distance between $\varphi_t(x)$ and Λ goes to zero, i.e. $\rho(\varphi_t(x), \Lambda) \to 0$ as $t \to \infty$.

The definitions of stability both refer to some neighbourhood of the invariant set Λ. Therefore, it is only natural that the definition of the attractor involves a special type of neighbourhood around it; the *trapping region*.

Definition 4.3.2. A subset N of a manifold M with a flow φ defined on it is called a *trapping region* if it is compact and $\varphi_t(N) \subseteq \mathrm{int}(N)$ for all $t > 0$.

In this Definition, the notation $\mathrm{int}(N)$ refers to the interior of the set N. Inside a trapping region, there exists some maximal invariant set, called an *attracting set*:

Definition 4.3.3. Suppose that M is a manifold with a flow φ defined on it. A set $\Lambda \subseteq M$ is an *attracting set* is there is a trapping region N such that $\Lambda \subseteq N$ and

$$\Lambda = \bigcap_{t>0} \varphi_t(N) . \tag{4.13}$$

Note that it follows from the definition that any attracting set is an invariant set. In addition, since N is closed and both φ_t and φ_t^{-1} are continuous, Λ is closed. Thus, the set $\{\varphi_t(N)\}$ is closed for every $t > 0$.

The following Lemma relates asymptotically stable sets and attracting sets:

Lemma 4.3.4.
Every attracting set is asymptotically stable. Conversely, if an asymptotically stable set is also compact, it is an attracting set.

For any attracting set, a maximal trapping region can be found, which is called the *basin of attraction* and is defined as follows:

Definition 4.3.5. Let M be a manifold with a flow φ defined on it and suppose that Λ is an invariant set. The *basin of attraction* $W^s(\Lambda)$ of Λ is the set

$$\{x \in M \,:\, \rho(\varphi_t(x), \Lambda) \to 0 \text{ as } t \to \infty\}. \tag{4.14}$$

In other words, it is the collection of all points that converge to Λ.

All the definitions above now finally allow us to define an attractor:

Definition 4.3.6. Let M be a manifold with a flow φ defined on it. A set $\Lambda \subseteq M$ is an *attractor* if it is an attracting set and there is some point $x \in M$ for which $\Lambda = \omega(x)$.

Remark. Every asymptotically stable equilibrium is an attractor, see Figure 4.1 for an example. However, not every asymptotically stable set is an attractor. An example of this is the following: consider a two-dimensional system given by

$$\begin{cases} \dot{x} = x(1 - x^2) \\ \dot{y} = -y, \end{cases} \tag{4.15}$$

where phase-space is $\mathbb{R}^2$. It can easily be shown that $\Lambda = \{(x, 0) \,:\, x \in [-1, 1]\}$ is an attracting set. However, a trajectory starting in $(\tilde{x}, 0)$ for $\tilde{x} \neq -1, 0, 1$ will eventually converge to either $x = 1$ or $x = -1$. Thus, Λ is not the ω-limit set of any $x \in \mathbb{R}^2$ and therefore cannot be an attractor.

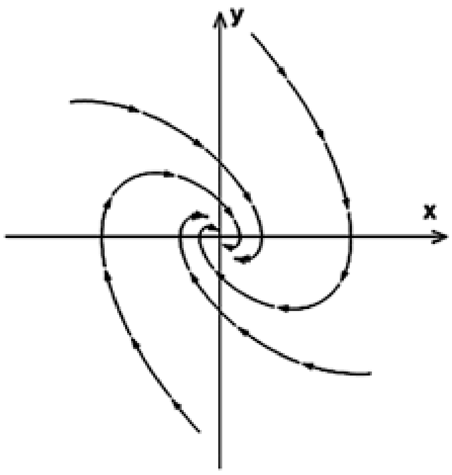

Figure 4.1: An attractor in a two-dimensional phase space. The attractor is the asymptotically stable fixed point $(0, 0)$. Since the flow spirals inward around the equilibrium, $(0, 0)$ is called a *sink*.
Credit: [Layek, 2024].

4.4 Preview to Chapter 6: on the Stability of Trajectories

In Section 6, scaling solutions will be studied. This is a specific type of solutions to multifield inflation models, for which the slow-roll parameter ϵ is constant along the trajectory. In particular, two-field models will be studied, resulting in a four-dimensional system of differential equations. In analysing this system, often the potential and/or metric on field-space will be chosen in a way such that two or three (k) of the differential equations 'decouple', i.e. such that they no longer depend on the variables of which the evolution is governed by the other equation(s). It will then follow that the phase-space of this subsystem consists of a k-dimensional invariant subspace of the original four-dimensional dynamical system.

Of this k-dimensional subsystem of decoupled equations, the critical points will be determined. Since the subsystem is part of a larger four-dimensional system, these critical points correspond to a trajectory or family of trajectories in the four-dimensional phase space. In principle, the stability of trajectories must be studied with Lyapunov exponents (see Section 4.2). However, we will not do this. Instead, we will either calculate the eigenvalues in the subsystem, or, if the eigenvalues depend on variables that can only be found in the original system, calculate the 'eigenvalues' in the original system. In the latter case, it will turn out that there are $4 - k$ zero eigenvalues, corresponding to eigenvectors that are orthogonal to the k-dimensional invariant subspace. We will assume that those zero eigenvalues correspond to the zero Lyapunov exponents along the trajectory, and that they therefore do not influence the stability of the solution. We will thus simply say that the solution is stable if all its nonzero eigenvalues have negative real part in the lower-dimensional subsystem, as is done in [Christodoulidis et al., 2019b]. Nonetheless, the exact relation between the eigenvalues of the critical points in the subsystem and the stability of the corresponding trajectory in the original system remains unclear and a further analysis of the correspondence is required.

Chapter 5

Attractors in Rapid-turn Inflation

Having gathered all prerequisite knowledge in the previous chapters, we are now in a position to start the analysis of two-field models of inflation. The goal of this chapter is to find an expression for an attractor solution that has a large, and slowly varying, turn rate ω. The turn rate is a quantity that measures the the rate at which the inflationary trajectory turns in field-space (see equations 5.11 and 5.13). To do this, we first (Section 5.2) determine a solution to the background equations of motion (equation 3.47 in Chapter 3) that satisfies the slow-roll conditions ($\epsilon, |\eta| \ll 1$) and has a large turn rate ($\omega^2 \gg \mathcal{O}(\epsilon)$) that slowly varies ($\nu := \mathcal{D}_N \ln \omega \ll 1$). See Figure 5.1 for an example of an inflationary trajectory with a large turn rate. After determining such a solution, we study its stability (Section 5.3) by considering spatially homogeneous perturbations. Finally, in Section 5.5 we compare the results of the analysis to several known models of inflation. Throughout this chapter, the analysis of the paper *Rapid-Turn Inflationary Attractors* [Bjorkmo, 2019] is followed. The derivation in Section 5.1 is inspired by [Bjorkmo and Marsh, 2019].

5.1 Deriving the Equations of Motion

Starting from the background equations of motion, we derive equations of motion for the scalar field velocities $\dot{\phi}_v$ and $\dot{\phi}_w$ (equations 5.4a and 5.4b) , which are defined using a *gradient basis* for the tangent spaces to field-space. First, recall that the background equations of motion for multifield inflation (in this case two-field) are given by

$$\mathcal{D}_t\dot{\phi}^a + 3H\dot{\phi}^a + \mathcal{G}^{ab}V_{,b} = 0, \tag{5.1}$$

and the two Friedmann equations by

$$\dot{H} = -\frac{1}{2}\mathcal{G}_{ab}\dot{\phi}^a\dot{\phi}^b, \tag{5.2a}$$

$$3H^2 = \frac{1}{2}\mathcal{G}_{ab}\dot{\phi}^a\dot{\phi}^b + V. \tag{5.2b}$$

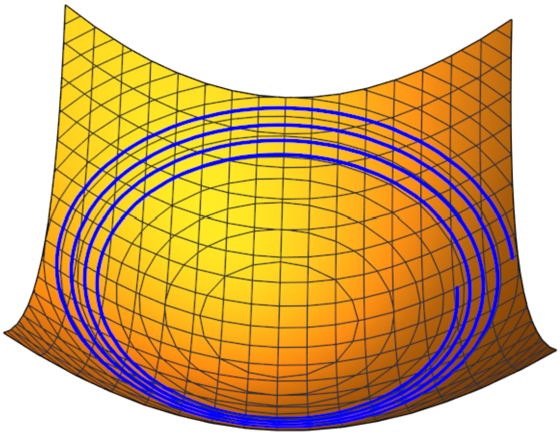

Figure 5.1: Example of a hypothetical inflationary trajectory. The inflationary trajectory (in blue), slowly spirals down towards the minimum of the potential $V(\rho,\theta) = (9x/2)^2 + (9y/2)^2$.

The (potential) gradient basis constitutes of two orthonormal basis vectors v^a and w^a, which are defined in the following way:

$$v^a = \frac{V^{,a}}{V_v}, \qquad \text{where } V_v = \sqrt{V^{,b}V_{,b}}. \tag{5.3}$$

The vector w^a is then defined as a vector that is orthonormal to v^a, i.e. $v^a w_a = \mathcal{G}_{ab} v^a w^a = 0$ and $w^a w_a = 1$. Using these basis vectors, we decompose the field velocity ϕ^a as $\dot{\phi}^a = v^a \dot{\phi}_v + w^a \dot{\phi}_w$, where

$$\dot{\phi}_v \coloneqq v_a \dot{\phi}^a, \tag{5.4a}$$

$$\dot{\phi}_w \coloneqq w_a \dot{\phi}^a. \tag{5.4b}$$

Using the background equations of motion (5.1) and the Friedmann equations (5.2a, 5.2b), we start with the derivation of equations of motion for the scalar field velocities $\dot{\phi}_v$ and $\dot{\phi}_w$. Using the chain rule, we see that

$$\ddot{\phi}_v = \mathcal{D}_t(\dot{\phi}_v) = v_a \mathcal{D}_t(\dot{\phi}^a) + \mathcal{D}_t(v_a)\dot{\phi}^a, \tag{5.5a}$$

$$\ddot{\phi}_w = \mathcal{D}_t(\dot{\phi}_w) = w_a \mathcal{D}_t(\dot{\phi}^a) + \mathcal{D}_t(w_a)\dot{\phi}^a. \tag{5.5b}$$

The $\mathcal{D}_t(\dot{\phi}^a)$ term can be readily rewritten in terms of H, $\dot{\phi}$ and V, but for the $\mathcal{D}_t(v_a)$ and $\mathcal{D}_t(w_a)$ terms, a little more work needs to be done. Using the definitions of the covariant directional derivative for dual vectors (equation 2.78 in Chapter 2) and the vector v^a (5.3) we see that

$$\mathcal{D}_t v_a \overset{(2.78)}{=} \dot{\phi}^b (\nabla_b v_a)$$

$$
\begin{aligned}
&\overset{(5.3)}{=} \dot{\phi}^b \left(\nabla_b \frac{V_{,a}}{V_v}\right) \\
&= \dot{\phi}^b \left(\frac{1}{V_v}\left(\nabla_b V_{,a}\right) - \frac{V_{,a}}{V_v^2}\nabla_b V_v\right) \\
&= \dot{\phi}^b \left(\frac{V_{,ab} - \Gamma^c_{ab} V_{,c}}{V_v} - \frac{V_{,a}}{V_v^2}\frac{1}{2\sqrt{V_{,b}V^{,b}}} \cdot 2V^{,c}\nabla_b V_{,c}\right) \\
&= \dot{\phi}^b \left(\frac{V_{;ab}}{V_v} - \frac{V_{,a}}{V_v^3} V^{,c} V_{;cb}\right) \\
&= \dot{\phi}^b \left(\frac{V_{;ab}}{V_v} - v_a v^c \frac{V_{;cb}}{V_v}\right) \\
&= \frac{V_{;ab}\dot{\phi}^b}{V_v} - v_a \frac{v^b V_{;bc}\dot{\phi}^c}{V_v}.
\end{aligned} \tag{5.6}
$$

To simplify this expression, we use the identity $v^a \cdot v_b + w^a \cdot w_b = \delta^a_b$. Note that the notation here is not meant to imply any summation, so the $v^a v_b$ term for example is just an ordinary multiplication of two (dual) vector components. This identity can easily be derived from the condition for an orthonormal basis of the tangent space, given by $\mathcal{G}\left(v, w\right) = \delta^v_w$. Thus, substituting $v_a v^b = \delta^a_b - w_a w^b$ yields

$$
\mathcal{D}_t v_a = \frac{V_{;ab}\dot{\phi}^b}{V_v} - \delta^a_b \frac{V_{;bc}\dot{\phi}^c}{V_v} + w_a \frac{w^b V_{;bc}\dot{\phi}^c}{V_v} = w_a \frac{w^b V_{;bc}\dot{\phi}^c}{V_v}. \tag{5.7}
$$

Now, differentiating the orthogonality condition $v_a w^a = 0$, we find that

$$
0 = \mathcal{D}_t(v_a) w^a + v_a \mathcal{G}^{ab}\mathcal{D}_t(w_b) = w^a w_a \frac{w^b V_{;bc}\dot{\phi}^c}{V_v} + v_a \mathcal{G}^{ab}\mathcal{D}_t(w_b), \tag{5.8}
$$

yielding

$$
\mathcal{D}_t(w_a) = -v_a \frac{w^b V_{;bc}\dot{\phi}^c}{V_v}, \tag{5.9}
$$

as $v_a v^a = 1$ and $w_a w^a = 1$. Finally, using equations 5.5a, 5.5b and 5.1, we can derive the equations of motion for the velocities $\dot{\phi}_v$ and $\dot{\phi}_w$:

$$
\begin{aligned}
\ddot{\phi}_v = \mathcal{D}_t(\dot{\phi}_v) &= v_a \left(-3H\dot{\phi}^a - V^{,a}\right) + \left(w_a \frac{w^b V_{;bc}\dot{\phi}^c}{V_v}\right)\dot{\phi}^a \\
&= -3H\dot{\phi}_v - V_v + \dot{\phi}_w \frac{w^a V_{;ab}\dot{\phi}^b}{V_v},
\end{aligned} \tag{5.10a}
$$

$$
\begin{aligned}
\ddot{\phi}_w = \mathcal{D}_t(\dot{\phi}_w) &= w_a \left(-3H\dot{\phi}^a - V^{,a}\right) + \left(-v_a \frac{w^b V_{;bc}\dot{\phi}^c}{V_v}\right)\dot{\phi}^a \\
&= -3H\dot{\phi}_w - \dot{\phi}_v \frac{w^a V_{;ab}\dot{\phi}^b}{V_v}.
\end{aligned} \tag{5.10b}
$$

A quantity that will be important for our analysis, is the turn rate, which is defined as

$$\omega := N_a \mathcal{D}_N T^a \tag{5.11}$$

Here $\mathcal{D}_N$ is the covariant directional derivative with respect to e-folds. Recall that $\mathcal{D}_N = H^{-1}\mathcal{D}_t$. In addition, we use that s^a and n^a are the two basis vectors of the kinematic basis (see Section 3.6.1), which is related to the potential gradient basis as [Bjorkmo and Marsh, 2019]

$$T^a = \dot{\phi}^a/\dot{\phi}, \quad \text{and} \quad N^a = (-\dot{\phi}_w v^a + \dot{\phi}_v w^a)/\dot{\phi}. \tag{5.12}$$

Using the potential gradient basis, we may rewrite the turn rate as follows:

$$\begin{aligned}
\omega &= \frac{-\dot{\phi}_w v_a + \dot{\phi}_v w_a}{H\dot{\phi}} \mathcal{D}_t \left(\frac{\dot{\phi}^a}{\dot{\phi}}\right) \\
&= \frac{1}{H\dot{\phi}^2}(-\dot{\phi}_w v_a + \dot{\phi}_v w_a)\mathcal{D}_t(\dot{\phi}^a) + \frac{1}{H\dot{\phi}}(-\dot{\phi}_w \dot{\phi}_v + \dot{\phi}_v \dot{\phi}_w)\mathcal{D}_t\left(\frac{1}{\dot{\phi}}\right) \\
&= \frac{1}{H\dot{\phi}^2}(-\dot{\phi}_w[-3H\dot{\phi}_v - V_v] + \dot{\phi}_v[-3H\dot{\phi}_w]) \\
&= \frac{\dot{\phi}_w V_v}{H\dot{\phi}^2}.
\end{aligned} \tag{5.13}$$

5.2 Finding a Candidate for an Attractor Solution

Equations 5.10a and 5.10b together form a two-dimensional dynamical system with many solutions. However, our goal is to find a solution that is consistent with existing insights into inflation. We do this in the following way: first a condition that guarantees the smallness of η is imposed, and (using $\omega^2 \gg \mathcal{O}(\epsilon)$), expressions for the field-space velocities are derived. Then, we require $\nu \ll 1$, such that the solution has a slowly varying turn rate. This allows us to derive a constraint on where in field-space (or phase space) the rapid-turning attractor might be. In Section 5.3, the stability of the solution that we have derived will be considered.

