On the Genus of Moduli Spaces of Rank 19 Lattice Polarized K3 Surfaces with a Level Structure

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Master's Thesis

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June 27th, 2024

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Contents

Acknowledgements

I am extremely grateful to my advisor, Emma Brakkee, for her vital guidance, support and insights throughout this research project. Her expertise and patience have been critical in shaping this thesis. In particular I would like to thank Emma for patiently reviewing my countless drafts and for providing me with detailed feedback which not only shaped this thesis but also impacted my academic growth. I would like to extend my heartfelt appreciation to Ronald van Luijk for his support and encouragement during my studies, which have been invaluable. To my family, I owe a debt of gratitude for their endless love, encouragement, and constant belief in my abilities. Their support has been the foundation of my mathematical journey, and I am deeply grateful for their presence in my life. Thank you for bearing with me.

Chapter 1

Introduction

The study of K3 surfaces holds a significant place in algebraic geometry due to their rich structure and intricate connections to various mathematical fields. These surfaces are two-dimensional complex manifolds with a trivial canonical bundle and vanishing first cohomology group $H^1(X, \mathcal{O}_X)$. This thesis focuses on the genus of moduli spaces of rank 19 lattice polarized K3 surfaces with a level structure, an area that intertwines lattice theory, modular forms, and algebraic geometry.

A key motivation for studying lattice polarized K3 surfaces lies in the work of Dolgachev as in [4], who first (defined and) studied the moduli space of lattice polarized K3 surfaces in detail. Lattice polarizations not only render the moduli space Hausdorff but also reduce its dimension, providing a more tractable framework for study. Adding level structures to these surfaces further refines the moduli spaces, converting them into fine moduli spaces.

In this thesis, we focus on a specific polarization with a lattice M_n, previously studied by Dolgachev. The moduli space of M_n -polarized K3 surfaces is particularly interesting because of its relation to the moduli space of elliptic curves with torsion points. This moduli space of M_n -polarized K3 surfaces is a modular curve of the form

$$X_0(n)^+ := \mathbb{H}^*/\langle \Gamma_0(n), F \rangle$$

where F denotes the Fricke involution (A.1) which was first introduced in [5]. It is known that $X_0(n) := \mathbb{H}^*/\Gamma_0(n)$ is a coarse moduli space of isomorphism classes of tuples (E, A) where E is an elliptic curve and A a cyclic subgroup of order n of E. The Fricke involution acts on this moduli space by sending a tuple (E, A) to $(E/A, E_n/A)$ where E_n is the n-torsion subgroup of E. For more details see Section 7 in [4].

The compactification of the moduli space of M_n-polarized K3 surfaces with level structure can be expressed as a quotient of the extended upper half-plane by a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Such a group is known as a Fuchsian group. This will be a modular curve of the form

$$X_0(n, l) := \mathbb{H}^*/\Gamma_0(nl) \cap \Gamma(l)$$

and our main goal is to compute a formula for its genus. This group is a normal subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and while the Fricke involution was not needed to compute its genus, in Appendix A we compute a bound for the genus of the modular curve given by extending the subgroup $\Gamma_0(nl) \cap \Gamma(l)$ by the Fricke involution. This is the modular curve

$$X_0(n, l)^+ := \mathbb{H}^*/\langle \Gamma_0(nl) \cap \Gamma(l), F \rangle.$$

While its geometric meaning in the context of K3 surfaces with level structures is not yet understood, it offers intriguing possibilities for the future.

1.1 Outline

This thesis is structured as follows.

Chapters 2 and 4 offer background on lattice theory and modular curves, respectively, following the work of [4] and [3]. These chapters also include certain tools which will be needed later on. Chapter 3 establishes the construction of the moduli space of M_n-polarized K3 surfaces and introduces the concept of level structure. In Chapter 5, we explicitly determine the Fuchsian subgroup that defines the modular curve which is isomorphic to the fine moduli space of M_n -polarized K3 surfaces with level l-structure. Chapter-6 computes the genus of this modular curve. Appendix A extends this computation by considering the genus when the group $\Gamma_0(nl) \cap \Gamma(l)$ is extended by the Fricke involution.

The Sage code for the computations regarding the ggenus formula can be found in Appendix B.

Chapter 2

Lattice Theory

In this section, we introduce the standard definitions on Lattice Theory. For this we follow closely the work of [8] in Chapter 14. Our later goal is to establish the lattice M_n which will provide the required polarization of K3 surfaces that we will study.

2.1 Background

Let V be a $\mathbb{Z}$-module. A *bilinear form* is a bilinear map $V \times V \to \mathbb{Z}$. It is *symmetric* if $(a.b) = (b.a)$. Furthermore, a bilinear form is *non-degenerate* if for a fixed $a \in V$ and for all b, we have that $(a.b) = 0$ then $a = 0$ and similarly if for a fixed b we have that for all $a, (a.b) = 0$ then $b = 0$.

Definition 2.1. We define a *lattice* Λ to be a free $\mathbb{Z}-$module of finite rank equipped with a non-degenerate symmetric bilinear form

$$(.) : \Lambda \times \Lambda \to \mathbb{Z}.$$

Definition 2.2. A lattice Λ is called *even* if for all $x \in \Lambda$ we have that

$$x^2 := (x.x) \in 2\mathbb{Z}.$$

Definition 2.3. The *signature* of a lattice Λ is the tuple (n_+, n_-) where n_+, n_- are the number of ± 1 entries after we diagonalize $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with only ± 1 in the diagonal.

Definition 2.4. Let Λ be a lattice with a basis $e_1, \ldots, e_n$. The matrix with entries $(e_i.e_j)_{i,j}$ is called the *intersection matrix* of the lattice. By *rank* of a lattice we mean the rank of Λ as a free $\mathbb{Z}$-module.

Definition 2.5. We have the following injection

$$\begin{aligned} i_\Lambda : \Lambda &\to \Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z}) \\ x &\mapsto (x.-). \end{aligned}$$

Injectivity follows from non-degeneracy of the bilinear form. The cokernel of the morphism $i_\Lambda : \Lambda \hookrightarrow \Lambda^*$ is called the *discriminant group* and is denoted by $A(\Lambda)$. We also consider Λ^* as the subset of $\Lambda \otimes \mathbb{Q}$ which contains all the elements $x \in \Lambda \otimes \mathbb{Q}$ such that $(x.\lambda) \subset \mathbb{Z}$ for all $\lambda \in \Lambda$.

Definition 2.6. We call a lattice *unimodular* if i_Λ defines an isomorphism.

Remark 2.7. We can see that if a matrix is unimodular then the discriminant group is trivial.

Definition 2.8. Let Λ_1 and Λ_2 be two lattices. We define the *direct sum* lattice $\Lambda_1 \oplus \Lambda_2$ to be the direct sum set theoretically. The form is given by

$$(x_1 + y_1.x_2 + y_2) := (x_1.x_2) + (y_1.y_2)$$

where we use both of the original bilinear forms and $x_i \in \Lambda_1$ and $y_i \in \Lambda_2$. We have an obvious isomorphism

$$A(\Lambda_1 \oplus \Lambda_2) \cong A(\Lambda_1) \oplus A(\Lambda_2).$$

Definition 2.9. We define a *morphism* between two lattices $f : \Lambda' \to \Lambda$ to be a $\mathbb{Z}$-linear map such that $(x.y)_{\Lambda'} = (f(x).f(y))_\Lambda$ for all $x, y \in \Lambda'$. It is an *isomorphism* if f is a $\mathbb{Z}$-module isomorphism.

Definition 2.10. Let $\Lambda_1 \hookrightarrow \Lambda$ be an injective morphism. It is called a *primitive embedding* if its cokernel is torsion free.

Definition 2.11. Let $\Lambda' \hookrightarrow \Lambda$ be an embedding of lattices. We denote by $\Lambda'^\perp$ the set

$$\Lambda'^\perp = \{\lambda \in \Lambda : \text{ for all } \ \lambda' \in \Lambda' \text{ we have that } (\lambda.\lambda') = 0\}.$$

We continue with some examples of lattices which will be needed in Chapter 3 in order to construct the K3 lattice and the lattice M_n.

Definition 2.12. We define the *hyperbolic plane*, U, to be the lattice of rank 2 given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and has generators $U \cong \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot g$ and has signature $(1, 1)$.

Definition 2.13. We define the lattice E_8 to be given by the intersection matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

It is an even, unimodular, positive definite lattice of rank 8.

Definition 2.14. We define the *twist by m* of a lattice Λ to be the lattice obtained by multiplying the intersection form $(.)$ of Λ by an integer m. As $\mathbb{Z}$-modules they satisfy $\Lambda = \Lambda(m)$ but we have that

$$(.)_{\Lambda(m)} := m \cdot (.)_{\Lambda}.$$

We have that $A(U(m)) \simeq (\mathbb{Z}/m\mathbb{Z})^2$ and the discriminant group is generated by the classes of $\frac{f}{m}$ and $\frac{g}{m}$ as we will see in Proposition 5.6. For this we will use the identification of Λ^* as the subset of $\Lambda_{\mathbb{Q}}$ which contains all the elements $x : (x.\Lambda) \subset \mathbb{Z}$ as seen in Definition 2.5.

Remark 2.15. We note that $U(-1) \cong U$ which can be seen by $f \mapsto -f$.

Definition 2.16. Let $m \in \mathbb{Z}$. We define the lattice $\langle m \rangle$ to be the lattice generated by a single generator e such that $(e.e) = m$.

Remark 2.17. We note that $\langle 2n \rangle$ be can embedded primitively into the hyperbolic plane by $U = \mathbb{Z}f + \mathbb{Z}g$ by sending $e \mapsto f + ng$.

Definition 2.18. We define the K3 lattice as the lattice

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

2.2 The Lattice M_n

In the rest of the thesis, we care about a specific lattice M_n. This lattice was introduced in Dolgachev's work in [4]. We also mention some important results which are needed for Chapter 6.

For n a positive integer we view $U \oplus \langle 2n \rangle$ as primitive sublattice of Λ_{K3} as folllows : Let U_i be the i-th copy of the hyperbolic plane inside Λ_{K3} and we denote by f, g the standard basis of U_2. Then we identify $U \oplus \langle 2n \rangle$ with $U_1 \oplus \langle f + ng \rangle$.

Definition 2.19. We denote by M_n the orthogonal complement of $U \oplus \langle 2n \rangle$ inside Λ_{K3}.

Explicitly, M_n is given by

$$M_n = E_8(-1)^{\oplus 2} \oplus U \oplus \langle (f - ng) \rangle.$$

This is a lattice of signature $(1, 18)$ and thus, of rank 19.

The following statements play an important role in determining the genus of $X_0(n, l)$ and it will be used in Section 5.7.

