Geometric quantization of symplectic toric manifolds
Es, Anton van

Geometric quantization of symplectic toric manifolds

THESIS

submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
in
MATHEMATICS

Author :	A.D. van Es
Supervisor :	Dr. F. Pasquotto
Second correctors :	Dr. B. Mesland
	Prof.dr. J.M. van Ruitenbeek

Leiden, The Netherlands, October 9, 2023

Geometric quantization of symplectic toric manifolds

A.D. van Es

Mathematisch Instituut, Leiden University

October 9, 2023

Contents

1 Introduction **3**

2 Preliminaries **7**

2.1 Formalism of classical mechanics 7
2.1.1 Symplectic geometry 7
2.1.2 Contact geometry 9
2.1.3 Completely integrable Hamiltonian systems 10
2.1.4 Symplectic reduction 11
2.1.5 Kähler manifolds 12
2.2 Formalism of quantum mechanics 16
2.2.1 Complex Hilbert spaces 16
2.2.2 Densely defined operators 18
2.2.3 Canonical quantization 19
2.2.4 Fock quantization 21

3 Prequantization **23**

3.1 Prequantum line bundles 23
3.2 Boothby-Wang prequantization 27
3.3 Existence of prequantum bundles 33
3.4 Examples 38
3.4.1 Harmonic oscillator 38
3.4.2 Free particle 39
3.4.3 Spin particle 41

4 Quantization **43**

4.1 Polarizations 43
4.2 Quantization of real polarization 46

4.3 Quantization of Kähler polarization 49
4.4 Examples 51
4.4.1 Harmonic oscillator 51
4.4.2 Free particle 54
4.4.3 Spin particle 56

5 Quantization of symplectic toric manifolds 59
5.1 Delzant's theorem 59
5.2 Quantization from the moment polytope 70
5.3 Examples 80
5.3.1 Harmonic oscillator 80
5.3.2 Spin particle 80

Bibliography 81

Chapter 1

Introduction

In the mathematical treatment of a macroscopic mechanical system, the assumption is made that Newton's laws hold. This so-called classical paradigm predicts the outcome of experiments well. However, since our universe is more fundamentally ruled by quantum mechanics, such a classical system should actually be a good approximation of some quantum-mechanical system. The question then arises how one could determine a quantum description of a system from the classical one.

The most convenient formalism of classical mechanics for this construction is that due to Hamilton. Here, the system is described in terms of its phase space, which is a symplectic manifold (M, ω) of which every point corresponds to a possible state of the classical system. An observable quantity of this system is represented by a function $f \in C^\infty(M)$: the value $f(p)$ is the outcome of the observation if the system is in state $p \in M$.

On the other hand, a quantum system is described by a complex Hilbert space $\mathfrak{H}$, and the states of the system are described by the unit elements of $\mathfrak{H}$. An observable quantity of this system is represented by a Hermitian operator A. If $\psi_\lambda \in \mathfrak{H}$ is a unit eigenvector of A with eigenvalue λ, the outcome of the observation will be λ if the system is in state ψ_λ. From the spectral theorem we conclude that more generally $\psi^* A \psi$ may be seen as the expectation value of the observation if the system is in state ψ.

These things considered, we are interested in constructing a Hilbert space endowed with some distinguished Hermitian operators from the geometric object (M, ω). This procedure is called ***geometric quantization***. After quantization the Poisson bracket of classical operators should correspond to the commutator bracket of quantum operators.

It turns out that if ω satisfies some integrality condition, it is not difficult to construct such a Lie algebra representation. However, the naive construction will give a representation on a Hilbert space that is 'too large' in some sense. To get the correct quantum Hilbert space some auxiliary structure on (M, ω) is required, namely a polarization.

We are especially interested in the case that (M, ω) is an effective Hamiltonian T^n-space, where T^n is the torus of half the dimension of M. Then M is called a ***symplectic toric manifold***, and it is a theorem by Delzant that a compact symplectic toric manifold is determined up to equivariant symplectomorphism by its moment polytope $\Delta \subseteq \mathbb{R}^n$. By the same construction we find that certain non-compact symplectic toric manifolds are also determined by their moment polyhedron.

Classical systems corresponding to symplectic toric manifold are especially nice, since they are completely integrable. We show that for such a system, both the integrality condition on ω and the complex Hilbert space associated with (M, ω) can be read from the lattice points in the moment polyhedron Δ. For compact symplectic toric manifolds this is a relatively well-known result, and a proof is found in [Ham08]. However, many mechanical systems that are of interest to physicists have a non-compact phase space. It turns out that the theorem generalizes the class of non-compact symplectic toric manifolds whose orbital moment map is an embedding. This gives us an easy way to quantize two of the most fundamental mechanical systems, namely the harmonic oscillator and the spin particle.

In this thesis, we start out by recalling some definitions and theorems from symplectic geometry and the connection with classical mechanics in Section 2.1. We then recall some basics on the formalism of quantum mechanics in Section 2.2.

In Chapter 3, we start by formulating a first attempt at quantization, called prequantization. As discussed in Section 3.1, the prequantization of (M, ω) relies on the choice of a complex line bundle with curvature two-form related to the symplectic form ω, and an admissible bundle is called a prequantum line bundle. The square-integrable sections of this bundle define a complex Hilbert space, and the Lie algebra of smooth functions on M has a representation as a Lie algebra of Hermitian operators on this space. In Section 3.2 we discuss an alternate approach, where the prequantum line bundles are replaced by certain principal $U(1)$-bundles called Boothby-Wang bundles. We show the two approaches are equivalent and result in isomorphic representations. Then in Section 3.3 we show that the condition for a symplectic manifold to admit a prequantum line bundle is an integrality condition on the symplectic form. Finally in Section 3.4 we give a treatment of a few simple mechanical systems.

It turns out that the resulting prequantum spaces are too big in some sense, and in Chapter 4 we discuss how to define quantization. In Section 4.1 we define the additional structure we need to endow (M, ω) with, namely a polarization. We find that we can only define quantization for a restricted class of classical observables. In Section 4.2 we explore the quantization procedure for a real polarization, and in Section 4.3 we show that a Kähler manifold (M, ω) admits a holomorphic prequantum line bundle which has a natural quantization. Then in Section 4.4 we revisit the examples of the previous chapter.

In Chapter 5 we specialize to the study of symplectic toric manifolds. We start in Section 5.1 by discussing a generalization of Delzants theorem which relates symplectic toric manifolds with the image of their moment map. In particular, we discuss a procedure to construct a symplectic toric manifold $(M_\Delta, \omega_\Delta)$ from a simple, rational and smooth polyhedron $\Delta \subseteq \mathbb{R}^n$. A slightly different construction gives M_Δ a complex structure. Then in Section 5.2 we get to the main part of the thesis. First Proposition 5.2.1 shows that a symplectic toric manifold is quantizable if the vertexes of its moment polyhedron Δ are in the lattice $\hbar\mathbb{Z}^n$, and Theorem 5.2.5 shows that in that case the quantization is given by the lattice points in Δ. Finally we show how this theorem leads to the quantization of the harmonic oscillator and the spin particle in Section 5.3.

Chapter 2

Preliminaries

2.1 Formalism of classical mechanics

In this section we will explain how classical mechanics can be formalized in terms of symplectic geometry, and we will recall some definitions and properties relating to symplectic manifolds. For details, we refer to [Can01] and [Lee03].

2.1.1 Symplectic geometry

A ***symplectic manifold*** is a smooth manifold M endowed with a non-degenerate closed two-form $\omega \in \Omega^2(M)$, called a ***symplectic form***. It follows from the non-degeneracy that M then must be of even dimension, say $\dim M = 2n$, and then ω^n is a volume form on M. The structure-preserving maps are smooth map that pull back the symplectic form on the codomain to that on the domain, and an symplecticly invertible symplectic map is called a ***symplectomorphism***.

At any point $p \in M$, the tangent space T_pM is a vector space endowed with the non-degenerate skew-symmetric bilinear form ω_p. This form gives rise to the ***symplectic complement***: for a subset $W \subseteq T_pM$ this is defined by

$$W^{\omega_p} := \{v \in T_pM \mid \forall w \in W\colon\ \omega_p(v,w) = 0\}.$$

A subspace $W \subseteq T_pM$ then is called ***isotropic*** if $W \subseteq W^{\omega_p}$ and ***Lagrangian*** if $W = W^{\omega_p}$. The Lagrangian spaces are precisely those who are maximal in the class of isotropic subspaces, and necessarily have dimension n. Now a submanifold $N \subseteq M$ is called ***Lagrangian*** if for any $p \in N$, the subspace $T_pN \subseteq T_pM$ is Lagrangian. The Lagrangian submanifolds play an important role in symplectic geometry.

In the Hamiltonian formalism of classical mechanics, a mechanical system is described by a ***Hamiltonian system***, that is: a symplectic manifold (M, ω) together with a distinguished smooth function $H \in C^\infty(M)$, called the ***Hamiltonian***. States of the system correspond to points in M, and the Hamiltonian is interpreted as assigning to any state its total energy. The dynamics of the system are given by the integral curves of the ***Hamiltonian vector field*** $X_H \in \mathfrak{X}(M)$, which is the unique vector field such that $\iota_{X_H}\omega = dH$. The existence and uniqueness of this vector field follows from the fact that a non-degenerate two-form $\omega_p \colon T_pM \times T_pM \to \mathbb{R}$ induces an isomorphism $T_pM \to T_p^*M$.

The definition of the Hamiltonian vector field of a smooth functions gives rise to the ***Poisson bracket*** on the space of smooth functions, given by $\{f, g\} := \omega(X_f, X_g)$. This bracket is bilinear, alternating and satisfies the Jacobi identify, making $(C^\infty(M), \{-,-\})$ into a Lie algebra. We recall some basic properties of the Poisson bracket with respect to Hamiltonian vector fields.

Proposition 2.1.1. *If (M, ω) is a smooth manifold and $f, g \in C^\infty(M)$, their Poisson bracket and Hamiltonian vector fields satisfy*

$$\{f, g\} = X_g f$$

and

$$X_{\{f,g\}} = -[X_f, X_g].$$

We also recall the coordinate representation of Hamiltonian vector fields and the Poisson bracket.

Proposition 2.1.2. *Let $\mathbb{R}^{2n}$ be the standard symplectic space with coordinates (q^j, p_j) and symplectic form $\omega = dq^j \wedge dp_j$. If $f, g \in C^\infty(\mathbb{R}^{2n})$ are smooth functions, then the Hamiltonian vector field of f is given by*

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$$

and the Poisson bracket of f and g is given by

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

Often a mechanical system is described by a ***configuration space***, a Riemannian manifold (Q, g) of all possible instantaneous configurations of the system. The ***phase space*** then is the cotangent bundle $(T^*Q, \omega^{\text{can}})$. The canonical symplectic form is defined as $\omega^{\text{can}} := -d\lambda$, where $\lambda \in \Omega^1(T^*Q)$ is the Liouville form, which on an open subset $U \subseteq Q$ admitting coordinates $(q^1, \dots, q^n)$ with standard coframe $(p_1, \dots, p_j)$ is given by

$$\lambda|_{T^*U} = \sum_{j=1}^{n} p_j dq^j.$$

For the canonical symplectic form, we then find

$$\omega|_{T^*U} = \sum_{j=1}^{n} dq^j \wedge dp_j.$$

If we endow Q with a potential energy function $V \in C^\infty(Q)$, and we consider the kinetic energy function

$$\begin{aligned} K\colon T^*Q &\longrightarrow \mathbb{R} \\ (q, p) &\longmapsto \frac{1}{2m} g_q(p, p), \end{aligned}$$

where m is a constant that represents the mass of the system. The Hamiltonian corresponding to this system then becomes

$$H\colon (q, p) \longmapsto K(q, p) + V(q), \tag{2.1}$$

A physicist may want to make a measurement on a mechanical system. In the most general sense, such a measurement gives a value that depends on the state of the system. In other words: to any observable quantity there is associated a function $f\colon M \to \mathbb{R}$. If we assume smoothness here, we may think of $C^\infty(M)$ as the space of observable quantities of the system. For example, the Hamiltonian H is the observable that corresponds to the measurement of the total energy of the system.

2.1.2 Contact geometry

Contact geometry is the odd-dimensional analogue to symplectic geometry. If M is a smooth manifold, a one-form $\alpha \in \Omega^1(M)$ defines a codimension-one distribution ξ by $\xi_p := \ker \alpha_p \subseteq T_pM$. Then α is called a ***contact form*** if $d\alpha$ restricts to a non-degenerate two-form on ξ, and we note that M then must be odd-dimensional. A smooth manifold M with a codimension-one distribution ξ then is called a ***contact manifold*** if ξ if locally defined by contact forms in the sense that for every open $U \subseteq M$ and $\alpha \in \Omega^1(U)$ with $\ker \alpha = \xi|_U$, α is a contact form.

If (M, ξ) is a contact manifold of dimension $2n+1$ and α is a global contact form, we have a canonical volume form of M given by $\alpha \wedge (d\alpha)^n$.

An important property of symplectic manifolds is that since the non-degenerate symplectic form gives a correspondence between smooth functions and certain vector fields. In contact geometry we have an analogous correspondence, albeit a little more complicated. First we see we already have a vector field associated with a contact form.

Proposition 2.1.3. *If (M, ξ) is a contact manifold with contact form α, there is a unique vector field $R_\alpha \in \mathfrak{X}(M)$ which satisfies $\iota_{R\alpha} d\alpha = 0$ and $\iota_{R\alpha} = 1$. We call this the **Reeb vector field** of α.*

The analogue to the construction of the Hamiltonian vector field of a smooth function on a symplectic manifold then is the following

Proposition 2.1.4. *If (M, ξ) is a contact manifold with contact form α and $f \in C^\infty(M)$ is a smooth function, there is a unique vector field $X_f \in \mathfrak{X}(M)$ which satisfies $\alpha(X_f) = f$ and $\iota_{X_f} d\alpha|_\xi = -df|_\xi$. We call this the **contact Hamiltonian vector field** of f.*

Proof. Note that since $d\alpha|_\xi$ is non-degenerate, there is a unique $Y_f \in \Gamma(\xi)$ such that $\iota_{Y_f} d\alpha|_\xi = df|_\xi$, and it is easy to verify that

$$X_f = -Y_f + fR_\alpha. \tag{2.2}$$

satisfies the defining properties of the contact Hamiltonian vector field. □

2.1.3 Completely integrable Hamiltonian systems

On a Hamiltonian system (M, ω, H), an observable $f \in C^\infty(M)$ is called an ***integral of motion*** if it is invariant under the flow of the Hamiltonian vector field X_H which gives the dynamics of the system: that is, if $X_H f = 0$. By Proposition 2.1.1 this happens if and only if $\{f, H\} = 0$. In particular, the Hamiltonian H is an integral of motion which corresponds to the principle of conservation of energy in physics.

If f is an integral of motion, the equations of motion restrict to the level sets of f. These are contact manifolds of dimension $2n-1$. If a system has multiple integrals of motion $f_1, \dots, f_k \in C^\infty(M)$, we can do even better by reducing to combined level sets.

Definition 2.1.5. A symplectic manifold (M, ω) with $\dim M = 2n$ is called ***completely integrable*** (in the Liouville sense) if there are smooth functions $f_1, \dots, f_n \in C^\infty(M)$, called a ***complete set of commuting functions***, that satisfy the following two conditions:

- they are ***independent*** in the sense that there exists some dense open $U \subseteq M$ such that $(df_1)_p, \dots, (df_n)_p$ are linearly independent in T_p^*M for all $p \in U$;
- they ***commute pairwise*** in the sense that for all $1 \leq j, k \leq n$ we have $\{f_j, k_k\} = 0$.

In particular, a Hamiltonian system (M, ω, H) is called ***completely integrable*** if (M, ω) is completely integrable with $f_1 = H$.

It should be noted here that for each $p \in M$, the subspace of T_pM spanned by the $(X_{f_j})_p$ is isotropic since

$$\omega_p(X_{f_j}, X_{f_k}) = \{f_j, k_k\}(p) = 0.$$

Using that under the identification $T^*M \cong TM$ given by ω the Hamiltonian vector field X_{f_j} corresponds to df_j, and by independence of the functions we conclude $\operatorname{span}\{(X_{f_j})_p\}_{j=1}^n$ has dimension n. Since it is isotropic of maximal dimension it is Lagrangian.

Example 2.1.6. On the standard symplectic space $\mathbb{R}^{2n}$ with the symplectic form $\omega = \sum_{j=1}^n dq^j \wedge dp_j$, both the sets $\{q^1, \dots, q^n\}$ and $\{p_1, \dots, p_n\}$ are complete sets of commuting functions.

2.1.4 Symplectic reduction

The idea of reducing the phase space of a constrained system is formalized as follows.

Definition 2.1.7. The action ϕ of a Lie group G on a symplectic manifold (M, ω) is called a ***Hamiltonian group action*** if G acts by symplectomorphisms, $\phi\colon G \to \mathrm{Sympl}(M, \omega)$, and there exists a map $\mu\colon M \to \mathfrak{g}^*$ that is equivariant with respect to the co-adjoint action of G on $\mathfrak{g}^*$ and that satisfies $d\langle \mu, \xi \rangle = \iota_{X_\xi}\omega$ for all $\xi \in \mathfrak{g}$, where $X_\xi \in \mathfrak{X}(M)$ denotes the fundamental vector field of ξ. The map μ is then called a ***moment map*** for the action, and the quintuple $(M, \omega, G, \phi, \mu)$ is called a ***Hamiltonian G-space***.

Theorem 2.1.8 (Marsden-Weinstein). *If $(M, \omega, G, \phi, \mu)$ is a Hamiltonian G-space for some compact Lie group G such that G acts freely on $Z := \mu^{-1}(0)$, then the orbit space Z/G, which is denoted by M_{red}, admits a unique smooth structure such that $\pi\colon Z \to M_{\text{red}}$ is a smooth submersion and a unique symplectic form ω_{red} such that*

$$\iota^*\omega = \pi^*\omega_{\text{red}},$$

where ι denotes the inclusion $Z \hookrightarrow M$.

2.1.5 Kähler manifolds

For quantization, it will turn out to be useful to study symplectic manifolds that have some sort of complex structure. This will locally be modelled by a complex structure in a real vector space.

Definition 2.1.9. If V is a finite-dimensional real vector space, an endomorphism J of V is called a ***complex structure*** if $J^2 = -I_V$.

We note that such a map can only exists if the dimension of V is even. The complex structure then gives rise to an actual complex vector space V_J such that $V_J = V$ as a set where scalar multiplication is given by

$$\begin{aligned} \mathbb{C} \times V_J &\longrightarrow V_J \\ (a + bi, v) &\longmapsto av + bJ(v). \end{aligned}$$

Another way to construct a complex vector space form V is by endowing the tensor product $V^{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V$ with the scalar product

$$\begin{aligned} \mathbb{C} \times V^{\mathbb{C}} &\longrightarrow V^{\mathbb{C}} \\ (\lambda, z \otimes v) &\longmapsto (\lambda z) \otimes v. \end{aligned}$$

Note that if V is of real dimension $2n$, then V_J is of complex dimension n and $V^{\mathbb{C}}$ is of complex dimension $2n$.

If J is a complex structure on V, it complex-linearly extends to an endomorphism $J^{\mathbb{C}}$ of $V^{\mathbb{C}}$. This operator has eigenvalues $\pm i$, and we denote by $V^{1,0}$ the eigenspace corresponding to i and by $V^{0,1}$ the eigenspace corresponding to $-i$. We have a $\mathbb{C}$-linear identification $V_J \cong_{\mathbb{C}} V^{1,0}$ since J corresponds to multiplication with i. Furthermore, complex conjugation gives a isomorphism of real vector spaces $V^{1,0} \cong V^{0,1}$.

Since $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, we get natural inclusions of the exterior algebras $\Lambda(V^{1,0}) \to \Lambda(V^{\mathbb{C}})$ and $\Lambda(V^{0,1}) \to \Lambda(V^{\mathbb{C}})$. We define $\Lambda^{k,l}(V)$ for $k, l \geq 0$ to be the subspace of $\Lambda(V^{\mathbb{C}})$ generated by

$$\{u \wedge v \mid u \in \Lambda^l(V^{1,0}),\ v \in \Lambda^k(V^{0,1})\}.$$

Definition 2.1.10. If (M, ω) is a symplectic manifold, then a section $J \in \Gamma(\mathrm{End}(TM))$ is called an ***almost complex structure*** on M if for every $p \in M$ the endomorphism J_p is a complex structure in T_pM.

Similar to the linear case an almost complex structure induces a decomposition of the complexified tangent bundle $T^{\mathbb{C}}M$, which is defined such that $T^{\mathbb{C}}_pM = (T_pM)^{\mathbb{C}}$, into two bundles consisting of pointwise eigenspaces. The bundle $T^{1,0}M$, which consists of the i-eigenspaces of J is called the ***holomorphic tangent bundle***, and the $T^{0,1}M$ which consists of the $-i$-eigenspaces of J and is called the ***anti-holomorphic tangent bundle***. For $k, l \geq 0$ this gives rise to the ***differential forms of type*** (k, l), $\Omega^{k,l}(M)$, which is the subspace of $\Gamma(\Lambda(T^{\mathbb{C}}M))$ generated by

$$\{\omega \wedge \eta \mid \omega \in \Gamma(\Lambda^k(T^{1,0}M)),\ \eta \in \Gamma(\Lambda^l(T^{1,0}M))\}.$$

We also have genuine complex manifolds.

Definition 2.1.11. If $n \geq 0$, a ***complex manifold*** of complex dimension n is a pair $(M, \mathcal{A})$ where M is a Hausdorff and second-countable topological space and $\mathcal{A}$ an atlas consisting of charts (U, ϕ) with $U \subseteq M$ an open subset and $\phi\colon U \to \mathbb{C}^n$ such that the transition maps are biholomorphic.

It is clear that a complex manifold M of dimension n also has the structure of a smooth manifold of dimension $2n$, when the complex coordinates $(z_1, \dots, z_n)$ are replaced by real coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ such that $z_j = q^j - ip_j$. The reader is warned that this is not the same convention as in [Can01]. However, it turns out that this is the natural identification for the quantization of the harmonic oscillator in Section 4.4.1.

Now M also has a canonical almost complex structure J, which is locally defined by

$$J_p \left(\frac{\partial}{\partial q^j}\bigg|_p \right) = -\frac{\partial}{\partial p_j}\bigg|_p \qquad J_p \left(\frac{\partial}{\partial p_j}\bigg|_p \right) = \frac{\partial}{\partial q^j}\bigg|_p .$$

If we define

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial q^j} + i\frac{\partial}{\partial p_j} \right) \qquad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial q^j} - i\frac{\partial}{\partial p_j} \right)$$

we can simply describe the holomorphic and anti-holomorphic tangent bundles by

$$T_p^{1,0}M = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_j} \;\middle|\; 1 \le j \le n \right\} \quad T_p^{0,1}M = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_j} \;\middle|\; 1 \le j \le n \right\},$$

and if we define $dz_j = dq^j - idp_j$ and $d\bar{z}_j = dq^j - idp_j$, we have that $\Omega^{k,l}(M)$ is generated by elements of the form

$$dz_{j_1} \wedge \cdots \wedge dz_{j_k} \wedge \bar{z}_{j_{k+1}} \wedge \cdots \wedge \bar{z}_{j_{k+l}}.$$

The constant $1/2$ in the definition of the holomorphic and anti-holomorphic derivations is chosen to ensure that for any complex-valued function $f \in C^\infty(M;\mathbb{C})$ we have

$$df = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \right).$$

Considering the projection operators

$$\pi^{k,l} \colon \Gamma(\Lambda(T^{\mathbb{C}}M)) \longrightarrow \Omega^{k,l}(M)$$

the complexification of the exterior derivative may be written as a sum $d^{\mathbb{C}} = \partial + \bar{\partial}$, where

$$\partial := \pi^{k+1,l} \circ d^{\mathbb{C}} \qquad \bar{\partial} := \pi^{k,l+1} \circ d^{\mathbb{C}}$$

are the so-called Dolbeault operators. From the vanishing of d^2, they satisfy:

$$\partial\partial = 0 \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0 \qquad \bar{\partial}\,\bar{\partial} = 0.$$

Definition 2.1.12. A complex manifold M, seen as a real manifold with almost complex structure J and endowed with a symplectic form ω, is called a ***Kähler manifold*** if

$$g(-,-) := \omega\big(J(-),-\big)$$

is a Riemannian metric on M.