For a sustained period of inflation, it is necessary for the slow-roll parameter η to be small. For this, we require that the scalar field velocities satisfy

$$\ddot{\phi}_i = \mathcal{O}(\epsilon)H\dot{\phi}_i \quad \text{for } i = v, w, \tag{5.14}$$

or, equivalently (as $\dot{\phi}^2 = \dot{\phi}_v^2 + \dot{\phi}_w^2$), that the total field velocity satisfies

$$\mathcal{D}_t\dot{\phi}^2 = \mathcal{O}(\epsilon)H\dot{\phi}^2. \tag{5.15}$$

From equation 5.15, it follows that

$$\eta := \frac{\mathrm{d}\ln\epsilon}{H\mathrm{d}t} = \frac{1}{H}\frac{2H^2}{\dot{\phi}^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\phi}^2}{2H^2}\right) = \frac{\mathcal{D}_t(\dot{\phi}^2)}{H\dot{\phi}^2} - 2\frac{\dot{H}}{H^2} = \frac{\mathcal{D}_t(\dot{\phi}^2)}{H\dot{\phi}^2} + 2\epsilon \overset{(5.15)}{=} \mathcal{O}(\epsilon). \tag{5.16}$$

Since $\epsilon \ll 1$, we find that η is indeed small. To somewhat simplify equations 5.10a and 5.10b, we define

$$\Omega_v \coloneqq \frac{w^a V_{;ab}\dot{\phi}^b}{V_v} \tag{5.17}$$

and

$$V_{\zeta\xi} \coloneqq \zeta^a \xi^b V_{;ab} \quad \text{for} \quad \zeta, \xi \in \{v, w\}. \tag{5.18}$$

Also note that we can thus write

$$\Omega_v = \frac{V_{vw}\dot{\phi}_v + V_{ww}\dot{\phi}_w}{V_v}. \tag{5.19}$$

Multiplying equations 5.10a and 5.10b by $\dot{\phi}_v$ and $\dot{\phi}_w$, respectively, we see that

$$\frac{1}{2}\mathcal{D}_t(\dot{\phi}_v^2) = -3H\dot{\phi}_v^2 - V_v\dot{\phi}_v + \Omega_v\dot{\phi}_w\dot{\phi}_v \tag{5.20a}$$

$$\frac{1}{2}\mathcal{D}_t(\dot{\phi}_w^2) = -3H\dot{\phi}_w^2 - \Omega_v\dot{\phi}_w\dot{\phi}_v. \tag{5.20b}$$

In addition, since $\dot{\phi}^2 = \dot{\phi}_v^2 + \dot{\phi}_w^2$, it follows that $\frac{1}{2}\mathcal{D}_t(\dot{\phi}^2) = \frac{1}{2}\mathcal{D}_t(\dot{\phi}_v^2) + \frac{1}{2}\mathcal{D}_t(\dot{\phi}_w^2)$. This results in the following:

$$\begin{aligned} \frac{1}{2}\mathcal{O}(\epsilon)H\dot{\phi}^2 = \frac{1}{2}\mathcal{D}_t(\dot{\phi}^2) &= -3H\dot{\phi}_v^2 - V_v\dot{\phi}_v + \Omega_v\dot{\phi}_w\dot{\phi}_v - 3H\dot{\phi}_w^2 - \Omega_v\dot{\phi}_w\dot{\phi}_v \\ &= -3H(\dot{\phi}_v^2 + \dot{\phi}_w^2) - V_v\dot{\phi}_v \\ &= -3H\dot{\phi}^2 - V_v\dot{\phi}_v. \end{aligned} \tag{5.21}$$

From this it straightforwardly follows that

$$\frac{\dot{\phi}_v V_v}{H\dot{\phi}^2} = -3 + \mathcal{O}(\epsilon). \tag{5.22}$$

Now, using the expression for the turn rate ω in equation 5.13, equation 5.22 and assuming $\omega^2 \gg \mathcal{O}(\epsilon)$, we may find expressions for the field-space velocities in terms of the turn rate. It holds that

$$\frac{\dot{\phi}_v^2 + \dot{\phi}_w^2}{\dot{\phi}^4} = \frac{(9 + \mathcal{O}(\epsilon))H^2}{V_v^2} + \frac{\omega^2 H^2}{V_v^2} = \left(9 + \omega^2 + \mathcal{O}(\epsilon)\right)\frac{H^2}{V_v^2}. \tag{5.23}$$

In addition, it holds that $(\dot{\phi}_v^2 + \dot{\phi}_w^2)/\dot{\phi}^2 = 1$, so $\left(9 + \omega^2 + \mathcal{O}(\epsilon)\right)(\dot{\phi}^2 H^2)/(V_v^2) = 1$. This implies that

$$\frac{\dot{\phi}H}{V_v} = \frac{1}{\sqrt{9 + \omega^2 + \mathcal{O}(\epsilon)}}, \qquad \text{or (using } w^2 \gg \mathcal{O}(\epsilon)), \quad \dot{\phi} = \frac{V_v}{H\sqrt{9 + \omega^2}}. \tag{5.24}$$

The expression for $\dot{\phi}_v$ now follows from 5.22:

$$\frac{\dot{\phi}_v}{\dot{\phi}} = (-3 + \mathcal{O}(\epsilon))\frac{\dot{\phi}H}{V_v} = (-3 + \mathcal{O}(\epsilon))\frac{1}{\sqrt{9 + \omega^2 + \mathcal{O}(\epsilon)}} \overset{\omega^2 \gg \mathcal{O}(\epsilon)}{=} -\frac{3}{\sqrt{9 + \omega^2}} + \frac{\mathcal{O}(\epsilon)}{\sqrt{9 + \omega^2}}. \tag{5.25}$$

Analogously, for $\dot{\phi}_w$ it follows that

$$\frac{\dot{\phi}_w}{\dot{\phi}} = \omega\frac{\dot{\phi}H}{V_v} = \frac{\omega}{\sqrt{9+\omega^2+\mathcal{O}(\epsilon)}} \overset{\omega^2 \gg \mathcal{O}(\epsilon)}{=} \frac{\omega}{\sqrt{9+\omega^2}} + \frac{\mathcal{O}(\epsilon)}{\sqrt{9+\omega^2}}. \tag{5.26}$$

Thus, we find the following expressions for $\dot{\phi}_v$ and $\dot{\phi}_w$ in terms of the turn rate and V_v:

$$\dot{\phi}_v = -\frac{(3-\mathcal{O}(\epsilon))V_v}{H(9+\omega^2)} = -\frac{3V_v}{H(9+\omega^2)}, \quad \dot{\phi}_w = \frac{(\omega+\mathcal{O}(\epsilon))V_v}{H(9+\omega^2)} = \frac{\omega V_v}{H(9+\omega^2)}. \tag{5.27}$$

Looking at the equation of motion for $\dot{\phi}_v$ (equation 5.10a) and substituting equations 5.24, 5.27, we get

$$\mathcal{O}(\epsilon)H\left(\frac{-3V_v}{H(9+\omega^2)}\right) = -3H\left(\frac{-3V_v}{H(9+\omega^2)}\right) - V_v + \Omega_v\frac{\omega V_v}{H(9+\omega^2)}, \tag{5.28}$$

which, after rearranging the terms, results in

$$\frac{\Omega_v}{H} = \omega + \mathcal{O}(\epsilon/\omega). \tag{5.29}$$

From this, it is possible to derive a constraint on the attractor solution in terms of the potential V and the turn rate ω. Later (see equation 5.51), we will see that this constraint can be written entirely in terms of the potential. Substituting equation 5.27 in the previous result we derived (equation 5.29), it follows that

$$\omega + \mathcal{O}(\epsilon/\omega) = \frac{V_{vw}\dot{\phi}_v + V_{ww}\dot{\phi}_w}{HV_v} = \frac{1}{HV_v}\left(\frac{-3V_vV_{vw}}{H(9+\omega^2)} + \frac{\omega V_vV_{ww}}{H(9+\omega^2)}\right). \tag{5.30}$$

After rearranging the terms, we find that

$$\frac{V_{ww}}{H^2} - \frac{3}{\omega}\frac{V_{vw}}{H^2} = 9+\omega^2+\mathcal{O}(\epsilon)+\mathcal{O}\big(\epsilon/\omega^2\big) \overset{\omega^2 \gg \mathcal{O}(\epsilon)}{=} \omega^2+9+\mathcal{O}(\epsilon). \tag{5.31}$$

Using the equation of motion for $\dot{\phi}_w$ (equation 5.10b), we can derive an equation analogous to 5.31:

$$\ddot{\phi}_w = \mathcal{O}(\epsilon)H\dot{\phi}_w = \mathcal{O}(\epsilon)H\frac{\omega V_v}{H(9+\omega^2)} = -3H\frac{\omega V_v}{H(9+\omega^2)} + \frac{3V_v\Omega_v}{H(9+\omega^2)}. \tag{5.32}$$

This results in the condition

$$\frac{\Omega_v}{H} = \omega + \mathcal{O}(\epsilon\omega). \tag{5.33}$$

We see that equation 5.31 is written entirely in terms of the potential derivatives V_{ww} and V_{vw} and the turn-rate ω. In order to derive a constraint for the attractor entirely in terms of potential derivatives, we need to express ω in terms of these quantities. For this, we now impose a second condition; the slowly varying turn rate ($\nu := \mathcal{D}_N \ln\omega$). Using the definition of the turn rate (equation 5.13), and the conditions on the field velocities (5.14, 5.15) we simplify this expression:

$$
\begin{aligned}
\nu := \mathcal{D}_N \ln\omega &= \frac{H\dot{\phi}^2}{\dot{\phi}_w V_v}\mathcal{D}_N\left(\frac{\dot{\phi}_w V_v}{H\dot{\phi}^2}\right) \\
&= \frac{H\dot{\phi}^2}{\dot{\phi}_w V_v}\left(\frac{\mathcal{D}_N(\dot{\phi}_w)V_v}{H\dot{\phi}^2} - \frac{\dot{\phi}_w V_v}{H^2\dot{\phi}^2}\mathcal{D}_N(H) - \frac{\dot{\phi}_w V_v}{H\dot{\phi}^4}\cdot 2\dot{\phi}\mathcal{D}_N(\dot{\phi}) + \frac{\dot{\phi}_w \mathcal{D}_N(V_v)}{H\dot{\phi}^2}\right). \\
&= \frac{H\dot{\phi}^2}{\dot{\phi}_w V_v}\left(\frac{V_v}{H^2\dot{\phi}^2}\ddot{\phi}_w - \frac{\dot{\phi}_w V_v}{H^3\dot{\phi}^2}\dot{H} - \frac{\dot{\phi}_w V_v}{H^2\dot{\phi}^4}\mathcal{D}_t(\dot{\phi}^2)\right) + \mathcal{D}_N \ln V_v \\
&\overset{(5.14),(5.15)}{=} \frac{H\dot{\phi}^2}{\dot{\phi}_w V_v}\left(\frac{V_v}{H^2\dot{\phi}^2}\left[\mathcal{O}(\epsilon)H\dot{\phi}_w\right] - \frac{\dot{\phi}_w V_v}{H\dot{\phi}^2}\frac{\dot{H}}{H^2} - \frac{\dot{\phi}_w V_v}{H^2\dot{\phi}^4}\left[\mathcal{O}(\epsilon)H\dot{\phi}^2\right]\right) + \mathcal{D}_N \ln V_v \\
&= \frac{H\dot{\phi}^2}{\dot{\phi}_w V_v}\left(\frac{V_v\dot{\phi}_w}{H\dot{\phi}^2}\mathcal{O}(\epsilon) + \frac{\dot{\phi}_w V_v}{H\dot{\phi}^2}\epsilon - \frac{\dot{\phi}_w V_v}{H\dot{\phi}^2}\mathcal{O}(\epsilon)\right) + \mathcal{D}_N \ln V_v \\
&= \mathcal{O}(\epsilon) + \epsilon + \mathcal{O}(\epsilon) + \mathcal{D}_N \ln V_v. \qquad (5.34)
\end{aligned}
$$

From this, we see that imposing $\nu = \mathcal{O}(\epsilon)$ immediately implies leads to the condition

$$\mathcal{D}_N \ln V_v = \mathcal{O}(\epsilon). \qquad (5.35)$$

Condition 5.35 then allows us to derive a second equation that relates the turn rate to the derivatives of the potential:

$$
\begin{aligned}
\mathcal{D}_N \ln V_v &= \frac{1}{HV_v}\mathcal{D}_t\sqrt{V^{,a}V_{,a}} \\
&= \frac{1}{HV_v}\frac{1}{2V_v}\dot{\phi}^b\nabla_b V_v \\
&= \frac{\dot{\phi}^b}{2HV_v^2}\cdot 2V^{,c}\nabla_b V_{,c} \\
&= \frac{V^{,b}\dot{\phi}^c V_{;bc}}{HV_v^2} \\
&= \frac{v^b}{HV_v}\left(v^b v^c V_{;bc}\dot{\phi}_v + v^b w^c V_{;bc}\dot{\phi}_w\right) \\
&= \frac{1}{HV_v}\left(V_{vv}\cdot\frac{-3V_v}{H(9+\omega^2)} + V_{vw}\cdot\frac{\omega V_v}{H(9+\omega^2)}\right) \\
&= \frac{-3V_{vv} + \omega V_{vw}}{H^2(9+\omega^2)} = \mathcal{O}(\epsilon). \qquad (5.36)
\end{aligned}
$$

This now results in

$$\frac{V_{vw}}{H^2} - \frac{3}{\omega}\frac{V_{vv}}{H^2} = \mathcal{O}(\epsilon\omega) + \mathcal{O}(\epsilon/\omega) \overset{\omega^2 \gg \mathcal{O}(\epsilon)}{=} \mathcal{O}(\epsilon\omega). \qquad (5.37)$$

We can also substitute this in equation 5.31 to get the convenient expression

$$\frac{V_{ww}}{H^2} - \frac{9}{\omega^2}\frac{V_{vv}}{H^2} = \omega^2 + 9 + \mathcal{O}(\epsilon) + \frac{3}{\omega}\mathcal{O}(\omega\epsilon) = \omega^2 + 9 + \mathcal{O}(\epsilon). \tag{5.38}$$

To derive an expression for the turn rate in terms of derivatives of the potential, we consider two cases: one where V_{vv} and V_{vw} are negligible, and one where they are not.

5.2.1 The Negligible Case

Suppose V_{vv} and V_{vw} are negligible, i.e. $V_{vv}/H^2 \lesssim \mathcal{O}(\omega^2\epsilon)$ and $V_{vw}/H^2 \lesssim \mathcal{O}(\omega\epsilon)$. Both equation 5.31 and 5.38 then straightforwardly yield

$$\omega^2 = \frac{V_{ww}}{H^2} - 9 + \mathcal{O}(\epsilon) \simeq \frac{V_{ww}}{H^2} - 9. \tag{5.39}$$

Thus, for the scalar field velocities (equation 5.27) we find

$$\dot{\phi}_v = -\frac{3V_v}{H(9+\omega^2)} = -\frac{3V_v}{H\left(\frac{V_{ww}}{H^2}\right)} = -\frac{3V_v H}{V_{ww}}, \tag{5.40}$$

and

$$\dot{\phi}_w = \frac{\omega V_v}{H(9+\omega^2)} = \frac{\sqrt{\frac{V_{ww}-9H^2}{H^2}}}{H\left(\frac{V_{ww}}{H^2}\right)} = V_v\sqrt{\frac{V_{ww}-9H^2}{V_{ww}^2}}. \tag{5.41}$$

This case is for example relevant for models of hyperinflation, see Section 5.5.

5.2.2 The Non-negligible Case

Now suppose that V_{vv} and V_{vw} are *not* negligible. The condition on V_v (equation 5.35) is equivalent to

$$\mathcal{D}_N \ln V_v = \frac{1}{HV_v}\mathcal{D}_t V_v = \frac{1}{HV_v}\dot{V}_v = \mathcal{O}(\epsilon), \tag{5.42}$$

since V_v is just a scalar-valued function. In addition, equation 5.36 gives us that $\dot{V}_v = V_{vv}\dot{\phi}_v + V_{vw}\dot{\phi}_w$, so we find

$$V_{vv}\dot{\phi}_v + V_{vw}\dot{\phi}_w \simeq 0. \tag{5.43}$$

This way, we can find two expressions for ω. Firstly, substituting the expressions for $\dot{\phi}_v$ and $\dot{\phi}_w$ of equation 5.27, we find

$$\dot{\phi}_w = \frac{\omega V_v}{H(9+\omega^2)} = -\frac{V_{vv}}{V_{vw}} \cdot \frac{-3V_v}{H(9+\omega^2)} = -\frac{V_{vv}}{V_{vw}}, \tag{5.44}$$

or

$$\omega = \frac{3V_{vv}}{V_{vw}}. \tag{5.45}$$

The second expression for ω can now be obtained using equation 5.38:

$$\begin{aligned}\omega^2 &= \frac{V_{ww}}{H^2} - \frac{9}{\omega^2}\frac{V_{vv}}{H^2} - 9 + \mathcal{O}(\epsilon) \\ &= \frac{V_{ww}}{H^2} - \frac{9}{\left(\frac{3V_{vv}}{V_{vw}}\right)^2}\frac{V_{vv}}{H^2} - 9 + \mathcal{O}(\epsilon) \\ &\simeq \frac{V_{ww}}{H^2} - \frac{V_{vw}^2}{V_{vv}^2}\frac{V_{vv}}{H^2} - 9. \end{aligned} \tag{5.46}$$

These two expressions for ω must then be matched in order for rapid-turn inflation to take place. Note that after a solution is found, it must be verified that the assumption $\omega^2 \gg \mathcal{O}(\epsilon)$ holds.

We can also use equations 5.45 and 5.46 to derive a 'constraint' on rapid-turn inflation, i.e. an equation that 'selects' a region of field-space in which rapid-turn inflation may take place (this constraint is also derived in [Wolters et al., 2024]). To derive this, we substitute 5.46 into equation 5.38:

$$\frac{V_{ww}}{H^2} - \frac{9}{\frac{V_{ww}}{H^2} - \frac{V_{vw}^2}{V_{vv}^2}\frac{V_{vv}}{H^2} - 9}\frac{V_{vv}}{H^2} = \frac{V_{ww}}{H^2} - \frac{V_{vw}^2}{V_{vv}^2}\frac{V_{vv}}{H^2} - 9 + \mathcal{O}(\epsilon), \tag{5.47}$$

which can be rewritten to give

$$\frac{9}{V_{ww} - \frac{V_{vw}^2}{V_{vv}} - 9H^2} = -\frac{V_{vw}^2}{V_{vv}H^2}, \tag{5.48}$$

or

$$9V_{vv} = \frac{V_{vw}^2}{V_{vv}H^2}\frac{V_{vw}^2}{H^2} - \left(\frac{V_{vw}^2}{V_{vv}}\right)^2\frac{1}{H^2} - \frac{9V_{vw}^2}{V_{vv}}. \tag{5.49}$$

We rearrange some terms and use that $V = (3-\epsilon)H^2$, which follows from the Friedmann equation (5.2a and 5.2b) as

$$V = 3H^2 - \dot{\phi}^2 = \left(3 - \frac{\dot{\phi}^2}{H^2}\right)H^2 = (3-\epsilon)H^2, \tag{5.50}$$

to finally obtain

$$\frac{V_{ww}}{V} = 3 + 3\left(\frac{V_{vv}}{V_{vw}}\right)^2 + \frac{V_{vw}^2}{V_{vv}V} + \mathcal{O}(\epsilon). \tag{5.51}$$

This is the desired constraint for rapid-turn inflation in case the 'potential derivatives' V_{vv} and V_{vw} are non-negligible.

5.3 Inspecting the Stability

In the previous section, we derived a constraint on rapid-turn inflation in terms of the background equations of motion. In other words, we derived in which part of field-space the desired form of inflation may take place. This therefore corresponds with finding the leading order parts of the solutions $\dot{\phi}_v = \bar{\dot{\phi}}_v + \delta\dot{\phi}_v$ and $\dot{\phi}_w = \bar{\dot{\phi}}_w + \delta\dot{\phi}_w$. In order to inspect the stability of the background solution, we therefore now need to look at precisely those terms $\delta\dot{\phi}_v$ and $\delta\dot{\phi}_w$. In this section, we impose the same conditions on the equations of motion for the perturbations as we did on the equations of motion for the scalar field velocities. This way, we may determine the stability of the candidate attractor solution that satisfies equation 5.51 (or equations 5.40 and 5.41).

When considering those perturbations, it is useful to use not the gradient basis $\{v^a, w^a\}$, but the (also orthonormal) kinematic basis $\{T^a, N^a\}$. These basis can be directly related to each other, see equation 5.12. In this basis, the equations of motion for the perturbations can be compactly written as (see e.g. [Achúcarro et al., 2011], [Bjorkmo and Marsh, 2019]), [Bjorkmo, 2019]

$$\delta\phi''^a + \left[(3-\epsilon)\delta^a_b - 2\omega\epsilon^a{}_b\right]\delta\phi'^b + C(k)^a{}_b\delta\phi^b = 0, \tag{5.52}$$

where the matrix $C(k)^a_b$ is given by

$$C(k)^a{}_b = \begin{pmatrix} \mu_n - \omega^2 + \frac{k^2}{a^2H^2} & \mu_\times - \omega(3-\epsilon+\nu) \\ \mu_\times + \omega(3-\epsilon+\nu) & \mu_s - \omega^2 + \frac{k^2}{a^2H^2} \end{pmatrix}. \tag{5.53}$$

In the previous two equations, we have denoted $\mathrm{d}/\mathrm{d}N$ by $'$s and defined

$$\mu_n = \frac{1}{H^2}T^aT^bM_{ab}, \quad \mu_\times = \frac{1}{H^2}T^aN^bM_{ab}, \quad \text{and} \quad \mu_s = \frac{1}{H^2}N^aN^bM_{ab}, \tag{5.54}$$

which are the projections of the dimensionless mass-matrix M_{ab} onto the kinematic basis vectors. This matrix M_{ab} is defined as

$$M_{ab} \coloneqq V_{;ab} - R_{acdb}\dot{\phi}^c\dot{\phi}^d + \frac{3-\epsilon}{H^2}\dot{\phi}_a\dot{\phi}_b + \frac{\dot{\phi}_aV_{,b} + V_{,a}\dot{\phi}_b}{H}, \tag{5.55}$$

where R_{acdb} is the Riemann tensor (acting on field-space).