Definition 2.20. Let be $\mathrm{Aut}(S)$ be the isomorphims of a lattice S. Then, we define the *orthogonal group*

$$O(S) := \{\sigma \in \mathrm{Aut}(S) : \forall\ x, y \in S, (\sigma x.\sigma y) = (x.y)\}.$$

This is the orthogonal group that consists of all $\mathbb{Z}$-linear automorphisms that preserve the bilinear form.

Definition 2.21. Let M be a any sublattice of Λ_{K3}. We define the following subset of $O(\Lambda_{K3})$

$$\Gamma(M) := \{\sigma \in O(\Lambda_{K3}) : \sigma(m) = m : \forall\ m \in M\}.$$

For a lattice N, there is a natural group homomorphism

$$O(N) \to \mathrm{Aut}(A(N))$$

defined as follows : For $g \in O(N)$, denote by g^* the induced automorphism on $N^* = \mathrm{Hom}(N, \mathbb{Z})$, then this acts on $A(N)$ by $[x] \mapsto [g^*(x)]$. We denote by $O(N)^*$ the kernel of

$$O(N) \to \mathrm{Aut}(A(N)).$$

Remark 2.22. For any lattice $M \subset \Lambda_{K3}$, we denote by N its orthogonal complement inside the lattice Λ_{K3}. Then we note that there is also a natural injective map

$$\Gamma(M) \to O(N)$$

given by $g \mapsto g|_N$.

Definition 2.23. We denote by Γ_M the image of $\Gamma(M)$ under the above map

$$\Gamma(M) \to O(N).$$

Proposition 2.24. ([8], Proposition 14.2.6, Lemma 14.2.5]) Let M be a primitive sublattice of Λ_{K3} and N its orthogonal complement. The map $\Gamma(M) \to O(N)$ given by $g \mapsto g|_N$ induces an isomorphism

$$\Gamma_M \simeq O(N)^*.$$

In this thesis we care about Γ_{M_n}. In the next chapter we continue by introducing K3 surface, lattice polarizations and level structure.

Chapter 3

K3 Surfaces and Level Structure

In this section we will define the basic theory of K3 surfaces following [8]. We will introduce lattice polarized K3 surfaces which have a coarse moduli space. Then we will add level structures which make the moduli space fine and Hausdorff.

3.1 K3 Surfaces

Throughout this background section let X be a scheme over an arbitrary field k.

Definition 3.1. A *variety* over an arbitrary field k is a separated, geometrically integral scheme of finite type over k.

Definition 3.2. We define a *K3 surface over a field k* to be a complete non-singular variety X which has dimension 2, and the following conditions are satisfied for the determinant bundle and the first cohomology group

$$\Omega^2_{X/k} \cong \mathcal{O}_X,\ H^1(X, \mathcal{O}_X) = 0$$

In this thesis, we will always assume that $k = \mathbb{C}$, and call X a *complex K3 surface.* We have that $X(\mathbb{C})$ with the Euclidean topology is a complex manifold which we also denote by X.

Theorem 3.3. ([7], Section 2.1) Let X be a complex K3 surface as above, then the second singular cohomology group $H^2(X, \mathbb{Z})$ equipped with the cup product is an even unimodular lattice of signature $(3, 19)$ and isometric to the K3 lattice

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

It turns out a K3 suface is determined by this lattice structure on $H^2(X, \mathbb{Z})$ together with what is called a Hodge structure on it. We explain what this means in the following definitions.

Let V be a finite $\mathbb{Z}$-module of finite rank.

Definition 3.4. A *Hodge structure of weight $n \in \mathbb{Z}$ on a $\mathbb{Z}$ module V* is defined to be the direct sum decomposition of the complexification $V_{\mathbb{C}}$

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

where we have that $\overline{V^{p,q}} = V^{q,p}$ for all $p + q = n$.

Definition 3.5. Let $r \in \mathbb{Z}$. We define a *morphism of Hodge structures* of bidegree (r, r) between two Hodge structures V and V' of weights $n, n + 2r$ respectively, to be a group homomorphism $\phi : V \to V'$ such that $\phi(V^{p,q}) \subset (V')^{p+r,q+r}$.

Remark 3.6. ([7], Section 2.1) Let X be a complex K3 surface, then we have the following Hodge decomposition for the second singular cohomology group

$$H^2(X, \mathbb{C}) \simeq H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

Furthermore, $H^{2,0}(X)$ and $H^{0,2}(X)$ are 1-dimensional and the cup product on $H^2(X, \mathbb{Z})$ extends to a symmetric bilinear pairing on $H^2(X, \mathbb{C})$.

Definition 3.7. Let V be a Hodge structure of weight 2. We say that V is of K3 *type* if we have that $\dim_{\mathbb{C}}(V^{2,0}) = 1$ and $V^{p,q} = 0$ for $|p - q| > 2$.

Theorem 3.8. (Weak Torelli Theorem) Let X, X' be two complex K3 surfaces. They are isomorphic if and only if there exists an isomorphism

$$H^2(X, \mathbb{Z}) \simeq H^2(X', \mathbb{Z})$$

of Hodge structures which respects the intersection pairing.

Finally, we introduce a tool which we will need to define lattice polarizations.

Definition 3.9. Let X be a complex K3 surface. Using the long exact sequence corresponding to the exponential sequence, we get an injective map

$$c_1 : \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z}).$$

The above map is injective since $H^1(X, \mathcal{O}_X) = 0$. The above injection is called the *first Chern class.*

3.2 Period Domain

In this section, we introduce some important tools based on Chapter 6 of [8] which will be needed in Section 3.4. Our lattices will have a hyperbolic plane embedded into them. We also denote by $\mathbb{H}$ the upper half plane.

Let Λ be lattice and let $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{C}} \mathbb{C}$ be the associated complex lattice. We consider the $\mathbb{C}$-linear extension of the bilinear form to get a $\mathbb{C}$-valued bilinear form on $\Lambda_{\mathbb{C}}$. Now this form will give us a homogeneous quadratic polynomial f in n variables where n is the rank of the original lattice. To do this, we choose a basis $e_1, \ldots, e_n$ of $\Lambda_{\mathbb{C}}$ and define $f \in \mathbb{C}[x_1, \ldots, x_n]$ by $f(\lambda_1, \ldots, \lambda_n) = (\lambda_1 e_1 + \cdots + \lambda_n e_n)^2$. We note that by construction $f(lx_1, \ldots, lx_n) = l^2 f(x_1, \ldots, x_n)$ and that it does indeed define a polynomial. For instance for $n = 2$, using bilinearity, we have that $f(x, y) = x^2(e_1.e_1)+2xy(e_1.e_2)+y^2(e_2.e_2)$.

Let Λ be a lattice and f be its associated homogeneous polynomial. Its vanishing locus inside the projective space $\mathbb{P}(\Lambda_{\mathbb{C}})$ is a smooth quadric as the form is non-degenerate.

Definition 3.10. We define the following open subset inside the aforementioned quadric

$$\mathcal{D}(\Lambda) := \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) : f(x) = 0, (x.\overline{x}) > 0\}.$$

We call this the *period domain associated with* Λ.

We assume $n_+ = 2, n_- > 0$ and let $\Lambda_{\mathbb{R}} = U_{\mathbb{R}} \oplus W$ be an orthogonal decomposition since we work over $\mathbb{R}$ for some lattice W. Let f, g be the usual generators of the hyperbolic plane U. Any point in $\mathbb{P}(\Lambda_{\mathbb{C}})$ is of the form $x_1 f + x_2 g + z \in U_{\mathbb{C}} \oplus W_{\mathbb{C}}$ and is denoted by $[x_1 : x_2 : z]$ for $x_i \in \mathbb{C}, z \in W_{\mathbb{C}}$. When $n_+ = 2$ we have that $\mathcal{D}(\Lambda)$ has two connected components as in [8] Remark 6.1.6. We denote one of them by $\mathcal{D}^+(\Lambda)$.

Remark 3.11. We note that there is an action of $O(\Lambda)$ on Λ which induces an action on the period domain $D(\Lambda)$ as we can see that for any $\sigma \in O(\Lambda)$ we have that $\sigma \cdot [x] = \sigma \cdot x$ satisfies that $(\sigma x)^2 = 0, (\sigma x, \sigma\overline{x}) > 0$. The action is independent of the representative x.

Definition 3.12. We define the associated *tube domain* to be

$$\mathcal{H} := \{z \in W_{\mathbb{C}} | (\mathrm{Im}(z))^2 > 0\}.$$

Proposition 3.13. ([8], Proposition 6.1.7) We have a biholomorphic map

$$\mathcal{H} \simeq \mathcal{D}(\Lambda)$$

given by $z \mapsto [1 : -(z)^2 : \sqrt{2}z]$.

Remark 3.14. ([8], Example 1.8) Let $n_+ = 2, n_- = 1$ so we may assume that $W_{\mathbb{C}} = \mathbb{C}$. Thus, $\mathcal{H} = \mathbb{H} \sqcup -\mathbb{H}$. This is because $(Im(z))^2$ is strictly positive which implies that $\mathrm{Im}(z)$ is either positive or negative which justifies the disjoint union. The previous biholomorphic map will give $\mathbb{H} \simeq \mathcal{D}^+(\Lambda)$.

When we select an arbitrary lattice Λ then as seen in [8], Proposition 6.1.2, the period domain of that lattice parametrizes Hodge structures of K3 type on this lattice. One can use this fact to show that the (coarse) moduli space of complex K3 surfaces is isomorphic to a quotient of $\mathcal{D}(\Lambda_{K3})$. As explained in [8] Remark 6.3.6 (iii), it is not Hausdorff. We are only interested in algebraic K3 surfaces. Hence, we will add lattice polarizations and level structures. This has the advantage that the moduli space of (ample) lattice polarized K3 surfaces is Hausdorff and fine as we will explain in Proposition 3.37.

3.3 Lattice Polarizations

We continue with the construction of the moduli space of lattice polarized K3 surfaces. The following definition is a motivation for Definition 3.15. For this we recall that the *Néron–Severi group* of a variety X is $c_1(\mathrm{Pic}(X)) \subseteq H^2(X, \mathbb{Z})$ which in the case of a K3 surface is isomorphic to $\mathrm{Pic}(X)$ as the first Chern Class is injective.

We want to define a generalization of the positive and ample cones of K3 surfaces as seen in Chapter 8 of [8]. For any lattice M we generalize this procedure as we will see right below.

Definition 3.15. For a lattice M, we define the *positive cone* $V(M)$ of M, as follows :

$$V(M) := \{x \in M_{\mathbb{R}} : (x.x) > 0\}.$$

We set $V(M)^+$ to be one of the two connected components of $V(M)$. Furthermore, we define the following set

$$\Delta(M) := \{\delta \in M : (\delta, \delta) = -2\}.$$

We select a subset $\Delta(M)^+$ such that $\Delta(M)$ can be split as $\Delta(M) = \Delta(M)^+ \bigsqcup \Delta(M)^-$ where $\Delta(M)^- := \{-\delta : \delta \in \Delta(M)^+\}$. We also demand that if $\delta_1, \ldots, \delta_k \in \Delta(M)^+$ and $\delta = \sum n_i \delta_i$ for some $n_i \in \mathbb{N}$ then $\delta \in \Delta(M)^+$.

