Toric-analytic construction of Kuga-Sato varieties

Tiersma, Samuel

Samuel Tiersma

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Thesis supervisor: dr. Eugenia Rosu

Universiteit Leiden
Mathematisch Instituut

Foreword

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Contents

Chapter 1

Introduction

Given the central role elliptic curves play in number theory and algebraic geometry, it is desirable to describe all isomorphism classes of elliptic curves, i.e. construct a moduli spaces for elliptic curves. In algebraic geometry, an elliptic curve E over a field k may be defined as a smooth projective connected curve of genus 1 over k together with a k-rational point $O \in E(k)$. They are most interesting because there is a unique structure of k-group variety on the curve E for which O is the identity section.

Over the field $\mathbb{C}$ of complex numbers elliptic curves can be studied from an analytic point of view as complex tori. A k-dimensional complex torus $\mathbb{C}^k/L$ is a compact Lie group that is the quotient of $\mathbb{C}^k$ by a co-compact lattice L inside $\mathbb{C}^k$, i.e. a discrete $\mathbb{Z}$-submodule L of $\mathbb{C}$ that is free of rank $2k$. To an elliptic curve $E \to \operatorname{Spec} \mathbb{C}$, the analytification functor defined in Section 2.6 attaches a compact Riemann surface E^{an}. The uniformization theorem asserts that E^{an} is isomorphic to a 1-dimensional complex torus $\mathbb{C}/\Lambda$ for a co-compact lattice Λ in $\mathbb{C}$ unique up to homothety (scaling by a nonzero complex number). In fact, $(\cdot)^{\text{an}}$ provides an equivalence of categories between that of elliptic curves over $\operatorname{Spec} \mathbb{C}$ and 1-dimensional complex tori. In the remainder of this introduction, the term complex elliptic curve will be used as synonym for 1-dimensional complex torus.

The uniformization theorem lies at the heart of the construction of the coarse moduli spaces of elliptic curves $Y(1)$, a (1-dimensional, connected, noncompact) complex manifold whose points are in bijection with isomorphism classes of elliptic curves (see Section 5.1 for a discussion of moduli spaces). Ideally there should exist a holomorphically varying family of elliptic curves over $Y(1)$, called a universal elliptic curve, which effectuates this bijection.

An obstacle hereto is that every elliptic curve E/k admits at least one nontrivial isomorphism, namely inversion, and even more if E has complex multiplication by $\mathbb{C}$. A fundamental technique in moduli theory is to rigidify the objects under study through the endowment of an extra structure. For a 1-dimensional complex torus $E = \mathbb{C}/\Lambda$ we have a canonical isomorphism $H_1(E;\mathbb{Z}) \cong \Lambda$. An H_1-structure on E is an isomorphism $\psi : \mathbb{Z}^2 \to H_1(E;\mathbb{Z})$. The set of H_1-structures on E is denoted $[H_1\text{–str}]_E$ and has a natural action of $\mathrm{SL}_2(\mathbb{Z}) = \operatorname{Aut}(\mathbb{Z}^2)$ given by

$$\mathrm{SL}_2(\mathbb{Z}) \times [H_1\text{–str}]_E \to [H_1\text{–str}]_E\,, \gamma\psi = \psi \circ \gamma^t$$

where γ^t is the transpose of γ.

Let Γ be a congruence subgroup. Then a Γ-structure on E is defined to be an element of the orbit space $\Gamma\backslash [H_1\text{–str}]_E$. We have that $\Gamma(N)$-structures correspond to isomorphisms $\psi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$ such that $e_N(\psi(0,1), \psi(1,0)) = \zeta_N$, where e_N is the Weil pairing on the N-torsion subgroup $E[N]$. We have that $\Gamma_1(N)$- and $\Gamma_0(N)$-structures correspond to points resp. cyclic subgroups of order N on E.

To formulate the existence of a fine moduli space for the moduli problem of elliptic curves with an H_1-structure, we are led to consider holomorphically varying families of elliptic curves. Given a complex manifold M, an elliptic curve over M is a proper submersive holomorphic map $f : E \to M$ with a section $e : M \to E$ such that for every point $m \in M$ the fibre $(E_m, e(m))$ is an elliptic curve. In Section 4.2 we describe a construction of such relative elliptic curves as quotients V/L of a holomorphic line bundle V on M by a co-compact lattice L inside V.

Again, an M-elliptic curve may admit nonidentity morphisms over M. In Section 4.1, following [Conc], we give a construction, functorial in elliptic curves E/M, of a local system $\underline{H}_1(E/M)$ of rank-2 free $\mathbb{Z}$-modules on M whose fibre over $m \in M$ is the first homology group of the fibre E_m with $\mathbb{Z}$-coefficients, that is, one has $\underline{H}_1(E/M)_m = H_1(E_m; \mathbb{Z})$. An H_1-trivialized elliptic curve over M is, by definition, an elliptic curve E/M together with an isomorphism $\mathbb{Z}^2 \times M \to \underline{H}_1(E/M)$ of local systems over M. In Section 5.2 we define a functor $[H_1\text{–str}] : \mathbb{C}\mathbf{Man}^{\text{op}} \to \mathbf{Set}$ sending M to the set of H_1-trivialized elliptic curves over M up to isomorphism over M.

Theorem 1.0.1. *The functor $[H_1\text{–str}]$ is represented by $(\mathcal{E} = V/L, \Psi) \to \mathfrak{H}$, where*

$$\mathbb{Z}^2 \times M \xrightarrow{\Psi} L := \left\{(m\tau + n, \tau) : (m,n) \in \mathbb{Z}^2, \tau \in \mathfrak{H}\right\} \subset V := \mathbb{C} \times \mathfrak{H},$$
$$((m,n),\tau) \mapsto (m\tau + n, \tau).$$

We paraphrase this result by saying that $(\mathcal{E}, \Psi) \to \mathfrak{H}$ is the universal H_1-trivialized elliptic curve.

Now let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. As in the absolute case we define a notion of Γ-structure on a relative elliptic curve $E \to M$, and interpret $\Gamma(N)$-, $\Gamma_1(N)$- and $\Gamma_0(N)$-structures in terms of N-torsion data for integers $N \geq 1$. All elliptic curves with a Γ-structure are rigid if and only if Γ is torsion-free. If this holds, a universal elliptic curve with a Γ-structure $(\mathcal{E}_\Gamma, \Gamma\Psi) \to Y(\Gamma)$ is constructed as the quotient of $(\mathcal{E}, \Gamma\Psi) \to \mathfrak{H}$ by Γ.

We next turn to constructing a complex surface $\mathcal{D}_\Gamma$ that compactifies $\mathcal{E}_\Gamma$. At a regular cusp of Γ, say of width h, the family of elliptic curves $\mathcal{E}_\Gamma \to Y(\Gamma)$ degenerates to a so-called Néron polygon

$$C_h = \left(\mathbb{CP}^1 \times \mathbb{Z}/h\mathbb{Z}\right)/((0,i) \sim (\infty, i+1))$$

a cycle of h transversally intersecting projective lines. This is a singular complex analytic space, whose regular locus $C_h^{\text{reg}} = \mathbb{C}^* \times \mathbb{Z}/h\mathbb{Z}$ has a group structure that extends to an action on C_h. A generalized elliptic curve is, roughly speaking, a family of elliptic curves but allowing Néron polygons as degenerate fibres; see Chapter Chapter 9 for a precise definition.

Theorem 1.0.2. *Let Γ be a torsion-free congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that all cusps of Γ are regular. Then there exists a generalized elliptic curve $\mathcal{D}_\Gamma \to X(\Gamma)$ extending $\mathcal{E}_\Gamma \to Y(\Gamma)$ whose fibre over a cusp $t \in \mathrm{Cusps}(\Gamma)$ is a Néron h_t-gon, where h_t is the width of t.*

The Shioda modular surfaces $\mathcal{D}_{\Gamma(N)}$ provide the following modular interpretation of the set of cusps of $X(N)$.

Theorem 1.0.3. *Let $N \geq 3$ be an integer. Then there exists a generalized elliptic curve $\mathcal{D}_{\Gamma(N)} \to X(N)$ whose singular fibres are Néron N-gons, together with an isomorphism $\Psi_N : (\mathbb{Z}/N\mathbb{Z})^2 \times X(N) \to \mathcal{D}_\Gamma^{\mathrm{sm}}[N]$, such that we have a bijection*

$$|X(N)| \to \{\textit{elliptic curves or Néron } N\textit{-gons with a level-}N\textit{ structure }\}/\cong .$$

We prove this result as Proposition 9.2.1, and the analogue for $\Gamma_1(N)$ as Proposition 9.3.2.

Suppose that $\tilde{\Gamma}$ is a subgroup of Γ. Then any $\tilde{\Gamma}$-structure defines also a Γ-structure, and there is a natural map $p_{\tilde{\Gamma},\Gamma} : Y(\tilde{\Gamma}) \to Y(\Gamma)$ given by $\tilde{\Gamma}\tau \mapsto \Gamma\tau$, which extends to a holomorphic map $X(\tilde{\Gamma}) \to X(\Gamma)$ mapping $\mathrm{Cusps}(\tilde{\Gamma})$ onto $\mathrm{Cusps}(\Gamma)$. We show this lifts to a holomorphic map of the corresponding Shioda modular surfaces.

Theorem 1.0.4. *Let $\tilde{\Gamma} \subset \Gamma$ be congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Suppose that Γ is torsion-free with no irregular cusps. Then there exists a holomorphic map $\mathcal{D}_{\tilde{\Gamma}} \to \mathcal{D}_\Gamma$ extending the natural map $\mathcal{E}_{\tilde{\Gamma}} = \tilde{\Gamma}\backslash\mathcal{E} \to \Gamma\backslash\mathcal{E} = \mathcal{E}_\Gamma$. The map*

$$\mathcal{D}_{\tilde{\Gamma}} \to \mathcal{D}_\Gamma \times_{X(\Gamma)} X(\tilde{\Gamma})$$

is a projective desingularization.

Now let $k \geq 2$ be an integer. Then the k-fold fibre power

$$\mathcal{E}_\Gamma^k = \mathcal{E}_\Gamma \times_{Y(\Gamma)} \mathcal{E}_\Gamma \times_{Y(\Gamma)} \cdots \times_{Y(\Gamma)} \mathcal{E}_\Gamma$$

is a complex torus of relative dimension k over $Y(\Gamma)$. Under the same hypothesis on Γ as in Theorem 9.1.1, we show there is a $(k+1)$-dimensional complex manifold $\mathcal{KS}_\Gamma^k$ compactifying $\mathcal{E}_\Gamma^k$, called the k-th *Kuga-Sato variety* attached to the congruence subgroup Γ.

Theorem 1.0.5. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a torsion-free congruence subgroup such that all cusps of Γ are regular. Let $k \geq 1$ be an integer. Then there exists a projective-algebraic $(k+1)$-dimensional complex manifold together with a proper holomorphic map $f : \mathrm{KS}_\Gamma^k \to X(\Gamma)$, and a multiplication map $m : \mathrm{KS}_\Gamma^{k,\mathrm{sm}} \times_{X(\Gamma)} \mathrm{KS}_\Gamma^k \to \mathrm{KS}_\Gamma^k$ that restricts over the open modular curve $Y(\Gamma) \subset X(\Gamma)$ to the $Y(\Gamma)$-complex torus $\mathcal{E}_\Gamma^k$ of relative dimension k that is the k-th power of the universal elliptic curve $\mathcal{E}_\Gamma \to Y(\Gamma)$.*

Moreover, we show in Theorem 1.0.4 that $\mathcal{KS}_\Gamma^k$ is projective-algebraic, i.e. it originates from a projective $\mathbb{C}$-variety. Furthermore, we show in Section 10.3 that if $\tilde{\Gamma} \subset \Gamma$ is an inclusion of groups satisfying the hypotheses of Theorem 1.0.5 then there is a natural holomorphic map $\mathcal{KS}_{\tilde{\Gamma}}^k \to \mathcal{KS}_\Gamma^k$.

We conclude this introduction by giving an overview of the method employed to construct the Kuga–Sato variety $\mathcal{KS}_\Gamma^k$. Chapter 1 lists the various objects we encounter on the way. Its first column indicates the category in which the objects in that row live. We start in the top row, and with each step move one row down until we arrive at the desired compact complex spaces in the bottom row.

The construction of a local model for $\mathcal{KS}_\Gamma^k$ at a regular cusp of width h is initiated in the algebraic category of algebraic varieties over $\mathbb{C}$. In Chapter 7 we discuss how in toric geometry one constructs complex toric varieties from combinatorial data consisting of cones in a finite-dimensional real vector space, called a rational partial polyhedral decomposition. This yields an equivalence of categories $F : \mathbf{RPPD} \to \mathbb{C}\mathbf{TorVrt}$ from the category of rational partial polyhedral decompositions to the category of complex toric varieties, used to pass from the first row to the second.

As the notation suggests, $\mathcal{G}^k = \mathcal{G} \times_{\mathcal{G}^0} \mathcal{G} \times_{\mathcal{G}^0} \cdots \times_{\mathcal{G}^0} \mathcal{G}$ is the k-fold fibre power of $\mathcal{G} \to \mathcal{G}^0$. This is a $(k+1)$-dimensional toric variety corresponding to a fan Σ^k in a $(k+1)$-dimensional real vector space, which is singular for $k \geq 2$. Following [Sch97] we perform a sequence of blowups centered at nonsingular closed subvarieties

$$\mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k\langle k-2\rangle \to \ldots \to \mathcal{G}^k\langle 1\rangle \to \mathcal{G}^k\langle 0\rangle = \mathcal{G}^k$$

corresponding to a sequence of star-subdivisions $\Sigma^k\langle k-1\rangle \to \ldots \to \Sigma^k\langle 1\rangle \to \Sigma^k\langle 0\rangle$, see Section 2.5 and Section 7.5 for the definition of the italicized terms.

The reduced analytic spaces in the third row are obtained from the algebraic objects in the second row by applying the analytification functor $(\cdot)^{\mathrm{an}}$ described in Section 2.6, restricting to the open unit disk Δ and then taking a suitable quotient. The objects in the final row are obtained by glueing the objects in the third row and the fourth row together, along the open cover of $X(\Gamma)$ provided in Theorem 3.5.4.

RPPD	(N^0, Σ^0)	(N^1, Σ^1)	(N^k, Σ^k)	$(N^k, \Sigma^k\langle l\rangle)$	$(N^k, \Sigma^k\langle k-1\rangle)$
$\mathbb{C}$**TorVrt**	$\mathcal{G}^0$	$\mathcal{G}$	$\mathcal{G}^k$	$\mathcal{G}^k\langle l\rangle$	$\mathcal{G}^k\langle k-1\rangle$
$\mathbb{C}$**AnSp** $/\Delta$	Δ	Tate_h	Tate_h^k	$\mathrm{Tate}_h^k\langle l\rangle$	KS_h^k
$\mathbb{C}$**Man** $/Y(\Gamma)$	$Y(\Gamma)$	$\mathcal{E}_\Gamma$	$\mathcal{E}_\Gamma^k$	$\mathcal{E}_\Gamma^k$	$\mathcal{E}_\Gamma^k$
$\mathbb{C}$**AnSp** $/X(\Gamma)$	$X(\Gamma)$	$\mathcal{D}_\Gamma$	$\mathcal{D}_\Gamma^k$	$\mathcal{D}_\Gamma^k\langle l\rangle$	$\mathcal{KS}_\Gamma^k$

Table 1.1: An overview of the various objects encountered in this thesis. The first column indicates in which category the objects live. Here $k \geq 1$ and $0 \leq l \leq k-1$ are integers.

Chapter 2

Preliminaries

In this section we discuss the background results from complex-analytic and algebraic geometry. The toric construction of Shioda modular surfaces and Kuga–Sato varieties runs through the category of algebraic $\mathbb{C}$-varieties and that of complex-analytic spaces. In Section 2.1 we introduce the category of (reduced) *complex(-analytic) spaces* $\mathbb{C}\mathbf{AnSp}$ and its full subcategory of *complex manifolds*, and defines Section 2.2 various properties of morphisms in $\mathbb{C}\mathbf{AnSp}$: proper, finite, flat and submersive. In Section 2.3 we discuss quotients by proper and free actions in (categories admitting a forgetful functor to) the category of topological spaces.

Then we flip to the algebraic category, introduce the category of $\mathbb{C}$-varieties. We define projective morphisms of schemes in Section 2.4, and define the Spec and Proj construction. In Section 2.5 we include a dicussion of desingularizations and blowups.

In Section 2.6 we show how a $\mathbb{C}$-variety X gives rise to a complex space X^{an} through a process called *analytification*. and show how a $\mathbb{C}$-variety functorially gives rise to a $\mathbb{C}$-analytic space.

2.1 Complex spaces

In this section we define complex(-analytic spaces), which for us will always be reduced. A more detailed treatment can be found in [Fis76][Chapter 0].

Definition 2.1.1. A $\mathbb{C}$-*ringed space* is a pair $X = (|X|, \mathcal{O}_X)$ consisting of a topological space $|X|$ and a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_X$ on $|X|$ whose stalk $\mathcal{O}_{X,p}$ at every point $p \in |X|$ is a local $\mathbb{C}$-algebra with trivial residue field $\mathbb{C}$.

A *morphism* $\phi : X \to Y$ of $\mathbb{C}$-ringed spaces is a pair $\left(|\phi|, \phi^{\#}\right)$ consisting of a continuous map $|\phi| : |X| \to |Y|$ and a homomorphism of sheaves of $\mathbb{C}$-algebras $\mathcal{O}_Y \to |\phi|_* \mathcal{O}_X$

For every $p \in |X|$ the homomorphism $\mathcal{O}_{Y,|\phi|(p)} \to \mathcal{O}_{X,p}$ is then a local homomorphism of local $\mathbb{C}$-algebras (i.e. ϕ is a morphism of locally ringed spaces).

Example 2.1.2. Every open subset $W \subset \mathbb{C}^n$, together with the sheaf $\mathcal{O}_W$ of holomorphic functions on W is a $\mathbb{C}$-ringed space, which by abuse of notation will also be denoted W.

Definition 2.1.3. Let $n \in \mathbb{Z}_{\geq 0}$. A *complex manifold* M of dimension n is a $\mathbb{C}$-ringed space $M = (|M|, \mathcal{O}_M)$ with $|M|$ Hausdorff and such that each point $p \in |M|$ admits an open neighborhood $p \in U \subset |M|$ with $(U, \mathcal{O}_M|_U)$ isomorphic to the $\mathbb{C}$-ringed spaces $(W, \mathcal{O}_W)$ constructed in Example 2.1.2 for an open subset W of $\mathbb{C}^n$.

Definition 2.1.4. Let $W \subset \mathbb{C}^n$ be an open subset. We say $A \subset W$ is an *analytic* subset if for every $p \in A$ there exist an open neighborhood $p \in U \subset W$ and $f_1, f_2, \ldots, f_n \in \mathcal{O}_W(U)$ such that

$$A \cap U = \{x \in U \mid f_1(x) = f_2(x) = \ldots = f_n(x) = 0\}\,.$$

Note that any analytic subset of W is closed in W.

Example 2.1.5. Let A be an analytic subset of an open subset W of $\mathbb{C}^n$. We define the sheaf of ideals $J_A \subset \mathcal{O}_W$ by letting for every open subset $U \subset W$

$$J_A(U) = \{f \in \mathcal{O}_W(U) \mid f(x) = 0 \text{ for all } x \in A \cap U\}\,.$$

Then we have $A := \operatorname{supp}(\mathcal{O}_W/J_A)$, and we set $\mathcal{O}_A = (\mathcal{O}_W/J_A)|_A$. Then $(A, \mathcal{O}_A)$ is a $\mathbb{C}$-ringed space, which by abuse of notation will also be denoted A.

Definition 2.1.6. A *complex-analytic space* or *complex space* is a $\mathbb{C}$-ringed space $X = (|X|, \mathcal{O}_X)$ with $|X|$ Hausdorff such that each $p \in |X|$ admits an open neighborhood $p \in U \subset |X|$ that is isomorphic to one of the $\mathbb{C}$-ringed spaces constructed in Example 2.1.5, i.e. there exists an isomorphism $(U, \mathcal{O}_M|_U) \cong (A, \mathcal{O}_A)$ for some analytic subset A of an open subset W of $\mathbb{C}^n$ for some $n \in \mathbb{Z}_{\geq 0}$.

Definition 2.1.7. We denote $\mathbb{C}$–**RSp** the category of $\mathbb{C}$-ringed spaces, and $\mathbb{C}$**Man** and $\mathbb{C}$**AnSp** its full subcategories of complex manifolds resp. (reduced) complex-analytic spaces.

Note that a morphism $\phi : X \to Y$ in $\mathbb{C}$**AnSp** is determined by $|\phi|$, because holomorphic functions separate the points of $\mathbb{C}^n$. Therefore we will also call a morphism in $\mathbb{C}$**AnSp** or $\mathbb{C}$**Man** a *holomorphic map*. An isomorphism in $\mathbb{C}$**AnSp** will be called a *biholomorphism*.