Equation 5.52 can also be written as a set of two equations; less compact but more insightful:

$$\delta\phi''^T + (3-\epsilon)\delta\phi'^T - 2\omega\delta\phi'^N + \left(\mu_n - \omega^2 + \frac{k^2}{a^2H^2}\right)\delta\phi^T + \left(\mu_\times - \omega(3-\epsilon+\nu)\right)\delta\phi^N = 0 \tag{5.56a}$$

$$\delta\phi''^N + (3-\epsilon)\delta\phi'^T + 2\omega\delta\phi'^T + \left(\mu_s - \omega^2 + \frac{k^2}{a^2H^2}\right)\delta\phi^N + \left(\mu_\times + \omega(3-\epsilon+\nu)\right)\delta\phi^T = 0. \tag{5.56b}$$

5.3.1 Deriving Expressions for the Projections of the Mass Matrix

First, we deduce expressions for $\mu_s, \mu_\times$ and μ_n in terms of the gradient basis. For μ_n we have

$$\begin{aligned}
\mu_T = \frac{1}{H^2} T^a T^b M_{ab} &= \frac{1}{H^2 \dot{\phi}^2} \left(v^a \dot{\phi}_v + w^a \dot{\phi}_w \right) \left(v^b \dot{\phi}_v + w^b \dot{\phi}_w \right) V_{;ab} \\
&\quad + \frac{1}{H^2 \dot{\phi}^2} \dot{\phi}^a \dot{\phi}^b \left(-R_{acdb} \dot{\phi}^c \dot{\phi}^d + \frac{3-\epsilon}{H^2} \dot{\phi}_a \dot{\phi}_b + \frac{\dot{\phi}_a V_{,b} + V_{,a} \dot{\phi}_b}{H} \right) \\
&= \frac{\dot{\phi}_v^2 V_{vv} + 2\dot{\phi}_v \dot{\phi}_w V_{vw} + \dot{\phi}_w^2 V_{ww}}{H^2 \dot{\phi}^2} + 0 + \frac{1}{H^2 \dot{\phi}^2} \left(\frac{3-\epsilon}{H^2} \dot{\phi}^4 + \dot{\phi}^4 \frac{2\dot{\phi}^b V_{,b}}{\dot{\phi}^2 H} \right) \\
&= \frac{\dot{\phi}_v^2 V_{vv} + 2\dot{\phi}_v \dot{\phi}_w V_{vw} + \dot{\phi}_w^2 V_{ww}}{H^2 \dot{\phi}^2} + \frac{\dot{\phi}^2}{H^2} \left(\frac{3-\epsilon}{H^2} + \frac{2\dot{\phi}^a V_{,a}}{\dot{\phi}^2 H} \right) \\
&= \frac{\dot{\phi}_v^2 V_{vv} + 2\dot{\phi}_v \dot{\phi}_w V_{vw} + \dot{\phi}_w^2 V_{ww}}{H^2 \dot{\phi}^2} + 2\epsilon \left(\frac{3-\epsilon}{H^2} + \frac{2\dot{\phi}^a V_{,a}}{\dot{\phi}^2 H} \right). \qquad (5.57)
\end{aligned}$$

In the second step of this derivation, the antisymmetry of the Riemann tensor was used (see Chapter 2). To simplify the last term in this equation, note that

$$\dot{\phi}^a V_{,a} = (v^a \dot{\phi}_v + w^a \dot{\phi}_w) v_a V_v = V_v \dot{\phi}_v, \qquad (5.58)$$

where we used the orthonormality of v^a and w^a. In addition, recall that the slow-roll parameter η can be written as (see also equation 5.16) $\eta = \frac{\mathcal{D}_t(\dot{\phi}^2)}{H\dot{\phi}^2} + 2\epsilon$. We now rewrite the first term in this equation. In doing so, recall that $\dot{\phi}^2 = \dot{\phi}_v^2 + \dot{\phi}_w^2$, so using the background equations of motion (5.10a and 5.10b), we find

$$\mathcal{D}_t \dot{\phi}^2 = 2\dot{\phi}_v \mathcal{D}_t \dot{\phi}_v + 2\dot{\phi}_w \mathcal{D}_t \dot{\phi}_w = -6H\dot{\phi}^2 - 2\dot{\phi}_v V_v. \qquad (5.59)$$

Thus, we obtain

$$\eta = \frac{\mathcal{D}_t(\dot{\phi}^2)}{H\dot{\phi}^2} + 2\epsilon = \frac{-6H\dot{\phi}^2 - 2\dot{\phi}_v V_v}{H\dot{\phi}^2} + 2\epsilon = -6 - 2\frac{\dot{\phi}_v V_v}{H\dot{\phi}^2} + 2\epsilon. \qquad (5.60)$$

Now, we may use equations 5.60 and 5.58, to give

$$2\epsilon \left(\frac{3-\epsilon}{H^2} + \frac{2\dot{\phi}^a V_{,a}}{\dot{\phi}^2 H} \right) = 2\epsilon \left(\frac{3-\epsilon}{H^2} - 6 - \eta + 2\epsilon \right), \qquad (5.61)$$

so that we obtain

$$\mu_n = \frac{\dot{\phi}_v^2 V_{vv} + 2\dot{\phi}_v \dot{\phi}_w V_{vw} + \dot{\phi}_w^2 V_{ww}}{H^2 \dot{\phi}^2} - 2\epsilon \left(\frac{3-\epsilon}{H^2} - 6 - \eta + 2\epsilon \right). \qquad (5.62)$$

For $\mu_\times$, the derivation is analogous but a little less involved:

$$\begin{aligned}
\mu_\times = \frac{1}{H^2}T^aN^aM_{ab} &= \frac{1}{H^2\dot\phi^2}\left(v^a\dot\phi_v + w^a\dot\phi_w\right)\left(-\dot\phi_w v^b + \dot\phi_v w^b\right)V_{;ab} \\
&\quad + \frac{1}{H^2\dot\phi^2}\dot\phi^a s^b\left(-R_{acdb}\dot\phi^c\dot\phi^d + \frac{3-\epsilon}{H^2}\dot\phi_a\dot\phi_b + \frac{\dot\phi_a V_{,b} + V_{,a}\dot\phi_b}{H}\right) \\
&= \frac{-\dot\phi_v\dot\phi_w V_{vv} + \dot\phi_v^2 V_{vw} - \dot\phi_w^2 V_{vw} + \dot\phi_v\dot\phi_w V_{ww}}{H^2\dot\phi^2} + 0 \\
&\quad + \frac{3-\epsilon}{H^2}\left(-\dot\phi_w v^b\dot\phi_b + \dot\phi_v w^b\dot\phi_b\right) + \frac{\dot\phi^2}{H}\left(-\dot\phi_w v^b V_{,b} + \dot\phi_v w^b V_{,b}\right) \\
&\quad + \frac{\dot\phi^a V_{,a}}{H}\left(-\dot\phi_w\dot\phi_v + \dot\phi_v\dot\phi_w\right) \\
&= \frac{(V_{ww}-V_{vv})\dot\phi_v\dot\phi_w + V_{vw}(\dot\phi_v^2 - \dot\phi_w^2)}{H^2\dot\phi^2} + \frac{\dot\phi^2}{H^3\dot\phi^2}\left(-\dot\phi_w v^b V_{,b}V_v + \dot\phi_v w^b v_b V_v\right) \\
&= \frac{(V_{ww}-V_{vv})\dot\phi_v\dot\phi_w + V_{vw}(\dot\phi_v^2 - \dot\phi_w^2)}{H^2\dot\phi^2} + \frac{\dot\phi^2}{H^2}\left(-\frac{\dot\phi_w V_v}{H\dot\phi^2} + \frac{1}{H\dot\phi^2}\cdot 0\right) \\
&= \frac{(V_{ww}-V_{vv})\dot\phi_v\dot\phi_w + V_{vw}(\dot\phi_v^2 - \dot\phi_w^2)}{H^2\dot\phi^2} - 2\epsilon\omega. \qquad (5.63)
\end{aligned}$$

Here, we used the definition of the turn rate ω in the last step. Now for the last projection term μ_N, we get

$$\begin{aligned}
\mu_N = \frac{1}{H^2}N^aN^bM_{ab} &= \frac{-\dot\phi_w^2 v^a v^b - \dot\phi_v\dot\phi_w(v^a w^b + v^b w^a) + \dot\phi_v^2 w^a w^b}{H^2\dot\phi}\left(V_{;ab} + \frac{3-\epsilon}{H^2}\dot\phi_a\dot\phi_b\right. \\
&\quad \left. + \frac{\dot\phi_a V_{,b} + \dot\phi_b V_{,a}}{H}\right) - \frac{\dot\phi^2}{H^2}s^a s^b n^c n^d R_{acdb} \\
&= \frac{V_{vv}\dot\phi_w^2 - 2\dot\phi_v\dot\phi_w V_{vw} + V_{vv}\dot\phi_w^2}{H^2\dot\phi^2} + \frac{1}{H^2\dot\phi^2}\cdot(0+0) + \frac{\dot\phi^4 s^a s^b n^c n^d R_{acdb}}{\dot\phi^2 H^2} \\
&= \frac{V_{vv}\dot\phi_w^2 - 2\dot\phi_v\dot\phi_w V_{vw} + V_{vv}\dot\phi_w^2 + \frac{R\dot\phi^4}{2}}{H^2\dot\phi^2}. \qquad (5.64)
\end{aligned}$$

For a better overview, we summarise the results obtained thus far:

$$\mu_T = \frac{\dot\phi_v^2 V_{vv} + 2\dot\phi_v\dot\phi_w V_{vw} + \dot\phi_w^2 V_{ww}}{H^2\dot\phi^2} - 2\epsilon\left(\frac{3-\epsilon}{H^2} - 6 - \eta + 2\epsilon\right), \qquad (5.65a)$$

$$\mu_\times = \frac{(V_{ww}-V_{vv})\dot\phi_v\dot\phi_w + V_{vw}(\dot\phi_v^2 - \dot\phi_w^2)}{H^2\dot\phi^2} - 2\epsilon\omega, \qquad (5.65b)$$

$$\mu_N = \frac{V_{vv}\dot\phi_w^2 - 2\dot\phi_v\dot\phi_w V_{vw} + V_{vv}\dot\phi_w^2 + \frac{R\dot\phi^4}{2}}{H^2\dot\phi^2}. \qquad (5.65c)$$

5.3.2 Rewriting the Perturbations

In the previous section, we found expressions for the components of the dimensionless mass matrix. Using these, we now impose the conditions for rapid-turn inflation on the equations of motion, in order to simplify those. Recall that $\omega = \dot\phi_w V_v / H\dot\phi^2$. We show that $\mathcal{D}_N\omega = -\mu_\times + \omega(-3+\epsilon-\eta)$:

$$\begin{aligned}
\mathcal{D}_N\omega &= \frac{\mathcal{D}_N(\dot\phi_w)V_v}{H\dot\phi^2} + \frac{\dot\phi_w \mathcal{D}_N(V_v)}{H\dot\phi^2} - \frac{\dot\phi_w V_v}{H^2\dot\phi^2}\mathcal{D}_N(H) - \frac{\dot\phi_w V_v}{H\dot\phi^4}\mathcal{D}_N\dot\phi^2 \\
&= \frac{V_v}{H^2\dot\phi^2}\ddot\phi_w + \frac{\dot\phi_w}{H^2\dot\phi^2}\mathcal{D}_t(\sqrt{V^{,a}V_{,a}}) - \frac{\dot\phi_w V_v}{H^3\dot\phi^2}\dot H - \frac{\dot\phi_w V_v}{H^2\dot\phi^4}\mathcal{D}_t(\dot\phi^2) \\
&\overset{(5.59)}{=} \frac{V_v}{H^2\dot\phi^2}\left(-3H\dot\phi_w - \Omega_v\dot\phi_v\right) + \frac{\dot\phi_w}{H^2\dot\phi^2}\left(V_{vv}\dot\phi_v + V_{vw}\dot\phi_w\right) \\
&\quad + \frac{\dot\phi_w V_v}{H\dot\phi^2}\cdot\frac{\dot\phi^2}{2H^2} - \frac{\dot\phi_w V_v}{H^2\dot\phi^4}\left(-6H\dot\phi^2 - 2\dot\phi_v V_v\right) \\
&= \frac{\dot\phi_w V_v}{H\dot\phi^2}\left(-3+\epsilon+6+\frac{2\dot\phi_v V_v}{\dot\phi^2}\right) + \frac{1}{H^2\dot\phi^2}\left(\Omega_v V_v\dot\phi_v + V_{vv}\dot\phi_v\dot\phi_w + V_{vw}\dot\phi_w^2\right) \\
&= \omega(-3+3\epsilon-\eta) + \frac{1}{H^2\dot\phi^2}\left(-(V_{vw}\dot\phi_v + V_{ww}\dot\phi_w)\dot\phi_v + V_{vv}\dot\phi_v\dot\phi_w + V_{vw}\dot\phi_w^2\right) \\
&= -\frac{1}{H^2\dot\phi^2}\left((V_{ww}-V_{vv})\dot\phi_v\dot\phi_w + V_{vw}(\dot\phi_v^2 - \dot\phi_w^2)\right) + \omega(-3+3\epsilon-\eta) \\
&= -\mu_\times + \omega(-3+\epsilon-\eta).
\end{aligned} \tag{5.66}$$

Requiring $\nu = \mathcal{D}_N \ln\omega \simeq \mathcal{O}(\epsilon)$ now yields

$$\mathcal{D}_N \ln\omega = \frac{-\mu_\times + \omega(-3+\epsilon-\eta)}{\omega} = -\frac{\mu_\times}{\omega} - 3 + \mathcal{O}(\epsilon) = \mathcal{O}(\epsilon), \tag{5.67}$$

from which it follows that

$$\mu_\times = -3\omega + \mathcal{O}(\omega\epsilon). \tag{5.68}$$

Using equations 5.37 and 5.38, we can derive an analogous condition for the μ_T component of the mass matrix:

$$\begin{aligned}
\mu_T &= \frac{\dot\phi_v^2 V_{vv} + 2\dot\phi_v\dot\phi_w V_{vw} + \dot\phi_w^2 V_{ww}}{H^2\dot\phi^2} - 2\epsilon\left(\frac{3-\epsilon}{H^2} - 6 - \eta + 2\epsilon\right) \\
&= \frac{V_{vv}}{H^2}\frac{\dot\phi_v^2}{\dot\phi^2} + 2\frac{V_{vw}}{H^2}\frac{\dot\phi_v\dot\phi_w}{\dot\phi^2} + \frac{V_{ww}}{H^2}\frac{\dot\phi_w^2}{\dot\phi^2} + \mathcal{O}(\epsilon) \\
&= \frac{V_{vv}}{H^2}\frac{\dot\phi_v^2}{\dot\phi^2} + 2\left(\frac{3}{\omega}\frac{V_{vv}}{H^2} + \mathcal{O}(\omega\epsilon)\right)\frac{\dot\phi_v\dot\phi_w}{\dot\phi^2} + \left(\frac{9}{\omega^2}\frac{V_{vv}}{H^2} + \omega^2 + 9 + \mathcal{O}(\epsilon)\right)\frac{\dot\phi_w^2}{\dot\phi^2} + \mathcal{O}(\epsilon) \\
&= \frac{V_{vv}}{H^2}\left(\frac{9}{9+\omega^2} - \frac{18\omega}{\omega(9+\omega^2)} + \frac{9\omega^2}{\omega^2(9+\omega^2)}\right) - \frac{6\omega\mathcal{O}(\omega\epsilon)}{9+\omega^2} + \frac{\omega^2(9+\omega^2)}{9+\omega^2} + \frac{\omega^2\mathcal{O}(\epsilon)}{9+\omega^2}
\end{aligned}$$

$$
\begin{aligned}
&= 0 + \mathcal{O}(\epsilon) + \omega^2 + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon) \\
&= \omega^2 + \mathcal{O}(\epsilon). && (5.69)
\end{aligned}
$$

Finally, we derive such a condition for the μ_N term of the mass matrix using equations 5.65c 5.37:

$$
\begin{aligned}
\mu_N &= \frac{V_{ww}}{H^2}\left(\frac{\dot{\phi}_v}{\dot{\phi}}\right)^2 - 2\frac{V_{vw}}{H^2}\frac{\dot{\phi}_v}{\dot{\phi}}\frac{\dot{\phi}_w}{\dot{\phi}} + \frac{V_{vv}}{H^2}\left(\frac{\dot{\phi}_w^2}{\dot{\phi}}\right)^2 + \frac{R\dot{\phi}^2}{2H^2} \\
&= \frac{V_{ww}}{H^2} frac99 + \omega^2 + 2\frac{V_{vw}}{H^2}\frac{3\omega}{9+\omega^2} + \frac{V_{vv}}{H^2}\frac{\omega^2}{9+\omega^2} + \frac{R\dot{\phi}^2}{2H^2} \\
&\overset{5.37}{=} \frac{9V_{ww}}{H^2(9+\omega^2)} + 2\frac{V_{vv}}{H^2}\frac{9}{9+\omega^2} + \frac{V_{vv}}{H^2}\frac{\omega^2}{9+\omega^2} + \frac{R\dot{\phi}^2}{2H^2} + \mathcal{O}(\epsilon) \\
&= \frac{9V_{ww}}{H^2(9+\omega^2)} + \frac{V_{vv}}{H^2}\left(1 + \frac{9}{9+\omega^2}\right) + \frac{R\dot{\phi}^2}{2H^2}. && (5.70)
\end{aligned}
$$

We now reconsider equations 5.56a and 5.56b using the results we just derived for $\mu_\times$ and μ_T. We define $\delta\pi^a = \delta\phi'^a$ for $a \in \{T, N\}$. The perturbations can then be written in matrix form as

$$
\begin{pmatrix} \delta\phi'_T \\ \delta\phi'_N \\ \delta\pi'_T \\ \delta\pi'_N \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -\left(\mu_T - \omega^2 + \frac{k^2}{a^2H^2}\right) & -\left(\mu_\times - \omega(3-\epsilon+\nu)\right) & -(3-\epsilon) & 2\omega \\ -\left(\mu_\times - \omega(3-\epsilon+\nu)\right) & -\left(\mu_N - \omega^2 + \frac{k^2}{a^2H^2}\right) & -2\omega & -(3-\epsilon) \end{pmatrix} \begin{pmatrix} \delta\phi_T \\ \delta\phi_N \\ \delta\pi_T \\ \delta\pi_N \end{pmatrix}. \tag{5.71}
$$

In this we follow the notation of [Bjorkmo, 2019] in denoting the components of the perturbations. Using the equations for μ_n and $\mu_\times$ we just derived, ignoring all $\mathcal{O}(\epsilon)$ corrections and setting $k = 0$, we find that

$$
\begin{pmatrix} \delta\phi'_T \\ \delta\phi'_N \\ \delta\pi'_T \\ \delta\pi'_N \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 6\omega + \mathcal{O}(\omega\epsilon) & -3 & 2\omega \\ \mathcal{O}(\omega\epsilon) & -\mu_N + \omega^2 & -2\omega & -3 \end{pmatrix} \begin{pmatrix} \delta\phi_T \\ \delta\phi_N \\ \delta\pi_T \\ \delta\pi_N \end{pmatrix}, \tag{5.72}
$$

seeing as

$$
\begin{aligned}
\mu_T - \omega^2 &= \omega^2 + \mathcal{O}(\epsilon) - \omega^2 = \mathcal{O}(\epsilon) && (5.73a) \\
\mu_\times - \omega(3-\epsilon+\nu) &= -3\omega - 3\omega + \mathcal{O}(\omega\epsilon) + \epsilon\omega - \omega\nu = -6\omega + \mathcal{O}(\omega\epsilon), && (5.73b) \\
\mu_\times + \omega(3-\epsilon+\nu) &= -3\omega + \mathcal{O}(\omega\epsilon) + 3\omega - \epsilon\omega + \omega\nu = \mathcal{O}(\omega\epsilon). && (5.73c)
\end{aligned}
$$

5.4 Discussion of Results

It is important to note that our results differ from the results in [Bjorkmo, 2019] at three points:

- For the TT-projection of the dimensionless mass matrix μ_T, the result stated in [Bjorkmo, 2019] is the following:

$$\mu_T = \frac{\dot{\phi}_v^2 V_{vv} + 2\dot{\phi}_v\dot{\phi}_w V_{vw} + \dot{\phi}_w^2 V_{ww}}{H^2\dot{\phi}^2} - 2\epsilon\left(3 - \epsilon + \eta\right), \tag{5.74}$$

which significantly differs from our result (see equation 5.65a) in the second term. Fortunately, the different results for this second term have no impact on the derivation of the condition on μ_T (see equation 5.69), as in both cases the second term is of $\mathcal{O}(\epsilon)$.