Definition 3.16. We define the following set

$$C(M)^+ := \{x \in V(M) \cap M : (x, \delta) > 0 \text{ for all } \delta \in \Delta(M)^+\}.$$

The lattice M_n satisfies this aforementioned assumption which follows from [8], Theorem 14.1.12 and Remark 14.1.13 (ii).

Proposition 3.17. The embedding $M_n \hookrightarrow \Lambda_{K3}$ is unique up to elements in $O(\Lambda_{K3})$.

We continue by introducing lattice polarizations on K3 surfaces. First we recall the definitions of nef and big line bundles.

Definition 3.18. Let X be a complete variety. Then we say that a class in $\mathrm{Pic}(X)$ is *nef* if $(L.C) \geq 0$ for all closed and irreducible curves C on X. A class is called *big* if it has positive self intersection and is also nef.

Definition 3.19. Let X be a K3 surface. We denote by $\mathrm{Pic}(X)^+ \subseteq \mathrm{Pic}(X)$ the big and nef classes and by $\mathrm{Pic}(X)^{++} \subseteq \mathrm{Pic}(X)$ we denote the ample classes of the Picard group of X.

Definition 3.20. We define an *M-polarized* K3 surface to be a pair (X, j) where X is a complex K3 surface and $j : M \hookrightarrow \mathrm{Pic}(X)$ is a primitive lattice embedding.

Definition 3.21. We call an M-polarized K3 surface (X, j) *quasi-ample* (resp. *ample*) if $j(C(M)^+) \cap \mathrm{Pic}(X)^+ \neq \emptyset$. We call (X, j) *ample* if $j(C(M)^+) \cap \mathrm{Pic}(X)^{++} \neq \emptyset$.

We continue with an assumption which we will carry through this chapter.

Assumption 3.22. Onwards, we will assume that we have an embedding of a lattice $i : M \hookrightarrow \Lambda_{K3}$ where M is even, non-degenerate lattice of signature $(1, n)$ with $n \leq 19$. The embedding of M inside the K3 lattice is assumed to be primitive and unique up to elements in $\mathrm{O}(\Lambda_{K3})$. We fix such an embedding $i : M \hookrightarrow \Lambda_{K3}$

Definition 3.23. We call two M-polarized K3 surfaces $(X, j), (X', j')$ *isomorphic* if there exists an isomorphism of K3 surfaces $f \colon X' \to X$ such that for the induced map on the Picard groups f^*, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r, "f^*"] & \mathrm{Pic}(X') \\
M \arrow[u, "j"] \arrow[ur, "j'"'] &
\end{tikzcd}$$

Definition 3.24. A *marked* M-polarized K3 surface is defined to be a tuple (X, ϕ) where ϕ is a lattice isomorphism

$$\phi\colon H^2(X, \mathbb{Z}) \to \Lambda_{K3}$$

such that $\phi^{-1}(M) \subseteq \mathrm{Pic}(X)$.

By setting $j = \phi^{-1}|_{i(M)} \circ i$ we make (X, j) an M-polarized K3 surface where i is the embedding as in Assumption 3.22.

Definition 3.25. Let $(X, j, \phi), (X', j', \phi')$ be two marked M-polarized K3 surfaces. We call them *isomorphic* if there exists an isomorphism of K3 surfaces, $f\colon X' \to X$ such that $\phi = \phi' \circ f^*$.

This means that (X, j) and (X', j') are isomorphic as M-polarized K3 surfaces.

Theorem 3.26. ([4], Section 3) There exists a fine moduli space, $\mathcal{K}_M$ of marked M-polarized K3-surfaces.

In Remark 3.4 of [4], Dolgachev uses the moduli space $\mathcal{K}_M$ to construct the moduli space of quasi-ample M-polarized K3 surfaces. However, this moduli space is only coarse. We will add level structures which has the advantage of making the moduli space fine, as we will sketch at the end of this chapter.

Definition 3.27. Let $N = i(M)^\perp$. Then we define the *period map* as follows

$$\begin{aligned}
&\mathcal{K}_M \to \mathcal{D}(N) \\
&(X, \phi) \mapsto [\phi(H^{2,0}(X))].
\end{aligned}$$

It is holomorphic as seen in Section 3 of [4].

Definition 3.28. We define $\mathcal{D}^\circ(N)$ to be the open set

$$\mathcal{D}^\circ(N) = \mathcal{D}(N) \setminus \bigcup_{\delta \in \Delta(N)} \delta^\perp$$

Definition 3.29. We denote by $\mathcal{K}_M^a$ the subset of $\mathcal{K}_M$ where the M-polarization is ample.

Proposition 3.30. ([4], Section 3) The period map induces a biholomorphic isomorphism

$$\mathcal{K}_M^a \to \mathcal{D}^\circ(N)$$

In order to get the respective moduli space of marked ample M-polarized K3 surfaces we consider the quotient $\mathcal{K}_M^a/\Gamma(M)$ in order to identify all markings on a surface. The action follows from Remark 3.11. In Section 3 of [4] it is shown that the period map gives a biholomorphism

$$\mathcal{K}_M^a/\Gamma(M) \simeq \mathcal{D}^\circ(N)/\Gamma_M$$

and $\mathcal{D}^\circ(N)/\Gamma_M$ is endowed with a structure of quasi-projective algebraic variety. If we select M of 19 then, $\mathcal{D}(N)$ is biholomorphic to the upper half plane $\mathbb{H}$ as seen in Remark 3.14.

3.4 Level Structures

We continue by introducing level structures on K3 surfaces. These were studied for K3 surfaces over general fields by [11] in Chapter 5.

Definition 3.31. Let l be a natural number. A *level l-structure* on a K3 surface X is an isomorphism

$$\phi : H^2(X, \mathbb{Z}/l\mathbb{Z}) \simeq \Lambda_{K3} \otimes \mathbb{Z}/l\mathbb{Z}$$

which respects the respective forms. We say that a K3 surface with a level l-structure is *compatible with an M-polarization* $j : M \hookrightarrow \mathrm{Pic}(X)$ if we have the following commutative diagram

$$\begin{tikzcd}
M \otimes \mathbb{Z}/l\mathbb{Z} \arrow[r, "i\otimes_{\mathbb{Z}/l\mathbb{Z}} id"] \arrow[d, "j\otimes_{\mathbb{Z}/l\mathbb{Z}} id"'] & \Lambda_{K3} \otimes \mathbb{Z}/l\mathbb{Z} \\
\mathrm{Pic}(X) \otimes \mathbb{Z}/l\mathbb{Z} \arrow[r, "c_1"'] & H^2(X, \mathbb{Z}/l\mathbb{Z}) \arrow[u, "\phi"']
\end{tikzcd}$$

where c_1 is the map modulo l induced by the first Chern class map as seen in Definition 3.9.

Remark 3.32. Let $N = i(M)^{\perp}$ in the K3 lattice Λ_{K3}. Then if the level structure ϕ is compatible with an M-polarization j, it induces an isomorphism

$$H^2(X, \mathbb{Z}/l\mathbb{Z}) \supset c_1(j(M))^{\perp} \simeq N \otimes \mathbb{Z}/l\mathbb{Z}.$$

From now on whenever we consider a K3 surface, we assume that the lattice polarization and the level structure are compatible with each other.

Definition 3.33. Let $(X, \phi), (X', \phi')$ be two K3 surfaces with a level l-structure. We say that they are *isomorphic* if there exists an isomorphism of K3 surfaces $f : X' \to X$ such that $\phi = \phi' \circ f^*$.

Following the strategy of [8], Section 6.4.2, one can prove the following theorem.

Theorem 3.34. Let $l > 2$. There exists a fine moduli space $\mathcal{K}_M^a(l)$ of ample M-polarized K3 surfaces with a level l-structure.

We explain the construction on the level of the period domain.
The following definition constructs a subgroup of Γ_M which we saw in Definition 2.23.

Definition 3.35. We define the subgroup $\Gamma_M(l)$ of Γ_M, to be

$$\Gamma_M(l) := \{g \in \Gamma_M : g \equiv 1 \mod l\}.$$

Proposition 3.36. ([9] , Theorem III.2.3 and Remark III.2.4) The group $\Gamma_M(l)$ acts freely on $\mathcal{D}(N)$ for $l > 2$.

Proposition 3.37. Let $l > 2$, then the moduli space $\mathcal{K}_M^a(l)$ of ample M-polarized K3 surfaces with level structure satisfies

$$\mathcal{K}_M^a(l) \simeq \mathcal{D}^{\circ}(N)/\Gamma_M(l).$$

Proof. (*Sketch*) We note that the above isomorphism is induced by the period map

$$\mathcal{K}_M^a(l) \to \mathcal{D}(N)$$

as seen in Definition 3.27. Note that the action of $\Gamma_M(l)$ on $\mathcal{D}(N)$ identifies two marked M-polarized K3 surfaces (X, j, φ) and (X', j', φ') exactly when (X, j) and (X', j') are isomorphic M-polarized K3 surfaces and $(X, \varphi \otimes \mathbb{Z}/l\mathbb{Z}), (X', \phi \otimes \mathbb{Z}/l\mathbb{Z})$ are isomorphic as $K3$ surfaces with a level structure. It follows that $\mathcal{D}(N)/\Gamma_M(l)$ parametrizes K3 surfaces together with an M-polarization and a level-l structure compatible with the polarization. Now the moduli space is also fine since by Proposition 3.36 we have that $\Gamma_M(l)$ acts freely on $\mathcal{K}_M$. Consequently, $\Gamma_M(l)$ acts freely on the universal family over $\mathcal{K}_M/\Gamma_M(l)$ and the quotient is a universal family over $\mathcal{D}(N)/\Gamma_M(l)$ which makes it into a fine moduli space. □

Note that the above moduli space is Hausdorff since it is the quotient of a Hausdorff space by a free action.

Remark 3.38. One can view the full quotient $\mathcal{D}(N)/\Gamma_M(l)$ as a coarse moduli space of quasi-ample M-polarized K3 surfaces with a level l-structure as seen in [4], Remark 3.4.

We continue to the next chapter with some background on modular curves which will be needed in Chapter 6. In Chapter 6 we compute the genus of the compactification of $\mathcal{K}^a_M(l)$.

Chapter 4

Modular Curves

We start with some background on modular curves following [12].