It turns out that ω can locally be written in terms of a scalar potential,

Lemma 2.1.13. *If (M, J, ω) is a Kähler manifold, and $p \in M$, then there exists a neighbourhood U of p and a smooth function $\rho \in C^\infty(U)$ such that*

$$\omega|_U = \frac{i}{2}\partial\bar{\partial}\rho.$$

Such a function ρ is called a Kähler potential. It immediately follows that ω must be of type $(1,1)$, and that the $(1,0)$-form

$$\theta := \frac{i}{2}\partial\rho$$

satisfies $\omega = -d\theta$.

Example 2.1.14. If we consider $\mathbb{C}^n \cong \mathbb{R}^{2n}$, the canonical symplectic form can be written as

$$\omega_0 = -\frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j. \tag{2.3}$$

The function

$$\rho(z_1, \ldots, z_n) = -\sum_{j=1}^{n} z_j \bar{z}_j$$

then is a global Kähler potential, and the corresponding symplectic potential is

$$\frac{i}{2}\partial\rho = -\frac{i}{2}\sum_{j=1}^{n} \bar{z}_j dz_j$$

We can write the Hamiltonian vector field in terms of the holomorphic and anti-holomorphic coordinates as follows.

Proposition 2.1.15. *If $f \in C^\infty(\mathbb{R}^{2n})$ is a smooth function, then the Hamiltonian vector field of f is given by*

$$X_f = 2i \cdot \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \overline{z}_j} \frac{\partial}{\partial z_j} - \frac{\partial f}{\partial z_j} \frac{\partial}{\partial \overline{z}_j} \right).$$

2.2 Formalism of quantum mechanics

In this section we provide some details on the formalism of quantum mechanics, which is formulated in terms of complex Hilbert spaces. We first discuss some concepts from functional analysis that form the mathematical framework of quantum mechanics. Then we discuss quantum mechanics itself and the canonical quantization of classical systems given by standard symplectic space.

There are many textbooks on quantum mechanics, and for example [SN17] gives a nice explanation of the physicists perspective. For a more mathematical treatment one might consult [Hal13] or the first few chapters of [Fol08], and for details on the functional analysis we refer to [Con90].

2.2.1 Complex Hilbert spaces

Since quantum mechanics is defined in terms of operators on a complex Hilbert space, let us first of all recall the definition of a Hilbert space.

Definition 2.2.1. If $\mathfrak{H}$ is a complex vector space, then an ***inner product*** is a map $\langle -, - \rangle \colon \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{C}$ such that

- For every $\psi \in \mathfrak{H}$ we have $\langle \psi, \psi \rangle \in \mathbb{R}$ and $\langle \psi, \psi \rangle \geq 0$ with equality if and only if $\psi = 0$;
- For every $\psi, \psi' \in \mathfrak{H}$ we have $\langle \psi, \psi' \rangle = \overline{\langle \psi', \psi \rangle}$;
- For every $\psi \in \mathfrak{H}$ the map $\psi' \mapsto \langle \psi, \psi' \rangle$ is linear.

A ***complex Hilbert space*** $(\mathfrak{H}, \langle -, - \rangle)$ then consists of a complex vector space together with a complex inner product such that $\mathfrak{H}$ is complete with respect to the norm $\|\psi\| := \sqrt{\langle \psi, \psi \rangle}$.

Remark 2.2.2. Here we follow the convention which is popular under physicists that the complex inner product is linear in the second term.

Lebesgue spaces are important examples of Hilbert spaces, and they also play an important roll in the formulation of quantum mechanics.

Example 2.2.3. If (X, Σ, μ) is a measure space, we consider the space $\mathcal{L}^2(X, \mu)$ consisting of the measurable functions $f\colon X \to \mathbb{C}$ such that the integral satisfies

$$\int_X |f|^2 d\mu \ < \ \infty.$$

Now we define an equivalence relation by setting $f \equiv g$ precisely if $f = g$ almost everywhere, and we define the ***Lebesgue space*** $L^p(X, \mu)$ to be the quotient space $\mathcal{L}^2(X, \mu)/\equiv$.

This space has a natural inner product: for $f, g \in L^2(X, \mu)$, we set

$$\langle f, g\rangle := \int_X f\overline{g} d\mu.$$

which is finite due to Hölder's inequality. The space is also complete, and thus an example of a Hilbert space.

In the rest of the thesis, we will mainly ignore this quotient and use language that refers to elements of the Lebesgue space as functions instead of equivalence classes.

Other Hilbert spaces that will play a role in this thesis are the so-called holomorphic Lebesgue spaces.

Example 2.2.4. If $U \subseteq \mathbb{C}^n$ is an open subset endowed with the standard Lebesgue measure dz and $\mu \in C^\infty(U)$ is a continuous and strictly positive function on U, we define the ***holomorphic Lebesgue space*** $HL^2(U, \alpha)$ to consist of the holomorphic functions $f\colon U \to \mathbb{C}$ such that

$$\int_X |f|^2 \mu dz \ < \ \infty.$$

This is a closed linear subspace of $L^2(U, \mu dz)$ and therefore is a Hilbert space.

Definition 2.2.5. If $\mathfrak{H}$ is a complex Hilbert space, then a linear map $A\colon \mathfrak{H} \to \mathfrak{H}$ is called a ***bounded linear operator*** if there exists a $k \in \mathbb{R}_{>0}$ such that for every $x \in \mathfrak{H}$,

$$\|A(x)\| \leq k \|x\|.$$

The set of all bounded linear operators on $\mathfrak{H}$ is denoted by $B(\mathfrak{H})$. It turns out that these are precisely the continuous linear endomorphisms of $\mathfrak{H}$.

We may similarly define bounded functionals to be bounded linear maps from $\mathfrak{H}$ to $\mathbb{C}$. The space of such maps is denoted by $\mathfrak{H}^*$ and called the ***continuous dual*** of $\mathfrak{H}$. One of the useful properties of Hilbert spaces is the following:

Theorem 2.2.6 (Riesz' representation)**.** *If $\mathfrak{H}$ is a complex Hilbert space, then the map*

$$\begin{aligned} \mathfrak{H} &\longrightarrow \mathfrak{H}^* \\ \psi &\longmapsto \big(\psi' \mapsto \langle \psi, \psi' \rangle\big) \end{aligned}$$

is well-defined and a conjugate linear isomorphism of vector spaces.

Among the consequences of this theorem is the definition of the Hilbert space adjoint of a bounded linear operator, and a notion of Hermitian — or, as a mathematician might prefer to call them, self-adjoint — operators.

2.2.2 Densely defined operators

The operators defined by quantization will, in general, not be bounded. We are therefore compelled to provide at least the definition of a Hermitian unbounded operator. However, since the aim of this thesis is to develop the geometric properties of quantization, we shall mostly ignore the issues of unboundedness in the later chapters.

Definition 2.2.7. If $\mathfrak{H}$ is a complex Hilbert space, then an ***operator*** on $\mathfrak{H}$ consists of a linear subspace $\operatorname{dom} A \subseteq \mathfrak{H}$, called the ***domain*** of the operator, together with a linear map $A\colon \operatorname{dom} A \to \mathfrak{H}$. The operator is called ***densely defined*** if $\operatorname{dom} A \subseteq \mathfrak{H}$ is dense.

Definition 2.2.8. If $\mathfrak{H}$ is a complex Hilbert space and A is an operator on $\mathfrak{H}$, then the ***graph*** of A is the linear subspace

$$\operatorname{gra}(A) := \big\{ \big(x, f(x)\big) \;\big|\; x \in \operatorname{dom} A \big\} \subseteq \mathfrak{H} \times \mathfrak{H}.$$

More generally, any linear subspace $G \subseteq \mathfrak{H} \times \mathfrak{H}$ such that for every $\psi \in \mathfrak{H}$, the set $(\{\psi\} \times \mathfrak{H}) \cap G$ has at most one element is called a ***graph*** on $\mathfrak{H}$. The operation gra then provides a bijective identification of the operators on $\mathfrak{H}$

and the graphs on $\mathfrak{H}$. This provides a partial ordering on the operators, and we say an operator B is an extension of A if $\operatorname{gra} A \subseteq \operatorname{gra} B$.

Definition 2.2.9. If $\mathfrak{H}$ is a complex Hilbert space and A is an operator on $\mathfrak{H}$, then A is called ***closable*** if $\operatorname{cl} \operatorname{gra} A$ is a graph and ***closed*** if $\operatorname{gra} A$ is closed.

Now we are interested in the construction of the adjoint of a densely defined operator A. First, we define

$$\operatorname{dom} A^* := \{\psi \in \mathcal{H} \mid \psi' \mapsto \langle \psi, A\psi' \rangle \text{ has a bounded extension}\}.$$

We note that since A is densely defined, this bounded extension must be unique. From Theorem 2.2.6 we then see that for $\psi \in \operatorname{dom} A^*$ we have a unique element of $\mathcal{H}$, which we will call $A^*\psi$, such that $\langle A^*\psi, \psi' \rangle = \langle \psi, A\psi' \rangle$ for all $\psi' \in \operatorname{dom} A$. This defines an operator A^*, which we call the ***adjoint*** of A.

The adjoint has the following properties:

Proposition 2.2.10. *If $\mathfrak{H}$ is a complex Hilbert space and A is a densely defined operator on $\mathfrak{H}$, then A^* is a closed operator. Moreover, A^* is densely defined if and only if A is closable, and in that situation the closure of A is A^{**}.*

This suggests the adjoint is well-behaved on the class of closed densely defined operators, which we shall denote by $C(\mathfrak{H})$.

Definition 2.2.11. If $\mathfrak{H}$ is a complex Hilbert space and A is an operator on $\mathfrak{H}$, then A is called ***Hermitian*** if $\langle A^*\psi, \psi' \rangle = \langle \psi, A\psi' \rangle$ for all $\psi, \psi' \in \operatorname{dom} A$.

Remark 2.2.12. This differs from the standard definition of self-adjointness of densely defined operators, where A is self-adjoint if $A = A^*$, since the equality of two operators is also a statement about their domains. What we call a Hermitian operator is referred to as a symmetric operator in most literature.

2.2.3 Canonical quantization

In the description of a quantum-mechanical system, the central object is a complex Hilbert space $\mathfrak{H}$. A state of the system is described by a unit vector $\psi \in \mathfrak{H}$. The obervables are describes by Hermitian operators. How an observation of such a system may be precisely describes is a contested topic, which we will not go into here. It suffices to say that if ψ is an eigenvector of an observable A with eigenvalue $\lambda_\psi \in \mathbb{C}$, the observation of A of a system in state ψ will be the value λ_ψ. If ψ is a linear combination of eigenvalues, $\langle \psi, A\psi \rangle$ may be interpreted as the expectation value of a measurement. This is also true if the spectrum of A is not discrete. Just as in the classical case, the equations of motion of a quantum system are given by the observable corresponding to the total energy of the system.

The classical systems with the most simple description are those with configuration space $Q = \mathbb{R}^n$, so that T^*Q is just the standard $2n$-dimensional symplectic space. It is not surprising that these are also among the most simple systems to give a quantum description. First let us consider the requirements for the operator algebra of the quantized system.

If $\mathcal{Q}_f$ denotes a quantum operator associated to some classical operator $f \in C^\infty(\mathbb{R}^{2n})$ Paul Dirac [Dir30] derived that the commutator of such operators must be related to the Poisson bracket by

$$[\mathcal{Q}_f, \mathcal{Q}_g] = i\hbar \mathcal{Q}_{\{f,g\}}. \tag{2.4}$$

Here $\hbar$ is the reduced Planck constant, which gives the scale at which energy is quantized. Explicitly, a quantum of electromagnetic radiation — called a photon — with angular frequency ω has energy $\hbar\omega$. The reduced Planck constant thus must have units of energy multiplied by time, and has the empirically determined value of $\hbar = 1.055 \cdot 10^{-34}\,\mathrm{J\,s}$. Since we prefer to keep the units implicit, in the rest of this thesis we will simply assume $\hbar \in \mathbb{R}_{>0}$.

If we also assume that

$$\mathcal{Q}_1 = I \tag{2.5}$$

as Dirac does implicitly, we find the so-called canonical commutation relations:

$$[\mathcal{Q}_{p_j}, \mathcal{Q}_{p_k}] = 0 \qquad [\mathcal{Q}_{q^j}, \mathcal{Q}_{q^k}] = 0 \qquad [\mathcal{Q}_{q^j}, \mathcal{Q}_{p_k}] = i\hbar\delta^j_k, \tag{2.6}$$

for all $1 \leq j, k \leq n$. Here we use a shorthand notation by writing a scalar $\lambda \in \mathbb{C}$ for the operator λI.

Operators satisfying these commutation relations may be realized on the Lebesgue space $\mathfrak{H} = L^2(\mathbb{R}^n)$ of square-integrable functions.

$$\mathcal{Q}_{q^j} \colon \psi \longmapsto q^j \psi \qquad \mathcal{Q}_{p_j} \colon \psi \longmapsto -i\hbar \frac{\partial \psi}{\partial q^j}. \tag{2.7}$$

This choice of operators is called ***canonical quantization***.

2.2.4 Fock quantization

There is another standard way to define the quantization of $Q = \mathbb{R}^n$ as functions on the total phase space. We note that we have canonical isomorphisms of the real manifolds $T^*Q \cong \mathbb{R}^{2n} \cong \mathbb{C}^n$. Following [Haloo], we consider the quantization of the complex coordinates $z_j = q^j + ip_j$ and the conjugate coordinates $\overline{z}_j = q^j - ip_j$. One might be concerned that we are now quantizing complex-valued classical observables. By $\mathbb{C}$-linear expansion of the quantization map this is well-defined, though the resulting operators no longer need be Hermitian.

From complex linearity, we get the operators

$$\mathcal{Q}_{z_j} = \mathcal{Q}_{q^j} + i\mathcal{Q}_{p_j} \qquad \mathcal{Q}_{\overline{z}_j} = \mathcal{Q}_{q^j} - i\mathcal{Q}_{p_j}$$

and since $\mathcal{Q}_{q^j}$ and $\mathcal{Q}_{p_j}$ are Hermitian, it is clear that we have $\mathcal{Q}_{z_j}^* = \mathcal{Q}_{\overline{z}_j}$. The canonical commutation relations (2.6) imply that for $1 \leq j,k \leq n$ we have

$$[\mathcal{Q}_{z_j}, \mathcal{Q}_{z_k}] = 0 \qquad [\mathcal{Q}_{\overline{z}_j}, \mathcal{Q}_{\overline{z}_k}] = 0 \qquad [\mathcal{Q}_{z_j}, \mathcal{Q}_{\overline{z}_k}] = 2\hbar\delta^j_k. \tag{2.8}$$

We now want to define a quantization in terms of the operators z_j and $\overline{z}_j$. From this one may recover the operators of Section 2.2.3 by

$$\mathcal{Q}_{q^j} = \frac{1}{2}\left(\mathcal{Q}_{z_j} + \mathcal{Q}_{\overline{z}_j}\right) \qquad \mathcal{Q}_{p_j} = -\frac{i}{2}\left(\mathcal{Q}_{z_j} - \mathcal{Q}_{\overline{z}_j}\right).$$

Vladimir Fock had the idea that to use the operators z_j and ∂_{z_j} up to some scaling to get the commutation relations (2.8), and Valentine Bargmann constructed the Hilbert space in which these operators are each others adjoints.

Definition 2.2.13. If $n \geq 0$ is an integer, the ***Segal-Bargmann space*** on $\mathbb{C}^n$ with parameter $\hbar$ is the holomorphic Lebesgue space $HL^2(\mathbb{C}^n, \mu_\hbar)$ as in Example 2.2.4 with the positive function $\mu_\hbar\colon \mathbb{C}^n \to \mathbb{R}_{>0}$ given by

$$\mu_\hbar(z) = \frac{1}{(2\pi\hbar)^n} e^{-|z|^2/2\hbar}.$$

Any holomorphic function can be written as a series of monomials, and after normalization we find that

$$\left\{ \prod_{j=1}^n \left(\frac{1}{\sqrt{k_j!(2\hbar)^{k_j}}} \right) z_1^{k_1} \cdots z_n^{k_n} \right\}_{k_1,\ldots,k_n \geq 0}$$

is an orthonormal basis of $HL^2(\mathbb{C}^n, \mu_\hbar)$.

On the Segal-Bargmann space $\mathfrak{H} = HL^2(\mathbb{C}^d, \mu_\hbar)$ ***Fock quantization*** is the choice of the operators

$$\mathcal{Q}_{z_j} : \psi \longmapsto 2\hbar \frac{\partial \psi}{\partial z_j} \qquad\qquad \mathcal{Q}_{\bar{z}_j} : \psi \longmapsto z_j \psi.$$

In $HL^2(\mathbb{C}^d, \mu)$ these operators are indeed adjoints of one another. These operators are related to the so-called creation and annihilation operators

$$a_j^* = \frac{1}{\sqrt{2\hbar}} \mathcal{Q}_{\bar{z}_j} \qquad\qquad a_j = \frac{1}{\sqrt{2\hbar}} \mathcal{Q}_{z_j}.$$

If we label the normalized basis vectors by $\{\psi_{k_1,\dots,k_d}\}_{k_1,\dots,k_n \geq 0}$, we see that

$$\begin{aligned} a_j^* \psi_{k_1,\dots,k_j,\dots,k_d} &= \sqrt{k_j + 1}\, \psi_{k_1,\dots,k_j+1,\dots,k_d} \\ a_j \psi_{k_1,\dots,k_j,\dots,k_d} &= \sqrt{k_j}\, \psi_{k_1,\dots,k_j-1,\dots,k_d}. \end{aligned}$$

The so-called number operator

$$N_j := a_j^* a_j = z_j \frac{\partial}{\partial z_j}$$

then clearly has any $\psi_{k_1,\dots,k_d}$ as eigenvector with eigenvalue k_j.

Chapter 3

Prequantization

3.1 Prequantum line bundles

We are interested in extending the notion of canonical quantization, as discussed in Section 2.2.3, to more general mechanical systems. As discussed in Section 2.1.1, these systems are mathematically described by a symplectic manifold (M, ω) and a Hamiltonian $H \in C^\infty(M)$.

A first attempt would be to construct a complex Hilbert space $\mathfrak{H}_M^{\text{pre}}$ of square-integrable complex-valued smooth functions on M endowed with a representation $\mathcal{Q}^{\text{pre}}$ of the Lie algebra of smooth functions on M as densely defined Hermitian operators on $\mathfrak{H}_M^{\text{pre}}$. However, it will turn out such a construction is bound to result in a Hilbert space that is too large to be a quantization theory. For this reason, the construction is called ***prequantization***. A nice description of this procedure is found in [Ler].

If this prequantization is to resemble the physical reality, we will require Dirac's quantization conditions (2.4) and (2.5) to hold:

$$[\mathcal{Q}_f^{\text{pre}}, \mathcal{Q}_g^{\text{pre}}] = i\hbar \mathcal{Q}_{\{f,g\}}^{\text{pre}}. \tag{3.1}$$

$$\mathcal{Q}_1^{\text{pre}} = I_{\mathfrak{H}_M^{\text{pre}}}. \tag{3.2}$$

Due to the identity $[X_f, X_g] = -X_{\{f,g\}}$ from Proposition 2.1.1, the choice $\mathcal{Q}_f^{\text{pre}} := -i\hbar X_f$ acting on $C^\infty(M)$ satisfies (3.1), but it does not satisfy (3.2) since $X_1 = 0$. We may adjust our prequantization map to $\mathcal{Q}_f^{\text{pre}} :=$

$-i\hbar X_f + m_f$, where the last term denotes pointwise multiplication with f. Then (3.2) is satisfied, but now we have

$$[-i\hbar X_f + m_f, -i\hbar X_g + m_g] = i\hbar\big(- i\hbar X_{\{f,g\}} + 2m_{\{f,g\}}\big).$$

However, this problem can be easily fixed by replacing the operator X_f on $C^\infty(M;\mathbb{C})$ with some exterior derivative ∇_{X_f} on a complex line bundle E with curvature $F(\nabla)$ such that

$$F(\nabla)(X_f, X_g) = i\hbar^{-1}\{f,g\} = i\hbar^{-1}\omega(X_f, X_g).$$

Definition 3.1.1. If (M,ω) is a symplectic manifold, a ***prequantum line bundle*** over M is a hermitian line bundle $\pi\colon E \to M$ together with a hermitian connection ∇ in E which has curvature $F(\nabla) = i\hbar^{-1}\omega$. The ***prequantum Hilbert space*** $\mathfrak{H}_M^{\text{pre}}$ of M with respect to this bundle is the Lebesgue space of sections $\psi \in \Gamma(E)$ which are square-integrable in the sense that

$$\int_M \langle\psi,\psi\rangle\omega^n < \infty,$$

and it is endowed with the inner product

$$\langle\langle\psi,\psi'\rangle\rangle := \int_M \langle\psi,\psi'\rangle\omega^n.$$

The ***prequantization map*** then is

$$\begin{aligned} \mathcal{Q}^{\text{pre}}\colon C^\infty(M) &\longrightarrow C\big(\mathfrak{H}_M^{\text{pre}}\big) \\ f &\longrightarrow -i\hbar\nabla_{X_f} + m_f, \end{aligned}$$

where we recall $C(\mathfrak{H}_M^{\text{pre}})$ is the space of closed densely defined operators on $\mathfrak{H}_M^{\text{pre}}$ as defined in Section 2.2.2.

Note that if M is compact, the situation is nicer since all measurable functions on a compact space are square-integrable. However, the cotangent bundles form an important class of spaces one might want to quantize, and such spaces are never compact.

Example 3.1.2. If the cotangent bundle T^*Q is endowed with the canonical symplectic form $\omega^{\text{can}} = -d\lambda$ defined in terms of the Liouville form λ, then there exists a canonical prequantum line bundle: the trivial bundle $\underline{\mathbb{C}}$ endowed with the connection

$$\nabla_X = X - i\hbar^{-1} m_{\lambda(X)}. \tag{3.3}$$

In the same way we can find a prequantum line bundle $(\underline{\mathbb{C}}, \nabla)$ on any manifold endowed with an exact symplectic form.

It is obvious that the prequantization map satisfies the condition (3.2). We now show it also satisfies the other quantization condition (3.1).