- The results of the NN-projection of M_{ab} also differ. The result found in [Bjorkmo, 2019] is

$$\mu_N = \frac{9V_{ww}}{H^2(9+\omega^2)} + \frac{V_{vv}}{H^2} + \frac{R\dot{\phi}^2}{2H^2}, \tag{5.75}$$

which again differs from our result (equation 5.70) in the second term. Although the results are different, the terms on which μ_N depends are still the same.

- A potentially more impactful difference is found in the equations of motion for the perturbations, in this analysis represented as a matrix in equation 5.72. In [Bjorkmo, 2019], the $ij = 41$ term is simply zero, while the $ij = 32$ term is 6ω. It is mentioned in [Bjorkmo, 2019] that the $\mathcal{O}(\epsilon)$ terms are neglected. However, per our analysis, this result can only be retrieved if either the $\mathcal{O}(\omega\epsilon)$ terms our neglected as well, or if the additional assumption is made that $\omega = \mathcal{O}(1)$ (which is rather more restrictive than only $\omega^2 \gg \mathcal{O}(\epsilon)$).

If one assumes that the analysis in [Bjorkmo, 2019] is correct, the eigenvalues of the evolution matrix are given by

$$\lambda_1 = -3, \quad \lambda_2 = 0, \quad \lambda_\pm = \frac{1}{2}\left(-3 \pm \sqrt{9 - 4\mu_N - 12\omega^2}\right), \tag{5.76}$$

where the eigenvector corresponding to the zero eigenvalue is $v_2 = (1,0,0,0)$, which thus points along the trajectory (by definition of the kinematic basis vector T^a). Assuming that these eigenvalues correspond to the local Lyapunov exponents of the trajectory (see the discussion in Section 4.4), the trajectory is thus stable if $\mu_N > -3\omega^2$. This should then imply that the solutions satisfying either of the constraints 5.39, 5.51 is an attractor.

Assuming that our analysis is correct, however, yields

$$\begin{aligned}
\lambda_{\pm,1} &= \frac{1}{2}\left(-3 \pm \sqrt{9 - 2\mu_N - 6\omega^2 - 2\sqrt{\mathcal{O}(\omega^2\epsilon^2) + \mathcal{O}(\omega^2\epsilon) + (\mu_N + 3\omega^2)^2}}\right), \\
\lambda_{\pm,2} &= \frac{1}{2}\left(-3 \pm \sqrt{9 - 2\mu_N - 6\omega^2 + 2\sqrt{\mathcal{O}(\omega^2\epsilon^2) + \mathcal{O}(\omega^2\epsilon) + (\mu_N + 3\omega^2)^2}}\right).
\end{aligned} \tag{5.77}$$

These eigenvectors are of course equal to the ones in equation 5.76 if $\mathcal{O}(\omega\epsilon)$ can be neglected. The key difference between the two sets of eigenvalues is that the ones in 5.77 not only depend on μ_N and ω, but also have $\mathcal{O}(\omega^2\epsilon^2)$, $\mathcal{O}(\omega^2\epsilon)$ terms.

To conclude the analysis of this chapter, we summarise the results that have been obtained. Starting with the background equations of motion and the Friedmann equations for two scalar fields, equations of motions for the scalar field velocities $\dot{\phi}_v$ and $\dot{\phi}_w$ (equations 5.10a and 5.10b) were obtained. Imposing the conditions for a rapidly turning solution (in particular $\omega^2 \gg \mathcal{O}(\epsilon)$), a constraint was derived, restricting the region in field-space where rapid-turn inflation may take place (equation 5.39) for the negligible case and equation 5.51 for the non-negligible case). After this, the equations of motion for spatially homogeneous perturbations (see 5.52) were simplified using the same conditions for a rapidly turning solution. In [Bjorkmo, 2019], it was then found that the solution satisfying either of the constraints is an attractor. In the present analysis, however, the eigenvalues of the evolution matrix were found to be more complicated. Therefore it remains unclear whether the 'attractor' solution is really an attractor.

5.5 Application to Hyperinflation

As an application of the analysis in this chapter, we discuss generalised hyperinflation. Hyperinflation, first introduced in [Brown, 2018], is a phase of inflation during which inflation occurs more rapidly then in standard slow-roll models. Usually, the metric corresponding to hyperinflation has a constant negative curvature, given by

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + L^2 \sinh^2\left(\frac{\rho}{L}\right)\mathrm{d}\theta^2. \tag{5.78}$$

The potential is taken to be a general function of θ only, i.e. $V(\rho,\theta) = V(\rho)$. In [Bjorkmo and Marsh, 2019], it was shown that hyperinflation may be generalised by simply requiring that the potential satisfies

$$V_{ww} = \frac{V_v}{L}, \quad V_{vw} \simeq 0, \quad V_{vv} \ll V_{ww}. \tag{5.79}$$

From these conditions on the potential, it follows that (generalised) hyperinflation falls under the 'negligible' case, as discussed in Section 5.2.1. Using equations 5.40 and 5.41, we find that the scalar field velocities as given by

$$\dot{\phi}_v = -\frac{3V_vH}{V_{ww}} = -\frac{-3V_vHL}{V_v} = -3HL, \tag{5.80a}$$

$$\dot{\phi}_w = V_v\sqrt{\frac{V_{ww}-9H^2}{V_{ww}^2}} = V_v\sqrt{\frac{\frac{V_v}{L}-9H^2}{\frac{V_v^2}{L^2}}} = \sqrt{V_vL-9HL^2}, \tag{5.80b}$$

and the total field velocity thus by

$$\dot{\phi}^2 = \dot{\phi}_v^2 + \dot{\phi}_w^2 = LV_v. \tag{5.81}$$

As a consistency check, we see that the conditions $\epsilon \ll 1$ and $\omega^2 \gg \mathcal{O}(\epsilon)$ thus correspond with

$$\epsilon = \frac{\dot{\phi}^2}{2H^2} = \frac{LV_v}{2H^2} \ll 1, \tag{5.82}$$

and thus

$$\omega^2 = \left(\frac{\dot{\phi}_w V_v}{H\dot{\phi}^2}\right)^2 = \frac{1}{L^2H^2}(V_vL - 9HL^2) \gg \mathcal{O}(\epsilon), \tag{5.83}$$

or $1/L^2(V_vL - 9HL^2) \gg 1/2LV_v$. Following the results of [Bjorkmo, 2019], stability of the attractor solution requires $\mu_N > -3\omega^2$. Using 5.75, we find that for hyperinflation $\mu_N = -\omega^2 + \mathcal{O}(\epsilon)$, so the background evolution is stable.

Chapter 6

Scaling Solutions and their Stability

In the previous section, a two-field model of inflation was studied and an equation was derived for a constraint on a possible attractor in the system. In this section, again a two-field model is studied. The approach, however, is different: the background equations of motion are transformed into a four-dimensional system of ordinary differential equations for the scalar coordinates ϕ^a and the associated velocities (with respect to e-folds) v^a. Then, assuming a general two-dimensional Riemannian metric, a transformation is applied to the system, inspired by the functions that are used to define the metric. With this transformed system, we will start the analysis of the system. In this analysis, we focus a so-called *scaling solutions*: solutions for which the slow-roll parameter ϵ is constant. We start with the scenario in which field-space is flat (Section 6.3). Then, we consider systems in which the metric has an isometry corresponding to shifts in the field-space coordinate θ (Sections 6.4 and 6.5). For this scenario, the analysis is divided amongst three cases, depending on the form of the potential $V(\rho, \theta)$ acting on field-space. Furthermore, the models of hyperinflation and angular inflation are briefly discussed.

6.1 Deriving the System

To start the analysis, recall the background equations for the scalar fields ϕ^a and the Friedmann equations. Recall that the background equations of motion are given by

$$\mathcal{D}_t\dot{\phi}^a + 3H\dot{\phi}^a + \mathcal{G}^{ab}V_{,b} = 0 \tag{6.1}$$

and the Friedmann equations are

$$3H^2 = \frac{1}{2}\mathcal{G}_{ab}\dot{\phi}^a\dot{\phi}^b + V. \tag{6.2a}$$

$$\dot{H} = -\frac{1}{2}\mathcal{G}_{ab}\dot{\phi}^a\dot{\phi}^b, \tag{6.2b}$$

We now transform the background equations of motion into a system of first order differential equations for the scalar coordinates ϕ^a and their velocities (with respect to e-folds) $\mathrm{v}^a := \frac{d\phi^a}{dN}$. Here, we use the notation v^a to distinguish this velocity from the basis vector v^a in the previous section.

$$\begin{aligned}
0 &= \mathcal{D}_t\dot{\phi}^a + 3H\dot{\phi}^a + \mathcal{G}^{ab}V_{,b} \\
&= \ddot{\phi}^a + \Gamma^a_{bc}\dot{\phi}^b\dot{\phi}^c + 3H^2\mathrm{v}^a + V^{,a} \\
&= \frac{d}{dt}\left(H\frac{d}{dN}\phi^a\right) + H^2\Gamma^a_{bc}\mathrm{v}^b\mathrm{v}^c + 3H^2\mathrm{v}^a + V^{,a} \\
&= \dot{H}\mathrm{v} + H^2\left(\mathrm{v}^a\right)' + H^2\Gamma^a_{bc}\mathrm{v}^b\mathrm{v}^c + \left(\frac{1}{2}\mathcal{G}_{bc}\dot{\phi}^b\dot{\phi}^c + V\right)\mathrm{v}^a + V^{,a} \\
&= -\frac{1}{2}H^2\mathrm{v}^a\mathcal{G}_{bc}\mathrm{v}^b\mathrm{v}^c + H^2\left(\mathrm{v}^a\right)' + H^2\Gamma^a_{bc}\mathrm{v}^b\mathrm{v}^c + \left(\frac{1}{2}H^2\mathcal{G}_{bc}\mathrm{v}^b\mathrm{v}^c + V\right)\mathrm{v}^a + V^{,a} \\
&= H^2\left(\mathrm{v}^a\right)' + H^2\Gamma^a_{bc}\mathrm{v}^b\mathrm{v}^c + (3-\epsilon)\,H^2\mathrm{v}^a + V^{,a}.
\end{aligned} \tag{6.3}$$

Rearranging these terms then yields

$$\begin{aligned}
\left(\mathrm{v}^a\right)' &= -\left(3-\epsilon\right)\mathrm{v}^a - \Gamma^a_{bc}\mathrm{v}^b\mathrm{v}^c - \frac{Vp^a}{H^2} \\
&= -\left(3-\epsilon\right)\left(\mathrm{v}^a + p^a\right) - \Gamma^a_{bc}\mathrm{v}^b\mathrm{v}^c,
\end{aligned} \tag{6.4}$$

where the definition $p_a = \frac{\partial(lnV)}{\partial\phi^a}$ has been used. This results in the following system:

$$\begin{cases} \left(\phi^a\right)' = \mathrm{v}^a \\ \left(\mathrm{v}^a\right)' = -\left(3-\epsilon\right)\left(\mathrm{v}^a + p^a\right) - \Gamma^a_{bc}\mathrm{v}^b\mathrm{v}^c. \end{cases} \tag{6.5}$$

Note that the slow-roll parameter ϵ can be rewritten is terms of the velocities as

$$\epsilon := -\frac{\dot{H}}{H^2} \overset{(6.2b)}{=} \frac{1}{2}\frac{\mathcal{G}_{ab}\dot{\phi}^a\dot{\phi}^b}{H^2} = \frac{1}{2}\frac{\mathcal{G}_{ab}H(\phi^a)'H(\phi^b)'}{H^2} = \frac{1}{2}\mathcal{G}_{ab}\mathrm{v}^a\mathrm{v}^b = \frac{1}{2}\mathrm{v}_a\mathrm{v}^a, \tag{6.6}$$

which will prove useful later on in the analysis.

6.2 Transforming the System

To make the system a bit more concrete, we will rewrite it in terms of the metric chosen for field-space. The most general two-dimensional Riemannian metric can be written as

$$\mathrm{d}s^2 = g^2(\rho,\theta)\mathrm{d}\rho^2 + f^2(\rho,\theta)\mathrm{d}\theta^2, \tag{6.7}$$

which follows from equation 2.45 in Chapter 2. The Christoffel symbols associated with this metric are given by

$$\Gamma^\rho_{\rho\rho} = \frac{g_{,\rho}}{g} \quad \Gamma^\rho_{\rho\theta} = \frac{g_{,\theta}}{g} \quad \Gamma^\rho_{\theta\theta} = -\frac{ff_{,\rho}}{g^2}$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{f_{,\theta}}{f} \quad \Gamma^{\theta}_{\rho\theta} = \frac{f_{,\rho}}{f} \quad \Gamma^{\theta}_{\rho\rho} = -\frac{g g_{,\theta}}{f^2}. \tag{6.8}$$

We introduce the coordinates $(x, y) := \left(gv^{\rho}, fv^{\theta}\right)$, so that the first slow-roll parameter can be written as

$$\epsilon = \frac{1}{2} v_a v^a = \frac{1}{2}\left(x^2 + y^2\right), \tag{6.9}$$

and the evolution equation for ϵ' becomes

$$\epsilon' = xx' + yy'. \tag{6.10}$$

In our analysis, we consider scaling solutions, so solutions for which $\epsilon' = 0$. As seen in the above equation, setting $x' = y' = 0$ corresponds to such scaling solutions and we will focus on this type of solutions.

Using the system 6.5, we can derive the evolution equations for x and y:

$$\begin{aligned}
x' &= (g)'v^{\rho} + g(v^{\rho})' \\
&= -g(3-\epsilon)(v^{\rho} + p^{\rho}) - g\Gamma^{\rho}_{bc}v^b v^c + \left(g_{,\rho}\rho' + g_{,\theta}\theta'\right)v^{\rho} \\
&= -(3-\epsilon)\left(x + \frac{p_{\rho}}{g}\right) - g\Gamma^{\rho}_{\rho\rho}v^{\rho}v^{\rho} - 2g\Gamma^{\rho}_{\rho\theta}v^{\theta}v^{\rho} - g\Gamma^{\rho}_{\theta\theta}v^{\theta}v^{\theta} + \left(g_{,\rho}v^{\rho} + g_{,\theta}v^{\theta}\right)v^{\rho} \\
&= -(3-\epsilon)\left(x + \frac{p_{\rho}}{g}\right) - \frac{g_{,\rho}}{g^2}x^2 - 2\frac{g_{,\theta}}{fg}xy + \frac{f_{,\rho}}{fg}y^2 + \frac{g_{,\rho}}{g^2}x^2 + \frac{g_{,\theta}}{fg}xy \\
&= -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)\left(x + \frac{p_{\rho}}{g}\right) - \frac{g_{,\theta}}{fg}xy + \frac{f_{,\rho}}{fg}y^2.
\end{aligned} \tag{6.11}$$

For y we analogously obtain

$$\begin{aligned}
y' &= (f)'v^{\theta} + f(v^{\theta})' \\
&= -f(3-\epsilon)(v^{\theta} + p^{\theta}) - f\Gamma^{\theta}_{bc}v^b v^c + \left(f_{,\rho}\rho' + f_{,\theta}\theta'\right)v^{\theta} \\
&= -(3-\epsilon)\left(y + \frac{p_{\theta}}{f}\right) - f\Gamma^{\theta}_{\theta\theta}v^{\theta}v^{\theta} - 2f\Gamma^{\theta}_{\rho\theta}v^{\rho}v^{\theta} - f\Gamma^{\theta}_{\rho\rho}v^{\rho}v^{\rho} + \left(f_{,\rho}v^{\rho} + f_{,\theta}v^{\theta}\right)v^{\theta} \\
&= -(3-\epsilon)\left(y + \frac{p_{\theta}}{f}\right) - \frac{f_{,\theta}}{f^2}y^2 - 2\frac{f_{,\rho}}{fg}xy + \frac{g_{,\theta}}{fg}x^2 + \frac{f_{,\rho}}{fg}xy + \frac{f_{,\theta}}{f^2}y^2 \\
&= -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)\left(y + \frac{p_{\theta}}{f}\right) - \frac{f_{,\rho}}{fg}xy + \frac{g_{,\theta}}{fg}x^2.
\end{aligned} \tag{6.12}$$

The full system is thus given by

$$\begin{cases}
\rho' = \dfrac{x}{g}, \\
x' = -\left(3 - \dfrac{1}{2}x^2 - \dfrac{1}{2}y^2\right)\left(x + \dfrac{p_{\rho}}{g}\right) - \dfrac{g_{,\theta}}{fg}xy + \dfrac{f_{,\rho}}{fg}y^2, \\
\theta' = \dfrac{y}{f}, \\
y' = -\left(3 - \dfrac{1}{2}x^2 - \dfrac{1}{2}y^2\right)\left(y + \dfrac{p_{\theta}}{f}\right) - \dfrac{f_{,\rho}}{fg}xy + \dfrac{g_{,\theta}}{fg}x^2.
\end{cases} \tag{6.13}$$

6.3 The Flat Field-space Case

We will now look at the simplest possible form of the system. We assume that field-space is flat, so $g, f \equiv 1$. Then metric is then simply the two-dimensional Euclidean metric

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{d}\theta^2, \tag{6.14}$$

and the dynamical system reduces to

$$\begin{cases} \rho' = x, \\ x' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(x + p_\rho) \\ \theta' = y \\ y' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(y + p_\theta). \end{cases} \tag{6.15}$$

Note in this system, the equations for x' and y' are quite symmetric, because there is no distinction between the two coordinates in the metric. We focus on the case where the potential is independent of θ, so where $p_\theta = 0$, i.e.

$$V(\rho, \theta) = V(\rho). \tag{6.16}$$

Specifically, we discuss two cases: the case in which p_ρ is a constant and the case in which it is not. If we were to choose a potential of the form $V(\theta)$, so a potential independent of ρ, the results of the analysis would be exactly the same (as the transformation $(\rho, x, \theta, y) \mapsto (\theta, y, \rho, x)$ does not intrinsically change the system).