Definition 4.1. We recall the following basic subgroups of $\mathrm{PSL}(2,\mathbb{R})$. Let $n \in \mathbb{Z}$.

$$\Gamma_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) : c \equiv 0 \mod n \right\}$$
$$\Gamma_1(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) : a \equiv d \equiv \pm 1 \mod n, c \equiv 0 \mod n \right\}$$
$$\Gamma(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) : a \equiv d \equiv \pm 1 \mod n, b \equiv c \equiv 0 \mod n \right\}$$

Definition 4.2. Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We define an action on $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ as follows :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

where $1/0$ will be ∞ and $1/\infty = 0$. We call an element of $\sigma \in \Gamma$ *parabolic* if it fixes exactly one element of $\mathbb{R} \cup \{\infty\}$ which is called a *cusp*. We call an element *elliptic* if it fixes one element in $\mathbb{H}$. One of them is a point z and the other its conjugate $\overline{z}$.

Definition 4.3. Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. This induces an action on $\mathbb{R} \cup \{\infty\}$ and thus on the extended upper half plane $\mathbb{H}^* := \mathbb{H} \cup \{\text{cusps of } \Gamma\}$. The quotient topological space $\mathbb{H}^*/\Gamma$ is called the *compactification of* Γ.

This quotient is in fact a Riemann Surface (see [12], Section 1.5).

Proposition 4.4. ([12], Proposition 1.30) Let Γ' be a finite-index subgroup of Γ in $\mathrm{PSL}(2,\mathbb{R})$. Then they have the same set of cusps.

Remark 4.5. We know that the cusps of the full modular group $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ are $\mathbb{Q} \cup \{\infty\}$ and are all in the same orbit. Thus, they are all equivalent to $\{\infty\}$. The action is given on a rational number $\frac{p}{q}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{p}{q} := \frac{ap+bq}{cp+dq}.$$

The point at infinity, ∞, gets mapped to a/c and $-d/c$ gets mapped to ∞ when $c \neq 0$. The point at infinity is mapped to itself when $c = 0$.

The main goal of the thesis is to determine the genus of the modular curve

$$X_0(n,l) = \mathbb{H}^*/\Gamma_0(nl) \cap \Gamma(l)$$

as we will see in Section 6. For this reason we introduce the theorem which we will use to compute that genus.

Theorem 4.6. ([12], Proposition 1.40) Let Γ be a finite-index subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. The genus g of the curve $\mathbb{H}^*/\Gamma$ is given by the formula

$$g = 1 + \frac{\mu}{12} - \frac{u_2}{4} - \frac{u_3}{3} - \frac{u_\infty}{2}$$

where μ is the index of $\Gamma_0(nl) \cap \Gamma(l)$ in $\mathrm{PSL}(2,\mathbb{Z})$, u_i the number of non-equivalent elliptic elements of order 2 and 3 respectively and u_∞ is the number of non-equivalent cusps.

Now we will mention some important tools which will be needed in computing the genus of $X_0(n,l)$.

Proposition 4.7. ([12], Proposition 1.39) If $n \geq 1$ then $\Gamma(n)$ has no elliptic elements.

The above implies that $u_2 = u_3 = 0$ for any subgroup of $\Gamma(l)$.

Definition 4.8. Let Γ, Γ' be two subgroups of $\mathrm{PSL}(2,\mathbb{R})$. We say that these subgroups are *conjugate* if there exists an invertible matrix in M_2, denoted by g, such that $g\Gamma g^{-1} = \Gamma'$.

Remark 4.9. It is clear that conjugate subgroups in $\mathrm{PSL}(2,\mathbb{R})$ have the same genus, number of cusps and elliptic points.

The numbers u_2, u_3 and u_∞ are well known for the subgroup $\Gamma_0(N)$ and $\Gamma(N)$. We will need them for computing the required genus.
The following proposition will be needed in Section 6 in order to compute the number of inequivalent cusps of a specific subgroup, $\Gamma_0(n,l)$, and also in Appendix A when we compute a bound for the genus of $X_0(n,l)^+$.
If Q is a group acting on a set X, we denote by Q_x the stabilizer at x and by Qx the orbit.

Proposition 4.10. Let X be a any set and Q a group which acts on X. Let $P \subseteq Q$ be a subgroup of finite index. Then for every element $x \in X$ we have that

$$|P\backslash Qx| = |P\backslash Q/Q_x|.$$

Additionally, when P is normal subgroup of Q we get that

$$|P\backslash Qx| = \frac{[Q:P]}{[Q_x:P_x]}.$$

Proof. We note the following bijection

$$\begin{aligned} Q/Q_x &\xrightarrow{1:1} Qx \\ [q] &\mapsto qx \end{aligned}$$

which is compatible with the action of P as

$$p \cdot [q] = [pq] \mapsto pqx = p \cdot qx$$

for any $p \in P$. Thus we get a bijection

$$P\backslash Q/Q_x \xrightarrow{1:1} P\backslash Qx$$

which proves that

$$|P\backslash Qx| = |P\backslash Q/Q_x|.$$

Now regarding the second claim, we assume that P is a normal subgroup of Q. It is clear that

$$P\backslash Q/Q_x \simeq Q/(P \cdot Q_x)$$

Using the third isomorphism theorem of groups we also get that

$$Q\backslash(P \cdot Q_x) \simeq (Q/P)/(PQ_x/P).$$

Furthermore, using the second isomorphism theorem for groups we get that

$$PQ_x/P \simeq Q_x/(Q_x \cap P).$$

Trivially we have that

$$Q_x/(Q_x \cap P) \simeq Q_x/P_x$$

Then the second statement follows. $\square$

In the next chapter we will determine the subgroup $\Gamma_0(n,l) := \Gamma_0(nl) \cap \Gamma(l)$. For technical reasons we want to work with a conjugate, $\Gamma_0(nl^2) \cap \Gamma_1(l)$ of this group which is explained in the following proposition.

Proposition 4.11. Let $m, l \in \mathbb{N}$. The subgroups $\Gamma_0(m) \cap \Gamma(l)$ and $\Gamma_0(ml) \cap \Gamma_1(l)$ are conjugate to each other by $\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m) \cap \Gamma(l)$. Then

$$\begin{pmatrix} 1/l & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b/l \\ cl & d \end{pmatrix}$$

which is obviously in $\Gamma_0(ml) \cap \Gamma_1(l)$.
Conversely, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(ml) \cap \Gamma_1(l)$. Then

$$\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1/l & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bl \\ c/l & d \end{pmatrix}$$

which is in $\Gamma_0(m) \cap \Gamma(l)$. $\square$

Lemma 4.12. The subgroup $\Gamma_0(nl) \cap \Gamma(l)$ is conjugate to $\Gamma_0(nl^2) \cap \Gamma_1(l)$

Proof. The proof follows from the previous propoposition applied to $m = nl$. $\square$

In order to use Proposition 4.10 we need the following result for the group $\Gamma_0(n,l)$.

Proposition 4.13. The subgroup $\Gamma_0(nl) \cap \Gamma(l)$ is a normal subgroup of $\Gamma_0(nl)$.

Proof. To show that $\Gamma_0(nl) \cap \Gamma(l)$ is a normal subgroup of $\Gamma_0(nl)$, we need to prove that for any $\gamma \in \Gamma_0(nl)$ and $\delta \in \Gamma_0(nl) \cap \Gamma(l)$, the conjugate $\gamma\delta\gamma^{-1}$ lies in $\Gamma_0(nl) \cap \Gamma(l)$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(nl)$ and $\delta = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \Gamma_0(nl) \cap \Gamma(l)$. We compute the conjugate:

$$\gamma\delta\gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since $\gamma \in \Gamma_0(nl)$, we have $c \equiv 0 \pmod{nl}$ and also since $\delta \in \Gamma(l)$, we have $e \equiv 1 \pmod{l}$, $f \equiv 0 \pmod{l}$, $g \equiv 0 \pmod{l}$, and $h \equiv 1 \pmod{l}$. Additionally, since $\delta \in \Gamma_0(nl)$, we have $g \equiv 0 \pmod{nl}$. We need to verify two things for $\gamma\delta\gamma^{-1}$:

- The $(2,1)$ entry of the resulting matrix should be $0 \pmod{nl}$.
- The resulting matrix should be congruent to the identity matrix modulo l.

The $(2,1)$ entry is $ced + d^2g - c^2f - cdh$ which is trivially $0 \mod nl$. Similarly for the entry $(1,2)$ we can see that is $0 \mod l$. The diagonal entries

$$(ae + bg)d - (af + bh)c$$
$$(ce + dg)(-b) + (cf + dh)a$$

are also easily seen to be $1 \mod l$, using the fact that $\det \delta = 1$. $\square$

The above proposition also implies that $\Gamma_0(nl^2) \cap \Gamma_1(l)$ is a normal subgroup of $\Gamma_0(nl^2)$ as $\Gamma_1(l)$ is a normal subgroup of $\Gamma(l)$. We will see in the next chapter that we care about computing the genus of the modular curve $\mathbb{H}^*/\Gamma_0(nl) \cap \Gamma(l)$. We want to use conjugation in order to work with the subgroup $\Gamma_0(nl^2) \cap \Gamma_1(l)$ which is easier to work with. For this we mention the following proposition.

Proposition 4.14. Let G be a group and H a subgroup. Let $g \in G$, then

$$[G : H] = [G : gHg^{-1}]$$

This means that conjugate subgroups have the same index.

Chapter 5

Determining the Fuchsian group

In this section, we will be following closely Dolgachev's work as in [4] and especially [4], Theorem 7.1. Throughout this section M_n is the lattice $(U \oplus \langle 2n \rangle)^{\perp} \subseteq \Lambda_{K3}$ and which satisfies Assumption 3.22. We define $N := M_n^{\perp} \subseteq \Lambda_{K3}$ as seen previously. In our case M_n, as seen in Definition 2.19, has signature $(1, 18)$ and the lattice is of rank 19. Let $l > 2$, from Theorem 3.34 we know that the moduli space of ample M_n-polarized K3 surfaces with level l-structure exists and is an open subvariety of a quotient of the upper half plane by a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. In this section we will determine this subgroup, which will be $\Gamma_0(n, l)$ as mentioned in at the end of Chapter 4.

5.1 Setup

Since M_n has $n_+ = 2$, the associated tube domain $\mathcal{D}(N)$ has two connected components. We have denoted one of them by $\mathcal{D}(N)^+$ and we know that is biholomorphic to $\mathbb{H}$. Also Γ_{M_n} as a subgroup of $O(N)$ acts on $D(N)$.

Definition 5.1. We denote by Γ'_{M_n} the subgroup of Γ_{M_n} that stabilizes $\mathcal{D}(N)^+$.

Definition 5.2. The *indefinite orthogonal group* $O(p, q)$ is the Lie group of linear transformations of an n-dimensional real vector space that leave invariant a nondegenerate, symmetric bilinear form of signature (p, q) where $n = p + q$.
We also define the *indefinite special orthogonal group* $SO(p, q)$ as the subgroup of $\mathrm{O}(p, q)$ consisting of all elements with determinant 1. This is not connected and has two connected components.

We note that over $\mathbb{R}$ two lattices with the same signature (p, q), define isomorphic indefinite orthogonal groups $O(p,q)$. With this in mind, we identify $O(2,1)$ with $O(N\otimes\mathbb{R})$. Then $O(2,1)$ acts on the period domain as seen in Remark 3.11. From now on, we assume that $l>2$.