Lemma 3.1.3. *If (M, ω) is a symplectic manifold and (E, ∇) a prequantum line bundle on M, then for any $f, g \in C^\infty(M)$ and $\psi \in \mathfrak{H}_M^{\text{pre}}$ in the domain of the relevant operators, the prequantization map satisfies*

$$[\mathcal{Q}_f^{\text{pre}}, \mathcal{Q}_g^{\text{pre}}]\psi = i\hbar \mathcal{Q}_{\{f,g\}}^{\text{pre}}\psi.$$

Proof. Note that per definition of the curvature two-form we have

$$\begin{aligned}
[\nabla_{X_f}, \nabla_{X_g}]\psi &= \nabla_{[X_f, X_g]}\psi + F(\nabla)(X_f, X_g)\psi \\
&= -\nabla_{X_{\{f,g\}}}\psi + i\hbar^{-1}\omega(X_f, X_g)\psi \\
&= -\nabla_{X_{\{f,g\}}}\psi + i\hbar^{-1}\{f, g\}\psi.
\end{aligned}$$

We calculate

$$\begin{aligned}
\mathcal{Q}_f^{\text{pre}}\mathcal{Q}_g^{\text{pre}}s &= (i\hbar)^2\nabla_{X_f}\nabla_{X_g}\psi - i\hbar\nabla_{X_f}g\psi - i\hbar f\nabla_{X_g}\psi + fgs \\
&= (i\hbar)^2\nabla_{X_f}\nabla_{X_g}\psi - i\hbar\psi X_f g - i\hbar g\nabla_{X_f}\psi - i\hbar f\nabla_{X_g}\psi + fgs
\end{aligned}$$

and similarly

$$\mathcal{Q}_g^{\text{pre}}\mathcal{Q}_f^{\text{pre}}s = (i\hbar)^2\nabla_{X_g}\nabla_{X_f}\psi - i\hbar\psi X_g f - i\hbar f\nabla_{X_g}\psi - i\hbar g\nabla_{X_f}\psi + gfs$$

Then using Proposition 2.1.1 we find

$$\begin{aligned}
[\mathcal{Q}_f^{\text{pre}}, \mathcal{Q}_g^{\text{pre}}]\psi &= (i\hbar)^2[\nabla_{X_f}, \nabla_{X_g}]\psi - i\hbar(\{g, f\} - \{f, g\})\psi \\
&= -(i\hbar)^2\nabla_{X_{\{f,g\}}}\psi - i\hbar\{f, g\}\psi + 2i\hbar\{f, g\}\psi \\
&= -(i\hbar)^2\nabla_{X_{\{f,g\}}}\psi + i\hbar\{f, g\}\psi \\
&= i\hbar\mathcal{Q}_{\{f,g\}}^{\text{pre}}\psi,
\end{aligned}$$

which proves the lemma. □

Lemma 3.1.4. *If (M, ω) is a symplectic manifold and (E, ∇) a prequantum line bundle on M, then for any $f \in C^\infty(M)$ the prequantization $\mathcal{Q}_f^{\text{pre}}$ is Hermitian.*

Proof. We need to show that for any $\psi' \in \mathfrak{H}_M^{\text{pre}}$ such that the linear map

$$\psi \longmapsto \langle\langle \psi', \mathcal{Q}_f^{\text{pre}} \psi \rangle\rangle = \int_M \Big(\langle \psi', -i\hbar \nabla_{X_f} \psi \rangle + \langle \psi', fs \rangle \Big) \omega^n$$

is bounded, we have

$$0 = \langle\langle \psi', \mathcal{Q}_f^{\text{pre}} \psi \rangle\rangle - \langle\langle \mathcal{Q}_f^{\text{pre}} \psi', \psi \rangle\rangle$$

for any $\psi \in \mathfrak{H}_M^{\text{pre}}$. We prove this by showing that, whenever the integrals under consideration are finite, both terms of the prequantization are Hermitian:

$$0 = \int_M \Big(\langle \psi', -i\hbar \nabla_{X_f} \psi \rangle - \langle -i\hbar \nabla_{X_f} \psi', \psi \rangle \Big) \, \omega^n$$
$$0 = \int_M \left(\langle \psi', f\psi \rangle - \langle f\psi', \psi \rangle \right) \omega^n$$

Note that $\langle \psi', f\psi \rangle = f \langle \psi', \psi \rangle = \langle f\psi', \psi \rangle$ holds since f is real-valued, so the latter integral clearly vanishes. For the former we calculate

$$\begin{aligned}
\int \Big(\langle \psi', -i\hbar \nabla_{X_f} \psi \rangle - \langle -i\hbar \nabla_{X_f} \psi', \psi \rangle \Big) \omega^n &= \int \Big(-i\hbar \langle \psi', \nabla_{X_f} \psi \rangle - i\hbar \langle \nabla_{X_f} \psi', \psi \rangle \Big) \omega^n \\
&= -i\hbar \int \Big(\langle \psi', \nabla_{X_f} \psi \rangle + \langle \nabla_{X_f} \psi', \psi \rangle \Big) \omega^n \\
&= -i\hbar \int X_f \langle \psi', \psi \rangle \omega^n \\
&= -i\hbar \int \mathcal{L}_{X_f} (\langle \psi', \psi \rangle \omega^n) \\
&= -i\hbar \int d(\langle \psi', \psi \rangle \iota_{X_f} \omega^n)
\end{aligned}$$

where we used the assumption that our connection is Hermitian and the last identity follows from Cartan's magic formula. Due to Stoke's theorem and the fact the manifold under consideration has no boundary, this vanishes. From these observation we conclude $\mathcal{Q}_f^{\text{pre}}$ is indeed Hermitian. □

Remark 3.1.5. This proof only checks that $\mathcal{Q}_f^{\text{pre}}$ acts as its adjoint on its domain, and so is not a proof of the self-adjointness but only of the symmetry of the operator: see the discussion in Remark 2.2.12

3.2 Boothby-Wang prequantization

Some texts, such as [SW76], describe prequantization in terms of certain principal U(1)-bundles called Boothby-Wang bundles. It turns out that this approach is completely equivalent with the prequantum line bundle-approach described above. However, both pictures have their advantages and while we will mainly work with line bundles, it is useful to also describe this approach and proof that it results in the same quantization as the line bundle approach.

Definition 3.2.1. If (M, ω) is a symplectic manifold, a ***Boothby-Wang bundle*** over M is a principal U(1)-bundle $p\colon P \to M$ together with an Ehresmann connection $\theta \in \Omega^1(P; i\mathbb{R})$ which has curvature $F(\theta) = i\hbar^{-1} p^*\omega$.

Note that we identify $\mathfrak{u}(1)$ with the imaginary numbers $i\mathbb{R}$, which is canonically identified with the space of skew-Hermitian elements in $\mathrm{End}(\mathbb{C})$. We choose a canonical generator $i\hbar^{-1} \in \mathfrak{u}(1)$ and accordingly we identify

$$\mathrm{U}(1) = \left\{ \frac{1}{2\pi\hbar} e^{2\pi i t} \;\middle|\; t \in \mathbb{R} \right\}.$$

This choice will ensure the contact structure on P is independent of $\hbar$.

Proposition 3.2.2. *If (P, θ) is a Boothby-Wang bundle, $\alpha := -i\hbar\theta$ is a contact form and the Reeb vector field R_α generates the* U(1)*-flow on P.*

Proof. Note that the connection form θ induces a horizontal distribution $\mathcal{H} = \ker\theta$ of co-dimension one, and it is obvious that α is a locally defining one-form for $\mathcal{H}$. Since $\dim \mathfrak{u}(1) = 1$ the Lie bracket of one-forms $[\theta, \theta]$ vanishes, and so $d\theta = F(\theta) = i\hbar^{-1} p^*\omega$, and then clearly $d\alpha = p^*\omega$. Therefore, at any $x \in P$ we clearly have $T_x p\colon \mathcal{H}_x \xrightarrow{\sim} T_{p(x)}M$ and we see $d\alpha|_{\mathcal{H}_x} = (T_x p)^*\omega_{p(x)}$ is non-degenerate. Thus α is a contact form on P, and $(P, \mathcal{H})$ is a contact manifold.

Since $\dim \mathfrak{u}(1) = 1$, its action is generated by the fundamental vector field $X_{i\hbar^{-1}}$ of the generator $i\hbar^{-1} \in \mathfrak{u}(1)$. By the properties of the Ehresmann connection θ, we have

$$\alpha(X_{i\hbar^{-1}}) = -i\hbar\theta(X_{i\hbar^{-1}}) = -i\hbar \cdot (i\hbar^{-1}) = 1.$$

We also note that since α is a contact form we have $\mathcal{V} = \ker d\alpha$, and since fundamental vector fields are vertical we see

$$d\alpha(X_{i\hbar^{-1}}) = 0.$$

From these two observations, it follows that $X_{i\hbar^{-1}} = R_\alpha$ is the Reeb vector field of α. □

Proposition 3.2.3. *Let (M, ω) be a symplectic manifold. Then there is a bijective correspondence between isomorphism classes of prequantum line bundles and isomorphism classes of Boothby-Wang bundles on M.*

Proof. We will give the construction of a Boothby-Wang bundle starting from a prequantum line bundle. It is left to the reader to convince themselves that the associated bundle-construction gives an inverse at the level of isomorphism classes of bundles with connection.

Let $(\pi\colon E \to M, \nabla)$ be a prequantum line bundle. By choosing local trivializations of E consisting of frames that are orthonormal with respect to the Hermitian metric on E, we reduce the structure group of E to the unitary group $\mathrm{U}(1)$. Therefore the orthonormal frame bundle $p\colon P \to M$, defined pointwise by

$$P_x = \{\phi_x\colon \mathbb{C} \to E_x \mid \phi_x \text{ is a linear isometry}\},$$

with the action defined by $\phi_x \cdot A := \phi_x \circ A$ for $A \in \mathrm{U}(1) \subseteq \mathrm{GL}(1;\mathbb{C})$ and $\phi_x \in P$ is the principal $\mathrm{U}(1)$-bundle associated with E.

Now consider the pullback bundle p^*E with pullback connection $p^*\nabla$. Note that this bundle admits a tautological global frame $(\mathbf{e}_1)$, since for $\phi_x \in P_x$ we have $(p^*E)_{\phi_x} = E_x$ and may take

$$\mathbf{e}_1(\phi_x) := \phi_x(1)$$

Therefore p^*E is canonically trivial, $p^*E \cong \underline{\mathbb{C}}$, and we have a canonical tautological connection ∇^{tau} in p^*E. The difference between the two connections is an $\mathrm{End}(p^*E)$-valued form,

$$\theta := p^*\nabla - \nabla^{\mathrm{tau}} \in \Omega^1(P; \mathrm{End}(p^*E)).$$

As the difference between two Hermitian connections, θ is in fact skew-

Hermitian-valued since for $X \in \mathfrak{X}(P)$, $\psi, \psi' \in \Gamma(p^*E)$ we have

$$\begin{aligned}\langle (\nabla_X - \nabla_X^{\text{tau}})\psi, \psi' \rangle &= \langle \nabla_X \psi, \psi' \rangle - \langle \nabla_X^{\text{tua}} \psi, \psi' \rangle \\ &= X\langle \psi, \psi' \rangle - \langle \psi, \nabla_S \psi' \rangle - X\langle \psi, \psi' \rangle + \langle \psi, \nabla_S^{\text{tau}} \psi' \rangle \\ &= -\langle \psi, \nabla_S \psi' \rangle + \langle \psi, \nabla_S^{\text{tau}} \psi' \rangle \\ &= -\langle \psi, (\nabla_X - \nabla_X^{\text{tau}})\psi' \rangle,\end{aligned}$$

so that we may consider $\theta \in \Omega^1(P; \mathfrak{u}(1))$.

We claim θ is an Ehresmann connection in P.

To show the curvature of θ agrees with the curvature of ∇, it is enough to show this is the case locally. So let $U \subseteq M$ be an open neighbourhood that admits a normalized trivialization of E, that is: a smooth section $\mathbf{z} \in \Gamma(U; E)$ with $\langle \mathbf{z}(x), \mathbf{z}(x) \rangle = 1$ for all $x \in U$. This gives identifications $E|_U \cong \underline{\mathbb{C}}_U$ by associating to any $x \in U$ and $e_x \in E_x$ the unique element $(x, \widetilde{e_x}) \in U \times \mathbb{C}$ such that $e_x = \widetilde{e_x}\mathbf{z}(x)$. The trivialization also gives an identification $P|_U \cong \underline{\mathrm{U}(1)}_U$ by associating to $\phi_x \colon \mathbb{C} \to E_x$ the unitary map

$$\widetilde{\phi_x} \colon z \longmapsto \widetilde{\phi_x(z)}.$$

Note that this gives rise to a canonical connection ∇^{can} in $\underline{\mathbb{C}}_U$. The tautological function

$$\tau \colon P|_U \cong \underline{\mathrm{U}(1)}_U \longrightarrow \mathrm{Mat}(1 \times 1; \mathbb{C})$$

given by the inclusion $\mathrm{U}(1) \hookrightarrow \mathrm{Mat}(1 \times 1; \mathbb{C})$ gives rise to the canonical Ehresmann connection

$$\theta^{\text{can}} = -d\tau \in \Omega^1(P|_U; \mathfrak{u}(1)) \subseteq \Omega^1(P|_U; \mathrm{Mat}(1 \times 1; \mathbb{C})),$$

and in fact this is the Ehresmann connection associated with ∇^{can}. Thus we see that if $\eta \in \Omega^1(M; \mathfrak{u}(1))$ is the one-form such that $\nabla|_U = \nabla^{\text{can}} + \eta$, then η also pulls back to the basic one-form $p^*\eta \in \Omega^1_{\text{bas}}(P; \mathfrak{u}(1))$ such that $\theta = \theta^{\text{can}} + p^*\eta$.

Since $\dim \mathfrak{u}(1) = 1$ all Lie brackets of one-forms vanish and and so we have

$$F(\theta) = F(\theta^{\text{can}} + p^*\eta) = d\theta^{\text{can}} + dp^*\eta = p^*d\eta = p^*F(\nabla).$$

and since we assumed $F(\nabla) = i\hbar^{-1}\omega$, we do indeed find $F(\theta) = i\hbar^{-1}p^*\omega$. □

The prequantization map can also be described at the level of Boothby-Wang bundles.

Definition 3.2.4. If (M, ω) is a symplectic manifold and (P, θ) is a Boothby-Wang bundle over M with contact structure defined by the contact form $\alpha = i\hbar\theta$, the the ***Boothby-Wang prequantum Hilbert space*** $\mathfrak{H}_M^{\mathrm{BW}}$ of M with respect to this bundle is the space of smooth functions on $s \in C^\infty(P; \mathbb{C})$ which are equivariant in the sense that

$$R_\alpha s = -i\hbar^{-1} s,$$

where R_α denoted the Reeb vector field of α, and square-integrable in the sense that

$$\int_P \bar{s} s \alpha \wedge (d\alpha)^n < \infty,$$

and it is endowed with the inner product

$$\langle\langle s, s' \rangle\rangle := \int_P \bar{s} s' \alpha \wedge (d\alpha)^n.$$

The ***Boothby-Wang prequantization map*** then is

$$\begin{aligned} \mathcal{Q}^{\mathrm{BW}} \colon C^\infty(M) &\longrightarrow C(\mathfrak{H}_M^{\mathrm{BW}}) \\ f &\longrightarrow X_{p^* f}, \end{aligned}$$

where $X_{p^* f} \in \mathfrak{X}(P)$ denotes the contact Hamiltonian vector field of $p^* f \in C^\infty(P)$.

This map is well-defined, in the sense that for any $f \in C^\infty(M)$ and any $s \in C^\infty(P)$ which is $\mathrm{U}(1)$-equivariant, the prequantization $\mathcal{Q}_f^{\mathrm{BW}} \psi$ is also equivariant. Note that the contact Hamiltonian vector field of $p^* f$ is a contact vector field that is invariant under the Reeb flow. Then we calculate

$$\begin{aligned} R_\alpha \mathcal{Q}_f^{\mathrm{BW}} \psi &= -i\hbar^{-1} R_\alpha X_{p^* f} s \\ &= -i\hbar^{-1} X_{p^* f} R_\alpha s \\ &= (-i\hbar^{-1})^2 X_{p^* f} s \\ &= -i\hbar^{-1} \mathcal{Q}_f^{\mathrm{BW}} \psi. \end{aligned}$$

In this paradigm, we could also show that $\mathcal{Q}^{\mathrm{BW}}$ satisfies the Dirac axioms of prequantization. However, we will find this holds as a corollary to the stronger statement that this representation of $C^\infty(M)$ is isomorphic to the prequantization as in Definition 3.1.1.

Proposition 3.2.5. *If (M, ω) is a symplectic manifold with (E, ∇) a prequantum line bundle over M and (P, θ) the corresponding Boothby-Wang bundle, then there is a canonical isometric isomorphism $T\colon \mathfrak{H}_M^{\text{pre}} \xrightarrow{\sim} \mathfrak{H}_M^{\text{BW}}$ satisfying, for any $f \in C^\infty(M)$,*

$$\mathcal{Q}_f^{\text{pre}} = T^{-1} \circ \mathcal{Q}_f^{\text{BW}} \circ T.$$

Proof. Recall the Boothby-Wang bundle associated with (E, ∇) is the orthonormal frame bundle,

$$P_x = \{\phi_x\colon \mathbb{C} \to E_x \mid \phi_x \text{ is a linear isometry}\}.$$

We get a map

$$\begin{aligned} \widetilde{T}\colon \Gamma(E) &\longrightarrow C^\infty(P) \\ \psi &\longrightarrow \big(s_\psi\colon \phi_x \mapsto \phi_x^{-1}(\psi(x))\big). \end{aligned}$$

which is linear and injective. Moreover, it takes values in the space of U(1)-equivariant functions, since for any $\psi \in \Gamma(E)$, $\phi_x \in P$ and $u \in \mathrm{U}(1)$ we have that

$$s_\psi(\phi_x u) = (\phi_x u)^{-1}(\psi(x)) = u^{-1} s_\psi(\phi_x).$$

Then writing

$$u(t) = \frac{1}{2\pi\hbar} e^{2\pi i t} \in \mathrm{U}(1) \qquad\qquad F_t(\phi_x) = \phi_x u(t),$$

for $t \in \mathbb{R}$, we may linearize this equation to

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} s_\psi(F_t(\phi_x)) &= \frac{d}{dt}\bigg|_{t=0} u(-t) s_\psi(\phi_x) \\ (R_\alpha s_\psi)(\phi_x) &= -i\hbar^{-1} s_\psi(\phi_x). \end{aligned}$$

so we conclude s_ψ is equivariant. Conversely, all equivariant functions are in the image of $\widetilde{T}$. Also, $\widetilde{T}$ is a pointwise isometry in the sense that

$$(\overline{s_\psi} s_\psi)(\phi_x) = \overline{\phi_x^{-1}(\psi(x))} \phi_x^{-1}(\psi(x)) = \phi_x^{-1}(\langle \psi, \psi \rangle(x)),$$

just as $\langle \psi, \psi \rangle \in C^\infty(M)$ we find the function

$$s_{\langle \psi, \psi \rangle}\colon \phi_x \longmapsto \phi_x^{-1}(\langle \psi, \psi \rangle(x))$$

is U(1)-invariant, and in fact $s_{\langle\psi,\psi\rangle} = p^*\langle\psi,\psi\rangle$. Globally we find

$$\begin{aligned}\int_P \overline{s_\psi} s_\psi \alpha \wedge (d\alpha)^n &= \int_P s_{\langle\psi,\psi\rangle}\alpha \wedge (d\alpha)^n \\ &= \int_M \left(\int_{\mathrm{U}(1)} s_{\langle\psi,\psi\rangle}\alpha\right)\omega^n \\ &= \int_M \langle\psi,\psi\rangle\omega^n.\end{aligned}$$

Thus it is clear that $\widetilde{T}$ defines an isometric isomorphism

$$T\colon\ \mathfrak{H}_M^{\mathrm{pre}} \xrightarrow{\sim} \mathfrak{H}_M^{\mathrm{BW}}.$$

Finally, let $f \in C^\infty(M)$ and $\psi \in \Gamma(E)$. Using (2.2) to rewrite the contact Hamiltonian vector field, we find

$$\begin{aligned}\big(\mathcal{Q}_f^{\mathrm{BW}} \circ T\big)(\psi) &= \mathcal{Q}_f^{\mathrm{BW}} s_\psi \\ &= i\hbar X_{p^*f} s_\psi \\ &= -i\hbar Y_{p^*f} s_\psi + i\hbar(p^*f)R_\alpha s_\psi \\ &= -i\hbar Y_{p^*f} s_\psi + (p^*f)s_\psi\end{aligned}$$

Where Y_{p^*f} is the unique horizontal vector field on P such that $(Tp)Y_{p^*f} = X_f$.

Now, let $\phi_x \in P$ and consider neighbourhood of ϕ_x that admits local coordinates $\{x^1,\dots,x^{2n},y\}$ such that the $x^1,\dots,x^{2n}$ descent down to local coordinates on M. Then we have local components $X_f^j \in C^\infty(M)$ such that $X_f = X_f^j\partial_{x^j}$, and we must locally have $Y_{p^*f} = (p^*X_f^j)\partial_{x^j}$. Moreover, we find

$$(\partial_{x^j} s_\psi)(\phi_x) = \phi_x^{-1}\big((\partial_{x^j}\psi)(x)\big).$$

and so

$$\begin{aligned}(Y_{p^*f} s_\psi)(\phi_x) &= \big((p^*X_f^j)\partial_{x^j} s_\psi\big)(\phi_x) \\ &= (p^*X_f^j)\phi_x^{-1}\big((\partial_{x^j}\psi)(x)\big) \\ &= \phi_x^{-1}\big(X_f^j(\partial_{x^j}\psi)(x)\big) \\ &= \phi_x^{-1}\big((X_f\psi)(x)\big) \\ &= s_{X_f\psi}(\phi_x).\end{aligned}$$

We also have $(p^*f)s_\psi = s_{p\psi}$. Continuing the calculation, then, we find

$$\begin{aligned}\big(\mathcal{Q}_f^{\mathrm{BW}} \circ T\big)(\psi) &= -i\hbar Y_{p^*f} s_\psi + (p^*f)s_\psi \\ &= -i\hbar s_{X_f\psi} + s_{f\psi} \\ &= -i\hbar T(X_f\psi) + T(f\psi) \\ &= T(-i\hbar X_f\psi + f\psi) \\ &= \big(T \circ \mathcal{Q}_f^{\mathrm{pre}}\big)(\psi)\end{aligned}$$

and since T is an isomorphism this gives the required identity. □

3.3 Existence of prequantum bundles

As we saw in Example 3.1.2, on a cotangent bundle it is easy to construct a prequantum line bundle in terms of the Liouville form. However, not every symplectic manifold admits a prequantum line bundle.

Consider a prequantum line bundle (E, ∇) on a symplectic manifold (M, ω) and note that the first Chern class of E is related to the De Rahm class of ω by

$$c_1(E) = \frac{1}{2\pi i}[F(\nabla)] = \frac{1}{2\pi\hbar}[\omega] \in H^2_{\mathrm{dR}}(M).$$

However, the Chern class $c_1(E)$ is integral, that is: it is in the image of the map

$$H^2(M;\mathbb{Z}) \hookrightarrow H^2(M;\mathbb{R}) \xrightarrow{\sim} H^2_{\mathrm{dR}}(M).$$

We will not prove this fact but instead refer to [Wel80], Proposition 4.3.

This places a restriction on the De Rahm class of the symplectic form of a symplectic manifold that admits a prequantization bundle. Note that if ω is exact, such as on a cotangent bundle T^*Q, we have $[\omega] = 0$ and so the Chern class vanishes. Indeed, as we saw in Example 3.1.2, such systems always admit a prequantum line bundle.

Definition 3.3.1. If M is a smooth manifold and $\omega \in \Omega^k(M)$ is a closed differential form, it is called ***ℏ-integral*** if the De Rahm class $[\omega]$ is in the image of

$$H^k(M;2\pi\hbar\mathbb{Z}) \hookrightarrow H^k(M;\mathbb{R}) \xrightarrow{\sim} H^k_{\mathrm{dR}}(M).$$

From the discussion above it is clear that if a symplectic manifold admits a prequantum line bundle, its symplectic form must be $\hbar$-integral. In fact, the converse is also true.

Proposition 3.3.2 (Weil integrality condition)**.** *If (M, ω) is a symplectic manifold such that ω is $\hbar$-integral, then it admits a prequantum line bundle.*

In order to prove this, we will follow the construction given by Weil in Section 5.3 [Wei58]. We do this by constructing the Čech cohomology element corresponding to the symplectic form and showing that this construction automatically gives rise to transition functions and local curvature forms of a prequantum line bundle. While Čech cohomology can be defined in a more general manner, we only need the following restricted definition. Let M be a topological space and $\mathcal{U} = \{U_j\}_{j \in I}$ an open cover of M with a well-ordered index set $(I, <)$, such that every intersection of finitely many elements of $\mathcal{U}$ is either the empty set or a contractible topological space. In particular, we require each U_i to be contractible but we do not require M to be contractible. An open cover is called ***good*** if it satisfies this property, and we note that a smooth manifold always admits a good open cover. To simplify notation, we write $U_{j_1, \ldots, j_n}$ for the intersection $U_{j_1} \cap \cdots \cap U_{j_n}$.

For A an Abelian group, we define the Čech cochain complex by

$$\check{C}^p(\mathcal{U}, A) := \prod_{\substack{j_0 < \ldots < j_p \\ U_{j_0, \ldots, j_n} \neq \emptyset}} A$$

with coboundary maps $\delta^p \colon \check{C}^p(\mathcal{U}, A) \longrightarrow \check{C}^{p+1}(\mathcal{U}, A)$ defined on components by

$$(\delta^p c)_{j_0, \ldots, j_{p+1}} = \sum_{k=0}^{p+1} (-1)^k c_{j_0, \ldots, \hat{j}_k, \ldots, j_{p+1}}$$

where the index with the hat is left out. This is a cochain complex, since the components of $(\partial^{p+1} \partial^p c)$ satisfy

$$\begin{aligned}
(\partial^{p+1} \delta^p c)_{j_0, \ldots, j_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k (\delta^p c)_{j_0, \ldots, \hat{j}_k, \ldots, j_{p+2}} \\
&= \sum_{l<k} (-1)^{k+l} c_{j_0, \ldots, \hat{j}_l, \ldots, \hat{j}_k, \ldots, j_{p+2}} - \sum_{k<l} (-1)^{k+l} c_{j_0, \ldots, \hat{j}_k, \ldots, \hat{j}_l, \ldots, j_{p+2}} \\
&= 0.
\end{aligned}$$

This complex then defines cohomology groups

$$\check{H}^n(\mathcal{U}, A) := \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}}$$

called the ***Čech cohomology groups*** of $\mathcal{U}$ with coefficients in A.