6.3.1 The Case of an Exponential Potential

We start with the assumption that p_ρ is constant, i.e. $p_\rho = c_2$ for some $c_2 \in \mathbb{R}$. Recall that p_ρ is defined as $\partial(\ln V)/\partial\rho$ and that we had assumed that $p_\theta = 0$. Thus, it follows that the potential must take the form of an exponential:

$$V(\rho) = c_1 e^{c_2 \rho} \quad \text{for } c_1 \in \mathbb{R}. \tag{6.17}$$

In this case, the equations for x' and y' are independent of both ρ and θ. This then implies that there is a two-dimensional subspace, spanned by x and y, that is invariant under the flow. The two-dimensional subsystem takes the form

$$\begin{cases} x' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(x + c_2), \\ y' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) y. \end{cases} \tag{6.18}$$

It is now easy to see that a fixed point of this system requires $x^2 + y^2 = 6$ or $x = -c_2$ and $y = 0$. We consider these two types of fixed points separately. Note that if $|c_2| \leq \sqrt{6}$ we also have a fixed point for $x = -c_2$ and $y = \pm\sqrt{6 - c_2^2}$. However, this is just a point on the circle $x^2 + y^2 = 6$, so we do not consider this critical point separately.

- First, we consider the fixed point $(-c_2, 0)$. For reasons that will be explained in Section 6.4.1, we call this the *gradient solution*. Using the relation $\epsilon = 1/2x^2 + 1/2y^2$, we find that the associated slow-roll parameter ϵ is given by $\epsilon = c_2^2/2$
- Next, consider the critical points given by

$$(x, y)_{circ} = (x_{circ}, y_{circ}) \quad \text{for which } x^2 + y^2 = 6. \tag{6.19}$$

For now, we call every such critical point a *circle solution*. It is easy to see that $\epsilon = 3$ for every circle solution.

Having described the critical points of the system, we now consider their stability. The Jacobian corresponding to the two-dimensional system 6.18 is given by

$$\mathcal{J}_{flat,2D} = \begin{pmatrix} x(x+c_2) - \left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) & y(x+c_2) \\ xy & y^2 - \left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) \end{pmatrix}. \tag{6.20}$$

We consider the two types of critical points separately.

- The eigenvalues that correspond to the gradient solution are given by $\lambda_{1,2,grad} = -3 + c_2^2/2$. Thus, the gradient solution is asymptotically stable if $|c_2| < \sqrt{6}$ and unstable if $|c_2| > \sqrt{6}$. If $|c_2| = \sqrt{6}$, the stability does not follow from the linearisation, since $\lambda_{1,2,grad} = 0$. However, we may directly consider the equation for x' in 6.18. From this, it follows that $(-c_2, 0)$ is unstable also if $|c_2| = \sqrt{6}$.

  Since $(-c_2, 0)$ is asymptotically stable for $|c_2| < \sqrt{6}$, it is an attractor of the two-dimensional system 6.18 (see Section 4.3). In fact, it can be shown that the basin of attraction is given by

$$\{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < 6\}. \tag{6.21}$$

- For the circle solutions, the corresponding eigenvalues are given by

$$\lambda_{1,circ} = 6 + c_2 x, \quad \lambda_{2,circ} = 0, \tag{6.22}$$

  Given that one eigenvalue is zero, it is necessary to consider the eigenvectors corresponding to it. It can be shown that the eigenvector $v_{2,circ}$ corresponding to the zero eigenvalue is always tangent to the circle $x^2 + y^2 = 6$, while the eigenvector $v_{1,circ}$ corresponding to the nonzero eigenvalue is not (except in a specific case, which we will reflect on later). Thus, for stability we only need to consider the value of $\lambda_{1,circ}$. We consider three separate cases: (a) $|c_2| = \sqrt{6}$, (b) $|c_2| < \sqrt{6}$ and (c) $|c_2| > \sqrt{6}$. We assume that $c_2 \geq 0$. The results for $c_2 \leq 0$ follow by the transformation $c_2 \mapsto -c_2$.

  (a) **If $\mathbf{c_2} = \sqrt{\mathbf{6}}$**, we have $\lambda_{1,circ} = 6 + \sqrt{6}x$. We see that $\lambda_{1,circ} < 0$ for $x < -\sqrt{6}$, so nowhere on the circle. For $x = \sqrt{6}$, $\lambda_{1,circ} = 0$, while $\lambda_{1,circ} > 0$ everywhere else on the circle. From the discussion about the stability of the gradient solution, it follows that the point $(\sqrt{6}, 0)$ is also unstable.

(b) **If $\mathbf{c_2 < \sqrt{6}}$**, we have $\lambda_{1,circ} \leq 0$ nowhere on the circle, so $\lambda_{1,circ} > 0$ everywhere. Thus, the whole circle is unstable.

(c) **If $\mathbf{c_2 > \sqrt{6}}$**, it holds that

$$\lambda_{1,circ} \begin{cases} < 0 & \text{for } x \in \left[-\sqrt{6}, -\dfrac{6}{c_2}\right) \\ = 0 & \text{for } x = -\dfrac{6}{c_2} \\ > 0 & \text{for } x \in \left(-\dfrac{6}{c_2}, \sqrt{6}\right] \end{cases}, \tag{6.23}$$

so the stability is clear for all $x \neq -6/c_2$. In the point $(x, y)_{circ} = (-6/c_2, \mp\sqrt{6 - 36/c_2^2})$, something interesting happens. Here, not only $v_{2,circ}$, but also $v_{1,circ}$ becomes tangent to the circle. However, we may still determine the stability of this point by looking at the y' equation in 6.18: for $x = -6/c_2$ and $y < -\sqrt{6 - 36/c_2^2}$ (so outside the circle), we have $y' < 0$, while for $-\sqrt{6 - 36/c_2^2} < y < 0$, we have $y' > 0$, so the point $(-6/c_2, -\sqrt{6 - 36/c_2^2})$ is unstable. It analogously follows that $(-6/c_2, \sqrt{6 - 36/c_2^2})$ is unstable.

The results of the stability analysis can be captured in Figures 6.1, 6.2 and 6.3. In each of these figure, the circle $x^2 + y^2 = \sqrt{6}$ of fixed points and the fixed point $(-c_2, 0)$ (in blue) are displayed. In Figure 6.3, the blue dashed line at $x = -c_2/6$ represents the transition of the stability of the points on the circle from unstable (dashed line) to stable (solid line).

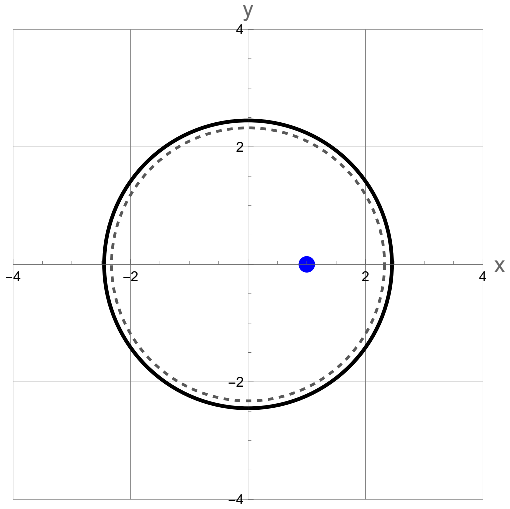

Figure 6.1: Fixed points for $|c_2| < \sqrt{6}$

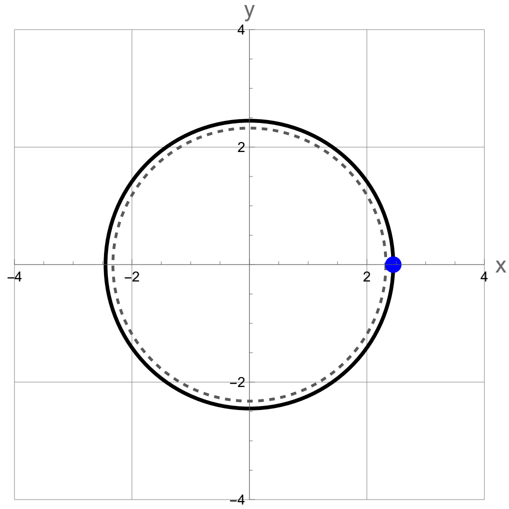

Figure 6.2: Fixed points for $|c_2| = \sqrt{6}$

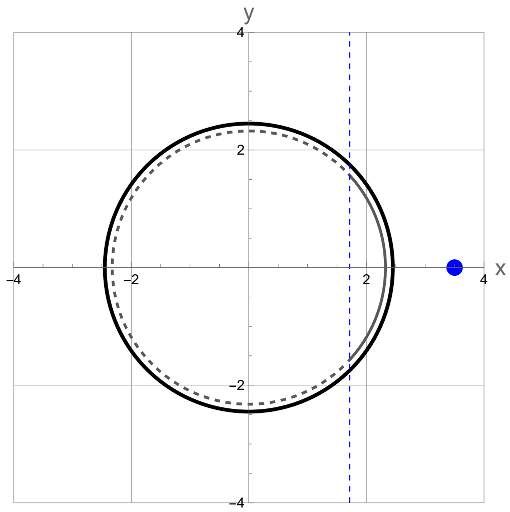

Figure 6.3: Fixed points for $|c_2| > \sqrt{6}$

The point $(c_2, 0)$ is asymptotically stable for $|c_2| < \sqrt{6}$. In the (ρ, θ)-space, this corresponds with a trajectory that only rolls down the potential in the ρ-direction and has no motion in the θ-direction. Thus, in the absence of field-space curvature, and for an exponential potential that only depends on one coordinate, the attractor $(x, y) = (-c_2, 0)$ (which can be seen as a kind of 'preferred trajectory') is just a direct generalisation of single field inflation. See Figure 6.4 for the inflationary trajectories corresponding to the attractor in the two-dimensional system.

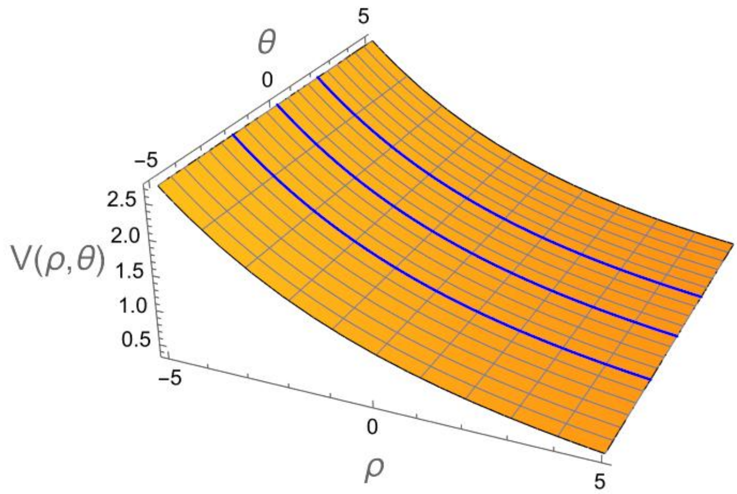

Figure 6.4: The potential $V(\rho,\theta) = e^{-0.2\rho}$ together with some trajectories (blue) corresponding to the attractor $(-c_2, 0)$.

6.3.2 The Case of Fixed Points

We now drop the assumption that p_ρ is constant. It follows that the equations for x' and y' are no longer independent of ρ, but still independent of θ. Thus, we now have a three-dimensional invariant subspace, spanned by ρ, x and y. The three-dimensional subsystem takes the form

$$\begin{cases} \rho' = x, \\ x' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(x + p_\rho), \\ y' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) y. \end{cases} . \tag{6.24}$$

We assume that the potential V is such that there are countable many isolated points ρ_{grad} for which $p_\rho(\rho_{ex}) = 0$ holds, i.e. for which V has an extremum, so we find the fixed points

$$(\rho, x, y)_2 = (\rho_{ex}, 0, \pm\sqrt{6}) \quad \text{and} \quad (\rho, x, y)_1 = (\rho_{ex}, 0, 0). \tag{6.25}$$

We will now determine the stability of these points. For $(\rho, x, y)_2$, one of the eigenvalues is $\lambda = 6$. Thus, this equilibrium is unstable.

For $(\rho, x, y)_2$, the eigenvalues are given by $\lambda_1 = -3$, $\lambda_\pm = 1/2\left(-3 \pm \sqrt{9 - 12\partial_\rho p_\rho}\right) = 1/2\left(-3 \pm \sqrt{9 - 12V_{,\rho\rho}}\right)$. Based on the value of $\partial_\rho p_\rho$, we have different scenarios:

- If $V_{,\rho\rho} > 0$, $\sqrt{9 - 12V_{,\rho\rho}} < 3$ or $\sqrt{9 - 12V_{,\rho\rho}} \in \mathbb{C}$, such that $\mathrm{Re}\{\lambda_\pm\} < 0$. Thus, $(\rho_{ex}, 0, 0)$ is asymptotically stable.
- If $V_{,\rho\rho} < 0$, we have that $\sqrt{9 - 12V_{,\rho\rho}} > 3$, so $\lambda_+ > 0$ and thus the fixed point is unstable.

6.4 Non-trivial Geometry for Specific Potentials

Having examined the system for a Euclidean metric, we now generalise the metric. Suppose the metric has a transitively acting isometry. It can then be brought into the form

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + f^2(\rho)\mathrm{d}\theta^2, \tag{6.26}$$

such that the isometry corresponds to shifts in θ. The flat plane can for example be written in this form, if one parametrises it in polar coordinates (so with $f = \rho$). Another example is the non-compact representation of the hyperbolic plane, with $f = L\sinh(\rho/L)$. With the example of the flat plane in mind, we refer to ρ as the *radial coordinate* and to θ as the *angular coordinate*. Using 7.12, we see that for this choice of metric the only non-zero Christoffel symbols are

$$\Gamma^{\rho}_{\theta\theta} = -\frac{ff_{,\rho}}{g^2} = -ff_{,\rho}, \quad \Gamma^{\theta}_{\rho\theta} = \frac{f_{,\rho}}{f}. \tag{6.27}$$

In addition, we assume that the scalar potential preserves the isometry of the metric. Since the metric has an isometry in the θ direction, this requires a shift-symmetry in the potential in this direction. It follows that $p_\theta = 0$. In this case, the system is given by

$$\begin{cases} \rho' = x; \\ x' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(x + p_\rho) + \frac{f_{,\rho}}{f}y^2; \\ \theta' = \frac{y}{f}; \\ y' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{f_{,\rho}}{f}x\right)y. \end{cases} \tag{6.28}$$

Note that the equations for ρ', x' and y' are all independent of θ. Having determined the form of the dynamical system, we can now start with the classification of the scaling solutions for which $x' = y' = 0$. We consider a variety of cases.

6.4.1 The Case of an Exponential Potential and Hyperbolic Metric

We start with the additional assumption that p_ρ and $f_{,\rho}/f$ are constants. A constant p_ρ corresponds to an exponential function

$$V(\rho, \theta) = c_1 e^{c_2\rho}, \tag{6.29}$$

such that $p_\rho \equiv c_2$, and a constant $f_{,\rho}/f$ corresponds to a hyperbolic space with $f(\rho) = e^{\rho/L}$, such that $f_{,\rho}/f = 1/L$. In this case, the equations for x' and y' are independent of both ρ and θ. Therefore, there exists an invariant two-dimensional subspace spanned by x and y, and we may consider the two-dimensional subsystem given by

$$\begin{cases} x' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(x + c_2) + \frac{1}{L}y^2; \\ y' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{L}x\right)y. \end{cases} \tag{6.30}$$

We can distinguish three types of critical points of this subsystem:

- If the velocity in the θ-direction vanishes, we find a scaling solution along the ρ-coordinate:
$$(x, y)_{grad} = (-p_\rho, 0) = (-c_2, 0). \tag{6.31}$$
This leads to $\epsilon = c_2^2/2$. The solution only has velocity in the radial direction, which corresponds to the gradient direction of the potential V. Therefore we refer to this scaling solution as the *gradient solution*. Note that this solution was also found in a flat field-space (Section 6.3.1).

- If, like for the gradient solution, the velocity in the θ-direction vanishes, the $(3 - 1/2x^2 - 1/2y^2)$ terms vanish if $x = \pm\sqrt{6}$. We refer to these solutions as *kinetic solutions*:
$$(x, y)_{kin} = (\pm\sqrt{6}, 0). \tag{6.32}$$
For these solutions, we have $\epsilon = 3$. We see that the kinetic solutions are just two special cases of the gradient solution.

- In case the angular velocity does not vanish, we find scaling solutions given by
$$(x, y)_{hyper} = \left(-\frac{6}{\frac{2}{L} + c_2}, \pm\frac{\sqrt{6}\sqrt{c_2^2 + 2\frac{c_2}{L} - 6}}{\frac{2}{L} + c_2}\right), \tag{6.33}$$
and
$$\epsilon = \frac{3Lc_2}{2 + Lc_2} = \frac{3Lp_\rho}{2 + Lp_\rho}. \tag{6.34}$$
Note that the hyperbolic solutions reduce to instances of the circle solutions found in Section 6.3.1 for $f \equiv 1$, i.e. for $1/L = 0$. The solutions only exist if the expressions in the roots are positive. Setting $c_2^2 - 2c_2/L - 6 = 0$ gives
$$c_{2,crit1} = \frac{1 - \sqrt{1 + 6L^2}}{L}, \quad c_{2,crit2} = \frac{1 + \sqrt{1 + 6L^2}}{L}. \tag{6.35}$$
It follows that the solutions exist for $c_2 \leq c_{2,crit1}$ and $c_2 \geq c_{2,crit2}$. These solutions move in the angular direction θ, while the radial coordinate ρ decreases along the trajectory. As we will see later, these solutions only exist if field space is hyperbolic, otherwise the solution would be ρ-dependent. Hence we call them *hyperbolic solutions.*

Next, we consider the stability of the critical points determined above. The Jacobian of the two-dimensional subsystem is given by
$$\mathcal{J}(x, y)_{2D} = \begin{pmatrix} -3 + \frac{3}{2}x^2 + \frac{1}{2}y^2 + c_2 x & \left(\frac{2}{L} + x + c_2\right) y \\ \left(-\frac{1}{L} + x\right) y & -3 - \frac{x}{L} + \frac{1}{2}x^2 + \frac{3}{2}y^2 \end{pmatrix}. \tag{6.36}$$

We can now determine the eigenvalues for each critical point we found before:

- For the gradient solution, we have that

$$\lambda_{1,grad} = \frac{1}{2}\left(c_2^2 - 6\right), \quad \lambda_{2,grad} = \frac{1}{2}(c_2^2 - 6) - \frac{c_2}{L}. \tag{6.37}$$

  The gradient solution is asymptotically stable if $\lambda_1, \lambda_2 < 0$. Solving these inequalities leads to:

  1. If $c_2 \in \left(-\sqrt{6}, 0\right]$, then the gradient solution is asymptotically stable if $L > 0$ or if $L < 0$ and $c_2 \in \left(-\frac{1}{L} - \sqrt{6 + \frac{1}{L^2}}, 0\right] = (c_{2,crit1}, 0]$.
  2. If $c_2 \in \left(0, \sqrt{6}\right)$, then the gradient solution is asymptotically stable if $L < 0$ or if $L > 0$ and $c_2 \in \left(0, -\frac{1}{L} + \sqrt{6 + \frac{1}{L^2}}\right) = (0, c_{2,crit2})$.