Consider local models $A \subset W \subset \mathbb{C}^n$ and $B \subset V \subset \mathbb{C}^m$ as in Example 2.1.5. Then a holomorphic map $\phi : A \to B$ is a map of sets $|\phi| : A \to B$ such that for every $p \in A$ there exists an open subset $p \in U \subset W$ and a holomorphic map $\tilde{\phi} : U \to \mathbb{C}^m$ with $\tilde{\phi}|_{A\cap U} = \phi|_{A\cap U}$.

Proposition 2.1.8. *The category* $\mathbb{C}$**AnSp** *has all finite fibre products.*

Proof. As in the category of schemes, this is a local question. Let $k \in \mathbb{Z}_{\geq 0}$ and for each $i \in \{1, 2, \ldots, k\}$ let A_i be an analytic subset of an open subset W_i of complex n_i-space $\mathbb{C}^{n_i}$ for some $n_i \in \mathbb{Z}_{\geq 0}$. Let X be a $\mathbb{C}$-analytic space and let $f_i : A_i \to X$ be holomorphic maps.

The cartesian product $W = W_1 \times W_2 \times \cdots \times W_k$ is an open subset of $\mathbb{C}^{n_1+n_2+\ldots+n_k}$ and the set-theoretic fibre product $A = A_1 \times_X A_2 \times_X \cdots \times_X A_k$ is an analytic subset of W. Thus we obtain the categorical fibre product of the morphisms f_i as in Example 2.1.5 from A and W. □

2.2 Some classes of holomorphic maps

In this section we discuss four propeties a holomorphic map of complex-analytic spaces could have: proper, finite, flat and submersive.

Definition 2.2.1. Let $\phi : X \to Y$ be a continuous map of topological spaces. We say ϕ is *connected* if for every $y \in Y$ the fibre $f^{-1}(y)$ is connected and nonempty.

Lemma-Definition 2.2.2 (proper, finite)**.** *Let $\phi : X \to Y$ be a continuous map of locally compact Hausdorff spaces. We call ϕ* proper *if it satisfies the following two equivalent conditions:*

(i) *for any compact subset K of Y the set $\phi^{-1}(K)$ is compact;*
(ii) *ϕ is closed and all fibres of ϕ are compact.*

We call ϕ finite *if it satisfies the following two conditions:*

(i) *ϕ is proper and every point $p \in X$ is an isolated point in its fibre $\phi^{-1}(\phi(p))$;*
(ii) *ϕ is closed and has finite fibres.*

Proof. See [Fis76][Def. 1.10] for a proof of these equivalences. □

Theorem 2.2.3. *Let $\phi : X \to Y$ be a holomorphic map of complex spaces, let $p \in |X|$ and set $q = |\phi|(p)$. Then the following are equivalent:*

(1) *$\mathcal{O}_{X,p}$ is a finitely generated $\mathcal{O}_{Y,q}$-module;*
(2) $\dim_{\mathbb{C}} (\mathcal{O}_{X,p}/\mathrm{m}_{Y,q}\mathcal{O}_{X,p}) < \infty$*;*
(3) *p is an isolated point of its fibre X_q.*

If p is an isolated point of $X_{\phi(p)}$ then there are open neighborhoods $U \subset X$ of p and $V \subset Y$ of q such that $\phi|_U : U \to V$ is finite.

We now define two technical properties of a morphism of complex spaces $\phi : X \to Y$, which will be part of the definition of a generalized elliptic curve in Section 6.5. The first, flatness, captures algebraically that the fibres $X_m = \phi^{-1}(m)$ of the morphism ϕ 'vary smoothly' with m. The second, having reduced fibres, encapsulates that no 'multiple fibres' occur. For a discussion of flatness and tensor products, see [MR86][§7].

Definition 2.2.4 (flatness, reduced fibres)**.** Let $\phi = \left(|\phi|, \phi^{\#}\right) : X \to Y$ be a morphism of $\mathbb{C}$-analytic space. Let $p \in |X|$ and set $q = |\phi|(p)$.

(1) We say ϕ is *flat at p* if the $\mathbb{C}$-algebra homomorphism $\mathcal{O}_{Y,|\phi|(p)} \to \mathcal{O}_{X,p}$ is flat. We say ϕ is *flat* if it is flat at every point of X.
(2) We say the fibre of ϕ over q is *reduced* at p if the $\mathbb{C}$-algebra $\mathcal{O}_{X,p} \otimes_{\mathcal{O}_{Y,q}} (\mathcal{O}_{Y,q}/\mathfrak{m}_{Y,q})$ is reduced. We say ϕ has *reduced fibres* if for every point p of X the fibre of ϕ over $|\phi|(p)$ is reduced at p.

In the remainder of this section, we consider a holomorphic map $\phi : X \to Y$ of complex-analytic spaces.

Proposition 2.2.5. *If ϕ is flat, then $|\phi|$ is open. If X and Y are complex manifolds, the converse holds.*

Proof. See [Fis76][Prop. 3.19 and Cor. 3.20] □

Lemma 2.2.6. *Let $\phi : X \to Y$ be a flat morphism with reduced fibres, of not necessarily reduced $\mathbb{C}$-analytic spaces. If Y is reduced, then so is X. If Z is a reduced $\mathbb{C}$-analytic space and $\psi : Z \to Y$ is holomorphic, then the fibre product $X \times_Y Z$ in the category of not necessarily reduced $\mathbb{C}$-analytic spaces is reduced.*

Proof. The first statement follows from [MR86][Thm. 23.9] which states that if $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is a flat local homomorphism of local rings, then A and $B/\mathfrak{m}_A B$ being reduced implies that B is reduced. □

Proposition 2.2.7. *If ϕ is finite, the following conditions are equivalent:*

(i) *ϕ is flat;*
(ii) *$\phi_* \mathcal{O}_X$ is locally free;*
(iii) *The function $Y \to \mathbb{Z}_{\geq 0}, q \mapsto \sum_{x \in X_q} \dim_{\mathbb{C}} \mathcal{O}_{X_q, x}$ is locally constant.*

Proof. See [Fis76][Prop. 3.13]. □

Definition 2.2.8 (submersive)**.** Let $\phi : X \to Y$ be a holomorphic map of complex spaces. Let $p \in |X|$ and set $q = |\phi|(p) \in |Y|$. We say ϕ is *submersive at* p if there exist open neighborhoods $p \in U \subset X$ and $q \in V \subset Y$, an open subset $Z \subset \mathbb{C}^k$ and a biholomorphism $U \to W \times V$ such that the following diagram commutes

$$\begin{array}{ccc} U & \cong & W \times V \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & V. & \end{array}$$

We say ϕ is *submersive* if it is submersive at every point of X. A *submersion* is a submersive holomorphic map.

Theorem 2.2.9. *The following conditions are equivalent:*

(i) *ϕ is a submersion at p;*
(ii) *ϕ is flat in p and $X_{\phi(p)}$ is a manifold at p;*

If X and Y are complex manifolds, then these conditions are also equivalent to:

(iii) *the map of tangent spaces $(d\phi)_p : T_p X \to T_{\phi(p)} Y$ is surjective.*

Proof. See [Fis76][Thm. 3.21] □

Proposition 2.2.10. *If ϕ is flat then the set $\{p \in X : X_{\phi(p)}$ is not a manifold in $p\}$ is an analytic subset of X.*

Proof. See [Fis76][Prop. 3.22]. □

Next, we show that the fibre product of two holomorphic maps of complex manifolds, taken in the category $\mathbb{C}$**AnSp** as in Proposition 2.1.8, is again a holomorphic manifold provided at least one of the maps is a submersion.

Lemma 2.2.11. *Let $f\colon X \to Y$ be a holomorphic submersion of complex manifolds. Then for any holomorphic map $T \to Y$ from a complex manifold T, the fibre product $X_T := X \times_Y T$ exists in the category of complex manifolds, and the pullback $f_T : X_T \to T$ of f is again a submersion.*

In particular, for every $y \in Y$ the fibre $X_y := f^{-1}(y)$ is a complex manifold. The assignment $y \mapsto \dim(X_y)$ defines a locally constant function on Y. If $\dim(X_y) = d$ for every $y \in Y$, then we say f has *relative dimension d*.

2.3 Quotients by proper actions

In this section we discuss quotients of holomorphic manifolds by group actions. If the action is particularly well behave, in the sense that it is proper and free, then the quotient always exists as a holomorphic manifold.

Recall that a map of topological spaces $\phi : X \to Y$ is called *proper* if for every compact subset $K \subset Y$ the inverse image $\phi^{-1}(K)$ is a compact subset of Y.

Definition 2.3.1. A continuous left action $\Gamma \times Y \to Y$, $(\gamma, y) \mapsto \gamma \cdot y$ of a topological group Γ on a topological space Y is called *proper* if the map

$$\begin{aligned} \Gamma \times Y &\to Y \times Y, \\ (\gamma, y) &\mapsto (\gamma \cdot y, y) \end{aligned}$$

is proper. If Y is Hausdorff, then it suffices to show for any compact subset $K \subset Y$ that

$$\Gamma_K := \{\gamma \in \Gamma : (\gamma \cdot K) \cap K \neq \emptyset\} \tag{2.1}$$

is a compact subset of Γ.

The *stabilizer* of a point $y \in Y$ is defined to be the group

$$\Gamma_y := \{\gamma \in \Gamma : \gamma \cdot y = y\}. \tag{2.2}$$

We say that the action is *free* if Γ_y is trivial for every $y \in Y$.

Proposition 2.3.2. *Let Γ be a discrete group and let Y be a locally connected locally compact Hausdorff topological space. Let $\Gamma \times Y \to Y, (\gamma, y) \mapsto \gamma y$ be a free left action of Γ on Y by homeomorphisms. Then the following statements are equivalent:*

(1) *the action of Γ on Y is proper;*

(2) *for any compact subset $K \subset Y$ the set $\{\gamma \in \Gamma : (\gamma \cdot K) \cap K \neq \emptyset\}$ is finite;*

(3) *the following two statements are true:*

  (i) *the quotient map $Y \to \Gamma\backslash Y$ is a covering map;*

  (ii) *the quotient space $\Gamma\backslash Y$ is Hausdorff.*

(4) *the following two statements are true:*

  (i) *every point $y \in Y$ has a neighborhood U such that $\gamma U \cap U = \emptyset$ for every nonidentity element $\gamma \in \Gamma\backslash\{1\}$;*

(ii) *if* $y \in Y$ *and* $y' \in Y$ *lie in distinct* Γ*-orbits, there exist neighborhoods* V *of* γ *and* V' *of* γ' *in* Y *such that* $(\gamma \cdot V) \cap V' = \emptyset$ *for all* $\gamma \in \Gamma$.

Proof. The equivalences are contained in [Lee11][Thm. 12.14, Prop. 12.21, Prop. 12.23, Prop. 12.24, Prop. 12.25, Thm. 12.26]. □

Lemma 2.3.3. *Let* Γ *be a topological group acting from the left on Hausdorff topological spaces* X *and* Y. *Suppose that* $f : X \to Y$ *is a* Γ*-equivariant continuous map. If the action of* Γ *on* Y *is proper (resp. free, resp. has finite stabilizers), the action of* Γ *and* X *is proper (resp. free, resp. has finite stabilizers).*

Proof. Let $x \in X$ and $\gamma \in \Gamma$ such that $\gamma \cdot x = x$. Since f is Γ-equivariant, we find that $\gamma \cdot f(x) = f(\gamma \cdot x) = f(x)$. Thus the isotropy group Γ_x of x is contained in the isotropy group $\Gamma_{f(x)}$ of $f(x)$. So Γ_y being trivial (resp. finite) for every $y \in Y$ implies that Γ_x is trivial (resp. finite) for each $x \in X$. We conclude that the action of Γ on X is free (resp. has finite stabilizers), if this holds for the Γ-action on Y.

Next, we claim that if $K \subset X$ is compact, then the set Γ_K defined by (2.1) is closed in Γ in any case. In fact, since X is Hausdorff and K is compact, the set K is closed in X. The inverse image of $X \setminus K$ under the continuous $\Gamma \times X \to X$ is an open subset $U = \{(\gamma, x) \in \Gamma \times X : \gamma \cdot x \notin K\}$ of $\Gamma \times X$. If $\gamma \in \Gamma \backslash \Gamma_K$ then $\{\gamma\} \times K \subset U$. Since K is compact, there exists a neighborhood V of K such that $V \times K \subset U$. Then $\gamma \in V \subset \Gamma \backslash \Gamma_K$, which shows that $\Gamma \backslash \Gamma_K$ is open in Γ, i.e. that Γ_K is closed in Γ.

Now suppose the action of Γ on Y is proper. Let $K \subset X$ be a compact subset. Since f is continuous, its image $L := f(K)$ in Y is compact as well. Since the action of Γ on Y is assumed to be proper, the set $\Gamma_L := \{\gamma \in \Gamma : (\gamma \cdot L) \cap L \neq \emptyset\}$ is compact. By Γ-equivariance of f, the set $\Gamma_K := \{\gamma \in \Gamma : (\gamma \cdot K) \cap K \neq \emptyset\}$ is a subset of Γ_L. Therefore Γ_K is a closed subset of the compact set Γ_L, hence Γ_K is compact, as we had to show. □

Lemma 2.3.4. *Let* Γ *(resp.* Γ'*) be a group acting freely by homeomorphisms from the left on a topological space* X *(resp.* X'*) such that the quotient map* $X \to \Gamma \backslash X$ *(resp.* $X' \to \Gamma' \backslash X'$*) is a covering space. Then the group* $\Gamma \times \Gamma'$ *acts freely by homeomorphisms on* $X \times X'$, *and the map* $X \times X' \to (\Gamma \backslash X) \times (\Gamma' \backslash X')$ *is a covering space that is a quotient map for the action of* $\Gamma \times \Gamma'$.

Let $p : X \to Y$ *be a covering map, and let* $f : E \to Y$ *be any continuous map. Then the pullback* $E \times_Y X \to E$ *of* p *along* f *is again a covering map.*

Proof. All statements follow directly from the definitions. □

Theorem 2.3.5. *Let* Γ *be a discrete group acting properly on a complex manifold* Y *via biholomorphisms. Assume either that* Y *is a Riemann surface, or that* Γ *acts freely. Then the topological space* $\Gamma \backslash Y$ *is Hausdorff, and admits a unique structure of complex manifold such that* $\pi : Y \to \Gamma \backslash Y$ *is holomorphic. The map* $\pi : Y \to \Gamma \backslash Y$ *is a categorical quotient in the category of complex manifolds for the action of* Γ *on* Y, *i.e. for any* Γ*-invariant holomorphic map* $\rho : Y \to Z$ *there exists a unique holomorphic map* $f : \Gamma \backslash Y \to Z$ *such that* $\rho = f \circ \pi$.

If Y *is a Riemann surface, so is* $\Gamma \backslash Y$, *and the ramification index* $e_\pi(y)$ *of* π *at* $y \in Y$ *is equal to* $\#\mathrm{image}\,(\Gamma_y \to \mathrm{Aut}\,(\mathcal{O}_{Y,y}))$.

If Γ acts freely, then π is a covering map and a local analytic isomorphism.
If the image of Γ in $\mathrm{Aut}(Y)$ is finite, then π is proper with finite fibres.

Proof. The case that Γ acts freely and properly is standard. For the case that Y is a Riemann surface one may consult [Cona][Prop. 3.1], whose proof determines the ramification indices of π to be as indicated. □

Remark 2.3.6. In a similar way way one shows the category of $\mathbb{C}$-analytic spaces admits a categorical quotient for a proper and free action of a discrete group Γ on a $\mathbb{C}$-analytic space Y.

Suppose $f : Y \to Z$ is a Γ-invariant holomorphic map to a $\mathbb{C}$-analytic space. The universal property of a categorical quotient yields a holomorphic map $\bar{f} : \Gamma\backslash Y \to Z$. For every $z \in Z$ the fibre $f^{-1}(z)$ is Γ-stable, and Γ acts properly and freely on this fibre. Further, we have an isomorphism

$$\bar{f}^{-1}(z) \cong \Gamma\backslash f^{-1}(z).$$

2.4 Projective morphisms of schemes

In this section we define projective morphisms of schemes.
We start with a general construction of the relative (homogeneous) spectrum.

Lemma 2.4.1. *Let S be a scheme.*
(a) Let $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Then functorially in S-schemes $f : T \to S$ we have bijections

$$\mathrm{Mor}_S(T, \mathrm{Spec}_S\,\mathcal{A}) \cong \mathrm{Hom}_{\mathcal{O}_S-\mathbf{Alg}}(\mathcal{A}, f_*\mathcal{O}_T).$$

(b) Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras that is locally finitely generated in degree 1*. Then functorially in S-schemes $f : T \to S$ we have bijections*

$$\mathrm{Mor}_S(T, \mathrm{Proj}_S\,\mathcal{A}) \cong \{(\mathcal{L}, \psi : f^*\mathcal{A} \to \bigoplus_{d\geq 0} \mathcal{L}^{\otimes d})\}/\cong,$$

where we consider pairs $(\mathcal{L}, \psi)$ consisting of an invertible $\mathcal{O}_T$-module $\mathcal{L}$ and and a homomorphism ψ of $\mathcal{O}_T$-algebras preserving the grading such that $\psi_1 : f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective.

Proof. [dJm][Lemmas 01LV and 01O4] □

If Ec is a coherent module on a Noetherian scheme, then using this construct we can attach a *projective bundle* to $\mathcal{E}$, with the following universal property.

Theorem-Definition 2.4.2. *Let Y be a Noetherian scheme, and let $\mathcal{E}$ be a coherent $\mathcal{O}_Y$-module. Then there exists a proper Y-scheme*

$$\mathbb{P}_Y(\mathcal{E}) = \mathrm{Proj}_Y(\mathrm{Sym}^\bullet\,\mathcal{E}), \tag{2.3}$$

called the projective bundle over Y *attached to $\mathcal{E}$, with the following universal property. Given a Y-scheme $f : X \to Y$, the Y-morphisms from X to $\mathbb{P}(\mathcal{E})$ correspond to isomorphism classes of pairs $(\mathcal{L}, \psi)$ with $\mathcal{L}$ a line bundle on T and $\psi : f^*\mathcal{E} \to \mathcal{L}$ an epimorphism.*

Proof. See [dJm][Section 01OA, Lemma 0104] □

Definition 2.4.3. Let Y be a Noetherian scheme and let $f : X \to Y$ be an Y-scheme. We say f is *projective* if there exists a coherent $\mathcal{O}_Y$-module $\mathcal{E}$ and a closed embedding $X \to \mathbb{P}(\mathcal{E})$ over Y.

In particular, for every coherent $\mathcal{O}_Y$-module $\mathcal{E}$ the structure morphism $\mathbb{P}(\mathcal{E}) \to Y$ is projective. Clearly any projective morphism is proper.

We now give a more intrinsic criterion for a proper morphism to be projective, in terms of so-called *(very) ample* line bundles.

Definition 2.4.4. Let $f : X \to Y$ be a proper morphism of Noetherian schemes. We say a line bundle $\mathcal{L}$ on X is f*-base point free* if the adjunction map $f^* f_* \mathcal{L} \to \mathcal{L}$ is an epimorphism.

Let $\mathcal{L}$ be an f-base point free line bundle on X. The universal property of $\mathbb{P}(f_*\mathcal{L})$ associates to the epimorphism $f^* f_* \mathcal{L} \to \mathcal{L}$ an Y-morphism denoted

$$\phi_{\mathcal{L}} : X \to \mathbb{P}(f_*\mathcal{L}).$$

Note that since f is proper, we have that $f_*\mathcal{L}$ is a coherent $\mathcal{O}_Y$-module by Grothendieck's Coherence Theorem, see e.g. [Vak][Theorem 18.9.1].

We say $\mathcal{L}$ is f*-very ample* if $\phi_{\mathcal{L}}$ is a closed embedding.

We say $\mathcal{L}$ is f*-ample* if for every quasi-compact open subset U of Y there exists $k_0 \geq 1$ such that $\mathcal{L}^{\otimes k}|_U$ is ample with respect to $f^{-1}(U) \to U$ for all $k \geq k_0$.

See [dJm][Section 01VG and Lemma 02NO] for other definitions of relative amplitude which are more customary in the literature. The notion of amplitude is local on the target scheme Y in the following sense.

Lemma 2.4.5. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes, and let $\mathcal{L}$ be a line bundle on X. Then the following are equivalent.*

(1) *$\mathcal{L}$ is (very) ample relative to $f : X \to Y$;*
(2) *there exists an open cover $\{U_i\}$ of X such that $\mathcal{L}|_{f^{-1}(U_i)}$ is (very) ample on $f^{-1}(U_i)/U_i$;*
(3) *for every affine open subset U of X we have that $\mathcal{L}|_{f^{-1}(U_i)}$ is (very) ample on $f^{-1}(U_i)/U_i$.*

Proof. This lemma follows from the definition of $\phi_{\mathcal{L}} : X \to \mathcal{P}_Y(f_*\mathcal{L})$ being local on Y in a suitable sense. □

Proposition 2.4.6. *Let Y be a Noetherian scheme and $f : X \to Y$ a proper morphism. Then f is projective if and only if there exists an f-very ample line bundle on X.*

Proof. Suppose that $\mathcal{L}$ is an f-very ample line bundle on X. By definition of relative very amplitude, $\mathcal{L}$ defines a closed embedding $\phi_L : X \to \mathbb{P}(f_*\mathcal{L})$ over Y. We conclude that f is projective.