In the proof of Proposition 3.3.2 we will need a notion of the components of a Čech cocycle with not necessarily increasing indices. Thus, for $a \in \check{C}^p(\mathcal{U}, A)$ and $j_0, \ldots, j_p \in I$ we define $a_{j_0,\ldots,j_p} := 0$ if there is a repeated index or if $U_{j_0,\ldots,j_p}$ is empty, and otherwise we define

$$a_{j_0,\ldots,j_p} := \operatorname{sgn}(\sigma) a_{j_{\sigma(0)},\ldots,j_{\sigma(p)}} \tag{3.4}$$

where $\sigma \in S_{p+1}$ is the unique permutation such that $j_{\sigma(0)} < \cdots < j_{\sigma(p)}$.

Čech cohomology with real coefficients turns out to agree with De Rahm cohomology.

Theorem 3.3.3. *If M is a smooth manifold and $\mathcal{U} = \{U_i\}_{i \in I}$ is a good open cover of M, then there is, for every $k \geq 0$, a canonical isomorphism*

$$H^k_{\mathrm{dR}}(M) \xrightarrow{\sim} \check{H}^k(\mathcal{U}, \mathbb{R})$$

of the De Rahm and Čech cohomology groups.

Proof. Following the construction in [Wei58], Section 5.2 we will only give the construction for $k = 2$, which is the situation we are interested in. The general proof is very similar but needs to be formulated inductively in k.

We will first give the construction of the map at the level of cocycles. If $\omega \in \Omega^2(M)$ is a closed two-form, we construct an element

$$\left(c_{j_0,j_1,j_2}\right)_{j_0,j_1,j_2} \in \prod_{\substack{j_0<j_1<j_2 \\ U_{j_0,j_1,j_2} \neq \emptyset}} \mathbb{R}.$$

Since for each $j_0 \in I$ the open set U_{j_0} is contractible, the De Rahm cohomology group $H^k_{\mathrm{dR}}(U_{j_0})$ vanishes and therefore $\omega|_{U_{j_0}}$ is exact. Thus we may choose a one-form $\lambda_{j_0} \in \Omega^1(U_{j_0})$ such that $d\lambda_{j_0} = \omega|_{U_{j_0}}$.

If $j_0 < j_1 \in I$ then are such that $U_{j_0,j_1} \neq \emptyset$, on this open set we have

$$d(\lambda_{j_0} - \lambda_{j_1}) = d\lambda_{j_0} - d\lambda_{j_1} = \omega - \omega = 0,$$

and since U_{j_0,j_1} is contractible we may choose a $f_{j_0,j_1} \in C^\infty(U_{j_0,j_1})$ such that $df_{j_0,j_1} = \lambda_{j_0} - \lambda_{j_1}$.

If finally $j_0 < j_1 < j_2 \in I$ are such that $U_{j_0,j_1,j_2} \neq \emptyset$, we note that on this open set

$$d(f_{j_0,j_1} - f_{j_0,j_2} + f_{j_1,j_2}) = \lambda_{j_0} - \lambda_{j_1} - \lambda_{j_0} + \lambda_{j_2} + \lambda_{j_1} - \lambda_{j_2} = 0,$$

and since U_{j_0,j_1,j_2} is connected such a function must be constant, so there exists a real number $c_{j_0,j_1,j_2} \in \mathbb{R}$ such that

$$f_{j_0,j_1} - f_{j_0,j_2} + f_{j_1,j_2} \equiv c_{j_0,j_1,j_2}.$$

This associates to ω a Čech cocycle, since

$$\begin{aligned}(\delta c)_{j_0,j_1,j_2,j_3} &= c_{j_1,j_2,j_3} - c_{j_0,j_2,j_3} + c_{j_0,j_1,j_3} - c_{j_0,j_1,j_2} \\ &= f_{j_1,j_2} - f_{j_1,j_3} + f_{j_2,j_3} - f_{j_0,j_2} + f_{j_0,j_3} - f_{j_2,j_3} \\ &\quad + f_{j_0,j_1} - f_{j_0,j_3} + f_{j_1,j_3} - f_{j_0,j_1} + f_{j_0,j_2} - f_{j_1,j_2} \\ &= 0.\end{aligned}$$

These forms λ_{j_0} and are uniquely defined up to a closed (and hence exact) form. In particular, any other choice can be expressed as $\lambda'_{j_0} = \lambda_{j_0} + dg_{j_0}$ for some family of functions $g_{j_0} \in C^\infty(U_{j_0})$. Now the f_{j_0,j_1} are uniquely defined up to a constant: any choice is of the form

$$f'_{j_0,j_1} = f_{j_0,j_1} + b_{j_0,j_1} - g_{j_0} + g_{j_1}$$

for constants $b_{j_0,j_1} \in \mathbb{R}$, which we note define a Čech cochain element $b := (b_{j_0,j_1})_{j_0<j_1} \in \check{C}^1(\mathcal{U};\mathbb{R})$. The f'_{j_0,j_1} then define the Čech cocycle

$$\begin{aligned}c'_{j_0,j_1,j_2} &= f'_{j_0,j_1} - f'_{j_0,j_2} + f'_{j_1,j_2} \\ &= f_{j_0,j_1} + b_{j_0,j_1} - g_{j_0} + g_{j_1} - f_{j_0,j_2} - b_{j_0,j_2} \\ &\quad + g_{j_0} - g_{j_2} + f_{j_1,j_2} + b_{j_1,j_2} - g_{j_1} + g_{j_2} \\ &= f_{j_0,j_1} - f_{j_0,j_2} + f_{j_1,j_2} + b_{j_0,j_1} - b_{j_0,j_2} + b_{j_1,j_2} \\ &= f_{j_0,j_1} - f_{j_0,j_2} + f_{j_1,j_2} + (\delta^1 b)_{j_0,\ldots,j_2}\end{aligned}$$

which agrees with the one we constructed earlier up to a coboundary. We thus have a well-defined map of cohomology groups. □

Proof of Proposition 3.3.2. Fix a good open cover $\mathcal{U} = \{U_j\}_{j\in I}$ of M. If ω is $\hbar$-integral, its De Rahm cohomology class corresponds under the isomorphism of Theorem 3.3.3 to a Čech cohomology class represented by some c with $c_{j_0,j_1,j_2} \in 2\pi\hbar\mathbb{Z}$ for all $j_0, j_1, j_2 \in I$. For $j_0, j_1 \in I$ with U_{j_0,j_1} nonempty, consider $f_{j_0,j_1} \in C^\infty(U_{j_0,j_1})$ as in the proof of Theorem 3.3.3 and define

$$\tau_{j_0,j_1} := \exp\left(i\hbar^{-1} f_{j_0,j_1}\right) : U_{j_0,j_1} \to S^1 \subseteq \mathbb{C}.$$

Due to (3.4) making elements of a Čech cycle alternating in their indices, it is clear that $\tau_{j_0,j_0} = 1$ and $\tau_{j_1,j_0} = \tau_{j_0,j_1}^{-1}$ (where the superscript denotes inversion in the group S^1, not inversion of the map). Moreover, on any nonempty U_{j_0,j_1,j_2} we have

$$\begin{aligned}
\tau_{j_0,j_1}\tau_{j_0,j_2}^{-1}\tau_{j_1,j_2} &= \exp\left(i\hbar^{-1} f_{j_0,j_1}\right)\exp\left(-i\hbar^{-1} f_{j_0,j_2}\right)\exp\left(i\hbar^{-1} f_{j_1,j_2}\right) \\
&= \exp\left(i\hbar^{-1}\left(f_{j_0,j_1} - f_{j_0,j_2} + f_{j_1,j_2}\right)\right) \\
&= \exp\left(i\hbar^{-1} c_{j_0,j_1,j_2}\right) \\
&= 1
\end{aligned}$$

since $\hbar^{-1} c_{j_0,j_1,j_2} \in 2\pi\mathbb{Z}$. Thus, this is a system of transition functions which defines a principal S^1-bundle P over M.

Finally consider the forms $\alpha_{j_0} := i\hbar^{-1}\lambda_{j_0} \in \Omega^1(U_{j_0}; i\mathbb{R})$ where the λ_{j_0} are as in the proof of Theorem 3.3.3. If $j_0, j_1 \in I$ are such that U_{j_0,j_1} is nonempty, on this intersection we calculate

$$d\tau_{j_0,j_1} = d\exp\left(i\hbar^{-1} f_{j_0,j_1}\right) = d(i\hbar^{-1} f_{j_0,j_1})\exp\left(i\hbar^{-1} f_{j_0,j_1}\right) = i\hbar^{-1} df_{j_0,j_1}\tau_{j_0,j_1}$$

so that we find

$$\begin{aligned}
\tau_{j_0,j_1}^{-1}\alpha_{j_1}\tau_{j_0,j_1} + \tau_{j_0,j_1}^{-1} d\tau_{j_0,j_1} &= \tau_{j_0,j_1}^{-1} i\hbar^{-1}\lambda_{j_1}\tau_{j_0,j_1} + \tau_{j_0,j_1}^{-1} i\hbar^{-1} df_{j_0,j_1}\tau_{j_0,j_1} \\
&= \tau_{j_0,j_1}^{-1} i\hbar^{-1}\left(\lambda_{j_1} + df_{j_0,j_1}\right)\tau_{j_0,j_1} \\
&= \tau_{j_0,j_1}^{-1} i\hbar^{-1}\left(\lambda_{j_1} + \lambda_{j_0} - \lambda_{j_1}\right)\tau_{j_0,j_1} \\
&= \tau_{j_0,j_1}^{-1} i\hbar^{-1}\lambda_{j_0}\tau_{j_0,j_1} \\
&= \alpha_{j_0}.
\end{aligned}$$

Thus the $(\alpha_{j_0})_{j_0\in I}$ satisfy the transition rules of local connection one-forms, and so they define a connection θ in P. Moreover, for any $j_0 \in I$ the curvature on U_{j_0} is given by

$$F(\theta)|_{U_{j_0}} = p^* d\alpha_{j_0} + p^*\alpha_{j_0} \wedge \alpha_{j_0} = p^* d(i\hbar^{-1}\lambda_{j_0}) = i\hbar^{-1} p^*\omega|_{U_{j_0}}$$

so that $F(\theta) = i\hbar^{-1}p^*\omega$.

Now (P, θ) defined a Boothby-Wang bundle over M, and due to Proposition 3.2.3 this is the same as a prequantum line bundle (E, ∇) over M. □

3.4 Examples

3.4.1 Harmonic oscillator

Let us first describe the prequantization of a system consisting of a one-dimensional harmonic oscillators with mass $m \in \mathbb{R}_{>0}$ and spring constants $c \in \mathbb{R}_{>0}$. Such a system has configuration space $Q = \mathbb{R}$ and phase space $T^*Q = \mathbb{R}^2$ with the standard symplectic form $\omega = dq \wedge dp$ and classical Hamiltonian

$$H(q, p) = \frac{p^2}{2m} + \frac{cq^2}{2m}. \tag{3.5}$$

It turns out that the prequantization has a nice form for the symplectic potential

$$\theta = \frac{1}{2}\left(pdq - qdp\right),$$

which is easily verified to satisfy $\omega = -d\theta$. The prequantum line bundle of this system as in Example 3.1.2 is the trivial bundle $\underline{\mathbb{C}}$ with connection

$$\nabla_X \psi = X\psi - i\hbar^{-1}\theta(X)\psi.$$

If $f \in C^\infty(\mathbb{R}^2)$ then is a classical observable, the prequantization acts on square-integrable $\psi \in \mathfrak{H}^{\text{pre}}_{\mathbb{R}^2}$ by

$$\mathcal{Q}^{\text{pre}}_f \psi = -i\hbar X_f \psi - \frac{1}{2}\left(pdq(X_f) - qdp(X_f)\right)\psi + f\psi.$$

Since we work in the standard symplectic space, we may apply Proposition 2.1.2 to find the coordinate representation of this operator:

$$\mathcal{Q}^{\text{pre}}_f \psi = -i\hbar \frac{\partial f}{\partial p}\frac{\partial \psi}{\partial q} + i\hbar \frac{\partial f}{\partial q}\frac{\partial \psi}{\partial p} - \frac{1}{2}\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)\psi + f\psi.$$

In particular for the Hamiltonian function H, we have

$$\frac{\partial H}{\partial q} = \frac{cq}{m} \qquad\qquad \frac{\partial H}{\partial p} = \frac{p}{m}.$$

The term $\theta(X_H)\psi$ then cancels with $H\psi$, so that

$$\mathcal{Q}_H^{\text{pre}}\psi = \frac{i\hbar}{m}\left(cq\frac{\partial\psi}{\partial p} - p\frac{\partial\psi}{\partial q}\right). \tag{3.6}$$

3.4.2 Free particle

Now we consider a free particle with mass $m \in \mathbb{R}_{>0}$ in a Riemannian manifold Q with metric tensor g. We then have potential $V \equiv 0$, so the classical Hamiltonian (2.1) on the phase space T^*Q of this mechanical system becomes

$$H\colon (q,p) \longmapsto \frac{g_q(p,p)}{2m}, \tag{3.7}$$

such that the Hamiltonian flow corresponds to the co-geodesic flow on (Q,g).

The prequantum line bundle of this system is again the trivial bundle $\underline{\mathbb{C}}$ with connection

$$\nabla_X\psi = X\psi - i\hbar^{-1}\lambda(X)\psi$$

as in (3.3).

If $f \in C^\infty(T^*Q)$ then is a classical observable, the prequantization acts on square-integrable $\psi \in \mathfrak{H}_{T^*Q}^{\text{pre}}$ by

$$\mathcal{Q}_f^{\text{pre}}\psi = -i\hbar X_f\psi - \lambda(X_f)\psi + f\psi.$$

For a local coordinate patch $U \subseteq Q$ with coordinates (q^j), we find local coordinates (q^j, p_j) on $T^*U \subseteq T^*Q$ such that $\lambda = p_j dq^j$ on this patch. By Proposition 2.1.2, on T^*U we have

$$\mathcal{Q}_f^{\text{pre}}\psi|_{T^*U} = -i\hbar\frac{\partial f}{\partial p_j}\frac{\partial\psi}{\partial q^j} + i\hbar\frac{\partial f}{\partial q^j}\frac{\partial\psi}{\partial p_j} - p_j\frac{\partial f}{\partial p_j}\psi + f\psi.$$

In particular for the Hamiltonian function H, we have

$$\frac{\partial H}{\partial q^j} = \frac{1}{2m}\frac{\partial g^{kl}}{\partial q^j}p_k p_l \qquad\qquad \frac{\partial H}{\partial p_j} = \frac{1}{m}g^{jk}p_k,$$

so that

$$\mathcal{Q}_H^{\text{pre}}\psi|_{T^*U} = \frac{1}{m}\left(-i\hbar g^{jk}p_k\frac{\partial\psi}{\partial q^j} + \frac{i\hbar}{2}\frac{\partial g^{kl}}{\partial q^j}p_k p_l\frac{\partial\psi}{\partial p_j} - \frac{1}{2}g^{jk}p_j p_k\psi\right).$$

Things become simpler when we take $Q = \mathbb{R}^n$ with the standard Euclidean inner product. The Hamiltonian then takes the form

$$H(q^1,\ldots,q^n,p_1,\ldots,p_n) \longmapsto \frac{1}{2m}\sum_{j=1}^{n}p_j^2, \tag{3.8}$$

with

$$\frac{\partial H}{\partial q^j} = 0 \qquad\qquad \frac{\partial H}{\partial p_j} = \frac{1}{m}p_j,$$

and prequantization of a classical observable $f \in C^\infty(\mathbb{R}^{2n})$ acts on square-integrable $\psi \in \mathfrak{H}^{\text{pre}}_{\mathbb{R}^{2n}}$ by

$$\mathcal{Q}_f^{\text{pre}}\psi = -i\hbar X_f\psi - p_j dq^j(X_f)\psi + f\psi,$$

which by Proposition 2.1.2 has coordinate representation

$$\mathcal{Q}_f^{\text{pre}}\psi = -i\hbar\frac{\partial f}{\partial p_j}\frac{\partial\psi}{\partial q^j} + i\hbar\frac{\partial f}{\partial q^j}\frac{\partial\psi}{\partial p_j} - p_j\frac{\partial f}{\partial p_j}\psi + f\psi.$$

Filling in the classical observables H, q^j and p_j give the following operators:

$$\mathcal{Q}_H^{\text{pre}}\psi = -i\hbar\frac{p_j}{m}\frac{\partial\psi}{\partial q^j} - H\psi \tag{3.9a}$$

$$\mathcal{Q}_{q^j}^{\text{pre}}\psi = i\hbar\frac{\partial\psi}{\partial p_j} + q^j\psi \tag{3.9b}$$

$$\mathcal{Q}_{p_j}^{\text{pre}}\psi = -i\hbar\frac{\partial\psi}{\partial q^j}. \tag{3.9c}$$

It can easily be verified that the canonical commutation relations (2.6) hold as expected.

However, the Hilbert space is too big in some sense. We make the observation that the space of sections that do not depend on the p_j-coordinates is an invariant subspace of both of these operators. On this linear subspace the prequantization of p and q agrees with the canonical quantization of these operators as given in (2.7). In the next chapter, we will formalize this procedure to arrive at a theory of geometric quantization.

3.4.3 Spin particle

We now calculate the prequantization of a spin particle with total angular momentum j. The classical phase space is the sphere $S^2 \subseteq \mathbb{R}^3$ with radius j, elements of which are denoted by J, and the symplectic form

$$\omega_J(v, w) = j^{-2} J \cdot (v \times w).$$

For any normal vector $e \in \mathbb{R}^3$, we get a smooth function $J_e \in C^\infty(S^2)$ defined by $J_e(J) = e \cdot J$, which is the classical observable corresponding to the angular momentum in the direction of e. This has Hamiltonian vector field

$$H_{J_e} \colon J \longmapsto e \times J \in T_J S^2.$$

For another normal vector e', the Poisson bracket of J_e and $J_{e'}$ is given by

$$\{J_e, J_{e'}\} = J_{e \times e'}.$$

In particular, with respect to the standard basis (x, y, z) of $\mathbb{R}^3$ we have the observables $J_x, J_y, J_z \in C^\infty(S^2)$ and

$$\{J_x, J_y\} = J_z \qquad \{J_y, J_z\} = J_x \qquad \{J_z, J_x\} = J_y. \tag{3.10}$$

By Proposition 3.3.2, this system admits a prequantization line bundle if and only if ω is $\hbar$-integral.

We consider spherical coordinates (θ, ϕ) on S^2 and claim ω is given by

$$\omega = j \sin \phi d\theta \wedge d\phi \tag{3.11}$$

in these coordinates. By symmetry, it is enough to show the two forms agree at a point. Take $(j, 0, 0)$, which corresponds to $\theta = 0$ and $\phi = \frac{1}{2}\pi$. We have

$$dy_{(j,0,0)} = j \sin\left(\frac{1}{2}\pi\right) \cos(0) d\theta + j \cos\left(\frac{1}{2}\pi\right) \sin(0) d\phi = j d\theta_{(0, \frac{1}{2}\pi)}$$
$$dz_{(j,0,0)} = j \sin\left(\frac{1}{2}\pi\right) d\phi = j d\phi_{(0, \frac{1}{2}\pi)}$$

and for element $v = (0, v_y, v_z)$ and $w = (0, w_y, w_z)$ of the tangent space at $(1, 0, 0)$, we verify

$$\begin{aligned} \omega_{(j,0,0)}(v, w) &= j \left(j^{-1} dy_{(j,0,0)} \wedge j^{-1} dz_{(j,0,0)}\right)(v, w) \\ &= j^{-1}\left(v_y w_z - v_z w_y\right) \\ &= j^{-2}(j, 0, 0) \cdot (v \times w). \end{aligned}$$

Now ω is $\hbar$-integral if the integral of ω over S^2 is an element of $2\pi\hbar\mathbb{Z}$. We now clearly have

$$\int_{S^2} \omega = 4\pi j,$$

and so we conclude this system is only prequantizable if and only if j is of the form $j = \hbar l/2$ for some integer l. This is in agreement with well-known results in quantum mechanics. Moreover, applying Lemma 3.1.3 to the classical brackets (3.10), we find the expected commutation relations

$$\begin{aligned}
\left[\mathcal{Q}^{\text{pre}}_{J_x}, \mathcal{Q}^{\text{pre}}_{J_y}\right] &= i\hbar \mathcal{Q}^{\text{pre}}_{J_z} \\
\left[\mathcal{Q}^{\text{pre}}_{J_y}, \mathcal{Q}^{\text{pre}}_{J_z}\right] &= i\hbar \mathcal{Q}^{\text{pre}}_{J_x} \\
\left[\mathcal{Q}^{\text{pre}}_{J_z}, \mathcal{Q}^{\text{pre}}_{J_x}\right] &= i\hbar \mathcal{Q}^{\text{pre}}_{J_y}.
\end{aligned}$$

We note that the prequantum Hilbert space $\mathfrak{H}^{\text{pre}}_{S^2}$ we constructed is infinite-dimensional. The correct calculation, such as described in Section 3.5 of [SN17], shows that the quantum Hilbert space has dimension $l + 1$. In the next chapter, we introduce the extra structure that is needed to get the correct dimensionality.

Chapter 4

Quantization

4.1 Polarizations

Comparing the examples in Section 3.4 to the results in any textbook on quantum mechanics, it is obvious that the prequantum Hilbert spaces are 'too big' in some sense. In order to get a quantization theory that agrees with the physical reality, we need to introduce an auxiliary structure on our symplectic manifold (M, ω), called a polarization. Recall that for a completely integrable Hamiltonian system as in Definition 2.1.5, the Hamiltonian flow is actually restricted to the leaves of a Lagrangian foliation. Defining the quantum Hilbert space to consists of sections that are constant on these leaves will give us some difficulties, but eventually lead to a theory of geometric quantization that agrees with physical reality.

First, we note that the existence of a complete set of commuting functions on (M, ω) is actually too restrictive: we only require the local analogue.

Definition 4.1.1. If (M, ω) is a symplectic manifold, a ***real polarization*** is an involutive distribution $\mathcal{P} \subseteq TM$ that is Lagrangian in the sense that for any $p \in M$, $\mathcal{P}_p \subseteq T_pM$ is a Lagrangian subspace.

Due to Frobenius' Theorem (see, for example, [Lee03] Theorem 19.21) , this is the same as a foliation of M by leaves that are Lagrangian subspaces. We will use the two pictures interchangeably and write $M/\mathcal{P}$ for the leaf space. Note that in general, this space may not admit the structure of a smooth manifold.

Example 4.1.2. We note that if (M, ω) is completely integrable with respect to some functions $(f_1, \ldots, f_n)$, the distribution $\mathcal{P}_p := \operatorname{span}\{(X_{f_j})_p\}_{j=1}^n$ is a polarization. The leaves are the level sets of the map $(f_1, \ldots, f_n)\colon M \to \mathbb{R}^n$.

Example 4.1.3. For a cotangent bundle $\pi\colon T^*Q \to Q$, we have a canonical vertical polarization given by $\mathcal{V}_p = \ker d\pi_p$. The leaves are the fibres $\pi^{-1}\{q\}$ for $q \in Q$, and the leaf space satisfies $T^*Q/\mathcal{V} \cong Q$.

It turns out the definition of a real polarization is still too restrictive. In general, we will consider polarizations to be distributions in the complexified tangent space

Definition 4.1.4. If (M, ω) is a symplectic manifold, a ***polarization*** is an involutive distribution $\mathcal{P} \subseteq T^{\mathbb{C}}M$ that is Lagrangian in the sense that for any $p \in M$, $\mathcal{P}_p \subseteq T_p^{\mathbb{C}}M$ is a Lagrangian subspace with respect to the complex-linear extension of ω_p, and such that $\mathcal{P} \cap \overline{\mathcal{P}}$ is of constant rank. The polarization is called ***completely real*** if $\mathcal{P} = \overline{\mathcal{P}}$ holds and ***completely complex*** if $\mathcal{P} \cap \overline{\mathcal{P}} = \{0\}$ holds.