- For the kinetic solutions, we find the eigenvalues

$$\lambda_{1,kin} = \sqrt{6}\left(\sqrt{6} \pm c_2\right), \quad \lambda_{2,kin} = \mp\frac{\sqrt{6}}{L}. \tag{6.38}$$

  Thus, for the positive kinetic solution we have asymptotic stability if $c_2 < -\sqrt{6}$ and $L > 0$. For the negative kinetic solution we have asymptotic stability if $c_2 > \sqrt{6}$ and $L < 0$.

- For the hyperbolic solution, the eigenvalues are a bit more complicated, so we shall not give them here. For both the positive and the negative solution, have $\lambda_{1,hyp}, \lambda_{2,hyp} < 0$, so asymptotic stability, in the following two cases:

$$c_2 \in \left(-\sqrt{6}, 0\right) \quad \wedge \quad L \in \left(-\frac{2c_2}{c_2^2 - 6}, \frac{27 - 8c_2^2}{4c_2(c_2^2 - 6)} - \frac{1}{4}\sqrt{3}\sqrt{\frac{243 - 16c_2^2}{c_2^2(c_2^2 - 6)^2}}\right] \tag{6.39a}$$

$$c_2 \in \left(0, \sqrt{6}\right) \quad \wedge \quad L \in \left[\frac{27 - 8c_2^2}{4c_2(c_2^2 - 6)} - \frac{1}{4}\sqrt{3}\sqrt{\frac{243 - 16c_2^2}{c_2^2(c_2^2 - 6)^2}}, -\frac{2c_2}{c_2^2 - 6}\right) \tag{6.39b}$$

We now discuss how the three critical points (or the solutions) are related. First, note that the kinetic solutions are just a special case of the gradient solution for the choices $c_2 = \pm\sqrt{6}$. We now inspect how the gradient solution is related to the hyperbolic solutions.

We first consider the behaviour of the system around the point $(c_{2,crit1}, 0)$. In the case that $L < 0$, we find a subcritical pitchfork bifurcation (see e.g. [Holmes, 2012] for a introduction to bifurcations) at this point, which can be seen in the following way: from equation 6.37, it was deduced that the gradient solution is asymptotically stable if $c_2 \in (c_{2,crit1}, 0]$. It can also be deduced that the gradient solution is unstable if $c_2 < c_{2,crit1}$. The hyperbolic solutions exist for $c_2 \leq c_{2,crit1}$ (see equation 6.35) and is symmetric is $y = 0$. From equation 6.39a, it may be deduced that both branches of the hyperbolic solution are stable on the interval $(a_1, c_{2,crit1})$ for some $a_1 < c_{2,crit1}$. Thus, at $(c_{2,crit1}, 0)$ we have a subcritical pitchfork bifurcation.

For the $(c_{2,crit2}, 0)$ the situation is similar. There is a supercritical pitchfork bifurcation at this point if $L > 0$: from equation 6.37, it followed that the gradient solution is asymptotically stable if $c_2 \in (0, c_{2,crit2})$. It also follows that the solution if unstable if $c_2 ? c_{2,crit2}$. From equation 6.35 is follows that the hyperbolic solutions exist for $c_2 \geq c_{2,crit2}$. In addition, they are symmetric in $y = 0$ and stable on the interval $(c_{2,crit2}, a_2)$ for some $a_2 > c_{2,crit2}$ (which can be deduced from equation 6.39b). Therefore, there is a supercritical pitchfork bifurcation at $(c_{2,crit2}, 0)$.

6.4.2 The Case of an Exponential Potential and Nonhyperbolic metric

We relax the assumptions of Section 6.4.1. We still use an exponential potential $V(\rho, \theta) = c_1 c^{c_2 \rho}$, but drop the assumption that $f_{,\rho}/f$ is constant. Looking at 6.28, we see that $p_\rho = c_2$. Although the equations for x' and y' are no longer independent of both ρ, they are still independent of θ. Therefore, there exists an invariant three-dimensional subspace spanned by x and y, and we may consider the three-dimensional subsystem given by

$$\begin{cases} \rho' = x; \\ x' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(x + c_2) + \frac{f_{,\rho}}{f}y^2; \\ y' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{f_{,\rho}}{f}x\right)y. \end{cases} \tag{6.40}$$

The ρ' equation only vanishes for $x = 0$. Using the equation for y', this implies that either $y = \pm\sqrt{6}$ or $y = 0$, which in turn implies (from the requirement that $x' = 0$) $f_{,\rho}/f(\rho) = 0$ and $c_2 = 0$, respectively. The case where $c_2 = 0$ is not very interesting, as it means that the potential is flat. The case $(f_{,\rho}/f)(\rho) = 0$ does yields fixed points of interest, namely

$$(\rho, x, y) = \left(\tilde{\rho}, 0, \pm\sqrt{6}\right), \tag{6.41}$$

where $\tilde{\rho}$ is some ρ satisfying $(f_{,\rho}/f)(\tilde{\rho}) = 0$.

The Jacobian associated with the three-dimensional system is given by

$$\mathcal{J}_{3D}(\rho, x, y) = \begin{pmatrix} 0 & 1 & 0 \\ \left(-\frac{f_{,\rho}^2}{f^2} + \frac{f_{,\rho\rho}}{f}\right)y^2 & -3 + \frac{3}{2}x^2 + \frac{1}{2}y^2 + c_2 x & (c_2 + x)y + 2\frac{f_{,\rho}}{f}y \\ \left(\frac{f_{,\rho}^2}{f^2} - \frac{f_{,\rho\rho}}{f}\right)xy & y\left(x - \frac{f_{,\rho}}{f}\right) & -3 + \frac{1}{2}x^2 + \frac{3}{2}y^2 - \frac{f_{,\rho}}{f}x. \end{pmatrix} \tag{6.42}$$

Evaluating at the fixed points $\left(\tilde{\rho}, 0, \pm\sqrt{6}\right)$ gives the following eigenvalues, which are the same for both the positive and the negative solutions:

$$\lambda_1 = 6, \quad \lambda_2 = \sqrt{6}\sqrt{\frac{f_{,\rho\rho}(\tilde{\rho})}{f(\tilde{\rho})}}, \quad \lambda_3 = -\sqrt{6}\sqrt{\frac{f_{,\rho\rho}(\tilde{\rho})}{f(\tilde{\rho})}}. \tag{6.43}$$

We see that the real part of the first eigenvalue is always positive. Thus is follows that all fixed points of this form are unstable.

From the analysis above, we conclude that there are no stable fixed points of interest in the three-dimensional subsystem (given by 6.40). However, the gradient solution (6.31) and the kinetic solutions (6.32) do not depend on ρ, so these solutions may be defined in the (x, y) subspace for any value of ρ. The Jacobian from equation 6.42 may be used to calculate the eigenvalues. We assume that these eigenvalues correspond to the local Lyapunov exponents (see the discussion in Chapter 4, Section 4.4).

- For the gradient solution, the eigenvalues are given by

$$\lambda_{1,grad} = 0, \quad \lambda_{2,grad} = \frac{1}{2}\left(-6 + c_2^2\right), \quad \lambda_{3,grad} = -3 + \frac{c_2}{L_\rho} + \frac{1}{2}c_2^2, \tag{6.44}$$

  with corresponding eigenvectors

$$v_{1,grad} = (1, 0, 0), \quad v_{2,grad} = \left(\frac{2}{c_2^2 - 6}, 1, 0\right), \quad v_{3,grad} = (0, 0, 1). \tag{6.45}$$

  Here, we defined

$$\frac{1}{L_\rho} = \frac{f_{,\rho}}{\rho}. \tag{6.46}$$

  Thus, we see that the eigenvalues are a direct generalisation of the eigenvalues corresponding to the gradient solution in the case that $1/L_\rho = 1/L$, see equation 6.37. From the eigenvectors corresponding to the eigenvalues, we see that the zero eigenvalue points only in the ρ-direction. This is a reflection of the fact that the gradient solution is ρ-independent. The stability of the gradient solution thus only depends on $\lambda_{2,grad}$ and $\lambda_{3,grad}$. The conditions on c_2 on L_ρ are exactly the same as for the case that field-space is hyperbolic (Section 6.4.1), only this time the value of L_ρ is different for each two-dimensional (x, y)-slice.

- The situation for the kinetic solutions is exactly the same as for the gradient solution. The eigenvalues are a direct generalisation (with the replacement $1/L \to 1/L_\rho$) of the eigenvalues in the hyperbolic case (see equation 6.38), with the addition of a zero eigenvalue of which the corresponding eigenvector points in the ρ-direction.

6.4.3 Case Study: Hyperinflation

Now that we have discussed several types of scaling solutions, both for a hyperbolic metric function and for a non-hyperbolic metric function $f(\rho)$, we consider the model of hyperinflation [Brown, 2018], and compare it to the hyperbolic solution that was found in Section 6.4.1. In models of hyperinflation, the curvature of field-space is negative and constant; recall that the metric can be written as

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + L^2 \sinh^2\left(\frac{\rho}{L}\right)\mathrm{d}\theta^2, \tag{6.47}$$

So with $f(\rho) = L\sinh(\rho/L)$ and thus

$$\frac{1}{L_\rho} = \frac{\coth(\rho/L)}{L}. \tag{6.48}$$

For large values of the radial coordinate ρ this metric function is approximately hyperbolic:

$$f(\rho) = L^2\sinh^2\left(\frac{\rho}{L}\right) = \frac{1}{2}L^2\left(e^{\rho/L} - e^{-\rho/L}\right) \overset{\rho \gg L}{\simeq} \frac{1}{2}L^2 e^{\rho/L}, \tag{6.49}$$

where we have assumed that $\rho/L > 0$. If $\rho/L < 0$, we get $f(\rho) \simeq 1/2L^2 e^{-\rho/L}$. Using equation 6.48, the expression for the x-component of the hyperbolic solution of Section 6.4.1 (equation 6.33) can be rewritten as

$$\dot{\rho} = \frac{-3HL}{\coth(\rho/L) + \frac{1}{2}c_2 L_\rho}. \tag{6.50}$$

We now assume that the field-space curvature is large, which corresponds to $L \ll 1$. As $\lim_{x\to\infty}\coth(\rho/L) = 1$, the radial velocity reduces to $\dot{\rho} \simeq -3HL$. This is precisely the expression of the 'attractor' for the hyperinflation model in [Brown, 2018].

6.5 Systems with an Isometry and Generic Potentials

In the previous section, we assumed that the potential only depends on the radial coordinate ρ. We now drop this assumption, such that we have a potential $V = V(\rho, \theta)$. We then find a slightly more general system then the one we considered in the previous section:

$$\begin{cases} \rho' = x; \\ x' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(x + p_\rho) + \frac{f_{,\rho}}{f}y^2; \\ \theta' = \frac{y}{f}; \\ y' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)\left(y + \frac{p_\theta}{f}\right) - \frac{f_{,\rho}}{f}xy. \end{cases} \tag{6.51}$$

In analysing this system, we focus on product-separable potentials with an exponential dependence on either ρ or θ.

6.5.1 Exponential Dependence on the Radial Coordinate

We start with the case of an exponential dependence on the radial coordinate ρ;

$$V(\rho, \theta) = h(\theta)e^{-c_2\rho}, \tag{6.52}$$

which is a direct generalisation of the exponential potential we considered before (see equation 6.29). In this case, we have $p_\rho = c_2$ and $p_\theta = h'(\theta)/h(\theta)$. Note that the equations for x', θ' and y' still depend on ρ through the terms $f_{,\rho}/f$ and p_θ/f. We distinguish two types of scaling solutions.

- Suppose that the function $h(\theta)$ has an extremum for some value θ_{ex}. We then have a scaling solution that is similar to the gradient solution of the previous section, given by
$$(\theta, x, y)_{grad} = (\theta_{ex}, -c_2, 0), \tag{6.53}$$
and with $\epsilon = \frac{1}{2}c_2^2$.
- We also retrieve the kinetic solution
$$(x, y)_{kin} = (\pm\sqrt{6}, 0), \tag{6.54}$$
with $\epsilon = 3$.

Having determined the critical points, we consider their stability. The Jacobian of the full system is given by

$$\mathcal{J}(x,y) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-f_{,\rho}^2+ff_{,\rho\rho}}{f^2}y^2 & -3+\epsilon+x(c_2+x) & 0 & (c_2+x+2\frac{f_{,\rho}}{f})y \\ -\frac{f_{,\rho}}{f^2}y & 0 & 0 & \frac{1}{f} \\ \frac{f_{,\rho}^2-ff_{,\rho\rho}}{f^2}xy - \frac{(-3+\epsilon)p_\theta f_{,\rho}}{f^2} & x(y+p_\theta)-\frac{yf_{,\rho}}{f} & \frac{(-3+\epsilon)\partial_\theta p_\theta}{f} & -3+\epsilon+y\left(y+\frac{p_\theta}{f}\right)-\frac{xf_{,\rho}}{f} \end{pmatrix}. \tag{6.55}$$

For the scaling solutions found above we then get the following:

- For the kinetic solution, the eigenvalues of the Jacobian are
$$\lambda_{1,kin} = \lambda_{2,kin} = 0, \quad \lambda_{3,kin} = \sqrt{6}(\sqrt{6} \pm c_2), \quad \lambda_{4,kin} = \mp\sqrt{6}\frac{f_{,\rho}}{f}, \tag{6.56}$$
where we have eigenvectors
$$v_{1,kin} = (1,0,0,0), \quad v_{2,kin} = (0,0,1,0), \tag{6.57}$$
The zero eigenvalues therefore correspond to the ρ and θ directions and therefore lie on the two-dimensional subspace consisting of kinetic solutions. It thus also follows that the stability of the kinetic solution is determined by $\lambda_{3,kin}$ and $\lambda_{4,kin}$.

  For the positive kinetic solution, we see that $\lambda_{3,kin} < 0$ if $\sqrt{6} + c_2 < 0$, so if $c_2 < -\sqrt{6}$. It follows that the positive kinetic solution is stable if $c_2 < -\sqrt{6}$ and $\frac{f_{,\rho}}{f} > 0$. For the negative kinetic solution, we have $\lambda_{3,kin} < 0$ if $\sqrt{6} - c_2 < 0$, so if $c_2 > \sqrt{6}$. Thus it is stable if $c_2 > \sqrt{6}$ and $\frac{f_{,\rho}}{f} < 0$.
- For the gradient solution, the eigenvalues are
$$\lambda_{1,grad} = 0, \quad \lambda_{2,grad} = -\frac{1}{2}\left(6 - c_2^2\right), \quad \lambda_{\pm,grad} = \frac{1}{2}\left(A_{grad} \pm \sqrt{A_{grad}^2 - B_{grad}}\right), \tag{6.58}$$

where we have defined

$$A_{grad} = -\frac{1}{2}\left(6 - c_2^2\right) + c_2\frac{f_{,\rho}}{f}, \tag{6.59}$$

$$B_{grad} = 4\left(\frac{1}{2}\left(6 - c_2^2\right)\frac{\partial_\theta p_\theta(\theta_{ex})}{f^2}\right) \overset{(h'(\theta)=0)}{=} 4\left(\frac{1}{2}\left(6 - c_2^2\right)\frac{V_{,\theta\theta}(\theta_{ex})}{f^2 V(\theta_{ex})}\right). \tag{6.60}$$

The eigenvalue corresponding to the zero eigenvalue is $v_{1,grad} = (1,0,0,0)$. Since it points in the ρ direction and the gradient solution does not depend on ρ, the stability depends on the other eigenvalues $\lambda_{2,grad}$ and $\lambda_\pm$. It is easy to see that the condition $\lambda_{2,grad} < 0$ implies $c_2 \in \left(-\sqrt{6}, \sqrt{6}\right)$. In addition, a necessary condition for $\lambda_{+,grad} < 0$ is $A_{grad} < 0$, which corresponds to

$$-\frac{1}{2}\left(6 - c_2^2\right) + \frac{c_2}{L_\rho} < 0, \text{or} \quad \tilde{c}_{2,crit1} < c_2 < \tilde{c}_{2,crit2}, \tag{6.61}$$

where $\tilde{c}_{2,crit1}$ and $\tilde{c}_{2,crit2}$ are defined as

$$\tilde{c}_{2,crit1} = -\frac{1}{L_\rho} - \sqrt{6 - \frac{1}{L_\rho^2}}, \quad \tilde{c}_{2,crit2} = -\frac{1}{L_\rho} - \sqrt{6 - \frac{1}{L_\rho^2}}. \tag{6.62}$$

These are thus the generalisation of the critical values defined in equation 6.35 for non-constant L_ρ. Stability furthermore requires that $\mathrm{Re}\{A_{grad} + \sqrt{A_{grad}^2 - B_{grad}}\} < 0$, so $B_{grad} > 0$. We already have the condition $|c_2| < \sqrt{6}$, so for $B_{grad} > 0$ the only condition is that $(V_{,\theta\theta}/V)(\theta_{ex}) > 0$. This is equivalent to the condition that $h(\theta)$ has a local minimum at θ_{ex}.

In short, for the gradient solution to be stable, we need

$$|c_2| < \sqrt{6}, \quad \tilde{c}_{2,crit1} < c_2 < \tilde{c}_{2,crit2}, \quad \text{and} \quad \frac{V_{,\theta\theta}}{V}(\theta_{ex}) > 0. \tag{6.63}$$

Compared to the stability conditions of the gradient solution in Section 6.4.2, this amounts to one extra condition.