The converse holds by [dJm][Lemma 01VR, (1) $\iff$ (4)]. □

2.5 Blowups of schemes

In this section we define the blowup of a scheme along a closed subscheme, which provides an example of closed embeddings. Our exposition closely follows [Vak].

Definition 2.5.1. Let X be a scheme. If Y is a closed subscheme of X, we denote $\mathcal{I}_Y = \ker(\mathcal{O}_X \to j_*\mathcal{O}_Y)$ the *sheaf of ideals* attached to Y, where $j : Y \hookrightarrow X$ is the closed embedding. If $g : Z \to X$ is any morphism, the *pullback* g^*Y *of* Y *by* g is the closed subscheme of Z defined by the *inverse image ideal sheaf* $g^{-1}\mathcal{I}_Y \cdot \mathcal{O}_Z := \mathrm{Image}(g^*\mathcal{I}_Y \to g^*\mathcal{O}_X \cong \mathcal{O}_Z)$.

Definition 2.5.2. 1. An open subset U of X is called *schematically dense* in X if the sequence $0 \to \mathcal{O}_X \to i_*\mathcal{O}_U$ is exact, where $i : U \hookrightarrow X$ is the open embedding.

2. A closed subscheme Y of X is called *locally principal* if X admits a cover by open affines $U = \operatorname{Spec} A$ such that $\Gamma(U, \mathcal{I}_Y) = fA$ for some $f \in A$.
3. An *effective Cartier divisor* on X is a closed subscheme D of X satisfying one, hence all, of the following three conditions:
   (a) the ideal sheaf $\mathcal{I}_D$ is an invertible sheaf of $\mathcal{O}_X$-modules;
   (b) X admits a cover by open affines $U = \operatorname{Spec} A$ such that $\Gamma(U, \mathcal{I}_D) = fA$ for some nonzerodivisor $f \in A$;
   (c) D is locally principal and the open subset $X \setminus D$ of X is schematically dense.

Definition 2.5.3. Let Y be a closed subscheme of X of finite presentation, i.e. such that $\mathcal{I}_Y$ is a finite type sheaf of ideals. The *blow-up of* X *along* Y, or with *center* Y is a final object in the category of X-schemes $p : Z \to X$ such that the pullback p^*Y of Y by p is an effective Cartier divisor on Z.

A blow-up of Y along X exists, is unique up to a unique morphism, and will be denoted $\beta : \mathrm{Bl}_Y X \to X$. The effective Cartier divisor β^*Y is called the *exceptional divisor* of $\mathrm{Bl}_Y X$ and we write $E_Y X$ for it.

The following lemma describes the behaviour of blow-ups under change of base.

Theorem 2.5.4. (1) *Let $g : Z \to X$ be any morphism, and put $W = g^*Y$. Let $j : \overline{Z} \hookrightarrow Z$ be the schematic closure of $(Z \setminus W) \times_X \mathrm{Bl}_Y X$ inside $Z \times_X \mathrm{Bl}_Y X$. The morphism $\mathrm{pr}_1 \circ j : \overline{Z} \to Z$ induced by the first projection is the blow-up of Z along W.*

(2) *Let $g : Z \to X$ be a flat morphism (e.g. an open immersion), and put $W = g^*Y$. The first projection $\mathrm{pr}_1 : Z \times_X \mathrm{Bl}_Y X \to Z$ is the blow-up of Z along W.*

Proof. (1) Denote $E_{\overline{Z}} = j^*(E_Y X)$ the pullback of $E_Y X$ to $\overline{Z}$; it is a locally principal closed subscheme being the pullback of one. Since $j : \overline{Z} \setminus E_{\overline{Z}} \to (Z \setminus W) \times_X \mathrm{Bl}_Y X$ is an isomorphism, $\overline{Z} \setminus E_{\overline{Z}}$ is schematically dense in $\overline{Z}$. We conclude that $E_{\overline{Z}}$ is an effective Cartier divisor on $\overline{Z}$.

To check the universal property, let $t : T \to Z$ be a morphism with $D := h^*W$ an effective Cartier divisor on T. Since $h^*W = (g \circ t)^*Y$, there exists a unique X-morphism $T \to \mathrm{Bl}_Y X$ by the universal property of $\mathrm{Bl}_Y X$. By the universal property of the fibre product, there exists a unique Z-morphism $h : T \to Z \times_X \mathrm{Bl}_Y X$. We are done if we can show h factors through $\overline{Z}$. Indeed, since we have that

$h^{-1}((Z \setminus W) \times_X \operatorname{Bl}_Y X) = T - D$, the pullback $h^*\overline{Z}$ contains the open subset $T - D$, which is schematically dense in T since D is an effective Cartier divisor on T. Hence $h^*\overline{Z} = T$, i.e. h factors through $\overline{Z}$.
(2) Let us show that if $f : U \to V$ is a flat morphism, and D is an effective Cartier divisor on V then f^*D is an effective Cartier divisor on U. Indeed, since f is flat it transforms the exact sequence $0 \to \mathcal{I}_Y \to \mathcal{O}_X$ into the exact sequence $0 \to f^*(\mathcal{I}_Y) \to f^*(\mathcal{O}_X)$. This implies that the ideal sheaf of f^*D, namely $f^{-1}\mathcal{I}_Y \cdot \mathcal{O}_Z$, is isomorphic to $f^*(\mathcal{I}_Y)$, which is an invertible sheaf since $\mathcal{I}_Y$ is one.

The second projection $\mathrm{pr}_2 : Z \times_X \operatorname{Bl}_Y X \to \operatorname{Bl}_Y X$ is flat, since it is pulled back from the flat morphism f. By what we have shown above, $\mathrm{pr}_2^*(E_Y X) = \mathrm{pr}_1^* W$ is an effective Cartier divisor on $Z \times_X \operatorname{Bl}_Y X$. The verification of the universal property is executed as in part (1). □

Theorem 2.5.5. *Let $\mathcal{I}$ be the ideal sheaf of a closed subscheme Y of X. Then the blowup of X along Y is the X-scheme*

$$\operatorname{Bl}_Y X = \operatorname{Proj}_X(\bigoplus_{d \geq 0} \mathcal{I}^d).$$

Proof. See [Liu06][Def. 8.1.11 and Prop 8.1.15]. □

Example 2.5.6. We compute the blow-up of affine n-space $X = \mathbb{A}^n_{\mathbb{Z}}$ along the origin Y. Let $A = \mathbb{Z}[X_1, \ldots, X_n]$ and $I = (X_1, \ldots, X_n) \subset A$, so $X = \operatorname{Spec} A$ and $Y = V(I)$. There is an isomorphism of graded rings

$$A[T_1, \ldots, T_n]/(X_iT_j - X_jT_i : 1 \leq i, j \leq n) \to \bigoplus_{d \geq 0} I^d, \qquad T_i \mapsto X_i$$

sending T_i to $X_i \in I$ in degree 1. We conclude that

$$\operatorname{Bl}_Y X = V_+(\{X_iT_j - X_jT_i : 1 \leq i, j \leq n\}) \subset \mathbb{P}^n_A, \tag{2.4}$$

with affine patches given by

$$\begin{aligned} D_+(T_1) &= \operatorname{Spec} \mathbb{Z}[X_1, S_2, \ldots, S_n]/(X_j - S_jX_1 : 1 \leq j \leq n) \\ &\cong \operatorname{Spec} \mathbb{Z}[X_1, X_2/X_1, \ldots, X_n/X_1] \cong \mathbb{A}^n_{\mathbb{Z}}. \end{aligned}$$

Example 2.5.7. Let B be a ring, and $J = (f_1, \ldots, f_n)$ be a finitely generated ideal. Set $Z = \operatorname{Spec} B$ and $W = V(J)$. We compute $\operatorname{Bl}_W Z$ using Theorem 2.5.4(1) and the previous Example 2.5.6, whose notation we keep in use. Let $g : Z \to X$ be the morphism corresponding to $A \to B, X_i \mapsto f_i$, so that $g^*Y = V(IB) = V(J) = W$. By the blowup-closure lemma Theorem 2.5.4(1) $\operatorname{Bl}_W Z$ is the schematic image of the open embedding

$$(Z \setminus W) \times_X \operatorname{Bl}_Y X \hookrightarrow \times_X \operatorname{Bl}_Y X = \operatorname{Proj} B[T_1, \ldots, T_n]/(f_iT_j - f_jT_i : 1 \leq i, j \leq n).$$

Since this morphism is quasi-compact, its schematic image may be computed working affine open by affine open [Vak][Theorem 8.3.4]. Let us compute the schematic image of

$$(Z \setminus W) \times_X D_+(T_1) \hookrightarrow \times_X \operatorname{Bl}_Y X = \operatorname{Spec} B[S_2, \ldots, S_n]/(f_j - S_jf_1) =: \operatorname{Spec} B_1.$$

The pullback of J to $\operatorname{Spec} B_1$ is the principal ideal generated by f_1, so this is the inclusion of the distinguished open subset $D(f_1)$ inside $\operatorname{Spec} B_1$. Since $(B_1)_{f_1} \cong A_{f_1}$, its schematic image is cut out by the kernel of the homomorphism $B_1 \to A_{f_1}$ sending S_j to f_j/f_1. Thus $\mathrm{Bl}_W Z \cap \mathrm{pr}_2^{-1}(D_+(T_i))$ is the spectrum of the sub-A-algebra of A_{f_1} generated by $\{f_j/f_1 : 2 \le j \le n\}$.

2.6 Géométrie Algébrique et Géometrie Analytique

Definition 2.6.1. A $\mathbb{C}$-*variety* is a reduced separated scheme locally of finite type over $\mathbb{C}$. The category of $\mathbb{C}$-varieties is denoted $\mathbb{C}$–**Vrt**.

The category $\mathbb{C}$–**Vrt** admits all finite fibre products. The final object is $\operatorname{Spec}\mathbb{C}$. The fibre product of X_1 and X_2 over X_0 taken in the category $\mathbb{C}$–**Vrt** of $\mathbb{C}$-varieties is the greatest reduced closed subscheme [Har77][Example 3.2.6] of the fibre product of X_1 and X_2 over Y taken in the category of $\mathbb{C}$-schemes [Har77][Thm. II.3.3].

In a functorial way one may attach to a $\mathbb{C}$-variety a complex-analytic spaces.

Lemma-Definition 2.6.2. *Let X be a $\mathbb{C}$-variety. Then there exists a complex-analytic space X^{an} and a morphism of $\mathbb{C}$-ringed spaces $j_X : X^{\mathrm{an}} \to X$ such that for every complex-analytic space Z and morphism of $\mathbb{C}$-ringed spaces $g : Z \to X$ there exists a unique holomorphic map $h : Z \to X^{\mathrm{an}}$ with $g = j_X \circ h$. This defines an* analytification functor

$$\begin{aligned}(\cdot)^{\mathrm{an}} : \mathbb{C}\text{–}\mathbf{Vrt} &\to \mathbb{C}\mathbf{AnSp},\\ X &\mapsto X^{\mathrm{an}},\\ f &\mapsto f^{\mathrm{an}}.\end{aligned} \tag{2.5}$$

Proof. For a detailed treatment of the analytification functor see [Ser56][§2]. □

Let us now give a more explicit description of the functor $(\cdot)^{\mathrm{an}}$. First consider an affine $\mathbb{C}$-variety $Y = \operatorname{Spec}\mathbb{C}[z_1, \dots, z_n] / (f_1, f_2, \dots, f_m)$. Then we have that $Y^{\mathrm{an}} := \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \ldots = f_m(x) = 0\}$, which is an analytic subspace of $\mathbb{C}^n$ because polynomial functions are holomorphic. A general $\mathbb{C}$-variety X is obtained by glueing affine $\mathbb{C}$-varieties Y_α, and this glueing data is used to glue the Y_α^{an} into the $\mathbb{C}$-analytic space X^{an}. We conclude that $j_X : X^{\mathrm{an}} \to X$ induces a bijection $|X^{\mathrm{an}}| \to X(\mathbb{C})$.

Secondly, consider a morphism of $\mathbb{C}$-varieties $f : X \to Y$, which by working locally we may assume to be affine: $X = \operatorname{Spec}\mathbb{C}[x_1, \dots, x_k] / (g_1, \dots, g_l)$ and $Y = \operatorname{Spec}\mathbb{C}[z_1, \dots, z_n] / (f_1, \dots, f_m)$. Then f is given by a homomorphism of $\mathbb{C}$-algebras $f^\# : \mathbb{C}[z_1, \dots, z_n] / (f_1, \dots, f_m) \to \operatorname{Spec}\mathbb{C}[x_1, \dots, x_k] / (g_1, \dots, g_l)$. For every $1 \le i \le n$ choose $h_i \in \mathbb{C}[x_1, \dots, x_k]$ such that $f^\#(\overline{z_i}) = \overline{h_i}$. Then the holomorphic map $(h_1, \dots, h_n) : \mathbb{C}^k \to \mathbb{C}^n$ restricts to the morphism of complex-analytic spaces $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$.

Lemma 2.6.3. *The analytification functor $(\cdot)^{an}$ is faithful and preserves all finite fibre products. The restriction of $(\cdot)^{\mathrm{an}}$ to the full subcategory of complete $\mathbb{C}$-varieties is fully faithful.*

Proof. Faithfulness holds because a morphism $f : X \to Y$ of (reduced) **C**-varieties is determined by the induced map on $\mathbb{C}$-points $X(\mathbb{C}) \to Y(\mathbb{C})$. The assertion about full faithfulness can be found in [GR71][Exposée XII, Corollaire 4.5]. Preservation of finite fibre products may be proved either using the universal property or the explicit description of $(\cdot)^{\mathrm{an}}$ given. □

Lemma 2.6.4. *Let X be a $\mathbb{C}$-variety, and let $f : X \to Y$ be a morphism of $\mathbb{C}$-varieties. Then the following equivalences hold:*

(1) *X is smooth over $\mathbb{C}$ if and only if X^{an} is a complex manifold;*
(2) *X is connected in the Zariski topology if and only if X^{an} is connected in the analytic topology;*
(3) *f is proper if and only if f^{an} is proper;*
(4) *f is flat if and only if f^{an} is flat;*
(5) *$X \to \operatorname{Spec}\mathbb{C}$ is proper if and only if X^{an} is compact.*
(6) *Suppose that X and Y are smooth over $\mathbb{C}$. Then f is smooth if and only if f^{an} is a holomorphic submersion.*
(7) *f has reduced fibres if and only if f^{an} has reduced fibres.*

Proof. See [GR71][Exposée XII, Prop. 3.1]. □

Remark 2.6.5. It follows from Lemma 2.6.4(1) that the analytification functor restricts to a functor

$$(\cdot)^{\mathrm{an}} : \mathbb{C}\text{–}\mathbf{Vrt}^{\mathrm{sm}} \to \mathbb{C}\mathbf{Man} \tag{2.6}$$

from the category of smooth $\mathbb{C}$-varieties to the category of complex manifolds.

Theorem 2.6.6. *Let X be a complete $\mathbb{C}$-variety. The functor that associates, to each coherent $\mathcal{O}_X$-module $\mathcal{F}$, its pullback $\mathcal{F}^{\mathrm{an}}$ on X^{an} by $j_X : X^{\mathrm{an}} \to X$ is exact and an equivalence of categories.*

Proof. See [GR71][Exposée XII, Théorème 4.4]. □

Theorem 2.6.7. *Let $f : X \to Y$ be a proper morphism of $\mathbb{C}$-varieties and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then for every integer $p \geq 0$ there is a canonical isomorphism*

$$(R^p f_* \mathcal{F})^{\mathrm{an}} \to R^p f_*^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}}).$$

Proof. See [GR71][Exposée XII, Théorème 4.2]. □

Theorem 2.6.8. *Let X be a complete $\mathbb{C}$-variety, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then for any integer $p \geq 0$ the canonical morphism $H^p(X, \mathcal{F}) \to H^p(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ is an isomorphism.*

Proof. See [GR71][Exposée XII, Corollaire 4.3]. □

2.7 Projective-algebraic complex spaces

In this section we give criteria for a complex space X to be *algebraic*, i.e. to be isomorphic to V^{an} for a $\mathbb{C}$-variety V. We will see this holds for compact Riemann surfaces in Theorem 2.7.2, and for analytic subspaces of a projective space $\mathbb{CP}^n$ as Chow's Theorem 2.7.3. Moreover, we prove a relative version of Chow's theorem in Corollary 2.7.8 in terms of relatively very ample analytic line bundles. Finally we show in Lemma 2.7.9 that the analytification functor preserves relative very amplitude of a line bundle. These results will be used in Chapter 10 to prove that the Kuga–Sato varieties we construct are actually algebraic.

Definition 2.7.1. We say a (reduced) complex space X is *algebraic* if it lies in the essential image of the analytification functor. We say a complex space Y is *projective-algebraic* if there exists a projective $\mathbb{C}$-variety V and a biholomorphism $X \cong V^{\text{an}}$.

Theorem 2.7.2. *Let X be a compact Riemann surface. Then X is projective-algebraic.*

Proof. See [Dem][Corollary VII.14.3]. □

Theorem 2.7.3. *Let X be an analytic subset of a complex projective space $\mathbb{CP}^n$, with homogeneous coordinates $(z_0 : z_1 : \ldots : z_n)$. Then X is the common zero set of finitely many homogeneous polynomials $P_j(z_0, ..., z_n)$ $(1 \leq j \leq t)$. In particular we have that $X = V_+(P_1, P_2, \ldots, P_t)^{\text{an}}$ is projective-algebraic.*

Proof. This is Chow's theorem, see [Dem][Theorem II.8.10]. □

Lemma 2.7.4. *Let $f_i : X_i \to Y$ $(1 \leq i \leq k)$ be holomorphic maps of complex spaces with common target Y. If each X_i is projective-algebraic, then the fibre product*

$$X_1 \times_Y X_2 \times_Y \cdots \times_Y X_k$$

is projective-algebraic.

Proof. By induction it suffices to treat the case $k = 2$. By definition $X_1 \times_Y X_2$ is the subset of $X_1 \times X_2$ which is the preimage under $f_1 \times f_2 : X_1 \times X_2 \to Y \times Y$ of the diagonal $\{(y, y) \in Y\} \subset Y \times Y$. The diagonal $\{(y, y) \in Y\}$ is locally given by analytic equations and is closed in $Y \times Y$ by virtue of the Hausdorff assumption on Y. Hence Δ is an analytic subset of $Y \times Y$, whence its preimage $X_1 \times_Y X_2$ is an analytic subset of $X_1 \times X_2$. Each X_i is projective-algebraic by assumption, so we may assume it is an analytic subset of $\mathbb{CP}^{n_i}$ for some integer $n_i > 0$. The Segre embedding provides a closed embedding $\mathbb{CP}^{n_1} \times \mathbb{CP}^{n_2} \to \mathbb{CP}^{(n_1+1)(n_2+1)} - 1$. We conclude that $X_1 \times_Y X_2$ is an analytic subset of $\mathbb{CP}^n$ with $n = (n_1 + 1)(n_2 + 1) - 1$, and an appeal to Chow's theorem concludes the proof. □

The projective bundle construction given in Theorem–Definition 2.4.2 also applies in the analytic category, as the following theorem asserts.

Theorem-Definition 2.7.5. *Let Y be a complex analytic space and let $\mathcal{E}$ be a coherent-analytic sheaf on Y. Then there exists a complex space $\mathcal{P}_Y(\mathcal{E})$ over Y such that for each holomorphic map $f : X \to Y$ of complex spaces, the Y-morphism from X into $\mathbb{P}_Y(\mathcal{E})$ correspond bijectively to isomorphism classes of pairs $(\mathcal{L}, \psi)$ consisting of a line bundles $\mathcal{L}$ on X and an epimorphism $\psi : f^*\mathcal{E} \to \mathcal{L}$. We call $\mathbb{P}_Y(\mathcal{E}) \to Y$ the* projective bundle over Y *attached to $\mathcal{E}$.*

Proof. See [Fis76][Def 1.9]. □

Lemma 2.7.6. *Let X be a projective manifold and $\mathcal{E}$ a coherent-analytic sheaf on X. Then $\mathbb{P}_X(\mathcal{E})$ is projective.*

Proof. Let X_0 be a projective $\mathbb{C}$-variety such that $X = X_0^{\mathrm{an}}$. By Theorem 2.6.6 there exists a coherent sheaf $\mathcal{E}_0$ on X_0 such that $\mathcal{E} = \mathcal{E}_0^{\mathrm{an}}$. Let $\mathbb{P}(\mathcal{E}_0^{\mathrm{an}})$ be the algebraic projective bundle of $\mathcal{E}_0$ over X_0. Then comparing the constructions of the projective bundles in the algebraic and analytic category yields that $\mathbb{P}(\mathcal{E}_0^{\mathrm{an}}) = \mathbb{P}(\mathcal{E}_0)^{\mathrm{an}}$. □

Definition 2.7.7. Let $f : X \to Y$ be a proper morphism of complex spaces. Let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{E} = f_*\mathcal{L}$ is a coherent sheaf on Y by Remmert's direct image theorem, see [Dem][Theorem IX.5.1]. We call $\mathcal{L}$ *very ample* on X/Y or *f-very ample* if the adjunction morphism $f^*\mathcal{E} = f^*f_*\mathcal{L} \to \mathcal{L}$ is an epimorphism and the induced map $X \to \mathbb{P}(\mathcal{E})$ is a closed embedding.