Definition 5.3. We denote by $O(2,1)'$ the subgroup of $\mathrm{O}(2,1)$ which stabilizes a connected component of $\mathcal{D}(N)$. Then we define $SO(2,1)'=SO(2,1)\cap O(2,1)'$.

We recall that $\Gamma_{M_n}(l)$ is the following subgroup of Γ_{M_n} :

$$\Gamma_{M_n}(l)=\{g\in\Gamma_{M_n}:g\equiv 1 \mod l\}.$$

We also define $\Gamma'_{M_n}(l)$ to be the subgroup of $\Gamma_{M_n}(l)$ which stabilizes a connected component of D_{M_n}.

Lemma 5.4. The subgroup $\Gamma'_{M_n}(l)$ is contained in $\mathrm{SO}(2,1)'$.

Proof. By definition we have that $\Gamma'_{M_n}(l)$ is contained in $\mathrm{O}(2,1)'$. Thus, an element of $\Gamma'_{M_n}(l)$ must have determinant ±1 as they are orthogonal transformations. We need to exclude the case where the determinant is -1. Indeed, let $g\in\Gamma'_{M_n}(l)$. If we had det $g=-1$ then this would mean that det $g\equiv -1 \mod l$, which is a contradiction as $l>2$. The statement follows. □

For our computations it is also important to look at the discriminant group of the orthogonal complement of M_n which is $U\oplus\langle 2n\rangle$.

Remark 5.5. We now look at $A(U\oplus\langle 2n\rangle)\simeq A(U)\oplus A(\langle 2n\rangle)$. We have seen that $A(U)$ is trivial. Let e be the generator of $\langle 2n\rangle$. Now, any linear functional $h:\langle 2n\rangle=\mathbb{Z}e\to\mathbb{Z}$ is given by $h(e)\left(\frac{e}{2n}.-\right)$ where the latter denotes the functional $x\mapsto h(e)\left(\frac{e}{2n}.x\right)$. Thus we have that $A(\langle 2n\rangle)=\mathbb{Z}/2n\mathbb{Z}[(\frac{e}{2n}.-)]$, i.e generated by the class of $\left(\frac{e}{2n}.-\right)$.

Proposition 5.6. Let σ be an element of $\mathrm{O}(U\oplus\langle 2n\rangle)^*$ and e the generator of $\langle 2n\rangle$. Then σ maps e to $e+2nx$ for some $x\in U\oplus\langle 2n\rangle$.

Proof. We have seen in Remark 5.5 that the discriminant group, $\mathrm{A}(U\oplus\langle 2n\rangle)$ is generated by the class of $\left(\frac{e}{2n}.-\right)$ under the natural identification with $\mathrm{Hom}(U\oplus\langle 2n\rangle,\mathbb{Z})$. Consider an element $\sigma\in O(U\oplus\langle 2n\rangle)^*$, then we have that $\sigma_{\mathbb{Q}}\left(\frac{1}{2n}e\right)\equiv\frac{e}{2n} \mod \Lambda$. By definition this means that there exists an $x\in\Lambda$ such that $\sigma_{\mathbb{Q}}\left(\frac{1}{2n}e\right)=\frac{1}{2n}e+x$. This implies that $\sigma(e)=e+2nx$ as required. □

Now we will adjust Theorem 7.1 from [4] to level structures.

5.2 Determining the subgroup

We recall that the moduli space of ample M_n-polarized K3 surfaces with level l-structure is an open subvariety of $\mathcal{D}(N)/\Gamma_{M_n}(l)$. Then we have that

$$\mathcal{D}(N)/\Gamma_{M_n}(l) \simeq \mathcal{D}(N)^+/\Gamma'_{M_n}(l)$$

where $\Gamma'_{M_n}(l)$ is as seen in Definition 5.3. The main result of this chapter is the following.

Theorem 5.7. Let l be a prime power and $l > 2$. Under the identification of $\mathcal{D}(N)^+$ with $\mathbb{H}$ as seen in Remark 3.14, we have that

$$\Gamma'_{M_n}(l) = \Gamma_0(nl) \cap \Gamma(l)$$

up to conjugation with an element of $\mathrm{PSL}(2, \mathbb{R})$.

Before continuing with the proof we will define certain notions and introduce a couple of propositions we will need throughout the proof.

Remark 5.8. Now similarly to Remark 3.14 we have an identification of $\mathbb{H}$ with $\mathcal{D}(N)^+$ given by

$$\begin{aligned} \phi\colon \mathbb{H} &\to \mathcal{D}(N)^+ \\ z &\mapsto \left(z^2 : -z/\sqrt{n} : 1\right) \end{aligned}$$

Proposition 5.9. Let n be a positive natural number. The map

$$A : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SO}(2, 1)$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto A(g) := \begin{pmatrix} a^2 & -2ab\sqrt{n} & b^2 \\ -ac/\sqrt{n} & ad+bc & -bd/\sqrt{n} \\ c^2 & -2\sqrt{n}cd & d^2 \end{pmatrix} \in SO(2,1)$$

induces an isomorphism between $\mathrm{PSL}(2, \mathbb{R})$ with $\mathrm{SO}(2, 1)'$.

We note that the map $A(g)$ is given with respect to a basis $(f, e, -g)$ of $U \oplus \langle 2n \rangle$

Proof. A straightforward computation shows that the matrix $A(g)$, is indeed an isometry. We will now check that the kernel of A is just $\{\pm 1\}$. If $A(g) = I_3$ then we get that $a = \pm 1, b = c = 0, d = \pm 1$ which proves exactly that $Ker(A) = \{\pm 1\}$. Thus, we get that $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm 1\}$ is isomorphic to Im(A). Now we will check that

$\mathrm{Im}(A) = SO(2,1)'$. By the identification of Remark 5.8 we get an induced isomorphism $\tilde{A} : PSL(2,\mathbb{Z}) = \mathrm{Aut}(\mathbb{H}) \simeq \mathrm{Aut}(\mathcal{D}(N)^+$ given by

$$g \mapsto \phi \circ g \circ \phi^{-1}.$$

For any $g \in \mathrm{Aut}(\mathbb{H})$ the map $\tilde{A}(g)$ is exactly $A(g)$. From this we get that $\mathrm{Im}(A) = \mathrm{Aut}(\mathcal{D}(N)^+)$. However we have that

$$\begin{aligned} SO(2,1)' &\subseteq \mathrm{Aut}(D(N)^+) \\ \mathrm{Im}(A) &\subseteq SO(2,1)'. \end{aligned}$$

From this we conclude that $\mathrm{Im}(A) = SO(2,1)'$. $\square$

Definition 5.10. As we have seen, we have that $\Gamma_{M_n}(l)'$ is contained in SO(2,1)$'$. We denote by $\Gamma'(l)$ the preimage of $\Gamma'_{M_n}(l)$ under A.

The proof of Theorem 5.7 follows from the following proposition.

Proposition 5.11. Let $l > 2$ be a power of a prime and $n > 0$. Then for the element

$$\gamma = \begin{pmatrix} 1/n^{1/4} & 0 \\ 0 & n^{1/4} \end{pmatrix}$$

we have that

$$\gamma\Gamma'(l)\gamma^{-1} = \left\{ \begin{pmatrix} a & b \\ nlc & d \end{pmatrix} \middle| \begin{array}{l} a,b,c,d \in \mathbb{Z}, \\ a \equiv d \equiv 1 \mod l, \\ b \equiv c \equiv 0 \mod l \end{array} \right\} = \Gamma_0(nl) \cap \Gamma(l)$$

Proof. For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'(l)$ we have that $A(g) \in O(N)$, which means that as an automorphism of a $\mathbb{Z}$-module it must have integer entries. By the form of the matrix $A(g)$ we get that a^2 is an integer. This implies that $a = a_1\sqrt{a_2}$ where the a_i are integers and a_2 is square-free. Similarly, $b = b_1\sqrt{b_2}$, $c = c_1\sqrt{c_2}$ and $d = d_1\sqrt{d_2}$ for positive integers $b_, c_i, d_i$ and square-free b_2, c_2, d_2.

We get that $2ab\sqrt{n}$ is an integer and thus, $ab\sqrt{n}$ is also an integer as it is a rational number which is algebraically integral. We write $n = k^2\tilde{n}$ with $\tilde{n}$ square-free. This means that

$$ab\sqrt{n} = a_1\sqrt{a_2}b_1\sqrt{b_2}\sqrt{n} = ka_1b_1\sqrt{a_2b_2\tilde{n}} \in \mathbb{Z}.$$

This forces $\sqrt{a_2b_2\tilde{n}}$ to be an integer. Specifically, $a_2b_2 = s^2\tilde{n}$ for some integer s. By the last equality, since a_2 and b_2 are square-free, we must have that s divides a_2 and s divides b_2 in order to get that s^2 divides a_2b_2. Furthermore, we have that no higher power of s divides a_2 or b_2. By construction, we have $a^2 = a_1^2a_2$. Since s divides a_2 but s^k does not divide a_2 for $k \geq 2$, it follows that s divides a_1^2 in order to satisfy the fact that a^2 is a square. Similarly, we get that s divides b_1^2. So s divides a_1^2, b_1^2, and $ab\sqrt{n}$. This means that s divides $\det A(g) = 1$ since $A(g) \in \mathrm{SO}(2,1)$. Therefore $s = 1$ and $a_2b_2 = \tilde{n}$. Similarly, using that $cd\sqrt{n}$ is an integer we find that $c_2d_2 = \tilde{n}$. Using the above results we get

$$ab\sqrt{n} = ka_1b_1a_2b_2.$$

Using that $A(g) \in O(N)$ we have that

$$\frac{ac}{\sqrt{n}} = \frac{a^2bc}{ka_1b_1a_2b_2} = \frac{a_1c_1}{k}\sqrt{\frac{c_2}{b_2}} \in \mathbb{Z}.$$

This forces $\sqrt{\frac{c_2}{b_2}}$ to be a rational number and k to divide a_1c_1 as c_2 and b_2 are square-free. In particular, $\sqrt{\frac{c_2}{b_2}}$ is actually an integer. If b_2 had a prime factor not dividing c_2 then since b_2 is square-free this would imply that $\sqrt{\frac{c_2}{b_2}}$ is not in $\mathbb{Q}$ which is a contradiction. Hence, $b_2 \mid c_2$ and we get that $\sqrt{\frac{c_2}{b_2}}$ is an integer. This also implies that $c_2 = b_2$. Similarly, using that using that $\frac{bd}{\sqrt{n}} \in \mathbb{Z}$, we get that

$$\sqrt{\frac{a_2}{d_2}} \in \mathbb{Z}, \quad k \mid b_1d_1.$$

This means that that $d_2 = a_2$. Combining all of the above we see that $A(g)$ looks as follows

$$A(g) = \begin{pmatrix} a_1^2a_2 & -2a_1a_2b_1b_2k & b_1^2b_2 \\ -a_1c_1/k & a_1d_1a_2 + b_1c_1b_2 & -b_1d_1/k \\ c_1^2b_2 & -2c_1d_1c_2b_2k & d_1^2a_2 \end{pmatrix}.$$