It is obvious that the real polarizations defined above correspond with the completely real polarizations in this more general context. The advantage of this more general picture will become clear in Section 4.3, when we discuss the Kähler polarization on a Kähler manifold.

We want the quantum space of a symplectic manifold with polarization to consist of the sections of the prequantum line bundle which are horizontal in the directions of the polarization.

Definition 4.1.5. If (M, ω) is a symplectic manifold with polarization $\mathcal{P}$, and $E \to M$ is a complex vector bundle with connection ∇, a section $\psi \in \Gamma(E)$ is called ***polarized*** if

$$\nabla_X \psi = 0$$

for every local section $X \in \Gamma(U; \mathcal{P})$. The algebra of polarized sections is denoted by $\Gamma_{\mathcal{P}}(E)$.

Note that by abuse of notation, we also use the symbol ∇ for the complex-linear extension $\mathfrak{X}^{\mathbb{C}}(M) \times \Gamma(E) \to \Gamma(E)$.

Example 4.1.6. It is clear that for a cotangent bundle T^*Q with the vertical polarization, the polarized sections are precisely the sections that are constant on the fibres of π.

In general, the prequantization of a smooth function does not respect this additional structure. Therefore we restrict the class of functions we may quantize.

Definition 4.1.7. If (M, ω) is a symplectic manifold with polarization $\mathcal{P}$, a smooth function $f \in C^\infty(M; \mathbb{C})$ is called ***polarization-preserving*** if the Hamiltonian vector field satisfies

$$[X_f, X] \in \Gamma(\mathcal{P})$$

for every section $X \in \Gamma(\mathcal{P})$, and the space of polarization-preserving functions is denoted by $C^\infty_{\mathcal{P}}(M; \mathbb{C})$.

By involutivity of the polarization $\mathcal{P}$, if $X_f \in \Gamma(\mathcal{P})$ then we must have that f is polarization-preserving. In particular, the constant function 1 has $X_1 = 0 \in \Gamma(\mathcal{P})$ and so is polarization-preserving. The space of polarization-preserving functions is a Lie subalgebra of $C^\infty(M; \mathbb{C})$ due to the following lemma:

Lemma 4.1.8. *If $f, g \in C^\infty_{\mathcal{P}}(M; \mathbb{C})$ are polarization-preserving functions, then so is their Poisson bracket $\{f, g\}$.*

Proof. Let $f, g \in C^\infty_{\mathcal{P}}(M; \mathbb{C})$ and $X \in \Gamma(\mathcal{P})$. We have

$$\begin{aligned} [X_{\{f,g\}}, X] &= [[X_g, X_f], X] \\ &= -[[X_f, X], X_g] - [[X, X_g], X_f] \\ &= [X_g, [X_f, X]] - [X_f, [X_g, X]]. \end{aligned}$$

and since f and g are polarization-preserving, both terms are in $\Gamma(\mathcal{P})$. □

This space then has a representation on the space of polarized section.

Lemma 4.1.9. *If (M, ω) is a symplectic manifold with prequantum line bundle E and polarization $\mathcal{P}$, $f \in C^\infty_{\mathcal{P}}(M; \mathbb{C})$ is a polarization-preserving function and $\psi \in \Gamma_{\mathcal{P}}(E)$ is a polarized section, then*

$$\mathcal{Q}^{\text{pre}}_f \psi \in \Gamma_{\mathcal{P}}(E).$$

Here, the quantization of a complex-valued function is defined by imposing complex linearity. It should be noted that the quantization of a complex-valued function needs not be Hermitian.

From the lemma we learn that the prequantum map is well-defined at the level of polarized sections. However for non-compact manifolds the definition of the quantum Hilbert space is still problematic.

Example 4.1.10. If Q is a manifold of positive dimension and we consider the phase space T^*Q with the vertical polarization, there are no non-zero square-integrable polarized section.

We need to go back to the drawing board and define a new Hilbert space consisting of polarized sections. In Section 4.2 we do this for a completely real polarization and in Section 4.3 for a Kähler polarization.

Even then, we only get a representation of the polarization-preserving functions. If the classical observable one wants to study does not preserve the polarization, it is sometimes possible to find another polarization which is preserved. If one is lucky, it is even possible to write down an isometric isomorphism between the two quantum Hilbert spaces associated to the system—but that is beyond the scope of this thesis. Clearly, the original goal to describe quantization as a representation of the Lie algebra $C^\infty(M)$ was too ambitious, since it turns out that not every classical observable has a quantum-mechanical counterpart.

4.2 Quantization of real polarization

Here we follow the approach of [Ler]. By considering Definition 4.1.1 in light of Frobenius' theorem, we see that a real polarization is the same as a foliation of M with Lagrangian leaves, and a section is polarized if and only if it is constant along the leaves. For systems such as cotangent bundles these leaves are non-compact, and so the only square-integrable polarized section will be the zero section.

A better strategy is to consider the leaf space $M/\mathcal{P}$. If it is a manifold and parallel transport is uniquely defined on the leaves, the prequantum line bundle pulls back to a line bundle over the leaf space. This is due to the following lemma:

Lemma 4.2.1. *If (M, ω) is a symplectic manifold and $\pi\colon E \to M$ is a prequantum line bundle with Hermitian connection ∇, and $\mathcal{P}$ is a real polarization of M by simply connected leaves such that $N := M/\mathcal{P}$ has the structure of a smooth manifold for which $M \to N$ is a submersion, then E quotients out to a bundle $E_{\mathcal{P}} \to N$ such that the sections of $E_{\mathcal{P}}$ correspond bijectively with the polarized sections of E.*

Proof. We define an equivalence relation on E by setting $e \sim e'$ if and only if there exists a leaf L of $\mathcal{P}$ and a piecewise smooth path $\gamma\colon [0,1] \to L$ such that $\pi(e) = \gamma(0)$, $\pi(e') = \gamma(1)$ and e' is the parallel transport of e along γ.

We show any equivalence class intersects a fibre of π at most once. Assume $e \sim e'$ with $\pi(e) = \pi(e') \in L$, then per definition there must be a loop γ in L such that the holonomy map $P_\gamma \in \mathrm{Aut}(E_{\pi(e)})$ has $P_\gamma(e) = e'$. Since L is simply connected such a loop must be path-homotopic to the constant loop, so that we find P_γ to be an element of the restricted holonomy group of $E|_L$ at $\pi(e)$. Since L is Lagrangian, the curvature $F(\nabla) = i\hbar\omega$ vanishes on L, and due to the Ambrose-Singer theorem ([AS53], Theorem 2) the restricted holonomy group then is trivial, so that $P_\gamma = I$ and $e = e'$.

As a consequence, for any two $l, l' \in L$ we get a well-defined isomorphism $E_l \xrightarrow{\sim} E_{l'}$ by parallel transport along any path from l to l' which agrees with the identification $\sim$.

Since $x \sim x'$ implies $\pi(x)$ and $\pi(x')$ lie in the same leaf of $\mathcal{P}$, the bundle map $\pi\colon E \to M$ quotients out to a map $\pi_{\mathcal{P}}\colon E_{\mathcal{P}} \to M/\mathcal{P}$ where we write $E_{\mathcal{P}} := E/{\sim}$. Due to the observation above the fibres of this map are again isomorphic to the model fiber $\mathbb{C}$ of E.

Now we need to show it is again a smooth bundle. Let $\mathcal{U} = (U_j)_{j\in I}$ be an open cover of M that admits local trivializations $\psi_j\colon U_j \to E$ and charts $\phi_j\colon U_j \to \mathbb{R}^{2n}$ with image the open cube $(-1,1)^{2n}$ that are flat for $\mathcal{P}$. Due to the constant rank level set theorem ([Lee03], Theorem 5.12) and the assumption that the map $M \to M/\mathcal{P}$ is a smooth submersion, we find the leaves are properly embedded in M so that we may choose the open cover in such a way that for any leaf L and any $U_j \in \mathcal{U}$ the intersection $L \cap U_j$ has at most one connected component. In coordinates, such an intersection then must be of the form

$$\phi_j(U_j \cap L) = \left\{(x^1, \ldots, x^n, y^1, \ldots, y^n) \in (-1,1)^{2n} \;\middle|\; y^k = q^k \text{ for all } 1 \le k \le n\right\}$$

for fixed $q^1, \ldots, q^n \in (-1,1)$. Then the $V_j := \{L \in M/\mathcal{P} \mid L \cap U_j \neq \emptyset\}$ with coordinate map $L \mapsto (q^1, \ldots, q^n)$ form an open cover by coordinate charts of $M/\mathcal{P}$, and in terms of these coordinates

$$\begin{aligned} \psi_{\mathcal{P},j} \colon V_j &\longrightarrow E_{\mathcal{P}} \\ (q^1, \ldots, q^n) &\longmapsto [\psi_j(0, \ldots, 0, q^1, \ldots, q^n)] \end{aligned}$$

is a nonvanishing local section of $E_{\mathcal{P}}$ which defines a local trivialization. We conclude this is a vector bundle.

For the final claim, note that per definition ψ is polarized if and only if $\nabla_X \psi = 0$ for all $X \in \Gamma(\mathcal{P})$. Since for $l \in L \subseteq M$ we have $T_l L = \mathcal{P}_l \subseteq T_l M$, a section is polarized precisely if on each leaf it is flat with respect to the connection ∇. In terms of our equivalence relation, this happens precisely when $\psi(e) \sim \psi(e')$ whenever e and e' are in the same leaf. Therefore we see a section of E quotients out to a section of $E_{\mathcal{P}}$ precisely if it is polarized. □

In order to define an inner product on the sections of $E_{\mathcal{P}}$, we need to integrate over the leaf space N. However there needs not be a natural top-level form on N, and N may not even be orientable. To circumvent this one could twist the prequantum line bundle by the densities on N, which may be integrated without reference to a top-level form. This approach is described in [Ler], but in this thesis it will suffice to assume N has an orientation form so that we may define quantization as follows.

Definition 4.2.2. If (M, ω, E, ∇) is a symplectic manifold together with a prequantum line bundle, and $\mathcal{P}$ is a real polarization with simply connected leaves such that $N := M/\mathcal{P}$ has the structure of a smooth manifold for which $M \to N$ is a submersion and N admits an orientation form $\eta \in \Omega^n(N)$, the ***quantum Hilbert space*** $\mathfrak{H}_{\mathcal{P}}$ with respect to this polarization is the space of polarized sections $\psi \in \Gamma_{\mathcal{P}}(E)$ which are square-integrable in the sense that

$$\int_N \langle \psi, \psi \rangle \eta < \infty$$

and it is endowed with the inner product

$$\langle\langle \psi_1, \psi_1 \rangle\rangle := \int_N \langle \psi_1, \psi_2 \rangle \eta.$$

The ***quantization map*** then is the restriction of the prequantization map to

$$\mathcal{Q} \colon C^\infty_{\mathcal{P}}(M; \mathbb{R}) \longrightarrow C(\mathfrak{H}_{\mathcal{P}}).$$

We note that quantization is well-defined due to Lemma 4.1.9, and this map satisfies Dirac's quantization conditions (2.4) and (2.5) since the prequantization does.

4.3 Quantization of Kähler polarization

In this section, we follow [GGK02], Section 6.5.

If (M, ω, J) is a Kähler manifold, the complexified tangent space at any $p \in M$ is the direct sum of the $\pm i$-eigenspaces of the operator

$$J_p^{\mathbb{C}} : T_p^{\mathbb{C}} M \to T_p^{\mathbb{C}} M.$$

Recall that this gives rise to the i-eigenbundle $T^{1,0}M$ called the ***holomorphic tangent bundle*** and the $-i$-eigenbundle $T^{0,1}M$ called the ***anti-holomorphic tangent bundle*** on M. Since ω is of type $(1,1)$, the complex-linear extension of ω vanishes on both of these bundles which therefore are totally complex polarizations. In this context, the anti-holomorphic tangent bundle is called the ***Kähler polarization***.

The polarized sections of a holomorphic vector bundle then are simply the holomorphic sections. In contrast to the smooth case, local holomorphic sections do not necessarily extend to global holomorphic sections. It is therefore customary to use the language of sheaves, and write $\mathcal{O}_E(U)$ or $\Gamma(U; \mathcal{O}_E)$ for the local holomorphic sections $\Gamma_{\mathcal{P}}(U; E)$ on $U \subseteq M$ open.

The quantum Hilbert space with respect to this polarization will simply consist of the square-integrable holomorphic sections.

Not every Kähler manifold admits a holomorphic prequantum line bundle and similar to the symplectic case in Proposition 3.3.2, the only obstruction is the cohomology class of the Kähler form.

Proposition 4.3.1. *If (M, ω) is a Kähler manifold such that ω is $\hbar$-integral, then it admits a holomorphic prequantum line bundle.*

We follow the proof given in Section V.4 of [Wei58], which relies on the following lemma from Section II.3:

Lemma 4.3.2. *If M is a complex manifold that satisfies $H^1_{\mathrm{dR}}(M) = 0$, and $F \in C^\infty(M; \mathbb{R})$ is a smooth real-valued function that satisfies $\partial\bar{\partial}F = 0$, then*

F is the real part of a holomorphic function on M which is determined by this condition up to an imaginary constant.

Proof. We have $d^{\mathbb{C}}(\partial F) = (\partial + \overline{\partial})\partial F = \partial\overline{\partial} F = 0$. By the vanishing of the first De Rahm cohomology group we find that ∂F is exact, and there must a function $f \in C^\infty(M; \mathbb{C})$ with $\partial F = d^{\mathbb{C}} f = (\partial + \overline{\partial}) f$. It follows that $\partial f = \partial F$ and $\overline{\partial} f = 0$, so f is holomorphic. Moreover, $g := \overline{F} - \overline{f}$ is holomorphic, and $F = f + \overline{g}$. Since F is real-valued we have $\overline{F} = F$ and so $f - g = \overline{f} - \overline{g}$. Setting $h := f - g$, we have that h is real-valued and holomorphic, and hence constant. We also see that $2f - h$ is holomorphic and the real part of this function is

$$\frac{1}{2}\left(2f - h + 2\overline{f} - \overline{h}\right) = f + \overline{f} - \overline{h} = f + \overline{f} - (\overline{f} - \overline{g}) = f + \overline{g} = F.$$

□

Now we are ready to prove the proposition.

Proof of Proposition 4.3.1. As in the proof of Proposition 3.3.2, we give a construction of the Čech cocycle associated with ω, and if this admits components in $2\pi\hbar\mathbb{Z}$ it will give rise to a prequantum line bundle.

Let $\mathcal{U} = \{U_j\}_{j \in I}$ be a good open cover of M. For $j \in I$, we have a Kähler potential $\rho_j \in C^\infty(U_j; \mathbb{R})$ such that $\omega|_{U_j} = \frac{i}{2}\partial\overline{\partial}\rho_j$. Note that on a non-empty intersection U_{j_0,j_1} we have $\partial\overline{\partial}(\rho_{j_1} - \rho_{j_0}) = 0$. Due to Lemma 4.3.2 the difference $\rho_{j_1} - \rho_{j_0}$ can be written as the real part of a holomorphic function on U_{j_0,j_1},

$$\rho_{j_1} - \rho_{j_0} = \frac{1}{2}\left(f_{j_0,j_1} + \overline{f_{j_0,j_1}}\right)$$

with f_{j_0,j_1} holomorphic. From

$$\overline{f_{j_0,j_1}} = 2\rho_{j_1} - 2\rho_{j_0} - f_{j_0,j_1}.$$

it follows that for $U_{j_0,j_1,j_2} \neq 0$ the holomorphic function

$$c_{j_0,j_1,j_2} := f_{j_0,j_1} - f_{j_0,j_2} + f_{j_1,j_2}$$

is imaginary-valued and hence constant on U_{j_0,j_1,j_2}.

We now claim that the real-valued Čech cocycle

$$\left(\frac{i}{4} c_{j_0,j_1,j_2}\right)_{j_0,j_1,j_2}$$

corresponds to ω under the isomorphism defined in Theorem 3.3.3 with both sides tensored by $\mathbb{C}$. This can be seen by setting $\lambda_j := -\frac{i}{2}\partial\rho_j \in \Omega^{1,0}(U_j)$. We then have $d\lambda_j = \omega|_{U_j}$ on U_{j_0} and

$$\lambda_{j_0} - \lambda_{j_1} = -\frac{i}{2}\left(\partial\rho_{j_0} - \partial\rho_{j_1}\right) = \frac{i}{4}\partial\left(f_{j_0,j_1} - \overline{f_{j_0,j_1}}\right) = \frac{i}{4}df_{j_0,j_1}$$

on a non-empty intersection U_{j_0,j_1}.

As in the proof of Proposition 3.3.2, the functions

$$\tau_{j_0,j_1} := \exp\left(\frac{1}{4\hbar}f_{j_0,j_1}\right)$$

are the transition functions of a prequantum line bundle over M if ω is $\hbar$-integral. Now it is clear that, since the f_{j_0,j_1} are holomorphic, these transition functions are as well and therefore the prequantum line bundle they define is holomorphic. □

4.4 Examples

4.4.1 Harmonic oscillator

We consider the classical system of a one-dimensional harmonic oscillators with mass $m \in \mathbb{R}_{>0}$ and spring constants $c \in \mathbb{R}_{>0}$, which we also studied in Section 3.4.1. The phase space $T^*\mathbb{R} = \mathbb{R}^2$ admits canonical vertical and horizontal polarizations,

$$\mathcal{V}_{(q,p)} := \operatorname{span}\{\partial_p\} \qquad \mathcal{H}_{(q,p)} := \operatorname{span}\{\partial_q\}.$$

However, the classical Hamiltonian (3.5) of this system has Hamiltonian vector field

$$X_H = \frac{p}{m}\partial_q - \frac{cq}{m}\partial_p$$

and so we find

$$[X_H, \partial_p] := \frac{p}{m}\partial_q\partial_p - \frac{cq}{m}\partial_p^2 - \partial_p\left(\frac{p}{m}\partial_q\right) + \frac{cq}{m}\partial_p^2 = -\frac{1}{m}\partial_q \notin \Gamma(M; \mathcal{V})$$

which means that H is not $\mathcal{V}$-preserving. By a similar calculation, it is also not $\mathcal{H}$-preserving. These polarizations therefore do not give a quantization of the Hamiltonian system $(\mathbb{R}^2, \omega, H)$.

Looking at the Hamiltonian vector field, however, suggests we consider

$$\mathcal{P}_{(q,p)} := \mathrm{span}\{p\partial_q - cq\partial_p\},$$

which is a polarization away from the zero-energy state $(0,0)$. Now H is clearly polarization-preserving, but the leaves now are not simply connected so that we cannot apply Lemma 4.2.1.

Now we try a complex polarization, following [Car18], Section 4.2. Without loss of generality we assume $c = 1$, by changing units such that the spring constant is one. We identify $\mathbb{R}^2 \cong \mathbb{C}$ using the complex coordinate $z = q - ip$ as in Example 2.1.14. In these coordinates, the Hamiltonian in given by

$$H = \frac{1}{2m} z\bar{z},$$

which has partial derivatives

$$\frac{\partial H}{\partial z} = \frac{1}{2m}\bar{z} \qquad\qquad \frac{\partial H}{\partial \bar{z}} = \frac{1}{2m} z$$

and the Hamiltonian vector field may be expressed as

$$X_H = \frac{i}{m}\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} \right).$$

From this expression it is clear that H preserves the Kähler polarization.

Now we calculate the quantum Hilbert space. We do this first with respect to the symplectic potential

$$\theta = -\frac{i}{2}\bar{z}dz$$

and then show how Fock quantization is recovered by quantization with respect to another symplectic potential.

Since $\omega = -d\theta$, the trivial complex line bundle $\underline{\mathbb{C}}$ together with the connection

$$\nabla^\theta_X \phi = X\phi - i\hbar^{-1}\theta(X)\phi$$

is a prequantum line bundle. From $\theta(\partial_{\bar{z}}) = 0$ it then follows that a section ϕ is polarized precisely if it is holomorphic as a complex-valued function.

Now we note that

$$\begin{aligned}\theta &= -\frac{i}{2}(p - iq)(dp + idq) \\ &= \frac{1}{2}(-ipdp - qdp + pdq - iqdq) \\ &= \frac{1}{2}(pdq - qdp) - d\left(\frac{i}{4}(q^2 + p^2)\right).\end{aligned}$$

Considering quantization with respect to the symplectic potential $\theta_0 = \frac{1}{2}(pdq - qdp)$, we have the connection

$$\nabla_X^{\theta_0}\psi = X\psi - i\hbar^{-1}\theta_0(X)\psi.$$

Since

$$\theta_0(X) = \theta(X) + d\left(\frac{i}{4}(q^2 + p^2)\right)(X) = \theta(X) + \frac{i}{4}X\left(z\bar{z}\right)$$

we have that

$$\nabla_X^{\theta_0} = X - i\hbar^{-1}\theta_0(X) = X - i\hbar^{-1}\theta(X) + \frac{1}{4\hbar}X\left(z\bar{z}\right) = \nabla_X^{\theta} + \frac{1}{4\hbar}X\left(z\bar{z}\right).$$

Hence a function ψ is polarized with respect to the connection ∇^{θ_0} if and only if it is of the form

$$\psi(z) = e^{-z\bar{z}/4\hbar}\phi(z) \tag{4.1}$$

for some ϕ polarized with respect to the connection ∇^{θ}. As we saw, this happens if and only if ϕ is holomorphic.

We see the quantum Hilbert space may be though of as consisting of the holomorphic functions ϕ such that

$$\int_{\mathbb{C}} \phi(z)\overline{\phi(z)}e^{-|z|^2/2\hbar}dz < \infty.$$

This is precisely the Segal-Bargmann space $HL^2(\mathbb{C}, \mu_\hbar)$ as in Definition 2.2.13, which is the Hilbert space of Fock quantization.

The corresponding prequantum line bundle is $(\underline{\mathbb{C}}, \nabla^{\theta_0})$. We find that prequantization is given by

$$\mathcal{Q}_f^{\text{pre}}\psi = -i\hbar X_f\psi - \frac{1}{2}(pdq - qdp)(X_f)\psi + f\psi.$$

Applying the prequantization of H (3.6) to a polarized section we find

$$\begin{aligned}\mathcal{Q}_H^{\text{pre}}\left(e^{-(p^2+q^2)/4\hbar}\phi\right) &= e^{-(p^2+q^2)/4\hbar}\frac{i\hbar}{m}\left(-\frac{qp}{4\hbar}\phi + q\frac{\partial\phi}{\partial p} + \frac{qp}{4\hbar}\phi - p\frac{\partial\phi}{\partial q}\right)\\ &= e^{-(p^2+q^2)/4\hbar}\frac{i\hbar}{m}\left(q\frac{\partial\phi}{\partial p} - p\frac{\partial\phi}{\partial q}\right)\\ &= e^{-(p^2+q^2)/4\hbar}\frac{i\hbar}{m}\left(-iq\left(\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial\bar{z}}\right) - p\left(\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial\bar{z}}\right)\right)\\ &= e^{-(p^2+q^2)/4\hbar}\frac{\hbar}{m}\left((q-ip)\frac{\partial\phi}{\partial z} - (q+ip)\frac{\partial\phi}{\partial\bar{z}}\right)\\ &= e^{-(p^2+q^2)/4\hbar}\frac{\hbar}{m}\left(z\frac{\partial\phi}{\partial z} - \bar{z}\frac{\partial\phi}{\partial\bar{z}}\right)\end{aligned}$$

Since ϕ is necessarily holomorphic, we conclude that the quantization of H acts on the Segal-Bargmann space by

$$\mathcal{Q}_H\phi = \frac{\hbar z}{m}\frac{\partial\phi}{\partial z}.$$

In terms of the number operator defined at the end of Section 2.2.4, we have $\mathcal{Q}_H = \frac{\hbar}{m}N$. Any monomial $\phi = z^k$ with $k \geq 0$ is an eigenvector, and the corresponding eigenvalue is $\frac{\hbar}{m}k$. Comparing this with a textbook on quantum mechanics, such as Section 2.3 in [SN17], we verify that we have found the correct eigenvectors.

Unfortunately, the eigenvalues are slightly off. To get the correct eigenvalues, which are $\frac{\hbar}{m}(k+\frac{1}{2})$, one has to consider the so-called metaplectic correction. This is outside the scope of this thesis, but an interested reader might consult Section 23.7 of [Hal13] for more details.