Our motivation for introducing relatively very ample line bundle is the following corollary, which will be used in Section 10.4 to prove that Kuga–Sato varieties are projective–algebraic.

Corollary 2.7.8. *Let $f : X \to Y$ be a proper morphism of complex spaces and suppose that $\mathcal{L}$ is an f-very ample line bundle. If Y is projective, then X is projective.*

Proof. Let $\mathcal{E} = f_*\mathcal{L}$ be the direct image of $\mathcal{L}$ by f. Then $\mathcal{E}$ is a coherent-analytic sheaf on Y. By Lemma 2.7.6 the projective bundle $\mathbb{P}(\mathcal{E})$ over Y is a projective-algebraic complex space. Since $\mathcal{L}$ is f-very ample, there exists a closed embedding of X into $\mathbb{P}(\mathcal{E})$. Since the composition of closed embeddings is a closed embedding, we conclude that X is isomorphic to a closed analytic subspace of a projective space. By Chow's theorem Theorem 2.7.3 we conclude that X is projective-algebraic. □

The analytification functor preserves very ample line bundles.

Lemma 2.7.9. *Let $f : X \to Y$ be a proper morphism of $\mathbb{C}$-varieties. Let $\mathcal{L}$ be an f-very ample line bundle on X. Then $\mathcal{L}^{\mathrm{an}}$ is an f^{an}-very ample line bundle on X^{an}.*

Proof. The first condition that the line bundle $\mathcal{L}$ be f-very ample states that the adjunction map $f^*f_*\mathcal{L} \to \mathcal{L}$ is an epimorphism in the category of coherent sheafs on X. We have to show the analogous statement that the adjunction map $(f^{\mathrm{an}})^*f_*^{\mathrm{an}}(\mathcal{L}^{\mathrm{an}})$ is an epimorphism in the category of coherent sheafs on X^{an}. Since the functor $(\cdot)^{\mathrm{an}} : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$ is exact by Theorem 2.6.6, it follows that $(f^*f_*\mathcal{L})^{\mathrm{an}} \to \mathcal{L}^{\mathrm{an}}$ is an epimorphism in the category of coherent sheafs on X^{an}. The definition of f^{an} implies that $(f^*\mathcal{F})^{\mathrm{an}} = (f^{\mathrm{an}})^*\mathcal{F}^{\mathrm{an}}$ for any coherent sheaf $\mathcal{F}$ on Y. Further, by Theorem 2.6.7 we have that $(f_*\mathcal{L})^{\mathrm{an}} = f_*^{\mathrm{an}}(\mathcal{L}^{\mathrm{an}})$. It follows that

$$(f^{\mathrm{an}})^*(f_*^{\mathrm{an}}(\mathcal{L}^{\mathrm{an}}) = (f^{\mathrm{an}})^*((f_*\mathcal{L})^{\mathrm{an}}) = (f^*f_*\mathcal{L})^{\mathrm{an}} \to \mathcal{L}^{\mathrm{an}}$$

is an epimorphism in the category of coherent analytic sheafs on X, as desired.

It remains to be checked that the Y^{an}-morphism $\phi_{\mathcal{L}^{\mathrm{an}}} : X^{\mathrm{an}} \to \mathbb{P}(f_*^{\mathrm{an}}\mathcal{L}^{\mathrm{an}})$ attached to the above epimorphism is a closed embedding. We saw in the proof of Lemma 2.7.6 that $\mathbb{P}(f_*\mathcal{L})^{\mathrm{an}} = \mathbb{P}((f_*\mathcal{L})^{\mathrm{an}})$ and in the first paragraph that one has $(f_*\mathcal{L})^{\mathrm{an}} = f_*^{\mathrm{an}}\mathcal{L}^{\mathrm{an}}$. Since $\mathcal{L}$ is f-very ample the Y-morphism $\phi_{\mathcal{L}} : X \to \mathbb{P}(f_*\mathcal{L})$ attached to the epimorphism $f^*f_*\mathcal{L} \to \mathcal{L}$ is a closed embedding. By we have that $(\phi_{\mathcal{L}})^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{P}(f_*\mathcal{L})^{\mathrm{an}}$ is a closed embedding. It is plain that via the isomorphism $\mathbb{P}(f_*\mathcal{L})^{\mathrm{an}} = \mathbb{P}((f_*\mathcal{L})^{\mathrm{an}}) = \mathbb{P}(f_*^{\mathrm{an}}\mathcal{L}^{\mathrm{an}})$ we have $(\phi_{\mathcal{L}})^{\mathrm{an}} = \phi_{\mathcal{L}^{\mathrm{an}}}$. We conclude that $\phi_{\mathcal{L}^{\mathrm{an}}}$ is a closed embedding, as desired. □

Chapter 3

Modular curves

This chapter reviews the classical construction of the complex-analytic open modular curve $Y(\Gamma)$ attached to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ as the quotient of the upper half-plane $\mathfrak{H}$ by the action of Γ via fractional linear transformations, and of its compactification $X(\Gamma)$ by adding a finite set of cusps $\mathrm{Cusps}\,(\Gamma)$. The end result of this procedure is encoded in Theorem 3.5.4 which identifies the compactified modular curve $X(\Gamma)$ with the pushout of a certain diagram of open embeddings of Riemann surfaces. This description of $X(\Gamma)$ lies at the basis of our construction in later chapters of various objects living over $X(\Gamma)$.

Examples of congruence subgroups which play a prominent role in the theory are $\Gamma(N), \Gamma_1(N)$ and $\Gamma_0(N)$ for integers $N \geq 1$. Therefore we will explicitly determine their sets of cusps $\mathrm{Cusps}(N)$, $\mathrm{Cusps}_1(N)$ resp. $\mathrm{Cusps}_0(N)$. The points of the modular curves $Y(N), Y_1(N)$ resp. $Y_0(N)$ attached to these congruence subgroups are in bijection with isomorphism classes of elliptic curves with a level-N structure resp. point of exact order N resp. cyclic subgroup of order N. For a general congruence subgroup Γ we also provide a modular description of $Y(\Gamma)$ as classifying so called Γ structures. In Section 5.3, we will define these structures in the relative setting, and discuss moduli spaces for them.

3.1 The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$

The story starts with an action of the special linear group of degree 2 over $\mathbb{Z}$

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

on the Poincaré upper half-plane

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}.$$

Lemma 3.1.1. *A proper action of* $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathfrak{H}$ *via biholomorphisms is given by*

$$\begin{aligned} &\mathrm{SL}_2(\mathbb{Z}) \times \mathfrak{H} \to \mathfrak{H} \\ &\left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \mapsto [\gamma](\tau) = \frac{a\tau + b}{c\tau + d} \end{aligned} \tag{3.1}$$

For every point $\tau \in \mathfrak{H}$*, the stabilizer* $\mathrm{SL}_2(\mathbb{Z})_\tau = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : [\gamma](\tau) = \tau\}$ *is finite.*

Proof. It is straightforward to verify associativity that the unit matrix operates as the identity. If $a, b, c, d \in \mathbb{R}$ are such that $ad - bc > 0$, then $\tau \mapsto (a\tau + b)/(c\tau + d)$ defines an automorphism of the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, called a fractional linear transformation or Möbius transformation, that leaves the open subset $\mathfrak{H} \subset \mathbb{C}$ invariant. This shows that $[\gamma] : \mathfrak{H} \to \mathfrak{H}$ is a biholomorphism for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

[Miy05][Thm. 1.5.3] asserts that $\mathrm{SL}_2(\mathbb{Z})$ acts properly on $\mathfrak{H}$. The stabilizers of points in $\mathfrak{H}$ are determined in [Miy05][Thm. 4.1.3] to be finite cyclic subgroups, of order 2, 4 or 6 . □

3.2 Congruence subgroups

Let $N \geq 1$ be an integer. The canonical map $\pi_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective [Miy05][Theorem 4.2.1(1)]. We define the principal congruence subgroup of level N to be the kernel of π_N,

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N \right\}.$$

More generally, a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if there exists an integer $N' \geq 1$ such that $\Gamma \supset \Gamma(N')$. The greatest common divisor $N = \gcd\{N' \geq 1 : \Gamma \supset \Gamma(N')\}$ of all such integers N' is called the level of Γ. For positive integers N_1 and N_2 it holds that $\Gamma(N_1) \cap \Gamma(N_2) = \Gamma(\mathrm{lcm}(N_1, N_2))$ and $\Gamma(N_1)\Gamma(N_2) = \Gamma(\gcd(N_1, N_2))$, so the level N of Γ is minimal with respect to the property that $\Gamma \supset \Gamma(N)$. Examples which will play a key role in the sequel include

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

the preimage under π_N of the upper triangular matrices, and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \bmod N, c \equiv 0 \bmod N \right\}$$

the preimage under π_N of the unipotent upper triangular matrices. In sum, all squares and rectangles in the following diagram are cartesian.

$$\begin{array}{ccccccc}
\Gamma(N) & \subset & \Gamma_1(N) & \subset & \Gamma_0(N) & \subset & \mathrm{SL}_2(\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} & \subset & \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} & \subset & \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} & \subset & \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).
\end{array}$$

The indices of any two horizontally aligned inclusions in the top and bottom row are equal, and determined as follows.

Lemma 3.2.1. *With p denoting a prime divisor of N, we have*

$$(\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)) = N \prod_{p|N} \left(1 + \frac{1}{p}\right),$$
$$(\Gamma_0(N) : \Gamma_1(N)) = N \prod_{p|N} \left(1 - \frac{1}{p}\right),$$
$$(\Gamma_1(N) : \Gamma(N)) = N.$$

Proof. This is the content of [Miy05][Thm. 4.2.1, Thm. 4.2.4(2) and Thm. 4.2.5]. □

For $N \leq 2$ we have $-1 \in \Gamma(N)$. For $N > 2$ we have $-1 \in \Gamma_0(N) \backslash \Gamma_1(N)$.

Proposition 3.2.2. *Let $N \geq 1$ be an integer.*

(1) *The congruence subgroup $\Gamma(N)$ is torsion-free if and only if $N \geq 3$.*
(2) *The congruence subgroup $\Gamma_1(N)$ is torsion-free if and only if $N \geq 4$.*
(3) *The congruence subgroup $\Gamma_0(N)$ is not torsion-free for every $N \geq 1$.*

Proof. See [Dia06][§3.9]. □

3.3 The open modular curve $Y(\Gamma) = \Gamma \backslash \mathfrak{H}$

Proposition 3.3.1. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. There exists a local analytic isomorphism*

$$p_\Gamma : \mathfrak{H} \to Y(\Gamma)$$

to a connected Riemann surface that is a categorical quotient for the action of Γ on $\mathfrak{H}$ in the category of complex manifolds.

Proof. By Lemma 3.1.1 the action of $\mathrm{SL}_2(\mathbb{Z})$ is proper with finite stabilizers, so that of the subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is as well. By Theorem 2.3.5 a categorical quotient for this Γ-action on the Riemann surface $\mathfrak{H}$ exists, which is a local analytic isomorphism $p_\Gamma : \mathfrak{H} \to Y(\Gamma)$ to a Riemann surface. Connectedness of $\mathfrak{H}$ implies that of the quotient $Y(\Gamma)$. □

Definition 3.3.2. The open modular curve attached to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is the Riemann surface

$$Y(\Gamma) := \Gamma \backslash \mathfrak{H}.$$

Suppose that $\tilde{\Gamma} \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ are two congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Since the map $p_\Gamma : \mathfrak{H} \to Y(\Gamma)$ is Γ-invariant, it is also $\tilde{\Gamma}$-invariant. Because $p_{\tilde{\Gamma}}$ is universal among $\tilde{\Gamma}$-invariant holomorphic maps out of $\mathfrak{H}$, there exists a unique holomorphic map

$$p_{\Gamma,\tilde{\Gamma}} : Y\left(\tilde{\Gamma}\right) \to Y(\Gamma) \tag{3.2}$$

such that $p_{\Gamma,\tilde{\Gamma}} \circ p_{\tilde{\Gamma}} = p_\Gamma$.

3.4 The set of cusps $\mathrm{Cusps}(\Gamma) = \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$

Using the same formula (3.1) as in Lemma 3.1.1, we define an action of $\mathrm{SL}_2(\mathbb{Z})$ on the projective line over $\mathbb{Q}$

$$\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}.$$

Lemma 3.4.1. *A transitive action of* $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathbb{P}^1(\mathbb{Q})$ *is given by the formula*

$$\begin{gathered}\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{P}^1(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q}) \\ \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, s\right) \mapsto \frac{as+b}{cs+d}.\end{gathered}$$

Proof. See [Dia06][§2.4]. □

The stabilizer of $\infty \in \mathbb{P}^1(\mathbb{Q})$ is the parabolic subgroup

$$P := \mathrm{SL}_2(\mathbb{Z})_\infty = \left\{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}. \tag{3.3}$$

Hence there is an isomorphism of $\mathrm{SL}_2(\mathbb{Z})$-sets

$$\begin{aligned}\mathrm{SL}_2(\mathbb{Z})\backslash P &\xrightarrow{\sim} \mathbb{P}^1(\mathbb{Q}), \\ \gamma P &\mapsto [\gamma](\infty) =: s_\gamma.\end{aligned} \tag{3.4}$$

Definition 3.4.2. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The set of Γ-orbits in $\mathbb{P}^1(\mathbb{Q})$ under the action Lemma 3.4.1 will be denoted

$$\mathrm{Cusps}(\Gamma) = \Gamma\backslash\mathbb{P}^1(\mathbb{Q}) = \left\{\Gamma s : s \in \mathbb{P}^1(\mathbb{Q})\right\}.$$

The elements of $\mathrm{Cusps}(\Gamma)$ will be called cusps of Γ.

The isomorphism of $\mathrm{SL}_2(\mathbb{Z})$-sets 3.5 induces a bijection

$$\begin{aligned}\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/P &\to \mathrm{Cusps}(\Gamma) \\ \Gamma\gamma P &\mapsto \Gamma[\gamma](\infty).\end{aligned} \tag{3.5}$$

Since the congruence subgroup Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, it follows that $\mathrm{Cusps}(\Gamma)$ is a finite set.

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The subgroup $P \cap \gamma^{-1}\Gamma\gamma$ of P depends only on on the double coset $\Gamma\gamma P$, so it makes sense to set

$$P_t = P_{\gamma,\Gamma} := P \cap \gamma^{-1}\Gamma\gamma, \qquad \text{where } t = \Gamma[\gamma](\infty). \tag{3.6}$$

Since the index of the congruence subgroup $\gamma^{-1}\Gamma\gamma$ in $\mathrm{SL}_2(\mathbb{Z})$ is finite, $P \cap \gamma^{-1}\Gamma\gamma$ is a finite-index subgroup of P. Because $P \cong \{\pm 1\} \times \mathbb{Z}$, the finite-index subgroups of P come in three families parametrized by an integer $h \in \mathbb{Z}_{\geq 1}$:

$$P_h = \left\{ \pm \begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$
$$P_h^+ = \left\{ \begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$
$$P_h^- = \left\{ \begin{pmatrix} (-1)^n & nh \\ 0 & (-1)^n \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

In pertaining to indices we have $h = (P : P_h)$ and $P_h = \{\pm 1\} \times P_h^+ = \{\pm 1\} \times P_h^-$.

Definition 3.4.3. The *width* of a cusp $t \in \mathrm{Cusps}(\Gamma)$ is the positive integer $h = h_t$ such that $P_t \in \left\{P_h, P_h^+, P_h^-\right\}$.

We call t a *regular cusp* if $P_t \in \left\{P_h, P_h^+\right\}$ and an *irregular cusp* if $P_t = P_h^-$.

For a cusp t of Γ having width h, we have $P_t = P_h$ if $-1 \in \Gamma$ and $P_t \in \left\{P_h^+, P_h^-\right\}$ if $-1 \notin \Gamma$ (e.g. if Γ is torsion-free). If $t = \Gamma s$ with $s \in \mathbb{P}^1(\mathbb{Q})$ then we have

$$h_t = (P : \pm P_t) = (\mathrm{SL}_2(\mathbb{Z})_s : \pm \Gamma_s).$$

Definition 3.4.4. For every $N \geq 1$, we denote the sets of cusps of the congruence subgroups of the shape $\Gamma(N), \Gamma_1(N)$ and $\Gamma_0(N)$ by

- $\mathrm{Cusps}(N) = \mathrm{Cusps}(\Gamma(N))$;
- $\mathrm{Cusps}_1(N) = \mathrm{Cusps}\left(\Gamma_1(N)\right)$;
- $\mathrm{Cusps}_0(N) = \mathrm{Cusps}\left(\Gamma_0(N)\right)$.

In the remainder of this section, we describe the cusps of the congruence subgroups $\Gamma(N), \Gamma_1(N)$ and $\Gamma_0(N)$ for every integer $N \geq 1$.

Definition 3.4.5. Let $N \in \mathbb{Z}_{\geq 1}$. We denote the image of the parabolic subgroup P under the canonical map $\pi_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ by

$$\bar{P}_N = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}/N\mathbb{Z} \right\}. \tag{3.7}$$

Since $\Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, setting $\Gamma = \Gamma(N)$ in (3.6) gives an isomorphism of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$-sets

$$\mathrm{Cusps}(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\bar{P}_N. \tag{3.8}$$

Definition 3.4.6. The images of the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$ under the canonical map $\pi_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ will be denoted respectively by

$$\bar{\Gamma}_1(N) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}/N\mathbb{Z} \right\},$$
$$\bar{\Gamma}_0(N) = \left\{ \begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix} : b \in \mathbb{Z}/N\mathbb{Z}, d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

Now (3.7) induces bijections

$$\mathrm{Cusps}_1(N) \cong \overline{\Gamma_1(N)}\backslash \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\bar{P}_N,$$
$$\mathrm{Cusps}_0(N) \cong \overline{\Gamma_0(N)}\backslash \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\bar{P}_N.$$

Proposition 3.4.7. *Let $N \geq 1$ be an integer and let $\Gamma \in \{\Gamma(N), \Gamma_1(N), \Gamma_0(N)\}$. Then the number of cusps of Γ is given for $N \leq 4$ by the table*

N	1	2	3	4
$\mathrm{Cusps}(N)$	1	3	4	6
$\mathrm{Cusps}_1(N)$	1	2	3	3
$\mathrm{Cusps}_0(N)$	1	2	2	3

and for $N \geq 5$ by

$$\begin{aligned}
\#\,\mathrm{Cusps}(N) &= \frac{1}{2}N^2 \prod_{p \mid N}\left(1 - \frac{1}{p^2}\right) = \frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{2N}, \\
\#\,\mathrm{Cusps}_1(N) &= \frac{1}{2}\sum_{0<d\mid N} \phi(d)\phi(N/d), \\
\#\,\mathrm{Cusps}_0(N) &= \sum_{0<d\mid N} \phi(\gcd(d, N/d)).
\end{aligned}$$

Proof. See [Dia06][§3.9]. □

Proposition 3.4.8. *Let $N \geq 1$ be an integer and let $\Gamma \in \{\Gamma(N), \Gamma_0(N), \Gamma_1(N)\}$. Then all cusps Γs of Γ are regular with the sole exception of $s = 1/2$ when $\Gamma = \Gamma_1(4)$.*

Proof. See [Dia06][§3.8]. □

3.5 The compactified modular curve $X(\Gamma) = \Gamma\backslash\mathfrak{H}^*$

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The aim in this section is to describe the classical compactification of the open modular curve $Y(\Gamma) = \Gamma\backslash\mathfrak{H}$ from Definition 3.3.2 by adding the set $\mathrm{Cusps}(\Gamma)$ from Definition 3.4.2 to it, which results in a compact connected Riemann surface denoted $X(\Gamma)$.

The compactification procedure is classical and can be found in any textbook on modular forms, see e.g. [Dia06][§2.4] or [Shi94][§1.5]. Therefore we have opted for a summarily treatment omitting many proofs. Our modest aim is to describe the end result $X(\Gamma)$ in Theorem 3.5.4 as the pushout of a certain diagram of open embeddings of Riemann surfaces involving $Y(\Gamma)$ and open disks in $\mathbb{C}$ centered at 0. This description of $X(\Gamma)$ as a pushout provides a uniform framework for the construction of a wide class of geometric objects living over $X(\Gamma)$, which is one of the principal aims of this thesis. It inevitably involves quite a bit of notation, which we will introduce in the present section and return to frequently in later chapters.

Theorem 3.5.1. *For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ there exists a (Hausdorff) compact Riemann surface $X(\Gamma)$ containing $Y(\Gamma)$ as the open complement of a finite set $\mathrm{Cusps}(\Gamma)$.*

If Γ_1 is a second congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with $\Gamma_1 \subset \Gamma$, then the map (3.2) extends to a holomorphic map

$$p_{\Gamma,\Gamma_1} : X(\Gamma_1) \to X(\Gamma) \tag{3.9}$$

mapping $\mathrm{Cusps}(\Gamma_1)$ surjectively onto $\mathrm{Cusps}(\Gamma)$.