By Proposition 5.6 and using the fact that $A(g) \in O(U \oplus \langle 2n \rangle)^*$ we get that $ad + bc \equiv 1$

mod $2n$ since the matrix $A(g)$ is with respect to the basis $(f, e, -g)$. Now we also have that $ad - bc = 1$. Combining the above we get that n divides bc. But $n = k^2\tilde{n} = k^2 a_2 b_2$ and $bc = b_1 c_1 \sqrt{b_2}\sqrt{c_2} = b_1 c_1 b_2$ so it follows that $k^2 a_2$ divides $b_1 c_1$.
Now let p be a prime number which divides a_2. Then p divides $k^2 a_2$ which in turn divides $c_1 b_1$. In the case of $p \mid b_1$ we have that p divides the first of row of $A(g)$ and thus divides the determinant which is 1. This forces p to be equal to 1 which is a contradiction. So p divides c_1 and as a result divides the third row of the matrix, which is a contradiction. Thus, a_2 is equal to 1 and similarly we get that $d_2 = 1$. This means that

$$\begin{aligned} a_2 = d_2 &= 1 \\ b_2 = c_2 &= \tilde{n} \\ k^2 &\mid b_1 c_1. \end{aligned}$$

Now let p^v be the highest power of a prime p, which divides k. Since $k^2 \mid b_1 c_1$ we have that $p^{2v} \mid b_1 c_1$. From the entries of $A(g)$, p^v also divides $a_1 c_1$ and $b_1 d_1$. Thus, p^v divides b_1 or c_1. If p^v divided b_1 then it would also divide c_1 because otherwise it would divide a_1 and therefore, the first row of $A(g)$ which is a contradiction. Similarly, if p^v divides c_1 then it also divides b_1. From this we can conclude that p^v divides both b_1 and c_1 and thus, k divides both b_1 and c_1 as well as p was arbitrary. We write $c_1 = kc_1'$ and $b_1 = kb_1'$ and we have that g is of the following form.

$$g = \begin{pmatrix} a_1 & kb_1'\sqrt{\tilde{n}} \\ kc_1'\sqrt{\tilde{n}} & d_1 \end{pmatrix}.$$

Now we know that the map A looks like

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a_1^2 & -2a_1 b_1 \tilde{n} k & b_1^2 \tilde{n} \\ \frac{-a_1 c_1}{k} & a_1 d_1 + b_1 c_1 \tilde{n} & \frac{b_1 d_1}{k} \\ c_1^2 \tilde{n} & -2c_1 d_1 \tilde{n}^2 k & d_1^2 \end{pmatrix}$$

where

$$
\begin{aligned}
&n = k^2\tilde{n} \text{ with } \tilde{n} \text{ square-free}\\
&a = a_1\\
&b = b_1\sqrt{\tilde{n}} = kb_1'\sqrt{\tilde{n}} = b_1'\sqrt{n}\\
&c = c_1\sqrt{\tilde{n}} = kc_1'\sqrt{\tilde{n}} = c_1'\sqrt{n}\\
&d = d_1
\end{aligned}
$$

and all the numbers are integers. Since $A(g)$ lies in $\Gamma_{M_n}(l)'$ we have the following conditions

$a_1^2 \equiv 1 \mod l$	$2a_1b_1\tilde{n}k \equiv 0 \mod l$	$b_1^2\tilde{n} \equiv 0 \mod l$
$a_1c_1' \equiv 0 \mod l$	$a_1d_1 + b_1c_1\tilde{n} \equiv 1 \mod l$	$\frac{b_1d_1}{k} \equiv 0 \mod l$
$c_1^2\tilde{n} \equiv 0 \mod l$	$2c_1d_1\tilde{n}^2k \equiv 0 \mod l$	$d_1^2 \equiv 1 \mod l$

We focus on the equations

$$c_1' \equiv 0 \mod l \tag{5.1}$$

$$c_1^2\tilde{n} \equiv 0 \mod l. \tag{5.2}$$

Since $a_1^2 \equiv 1 \mod l$ we have that $(a_1, l) = 1$ and by equation 5.1 we get that $c_1' \equiv 0 \mod l$.
Since l is an odd prime power, we get that

$$a_1 \equiv \pm 1 \mod l$$

Similarly we can get that $d_1 \equiv \pm 1 \mod l$. Because we have that $a_1d_1 \equiv 1 \mod l$, this means that $a_1 \equiv d_1 \equiv 1 \mod l$ or $a_1 \equiv d_1 \equiv -1 \mod l$. Also in this case they have the same sign $\mod l$. We gather all our relations to get that

$$
\begin{aligned}
a_1 \equiv d_1 \equiv \pm 1 \mod l\\
b_1' \equiv c_1' \equiv 0 \mod l.
\end{aligned}
$$

using that Equation 5.1 and the fact that $a_1 \equiv \pm 1 \mod l$. Now we note that

$$\begin{pmatrix} 1/n^{1/4} & 0 \\ 0 & n^{1/4} \end{pmatrix} \begin{pmatrix} a_1 & b_1' \sqrt{n} \\ c_1' \sqrt{n} & d_1 \end{pmatrix} \begin{pmatrix} 1/n^{1/4} & 0 \\ 0 & n^{1/4} \end{pmatrix}^{-1} = \begin{pmatrix} a_1 & b_1' \\ n c_1' & d_1 \end{pmatrix}.$$

and this proves the proposition. □

It follows from Theorem 5.7 that $\mathcal{D}(N)/\Gamma_{M_n}(l)$, which contains the moduli space $\mathcal{K}^a_{M_n}(l)$ as an open subvariety is isomorphic to

$$\mathbb{H}/\Gamma_0(n,l) = \mathbb{H}/\Gamma_0(nl) \cap \Gamma(l)$$

Definition 5.12. We denote by $X_0(n,l)$ the natural compactification of $\mathbb{H}/\Gamma_0(n,l)$

$$X_0(n,l) := \mathbb{H}^*/\Gamma_0(n,l).$$

In the next chapter we will compute the genus of $X_0(n,l)$.

Chapter 6

The genus of $X_0(n, l)$

Let $n \geq 1$ and l an odd prime power. We recall $X_0(n, l)$ is modular curve given as

$$X_0(n, l) = \mathbb{H}^*/\Gamma_0(nl) \cap \Gamma(l)$$

and $\Gamma_0(n, l) = \Gamma_0(nl) \cap \Gamma(l)$. We will compute its genus using the Riemann Hurwitz formula. From now on the genus will be denoted as $g_0(n, l)$ and we know that it satisfies

$$g_0(n, l) = 1 + \frac{\mu}{12} - \frac{u_2}{4} - \frac{u_3}{3} - \frac{u_\infty}{2}$$

where μ is the index $[\mathrm{PSL}(2, \mathbb{Z}) : \Gamma_0(n, l)]$, u_2, u_3 the number of non-equivalent elliptic elements of order 2 and 3 respectively and u_∞ the number of non-equivalent cusps. Using that $\Gamma_0(nl) \cap \Gamma(l)$ is a subgroup of $\Gamma(l)$ we can deduce, using Proposition 4.7, that $u_2 = u_3 = 0$. So it all comes down to computing the index and the number of non-equivalent cusps. Firstly, we will compute the index. For this purpose, we need the following tools.

Proposition 6.1. ([14], Remark 2.12) Let a be a natural number. The index of $\Gamma_0(a)$ in $\mathrm{PSL}(2, \mathbb{Z})$ is :

$$[\mathrm{PSL}(2, \mathbb{Z}) : \Gamma_0(a)] = a \prod_{\substack{p \text{ prime} \\ p|a}} \left(1 + \frac{1}{p}\right)$$

Remark 6.2. Using Proposition 5.11, we have that $\Gamma_0(nl) \cap \Gamma(l)$ is congruent to $\Gamma_0(nl^2) \cap \Gamma_1(l)$. Since congruent subgroups have the same genus, we can focus on the latter.

In the following lemma, we will work with subgroups of $\mathrm{SL}(2,\mathbb{Z})$ which we distinguish by their image in $\mathrm{PSL}(2,\mathbb{Z})$ with tilde. Specifically, $\widetilde{\Gamma_0(nl^2)}$, $\widetilde{\Gamma_1(l)}$ are the usual subgroups of $\mathrm{SL}(2,\mathbb{Z})$.

Lemma 6.3. Assume that $nl^2 \geq 3$, then we have that

$$[\widetilde{\Gamma_0(nl^2)} : \widetilde{\Gamma_0(nl^2)} \cap \widetilde{\Gamma_1(l)}] = \phi(l)$$

where ϕ denotes the Euler's totient function.

Proof. First of all we note that

$$[\widetilde{\Gamma_0(nl^2)} : \widetilde{\Gamma_0(nl^2)} \cap \widetilde{\Gamma_1(l)}] = [\widetilde{\Gamma_0(nl^2)}/\widetilde{\Gamma(nl^2)} : (\widetilde{\Gamma_0(nl^2)} \cap \widetilde{\Gamma_1(l)})/\widetilde{\Gamma(nl^2)}].$$

In order to compute the above index , we note that

$$0 \to \widetilde{\Gamma(nl^2)} \xrightarrow{f} \mathrm{SL}(2,\mathbb{Z}) \xrightarrow{g} \mathrm{SL}(2,\mathbb{Z}/nl^2\mathbb{Z}) \to 0$$

is a short exact sequence as in Section 1.6 of [12]. We have that the index $[\widetilde{\Gamma_0(nl^2)}/\widetilde{\Gamma(nl^2)}]$ is equal to the cardinality of the image of $\widetilde{\Gamma_0(nl^2)}$ under the map g, and similarly the index $[(\widetilde{\Gamma_0(nl^2)} \cap \widetilde{\Gamma_1(l)})/\widetilde{\Gamma(nl^2)}]$ is equal to the cardinality of the image of $\widetilde{\Gamma_0(nl^2)} \cap \widetilde{\Gamma_1(l)}$. We note that for the first index, its image has matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

where $a \in \mathbb{Z}/nl^2\mathbb{Z}$ is a unit and $b \in \mathbb{Z}/nl^2\mathbb{Z}$. This means that the image has cardinality $nl^2\phi(nl^2)$. Now regarding the second index, the image contains the same matrices but this time we also require $a \equiv 1 \mod l$. Therefore, the cardinality is equal to $nl^2 \cdot |a \in (\mathbb{Z}/nl^2\mathbb{Z})^* : a \equiv 1 \mod l|$. The latter set is seen to have cardinality $\dfrac{\phi(nl^2)}{\phi(l)}$. The term nl^2 corresponds the nl^2 free choices of b. This proves the required claim. □

From the above lemma we can conclude that in $\mathrm{PSL}(2,\mathbb{Z})$, we have

$$[\Gamma_0(nl^2) : \Gamma_0(nl^2) \cap \Gamma_1(l)] = \frac{\phi(l)}{2}$$

Corollary 6.4. The index $[PSL(2,\mathbb{Z}) : \Gamma_0(nl^2) \cap \Gamma_1(l)]$ satisfies

$$[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma_0(nl^2) \cap \Gamma_1(l)] = \frac{\phi(l)nl^2}{2} \cdot \prod_{\substack{p \text{ prime} \\ p|nl^2}} \left(1 + \frac{1}{p}\right)$$

Proof. We have that

$$[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma_0(nl^2) \cap \Gamma_1(l)] = [\Gamma_0(nl^2) : \Gamma_0(nl^2) \cap \Gamma_1(l)] \cdot [PSL(2,\mathbb{Z}) : \Gamma_0(nl^2)]$$

and thus the required index satisfies

$$[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma_0(nl^2) \cap \Gamma_1(l)] = \frac{\phi(l)}{2} \cdot nl^2 \cdot \prod_{\substack{p \text{ prime} \\ p|nl^2}} \left(1 + \frac{1}{p}\right)$$

□

Now we will focus on computing the number of non-equivalent cusps of $\Gamma_0(nl^2) \cap \Gamma_1(l)$.