4.4.2 Free particle

As in Section 3.4.2 we consider a free particle with mass $m \in \mathbb{R}_{>0}$ with configuration space given by $Q = \mathbb{R}^n$ with the standard Euclidean inner product. The phase space then is $T^*Q = \mathbb{R}^{2n}$, and as in Example 3.1.2 a prequantum line bundle is given by the trivial line bundle $\underline{\mathbb{C}}$ with connection

$$\nabla_X\psi = X\psi - i\hbar^{-1}\left(\sum_{j=1}^n p_j dq_j\right)(X)\psi.$$

The phase space admits canonical vertical and horizontal polarizations,

$$\mathcal{V}_{(q,p)} := \text{span}\{\partial_{p_1}, \ldots, \partial_{p_n}\} \qquad \mathcal{H}_{(q,p)} := \text{span}\{\partial_{q^1}, \ldots, \partial_{q^n}\}.$$

The Hamiltonian (3.8) has Hamiltonian vector field

$$X_H = \frac{1}{m} p_j \partial_{q^j},$$

and we see that H preserves both of these polarizations.

First we describe the quantization with respect to the polarization $\mathcal{V}$. If $\psi \in \Gamma_{\mathcal{V}}(\underline{\mathbb{C}})$ is a $\mathcal{V}$-polarized section of the prequantum line bundle, we have

$$0 = \nabla_{\partial_{p_j}} \psi = \partial_{p_j} \psi$$

In other words, the polarized sections are just the sections that are independent of the p_j-coordinates. We thus have $\mathfrak{H}_{\mathcal{V}} = L^2(Q) = L^2(\mathbb{R}^n)$. Now applying the results of prequantization we found in (3.9) to a polarized section ψ, we find

$$\begin{aligned} \mathcal{Q}^{\text{pre}}_H \psi &= H\psi \\ \mathcal{Q}^{\text{pre}}_{q^j} \psi &= q^j \psi \\ \mathcal{Q}^{\text{pre}}_{p_j} \psi &= -i\hbar \frac{\partial \psi}{\partial q^j}. \end{aligned}$$

These agree with the operators from canonical quantization (2.7).

We can also describe quantization with respect to the polarization $\mathcal{H}$. If $\psi \in \Gamma_{\mathcal{H}}(\underline{\mathbb{C}})$ is a $\mathcal{H}$-polarized section of the prequantum line bundle, we have

$$0 = \nabla_{\partial_{q^j}} \psi = \partial_{q^j} \psi - i\hbar^{-1} p_j \psi$$

In other words, the polarized sections are of the form

$$\psi(q, p) = e^{i\langle q,p\rangle/\hbar} \phi(p)$$

for $\phi \in C^\infty(\mathbb{R}^n)$. We thus have $\mathfrak{H}_{\mathcal{H}} = L^2(\mathbb{R}^n)$. Comparing this Lemma 4.2.1, we find that the leaf space is identified with $\mathbb{R}^n$.

Applying this to the results (3.9) we find that for a polarized section we have

$$\mathcal{Q}_H^{\text{pre}}\left(e^{i\langle q,p\rangle/\hbar}\phi\right) = e^{i\langle q,p\rangle/\hbar}\left(H\phi\right)$$

$$\mathcal{Q}_{q^j}^{\text{pre}}\left(e^{i\langle q,p\rangle/\hbar}\phi\right) = e^{i\langle q,p\rangle/\hbar}\left(i\hbar\frac{\partial\phi}{\partial p_j}\right)$$

$$\mathcal{Q}_{p_j}^{\text{pre}}\left(e^{i\langle q,p\rangle/\hbar}\phi\right) = e^{i\langle q,p\rangle/\hbar}\left(p_j\phi\right).$$

Since the quantum operators act on the position-independent functions, this is called the momentum representation of the quantum system. It is isomorphic to the position representation obtained from the polarization $\mathcal{V}$.

4.4.3 Spin particle

As in Section 3.4.3, we will consider a spin particle with total angular momentum j, which has phase space S^2 endowed with the symplectic form (3.11). Recall that this system is quantizable if we have $j = \hbar l/2$ for some integer l. We will construct the quantization of this space using the natural Kähler polarization.

The Kähler structure of S^2 comes from the stereographic charts $U_N = S^2 \setminus \{N\} \to \mathbb{C}$ and $U_S = S^2 \setminus \{S\} \to \mathbb{C}$, where N and S denote the north and south pole of S^2, respectively. On U_N the spherical coordinates are related to the coordinates $(z, \bar{z})$ on $\mathbb{C}$ by

$$z = \frac{\sin\phi}{1-\cos\phi}e^{i\theta} \qquad \bar{z} = \frac{\sin\phi}{1-\cos\phi}e^{-i\theta}.$$

A calculation verifies that the two-form

$$\omega_N = i\hbar l\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2}$$

agrees with the symplectic form in (3.11) for $j = \hbar l/2$.

Similarly, on U_S the identification is

$$z' = \frac{\sin\phi}{1+\cos\phi}e^{-i\theta} \qquad \bar{z}' = \frac{\sin\phi}{1+\cos\phi}e^{i\theta}.$$

and in these coordinates, the symplectic form is also expressed as

$$\omega_S = i\hbar l\frac{dz'\wedge d\bar{z}'}{(1+z'\bar{z}')^2}.$$

Now Proposition 4.3.1 allows us to calculate the transition functions of the holomorphic prequantum line bundle. We see the Kähler potentials of the two forms are

$$\rho_N(z) = 2\hbar l \log(1 + z\bar{z}) \qquad \rho_S(z) = 2\hbar l \log(1 + z'\bar{z}').$$

On the intersection U_{NS} the coordinates on U_N and U_S are related by

$$z' = \frac{1}{z} \qquad \bar{z}' = \frac{1}{\bar{z}}.$$

Thus in the $(z, \bar{z})$-coordinates we find

$$\rho_S - \rho_N = 2\hbar l \left(\log\left(1 + \frac{1}{z\bar{z}}\right) - \log(1 + z\bar{z}) \right).$$

This is the real part of the holomorphic function

$$f_{NS} = 2\hbar l \left(\log\left(1 + z^{-2}\right) - \log(1 + z^2) \right)$$

giving us the transition function

$$\tau_{NS}(z) = \exp\left(\frac{l}{2} \left(\log\left(1 + z^{-2}\right) - \log(1 + z^2) \right) \right) = \left(\frac{1 + z^{-2}}{1 + z^2} \right)^{l/2} = z^{-l}.$$

A holomorphic section of the prequantum line bundle then consists of a holomorphic function f on $\mathbb{C}$ identified with U_N such that $\tau_{NS}(z) f(z)$ is also holomorphic.

Let $k \in \mathbb{Z}$, then a monomial $f(z) = z^k$ is holomorphic if $k \geq 0$. Using the transition function, we find

$$\tau_{NS}(z) f(z) = z^{-l} z^k = z^{k-l},$$

so the corresponding smooth function on U_S is the monomial z'^{l-k} which is holomorphic if $k \leq l$. From this, we see that the monomials that correspond to holomorphic sections are z^k with $k \in \{0, 1, \ldots, l\}$. Thus the quantum Hilbert space is of dimension $l + 1$. This agrees with known physics, and an interested reader might consult Section 3.5 of [SN17] to see the physicists perspective on this example.

Chapter 5

Quantization of symplectic toric manifolds

5.1 Delzant's theorem

In Chapter 4 we introduced the notion of polarizations in order to find the quantum Hilbert space associated to a symplectic manifold. However, symplectic manifolds may admit many different polarization. In this chapter, we will specialize to Kähler polarizations that naturally emerge from a symmetry of the symplectic manifold (M, ω) that is given by a Hamiltonian action of a Lie group G. In Definition 2.1.7, we saw that a manifold admitting such an action and a moment map μ is called a Hamiltonian G-space. A nice class of these spaces are those where G is a torus of half the dimension of M, which leads to the following definition.

Definition 5.1.1. A ***symplectic toric manifold*** is a Hamiltonian torus space $(M, \omega, T^n, \phi, \mu)$ where $T^n = (S^1)^n$ is the n-torus, M is a connected smooth manifold of dimension $2n$ and the action ϕ is effective in the sense that any $t \in T^n$ acts non-trivially on M.

In many sources symplectic toric manifolds are also assumed to be compact. Since many of the symplectic manifolds corresponding to physical systems are non-compact, we will work in a more general context.

We note that the Lie algebra of T^n is simply $\mathbb{R}^n$, and after identifying $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ we consider the moment map to be a map $\mu\colon M \to \mathbb{R}^n$. Since the adjoint action of T^n on $\mathbb{R}^n$ is trivial, the equivariance condition on the moment map becomes the condition that μ be T^n-invariant.

If M is a compact symplectic toric manifold, the quotient of the moment map is always an embedding and the image of this map turns out to be a polytope and a theorem by Delzant asserts that these polytopes characterize the compact symplectic toric manifolds up to equivariant symplectomorphism. We will first discuss polytopes, using [Zie07] as a reference, and then discuss this correspondence.

Definition 5.1.2. A subset $\Delta \subseteq \mathbb{R}^n$ is called a ***polytope*** if it is the convex hull of a finite number of points in $\mathbb{R}^n$ and is called a ***polyhedron*** if it is the intersection of a finite number of affine closed half-spaces.

We recall that an affine closed half-space is a space of the form

$$H(v,c) = \{x \in \mathbb{R}^n \mid \langle v, x\rangle \leq c\}$$

for $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

The two concepts are closely related, as shown by the following theorem.

Theorem 5.1.3 (Weyl-Minkowski)**.** *A subset $\Delta \subseteq \mathbb{R}^n$ is a polytope if and only if it is a bounded polyhedron.*

Definition 5.1.4. If $\Delta \subseteq \mathbb{R}^n$ is a polyhedron, a ***face*** of Δ is a set of the form

$$F = \{x \in P \mid \langle v, x\rangle = c\},$$

where $v \in \mathbb{R}^n$, $c \in \mathbb{R}$ are such that for all $x \in \Delta$ we have $\langle v, x\rangle \leq c$. The vector v is called an ***outward-pointing normal*** of F. A face is called a ***vertex*** if it is of dimension zero, an ***edge*** if it is of dimension one and a ***facet*** if it is of dimension $(\dim \Delta - 1)$.

Not every polyhedron can be realized as the image of the moment map of a symplectic toric manifold. In order to classify the polyhedra that are the image of a moment map, we introduce the following terminology.

Definition 5.1.5. If $\Delta \subseteq \mathbb{R}^n$ is a polyhedron, it is called ***simple*** if there are exactly n edges meeting at any vertex, ***rational*** if any edge e meeting a vertex $*$ is of the form $\{* + tu \mid t \geq 0\} \cap \Delta$ for some $u \in \mathbb{Z}^n$ and ***smooth*** if moreover for each vertex $*$ meeting edges $e_1, \ldots, e_n$, the $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Note that if $u_1, \ldots, u_n$ are a $\mathbb{Z}$-basis, they are necessarily primitive elements of $\mathbb{Z}^n$ in the sense that for each j the equation $u_j = nu$ with $n \in \mathbb{Z}$, $u \in \mathbb{Z}^n$ implies $n \in \{\pm 1\}$ since otherwise the element u could not be written as a linear combination of the $\{u_j\}_{1 \le j \le n}$.

Definition 5.1.6. A polytope that is simple, rational and smooth is called a ***Delzant polytope***. A polyhedron that is simple, rational, smooth and has at least one vertex is called a ***Delzant polyhedron***.

It should be noted here that, while 'Delzant polytope' is a term commonly used in the literature, this definition of a 'Delzant polyhedron' is non-standard.

Example 5.1.7. In Figure 5.1 we exhibit some examples of polytopes and polyhedra in $\mathbb{R}^2$. Of these, Figures 5.1a, 5.1b and 5.1c depict Delzant polytopes and Figures 5.1d and 5.1e depict Delzant polyhedra. The polyhedron in Figure 5.1f fails to have at least one vertex, the polytope in Figure 5.1g is not smooth since the primitive vectors corresponding to the edges that meet $(0,0)$ do not span $\mathbb{Z}^2$, and the polytope in Figure 5.1h is not rational since one of its edges is of the form $\{t \cdot (1, \sqrt{3}) \mid t \in \mathbb{R}\} \cap \Delta$.

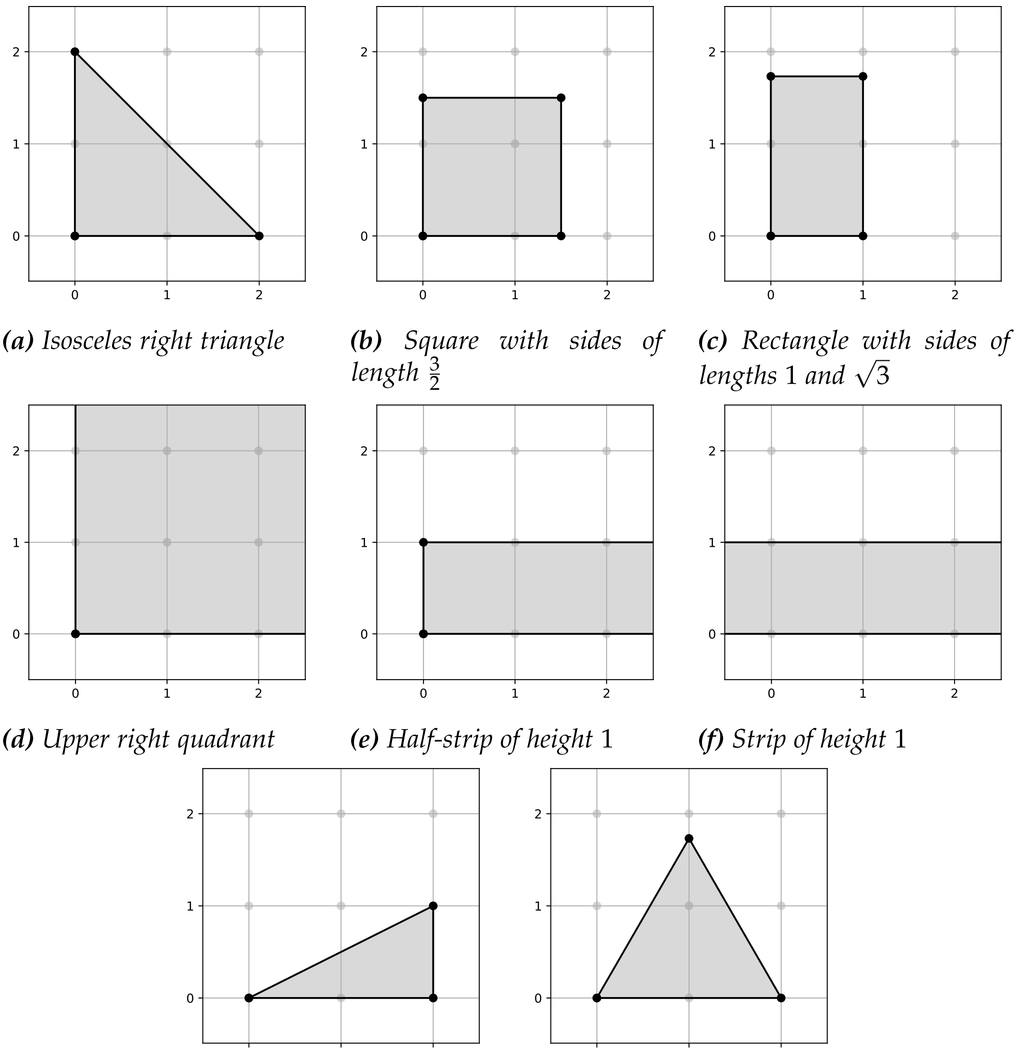
Now we are ready to discuss the classification theorem for compact symplectic toric manifolds due to Delzant [Del88].

Theorem 5.1.8 (Delzant)**.** *There is a bijective correspondence of compact symplectic toric manifolds up to equivariant symplectomorphism and Delzant polytopes, given by*

$$(M, \omega, T^n, \phi, \mu) \longmapsto \mu(M).$$

The proof of this theorem relies on the construction of a symplectic toric manifold M_Δ from a Delzant polytope Δ, and then showing this construction is the inverse of the operation of taking the image of the moment map. We do not provide a full proof here but we will work out the construction of M_Δ in the more general setting of the following theorem due to Yael Karshon and Eugene Lerman ([KL15], Theorem 6.7).

Theorem 5.1.9. *A symplectic toric manifold $(M, \omega, T^n, \phi, \mu)$ is isomorphic to a regular symplectic quotient of $\mathbb{C}^d$ by a subtorus of T^d if and only if the quotient of the moment map by the torus action $\overline{\mu}\colon M/T^n \to \mathbb{R}^n$ is an embedding and its image is a Delzant polyhedron with at most d facets.*



(a) *Isosceles right triangle*

(b) *Square with sides of length* $\frac{3}{2}$

(c) *Rectangle with sides of lengths* 1 *and* $\sqrt{3}$

(d) *Upper right quadrant*

(e) *Half-strip of height* 1

(f) *Strip of height* 1

(g) *30-60-90 right triangle*

(h) *Equilateral triangle*

Figure 5.1: *A few examples of polyhedra in* $\mathbb{R}^2$.

Remark 5.1.10. Not every (non-compact) symplectic toric manifold is of this form, and accordingly $\overline{\mu}$ may in general not be an embedding and its image may not be convex.

We exhibit the construction of M_Δ from a Delzant polyhedron Δ, based on Delzants construction as presented in [Del88], Section 2. This construction is also spelled out in [Can01], Section 28 and Section 29.

Construction for Theorem 5.1.9. Without loss of generality, we assume Δ has precisely d facets.

Fix a vertex $*$ of Δ and let $F_1, \ldots, F_d$ denote the facets of Δ, ordered such that $F_1, \ldots, F_d$ are the facets meeting $*$. By smoothness of Δ, we may choose primitive outward-pointing normals $v_1, \ldots, v_d \in \mathbb{Z}^n$ such that $v_1, \ldots, v_n$ are a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Let $c_1, \ldots, c_d \in \mathbb{R}$ be the corresponding constants such that

$$\Delta = \bigcup_{j=1}^{d} H(v_j, c_j).$$

Let $e_1, \ldots, e_d \in \mathbb{Z}^d$ denote the standard basis. Sending e_j to v_j defines a surjective linear map

$$\mathbb{R}^d \longrightarrow \mathbb{R}^n \tag{5.1}$$

which maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$ and thus quotients to a map

$$\phi\colon T^d \longrightarrow T^n$$

of tori. We write $N := \ker \phi$, which is a $(d-n)$-dimensional Lie subgroup of T^d. This gives an exact sequence of Lie groups

$$1 \longrightarrow N \xrightarrow{\iota} T^d \xrightarrow{\phi} T^n \longrightarrow 1 \tag{5.2}$$

which induces an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{d\iota_e} \mathbb{R}^d \xrightarrow{d\phi_e} \mathbb{R}^n \longrightarrow 0, \tag{5.3}$$

where $d\phi_e\colon \mathbb{R}^d \to \mathbb{R}^n$ is the same as the map (5.1). Dually this becomes

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{(d\phi_e)^*} (\mathbb{R}^d)^* \xrightarrow{L} \mathfrak{n}^* \longrightarrow 0,$$

where we write L for the map $(d\iota_e)^*$ to ease notation.

Now we consider the complex space $\mathbb{C}^d$ with symplectic form ω_0 as in (2.3) and T^d-action

$$(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d}) \cdot (z_1, \ldots, z_d) = (e^{2\pi i t_1} z_1, \ldots, e^{2\pi i t_d} z_d), \tag{5.4}$$

where we choose the moment map

$$\nu\colon (z_1, \ldots, z_d) \longmapsto \frac{1}{2}\left(|z_1|^2, \ldots, |z_d|^2\right) - (c_1, \ldots, c_d). \tag{5.5}$$

Then N acts on $\mathbb{C}^d$ by restriction, and this action is Hamiltonian with moment map $L \circ \nu\colon \mathbb{C}^d \to \mathfrak{n}$. We write $Z := (L \circ \nu)^{-1}(0)$ with embedding $\iota\colon Z \to \mathbb{C}^d$.

We show the action of N on Z is free. Let $z \in Z$ and let $I_z = \{j_1, \ldots, j_r\}$ be the vanishing index-set, defined such that $z_j = 0 \Leftrightarrow j \in I_z$. Then the intersection $\bigcup_{k=1}^r F_{j_k}$ of the corresponding facets of Δ is non-empty, since its image under $(d\phi_e)^*$ contains $\nu(z)$. Let a be a vertex of Δ that meets these faces, and let $F_{j_{r+1}}, \ldots, F_{j_n}$ be the other faces that meet a. Then the primitive outward-pointing normals $v_{j_1}, \ldots, v_{j_n}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Now let $j_{n+1}, \ldots, j_d$ be the remaining indices, so that in this renumbered system of indices the map (5.1) takes the form

$$[d\phi_e]^{\mathcal{E}}_{\mathcal{V}} = \left(\begin{array}{c|ccc} & | & & | \\ \mathrm{I}_n & v_{j_{n+1}} & \cdots & v_{j_d} \\ & | & & | \end{array}\right)$$

with respect to the bases $\mathcal{E} = (e_{j_1}, \ldots, e_{j_d})$ of $\mathbb{R}^d$ and $\mathcal{V} = (v_{j_1}, \ldots, v_{j_n})$ of $\mathbb{R}^n$. Any nontrivial element of N is of the form $(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d})$ for some nonzero element $t \in \mathbb{R}^d$ of the kernel the map (5.1). Assume $t \in [0, 1)$, and note that from linear algebra it is clear that such a t must have at least one nonvanishing component $t_{j_k} \neq 0$ for some $n < k \leq d$. Per assumption we also have $z_{j_k} \neq 0$, and so $(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d}) \cdot z \neq z$. From this we conclude the stabilizer of z for the N-action is trivial, and since this holds for any $z \in Z$ we conclude that N acts freely on Z.

By Theorem 2.1.8, there exists a quotient symplectic manifold Z/N, denoted by $(M_\Delta, \omega_\Delta)$, with projection $p\colon Z \to M_\Delta$ such that

$$p^*\omega_\Delta = \iota^*\omega_0 \tag{5.6}$$

Finally construct the T^n-action on M_Δ. The short exact sequence (5.3) is split and so we write $\mathbb{R}^d = \mathfrak{n} \oplus \mathbb{R}^n$. Now (5.2) becomes

$$1 \longrightarrow N \longrightarrow N \oplus T^n \longrightarrow T^n \longrightarrow 1.$$

The torus T^n thus acts on $\mathbb{C}^d$ by restriction of (5.4). It is clear the actions of N and T^d commute, and so the T^n-action descends to M_Δ.

The moment map $\nu\colon \mathbb{C}^d \to \mathbb{R}^d = \mathfrak{n} \oplus \mathbb{R}^n$ now decomposes as $\nu = \nu_{\mathfrak{n}} \oplus \nu_{\mathbb{R}^n}$. Since $\nu_{\mathbb{R}^n} \circ \iota$ is N-invariant, it quotients out to the moment map $\mu_\Delta\colon M_\Delta \to \mathbb{R}^n$. We have

$$\operatorname{im} \mu_\Delta = \operatorname{im}(\mu_\Delta \circ p) = \operatorname{im}(\nu_{\mathbb{R}^n} \circ \iota) = \nu_{\mathbb{R}^n}(Z) = \Delta.$$

□

Remark 5.1.11. From the discussion about the N-stabilizer at a point $p \in Z$, it follows that the stabilizer of a point $p \in Z$ for the T^d-action has infinitesimal generators corresponding to the indices $j_1, \ldots, j_r$. For the quotient, this means that an element $p \in M_\Delta$ has a stabilizer under the T^n-action isomorphic to T^r, where r is the codimension of the minimal face F that contains $\mu_\Delta(p)$ in $\mathbb{R}^n$. Note also that this same subtorus then acts trivially on F.

Example 5.1.12. Let $T > 0$ and consider $\Delta = [0, T] \subseteq \mathbb{R}$ the interval of length T. It is clear that this is a Delzant polytope, and we proceed with the construction of a symplectic toric manifold $(M_\Delta, \omega_\Delta)$ which has Δ as the image of its moment map.

We fix the vertex 0, which itself is a face with primitive outward-pointing normal $v_1 = -1$ and $c_1 = 0$. For the other face we have $v_2 = 1$ and $c_2 = T$, and we verify $\Delta = H(-1, 0) \cap H(1, T)$.