We have constructed actions of $\mathrm{SL}_2(\mathbb{Z})$ on the sets $\mathfrak{H}$ and $\mathbb{P}^1(\mathbb{Q})$. In [Miy05][§1.7] it is shown that the union of these two $\mathrm{SL}_2(\mathbb{Z})$-sets

$$\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \tag{3.10}$$

admits a unique topology for which

- $\mathcal{H}$ is an open subset with its usual topology;
- $\mathbb{P}^1(\mathbb{Q})$ is a discrete closed subset;
- a fundamental system of neighborhoods of ∞ is given by the sets $\{z \in \mathbb{C} : \Im z > l\}$ for $l \in \mathbb{R}_{\geq 1}$;
- the induced action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}^*$ is via homeomorphisms.

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and form the quotient topological space

$$X(\Gamma) := \Gamma \backslash \mathfrak{H}^*.$$

Lemma 3.5.2. *The topological space $X(\Gamma)$ is compact and Hausdorff.*

Proof. See [Miy05][Lemma 1.7.7, Theorem 4.1.2(2), Corollary 1.9.2]. The compactness of $X(\Gamma)$ is stated in this reference by saying that Γ is a Fuchsian group of the first kind). □

Since the decomposition (3.10) is Γ-stable, we have that

$$X(\Gamma) = Y(\Gamma) \sqcup \mathrm{Cusps}(\Gamma)$$

is the union of an open subset $Y(\Gamma)$ and a finite closed set $\mathrm{Cusps}(\Gamma)$.

Next we make the topological space $X(\Gamma)$ into a Riemann surface by endowing it with a complex structure. By Proposition 3.3.1 we already have a complex structure on the open subset $Y(\Gamma) = \Gamma \backslash \mathfrak{H}$. So it remains to give complex charts centered at the cusps. First we define certain (punctured) neighbhorhoods of the cusps in $X(\Gamma)$.

Consider the following punctured neighborhood resp. neighborhood of the cusp ∞ in $\mathfrak{H}^*$

$$U_\infty = \{\tau \in \mathbb{C} : \Im(\tau) > 1\} \subset \mathfrak{H}, \quad U_\infty^* = U_\infty \cup \{\infty\} \subset \mathfrak{H}^*$$

For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ one has that $[\gamma](U_\infty) \cap U_\infty \neq \emptyset$ only if $[\gamma](\infty) = (\infty)$. Since P acts on $\mathfrak{H}$ via translation by some integer, U_∞ is stable under P.
Let $s \in \mathbb{Q} = \mathbb{P}^1(\mathbb{Q}) \backslash \{\infty\}$ be another cusp. We set $U_s := [\gamma](U_\infty)$, where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is any element such that $[\gamma](\infty) = s$. Then U_s does not depend on the choice of γ, since any other $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ with $[\gamma'](\infty) = s$ is of the shape $\gamma' = \gamma\delta$ for some $\delta \in P$, so that $[\gamma'](U_\infty) = [\gamma]([\delta](U_\infty)) = [\gamma](U_\infty)$. An open neighborhood of the cusp $t = \Gamma s \in \mathrm{Cusps}(\Gamma)$ in $X(\Gamma)$ is given by $\Gamma_s \backslash U_s^* = \{t\} \sqcup \Gamma_s \backslash U_s$.

Note also that $U_s \cap U_{s'} = \emptyset$ for two distinct cusps $s, s' \in \mathbb{P}^1(\mathbb{Q})$ (cf. [Miy05][§1.7]). Therefore an open neighbhorood of the set $\mathrm{Cusps}(\Gamma)$ in $X(\Gamma)$ is given by

$$\bigsqcup_s \Gamma_s \backslash U_s^* = \mathrm{Cusps}(\Gamma) \cup \bigsqcup_s \Gamma_s \backslash U_s,$$

where $s \in \mathbb{P}^1(\mathbb{Q})$ ranges over a system of orbit representatives for $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$.

Notation 3.5.3. For a positive integer h set

$$V_h := \{q_h \in \mathbb{C} : 0 < |q_h| < \exp(-2\pi/h)\},$$
$$V_h^* := \{q_h \in \mathbb{C} : |q_h| < \exp(-2\pi/h)\} = V_h \cup \{0\}.$$

Let $P_h' \in \{P_h, P_h^+, P_h^-\}$. There exists an analytic isomorphism

$$\begin{aligned} e_h : P_h' \backslash U_\infty &\xrightarrow{\sim} V_h^* \\ P_h'\tau &\mapsto \exp(2\pi \mathrm{i} \tau / h) \end{aligned} \tag{3.11}$$

which extends to a homeomorphism

$$\begin{aligned} e_h : P_h' \backslash U_\infty^* &\xrightarrow{\sim} V_h \\ P_h'\infty &\mapsto 0. \end{aligned}$$

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and set $s = [\gamma](\infty)$ and $t = \Gamma s$. Recall that in (3.6) we defined

$$P_t = P_{\gamma,\Gamma} = \gamma^{-1}\Gamma_s\gamma = P \cap \gamma^{-1}\Gamma\gamma.$$

Let h be the width of t, so $P_t \in \{P_h, P_h^+, P_h^-\}$. We take as a complex chart of $X(\Gamma)$ centered at t the composite

$$\Gamma_s \backslash U_s^* \xrightarrow{[\gamma^{-1}]} P_t \backslash U_\infty^* \xrightarrow{e_h} V_h. \tag{3.12}$$

Since (3.11) is a biholomorphism, this chart is compatible with those in the complex atlas on $Y(\Gamma)$. Thus we have constructed a complex atlas on $X(\Gamma)$, and the construction of the Riemann surface $X(\Gamma)$ is complete. We encode the end result of this procedure in the following theorem.

Theorem 3.5.4. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and R be a set of double coset representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/P$. Then the Riemann surface $X(\Gamma) = \Gamma\backslash\mathfrak{H}^*$ in Theorem 3.5.1(1) is the pushout of the following diagram of topological spaces, in which we set $h = h_\gamma$ and $s = s_\gamma$*

$$\begin{array}{ccc} \bigsqcup_{\gamma\in R} P_{\gamma,\Gamma}\backslash U_\infty & \overset{\sqcup[\gamma]}{\cong} & \bigsqcup_{\gamma\in R} \Gamma_{s_\gamma}\backslash U_s \\ \downarrow{\scriptstyle \sqcup e_h} & & \downarrow \\ \bigsqcup_{\gamma\in R} V_{\tilde{h}} & & Y(\Gamma) \\ & \searrow \quad \swarrow & \\ & X(\Gamma). & \end{array} \tag{3.13}$$

Proof. This follows from Theorem 3.5.1(1) and the preceding discussion. ☐

Notation 3.5.5. (1) Let $h \in \mathbb{Z}_{\geq 1}$. There is a unique action $P \to \mathrm{Aut}(V_h)$, $\delta \mapsto [\delta]_h$ of P on V_h via biholomorphism such that for each $\delta \in P$ the following diagram commutes

$$\begin{array}{ccc} U_\infty & \xrightarrow{e_h} & V_h^* \\ {\scriptstyle [\delta]}\downarrow & & \downarrow{\scriptstyle [\delta]_h} \\ U_\infty & \xrightarrow{e_h} & V_h^*. \end{array}$$

For $z \in V_h$ and $n \in \mathbb{Z}$ we have

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]_h (z) = z, \quad \left[\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right] (z) = \zeta_N^n z. \tag{3.14}$$

(2) Let $d \in \mathbb{Z}_{\geq 1}$ and set $\tilde{h} = dh$, so that $\tilde{h}$ is a multiple of h. Denote $p_d : \Delta \to \Delta$ the map given by $p_d(z) = z^d$, which restricts to a map

$$\begin{aligned} p_d : V_{dh} &\to V_h, \\ z &\mapsto z^d. \end{aligned} \tag{3.15}$$

Note that $[\delta]_h \circ p_d = p_d \circ [\delta]_{\tilde{h}}$ for each $\delta \in P$. If $P'_{\tilde{h}} \in \left\{ P_{\tilde{h}}, P^+_{\tilde{h}}, P^-_{\tilde{h}} \right\}$ is a subgroup of P'_h and $\delta \in P$, then the following diagram commutes:

$$\begin{array}{ccc} P'_{\tilde{h}} \backslash U^*_\infty & \xrightarrow{e_{\tilde{h}}} & V_{\tilde{h}} \\ {\scriptstyle [\delta]}\big\downarrow & & \big\downarrow{\scriptstyle [\delta]_h \circ p_d} \\ P'_h \backslash U^*_\infty & \xrightarrow{e_h} & V_h. \end{array} \tag{3.16}$$

Theorem 3.5.6. *Let $\Gamma \subset \tilde{\Gamma}$ be two congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Choose sets R and $\tilde{R}$ of double coset representatives for $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z}) / P$ resp. $\tilde{\Gamma} \backslash \mathrm{SL}_2(\mathbb{Z}) / P$. Then for each $\beta \in \tilde{R}$ there exist $\alpha(\beta) \in \Gamma$,$\gamma(\beta) \in R$ and $\delta(\beta) \in P$ with $\alpha(\beta)\beta = \gamma(\beta)\delta(\beta)$. For every $\beta \in \tilde{R}$ we have that $P_{\beta,\tilde{\Gamma}} \subset P_{\gamma(\beta),\Gamma}$ and $d(\beta) := h_{\beta,\tilde{\Gamma}} / h_{\gamma(\beta),\Gamma}$ is an integer. The quotient map $p_{\Gamma,\tilde{\Gamma}} : X(\tilde{\Gamma}) = \tilde{\Gamma} \backslash \mathfrak{H} \to X(\Gamma) = \tilde{\Gamma} \backslash \mathfrak{H}$ in Theorem 3.5.1(2) is the pushout of the following morphism between the rows of the diagram below induced by $\gamma : \tilde{R} \to R$, in which we set $s = [\gamma](\infty)$, $h = h_{\Gamma s}$ $\tilde{s} = s_\beta$ and $\tilde{h} = h_{\tilde{\Gamma}\tilde{s}}$:*

$$\begin{array}{ccccccc} \bigsqcup_{\beta \in \tilde{R}} V^*_{\tilde{h}} & \xleftarrow{\sqcup e_{\tilde{h}}} & \bigsqcup_{\beta \in \tilde{R}} P_{\beta,\tilde{\Gamma}} \backslash U_\infty & \xrightarrow{\sqcup [\beta]} & \bigsqcup_{\beta \in \tilde{R}} \tilde{\Gamma}_{\tilde{s}} \backslash U_{\tilde{s}} & \longrightarrow & Y(\tilde{\Gamma}) \\ {\scriptstyle \sqcup([\delta(\beta)]_h \circ p_{d(\beta)})}\big\downarrow & & \big\downarrow{\scriptstyle \sqcup[\delta(\beta)]} & & \big\downarrow{\scriptstyle \sqcup[\alpha(\beta)]} & & \big\downarrow{\scriptstyle p_{\Gamma,\tilde{\Gamma}}} \\ \bigsqcup_{\gamma \in R} V^*_h & \xleftarrow{\sqcup e_h} & \bigsqcup_{\gamma \in R} P_{\gamma,\Gamma} \backslash U_\infty & \xrightarrow{\sqcup [\gamma]} & \bigsqcup_{\gamma \in \mathbb{R}} \Gamma_s \backslash U_s & \longrightarrow & Y(\Gamma). \end{array} \tag{3.17}$$

Proof. Let $\beta \in \tilde{R}$. We have $\alpha(\beta)\beta = \gamma(\beta)\delta(\beta)$ with $\alpha(\beta) \in \Gamma$ and $\delta(\beta) \in P$, so $\Gamma\beta P = \Gamma\gamma(\beta)P$, i.e. β and γ define the same cusp $\Gamma\tilde{s} = \Gamma s$ of $X(\Gamma)$. Recall from (3.6) that for $\epsilon \in \mathrm{SL}_2(\mathbb{Z})$ the subgroup $P_{\epsilon,\Gamma}$ depends only on the cusp of $X(\Gamma)$ defined by ϵ; in particular we have $P_{\beta,\Gamma} = P_{\gamma,\Gamma}$. Hence the inclusion $\tilde{\Gamma} \subset \Gamma$ implies that $P_{\beta,\tilde{\Gamma}} \subset P_{\beta,\Gamma} = P_{\gamma,\Gamma}$. This in turn implies that

$$d(\beta) = h_{\beta,\tilde{\Gamma}} / h_{\gamma(\beta),\Gamma} = \left(P : P_{\beta,\tilde{\Gamma}}\right) / \left(P : P_{\gamma(\beta),\Gamma}\right) = \left(P_{\gamma(\beta),\Gamma} : P_{\beta,\tilde{\Gamma}}\right)$$

is an integer.

Next we show there is a morphism between the rows of the diagrams as asserted. Let again $\beta \in \tilde{R}$ be given. Since $s = [\alpha(\beta)](\tilde{s})$ we have $\tilde{\Gamma}_{\tilde{s}} \subset \Gamma_{\tilde{s}} = \alpha(\beta)^{-1}\Gamma_s\alpha(\beta)$. Therefore $P_{\beta,\tilde{\Gamma}} \subset P_{\gamma,\Gamma}$ and $\alpha(\beta)\tilde{\Gamma}_{\tilde{s}} \subset \alpha(\beta)\Gamma_s$, whence there are well-defined vertical maps in the middle of diagram. We have seen that the left square commutes in (3.16). The identity $\alpha(\beta)\beta = \gamma(\beta)$ implies that the middle square commutes. The right square commutes since $\alpha(\beta) \in \Gamma$. This concludes the proof. □

Remark 3.5.7. In particular, taking $\tilde{\Gamma} = \Gamma$, we see that the diagrams (3.13) for various choices of a set R of double coset representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/P$ are each isomorphic, hence the pushouts of these diagrams are also isomorphic. Therefore we could have defined $X(\Gamma)$ as the pushout of any of these squares.

3.6 Modular curves as moduli spaces of elliptic curves

This section recollects some classical facts about the torsion of complex elliptic curves. Without proof we give the modular description of the points of the Riemann surfaces $Y(N), Y_1(N)$ and $Y_0(N)$ for integers $N \geq 1$ as parametrizing isomorphism classes of complex elliptic curve with some additional structure on their N-torsion subgroup, see Theorem 3.6.2. These definitions and results will be generalized in Chapter 4 to the relative setting of an elliptic curve living over a general base complex manifold, as opposed to the final object $\{*\}$ of $\mathbb{C}\mathbf{Man}$.

A *complex elliptic curve* may be defined as a compact connected complex Lie group of dimension 1. By the uniformization theorem, there exists a co-compact lattice $\Lambda \subset \mathbb{C}$ (that is, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for some $\mathbb{R}$-basis (ω_1, ω_2) of $\mathbb{C}$) and an isomorphism $E \cong \mathbb{C}/\Lambda$. We fix such an isomorphism, and work in the sequel with $E = \mathbb{C}/\Lambda$. A point of E is a coset $z + \Lambda$ with $z \in \mathbb{C}$, but when it is clear from the context that a point of E is meant, we may simply write z instead of $z + \Lambda$.

The universal covering space of E is the projection map $\mathbb{C} \to \mathbb{C}/\Lambda = E$. Let $H_1(E;\mathbb{Z})$ be the first singular homology group of E with $\mathbb{Z}$-coefficients. The map $\Lambda \to H_1(E;\mathbb{Z})$ sending $\lambda \in \Lambda$ to the homology class of the loop $[0,1] \to E$, $t \mapsto t\lambda$ is an isomorphism, whence $H_1(E;\mathbb{Z})$ is a free rank-2 abelian group. An H_1-trivialization of E is a choice of isomorphism $\psi : \mathbb{Z}^2 \to H_1(E;\mathbb{Z})$.

For an integer $N \geq 1$, the N-torsion subgroup of E is described as

$$E[N] = \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda = H_1(E;\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/N\mathbb{Z}).$$

A *level-N structure ϕ on E* is an isomorphism of groups $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$. It is equivalent data to give the $(\mathbb{Z}/N\mathbb{Z})$-basis $(\phi(1,0), \phi(0,1))$ of $E[N]$. A *point of exact order N of E* is an element of E generating a subgroup of order N, i.e. the image of $1 \in \mathbb{Z}/N\mathbb{Z}$ under an injective homomorphism $\mathbb{Z}/N\mathbb{Z} \to E$. A *cyclic subgroup of order N of E* is a subgroup of E isomorphic to $\mathbb{Z}/N\mathbb{Z}$, i.e. the image of an injective homomorphism $\mathbb{Z}/N\mathbb{Z} \to E$.

There exists an alternating non-degenerate bilinear map $e_N : E[N] \times E[N] \to \mu_N$, called the *Weil e_N-pairing on E*, determined by the condition that for every $\mathbb{Z}$-basis (ω_1, ω_2) of Λ we have

$$e_N\left(\frac{1}{N}\omega_1, \frac{1}{N}\omega_2\right) = \zeta_N^{\mathrm{sgn}(\Im(\omega_2/\omega_1))},$$

where $\zeta_N = \exp(2\pi\mathrm{i}/N)$. If ϕ is a level-N structure, then $\zeta = e_N(\phi(0,1), \phi(1,0))$ is a primitive N-th root of unity, and we say that *ϕ has Weil pairing $\zeta \in \mu_N^\times$*.

Example 3.6.1. Let $\tau \in \mathfrak{H}$. Consider the co-compact lattice $\Lambda_\tau := \mathbb{Z}\tau + \mathbb{Z}$, and set $E_\tau = \mathbb{C}/\Lambda_\tau$. Then a level- N structure with Weil pairing ζ_N on E is given by $\Psi_\tau = (\tau/N, 1/N) : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$, $(m, n) \mapsto (m\tau + n)/N$. The point $\frac{\tau}{N} \in E[N]$ has exact order N, and it generates a cyclic subgroup $(\mathbb{Z}/N\mathbb{Z}) \cdot \frac{\tau}{N}$ of order N of E.

Theorem 3.6.2. *Let $N \geq 1$ be an integer.*

(1) *The following map is a well-defined bijection between the set of points of the Riemann surface $Y(N) = \Gamma(N)\backslash\mathfrak{H}$ and the set of isomorphism classes of complex elliptic curves with a level-N structure with Weil pairing ζ_N:*

$$\begin{aligned} |Y(N)| &\xrightarrow{\sim} \left\{ \begin{matrix} \textit{complex elliptic curves with a level-}N \\ \textit{structure with Weil pairing } \zeta_N \end{matrix} \right\} / \cong, \\ \Gamma(N)\tau &\mapsto \left(E_\tau, \Psi_\tau = \left(\frac{\tau}{N}, \frac{1}{N} \right) \right). \end{aligned} \tag{3.18}$$

(2) *The following map is a well-defined bijection between the set of points of the Riemann surface $Y_1(N) = \Gamma_1(N)\backslash\mathfrak{H}$ and the set of isomorphism classes of complex elliptic curves with a point of exact order N:*

$$\begin{aligned} |Y_1(N)| &\xrightarrow{\sim} \{\textit{complex elliptic curves with a point of exact order } N\} / \cong, \\ \Gamma_1(N)\tau &\mapsto \left(E_\tau, \frac{\tau}{N} \right). \end{aligned} \tag{3.19}$$

(3) *The following map is a well-defined bijection between the set of points of the Riemann surface $Y_0(N) = \Gamma_0(N)\backslash\mathfrak{H}$ and the set of isomorphism classes of complex elliptic curves with a cyclic subgroup of order N:*

$$\begin{aligned} |Y_0(N)| &\xrightarrow{\sim} \{\textit{complex elliptic curves with a cyclic subgroup of order } N\} / \cong, \\ \Gamma_0(N)\tau &\mapsto \left(E_\tau, (\mathbb{Z}/N\mathbb{Z}) \cdot \frac{\tau}{N} \right). \end{aligned} \tag{3.20}$$

Proof. See [Dia06][Thm. 1.5.1] □

Chapter 4

Relative elliptic curves

In Section 3.6 we considered a complex elliptic curve, which we defined as a compact connected complex Lie group of dimension 1. The Uniformization Theorem gives an isomorphism $E \cong \mathbb{C}/\Lambda$, where Λ is a co-compact lattice in $\mathbb{C}$ that is naturally identified with the first singular homology group $H_1(E;\mathbb{Z})$.

This chapter relativizes this theory by considering holomorphically varying families of elliptic curves. Let M be a holomorphic manifold. An elliptic curve over M is, roughly speaking, a holomorphic manifold E with a map to M such that the fibre E_m over every point $m \in M$ is elliptic curves. See Section 4.1 for precise definitions of various kinds of geometric objects living over M. We state in Theorem 4.2.6 the analogue of the uniformization theorem for elliptic curves over M.

There exists a natural pairing on the N-torsion $E[N]$ of E with values in the group μ_N of complex N-th roots of unity, called the Weil pairing and defined in Section 4.5. We define this pairing by uniformizing E as the quotient of a holomorphic line bundle by a lattice. In Theorem 4.3.1 we construct this lattice by assembling the fibral homology groups $H_1(E_m;\mathbb{Z})$ into a local system $\underline{H}_1(E/M)$ of rank-2 free $\mathbb{Z}$-modules on M. This local system plays a pivotal role in Chapter 5, where it is used to define various enrichments of elliptic curves.