Proposition 6.5. The number of non-equivalent cusps of $\Gamma_0(nl^2) \cap \Gamma_1(l)$ is equal to

$$\sum_{a/c} \frac{[\Gamma_0(nl^2) : \Gamma_0(nl^2) \cap \Gamma_1(l)]}{[\Gamma_0(nl^2)_{a/c} : (\Gamma_0(nl^2) \cap \Gamma_1(l))_{a/c}]}$$

where the sum is taken over all the non-equivalent cusps of $\Gamma_0(nl^2)$ and the subgroups $\Gamma_0(nl^2)_{a/c}, (\Gamma_0(nl^2) \cap \Gamma_1(l))_{a/c}$ denote respective stabilizer subgroups.

Proof. The proof follows directly from Proposition 4.10 and Proposition 4.4. □

Before we proceed with the computations of this number, we note a set of representatives of the cusps of $\Gamma_0(nl^2)$. For $a, b \in \mathbb{Z}$ we denote (a, b) the greatest common divisor of a and b.

Proposition 6.6. ([14], Theorem 2.25) The non-equivalent cusps of $\Gamma_0(n)$ can be given by

$$\left[\frac{a_{i,j}}{c_i}\right]$$

such that $(a_{i,j}, c_i) = 1$, c_i is a divisor of n, $0 \leq a_{i,j} < c_i$ and we have that $a_{i,j}$ is not equivalent to $a_{i,j'}$ modulo $\left(\frac{n}{c_i}, c_i\right)$ when $j \neq j'$ for all i.

Since we now have a complete list of cusp representatives we can compute the index of the respective stabilizers in the form given by the above proposition. For this we will need the following proposition.

Proposition 6.7. Let a, c be integers such that $(a, c) = 1$ and c divides nl^2. We have that

$$[\Gamma_0(nl^2)_{a/c} : (\Gamma_0(nl^2) \cap \Gamma_1(l))_{a/c}] = 1.$$

Proof. Proposition 2.31 in [14] mentions that for every subgroup Γ of $\mathrm{PSL}(2, \mathbb{Z})$ and a/c a cusp of this group, $\Gamma_{a/c}$ is generated by the element

$$A = \begin{pmatrix} 1 - act & a^2x \\ -c^2t & 1 + act \end{pmatrix}$$

for integers a, b, c, d with $ad - bc = 1$ and t being the smallest positive integer such that $A \in \Gamma$. Thus, we see that the index

$$[\Gamma_0(nl^2)_{a/c} : (\Gamma_0(nl^2) \cap \Gamma_1(l))_{a/c}]$$

is equal to smallest positive integer t such that

$$\frac{actnl^2}{(nl^2, c^2)} \equiv 0 \mod l.$$

Let $c = c_0(l, c)$ for some integer $c_0 \in \mathbb{Z}$. Then we have that

$$(nl^2, c^2) = (l, c)^2 \cdot (n, c_0^2).$$

This means that

$$\frac{acnl^2}{(nl^2, c^2)} = \frac{acnl^2}{(l, c)^2 (n, c_0^2)} = \frac{c}{(l, c)} \cdot \frac{l}{(l, c)} \cdot \frac{n}{(n, c_0^2)} \cdot al$$

From the above, we note that $\frac{acnl^2}{(nl^2, c^2)}$ is divisible by l. This directly implies that the required index is 1. $\square$

Proposition 6.8. ([3], Section 8, page 103) Let Ω_{nl^2} denote the set of non-equivalent cusps of $\Gamma_0(nl^2)$. Then

$$|\Omega_{nl^2}| = \sum_{d|nl^2} \phi((d, nl^2/d))$$

Using Proposition 6.5, 6.7 and 6.8 we can conclude that the number of non-equivalent cusps of $\Gamma_0(nl) \cap \Gamma(l)$ is equal to

$$\frac{\phi(l)}{2} \cdot \sum_{d|nl^2} \phi((d, nl^2/d)).$$

From the above we get the following theorem.

Theorem 6.9. Let $l > 2$. The genus of $\mathbb{H}^*/\Gamma_0(nl^2) \cap \Gamma_1(l)$ is given by the formula

$$\begin{aligned} g_0(n,l) &= 1 + \frac{\phi(l)nl^2}{24} \cdot \prod_{\substack{p \text{ prime} \\ p|nl}} \left(1 + \frac{1}{p}\right) - \sum_{a/c} \frac{\phi(l)}{4} \\ &= 1 + \frac{\phi(l)nl^2}{24} \cdot \prod_{\substack{p \text{ prime} \\ p|nl}} \left(1 + \frac{1}{p}\right) - \sum_{d|nl^2} \phi((d, nl^2/d)) \cdot \frac{\phi(l)}{8} \end{aligned}$$

where the sum, $\sum_{a/c}$, is over the set of non-equivalent cusps of $\Gamma_0(nl^2)$.

The following contains example values for $g_0(n,l)$ for low values of n, l. The Sage code used for this is included in Appendix B.

l	Genus for $n=2$	Genus for $n=3$	Genus for $n=4$	Genus for $n=5$	Genus for $n=6$	Genus for $n=7$
3^1	0	1	1	3	4	5
3^2	46	55	109	127	190	181
3^3	1864	1945	3889	4051	6076	5509
3^4	56134	56863	113725	115183	172774	154549
3^5	1568080	1574641	3149281	3162403	4743604	4225285
5^1	4	9	13	16	37	29
5^2	1576	2201	3301	2876	6901	4701
5^3	226876	305001	457501	384376	922501	617501
5^4	29109376	38875001	58312501	48671876	116812501	77937501
5^5	3657421876	4878125001	7317187501	6099609376	14639062501	9760937501
7^1	19	33	49	61	121	78
7^2	13231	18033	27049	27637	55273	32586
7^3	4883635	6530721	9796081	9824893	19649785	11479182
7^4	1692027919	2256978417	3385467625	3386879413	6773758825	3952182858
7^5	581195707219	774973728033	1162460592049	1162529769661	2325059539321	1356325084878

Table 6.1: Genus Results for small n and $l = 3^k, 5^k, 7^k$

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Appendix A

The genus of $X_0(n,l)^+$

In this section we expand the subgroup $\Gamma_0(n,l) = \Gamma_0(nl) \cap \Gamma(l)$ by the Fricke involution F_n, and we compute a bound for the genus of the modular curve

$$\mathbb{H}^*/\langle \Gamma_0(n,l), F_n \rangle.$$

for $l > 2$.

Definition A.1. We define the *Fricke involution* as the element

$$F_n = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}.$$

This is an order 2 element of $\mathrm{PSL}(2,\mathbb{R})$. The subgroup of $\mathrm{PSL}(2,\mathbb{R})$ given by $\langle \Gamma_0(n), F_n \rangle$ is called the *Fricke modular group of level n* and is denoted by $\Gamma_0(n)^+$.

We recall that the subgroups $\Gamma_0(nl^2) \cap \Gamma_1(l)$ and $\Gamma_0(nl) \cap \Gamma(l)$ are conjugate and since they have the same genus it is enough to work with either of them. When we conjugate to this new subgroup, we must also conjugate the Fricke involution F_n which in turn becomes F_{nl^2} after conjugating with

$$\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$$

as seen in Proposition 4.11.

Definition A.2. We define $X_0(n,l)^+$ to be the modular curve

$$X_0(n,l)^+ := \mathbb{H}^*/\langle \Gamma_0(nl^2) \cap \Gamma_1(l), F_{nl^2} \rangle.$$

and we also define the conjugate

$$\Gamma_0(n,l)' = \Gamma_0(nl^2) \cap \Gamma_1(l)$$

We want to compute a bound for the the genus of $X_0(n,l)^+$ and to this goal, we will need the Riemann-Hurwitz genus formula.

Theorem A.3. (Riemann-Hurwitz formula, [6], Theorem 8.5) Let $f : C' \to C$ be a morphism of smooth complete curves of degree N then

$$2g(C') - 2 = N(2g(C) - 2) + \sum_{p \in C'} (e_f(p) - 1)$$

where g are the respective genera, and $e_f(p)$ the ramification index of f the point p.

Consider the morphism of curves

$$\phi : \mathbb{H}^*/\Gamma_0(n,l)' \to \mathbb{H}^*/\langle(\Gamma_0(n,l)', F_{nl^2}\rangle. \tag{A.1}$$

One can check that $\Gamma_0(n,l)'$ is a normal subgroup of $\langle\Gamma_0(n,l)', F_{nl^2}\rangle$. Hence, since F_{nl^2} is of order 2 the degree of ϕ is 2. We know then that the ramification index will be either 1 or 2 for each point. We can see that only the points of ramification index 2 will contribute to the Riemann-Hurwitz genus formula. The points of ramification index 2 correspond to the fixed points of the Fricke involution in $\mathbb{H}^*/\Gamma_0(n,l)'$. Cusp-wise, we will see in Proposition A.4 that it fixes no cusp, since $l > 2$, so we will bound the number of fixed points of F_{nl^2} in $\mathbb{H}^*/\Gamma_0(n,l)'$ and we can compute a bound for the genus of $X_0(n,l)^+ = \mathbb{H}^*/\langle(\Gamma_0(n,l)', F_{nl^2}\rangle$. For this we will use the morphism (A.1) and Riemann-Hurwitz. We continue with checking how the Fricke involution acts on the cusps of $\Gamma_0(n,l)'$, and then we turn to the fixed points.

Proposition A.4. Let F_{nl^2} be the Fricke involution. Then F_{nl^2} acts on the cusps of $X_0(n,l)$ without fixed points.