The map $\mathbb{R}^2 \longrightarrow \mathbb{R}^1$ in (5.1) is given by the matrix

$$\begin{pmatrix} -1 & 1 \end{pmatrix},$$

and at the level of tori this is the map

$$\begin{aligned} \phi\colon T^2 &\longrightarrow T^1 \\ (e^{2\pi i t_1}, e^{2\pi i t_2}) &\longmapsto e^{i(t_2 - t_1)}. \end{aligned}$$

Then $N := \ker \pi$ is the diagonal torus $\{(e^{2\pi i t}, e^{2\pi i t}) \mid e^{2\pi i t} \in T^1\} \subseteq T^2$, and $\mathfrak{n} \cong \mathbb{R}$ is included in $\mathbb{R}^2$ by the diagonal map $d\iota_e\colon x \mapsto (x, x)$. The dual of this map is $L\colon (x, y) \mapsto x + y$.

The T^2-action (5.4) on $\mathbb{C}^2$ has moment map (5.5)

$$\nu\colon (z_1, z_2) \longrightarrow \left(\frac{1}{2}|z_1|^2, \frac{1}{2}|z_2|^2 - T\right),$$

and the restriction to the diagonal action of T^1 has moment map $(z_1, z_2) \mapsto \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2 - T$. We then have

$$Z = (L \circ \nu)^{-1}(0) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; |z_1|^2 + |z_2|^2 = 2T \right\}.$$

which is the three-sphere of radius $\sqrt{2T}$. Taking the quotient $Z/T^1 = S^3/S^1$, we find that M_Δ is the two-sphere S^2.

Now we calculate the symplectic form M_Δ. We introduce the complex protective space $\mathbb{CP}^1$ with $\pi \colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ the canonical projection map and $\phi \colon S^2 \xrightarrow{\sim} \mathbb{CP}^1$ the canonical diffeomorphism. This gives a commutative diagram

$$\begin{array}{ccc} S^3_{\sqrt{2T}} & \xrightarrow{\iota} & \mathbb{C}^2 \setminus \{0\} \\ \Big\downarrow p & & \pi \Big\downarrow \\ S^2 & \xrightarrow{\phi} & \mathbb{CP}^1 \end{array}$$

We have the Fubini-Study form ω_{FS} on $\mathbb{CP}^1$, which pulls back to the form

$$\tilde{\omega}_{\mathrm{FS}} := \pi^* \omega_{\mathrm{FS}} = \frac{i}{2|z_1\bar{z}_1 + z_2\bar{z}_2|^2} \left(\bar{z}_2 dz_1 + \bar{z}_1 dz_2 \right) \wedge \left(z_2 d\bar{z}_1 + z_1 d\bar{z}_2 \right).$$

We claim $\iota^*\omega_0 = 2T \cdot \iota^*\tilde{\omega}_{\mathrm{FS}}$, and by symmetry it suffices to check this at a single point. Consider the north pole of S^3, which corresponds to the point $(0, \sqrt{2T}\, i) \in \mathbb{C}^2$. There we have $z_1 = \bar{z}_1 = 0$, so the Fubini-Study form reads

$$\tilde{\omega}_{\mathrm{FS},(0,\sqrt{2T}i)} = \frac{i}{2|z_2\bar{z}_2|^2} z_2\bar{z}_2 dz_1 \wedge d\bar{z}_1 = \frac{i}{4T} dz_1 \wedge d\bar{z}_1.$$

On the tangent space $T_{(0,\sqrt{2T}i)}\iota(S^3)$ the form $dz_2 \wedge d\bar{z}_2$ vanishes, so the standard symplectic form restricted to that space is

$$\omega_0|_{T_{(0,\sqrt{2T}i)}\iota(S^3)} = \frac{i}{2} dz_1 \wedge d\bar{z}_1.$$

From these two assertions, we see $\iota^*\omega_0 = 2T \cdot \iota^*\tilde{\omega}_{\mathrm{FS}}$ holds at the north pole, and thus holds globally. Also note that the Fubini-Study form is related to the standard form on S^2 by

$$\phi^*\omega_{\mathrm{FS}} = \frac{1}{4} \sin\phi d\theta \wedge d\phi.$$

We then claim that the symplectic form of the quotient M_Δ is given by

$$\omega_\Delta = \frac{T}{2} \sin\phi d\theta \wedge d\phi \tag{5.7}$$

and we verify this by the calculation

$$p^*\omega_\Delta = 2T \cdot p^*\phi^*\omega_{\text{FS}} = 2T \cdot \iota^*\pi^*\omega_{\text{FS}} = 2T \cdot \iota^*\tilde{\omega}_{\text{FS}} = \iota^*\omega_0.$$

Example 5.1.13. Let $T > 0$ and consider $\Delta = \mathbb{R}_{>0} \times [0, T] \subseteq \mathbb{R}^2$ the half-strip of height T such as depicted for $T = 1$ in Figure 5.1e. It is easily verified that this is a Delzant polyhedron, and we proceed with the construction of a symplectic toric manifold $(M_\Delta, \omega_\Delta)$ which has Δ as the image of its moment map.

We fix the vertex $(0, 0)$, which meets two faces: the left-hand face with primitive outward-pointing normal $v_1 = (-1, 0)$ and $c_1 = 0$ and the bottom face with $v_2 = (0, -1)$ and $c_2 = 0$. The only other face is the top face with $v_3 = (0, 1)$ and $c_3 = T$, and we verify $\Delta = H\big((0, -1), 0\big) \cap H\big((-1, 0), 0\big) \cap H\big((0, 1), T\big)$.

The map $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ in (5.1) is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

and at the level of tori this is the map

$$\begin{aligned} \phi\colon T^3 &\longrightarrow T^2 \\ (e^{2\pi i t_1}, e^{2\pi i t_2}, e^{2\pi i t_3}) &\longmapsto (e^{-2\pi i t_1}, e^{2\pi i (t_3 - t_2)}). \end{aligned}$$

Then $N := \ker \pi$ is the torus $\{(1, e^{2\pi i t}, e^{2\pi i t}) \mid e^{2\pi i t} \in T^1\} \subseteq T^3$, $\mathfrak{n} \cong \mathbb{R}$ is included in $\mathbb{R}^3$ by the map $d\iota_e\colon x \mapsto (0, x, x)$, which has dual $L\colon (x_1, x_2, x_3) \mapsto x_2 + x_3$.

Writing ν for the moment map of the T^3-action (5.4) on $\mathbb{C}^3$, we find the action of N has moment map (5.5)

$$(z_1, z_2, z_3) \longmapsto L(\nu(z_1, z_2, z_3)) = \frac{1}{2}|z_2|^2 + \frac{1}{2}|z_3|^2 - T.$$

We then have

$$Z = (L \circ \nu)^{-1}(0) = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \;\middle|\; |z_2|^2 + |z_3|^2 = 2T \right\}.$$

which is $\mathbb{C} \times S^3$. Taking the quotient $Z/T^1 = (\mathbb{C} \times S^3)/S^1$, where S^1 acts trivially on $\mathbb{C}$, we find that M_Δ is $\mathbb{R}^2 \times S^2$.

We can now also show that symplectic toric manifolds are examples of completely integrable systems.

Lemma 5.1.14. *If M is a symplectic toric manifold satisfying the conditions in Theorem 5.1.9, then it is a completely integrable system. In particular, if $(e_j)_{1\le j\le n}$ denotes the standard basis of $\mathbb{R}^n$ the components of the moment map $\mu_j\colon p \to \langle \mu(p), e_j\rangle$ are a complete set of commuting functions.*

Proof. Per definition of the moment map we have

$$d\mu_j = \iota_{X_{e_j}}\omega,$$

where X_{e_j} is the fundamental vector field of the Lie algebra element e_j, and so $X_{\mu_j} = X_{e_j}$. These are linearly independent on the dense, open set of points in M with trivial stabilizer. Moreover since the moment map is constant on the orbits of the T^n-action, for $1 \le k \le n$ the component X_{μ_k} satisfies $d\mu_j(X_{\mu_k}) = 0$ and so $\{\mu_k, \mu_j\} = 0$. □

There is another way to construct the manifold M_Δ from the Delzant polyhedron Δ, which gives M_Δ a complex instead of symplectic structure. It turns out that the two structures are compatible, and M_Δ becomes a Kähler manifold. This gives M_Δ a natural Kähler polarization and we will be able to define the quantization of M_Δ with respect to this polarization.

Proposition 5.1.15. *If $\Delta \subseteq \mathbb{R}^n$ is a Delzant polyhedron, then the symplectic toric manifold M_Δ as in the construction for Theorem 5.1.9 admits a complex structure for which the T^n-action is holomorphic.*

For this construction, we follow Section 2.2 of [Ham08], Sections VII.1 and VII.2 of [Aud04] and Section 5.5 of [GGK02].

Proof. Say we have the same data as in the construction for Theorem 5.1.9. In particular, recall that $F_1, \dots, F_d$ are the facets of Δ. We construct the set

$$\Sigma := \left\{ I \subseteq \{1, \dots, d\} \;\middle|\; \bigcap_{j\in I} F_j \neq \emptyset \right\}$$

where, in particular, we do have $\emptyset \in \Sigma$. This set may be thought of as the cone associated with Δ, but we shall not formalize this statement. From this, we construct for any $I \in \Sigma$ the open set

$$U_I := \left\{ z = (z_1, \dots, z_d) \in \mathbb{C}^d \;\middle|\; \forall j \notin I\colon\; z_j \neq 0 \right\},$$

and define

$$U_\Sigma := \bigcup_{I \in \Sigma} U_I.$$

Unpacking the definition, we see a point $z \in \mathbb{C}^d$ is in U_Σ if and only if the vanishing components $z_{j_1}, \ldots, z_{j_k}$ of z correspond to faces $F_{j_1}, \ldots, F_{j_k}$ of Δ that have a nonempty intersection.

We use the notation U_Σ to emphasize that this set only depend on the fan Σ associated to Δ. In particular, it is independent of the choice of the coefficients $c_1, \ldots, c_d$ which determine the position of the facets of Δ. As such the symplectic form ω_Δ may not be recovered from U_Σ.

Now we complexify (5.2) to get

$$1 \longrightarrow N_\mathbb{C} \xrightarrow{\iota} T^d_\mathbb{C} \xrightarrow{\phi} T^n_\mathbb{C} \longrightarrow 1,$$

where, to be explicit, the complex tori are defined by $T^n_\mathbb{C} = (\mathbb{C}^\times)^n$, $T^d_\mathbb{C} = (\mathbb{C}^\times)^d$ and $N_\mathbb{C} = N \cdot \exp(i\mathfrak{n}) \subseteq (\mathbb{C}^\times)^d$.

Recall the set $Z \subseteq \mathbb{C}^d$ from the symplectic construction. We claim that $Z \subseteq U_\Sigma$. Say $p \in Z$ and $I_p \subseteq \{1, \ldots, d\}$ such that $p_j = 0 \Leftrightarrow j \in I$. In showing N acts freely on Z, we already proved that in this situation the intersection $\bigcup_{j \in I_p} F_j$ is nonempty, and hence $I_p \in \Sigma$. Thus we find $p \in U_{I_p} \subseteq U_\Sigma$, and since this holds for any $p \in Z$ this proves our claim.

We also claim that U_Σ is related to the set $Z \subseteq \mathbb{C}^d$ from the Delzant construction by $U_\Sigma = N_\mathbb{C} \cdot Z$, and we refer to Theorem 5.18 in [GGK02] for the proof of this fact.

In this situation we have a diffeomorphism of real manifolds $Z/N \xrightarrow{\sim} U_\Sigma/N_\mathbb{C}$, so that we do indeed have

$$M_\Delta = U_\Sigma/N_\mathbb{C} \tag{5.8}$$

as a manifold. The proof uses G.I.T. quotients and is beyond the scope of this thesis, so for more details we refer to Section 5.4 of [GGK02].

As the quotient of a complex manifold by a proper, free and holomorphic complex Lie group action, $U_\Sigma/N_\mathbb{C}$ itself has the structure of a complex manifold. □

Remark 5.1.16. It is Remark 5.12 in [GGK02] that the symplectic form ω_Δ is compatible with this complex structure, and so M_Δ becomes a Kähler manifold.

Example 5.1.17. We consider the polytope $\Delta = [0, T] \subseteq \mathbb{R}$ for some $T > 0$ from Example 5.1.12. It has faces $F_1 = 0$ and $F_2 = T$ with $F_1 \cap F_2 = \emptyset$, and thus the cone is $\Sigma = \{\emptyset, \{1\}, \{2\}\}$. The corresponding open subsets of $\mathbb{C}^2$ are

$$U_\emptyset = \mathbb{C}^* \times \mathbb{C}^* \qquad U_{\{1\}} = \mathbb{C} \times \mathbb{C}^* \qquad U_{\{2\}} = \mathbb{C}^* \times \mathbb{C},$$

and so $U_\Sigma = \mathbb{C}^2 \setminus \{(0,0)\}$. The complex torus $N_\mathbb{C} = \mathbb{C}^*$ acts diagonally, and we find

$$U_\Sigma / N_\mathbb{C} = \big(\mathbb{C}^2 \setminus \{(0,0)\}\big)/\mathbb{C}^* = \mathbb{CP}^1.$$

As expected, this is a complex manifold diffeomorphic to S^2.

Example 5.1.18. We consider the polyhedron $\Delta = \mathbb{R}_{>0} \times [0, T] \subseteq \mathbb{R}^2$ for some $T > 0$ as in Example 5.1.13. Labelling the left-hand face, bottom face and top face by F_1, F_2 and F_3 respectively, we get the cone $\Sigma = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$. The corresponding open subsets of $\mathbb{C}^3$ are

$$\begin{aligned} U_\emptyset &= \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* & U_{\{1\}} &= \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* & U_{\{2\}} &= \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^* \\ U_{\{3\}} &= \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C} & U_{\{1,2\}} &= \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* & U_{\{1,3\}} &= \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}. \end{aligned}$$

From this we read $U_\Sigma = \mathbb{C}^3 \setminus \{(z_1, 0, 0) \mid z_1 \in \mathbb{C}\}$. The quotient by the complex torus $N_\mathbb{C} = \mathbb{C}^*$ then becomes

$$U_\Sigma / N_\mathbb{C} = \big(\mathbb{C}^3 \setminus \{(z_1, 0, 0) \mid z_1 \in \mathbb{C}\}\big)/\mathbb{C}^* = \mathbb{C} \times \mathbb{CP}^1.$$

5.2 Quantization from the moment polytope

We saw in Section 3.3 that only the $\hbar$-integral symplectic manifold admit prequantum line bundles. It turns out that that for symplectic toric manifolds Weil's integrality condition can easily be read from the moment polytope.

Proposition 5.2.1. *If $\Delta \subseteq \mathbb{R}^n$ is a Delzant polyhedron, then the associated symplectic toric manifold $(M_\Delta, \omega_\Delta)$ as in Theorem 5.1.9 has a $\hbar$-integral symplectic form if and only if the difference between any two vertices is an element of $\hbar\mathbb{Z}^n$.*

Recall that $(M_\Delta, \omega_\Delta)$ admits a prequantum line bundle if and only if ω_Δ is $\hbar$-integral. Therefore, we call a Delzant polyhedron ***quantizable*** if it satisfies the condition in this proposition.

For the proof, we will follow the proof of [Del88], Proposition 3.1.

Proof. First, let us assume that ω_Δ is $\hbar$-integral, that is, it is in the image of the map $H^2(M; 2\pi\hbar\mathbb{Z}) \to H^2_{\mathrm{dR}}(M)$ under the identification of De Rahm cohomology and singular cohomology with real coefficients.

Let $a, b \in \Delta$ be two vertices, and without loss of generality assume they are connected by an edge e of Δ. Since the polyhedron is rational, there exits a primitive vector $u \in \mathbb{Z}^n$ and a $T \in \mathbb{R}_{>0}$ such that $e = \{a + ut \mid t \in [0, T]\}$.

Consider the submanifold $\iota\colon \mu^{-1}(e) \to M_\Delta$. From Remark 5.1.11 it follows that the T^n-action on M_Δ restricts to an effective circle action on $\mu^{-1}(e)$ with moment map $\nu\colon \mu^{-1}(e) \to \mathbb{R}$ defined such that $\mu(p) = a + u\nu(p)$ for $p \in \mu^{-1}(e)$. By Theorem 5.1.8 we must then have $(\mu^{-1}(e), \iota^*\omega) \cong (M_{[0,T]}, \omega_{[0,T]})$. Now from commutativity the diagram

$$\begin{array}{ccc} \check{H}^2(M; 2\pi\hbar\mathbb{Z}) & \xrightarrow{\iota^*} & \check{H}^2(\mu^{-1}(e); 2\pi\hbar\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^2_{\mathrm{dR}}(M) & \xrightarrow{\iota^*} & H^2_{\mathrm{dR}}(\mu^{-1}(e)) \end{array}$$

it follows that $\iota^*\omega$ is a $\hbar$-integral form on $\mu^{-1}(e)$, and so we must have

$$\int_{\mu^{-1}(e)} \iota^*\omega = 2\pi k$$

for some integer $k \in \mathbb{Z}$. Comparing this with (5.7) in the construction of the toric manifold with the line segment as moment polytope, we must have $T = \hbar k$. We see then that a and b differ by $\hbar k u \in \hbar\mathbb{Z}^n$.

The proof of the converse uses Morse theory, which is beyond the scope of this thesis, to show that the second homology group of M_Δ is generated by the inverse images of the edges of Δ. The statement then follows from the same calculation as above. □

Example 5.2.2. Let us here take $\hbar = 1$. Of the Delzant polyhedra in Figure 5.1, those depicted in Figure 5.1a, 5.1d and 5.1e are quantizable since their vertexes are in the lattice $\mathbb{Z}^2$. The polytopes in Figures 5.1b and 5.1c are not quantizable. The polytope in Figure 5.1b is quantizable for $\hbar = \frac{1}{2}$ since its vertexes are in the lattice $\frac{1}{2}\mathbb{Z}^2$, but there is no value for $\hbar$ for which the polytope in Figure 5.1c is quantizable since $\sqrt{3}$ is not rational.

Considering that M_Δ is a Kähler manifold, it is natural to study the Kähler quantization. For this we do require one other ingredient, namely a holomorphic prequantum line bundle over M_Δ. By Proposition 5.2.1 and Proposition 4.3.1, we know that if Δ is quantizable, the symplectic manifold M_Δ must admit a holomorphic prequantum line bundle. We now explicitly construct this bundle using the Delzant construction.

To do this, we first note that we can easily construct prequantum line bundles over $\mathbb{C}^d$. Since M_Δ is constructed in Theorem 5.1.9 as the symplectic reduction of $\mathbb{C}^d$ by N, we may hope that such a bundle with connection gives rise to a bundle and connection over the reduced space M_Δ. In fact for compact manifolds this is an important theorem referred to as 'quantization commutes with reduction'. We will adapt the proof of this theorem as presented in [Ren20] and Section 3 of [GS82] to show it also holds in the situation we are interested in.

We do this as follows: first, we endow the total space of the prequantum line bundle over $\mathbb{C}^d$ with a torus action. If Δ is quantizable we can choose this action such that it interacts nicely with the connection in the prequantum line bundle. Then the prequantum line bundle on $\mathbb{C}^d$ gives rise to a prequantum line bundle over the reduced space M_Δ.

Recall that we have $\mathbb{C}^d \cong \mathbb{R}^{2d}$ with coordinated such that $z_j = q^j - ip_j$. Consider the symplectic potential

$$\theta_0 = \frac{1}{2}\left(p_j dq^j - q^j dp_j\right),$$

which satisfies $\omega_0 = -d\theta_0$. As in Example 3.1.2 this gives rise to a connection ∇^{θ_0} in the trivial line bundle $\underline{\mathbb{C}}$ over $\mathbb{C}^d$ which satisfies $F(\nabla^{\theta_0}) = i\hbar^{-1}\omega_0$. Explicitly, we have

$$\nabla^{\theta_0}_X = X - i\hbar^{-1} m_{\theta_0(X)}.$$

Recall that the torus N acts on $\mathbb{C}^d$ with moment map $\nu\colon \mathbb{C}^d \to \mathfrak{n}^*$ as in (5.5). This gives rise to a representation of the Lie algebra $\mathfrak{n}$ on the space of sections $\Gamma(\underline{\mathbb{C}}) \cong C^\infty(\mathbb{C}^d; \mathbb{C})$. We write $\xi = (\xi_1, \ldots, \xi_d) \in \mathfrak{n}$ for an element of the Lie algebra and $e^{i\xi}$ for the corresponding Lie group element $(e^{i\xi_1}, \ldots, e^{i\xi_d}) \in N$. Then $\mathfrak{n}$ acts on $\Gamma(\underline{\mathbb{C}})$ by

$$\xi \cdot \psi := \nabla^{\theta_0}_{X_\xi}\psi + i\hbar^{-1}\langle \nu, \xi\rangle \psi = X_\xi \psi - i\hbar^{-1}\big(\theta_0(X_\xi) - \langle \nu, \xi\rangle\big)\psi. \qquad (5.9)$$

Here ν is the moment map defined in (5.5) and $X_\xi \in \mathfrak{X}(\mathbb{C}^d)$ denotes the fundamental vector field associated with X_ξ. A calculation verifies

$$X_\xi = \sum_{j=1}^{d} \left(\xi_j p_j \frac{\partial}{\partial q^j} - \xi_j q^j \frac{\partial}{\partial p_j} \right) \tag{5.10a}$$

$$= \sum_{j=1}^{d} \left(i\xi_j z_j \frac{\partial}{\partial z_j} - i\xi_j \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right). \tag{5.10b}$$

Now the connection ∇^{θ_0} is called N-invariant if there is some N-action on the total space $\mathbb{C}^d \times \mathbb{C}$, lifting the action on $\mathbb{C}^d$, such that (5.9) is the induced infinitesimal action on the sections.

Let us consider the action of T^d on $\mathbb{C}$ with weights $k_1, \ldots, k_d \in \mathbb{Z}$, which is explicitly given by

$$\left(e^{i\xi_1}, \ldots, e^{i\xi_d}\right) \cdot z := e^{ik_1\xi_1 + \cdots + ik_d\xi_d} z.$$

This restricts to an action on N on $\mathbb{C}$, and makes $\underline{\mathbb{C}}$ into an equivariant N-bundle with respect to the N-action om the total space given by

$$\left(e^{i\xi_1}, \ldots, e^{i\xi_d}\right) \cdot \big((z_1, \ldots, z_d), z\big) = \Big(\left(e^{i\xi_1} z_1, \ldots, e^{i\xi_d} z_d\right), e^{ik_1\xi_1 + \cdots + ik_d\xi_d} z\Big).$$

At the level of sections, which we here identify with the smooth complex valued function, the action of $e^{i\xi} \in N$ on $\psi \in C^\infty(\mathbb{C}^d, \mathbb{C})$ gives the section

$$z \longmapsto e^{-i\xi} \cdot \psi(e^{i\xi} \cdot z)$$

Taking the derivative of the action $N \to \mathrm{GL}(\Gamma(\underline{\mathbb{C}}))$ gives an action of the Lie algebra $\mathfrak{n}$ on $\Gamma(\underline{\mathbb{C}})$. We then have

$$\begin{aligned}
(\xi \cdot \psi)(z) &= \frac{d}{dt}\bigg|_{t=0} e^{-it\xi} \cdot \psi(e^{it\xi} \cdot z) \\
&= \frac{d}{dt}\bigg|_{t=0} e^{-itk_1\xi_1 - \cdots - itk_d\xi_d} \cdot \psi(e^{it\xi_1} z_1, \ldots, e^{it\xi_d} z_d) \\
&= (X_\xi \psi)(z) - i(k_1\xi_1 + \cdots + k_d\xi_d)\psi(z)
\end{aligned}$$

Comparing with (5.9), we see that the condition for ∇^{θ_0} to be N-invariant is the existence of integers $k_1, \ldots, k_d$ such that

$$\theta_0(X_\xi) - \langle \nu, \xi \rangle = \hbar(k_1\xi_1 + \cdots + k_d\xi_d)$$

Applying θ_0 to the fundamental vector field (5.10a), we find

$$\theta_0(X_\xi) = \frac{1}{2}\sum_{j=1}^{d} \xi_j \left(p_j^2 + (q^j)^2\right) = \frac{1}{2}\sum_{j=1}^{d} \xi_j |z_j|^2 = \left\langle \frac{1}{2}(|z_1|^2, \dots, |z_d|^2), \xi \right\rangle.$$

Comparing this with (5.5) we see that if the connection ∇^{θ_0} is N-invariant, we must have $c_j \in \hbar\mathbb{Z}$ and $k_j = \hbar^{-1} c_j$ for all $1 \leq j \leq d$. Note that this is precisely the condition for Δ to be quantizable due to Proposition 5.2.1.