4.1 Elliptic curves over a complex manifold

Let M be a complex manifold.

Definition 4.1.1. (1) A *complex manifold over M* or *M-complex manifold* is a complex manifold X together with a holomorphic map $f : X \to M$. We let $\mathbb{C}\mathbf{Man}_{/M}$ be the category of complex manifolds over M, in which the morphisms are commutative triangles

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & M. & \end{array}$$

(2) The full subcategory of $\mathbb{C}\mathbf{Man}$ consisting of those holomorphic maps $f : X \to M$ which are submersive is denoted $\mathcal{C}_{/M}$.

The category $\mathcal{C}_{/M}$ admits all finite fibered products, by Lemma 2.2.11 and the fact that identity maps and compositions of holomorphic submersions are submersive.

Definition 4.1.2. (1) An *M-group* is a complex manifold G with a submersion $f : G \to M$, a multiplication map $m : G \times_M G \to G$ and an identity section $e : M \to G$ such that $(f : G \to M, m, e)$ is a group object of the category $\mathcal{C}_{/M}$.

(2) Let $g \in \mathbb{Z}_{\geq 0}$. An *$M$-complex torus of relative dimension g* is a proper, connected, holomorphic submersion $f : G \to M$ of relative dimension g with a structure of commutative M-group.

(3) An *M-elliptic curve* is an M-complex torus of relative dimension 1.

By abuse of notation, often we will refer to an M-group $(f : G \to M, m, e)$ simply by the underlying holomorphic map $f : G \to M$, or even the underlying complex manifold G.

Definition 4.1.3. Let $h : M' \to M$ be a holomorphic map of complex manifolds.

If $f' : X' \to M'$ and $f : X \to M$ are complex manifolds over M' respectively M, then we say a holomorphic map $\tilde{h} : X' \to X$ *covers* h if $f \circ \tilde{h} = h \circ f'$.

If X' is an M'-group and X is an M-group, then a *homomorphism of M-groups* from X' to X *covering* h is a holomorphic map $\tilde{h} : X' \to X$ covering h such that for every $m' \in M'$ the map $X'_{m'} \to X_{h(m')}$ is a homomorphism of groups.

This defines a category of relative Lie groups over $\mathbb{C}$**Man**. We denote $\mathbb{C}$**Tor** resp. **Ell** its full subcategory spanned by relative complex tori resp. elliptic curves.

Example 4.1.4. If $M = \{*\}$ is a point, then an M-complex torus resp. M-elliptic curve is a classical complex torus resp. complex elliptic curve.

Example 4.1.5. Let $k \in \mathbb{Z}_{\geq 0}$ and for each $i \in \{1, 2, \ldots, k\}$ let $f_i : G_i \to M$ be an M-group. Then we can form a *product M-group* $f : G = G_1 \times_M G_2 \times_M \cdots \times_M G_k \to M$ as the product of the group objects $f_i : G_i \to M$ in $\mathcal{C}_{/M}$.

If each f_i is an M-complex torus of relative dimension, then f is an M-complex torus of relative dimension $g_1 + g_2 + \ldots + g_k$.

Now let us provide context to Example 4.1.5(2) by giving an alternative description of M-elliptic curves. We first discuss the classical equivalence between elliptic curves and pointed genus-1 Riemann surfaces.

Theorem-Definition 4.1.6. *Let C be a connected compact Riemann surface. Then there exists an integer $g = \mathrm{g}(C) \in \mathbb{Z}_{\geq 0}$, called the genus of C such that the following are true:*

(1) *As a topological space C is homeomorphic to a g-holed torus. Here a 0-holed torus is understood to be the 2-sphere S^2.*
(2) *The first singular homology group of C with $\mathbb{Z}$-coefficients is free of rank $2g$, i.e. $H_1(C; \mathbb{Z}) \cong \mathbb{Z}^{2g}$.*
(3) *One has $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = g$, where $\mathcal{O}_C$ is the structure sheaf of $\mathcal{O}_C$.*
(4) *One has $\dim_{\mathbb{C}} H^0(C, \Omega^1_C) = g$, where Ω^1_C is the sheaf of holomorphic 1-forms on C.*

(5) *One has* $\deg K_C = 2g - 2$*, where* K_C *is the canonical divisor class on* C.

Proof. See any introductory textbook on Riemann surfaces, e.g. [Mir95] □

To sum up, the genus $\mathrm{g}(C)$ of a compact connected Riemann surface C satisfies

$$\mathrm{g}(C) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = \dim_{\mathbb{C}} H^0\left(C, \Omega^1_C\right) = \frac{1}{2}\left(\deg\left(K_C\right) + 1\right).$$

Let X be a compact connected Riemann surface. From the classical theory of elliptic curves we know that X admits a structure of complex Lie group if and only if $\mathrm{g}(X) = 1$, and when $\mathrm{g}(X) = 1$ for each point $e \in X$ there exists a unique group law on X for which e is the identity element. Furthermore, a holomorphic map $h : X \to X'$ of elliptic curves is a homomomorphsim if and only if $h(e) = e'$, where e and e' denote the identity elements of X resp. X'.

Theorem 4.1.7. *The assignment* $(X, m, e) \mapsto (X, e)$ *defines an equivalence of categories from the category of elliptic curves to the category of compact connected Riemann surfaces of genus* 1 *endowed with a point.*

Proof. See [Mil06][Chapter II, §1]. □

Example 4.1.8. Let $a, b \in \mathbb{C}$ such that $4a^3 + 27b^2 \neq 0$. Then the *Weierstrass equation*

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3$$

defines a complex submanifold of $\mathbb{CP}^2$ with homogeneous coordinates $(X : Y : Z)$, which is a compact connected Riemann surface E of genus 1. The choice of the point $O = (0 : 1 : 0)$ yields a complex Lie group law on E with identity element O.

We have the following relative version of Theorem 4.1.7. We define an *M-curve of genus* 1 to be a proper holomorphic submersion $f : X \to M$ of complex manifolds such that each fibre X_m is a compact connected Riemann surface of genus 1.

Theorem 4.1.9. *The assignment* $(f : X \to M, m, e) \mapsto (X, e)$ *defines an equivalence of categories from the category of* M*-elliptic curves to the category of* M*-curves of genus* 1 *endowed with a section.*

Proof. This is a relative version of Abel's theorem, see [Conc][4.19]. □

Example 4.1.10. Let $a, b : M \to \mathbb{C}$ be holomorphic functions such that in each point $m \in M$ one has $4a(m)^3 + 27b(m)^2 \neq 0$. Then the *Weierstrass equation*

$$E : Y^2Z = X^3 + a(m)XZ^2 + b(m)Z^3$$

defines a complex submanifold of the M-complex manifold $\mathbb{CP}^2 \times M \to M$, with homogeneous coordinates $(X : Y : Z)$ on the fibres, which is an M-curve $E \to M$ of genus 1. The choice of the section $O : m \mapsto \{((0 : 1 : 0), m)\} \in E_m$ makes $E \to M$ into an M-elliptic curve with identity section O.

4.2 Quotients modulo relative lattices

Recall from the classical theory of complex tori that each compact connected complex Lie group X of dimension, say g, is isomorphic to $\mathbb{C}^g/\Lambda$, where Λ is a *co-compact lattice* in $\mathbb{C}^g$, i.e. Λ is a free $\mathbb{Z}$-submodule of $\mathbb{C}^g$ such that the natural map $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{C}^g$ is an isomorphism. On the level of morphisms, homomorphisms $h : \mathbb{C}^g/\Lambda \to \mathbb{C}^{g'}/\Lambda'$ of complex tori correspond bijectively to $\mathbb{C}$-linear maps $\psi : \mathbb{C}^g \to \mathbb{C}^{g'}$ with $\psi(\Lambda) \subset \Lambda'$.

Theorem 4.2.1. *There exists an equivalence of categories between the category of complex tori and the category of inclusions of a free $\mathbb{Z}$-module as a co-compact lattice in a finite-dimensional $\mathbb{C}$-vector space.*

The aim of this section is to state a relative version of this result, which is proved in [Ric21]. First, we need a relative notion of free $\mathbb{Z}$-module and of finite-dimensional $\mathbb{C}$-vector space over a complex manifold.

Definition 4.2.2. Let M be a complex manifold, and let $r, g \in \mathbb{Z}_{\geq 0}$.

(1) A *local system of rank-r free $\mathbb{Z}$-modules over M* is an M-complex manifold L with a structure of $\mathbb{Z}$-module on each fibre L_m such that each point $m \in M$ has an open neighborhood U with $L \times_M U \cong \mathbb{Z}^r \times U$.

(2) A *holomorphic rank-g vector bundle over M* is an M-complex manifold V with a structure of $\mathbb{C}$-vector space on each fibre V_m such that each point $m \in M$ has an open neighborhood U with $V \times_M U \cong \mathbb{C}^g \times U$.

Note that L and V as in the above definition are M-groups. Next, we need a relative notion of co-compact lattice.

Definition 4.2.3. Let M be a complex manifold.

(0) A *co-compact lattice inclusion over M* is a homomomorphism of M-groups $j : L \to V$ with L and V as in Definition 4.2.2 such that $r = 2g$ and the scalar-extension $j_{\mathbb{R}} : L \otimes_{\mathbb{Z}} \mathbb{R} \to V$ is an isomorphism of real C^∞-vector bundles.

(1) Let $h : M' \to M$ be a morphism in $\mathbb{C}$**Man**, and $j : L \to V$ and $k : \Lambda \to W$ be co-compact lattice inclusions over M resp. M'. A *homomorphism of relative co-compact lattice inclusions* from $(k : \Lambda \to W)$ to $(j : L \to V)$ *covering* $h\colon M' \to M$ is a pair (χ, ψ) of homomorphism of relative Lie groups covering h fitting in a commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\chi} & L \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle j} \\ V & \xrightarrow{\psi} & W. \end{array}$$

(2) We denote **Lat** the category of relative co-compact lattice inclusions and morphisms of such. We define the functor **Lat** $\to \mathbb{C}$**Man** by sending an object j as in part (0) to M, and a morphism (χ, ψ) as in part (1) to $h : M' \to M$. We denote the fiber category over the complex manifold M by $\mathbf{Lat}_{/M}$. For $g \in \mathbb{Z}_{\geq 0}$ we write $\mathbf{Lat}^g$ for the full subcategory of **Lat** spanned by those objects $j : L \to V$ for which V has rank g.

The proposition below defines a functor **Lat** $\to$ $\mathbb{C}$**Tor** by taking quotients modulo relative lattices.

Proposition 4.2.4. (0) *Let $j : L \to V$ over M be as in Definition 4.2.2(0). Let $q : V \to V/L$ be the set-theoretic quotient map for the equivalence relation $\sim_L$ on the set V such that $v \sim_L v'$ if and only if v and v' lie over the same point $m \in M$ and $v - v' \in L_m$ inside V_m. Then there exists a unique structure of M-complex torus on V/L such that there is an exact sequence of M-groups*

$$0 \to L \to V \xrightarrow{q} V/L \to 0. \tag{4.1}$$

(1) *Let (χ, ψ) covering $h : M' \to M$ be as in Definition 4.2.2(1). Then there exists a unique homomorphism of relative complex tori $\overline{\chi} : W/N \to V/L$ giving a homomorphism of short exact sequences of relative Lie groups covering $h : M' \to M$.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{k} & W & \longrightarrow & W/N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \overline{\psi}} & & \\ 0 & \longrightarrow & L & \xrightarrow{j} & V & \longrightarrow & V/L & \longrightarrow & 0. \end{array}$$

Proof. We mention the various steps in the construction of a structure of M-complex torus on V/L, referring to [Cond] for proofs of the claims made. Endow V/L with the quotient topology for $\sim_L$. Then $q : V \to V/L$ is universal among $\sim_L$-invariant continuous maps out of V, whence there is a natural continuous map $V/L \to M$. In [ibid., Prop 2.2] it is shown that $q : V \to V/L$ is a covering map, and that $V/L \to M$ is proper.

According to [ibid., Thm. 3.2] there is a unique complex manifold structure on V/L relative to which $q : V \to V/L$ is a local analytic isomorphism, and this makes $V/L \to M$ into a submersion. Finally [ibid., Cor 3.3] shows there is a unique structure of M-group V/L that makes $q\colon V \to V/L$ into an M-group homomorphism. Since each fibre of $V/L \to M$ is a connected complex manifold of dimension g, we conclude that $V/L \to M$ is an M-complex torus.

The reader may find part (1) in [ibid., Thm. 3.2], or verify it by herself. □

Remark 4.2.5. Let $j : L \to V$ be a co-compact lattice inclusion over M. The construction of V/L is compatible with base change by a morphism $h : M' \to M$ in $\mathbb{C}$**Man**, in the following sense (for proofs, see again [ibidem, Thm 3.2 and Cor. 3.3]).

Denote $j' : L' \to V'$ the base change of $j : L \to V$ by h. Then j' is a co-compact lattice inclusion over M' and the natural map $V'/L' \to (V/L) \times_M M'$ is an M'-group isomorphism.

Theorem 4.2.6. *The constructions in Proposition 4.2.4 define an equivalence of categories over* $\mathbb{C}$**Man**,

$$\mathbf{Lat} \to \mathbb{C}\mathbf{Tor},$$
$$(j : L \to V) \to V/L,$$
$$(\chi, \psi) \mapsto \overline{\psi}.$$

Proof. It follows from Remark 4.2.5 that the functor $\mathbf{Lat} \to \mathbb{C}\mathbf{Tor}$ is a morphism of fibered categories over $\mathbb{C}\mathbf{Man}$ in the sense of [Ols16][Def. 3.1.3(ii)]. By [ibidem, Prop. 3.1.10] a morphism of fibered categories is an equivalence if and only if it induces an equivalence on each fibre category. Thus it suffices to show that $\mathbf{Lat}_{/M} \to \mathbb{C}\mathbf{Tor}_{/M}$ is an equivalence for every object M of $\mathbb{C}\mathbf{Man}$, which is [Ric21][Theorem 4.3.2]. □

4.3 Relative homology and tangent bundles

Theorem 4.3.1 (Ehresmann)**.** *Let $f : E \to M$ be a proper holomorphic submersion of complex manifolds. For every contractible open subset $U \subset M$, there exists a C^∞-manifold F and a C^∞-diffeomorphism $E \times_M U \cong F \times U$ over U.*

Proof. See [Voi02][Theorem 9.3]. □

Now let $\pi : E \to M$ be a proper holomorphic submersion of complex manifolds, and let $p \in \mathbb{Z}_{\geq 0}$. For every $m \in M$ the singular homology group of the fibre E_m with $\mathbb{Z}$-coefficients $H_p(E_m; \mathbb{Z})$. Our aim is to glue the fibral homology together into a global object.

Proposition 4.3.2. *Let $p \in \mathbb{Z}_{\geq 0}$. Let $f : X \to M$ be a proper holomorphic submersion. Then there exists a local system $\underline{H}_p(E/M)$ of $\mathbb{Z}$-modules whose stalk at $m \in M$ is given by*

$$\underline{H}_p(E/M)_m = H_p(E_m; \mathbb{Z}) \tag{4.2}$$

Proof. Every complex manifold is locally contractible, so the contractible open subsets of M form a basis for the topology on M. We will construct a sheaf on this basis for M, in the sense of [dJm][Def. 009J]

Consider a contractible open subset $U \subset M$. By Ehresmann's fibration Theorem 4.3.1 there exists a C^∞ manifold F and a C^∞ diffeomorphism $f^{-1}(U) \cong F \times U$ over U. Using this diffeomorphism and the contractibility of U, for every $m \in U$ we see that $\iota_{U,m} : E_m \to E \times_M U$ admits a retraction, hence is a homotopy equivalence. Thus $(\iota_{U,m})_* : H_p(E_m; \mathbb{Z}) \to H_p(f^{-1}(U); \mathbb{Z})$ is an isomorphism. Similarly, if $V \subset U$ is a second contractible open subset, then the inclusion $\iota_{U,V} : E \times_M V \to E \times_M U$ is a homotopy equivalence, and $(\iota_{U,V})_* : H_p(E \times_M V; \mathbb{Z}) \to H_p(E \times_M U; \mathbb{Z})$ is an isomorphism

The sections of the sheaf $\underline{H}_p(E/M)$ over a contractible open subset U of M are defined to be

$$\Gamma(U, \underline{H}_p(E/M)) = \mathrm{Image}(H_p(E \times_M U; \mathbb{Z}) \to \prod_{m \in U} H_p(E_m; \mathbb{Z}))$$
$$s \mapsto (\iota_{U,m})_*^{-1}(s).$$

Now consider an inclusion $V \subset U$ is of contractible open subsets. We contend that the projection map $\prod_{m \in U} H_p(E_m; \mathbb{Z}) \to \prod_{m \in V} H_p(E_m; \mathbb{Z})$ send $\Gamma(U, \underline{H}_p(E/M))$ into $\Gamma(V, \underline{H}_p(E/M))$. This follows from $\iota_{U,V} : H_p(f^{-1}(V); \mathbb{Z}) \to H_p(f^{-1}(U); \mathbb{Z})$ being an isomorphism and the fact that $\iota_{U,V} \circ \iota_{V,m} = \iota_{U,m}$ for each $m \in M$. It is straightforward to verify this defines a sheaf on the basis of contractible open subsets of M. Using [ibid.][Lemma 009N] it defines a locally constant sheaf

$\underline{H}_p(E/M)$ of free $\mathbb{Z}$-modules on M, whose stalk at $m \in M$ is by construction $H_p(E/M)_m = H_p(E_m; \mathbb{Z})$. □

Lemma 4.3.3. *Consider a commutative square in* $\mathbb{C}$**Man***, in which* f *and* f' *are proper submersions:*

$$\begin{array}{ccc} X' & \xrightarrow{\tilde{h}} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \xrightarrow[h]{} & M. \end{array}$$

Then for each $p \in \mathbb{Z}_{\geq 0}$ *there exists a natural homomorphism of relative Lie groups* $\underline{H}_p(E'/M') \to \underline{H}_p(E/M)$ *covering* $h : M' \to M$ *given in a point* $m' \in M$ *by the map* $H_p(E'_{m'}; \mathbb{Z}) \to H_p(E_{h(m)}; \mathbb{Z})$ *on singular homology groups induced by the restriction* $\tilde{h} : E'_{m'} \to E_{h(m')}$.

Proof. For every pair of contractible open subsets $U' \subset M'$ and $U \subset M$ such that $h(U') \subset U$, the map $\underline{H}_1(\tilde{h}/h)$ is defined as

$$\Gamma\left(U, \underline{H}_p\left(E'/M'\right)\right) = H_{1p}\left(\left(\pi'\right)^{-1}\left(U'\right); \mathbb{Z}\right) \xrightarrow{\tilde{h}_*} H_p\left(\pi^{-1}(U); \mathbb{Z}\right) = \Gamma\left(U, \underline{H}_p(E/M)\right).$$

In a similar way to Proposition 4.3.2 we obtain from this the required homomorphism of sheaves. □

Let us record a special case of this proposition, to be used frequently in Chapter 5.

Corollary 4.3.4. *A homomorphism of elliptic curves*

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{h}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{h} & M \end{array}$$

induces a homomorphism of local systems of rank-2 free $\mathbb{Z}$*-modules on* M'

$$\underline{H}_1(\tilde{h}/h) : \underline{H}_1\left(E'/M'\right) \to h^*\underline{H}_1(E/M) \tag{4.3}$$

whose stalk at $m' \in M'$ *is the induced map on singular homology*

$$\underline{H}_1(\tilde{h}/h)_{m'} : H_1\left(E'_{m'}; \mathbb{Z}\right) \to H_1\left(E_{h(m')}; \mathbb{Z}\right). \tag{4.4}$$

Remark 4.3.5. For any $m' \in M'$, the map $\underline{H}_1(\tilde{h}/h)_{m'}$ is either zero or injective with image of finite index. The *degree* of $\tilde{h}$ at m', denoted $\deg(\tilde{h})\,(m')$, is set to be 0 if the map is zero, and the index of the image if the map is nonzero. The function $\deg(\tilde{h}) : M' \to \mathbb{Z}$ thus defined is locally constant and called the degree of $\tilde{h}$. We say $\tilde{h}$ is *cartesian* if $\deg(\tilde{h}) = 1$, i.e. if $\underline{H}_1(\tilde{h}/h)$ is an isomorphism.

Now consider a holomorphic submersion $f : X \to M$, not necessarily proper. We can glue the tangent bundles $T(X_m)$ of the fibres $X_m = f^{-1}(m)$ into a holomorphic vector bundle over X.