Proof. Consider the morphism

$$f : \mathbb{H}^*/\Gamma_0(n,l)' \to \mathbb{H}^*/\Gamma_0(nl^2).$$

If z is a fixed point in $\mathbb{H}^*/\Gamma_0(n,l)''$ of F_{nl^2} then $f(z)$ is a fixed point of F_{nl^2} in $\mathbb{H}^*/\Gamma_0(nl^2)$. Thus, every cusp point of $X_0(n,l)$ that F_{nl^2} fixes has to lie above a cusp point of $X_0(nl^2) = \mathbb{H}^*/\Gamma_0(nl^2)$ that F fixes. However, F_{nl^2} acts without fixed points on the cusp points of $X_0(nl^2)$ as mentioned in [4], Proposition 7.3 since $l > 2$ implies that $nl^2 \neq 4$. $\square$

Now we continue to the case of the fixed points. We will need the following theorem to classify the fixed points of the Fricke involution in $\mathbb{H}^*/\Gamma_0(n,l)$. The following theorem holds for a general m but we will focus on $m = nl^2$.

Theorem A.5. ([4], Theorem 7.3) For a general $m \in \mathbb{N}$, let $S \subset \mathbb{H}^*/\Gamma_0(m)^+$ be the set of orbits of points of the form $\frac{c}{b} + \frac{i}{b\sqrt{m}}$ for $b, c \in \mathbb{Z}$ and $b \mid cm^2 + 1$. Then these are the classes of the fixed points of the Fricke involution in $\mathbb{H}^*/\Gamma_0(m)$ and

$$|S| = \begin{cases} 1 & \text{if } m \leq 4 \\ 2h(-4m) & \text{if } m \equiv 7 \mod 8 \\ 4h(-4m)/3 & \text{if } m \equiv 3 \mod 8, m \geq 4 \\ h(-4m) & \text{otherwise} \end{cases}$$

where $h(k)$ denotes the number of classes of primitive binary quadratic forms of discriminant k.

From now on we denote by S the above set for $m = nl^2$. The fixed points of F_{nl^2} in $\mathbb{H}^*/\Gamma_0(nl^2)$ will split into possibly more points in $X_0(n,l)$ and not all of them will be necessarily fixed points whose cardinality bounds the number of fixed points in $X_0(n,l)^+$. Meaning that for the morphism

$$\mathbb{H}^*/\Gamma_0(n,l)' \xrightarrow{f} \mathbb{H}^*/\Gamma_0(nl^2).$$

a fixed point of F_{nl^2} in $\mathbb{H}^*/\Gamma_0(nl^2)$ will split into at most $f^{-1}(z)$ many fixed points in $X_0(n,l)$. To this goal, we want to apply Proposition 4.10 to the above set S. Thus, we will need to compute the stabilizers at the points of the set S for the subgroups $\Gamma_0(nl^2), \Gamma_0(n,l)'$. For these, we see that the points of S are actually of a special form which helps in our calculations as seen below.

Remark A.6. According to Section 1.4 in [12], page 15, all elliptic points of $\Gamma(1)$ which are of order 2 are equivalent to i and those of order 3 are equivalent to $e^{2\pi i/3}$. From this we deduce that

$$z = \frac{a}{c} + \frac{i}{c}, \text{ if } z \text{ is fixed by an order 2 element}$$
$$z = \frac{2a \pm 1}{2c} + \frac{\sqrt{3}}{2c}, \text{ if } z \text{ is fixed by an order 3 element}$$

for $a, c \in \mathbb{Z}$. Thus, the stabilizers H_z for H any subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ are either trivial or of order 2 or 3. From Theorem A.5 one sees that the points of S can only be of the above form when $nl^2 = 3$ which does not happen when $l > 2$. Thus for any $z \in S$ we have that $\mathrm{PSL}(2,\mathbb{Z})_z$ is trivial, and the stabilizer of any subgroup at this point is also trivial. Specifically, $\Gamma_0(nl^2)_z$ and $\Gamma_0(n,l)'_z$ are trivial.

Proposition A.7. The morphism f

$$\mathbb{H}^*/\Gamma_0(n,l)' \xrightarrow{f} \mathbb{H}^*/\Gamma_0(nl^2).$$

is unramified above the fixed points of the Fricke involution in $\mathbb{H}^*/\Gamma_0(nl^2)$.

Proof. We note that in the above equation, if z is a fixed point in $\mathbb{H}^*/\Gamma_0(n,l)'$ of F_{nl^2} then $f(z)$ is a fixed point in $\mathbb{H}^*/\Gamma_0(nl^2)$ of F_{nl^2}. Let S denote the set of fixed points of F in $\mathbb{H}^*/\Gamma_0(nl^2)$ which are of the form given in Theorem A.5. From Proposition 4.10 we get that if $x \in S$

$$\begin{aligned} f^{-1}(x) &= \frac{[\Gamma_0(nl^2) : \Gamma_0(n,l)']}{[\Gamma_0(nl^2)_z : \Gamma_0(n,l)'_z]} \\ &= [\Gamma_0(nl^2) : \Gamma_0(n,l)'] \\ &= \deg f \end{aligned}$$

by also using Remark A.6. Thus, f is unramified at x. □

Now we know that the number of fixed points of F in $\mathbb{H}^*/\Gamma_0(n,l)'$ is at most $|f^{-1}(S)|$.

Corollary A.8. The cardinality of $f^{-1}(S)$ is given by

$$|f^{-1}(S)| = [\Gamma_0(nl^2) : \Gamma_0(nl^2) \cap \Gamma_1(l)]|S|.$$

Proof. The above follows by Proposition A.7 and the fact that $[\Gamma_0(nl^2) : \Gamma_0(n,l)'] = \deg f$ □

The index $[\Gamma_0(nl^2) : \Gamma_0(nl^2) \cap \Gamma_1(l)]$ has been computed in Lemma 6.3. The Fricke involution has at most $|f^{-1}(S)|$ many fixed points in $\mathbb{H}^*/\Gamma_0(n,l)'$ and now we have a way to compute this number.

Proposition A.9. The cardinality of $f^{-1}(S)$ is given by

$$\frac{\phi(l)}{2} \cdot |S|$$

and more specifically,

$$|f^{-1}(S)| = \begin{cases} \dfrac{\phi(l)}{2} & \text{if } nl^2 = 2, 3 \\ \phi(l) \cdot h(-4nl^2) & \text{if } nl^2 \equiv 7 \mod 8 \\ \dfrac{2\phi(l)}{3} \cdot h(-4nl^2) & \text{if } nl^2 \equiv 3 \mod 8, nl^2 > 4 \\ \dfrac{\phi(l)}{2} \cdot h(-4nl^2) & \text{otherwise} \end{cases}$$

Proof. This follows from Corollary A.8 and Lemma 6.3. □

We note that the first condition happens exactly when $l = 1$ and $n = 2, 3$ respectively which we exclude. As we mentioned before we know that the number of fixed points of F in $\mathbb{H}^*/\Gamma_0(n, l)$ lies between 0 and $|f^{-1}(S)|$. Now we have everything we need to bound the genus of $X_0(n, l)^+$ using the Riemann-Hurwitz formula.

Theorem A.10. The genus of $X_0(n, l)^+$, when l is of the form p^k for some prime $l > 2$ and $k \in \mathbb{Z}$, has a lower bound given by

$$g_0(n, l)^+ \geq \frac{g_0(n, l)}{2} + \frac{1}{2} - \frac{|f^{-1}(S)|}{4}$$

where $g_0(n, l)$ is the genus of $X_0(n, l)$. To be precise

$$g_0(n, l)^+ \geq \begin{cases} 1 + \frac{\phi(l)nl^2}{48} \cdot \prod\limits_{\substack{p \text{ prime} \\ p \mid nl}} \left(1 + \dfrac{1}{p}\right) - \sum_{d \mid nl^2} \phi((d, nl^2/d)) \cdot \frac{\phi(l)}{16} & \\ \quad - \frac{\phi(l)}{4} \cdot h(-4nl^2) & \text{if } nl^2 \equiv 7 \text{ mod } 8 \\ 1 + \frac{\phi(l)nl^2}{48} \cdot \prod\limits_{\substack{p \text{ prime} \\ p \mid nl}} \left(1 + \frac{1}{p}\right) - \sum_{d \mid nl^2} \phi((d, nl^2/d)) \cdot \frac{\phi(l)}{16} & \\ \quad - \frac{\phi(l)}{6} \cdot h(-4nl^2) & \text{if } nl^2 \equiv 3 \text{ mod } 8, nl^2 \geq 5 \\ 1 + \frac{\phi(l)nl^2}{48} \cdot \prod\limits_{\substack{p \text{ prime} \\ p \mid nl}} \left(1 + \frac{1}{p}\right) - \sum_{d \mid nl^2} \phi((d, nl^2/d)) \cdot \frac{\phi(l)}{16} & \\ \quad - \frac{\phi(l)}{8} \cdot h(-4nl^2) & \text{otherwise} \end{cases}$$

Proof. This follows directly from the Riemann-Hurwitz formula and Proposition A.9. □

Additionally, we also have an upper bound for the genus $g_0(n, l)^+$.

Proposition A.11. The genus $g_0(n,l)^+$ satisfies

$$g_0(n,l)^+ \leq \frac{g_0(n,l)}{2} + \frac{1}{2} = 1 + \frac{\phi(l)nl^2}{24} \cdot \prod_{\substack{p \text{ prime} \\ p|nl}} \left(1 + \frac{1}{p}\right) - \sum_{d|nl^2} \phi((d, nl^2/d)) \cdot \frac{\phi(l)}{8}.$$

Proof. From the Riemann Hurwitz formula we know that

$$2(g_0(n,l) - 2) = 2(2g_0(n,l)^+ - 2) + \sum_p (e_F(p) - 1) \geq 2(2g_0(n,l)^+ - 2)$$

and thus the result follows from Theorem 6.9. □

Appendix B

Sage Code

Following is the Sage code used to calculate the genus of $X_0(n, l)$ and then the values of Table 6.1.

```
#We define a function which we will call in the print below
def genus_formula(n, l):

   #We compute a product term which will be needed in the genus
      formula of X_0(n,l)

    product=1
    for d in prime_divisors(n*l):
        product*= (1+1/d)

    # number of cusps of Gamma_0(nl^2)

    G = Gamma0(n*(l^2)).ncusps()

    # genus formula
    genus = 1 + (1/24)*euler_phi(l)*n*(l^2)*product - (G/4)*
       euler_phi(l)

    return genus

primes = [3, 5, 7]

# n from 2 to 7
for n in range(2,8):
    print(f"Results for n = {n}:")
```

```
for l_base in primes:
    for k in range(1, 6):
        l = l_base^k
        print(f"Result for n = {n}, l = {l_base}^{k} (l = {l})
            : {genus_formula(n, l)}")
    print()
print()
```