We have constructed the N-action for with ∇^{θ_0} is N-invariant. The following theorem tells us that now the prequantum line bundle $(\underline{\mathbb{C}}, \nabla^{\theta_0})$ descents to a prequantum line bundle on M_Δ.

Theorem 5.2.3. *If the polyhedron*

$$\Delta = \bigcup_{j=1}^{d} H(v_j, c_j) \subseteq \mathbb{R}^n.$$

is Delzant and quantizable with $c_j \in \hbar\mathbb{Z}^n$, let us denote by E_0 the equivariant trivial complex line bundle over $\mathbb{C}^d$ where N acts on the fibre with weights $(\hbar^{-1}c_1, \dots, \hbar^{-1}c_d)$. Let us also consider the maps $\iota\colon Z \to \mathbb{C}^d$ and $p\colon Z \to M_\Delta$ as in the construction for Theorem 5.1.9. If ∇^{θ_0} is a N-invariant connection in E_0, then there exists a unique complex line bundle with connection $(E_\Delta, \nabla^{\Delta,\theta_0})$ over M_Δ such that $p^*E_\Delta = \iota^*E_0$ and $p^*\nabla^{\Delta,\theta_0} = \iota^*\nabla^{\theta_0}$.

Proof. Since vector bundles are defined by their global sections, there is a unique bundle E_Δ with global sections corresponding to the N-equivariant sections of ι^*E_0.

Recall that a connection in a vector bundle $E \to M$ is a $C^\infty(M)$-linear map

$$\Gamma(E) \longrightarrow \Omega^1(M; E).$$

Moreover, recall that the quotient map p induces isomorphism

$$p^*\colon \Omega^*(M_\Delta; E_\Delta) \xrightarrow{\sim} \Omega^*_{\mathrm{bas}}(Z; \iota^*E).$$

From these facts, it follows that if $(\iota^*\nabla^{\theta_0})(p^*\psi)$ is a basic one-form for any $\psi \in \Gamma(E_\Delta)$, the map

$$\nabla^{\Delta,\theta_0}\colon \psi \longmapsto (p^*)^{-1}\big((\iota^*\nabla^{\theta_0})(p^*\psi)\big)$$

is a well-defined connection in E_Δ. It is clear that $(\iota^*\nabla^{\theta_0})(p^*\psi)$ is G-invariant. To show it is horizontal, note that for any $\xi \in \mathfrak{n}$ we have $\xi \cdot p^*\psi = 0$ since $p^*\psi$ is G-invariant. Since the moment map ν vanishes on Z, (5.9) implies that $\nabla^{\theta_0}_{X_\xi} p^*\psi = 0$. From this we see that $(\iota^*\nabla^{\theta_0})(p^*\psi)$ is basic, which finishes the proof. □

In fact, this is a holomorphic vector bundle. In light of Proposition 5.1.15 the complex structure of M_Δ arises from the quotient $M_\Delta = U_\Sigma / N_\mathbb{C}$. Let the complexified torus $T^d_\mathbb{C}$ act on $\mathbb{C}$ with weights $k_1, \dots, k_n \in \mathbb{Z}$ by

$$(w_1, \dots, w_d) \cdot z = {w_1}^{c_1} \cdots {w_d}^{c_d} z.$$

This restricts to an action of $N_\mathbb{C}$ on $\mathbb{C}$, and this makes $\underline{\mathbb{C}}$ into an equivariant $N_\mathbb{C}$-bundle with respect to the $N_\mathbb{C}$-action om the total space given by

$$(w_1, \dots, w_d) \cdot \big((z_1, \dots, z_d), z\big) = \big((w_1 z_1, \dots, w_d z_d), {w_1}^{c_1} \cdots {w_d}^{c_d} z\big).$$

The quotient of this bundle by $N_\mathbb{C}$ is a holomorphic line bundle on M_Δ, which is isomorphic to E_Δ as a vector bundle. This gives E_Δ a holomorphic structure.

Corollary 5.2.4. $(E_\Delta, \nabla^{\Delta,\theta_0})$ *is a holomorphic prequantum line bundle on* M_Δ*.*

Proof. Since we are considering connections in line bundles, the local curvature forms are just the exterior derivative of the local connection forms. Thus in this setting curvature commutes with pullback. Hence

$$p^* F(\nabla^{\Delta,\theta_0}) = \iota^* F(\nabla^{\theta_0}) = i\hbar^{-1} \iota^* \omega_0 = i\hbar^{-1} p^* \omega_\Delta,$$

where the last equality is (5.6). It then follows that $F(\nabla^{\Delta,\theta_0}) = i\hbar^{-1}\omega_\Delta$, and we find this is a prequantum line bundle. □

Since M_Δ is a Kähler manifold, it is natural to consider quantization with respect to the Kähler polarization. Recall that this is most readily defined in terms of the symplectic potential that emerges as the holomorphic exterior derivative of the Kähler form. From Example 2.1.14 we recall that on $\mathbb{C}^d$ this is the form

$$\theta = -\frac{i}{2} \sum_{j=1}^{n} \bar{z}_j dz_j.$$

Comparing this with the fundamental vector field X_ξ form (5.10b), we find that just as for θ_0 we have

$$\theta(X_\xi) = \frac{1}{2}\sum_{j=1}^{d} \xi_j z_j \bar{z}_j = \frac{1}{2}\sum_{j=1}^{d} \xi_j |z_j|^2 = \left\langle \frac{1}{2}(|z_1|^2, \dots, |z_d|^2), \xi \right\rangle.$$

Following the same argument as above, we see that the prequantum line bundle over $\mathbb{C}^d$ with connection ∇^θ also defines a prequantum line bundle over M_Δ with connection denoted by $\nabla^{\Delta,\theta}$.

We are then ready to state a theorem that shows that for a symplectic toric variety, the quantization is given by the points of integer coefficients in the moment polyhedron. The slightly more restricted version for compact symplectic toric manifolds is found in [Ham08].

Theorem 5.2.5. *If $(M, \omega, T^n, \phi, \mu)$ is a symplectic toric manifold that satisfies the condition of Theorem 5.1.9 and the moment map μ is such that the vertices of the moment polyhedron Δ have coefficients in $\hbar\mathbb{Z}$, the quantum Hilbert space $\mathfrak{H}$ of the prequantum line bundle from Corollary 5.2.4 has a canonical Hilbert basis corresponding to the points in $\Delta \cap \hbar\mathbb{Z}^n$.*

Proof. By Theorem 5.1.9 we have $M = M_\Delta$, and by the discussion surrounding Theorem 5.2.3 we see the sections of the prequantum line bundle E_Δ correspond to the N-equivariant functions on Z, where N acts on $\mathbb{C}$ with weights $(\hbar^{-1}c_1, \dots, \hbar^{-1}c_d)$.

Claim. *A section of E_Δ is polarized with respect to the connection ∇^{Δ,θ_0} if and only if it corresponds to a function $\psi(z) = f(z)e^{-|z|^2/2\hbar}$ on Z where f corresponds to a holomorphic section.*

Thus we are tasked with determining the holomorphic sections of E_Δ. By the discussion below Theorem 5.2.3 we see the holomorphic sections of the prequantum line bundle E_Δ correspond to the $N_\mathbb{C}$-equivariant holomorphic functions $U_\Sigma \to \mathbb{C}$, where $N_\mathbb{C}$ also acts on $\mathbb{C}$ with weights $(\hbar^{-1}c_1, \dots, \hbar^{-1}c_d)$.

Recall that $\mathbb{C} \setminus U_\Sigma$ is the union of submanifolds of complex codimension at least two. A version of Hartogs theorem (found in, for example, Theorem 2.5B in [Whi72]) ensures that such a holomorphic function on U_Σ uniquely extends to a holomorphic function on $\mathbb{C}^d$. Thus the holomorphic sections of E_Δ correspond to the holomorphic $N_\mathbb{C}$-equivariant functions $\mathbb{C}^d \to \mathbb{C}$.

Now let $f\colon \mathbb{C}^d \to \mathbb{C}$ be holomorphic. We write f as the power series

$$f = \sum_{k_1,\ldots,k_d \in \mathbb{Z}_{\geq 0}} a_{k_1,\ldots,k_d} z_1^{k_1} \cdots z_d^{k_d}$$

for some $a_{k_1,\ldots,k_d} \in \mathbb{C}$. Now fix $z = (z_1, \ldots, z_d) \in \mathbb{C}$ and $w = (w_1, \ldots, w_d) \in N_{\mathbb{C}} \subseteq T^d_{\mathbb{C}}$. From the $T^d_{\mathbb{C}}$-action on $\mathbb{C}^d$ we have

$$\begin{aligned} f(w \cdot z) &= f(w_1 z_1, \ldots, w_d z_d) \\ &= \sum_{k_1,\ldots,k_d \in \mathbb{Z}_{\geq 0}} a_{k_1,\ldots,k_d} w_1^{k_1} \cdots w_d^{k_d} z_1^{k_1} \cdots z_d^{k_d} \end{aligned}$$

and from the action on $\mathbb{C}$

$$\begin{aligned} w \cdot f(z) &= w_1^{\hbar^{-1}c_1} \cdots w_d^{\hbar^{-1}c_d} \sum_{k_1,\ldots,k_d \in \mathbb{Z}_{\geq 0}} a_{k_1,\ldots,k_d} z_1^{k_1} \cdots z_d^{k_d} \\ &= \sum_{k_1,\ldots,k_d \in \mathbb{Z}_{\geq 0}} a_{k_1,\ldots,k_d} w_1^{\hbar^{-1}c_1} \cdots w_d^{\hbar^{-1}c_d} z_1^{k_1} \cdots z_d^{k_d} \end{aligned}$$

Comparing these expressions, we see that f is $T^d_{\mathbb{C}}$-equivariant if and only if for $k_1, \ldots, k_d \in \mathbb{Z}_{\geq 0}$, the factor $a_{k_1,\ldots,k_d}$ vanish unless

$$w_1^{k_1} \cdots w_d^{k_d} = w_1^{\hbar^{-1}c_1} \cdots w_d^{\hbar^{-1}c_d} \tag{5.11}$$

for all $w \in N_{\mathbb{C}}$. Since this equation is invariant under scalar multiplication of the w_j, we may restrict to the real torus $N \subseteq T^d$. Then we see condition (5.11) is met if and only if

$$L(\hbar k_1, \ldots, \hbar k_d) = (c_1, \ldots, c_d),$$

where $L = (d\iota_e)^*$, for $\iota\colon N \to T^d$ the inclusion map, is as in the construction for Theorem 5.1.9. Thus, a Hilbert basis of the space of holomorphic $T^d_{\mathbb{C}}$-equivariant functions $\mathbb{C}^d \to \mathbb{C}$ is given by

$$\left\{ z_1^{k_1} \cdots z_d^{k_d} \;\middle|\; (\hbar k_1, \ldots, \hbar k_d) \in L^{-1}(c_1, \ldots, c_d) \cap \hbar \mathbb{Z}^d_{\geq 0} \right\}.$$

Writing $\Delta' := L^{-1}(c_1, \ldots, c_d) \cap \mathbb{R}^d_{\geq 0}$, it is only left to show that we have a natural identification of $\Delta' \cap \hbar\mathbb{Z}^d$ with $\Delta \cap \hbar\mathbb{Z}^n$. We consider the map

$$(c_1, \ldots, c_d) - (d\phi_e)^*\colon \Delta \xrightarrow{\sim} \Delta'. \tag{5.12}$$

Note that the map $d\phi_e \colon \mathbb{R}^d \to \mathbb{R}^n$, which is the same as the one defined in (5.1), has matrix elements in $\mathbb{Z}$, and so does its dual. Now the c_j are elements of $\hbar\mathbb{Z}$, so the map in (5.12) send $\hbar\mathbb{Z}^n$ to $\hbar\mathbb{Z}^d$. Then it is only left to show that any $a \in \Delta' \cap \hbar\mathbb{Z}^d$ comes from an element of Δ with components in $\hbar\mathbb{Z}$. Consider the first n columns of the matrix of $d\phi_e$. Since we assumed these to correspond to the faces meeting a single vertex of Δ, these columns are a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ and so this is an invertible matrix with determinant ± 1. Thus, the inverse of this matrix also has integer elements. In particular, applying this inverse matrix to the first n components of a then shows the inverse image of a to lie in $\hbar\mathbb{Z}^n$. $\square$

Proof of claim. Similar to Section 4.4.1, we have

$$\theta_0(X) = \theta(X) + X\left(\frac{i}{4}\sum_{j=1}^{d} z_j\bar{z}_j\right)$$

for all $X \in \mathfrak{X}(\mathbb{C}^d)$. The connections with respect to these forms are thus related by

$$\nabla_X^{\theta_0} = \nabla_X^{\theta} + \frac{1}{4\hbar}X\left(\sum_{j=1}^{d} z_j\bar{z}_j\right).$$

Now a smooth function ψ on $\mathbb{C}^d$ is polarized with respect to the connection ∇^{θ_0} if and only if it is of the form

$$\psi(z) = f(z)e^{-|z|^2/4\hbar}$$

for some f polarized with respect to the connection ∇^{θ}, and since $\theta(\partial_{\bar{z}}) = 0$ this happens if and only if f is holomorphic. After descending to the reduced space we get that the claim holds. $\square$

Remark 5.2.6. From the proof we see that the quantum Hilbert space consists of holomorphic functions f such that

$$\int_{\mathbb{C}} f(z)\overline{f(z)}e^{|z|^2/2\hbar}dz < \infty.$$

In fact $\mathfrak{H}_{M_\Delta}$ is the closed subspace the Segal-Bargmann space $HL^2(\mathbb{C}^d, \mu_\hbar)$ generated by the monomials $z_1^{k_1}\cdots z_d^{k_d}$ such that $(\hbar k_1, \ldots, \hbar k_d)$ is a lattice point in Δ'.

Finally, we compute the quantization of the components of the moment map. Assume as we did in the construction for Theorem 5.1.9 that we order the faces such that $v_1, \ldots, v_n$ span $\mathbb{R}^n$. Consider the basis of $\{e'_j\}_{1 \le j \le n}$ of $\mathbb{R}^n$ with $e'_j = -v_j$.

Now fix a $1 \le j \le n$ and consider $\mu_j \colon p \to \langle \mu(p), e'_j \rangle$. Let $v'_j \in \mathbb{R}^d$ be a vector such that $(d\phi_e)(v'_j) = v_j$. Note that $\nu_j \colon z \mapsto -\langle \nu(z), v'_j \rangle$ is a N-invariant function on $\mathbb{C}^d$ such that $\nu_j|_Z$ factors via μ_j.

We have

$$\nu_j(z) = -\langle \nu(z), v'_j \rangle = -\sum_{l=1}^{d} (v'_j)_l \left(\frac{1}{2}|z_l|^2 - c_l \right)$$

For fixed l, we recall that $\frac{1}{2}|z_l|^2$ is just the Hamiltonian of a harmonic oscillator. We already quantized this in Section 4.4.1. By linearity then the quantization of ν_j on $\mathbb{C}^d$ becomes

$$\mathcal{Q}_{\nu_j} = \sum_{l=1}^{d} (v'_j)_l \left(c_l - \hbar z_l \frac{\partial}{\partial z_l} \right).$$

Now we let $\mathcal{Q}_{\nu_j}$ act on a basis vector of $\mathfrak{H}_{M_\Delta}$. If $(\hbar k_1, \ldots, \hbar k_d) \in \Delta$ is a lattice point, under (5.12) it corresponds to the lattice point

$$\big(c_1 - \hbar\langle k, v_1 \rangle, \ldots, c_d - \hbar \langle k, v_d \rangle\big) \in \hbar\mathbb{Z}^d.$$

By the proof of Theorem 5.2.5 corresponds to a monomial

$$z_1^{\hbar^{-1}c_1 - \langle k, v_1 \rangle} \ldots z_d^{\hbar^{-1}c_d - \langle k, v_d \rangle} \in HL^2(\mathbb{C}^d, \mu_\hbar)$$

Note that for any $1 \le l \le d$ this is an eigenvector of $\hbar z_l \frac{\partial}{\partial z_l}$ with eigenvalue $c_l - \hbar\langle k, v_l \rangle$. Hence it is an eigenvector of $\mathcal{Q}_{\nu_j}$, and we calculate the eigenvalue to be

$$\sum_{l=1}^{d} (v'_j)_l \left(c_l - (c_l - \hbar\langle k, v_l \rangle)\right) = \hbar \sum_{l=1}^{d} (v'_j)_l \langle k, v_l \rangle = \hbar \sum_{m=1}^{n} \delta_{jm} k_m = \hbar k_j.$$

If we denote by $\psi_{\hbar k_1, \ldots, \hbar k_n}$ the vector in $\mathfrak{H}_{M_\Delta}$ corresponding to the lattice point $(\hbar k_1, \ldots, \hbar k_n) \in \Delta$, it is an eigenvector of the quantization $\mathcal{Q}_{\mu_j}$ with eigenvalue $\hbar k_j$.

5.3 Examples

5.3.1 Harmonic oscillator

Recall the one-dimensional harmonic oscillator from Sections 3.4.1 and 4.4.1, and for simplicity assume $k = 1$. The phase space is $T^*\mathbb{R} = \mathbb{R}^2$ which has a natural S^1-action by rotation around the origin,

$$e^{i\theta} \cdot (q, p) = (q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta),$$

which admits a moment map $\mu(p, q) = \sqrt{p^2 + q^2}$. The moment polyhedron then is $\mathbb{R}_+ \subseteq \mathbb{R}$. Since this polytope has only a single vertex, the condition of Proposition 5.2.1 is automatically satisfied, so we see that this phase space must be quantizable. Due to Theorem 5.2.5 the corresponding Hilbert space then has a natural basis $\{\psi_{\hbar k}\}_{k \in \mathbb{Z}_{\geq 0}}$ corresponding to the positive integers. From the discussion below the theorem we learn that these are eigenvectors of the quantization of $H = \frac{1}{2m} z\bar{z} = \frac{1}{m}\mu$ and $\mathcal{Q}_H \psi_{\hbar k} = \frac{\hbar}{m} k \psi_{\hbar k}$. This agrees with the quantization we found in Section 4.4.1.

5.3.2 Spin particle

Recall the spin particle with total angular momentum j from Section 3.4.3, which has phase space (S^2, ω) where ω is as in (3.11). Comparing this with (5.7) in Example 5.1.12, we see the moment polytope must be the line segment $[0, T]$ such that

$$j \sin\phi d\theta \wedge d\phi = \frac{T}{2} \sin\phi d\theta \wedge d\phi.$$

In terms of the total angular momentum j, the moment polytope thus is $[0, 2j]$.

As this is the intersection of the half-lines $H(-1, 0)$ and $H(1, 2j)$, it follows from Proposition 5.2.1 that this is quantizable precisely if we can write $j = \hbar l/2$ is a for some integer l. From Theorem 5.2.5 we read that the corresponding Hilbert space has a basis corresponding to the lattice points $0, \hbar, \ldots, \hbar l = 2j$. This is in agreement with the holomorphic quantization in Section 4.4.3. If we denote these basis elements by $\psi_{\hbar k}$ for $0 \leq k \leq 2j$, we now also find that the quantization of $H = J_z = \mu$ satisfies $\mathcal{Q}_H \psi_{\hbar k} = \hbar k \psi_{\hbar k}$.

Bibliography

[AS53] W. Ambrose and I. M. Singer. "A Theorem on Holonomy." In: *Transactions of the American Mathematical Society* 75.3 (1953), pp. 428–443. ISSN: 00029947. DOI: 10.2307/1990721.

[Aud04] Michèle Audin. *Torus Actions on Symplectic Manifolds*. 2nd ed. Vol. 93. Progress in Mathematics. Basel: Birkhäuser, 2004. ISBN: 978-3-0348-7960-6.

[Can01] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Vol. 1764. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001. ISBN: 978-3-540-42195-5. DOI: 10.1007/978-3-540-45330-7.

[Car18] Andrea Carosso. *Geometric Quantization*. 2018. arXiv: 1801.02307 [math-ph].

[Con90] John B. Conway. *A Course in Functional Analysis*. 2nd ed. Graduate Texts in Mathematics 96. New York: Springer, 1990. ISBN: 0-387-97245-5.

[Del88] Thomas Delzant. "Hamiltoniens Périodiques et Images Convexes de l'Application Moment." In: *Bulletin de la Société mathématique de France* 116.3 (1988), pp. 315–339. ISSN: 0037-9484. DOI: 10.24033/bsmf.2100.

[Dir30] Paul A.M. Dirac. *The Principles of Quantum Mechanics*. Oxford: Oxford University Press, 1930.

[Fol08] Gerald B. Folland. *Quantum Field Theory. A Tourist Guide for Mathematicians*. Vol. 149. Mathematical Surveys and Monographs. Providence: American Mathematical Society, 2008. ISBN: 0-8218-4705-8.

[GGK02] Victor Guillemin, Viktor Ginzburg, and Yael Karshon. *Moment Maps, Cobordisms, and Hamiltonian Group Actions*. Vol. 98. Mathematical Surveys and Monographs. Providence: American Mathematical Society, 2002. ISBN: 978-0-8218-0502-2.

[GS82] Victor Guillemin and Shlomo Sternberg. "Geometric Quantization and Multiplicities of Group Representations." In: *Inventiones mathematicae* 67.3 (1982), pp. 515–538. ISSN: 0020-9910.

[Hal00] Brian C. Hall. "Holomorphic Methods in Mathematical Physics." In: *First Summer School in Analysis and Mathematical Physics. Quantization, the Segal-Bargmann Transform and Semiclassical Analysis*. Vol. 260. Contemporary Mathematics. Providence: American Mathematical Society, 2000, pp. 1–59.

[Hal13] Brian C. Hall. *Quantum Theory for Mathematicians*. New York: Springer, 2013. ISBN: 1-4614-7116-8.

[Ham08] Mark D. Hamilton. "The Quantization of a Toric Manifold Is Given by the Integer Lattice Points in the Moment Polytope." In: *Toric Topology. International Conference May 28–June 3, 2006, Osaka City University, Osaka, Japan*. Vol. 460. Contemporary Mathematics. Providence: American Mathematical Society, 2008, pp. 131–140.

[KL15] Yael Karshon and Eugene Lerman. "Non-Compact Symplectic Toric Manifolds." In: *Symmetry, Integrability and Geometry: Methods and Applications* (July 2015). ISSN: 1815-0659. DOI: 10.3842/sigma.2015.055.

[Lee03] John M. Lee. *Introduction to Smooth Manifolds*. Vol. 218. Graduate Texts in Mathematics. New York: Springer, 2003. ISBN: 978-0-387-21752-9.

[Ler] Eugene Lerman. *Geometric Quantization. A Crash Course*. Providence.

[Ren20] Qiuyu Ren. *Quantization Commutes with Reduction*. 2020.

[SN17] J.J. Sakurai and Jim Napolitano. *Modern Quantum Mechanics*. 2nd ed. Cambridge: Cambridge University Press, 2017. DOI: 10.1017/9781108499996.

[SW76] David J. Simms and Nicholas M. J. Woodhouse. *Lectures on Geometric Quantization*. Vol. 53. Lecture Notes in Physics. Berlin, Heidelberg, and New York: Springer-Verlag, 1976. ISBN: 3-540-07860-6.

[Wei58] André Weil. *Introduction à l'Étude des Variétés Kählériennes*. Actualités Scientifiques et Industrielles 1267. Paris: Hermann, 1958.

[Wel80] R. O. Wells. *Differential Analysis on Complex Manifolds*. New York: Springer, 1980. ISBN: 978-1-4757-3946-6.

[Whi72] Hassler Whitney. *Complex Analytic Varieties*. Addison-Wesley Series in Mathematics. Reading: Addison-Wesley Publishing Company, 1972.

[Zie07] Günter M. Ziegler. *Lectures on Polytopes*. 7th ed. Vol. 152. Graduate Texts in Mathematics. New York: Springer, 2007. ISBN: 978-1-4613-8431-1.