Proposition-Definition 4.3.6. (1) *Let $f : X \to M$ be a submersive holomorphic map of complex manifolds, say of relative dimension g. Then there exists a holomorphic vector bundle $T_{X/M}$ of rank g over X whose stalk at $x \in X$ is the kernel of $(df)_x : T_xX \to T_{f(x)}M$. We call $T_{X/M}$ the* relative tangent bundle *of X/M.*

(2) *Let $f' \colon X' \to M'$ be a second holomorphic submersion of complex manifolds, and let $\tilde{h} : X' \to X$ be a holomorphic map covering $h : M' \to M$. Then there is a homomorphism of holomorphic vector bundles $T_{X'/M'} \to T_{X/M}$ covering h.*

Proof. See [Fis76][§2.7 and Theorem 2.19] □

4.4 Uniformization of M-complex tori

Theorem 4.4.1. *Let X be a classical complex torus. Then there exists a short exact sequence of complex Lie groups*

$$0 \to H_1(X;\mathbb{Z}) \xrightarrow{\iota} T_eX \xrightarrow{\exp} X \to 0, \tag{4.5}$$

where
(i) $\exp$ *is the unique holomorphic homomorphism such that* $(d\exp)_0 = \mathrm{id}$ *if we identify* $T_0(T_{X,e}) \cong T_{x,e}$
(ii) the map $\iota^{-1} : (\ker\exp) \to H_1(X;\mathbb{Z})$ *sends* $v \in T_eX$ *with* $\exp(v) = e$ *to the homology class of the loop* $[0,1] \to X$, $t \mapsto \exp(tv)$.

Proof. See [Mum70][Section 1, (2)]. □

Corollary 4.4.2. *Let $X = V/L$ be a complex torus, given as the quotient of a co-compact lattice L inside a finite-dimensional $\mathbb{C}$-vector space V. Then there is an isomorphism of short exact sequences of complex Lie groups*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & V & \xrightarrow{q} & V/\Lambda & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle (dq)_0} & & \downarrow & & \\ 0 & \longrightarrow & H_1(X;\mathbb{Z}) & \longrightarrow & T_eX & \xrightarrow{\exp} & X & \longrightarrow & 0, \end{array}$$

where the middle map is the isomorphism $(dq)_0 : V = T_0V \to T_0(V/L) = T_0X$ of tangent spaces at 0 induced by the local analytic isomorphism $q : V \to X$.

Proof. The homomorphism of complex Lie group $q \circ (dq)_0^{-1} : T_eX \to X$ induces on tangent spaces at 0 the map $(dq)_0 \circ (dq)_0^{-1} = \mathrm{id}$. Thus $q\circ$ it coincides with exponential map $\exp : T_eX \to X$ since it satisfies the characterising property.

It follows that $(dq)_0$ restricts to an isomorphism $\ker q \cong \ker\exp$, whence there is a unique isomorphism $L \to H_1(X;\mathbb{Z})$ making the diagram commute. □

We formulate a relative version of this result for complex tori over a complex manifold, as Theorem 4.4.3.

Theorem 4.4.3. *Let $(X, e) \to M$ be a complex torus over M. Then there exists a short exact sequence of M-groups*

$$0 \to \underline{H}_1(E/M) \to e^*T_{X/M} \xrightarrow{\exp} X \to 0 \tag{4.6}$$

whose fibre above $m \in M$ is

$$0 \to H_1(X_m; \mathbb{Z}) \to T_e(X_m) \overset{\exp}{\to} X_m \to 0, \tag{4.7}$$

Proof. We define (4.6) as the disjoint union of the short exact sequences (4.7) for the various m. All properties are stalkwise, except for holomorphicity of the maps in (4.6). In view of the essential surjectivity of the functor Theorem 4.2.6, it is harmless to assume that $X = V/L$ is the quotient modulo a relative co-compact lattice inclusion $j : L \to V$.

Since $q : V \to X$ is a local analytic isomorphism over M, there is an isomorphism $e^*(dq) : V = e^*T_{V/M} \cong e^*T_{X/M}$ of holomorphic vector bundles over M, which is the middle vertical map in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & V & \longrightarrow & V/L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{H}_1(X/M; \mathbb{Z}) & \longrightarrow & e^*T_{X/M} & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

It follows that $e^*(dq)$ restricts to an isomorphism $\ker q \cong \ker \exp$, whence there is a unique isomorphism $L \to \underline{H}_1(X/M; \mathbb{Z})$ of local systems of free $\mathbb{Z}$-modules making the above diagram commute. □

4.5 Weil pairing on abelian varieties

Let $X \to M$ be an M-elliptic curve, or more generally a principally polarized M-complex torus to be defined in Definition 4.5.7. For each $N \in \mathbb{Z}_{\geq 1}$ Definition 4.5.9 constructs a nondegenerate alternating bilinear map $e_N : X[N] \times_M X[N] \to \mu_N$ on the N-torsion $X[N]$ of X with values in the group μ_N of complex N-th roots of unity, called the *Weil e_N-pairing*.

Proposition-Definition 4.5.1. *Let $N \geq 1$ be an integer and $X \to M$ be a complex torus of relative dimension g. Then there exists an isomorphism*

$$\underline{H}_1(X/M) \otimes (\mathbb{Z}/N\mathbb{Z}) \to X[N] \tag{4.8}$$

and $X[N]$ is a local system of rank-$2g$ free $(\mathbb{Z}/N\mathbb{Z})$-modules. If N_1 is a positive divisor of N, and $\mathrm{can}: \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N_1\mathbb{Z}$ is the canonical map, then there are commutative diagrams

$$\begin{array}{ccc} \underline{H}_1(X/M) \otimes (\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & X[N] \\ \downarrow{\scriptstyle 1\otimes\mathrm{can}} & & \downarrow{\scriptstyle [N/N_1]_X} \\ \underline{H}_1(X/M) \otimes (\mathbb{Z}/N_1\mathbb{Z}) & \longrightarrow & X[N_1], \end{array} \qquad \begin{array}{ccc} \underline{H}_1(X/M) \otimes (\mathbb{Z}/h\mathbb{Z}) & \longrightarrow & X[N_1] \\ \downarrow{\scriptstyle 1\otimes[N/N_1]_X} & & \cap \\ \underline{H}_1(X/M) \otimes (\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & X[N]. \end{array}$$

Proof. Let $V/L \cong X$ be a uniformization of X, so that $L = \underline{H}_1(X/M)$. We find that

$$X[N] = (V/L)[N] = \frac{1}{N}L/L \xrightarrow[\sim]{[N]} L/NL = \underline{H}_1(X/M) \otimes (\mathbb{Z}/N\mathbb{Z}.)$$

Now the verification of the commutativity of the above diagrams comes down to that of the diagrams below, which is plain:

$$\begin{array}{ccc} \frac{1}{N_1}L/L & \xrightarrow{[N_1]} & L/N_1L \\ \downarrow{\scriptstyle [1]} & & {\scriptstyle [N_1/N]}\downarrow \\ \frac{1}{N}L/L & \xrightarrow{[N]} & L/NL, \end{array} \qquad \begin{array}{ccc} \frac{1}{N}L/L & \xrightarrow{[N]} & L/NL \\ {\scriptstyle [N/N_1]}\downarrow & & \downarrow{\scriptstyle [1]} \\ \frac{1}{N_1}L/L & \xrightarrow{[N_1]} & L/N_1. \end{array}$$

□

Definition 4.5.2. Let $V \to M$ be a holomorphic vector bundle over a complex manifold M. A *Hermitian metric on* V is the assignment to each point $m \in M$ of a complex inner product $H_m : V_m \times V_m \to \mathbb{C}$ such that for any two C^∞ local sections s_1 and s_2 of V the function $m \mapsto H_m(s_1(m), s_2(m))$ is C^∞ smooth.

Lemma 4.5.3. *Any holomorphic vector bundle admits a Hermitian metric.*

Proof. This is proved using partitions of unity subordinate to a trivializing open cover of the base manifold, see [Huy05][Prop. 4.1.4] □

Definition 4.5.4. Let L be a $\mathbb{Z}$-module. A *symplectic form* ω on L is a unimodular nondegenerate alternating bilinear map $\omega : L \times L \to \mathbb{Z}$, meaning that for all $v \in L$ the map $\omega(v, \cdot) : L \to \mathbb{Z}$ defined by $\omega(mv, \cdot)(w) = \omega(v, w)$ is $\mathbb{Z}$-linear and satisfies $\omega(v, v) = 0$, and that the resulting map $L \mapsto \mathrm{Hom}(L, \mathbb{Z}), v \mapsto \omega(v, \cdot)$ is an isomorphism of $\mathbb{Z}$-modules.

Example 4.5.5. Let $g \geq 0$ be an integer. We define the standard symplectic form ω_g on $\mathbb{Z}^{2g}$ to be the non-degenerate alternating bilinear form $\omega_g : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}$ that is given on the standard basis of $\mathbb{Z}^{2g}$ by the matrix

$$\omega_g = \begin{pmatrix} 0 & -\mathbb{1}_g \\ \mathbb{1}_g & 0 \end{pmatrix} \tag{4.9}$$

where $\mathbb{1}_g$ is the $g \times g$-identity matrix.

Lemma 4.5.6. *Let ω be a symplectic form on a $\mathbb{Z}$-module L. Then there exists an integer $g \geq 0$ and an isomorphism $(\mathbb{Z}^{2g}, \omega_g) \cong (L, \omega)$ of $\mathbb{Z}$-modules endowed with a symplectic form.*

Proof. See [Bou59][Chap. 9, §5, Theorème 1]. □

Definition 4.5.7. Let $f : X \to M$ be an M-complex torus, with canonical uniformization $X \cong V/L$, where $L = \underline{H}_1(E/M)$ and $V = e^*T_{X/M}$. A *principal polarization* on X is a Hermitian metric $H : V \times_M V \to \mathbb{C}$ on V whose imaginary part $\Omega = \Im H : V \times_M V \to \mathbb{R}$ restricts to a symplectic form $L \times_M L \to \mathbb{Z}$ on L.

Proposition 4.5.8. *Let $f\colon E \to M$ be an M-elliptic curve. Then there exists a unique prinicpal polarization on E.*

Proof. Let $E = V/L$ be the canonical uniformization of E as in Definition 4.5.7. By virtue of the asserted uniqueness, we can work locally on M. Thus we may assume there exists a global frame (τ_1, τ_2) for the local system of rank-2 free $\mathbb{Z}$-modules L. Choose a Hermitian metric H on V using Lemma 4.5.3. Since V has rank 1, the Hermitian metric H is unique up to a positive real-valued C^∞ function on M. Then H is a principal polarization if and only if $|H_m(\tau_1(m), \tau_2(m))| = 1$ for every $m \in M$. Since $H(\tau_1, \tau_2) : m \mapsto H_m(\tau_1(m), \tau_2(m))$ is a nowhere vanishing C^∞ function on M, $H/|H(\tau_1, \tau_2)|$ is the desired principal polarization on E. $\square$

Definition 4.5.9. (Weil pairing) Let $f : X \to M$ be a principally polarized M-complex torus. Write Ω for the symplectic form on $\underline{H}_1(X/M)$ corresponding to the principal polarization. Let $N \in \mathbb{Z}_{\geq 1}$ and identify $\mathbb{Z}/N\mathbb{Z} \cong \mu_N$ via $a + N\mathbb{Z} \mapsto \zeta_N^a$. Then the *Weil e_N-pairing* is the unique nondegenerate alternating bilinear map

$$e_N : X[N] \times_M X[N] \to X[N] \tag{4.10}$$

such that the following diagram commutes:

$$\begin{array}{ccc}
(\underline{H}_1(X/M) \times_M \underline{H}_1(X/M)) \otimes (\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & X[N] \times_M X[N] \\
\downarrow{\scriptstyle \Omega \otimes \mathrm{id}_{\mathbb{Z}/N\mathbb{Z}}} & & \downarrow{\scriptstyle e_N} \\
\underline{\mathbb{Z}/N\mathbb{Z}}_M & \xrightarrow{a+N\mathbb{Z} \mapsto \zeta_N^a} & \underline{\mu_N}_M.
\end{array}$$

4.6 Quotients of elliptic curves

We now discuss quotients for the action of a group on a relative elliptic curve, which we will define in Definition 4.6.1 below. The results of this section will be used in Chapter 5 to construct from a 'universal' elliptic curve over the upper half plane $\mathfrak{H}$ ones over open modular curves $Y(\Gamma) = \Gamma \backslash \mathfrak{H}$.

Definition 4.6.1. Let G be a discrete group, and let $E \to M$ be an elliptic curve. Then an *action* of G on $E \to M$ is a homomorphism $G \to \mathrm{Aut}_{\mathbf{Ell}}(E/M)$, i.e. a pair of homomorphisms $G \to \mathrm{Aut}(E)$, $g \mapsto [g]_E$ and $G \to \mathrm{Aut}(M)$, $g \mapsto [g]$ making for every $g \in G$ an isomorphism of elliptic curves

$$\begin{array}{ccc}
E & \xrightarrow{[g]_E} & E \\
\downarrow & & \downarrow \\
M & \xrightarrow{[g]} & M.
\end{array}$$

Lemma 4.6.2. *Let G be a discrete group acting on an elliptic curve $E \to M$. Assume the action of G on M is proper and free. Then there exists a unique structure $G\backslash E \to G\backslash M$ of elliptic curve on the map of topological spaces $G\backslash E \to G\backslash M$ such that*

$$\begin{array}{ccc}
E & \xrightarrow{\pi_E} & G\backslash E \\
\downarrow & & \downarrow \\
M & \xrightarrow{\pi_M} & G\backslash M
\end{array}$$

is a cartesian homomorphism of elliptic curves and a categorical quotient in the category of relative elliptic curves for the action of G on E/M.

Proof. Since G acts freely and properly on M, it also acts freely and properly on E by Lemma 2.3.3. It follows from Theorem 2.3.5 that orbit spaces $G\backslash E$ resp. $G\backslash M$ admits a unique structure of complex manifold such that π_E resp. π_M is a covering map and a local analytic isomorphism. Since the map $E \to M$ is G-equivariant and the map $M \to G\backslash M$ is G-invariant, their composite $E \to G\backslash M$ is G-invariant. Since $E \to G\backslash E$ is initial among G-invariant maps out of E, there exists a unique holomorphic map $G\backslash E \to G\backslash M$ which makes the diagram commute.

We transport the structure of elliptic curve on $E \to M$ to the induced map on G-orbit spaces $G\backslash E \to G\backslash M$ via (π_E, π_M). Since π_M is a covering map, we can cover $G\backslash M$ by open subsets V for which there exists an open subset $U \subset M$ such that $\pi_M^{-1}(V) = \bigsqcup_{g\in G} G \cdot U$, and then for each such U and $g \in G$ we have that $([g]_E, [g]) : E_U/U \to E_{g\cdot U}/(g \cdot U)$ is an isomorphism. Therefore there is a unique elliptic curve structure on $(G\backslash E)_V \to V$ having the property that the pair $(\pi_E, \pi_M) : E_{g\cdot U}/(g \cdot U) \to (G\backslash E)_V/V$ is an isomorphism for every $g \in U$, equivalently, such that $(\pi_E, \pi_M) : E_{\pi_M^{-1}(V)}/\pi_M^{-1}(V) \to (G\backslash E)_V/V$ is an isomorphism, and on varying V over a covering of $G\backslash M$ these glue together to a unique elliptic curve structure on $G\backslash E \to G\backslash M$ making the map $(\pi_E, \pi_M) : E/M \to (G\backslash E)/(G\backslash M)$ an isomorphism.

It remains to be shown that (π_E, π_M) is a quotient map for the G-action in the category of elliptic curves. Let $(\tilde{\rho}, \rho) : E/M \to X/Y$ be a G-invariant homomorphism of ellptic curves. Since π_E resp. π_M is a quotient map for the G-action on E resp. M in the category of complex manifolds, there exists a unique holomorphic map $\tilde{f}\colon G\backslash E \to X$ resp. $f\colon G\backslash M \to Y$ such that $\tilde{\rho} = \tilde{f} \circ \pi_E$ resp. $\rho = f \circ \pi_M$. To show that $(\tilde{f}, f)$ is a homomorphism of elliptic curves, we may and do work locally over an open subset $V \subset G\backslash M$ for which there exists an open subset $U \subset M$ such that $(\pi_E, \pi_M) : E_U/U \to (G\backslash E)_V/V$ is an isomorphism. Then the fact that $(\tilde{f}, f) \circ (\pi_E, \pi_M) = (\tilde{\rho}, \rho)$ is a homomorphism implies that $(\tilde{f}, f)$ is a homomorphism over V, as remained to be shown. □

Remark 4.6.3. Mutatis mutandis the proof of Lemma 4.6.2 goes through for any kind of 'object over a base manifold' provided the definition is local on the base, such as complex tori, holomorphic vector bundles, local systems of $\mathbb{Z}$-modules, relative Lie groups, and also for the following kinds of 'enriched' elliptic curves encountered in Chapter 5: elliptic curves with an H_1-structure, with a Γ-structure, with an N-structure, with a point of exact order N and with a cyclic subgroup of order N.

Chapter 5

Universal elliptic curves

5.1 Moduli spaces

We denote the opposite of a category $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$.

Definition 5.1.1 (presheaf, sheaf)**.** A *presheaf* on the category of complex manifolds $\mathbb{C}\mathbf{Man}$ is a functor

$$F : \mathbb{C}\mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}\,.$$

We say the presheaf F is a *sheaf* if for every complex manifold M and open covering $\{U_i\}_{i\in I}$ of M the restriction maps define a bijection

$$\begin{aligned}\mathcal{F}(M) \to \left\{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i) : s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \text{ for all } (i,j) \in I\times I\right\},\\ s \mapsto \left(s|_{U_i}\right)_{i\in I}.\end{aligned}$$

Thus the functor $F : \mathbb{C}\mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf precisely when for every complex manifold M the restriction $F|_M$ of F to the category of open subsets of M is a sheaf on the topological space M in the usual sense.

Example 5.1.2. Let $\mathcal{M}$ be a complex manifold. The functor $\mathrm{h}_{\mathcal{M}} : \mathbb{C}\mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ sending an object M to the set $\mathrm{h}_{\mathcal{M}}(M) = \{\phi\colon M \to \mathcal{M} \mid \phi \text{ is holomorphic }\}$ and a morphism $h\colon M' \to M$ to the map $\mathrm{h}_{\mathcal{M}}(h)\colon \mathrm{h}_{\mathcal{M}}(M) \to \mathrm{h}_{\mathcal{M}}(M')$, $\phi \mapsto \phi\circ h$ is a sheaf on $\mathbb{C}\mathbf{Man}$. The sheaf axiom is verified because holomorphicity is a local condition.

Definition 5.1.3. (fine moduli space) Let F be a presheaf on $\mathbb{C}\mathbf{Man}$. A *fine moduli space* for F is a complex manifold $\mathcal{M}$ together with a natural isomorphism $\theta : F \to h_{\mathcal{M}}$ of functors $\mathbb{C}\mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$.

Remark 5.1.4. Suppose the functor F classifies some kind of 'structures' living over a base manifold, with restriction maps being given by pullback. Then the data of a fine moduli space $\theta : F \to h_{\mathcal{M}}$ is equivalent to a complex manifold $\mathcal{M}$ together with a 'structure' α^{univ} on $\mathcal{M}$ which is *universal* in the sense that for any other 'structure' α over some complex manifold M there is a unique holomorphic map $\phi : M \to \mathcal{M}$ pulling back α^{univ} on $\mathcal{M}$ to α on M.

Example 5.1.5 (Grassmannian)**.** Let $0 \leq k \leq n$ be integers. Define the sheaf $G_{k,n}$ on $\mathbb{C}$**Man** by letting $G_{k,n}(M)$ be the set of rank-k holomorphic subbundles $E \subset \mathbb{C}^n \times M$ of the trivial rank-n vector bundle over M. If $h : M' \to M$ is a holomorphic map and $E \in F(M)$, then $F(h)(E) := h^*E$ is defined to be the preimage $h^*E \subset \mathbb{C}^n \times M'$ of $E \subset \mathbb{C}^n \times M$ under $\mathrm{id}_{\mathbb{C}^n} \times h : \mathbb{C}^n \times M' \to \mathbb{C}^n \times M$.

In classical algebraic geometry one constructs (see e.g. [GH78][Ch. 1, Section 5]) a projective manifold $G(k,n)$, called a *Grassmannian*, and a natural isomorphism $\theta : G_{k,n} \to \mathrm{h}_{G(k,n)}$, i.e., a fine moduli space for $G_{k,n}$. In particular, $\theta(*)$ is a bijection from the set $G_{k,n}(*)$ of k-dimensional linear subspaces of $\mathbb{C}^n$ to the set $h_{G(k,n)}(*)$ of points of $G(k,n)$. Furthermore $\theta(G(k,n))^{-1}\left(\mathrm{id}_{G(k,n)}\right) \in G_{k,n}(G(k,n))$ corresponds to a rank-k holomorphic subbundle $\mathcal{E} \subset \mathbb{C}^n \times G(k,n)$ that is universal. That is, for every holomorphic manifold M and rank-k holomorphic subbundle $E \subset \mathbb{C}^n \times M$ there exists a unique holomorphic map $f = \theta(E) : M \to G(k,n)$ such that $E = f^*\mathcal{E}$.

Definition 5.1.6 (coarse moduli space)**.** Let $F : \mathbb{C}\mathbf{Man} \to \mathbf{Set}$ be a functor. A *coarse moduli space* for F is a complex manifold $\mathcal{M}$ with a natural transformation $\theta : F \to h_{\mathcal{M}}$ that is bijective on $*$-valued points, i.e. the map $\theta_* : F(*) \to h_{\mathcal{M}}(*)$ to the underlying set of $\mathcal{M}$ is bijective.

Note that every fine moduli space for a sheaf F on $\mathbb{C}$**Man** is also a coarse moduli space.