6.5.2 Exponential Dependence on the Angular Coordinate

Having considered a variety of scaling solutions for which p_ρ is constant, we turn to scaling solutions for which p_θ is constant. This corresponds with a potential that shows exponential dependence on the angle-like coordinate θ:

$$V(\rho, \theta) = k(\rho)e^{c_3\theta}. \tag{6.64}$$

In this case, we have $p_\theta = c_3$ and $p_\rho = k'(\rho)/k(\rho)$. Since p_θ is constant, we have a three-dimensional invariant subspace, spanned by ρ, x and y. The three-dimensional subsystem is then

$$\begin{cases} \rho' = x, \\ x' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)(x + p_\rho) + \frac{f_{,\rho}}{f}y^2, \\ y' = -\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)\left(y + \frac{c_3}{f}\right) - \frac{f_{,\rho}}{f}xy. \end{cases} \tag{6.65}$$

From the equation for ρ', it is immediate that any critical point requires $x = 0$. Looking at the equations for x' and y', we can distinguish a number of cases. For $y' = 0$, we require either $y = \pm\sqrt{6}$, or $y = -c_3/f$. If $y = -c_3/f$, the equation for x' yields

$$-\left(3 - \frac{1}{2}\frac{c_3^2}{f^2}\right)p_\rho + \frac{f_{,\rho}}{f}\frac{c_3^2}{f^2} = 0, \tag{6.66}$$

which can be solved for f^2 to give

$$f^2 = \frac{c_3^2}{6}\left(1 + \frac{2}{p_\rho}\frac{f_{,\rho}}{f}\right). \tag{6.67}$$

Note that equation 6.67 may admit any number of solutions. Motivated by already known models of inflation (such as shift-symmetric orbital inflation, see e.g. [Achúcarro et al., 2020]), we assume that the equation has either a maximum of two distinct solutions, or a continuous curve of solutions. Looking at the equation for x' (for $x = 0$), we see that an additional critical point is found if both p_ρ and f_ρ vanish for some value ρ_{extr}, i.e. if f and $k(\rho)$ have an extremum at this point. In this case, the condition $y' = 0$ gives $y = \pm\sqrt{6}$ or $y = -c_3/f$. Summarising the above, we have the following critical points of the three-dimensional subsystem:

- If we assume that equation 6.67 has a maximum of up to two solutions ρ_0, we obtain a critical point

$$(\rho, x, y)_{froz} = \left(\rho_0, 0, -\frac{c_3}{f}\right). \tag{6.68}$$

  Since $x = 0$, the ρ-coordinate is constant along the trajectory corresponding to this critical point, i.e. this coordinate is 'frozen'. Therefore we call this the *frozen solution*. Using equation 6.67 again, the value of the slow-roll parameter ϵ can be written independently of the value of c_3, i.e. the gradient of the potential in the θ direction:

$$\epsilon = \frac{1}{2}x^2 + \frac{1}{2}y^2 = 0 + \frac{1}{2}\frac{c_3^2}{f^2} = \frac{3p_\rho L_\rho}{2 + p_\rho L_\rho}. \tag{6.69}$$

  Note that this is (almost) exactly the same expression for ϵ as the one that was obtained for the hyperbolic solution in Section 6.4.1, see equation 6.34. This is no coincidence, as explained in 6, Section 5.3.

- Another solution is obtained if we assume that equation 6.67 has an infinite number of solutions, such that the critical points $(\rho_{crit}, 0, -c_3/f)$ form a continuous curve in ρ. This also implies that we have a continuous curve of slow-roll parameters

$$\epsilon = \frac{c_3^2}{2f(\rho_{crit})}, \tag{6.70}$$

  as f is continuous and unequal to zero everywhere.

- The assumption that $f_{,\rho}$ and p_ρ both vanish at the same ρ-coordinate corresponds with f and V having an extremum at the same value ρ_{extr}. This solution is very similar to the gradient solution of Sections 6.4.1, 6.4.2 and 6.5.1. This implies the existence of a critical point

$$(\rho, x, y)_{extr} = \left(\rho_{extr}, 0, -\frac{c_3}{f}\right). \tag{6.71}$$

  The value of the slow-roll parameter ϵ corresponding to this solution is

$$\epsilon = \frac{c_3^2}{2f(\rho_{extr})^2}. \tag{6.72}$$

- Setting $y' = 0$ also yields $y = \pm\sqrt{6}$, which in turn results in $f_{,\rho} = 0$. Thus, if there is some ρ_{extr} for which f has an extremum, we find a type of kinetic solutions given by

$$(\rho, x, y)_{kin} = \left(\rho_{extr}, 0, \pm\sqrt{6}\right). \tag{6.73}$$

  The assumption that $f_{,\rho}$ and p_ρ both vanish at ρ_{extr}, as was done to find the extremum solution, also implies the existence of these kinetic solutions with $\epsilon = 3$.

Having determined the critical points, we consider their stability. The Jacobian of the three-dimensional subsystem is given by

$$\mathcal{J}(x,y) = \begin{pmatrix} 0 & 1 & 0 \\ -y^2\left(\frac{f_{,\rho}^2}{f^2} - \frac{f_{,\rho\rho}}{f}\right) - (3-\epsilon)\partial_\rho p_\rho & -(3-\epsilon) + x(p_\rho + x) & y(p_\rho + x) + 2\frac{f_{,\rho}}{f}y \\ xy\left(\frac{f_{,\rho}^2}{f^2} - \frac{f_{,\rho\rho}}{f}\right) + \frac{(3-\epsilon)c_3 f_{,\rho}}{f^2} & x\left(y + \frac{c_3}{f}\right) - \frac{yf_{,\rho}}{f} & -(3-\epsilon) + y\left(y + \frac{c_3}{f}\right) - \frac{xf_{,\rho}}{f} \end{pmatrix}. \tag{6.74}$$

We determine conditions for the stability of all critical points found above.

- Making the substitutions $x = 0$ and $y = -c_3/f$ allows us to calculate the eigenvalues corresponding to the frozen solution. These are given by

$$\lambda_{1,froz} = -A_{froz}, \quad \lambda_{\pm,froz} = -\frac{1}{2}\left(A_{froz} \pm \sqrt{A_{froz}^2 - B_{froz}}\right), \tag{6.75}$$

  where we have defined

$$A_{froz} = 3 - \epsilon = 3 - \frac{c_3^2}{2f^2} = 3 - \frac{3p_\rho L_\rho}{2 + p_\rho L_\rho}, \tag{6.76a}$$

$$B_{froz} = 4c_3^2 p_\rho \frac{f_{,\rho}}{f^3} + 12c_3^2 \frac{f_{,\rho}^2}{f^4} - 2c_3^2 \frac{\partial_\rho p_\rho}{f^2} + 12\partial_\rho p_\rho - 4c_3^2 \frac{f_{,\rho\rho}}{f^3}. \tag{6.76b}$$

Of course, ρ_0 must be substituted for every instance of L_ρ, p_ρ and f and their derivatives. For the frozen solution to be stable, we require that $\lambda_{1,froz}, \lambda_{\pm,froz} < 0$. As the fraction $3p_\rho L_\rho/(2 + p_\rho L_\rho)$ is smaller than 3 for all values of p_ρ and L_ρ, it follows that $A_{froz} > 0$, and thus $\lambda_{1,froz} < 0$. The eigenvalue $\mathrm{Re}\{\lambda_{+,froz}\}$ (and thus also $\mathrm{Re}\{\lambda_{-,froz}\}$) is negative if and only if $B_{froz} > 0$.

We specifically consider the case where $V(\rho, \theta)$ is a product exponential and f is exponential in ρ. As before, this means that $p_\rho = c_2$ and $1/L_\rho = 1/L$. It also follows that $\partial_\rho p_\rho = 0$ and $\partial_\rho(1/L_\rho) = f_{,\rho\rho}/f - f_{,\rho}^2/f^2 = 0$. Using the expression for B_{froz} (equation 6.76b), it follows that

$$\begin{aligned}
B_{froz} &= \frac{1}{f^2}\left(4c_3^2 c_2 \frac{f_{,\rho}}{f} + 12c_3^2 \frac{f_{,\rho}^2}{f^2} - 0 + 0 - 4c_3^2 \frac{f_{,\rho\rho}}{f}\right) \\
&= \frac{c_3^2}{f^2}\left(4c_2 \frac{1}{L} + 8\frac{1}{L^2} + 4\left(-\frac{f_{,\rho\rho}}{f} + \frac{f_{,\rho}^2}{f^2}\right)\right) \\
&= \frac{c_3^2}{f^2 L^2}\left(4c_2 L + 8\right) \\
&\overset{(6.67)}{=} \frac{1}{L^2} \frac{6}{1 + \frac{2}{c_2 L}} \left(4c_2 L + 8\right) \\
&= 24c_2 L.
\end{aligned} \tag{6.77}$$

In short, the stability of the frozen solution only requires $B_{froz} > 0$, which is in general given by the expression in equation 6.76b. In the specific case that $V(\rho, \theta)$ is a product exponential and field space is hyperbolic (which corresponds to an exponential f), we find that the frozen solution is stable is $p_\rho L_\rho > 0$.

- For the case that the solutions to 6.67 form a continuous curve in ρ, the results of the stability analysis do not differ significantly from the results of the stability analysis for the frozen solution. However, one of the eigenvalues will be zero, with an eigenvector that points along the curve. Per the results above, every critical point on this curve with $p_\rho L_\rho > 0$ will thus be stable, but not asymptotically stable. Note that some of the critical points on the curve may have $p_\rho L_\rho \geq 0$. However, for every critical point with $p_\rho L_\rho > 0$ we can by continuity find a neighbourhood such that $p_\rho L_\rho > 0$ for all other critical points on the curve in this neighbourhood, such that the critical point is stable.
- For the kinetic solutions, the eigenvalues are given by

$$\lambda_{1,kin} = \frac{\sqrt{6}}{f}\left(\sqrt{6}f \pm c_3\right), \quad \lambda_{\pm,kin} = \sqrt{6}\sqrt{\frac{f_{,\rho\rho}}{f} - \frac{f_{,\rho}^2}{f^2}}. \tag{6.78}$$

We see that $\mathrm{Re}\{\lambda_{+,kin}\} > 0$ if $(\partial_\rho(1/L_\rho))(\rho_{extr}) \neq 0$, so the kinetic solutions are always unstable, except possibly in the special case that $\partial_\rho(1/L_\rho))(\rho_{extr}) = 0$. Note that the eigenvalues do not depend on p_ρ, so that it does not matter whether we assume $p_\rho(\rho_{extr}) = 0$ or not.

- Lastly, the eigenvalues corresponding with the extremum solution are given by

$$\lambda_{1,extr} = -A_{extr}, \quad \lambda_{\pm,extr} = -\frac{1}{2}\left(A_{extr} \pm \sqrt{A_{extr}^2 - B_{extr}}\right), \tag{6.79}$$

where we have introduced

$$A_{extr} = -(3-\epsilon) = -\left(3 - \frac{c_3^2}{2f^2}\right) \tag{6.80}$$

$$B_{extr} = 4c_3^2\frac{f_{,\rho}^2}{f^4} - 2c_3^2\frac{\partial_\rho p_\rho}{f^2} + 12\partial_\rho p_\rho - 4c_3^2\frac{f_{,\rho\rho}}{f^3}, \tag{6.81}$$

analogously to A_{froz} and B_{froz} for the frozen solution. We see that $\lambda_{1,extr} < 0$ if $A_{extr} > 0$, so (from 6.80) if $c_3^2/2f^2 < 3$. In addition, we require for stability that $\lambda_{+,extr} < 0$ (from which it follows that $\lambda_{-,extr}$). Thus, the conditions for stability of the extremum solution are $c_3^2/2f^2 < 3$ and $B_{extr} > 0$.

6.5.3 Case Study: Angular Inflation

In general, models of inflation cannot be described by product-separable potentials. However, there are systems with more general potentials that do exhibit behaviour similar to the frozen solution that we found in Section 6.5.2. We specifically consider a model of angular inflation, in which the inflationary trajectory is predominantly in the angular θ direction. In [Christodoulidis et al., 2019a], a model for angular inflation described, where field-space is taken to be a so-called *Poincare disc* and the potential is not a product-separable potential. This can be described in polar coordinates as

$$\mathrm{d}s^2 = \frac{6\alpha}{(1-\rho^2)^2}\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2\right), \tag{6.82}$$

for some $\alpha \in \mathbb{R}$. We transform the system by setting $\rho = \tanh(\rho/\sqrt{6\alpha})$, so that we obtain

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \frac{3\alpha}{2}\sinh^2\left(\sqrt{\frac{2}{3\alpha}}\rho\right). \tag{6.83}$$

Note that this is the same metric as in Section 6.4.3, under the identification $L = \sqrt{3\alpha/2}$. Following [Christodoulidis et al., 2019a], the velocity in the angular direction is found to be

$$\dot{\theta} \simeq -\frac{V_{,\theta}}{3Hf^2}. \tag{6.84}$$

Using the slow-roll approximation ($\epsilon \ll 1$), we have $3H^2 = V$, so we obtain

$$f\theta' = y = -\frac{V_{,\theta}}{3H^2 f} = -\frac{V_{,\theta}}{Vf} = -\frac{p_\theta}{f}. \tag{6.85}$$

In addition, it can be derived that in the slow-roll approximation, the radial coordinate ρ must satisfy

$$f^2 = \frac{2p_\theta^2}{6p_\rho L_\rho}. \tag{6.86}$$

This equals equation 6.67, that must be satisfied by the radial coordinate for the frozen solution if $1 \ll 2/(p_\rho L_\rho)$, i.e. if $p_\rho L_\rho \ll 1$, and p_θ is non-constant in equation 6.67. The condition $p_\rho L_\rho \ll 1$ holds if the hyperbolic field-space, described by 6.82, is strongly curved. It now follows that the inflationary trajectory of angular inflation is an approximate frozen solution, albeit with a slowly varying radius (so $x \neq 0$).

6.6 Discussion of Results

In this Chapter, two-field models of inflation were studied by transforming the background equations of motion into a four-dimensional dynamical system with two equations for the scalar coordinates ρ and θ and two equations for the associated 'velocities' $x = gv^\rho$ and $y = fv^\theta$.

The first case that was studied was a Euclidean metric (Section 6.3). Most notably, under the additional assumption that $V(\rho, \theta) = V(\rho)$ has an exponential dependence on ρ, it was found that if $|c_2| < \sqrt{6}$, the equilibrium $(-c_2, 0)$ is an attractor in the two-dimensional subsystem that consists of equations for x and y.

Next, it was assumed that the metric has an isometry corresponding to shifts in the θ-coordinate and that $V(\rho)$ has an exponential dependence on ρ. When the additional assumption of $f_{,\rho}/f = const.$ was made, gradient, kinetic and hyperbolic solutions were found, with the hyperbolic solutions being a generalised instance of the circle solution in Section 6.3.1. Pitchfork bifurcations were found at two critical points $c_{2,crit1}$, $c_{2,crit2}$. In addition, it was shown that hyperinflation may be approximated by a hyperbolic solution.

Finally, more general potentials $V(\rho, \theta)$ were studied. Assuming an exponential dependence on θ led to a new types of solutions; the frozen and extremum solutions. In the context of those frozen solutions, angular inflation was briefly discussed.

Important to note is that some of the solutions that were found as scaling solutions violate the slow-roll conditions. The most obvious examples are the circle and and kinetic solutions, as we found that $\epsilon = 3$ for these solutions. For other solutions, such as the hyperbolic solutions, the slow-roll conditions are also violated for some choices of the parameters (in this instance specifically c_2 and L). If the slow-roll conditions are violated for some solution, this solution does not correspond to anything studied in Chapter 5, as there is was assumed that $\epsilon \ll 1$. In particular, the slow-roll parameter corresponding to the attractor $(-c_2, 0)$ is $\epsilon = c_2^2/2$, and thus is still an attractor if the slow-roll condition $\epsilon \ll 1$ is satisfied.

Chapter 7

Comparison of the approaches

In Chapter 5, conditions for an attractor were derived, both for the case that V_{vv} and V_{vw} are negligible and for the case when they are not. In the first case, the only constraint on the attractor is a contraint on the velocities $\dot{\phi}_v, \dot{\phi}_w$, given by

$$\dot{\phi}_v = \frac{-3V_v}{H(9+\omega^2)}, \quad \dot{\phi}_w = \frac{\omega V_v}{H(9+\omega^2)}, \tag{7.1}$$

where ω is given by

$$\omega^2 \simeq \frac{V_{ww}}{H^2} - 9. \tag{7.2}$$

Although there is a constraint for the velocity that the attractor might have, it may be located anywhere is field-space. Thus, equation 7.1 restricts the attractor to a two-dimensional subspace of the 4-dimensional phase space of solutions. In the non-negligible case, equation 7.1 still holds, but an additional constraint was derived in the form of a relation between the field-space coordinates ρ and θ:

$$\frac{V_{ww}}{V} = 3 + 3\left(\frac{V_{vv}}{V_{vw}}\right) + \frac{V_{vv}}{V}\left(\frac{V_{vw}}{V_{vv}}\right)^2 + \mathcal{O}(\epsilon). \tag{7.3}$$

On the other hand, in Chapter 6, stability analyses were done for specific choices for the metric $\mathcal{G}_{ab}$ on field-space and potential $V(\rho,\theta)$. The aim of this chapter is therefore to relate the results of Chapters 5 and 6 to each other. In Section 7.1, we rewrite the conditions for the attractor in terms of a general metric and potential. In addition, the turn rates of the critical points/solutions of Chapter 6 are determined in Section 7.2.

7.1 Rewriting the Attractor Conditions

Although equation 7.3 is a very elegant expression, one first has to determine the terms V_{vv}, V_{vw} and V_{ww} to get any meaningful result. Our aim in this section is therefore to 'unpack' the result for any concrete choice of metric and potential. As in Chapter 6, the most general metric for a two-dimensional field-space manifold is given by

$$\mathrm{d}s^2 = g^2(\rho,\theta)\mathrm{d}\rho^2 + f^2(\rho,\theta)\mathrm{d}\theta^2. \tag{7.4}$$

In addition, we assume a general potential $V = V(\rho,\theta)$. Recalling the definition of for example V_{vw} as $V_{vw} = v^a w^b V_{;ab}$, our first goal will be to derive explicit expressions for the orthonormal gradient basis vectors v^a and w^a. For future reference, note that

$$g_{ab} = \begin{pmatrix} g^2(\rho,\theta) & 0 \\ 0 & f^2(\rho,\theta) \end{pmatrix} \quad \text{and} \quad g^{ab} = \begin{pmatrix} \frac{1}{g^2(\rho,\theta)} & 0 \\ 0 & \frac{1}{f^2(\rho,\theta)} \end{pmatrix}. \tag{7.5}$$

Starting out with v^a, we first derive that

$$V_v := \sqrt{V^{,a}V_{,a}} = \sqrt{g^{ab}V_{,a}V_{,b}} = \sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}, \tag{7.6}$$

such that

$$v^\rho := \frac{V^{,\rho}}{V_v} = \frac{g^{\rho b}V_{,b}}{V_v} = \frac{V_{,\rho}}{g^2\sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}}, \quad v^\theta := \frac{V^{,\theta}}{V_v} = \frac{g^{\theta b}V_{,b}}{V_v} = \frac{V_{,\theta}}{f^2\sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}}. \tag{7.7}$$

The orthogonality condition $v^a w_a = 0$ gives that

$$g_{ab}v^a w^b = g^2 v^\rho w^\rho + f^2 v^\theta w^\theta = \frac{V_{,\rho}}{\sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}} w^\rho + \frac{V_{,\theta}}{\sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}} w^\theta = 0, \tag{7.8}$$

or

$$w^\theta = -\frac{V_{,\rho}}{V_{,\theta}} w^\rho. \tag{7.9}$$

Normalising w^a, i.e. using that $g_{ab}w^a w^b = 1$, then gives

$$w^\rho = \frac{1}{\sqrt{g^2 + f^2\left(\frac{V_{,\rho}}{V_{,\theta}}\right)^2}} = \frac{V_{,\theta}}{\sqrt{g^2V_{,\theta}^2 + f^2V_{,\rho}^2}}, \tag{7.10a}$$

$$w^\theta = -\frac{\frac{V_{,\rho}}{V_{,\theta}}}{\sqrt{g^2 + f^2\left(\frac{V_{,\rho}}{V_{,\theta}}\right)^2}} = -\frac{V_{,\rho}}{\sqrt{g^2V_{,\theta}^2 + f^2V_{,\rho}^2}}. \tag{7.10b}$$

Next, we will derive the covariant derivatives of the potential $V_{;\rho\rho}$, $V_{;\rho\theta} = V_{;\theta\rho}$ and $V_{;\theta\theta}$, using the definition

$$V_{;ab} = V_{,ab} - \Gamma^c_{ab}V_{,c}, \tag{7.11}$$

and the Christoffel symbols (see also section 6):

$$\begin{aligned}
\Gamma^{\rho}_{\rho\rho} &= \frac{g_{,\rho}}{g}, \quad \Gamma^{\rho}_{\rho\theta} = \frac{g_{,\theta}}{g}, \quad \Gamma^{\rho}_{\theta\theta} = -\frac{ff_{,\rho}}{g^2}, \\
\Gamma^{\theta}_{\theta\theta} &= \frac{f_{,\theta}}{f}, \quad \Gamma^{\theta}_{\rho\theta} = \frac{f_{,\rho}}{f}, \quad \Gamma^{\theta}_{\rho\rho} = -\frac{gg_{,\theta}}{f^2}.
\end{aligned} \tag{7.12}$$

This gives us

$$V_{;\rho\rho} = V_{,\rho\rho} - \Gamma^{c}_{\rho\rho}V_{,c} = V_{,\rho\rho} - \frac{g_{,\rho}}{g}V_{,\rho} + \frac{gg_{,\theta}}{f^2}V_{,\theta}; \tag{7.13a}$$

$$V_{;\rho\theta} = V_{;\theta\rho} = V_{,\rho\theta} - \Gamma^{c}_{\rho\theta}V_{,c} = V_{,\rho\theta} - \frac{g_{,\theta}}{g}V_{,\rho} - \frac{f_{,\rho}}{f}V_{,\theta}; \tag{7.13b}$$

$$V_{;\theta\theta} = V_{,\theta\theta} - \Gamma^{c}_{\theta\theta}V_{,c} = V_{,\theta\theta} + \frac{ff_{,\rho}}{g^2}V_{,\rho} - \frac{f_{,\theta}}{f}V_{,\theta}. \tag{7.13c}$$

This now finally allows us to calculate V_{vv}, V_{vw} and V_{ww}. We have

$$\begin{aligned}
V_{vv} &= v^a v^b V_{;ab} = (v^\rho)^2 V_{;\rho\rho} + 2v^\rho v^\theta V_{;\rho\theta} + (v^\theta)^2 V_{;\theta\theta} \\
&= \frac{1}{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}} \left(\frac{V_{,\theta}^2}{f^4}\left(V_{,\theta\theta} - \frac{f_{,\theta}V_{,\theta}}{f} + \frac{ff_{,\rho}V_{,\rho}}{g^2}\right) + 2\frac{V_{,\theta}V_{,\rho}}{f^2g^2}\left(V_{,\rho\theta} - \frac{g_{,\theta}V_{,\rho}}{g} - \frac{f_{,\rho}V_{,\theta}}{f}\right) \right. \\
&\quad \left. + \frac{V_{,\rho}^2}{g^4}\left(V_{,\rho\rho} - \frac{g_{,\rho}V_{,\rho}}{g} + \frac{gg_{,\theta}V_{,\theta}}{f^2}\right)\right);
\end{aligned} \tag{7.14a}$$

$$\begin{aligned}
V_{vw} &= v^a w^b V_{;ab} = v^\rho w^\rho V_{;\rho\rho} + \left(v^\rho w^\theta + v^\theta w^\rho\right) V_{;\rho\theta} + v^\theta w^\theta V_{;\theta\theta} \\
&= \frac{1}{\sqrt{g^2V_{,\theta}^2 + f^2V_{,\rho}^2}\sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}} \left(-\frac{V_{,\theta}V_{,\rho}}{f^2}\left(V_{,\theta\theta} - \frac{f_{,\theta}V_{,\theta}}{f} + \frac{ff_{,\rho}V_{,\rho}}{g^2}\right) \right. \\
&\quad \left. + \left(\frac{V_{,\theta}^2}{f^2} - \frac{V_{,\rho}^2}{g^2}\right)\left(V_{,\rho\theta} - \frac{g_{,\theta}}{g}V_{,\rho} - \frac{f_{,\rho}}{f}V_{,\theta}\right) + \frac{V_{,\theta}V_{\rho}}{g^2}\left(V_{,\rho\rho} - \frac{g_{,\rho}}{g}V_{,\rho} + \frac{gg_{,\theta}}{f^2}V_{,\theta}\right)\right);
\end{aligned} \tag{7.14b}$$

$$\begin{aligned}
V_{ww} &= w^a w^b V_{;ab} = (w^\rho)^2 V_{;\rho\rho} + 2w^\rho w^\rho V_{;\rho\theta} + (w^\theta)^2 V_{;\theta\theta} \\
&= \frac{1}{g^2V_{,\theta}^2 + f^2V_{,\rho}} \left(V_{,\rho}^2\left(V_{,\theta\theta} + \frac{ff_{,\rho}}{g^2}V_{,\rho} - \frac{f_{,\theta}}{f}V_{,\theta}\right) - 2V_{,\theta}V_{,\rho}\left(V_{,\rho\theta} - \frac{g_{,\theta}}{g}V_{,\rho} - \frac{f_{,\rho}}{f}V_{,\theta}\right) \right. \\
&\quad \left. + V_{,\theta}^2\left(V_{,\rho\rho} - \frac{g_{,\rho}}{g}V_{,\rho} + \frac{gg_{,\theta}}{f^2}V_{,\theta}\right)\right).
\end{aligned} \tag{7.14c}$$

7.1.1 Constraints for the Non-negligible Case

Combining the constraint in equation 7.3 with the expressions derived in 7.14a, 7.14b and 7.14c, we obtain

$$\frac{1}{V\left(g^2V_{,\theta}^2 + f^2V_{,\rho}^2\right)}\left(V_{;\theta\theta}V_{,\rho}^2 - 2V_{;\rho\theta}V_{,\theta}V_{,\rho} + V_{;\rho\rho}V_{\theta}^2\right)$$

$$
\begin{aligned}
&= 3 + 3\frac{g^2V_{,\theta}^2 + f^2V_{,\rho}^2}{\frac{V_{,\theta}^2}{f^2} + \frac{V_{,\rho}^2}{g^2}} \left(\frac{V_{\theta\theta}\frac{V_{,\theta}^2}{f^4} + 2V_{;\rho\theta}\frac{V_{,\theta}V_{,\rho}}{f^2g^2} + V_{;\rho\rho}\frac{V_{,\rho}^2}{g^4}}{-V_{;\theta\theta}\frac{V_{,\theta}V_{,\rho}}{f^2} + V_{;\rho\theta}\left(\frac{V_{,\theta}^2}{f^2} - \frac{V_{,\rho}^2}{g^2}\right) + V_{;\rho\rho}\frac{V_{,\theta}V_{,\rho}}{g^2}} \right)^2 \\
&+ \frac{1}{V}\frac{1}{g^2V_{,\theta}^2 + f^2V_{,\rho}^2} \frac{\left(V_{;\theta\theta}\frac{V_{,\theta}V_{,\rho}}{f^2} + V_{;\rho\theta}\left(\frac{V_{,\theta}^2}{f^2} - \frac{V_{,\rho}^2}{g^2}\right) + V_{;\rho\rho}\frac{V_{,\theta}V_{,\rho}}{g^2}\right)^2}{-V_{;\theta\theta}\frac{V_{,\theta}^2}{f^4} + 2V_{;\rho\theta}\frac{V_{,\theta}V_{,\rho}}{f^2g^2} + V_{;\rho\rho}\frac{V_{,\rho}^2}{g^4}}
\end{aligned}
\tag{7.15}
$$

Here, $V_{;\rho\rho}$, $V_{;\rho\theta}$ and $V_{;\theta\theta}$ are as in equations 7.13a,7.13b and 7.13c.

Having worked out the equation for the constraint on the field-space coordinates ρ and θ, we see that this equation is quite involved. However, the worked-out equation 7.15 is very useful for calculation purposes, as much of the terms immediately simplify upon making a certain choice for metric and potential.

Furthermore, additional constraints can be derived for the field-space velocities $\dot{\phi}_v$ and $\dot{\phi}_w$. Using equations 5.27 and the expressions for the turn rate 5.45 and 5.46 in Chapter 5, we obtain

$$
\dot{\phi}_v = \frac{-3V_v}{H(9+\omega^2)} = \frac{-3V_v}{H\left(9 + \frac{V_{ww}}{H^2} - \frac{V_{vw}^2}{V_{vv}^2}\frac{V_{vv}}{H^2} - 9\right)} = \frac{-3V_v}{H}\frac{1}{V_{ww} - \frac{V_{vw}^2}{V_{vv}}}, \tag{7.16}
$$

and

$$
\dot{\phi}_w = \frac{\omega V_v}{H(9+\omega^2)} = \frac{3V_v}{H}\frac{V_{vv}}{V_{vw}V_{ww} - \frac{V_{vw}^3}{V_{vv}}}. \tag{7.17}
$$

Here, we do not fill in the terms V_v, V_{vv}, V_{vw} and V_{ww} for the sake of clarity.

7.1.2 Constraints for the Negligible Case

In the case where $V_{vv}/H^2 \lesssim \mathcal{O}(\omega^2\epsilon)$ and $V_{vw}/H^2 \lesssim \mathcal{O}(\omega\epsilon)$, we had that

$$
\omega^2 \simeq \frac{V_{ww}}{H^2} - 9, \tag{7.18}
$$

which leads to constraints on the field-space velocities that are given by

$$
\dot{\phi}_v = \frac{-3V_v}{H(9+\omega^2)} = \frac{-3HV_v}{V_{ww}}, \tag{7.19}
$$

and

$$
\dot{\phi}_w = \frac{\omega V_v}{H(9+\omega^2)} = \pm\frac{V_v}{\sqrt{V_{ww}}}, \tag{7.20}
$$

where we again do not fill in the terms V_v and V_{ww} for the sake of clarity. Unlike the non-negligible case, the negligible case does not lead to any constraint on the field-space coordinates ρ and θ, other than the assumption that V_{vv} and V_{vw} be negligible.

7.2 Scaling Solutions and their Turn Rates

A quantity that played an important role in the analysis of Chapter 5, was the turn rate ω. We determine the turn rates of the scaling solutions found in Chapter 6. Recall that ω can be written as

$$\omega = \frac{\dot{\phi}_w V_v}{H\dot{\phi}^2}. \tag{7.21}$$

Using the definitions of $\dot{\phi}_w$, V_v and $\dot{\phi}^2$, we find that

$$\begin{aligned}\omega &= \frac{\mathcal{G}_{ab} w^a \dot{\phi}^b \sqrt{\mathcal{G}^{ab} V_{,a} V_{,b}}}{H \mathcal{G}_{ab} \dot{\phi}^a \dot{\phi}^b} \\ &= \frac{\left(g^2 w^\rho \dot{\rho} + f^2 w^\theta \dot{\theta}\right) \sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}}{H\left(g^2 \dot{\rho}^2 + f^2 \dot{\theta}^2\right)} \\ &= \frac{\left(gxV_{,\theta} - fyV_{,\rho}\right) \sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}}{H^2\left(x^2+y^2\right)\sqrt{g^2 V_{,\theta}^2 + f^2 V_{,\rho}^2}} \\ &= \frac{\left(gxV_{,\theta} - fyV_{,\rho}\right)\left(3 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) \sqrt{\frac{V_{,\rho}^2}{g^2} + \frac{V_{,\theta}^2}{f^2}}}{V\left(x^2+y^2\right)\sqrt{g^2 V_{,\theta}^2 + f^2 V_{,\rho}^2}}. \end{aligned} \tag{7.22}$$

As the scaling solutions were found in terms of (ρ, θ,) x and y, we may use equation 7.22 to determine the corresponding turn rates.

The scaling solutions with non-zero turn rates are the following:

- In Section 6.4.1, the turn rate corresponding to the hyperbolic solution is given by

  $$\omega^2 = \frac{6(2c_2 + L(c_2^2 - 6))}{L(2 + c_2 L)} = \frac{6(c_2^2 + 2\frac{c_2}{L} - 6)}{2 + Lc_2}. \tag{7.23}$$

  Recall that c_2 is a constant that comes from the potential $V(\rho) = c_1 e^{c_2 \rho}$. Recall that the first slow-roll parameter of the hyperbolic solution is given by

  $$\epsilon = \frac{3Lc_2}{2 + Lc_2}. \tag{7.24}$$

  In Chapter 5, a minimal requirement for the attractor was that $\omega^2 \gg \mathcal{O}(\epsilon)$. Thus, for the hyperbolic solution we require $6(c_2^2 + 2c_2/L - 6) \gg 3Lc_2$.

- In Section 6.5.2, the turn rate corresponding to the frozen solution $(\rho_0, 0, -c_3/f)$ is

  $$\omega^2 = \frac{\left(c_3^2 - 6f^2\right)^2 \left(c_3 k(\rho) + k'(\rho)\right)^2}{4c_3^2 f^2 k^2(\rho)}$$

$$
\begin{aligned}
&\overset{6.67}{=} \frac{\left(c_3^2 - c_3^2 - \frac{2c_3^2}{p_\rho}\frac{1}{L_\rho}\right)^2 (c_3 k(\rho) + k'(\rho))^2}{4c_3^2 f^2 k^2(\rho)} \\
&= \frac{\left(\frac{4c_3^4}{p_\rho^2}\frac{1}{L_\rho^2}\right)(c_3 k(\rho) + k'(\rho))^2}{4c_3^2 f^2 k^2(\rho)} \\
&= \frac{\frac{c_3^2}{p_\rho^2}\frac{1}{L_\rho^2}\left(c_3^2 k^2(\rho) + 2c_3 k(\rho)k'(\rho) + (k'(\rho))^2\right)}{f^2 k^2(\rho)} \\
&= \frac{c_3^2}{p_\rho^2}\frac{1}{L_\rho^2}\left(\frac{c_3^2}{f^2} + \frac{1}{f^2}\left(2c_3 p_\rho + p_\rho^2\right)\right) \\
&= \frac{c_3^2}{p_\rho^2}\frac{1}{L_\rho^2}\left(\frac{6L_\rho p_\rho}{2 + p_\rho L_\rho} + \frac{1}{f^2}\left(2c_3 p_\rho + p_\rho^2\right)\right) \\
&= \frac{c_3^2}{p_\rho^2}\frac{1}{L_\rho^2}\left(\frac{6L_\rho p_\rho + \frac{1}{f^2}\left(4\frac{c_3}{p_\rho} + 2c_3 L_\rho + p_\rho L_\rho + 2\right)}{2 + p_\rho L_\rho}\right)
\end{aligned} \tag{7.25}
$$

Recall that $k(\rho)$ and c_3 are both part of the potential $V(\rho, \theta) = k(\rho)e^{c_3\theta}$. In addition, the first slow-roll parameter corresponding to the frozen solution was found to be

$$
\epsilon = \frac{3p_\rho L_\rho}{2 + p_\rho L_\rho}. \tag{7.26}
$$

Therefore, a minimum requirement for the frozen solution to be an attractor is

$$
\frac{c_3^2}{p_\rho^2}\frac{1}{L_\rho^2}\left(6L_\rho p_\rho + \frac{1}{f^2}\left(4\frac{c_3}{p_\rho} + 2c_3 L_\rho + p_\rho L_\rho + 2\right)\right) \gg 3p_\rho L_\rho. \tag{7.27}
$$

As highlighted before, the hyperbolic solutions can in fact be seen as a special case of the frozen solution. Therefore, it is not surprising that precisely these two types of solutions have non-zero turn rates.

Chapter 8

Conclusion

In this thesis, an extensive introduction has been given into general relativity, and in particular into cosmology of inflation. In Section 2, an overview was given of a number of topics in differential geometry that form the basis of the theory of general relativity, such as (co)tangent spaces, tensors, pseudo-Riemannian metrics, and the Levi-Civita connection. Some of the physical concepts of general relativity, and specifically of cosmology, were introduced in Section 3. For example the Einstein field equation was introduced, and the FLRW metric was motivated and introduced. Furthermore, single-field inflation and the more general concept of multifield inflation were introduced in this section. In Section 4, a number of mathematical concepts from dynamical systems theory, needed to study the system of differential equations in multifield inflation, were defined.

Besides having introduced the required background knowledge, two-field models of inflation have been studied in two different ways and the different approaches have been compared. In Section 5, a constraint was derived which must be satisfied by a possible attractor in rapid-turn inflation, following the approach of [Bjorkmo, 2019]. In addition, the stability of this possible attractor was studied by considering the equations of motion for spatially homogeneous perturbations. However, a number of results have been found that differ from the results in [Bjorkmo, 2019]. In [Bjorkmo, 2019], insightful results about the stability of the background solution were found by calculating the eigenvalues of the evolution matrix of the perturbations. However, in our analysis, different results were recovered, making these eigenvalues significantly less insightful. Therefore, the results of the stability analysis remain unclear as of yet. In addition, it is not yet clear if the eigenvalues of the evolution matrix indeed correspond to the local Lyapunov exponents of the inflationary trajectory, as claimed in [Bjorkmo, 2019].

In Section 6, two-field models of inflation were studied in a different way. By deriving a four-dimensional dynamical system from the background equations of motion, the equations of motion for the scalar field coordinates and the associated velocities could be studied for concrete choices of metrics and potentials on field-space, following the approach of [Christodoulidis et al., 2019b]. Different types of solutions, characterised as critical points of two- or three-dimensional subsystems, were found, such as gradient solutions, hyperbolic solutions, frozen solutions and kinetic solutions. However, as for the analysis in Chapter 5, the exact stability of these solutions remains unclear. In [Christodoulidis et al., 2019b], it is claimed that the eigenvalues of the critical points in the subsystem correspond to the local Lyapunov exponents of the trajectory associated with the critical points in the original four-dimensional dynamical system. However, this is not necessarily the case and further research is needed to investigate this claim (see the discussion in Chapter 4, Section 4.4).

Finally, in Chapter 7, the two methods of studying the models of inflation were compared. The constraints for the attractor found in Chapter 5 were written in terms of the most general two-dimensional Riemannian metric. In addition, the turn rates of the scaling solutions found in Chapter 6 were calculated.

For further research, it would be interesting to make a more extensive comparison between the two approaches we have described. For example, in Section 5.5, (generalised) hyperinflation was considered, using the constraint derived for the possible attractor. It would be useful to consider the result of this analysis for hyperinflation in the four-dimensional system used in Chapter 6, so that the possible stability can be studied more directly. Another interesting topic would be to investigate how the local Lyapunov exponents are related to the eigenvalues of the evolution matrix for spatially homogeneous perturbations (Chapter 5), and to the eigenvalues of critical points of a two- or three dimensional subsystem (Chapter 6). Furthermore, it would be interesting to study how the results of our stability analysis in Chapter 5 (as opposed to the results of [Bjorkmo, 2019]), impact the stability of the background solution that satisfies the constraint. Finally, the analysis can be extended to higher dimensional multifield inflation models, although the approach for this is not immediately clear.

In conclusion, we have gained more insight into two-field models of inflation and have established a solid foundation in general relativity, differential geometry and dynamical systems. The analysis and comparisons done in this work contribute to a better understanding of the different approaches in analysing this crucial era in the very early universe.

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