Remark 5.1.7. In the literature one often imposes the additional condition for the functor $\theta : F \to h_{\mathcal{M}}$ to be a coarse moduli space, viz. that for any complex manifold $\mathcal{N}$ and natural transformation $\eta : F \to h_{\mathcal{N}}$ there exists a unique holomorphic map $\phi : \mathcal{M} \to \mathcal{N}$ such that $\eta = h_\phi \circ \theta$. Since this condition will not play a role in the sequel, we have chosen to omit it.

5.2 H_1-structures

In this section, we introduce H_1-trivialized elliptic curves in Definition 5.2.1, and show in Theorem 5.2.3 that the functor $[H_1\text{–str}]$ classifying H_1-trivialized elliptic curves admits a fine moduli space $\mathcal{E} \to \mathfrak{H}$.
Let $\omega_1 : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ be the standard symplectic form on $\mathbb{Z}^2$ represented on the standard basis by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For any complex manifold M this defines a *constant* symplectic form on the local system of rank-2 free $\mathbb{Z}$-modules $\mathbb{Z}^2 \times M$ over M, which we also denote ω_1.

Definition 5.2.1. (1) Let M be a complex manifold and E be an M-elliptic curve. An H_1*-structure* on E/M is an isomorphism

$$\psi : \left(\mathbb{Z}^2 \times M, \omega_1\right) \to \left(\underline{H}_1(E/M), \bullet\right) \tag{5.1}$$

of local systems of rank-2 free $\mathbb{Z}$-modules which carries the constant standard symplectic form ω_1 on $\mathbb{Z}^2 \times M$ to the intersection pairing $\bullet$ on $\underline{H}_1(E/M)$.

(2) An H_1*-trivialized elliptic curve* is a relative elliptic curve together with an H_1-structure.

(3) A *homomorphism* $(E' \to M', \psi') \to (E \to M, \psi)$ *of* H_1*-trivialized elliptic curves* is a homomorphism $(\tilde{h}\colon E' \to E, h\colon M' \to M)$ of elliptic curves such that $\psi \circ (\mathbb{1}_{\mathbb{Z}^2} \times h) = \underline{H}_1(\tilde{h}/h) \circ \psi'$ (latter condition implies that $(\tilde{h}, h)$ is cartesian). If h is the identical map on $M' = M$, we say it is an *isomorphism over* M and write $(E' \to M, \psi') \cong_M (E \to M, \psi)$.

Let $(E \to M, \psi)$ be an H_1-trivialized elliptic curve, and let $h : M' \to M$ be a holomorphic map. Then there exists up to isomorphism over M' a unique H_1-trivialized elliptic curve $(E' \to M', \psi')$ over M' such that there exists a homomorphism $\left(\tilde{h} : E' \to E, h : M' \to M\right)$ of H_1-trivialized elliptic curves. We refer to $h^*(E \to M, \psi) := (E' \to M', \psi')$ as the *pullback of* $(E \to M, \psi)$ *by* h. It is plain that $\mathrm{id}_M^*(E \to M, \psi) \cong (E \to M, \psi)$ and that for a further holomorphic map $g : M'' \to M'$ we have $g^*\left(h^*(E \to M, \psi)\right) \cong (h \circ g)^*(E \to M, \psi)$. This allows us to define the following functor $\mathbb{C}\mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$.

Definition 5.2.2. Let $[H_1\text{–str}] : \mathbb{C}\mathbf{Man} \to \mathbf{Set}$ be the functor, assigning to a complex manifold M the set

$$[H_1\text{–str}]\,(M) = \{H_1\text{-trivialized elliptic curves over } M\} \,/ \cong_M \tag{5.2}$$

and to a holomorphic map $M' \to M$ the map

$$\begin{aligned} [H_1\text{–str}]\,(h) : [H_1\text{–str}]\,(M) &\to [H_1\text{–str}]\,(M')\,, \\ (E \to M, \psi) &\mapsto h^*(E \to M, \psi). \end{aligned}$$

We will now construct a fine moduli space for the functor $[H_1\text{–str}]$. Let $V = \mathbb{C} \times \mathfrak{H}$ be the trivial holomorphic line bundle over the upper half-plane $\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}$. It contains the trivial local system of rank-2 free $\mathbb{Z}$-modules

$$L := \left\{(m\tau + n, \tau) : \tau \in \mathfrak{H}, (m, n) \in \mathbb{Z}^2\right\} \subset \mathbb{C} \times \mathfrak{H} \tag{5.3}$$

as a co-compact relative lattice. We let $\mathcal{E} := V/L \to \mathfrak{H}$ be the corresponding quotient elliptic curve, and endow it with the H_1-structure

$$\begin{aligned} \Psi = (\tau, 1) : \mathbb{Z}^2 \times \mathfrak{H} &\xrightarrow{\sim} L = \underline{H}_1(\mathcal{E}/\mathfrak{H}), \\ ((m, n), \tau) &\mapsto (m\tau + n, \tau). \end{aligned} \tag{5.4}$$

Theorem 5.2.3. *The elliptic curve with* H_1*-structure* $(\mathcal{E} \to \mathfrak{H}, \Psi)$ *is a fine moduli space for the functor* $[H_1\text{–}str]$*. In other words, given an* H_1*-trivialized elliptic curve* $(E \to M, \psi)$ *there exists a unique holomorphic map* $h : M \to \mathfrak{H}$ *fitting into a Cartesian square*

$$\begin{array}{ccc} (E, \psi) & \xrightarrow{\tilde{h}} & (\mathcal{E}, \Psi) \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & \mathfrak{H}. \end{array}$$

Proof. This is [Conc][Theorem 6.7]. ∎

5.3 The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{E} \to \mathfrak{H}$

In this section we define a natural action of $\mathrm{SL}_2(\mathbb{Z})$ on the functor $[H_1\text{–str}]$. Since $\mathcal{E} \to \mathfrak{H}$ is a fine moduli space for $[H_1\text{–str}]$, via Yoneda's lemma we obtain an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{E} \to \mathfrak{H}$.

Fix an elliptic curve $E \to M$. Let $[H_1\text{–str}]_{E/M}$ be the sheaf of sets on M that assigns to an open subset $U \subset M$ the set of H_1-structures on $E_U := E \times_M U \to U$, with the obvious restriction maps. The sheaf $[H_1\text{–str}]_{E/M}$ locally admits a section. This is merely a reformulation of the fact that $\underline{H_1}(E/M)$ is a local system of rank-2 free $\mathbb{Z}$-modules.

Let $U \subset M$ be an open subset and $\psi : \mathbb{Z}^2 \times U \to \underline{H_1}\,(E_U/U)$ be an H_1-trivialization of E_U/U, i.e. $\psi \in [H_1\text{–str}]_{E/M}\,(U)$. Let $\underline{\mathrm{SL}_2(\mathbb{Z})}_M$ be the constant M-group assigned to the abstract group $\mathrm{SL}_2(\mathbb{Z})$, so a section γ of $\underline{\mathrm{SL}_2(\mathbb{Z})}_M$ over U is a locally constant function $\gamma : U \to \mathrm{SL}_2(\mathbb{Z})$. The composite

$$\psi \circ \gamma^t : \mathbb{Z}^2 \times U \xrightarrow{\gamma^t \times \mathbb{1}_U} \mathbb{Z}^2 \times U \xrightarrow{\psi} \underline{H_1}\,(E_U/U)\,.$$

defines one more H_1-structure on E_U/U. This defines an action of $\underline{\mathrm{SL}_2(\mathbb{Z})}_M$ on $[H_1\text{–str}]_{E/M}$:

$$\gamma \cdot \psi = \psi \circ \gamma^t, \quad \psi \in [H_1\text{–str}]_{E/M}\,(U),\ \gamma \in \underline{\mathrm{SL}_2(\mathbb{Z})}_M(U).$$

Since $\mathrm{SL}_2(\mathbb{Z})$ is the full automorphism group of $(\mathbb{Z}^2, \bullet)$ we have an isomorphism of sheaves on M

$$\begin{aligned} \underline{\mathrm{SL}_2(\mathbb{Z})}_M \times [H_1\text{–str}]_{E/M} &\to [H_1\text{–str}]_{E/M} \times [H_1\text{–str}]_{E/M}\,, \\ (\gamma, \psi) &\mapsto (\gamma \cdot \psi, \psi). \end{aligned}$$

Lemma 5.3.1. *The sheaf $[H_1\text{–}str]_{E/M}$ is an $\mathrm{SL}_2(\mathbb{Z})$-torsor over M.*

Proof. This follows from (5.3) and $[H_1\text{–str}]_{E/M}$ locally admitting sections. ☐

In a similar way we define an action of $\mathrm{SL}_2(\mathbb{Z})$ on the functor $[H_1\text{–str}]_{E/M}$. By Theorem 5.2.3 we have $[H_1\text{–str}] \cong \mathrm{h}_{\mathfrak{H}}$. Thus by Yoneda's lemma there is an induced action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$, denoted $\gamma \mapsto [\gamma]$, as well as a lift $\gamma \mapsto [\gamma]_{\mathcal{E}}$ of this action to $\mathcal{E}$, that does not however respect the H_1-trivialization Ψ. Instead, the maps $[\gamma]$ and $[\gamma]\mathcal{E}$ fit into the following cartesian square afforded by Theorem 5.2.3

$$\begin{array}{ccc} (\mathcal{E}, \Psi \circ \gamma^t) & \xrightarrow{[\gamma]_{\mathcal{E}}} & (\mathcal{E}, \Psi) \\ \downarrow & & \downarrow \\ \mathfrak{H} & \xrightarrow{[\gamma]} & \mathfrak{H}. \end{array} \tag{5.5}$$

Lemma 5.3.2. *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. The automorphism $[\gamma]$ of $\mathfrak{H}$ is given by*

$$[\gamma] : \tau \mapsto \frac{a\tau + b}{c\tau + d} \tag{5.6}$$

and the map $[\gamma]_{\mathcal{E}}$ is induced by the line bundle automorphism of $V = \mathbb{C} \times \mathfrak{H}$, which leaves L invariant and covers $[\gamma]$, defined by

$$[\gamma]_{\mathcal{E}} : (\tau, z) \mapsto \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \tag{5.7}$$

Proof. We refer the reader to [Conb][Prop. 2.1] for this verification. □

In this way we recover the classical left action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$ via fractional linear transformations or Möbius transformations defined in Lemma 3.1.1.

5.4 Γ-structures

Throughout this section, we let $\Gamma \subset SL_2(\mathbb{Z})$ denote a congruence subgroup. We define Γ-structures on relative elliptic curves in Definition 5.4.1, and show in Proposition 5.4.3 that the open modular curve $Y(\Gamma)$ is a coarse moduli space for the functor $[\Gamma\text{–str}]$ classifying elliptic curves with a Γ-structure.

Definition 5.4.1. Let $E \to M$ be a relative elliptic curve.
(1) We denote by $[\Gamma\text{–str}]_{E/M}$ the quotient sheaf $\Gamma\backslash [H_1\text{–str}]_{E/M}$, i.e. the sheafification of the presheaf on M given by $U \mapsto \Gamma\backslash([H_1\text{–str}]_{E/M}(U))$.
(2) A Γ-*structure* on an elliptic curve $E \to M$ is a global section $\alpha \in [\Gamma\text{–str}]_{E/M}(M)$.

Each H_1-structure ψ on E/M defines naturally a Γ-structure on E/M denoted $\Gamma\psi$. However, the map $[H_1\text{–str}]_{E/M}(M) \to [\Gamma\text{–str}]_{E/M}(M)$ is not surjective in general. In the same vein, $\Gamma \subset \Gamma' \subset SL_2(\mathbb{Z})$ are congruence subgroups, then a Γ-structure on E/M naturally defines a Γ'-structure on E/M. Again, it is quite possible that the map $[\Gamma\text{–str}]_{E/M}(M) \to [\Gamma'\text{–str}]_{E/M}(M)$ is not surjective.

Now let $\tilde{h} : E' \to E$ be a cartesian morphism of elliptic curves covering $h : M' \to M$. Pullback along h defines an $SL_2(\mathbb{Z})$-equivariant morphism of sheafs on M

$$\tilde{h}^* : [H_1\text{–str}]_{E/M} \to h_* [H_1\text{–str}]_{E'/M'} \tag{5.8}$$

which induces a morphism of sheafs

$$\tilde{h}^* : [\Gamma\text{–str}]_{E/M} \to h_*[\Gamma\text{–str}]_{E'/M'}.$$

Definition 5.4.2. We say two elliptic curves with a Γ-structure $(E \to M, \alpha)$ and $(E' \to M, \alpha')$ are *isomorphic over* M if there exists an isomorphism $\tilde{h} : E' \to E$ of elliptic curves over M such that $\tilde{h}^*(\alpha) = \alpha'$. We denote $[\Gamma\text{–str}](M)$ the set of M-elliptic curves with a Γ-structure $(E \to M, \alpha)$ up to isomorphism over M. Given a holomorphic map $h : M' \to M$, *pullback* defines restriction maps

$$\begin{aligned}[\Gamma\text{–str}](h) : [\Gamma\text{–str}](M) &\to [\Gamma\text{–str}](M'), \\ (E \to M, \alpha) &\mapsto (h^*E \to M', h^*\alpha),\end{aligned}$$

so that $[\Gamma\text{–str}] : \mathbb{C}\mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ becomes a functor. Furthermore, equivariance of the map (5.2) allows us to define an action of the abstract group $SL_2(\mathbb{Z})$ on the sheaf $[H_1\text{–str}] : \mathbb{C}\mathbf{Man} \to \mathbf{Set}$, by $\gamma \cdot (E \to M, \psi) = (E \to M, \gamma \cdot \psi)$.

Proposition 5.4.3. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. The presheaf $[\Gamma\text{–}str]$ on* $\mathbb{C}$**Man** *admits a coarse moduli space*

$$\theta_\Gamma : [\Gamma\text{–}str] \to \mathrm{h}_{Y(\Gamma)}, \tag{5.9}$$

where $Y(\Gamma) = \Gamma\backslash\mathfrak{H}$ is a connected Riemann surface that is the categorical quotient in $\mathbb{C}$**Man** *for the action of Γ on $\mathfrak{H}$, and the inverse of the bijection $\theta_\Gamma(*)$ is given by*

$$\begin{aligned} \theta_\Gamma(*)^{-1} : Y(\Gamma) &\to [\Gamma\text{–}str](*), \\ \Gamma\tau &\mapsto [(\mathcal{E}_\tau, \Gamma\Psi_\tau)] \,. \end{aligned}$$

In other words, the points of $Y(\Gamma)$ are in bijection with isomorphism classes of complex elliptic curves with a Γ-structure in such a way that for every elliptic curve with a Γ-structure $(E \to M, \alpha)$ the map $M \to Y(\Gamma)$, assigning to a point $m \in M$ the point of $Y(\Gamma)$ corresponding to the isomorphism class of the fibre (E_m, α_m) is holomorphic.

Proof of Proposition 5.4.3. By Lemma 3.1.1 the action of $\mathrm{SL}_2(\mathbb{Z})$ is proper with finite stabilizers, so a forteriori the same holds for the action of the subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. By Theorem 2.3.5 a categorical quotient for this Γ-action on the Riemann surface $\mathfrak{H}$ exists, which is a local analytic isomorphism $\mathfrak{H} \to \Gamma\backslash\mathfrak{H} =: Y(\Gamma)$ to a Riemann surface. Connectedness of $\mathfrak{H}$ implies that of its quotient $Y(\Gamma)$.

For every $\gamma \in \Gamma$ the H_1-structures $\Psi \circ \gamma^t$ and Ψ on $\mathcal{E}$ define the same Γ-structure. Hence the isomorphism $([\gamma]_{\mathcal{E}}, [\gamma])$ in (5.5)) is an automorphism of the elliptic curve with a Γ-structure $(\mathcal{E}, \Gamma\Psi) \to \mathfrak{H}$. In particular we have for every $\tau \in \mathfrak{H}$ an isomorphism $(\mathcal{E}_\tau, \Gamma\Psi_\tau) \cong \left(\mathcal{E}_{[\gamma](\tau)}, \Gamma\Psi_{[\gamma](\tau)}\right)$ showing that the map $\theta_\Gamma(*)^{-1}$ in the statement of the proposition is well-defined. Conversely, any classical elliptic curve with a Γ-structure is isomorphic to $(\mathcal{E}_\tau, \Gamma\Psi_t)$ for a point $\tau \in \mathfrak{H}$ unique up to the Γ-action, so we see that $\theta_\Gamma(*)^{-1}$ is bijective.

Let $(E, \alpha) \to M$ be an elliptic curve with a Γ-structure over a complex manifold M. We have a $\underline{\Gamma}_M$-equivariant homomorphism of sheafs on M

$$[H_1\text{–str}]_{E/M} \to [H_1\text{–str}]\,|_M \cong (\mathrm{h}_{\mathfrak{H}})|_M.$$

The quotient map $\mathfrak{H} \to Y(\Gamma)$ is Γ-invariant, so one sees that it induces a morphism $\underline{\Gamma}_M\backslash \mathrm{h}_{\mathfrak{H}}|_M \to \mathrm{h}_{Y(\Gamma)}|_M$ of sheaves on M. We defined $[\Gamma\text{–str}]_{E/M} = \underline{\Gamma}_M\backslash\,[H_1\text{–str}]_{E/M}$, so there is an induced homomorphism

$$[\Gamma\text{–str}]_{E/M} = \underline{\Gamma}_M\backslash\,[H_1\text{–str}]_{E/M} \to \underline{\Gamma}_M\backslash \mathrm{h}_{\mathfrak{H}}|_M \to \mathrm{h}_{Y(\Gamma)}|_M.$$

We let $\theta_\Gamma([(E,\alpha) \to M]) \in \mathrm{h}_{Y(\Gamma)}(M)$ be the image of $\alpha \in [\Gamma\text{–str}]_{E/M}(M)$ under this map in $\mathrm{h}_{Y(\Gamma)}(M)$. It is straightforward to see this assignment yields a well-defined natural transformation θ_Γ that gives on $*$-valued points the bijection detailed in the statement of the proposition. □

5.5 The universal elliptic curve $\mathcal{E}_\Gamma \to Y(\Gamma)$ for a torsion-free Γ

In the previous section, we considered a congruence subgroup Γ and showed that the presheaf $[\Gamma\text{–str}]$ admits a coarse moduli space $Y(\Gamma)$. In this section, we prove that if Γ is torsion-free, then in fact $[\Gamma\text{–str}]$ admits a *fine* moduli space $\mathcal{E}_\Gamma \to Y(\Gamma)$.

Definition 5.5.1 (rigidity)**.** We say an elliptic curve $(E \to M, \alpha)$ with a Γ-structure is *rigid* if its only automorphism over M is the identity morphism.

Lemma 5.5.2. *For a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, the following are equivalent:*

(1) *every elliptic curve $(E \to M, \alpha)$ with a Γ-structure is rigid;*

(2) *Γ acts freely on $\mathfrak{H}$;*

(3) *Γ is torsion-free.*

Proof. See [Cona][Prop. 4.2] □

Example 5.5.3. In Proposition 3.2.2 we saw that $\Gamma(N)$ is torsion-free if $N \geq 3$, and that $\Gamma_1(N)$ is torsion-free if $N \geq 4$.

Theorem 5.5.4. *Let the notation and hypotheses be as in Proposition 5.4.3 and assume in addition that $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is torsion-free. Then the natural transformation $\theta_\Gamma : [\Gamma\text{–}str] \to h_{Y(\Gamma)}$ is a fine moduli space for $[\Gamma\text{–}str]$. The corresponding universal elliptic curve with a Γ-structure is (up to isomorphism) the categorical quotient $(\mathcal{E}_\Gamma \to Y(\Gamma), \Gamma\Psi)$ for the action of Γ on $(\mathcal{E} \to \mathfrak{H}, \Gamma\Psi)$ in the category of elliptic curves with a Γ-structure.*

Proof. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$ is free and proper in view of Lemma 5.5.2 resp. Lemma 3.1.1. In Proposition 5.4.3 we showed the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathcal{E} \to \mathfrak{H}$ from Section 5.3 restricts to an action of Γ on the elliptic curve with a Γ-structure $(\mathcal{E} \to \mathfrak{H}, \Gamma\Psi)$ in the sense of Definition 4.6.1. By Lemma 4.6.2 and Remark 4.6.3 there exists a categorical quotient for the Γ-action in the category of elliptic curve with a Γ-structure, which we denote $(\mathcal{E}_\Gamma := \Gamma\backslash\mathcal{E}, \Gamma\Psi) \to Y(\Gamma) := \Gamma\backslash\mathfrak{H}$, which makes the right square in the diagram below caresian:

$$\begin{array}{ccccc} (E,\alpha) & \longrightarrow & (\mathcal{E},\Gamma\Psi) & \longrightarrow & (\mathcal{E}_\Gamma,\Gamma\Psi) \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & \mathfrak{H} & \longrightarrow & Y(\Gamma). \end{array} \tag{5.10}$$

Now we turn to proving that θ_Γ is an isomorphism. Let $(E, \alpha) \to M$ be an elliptic curve with a Γ-structure. It is to be demonstrated that there is a unique cartesian diagram making up the outer rectangle of the above diagram. By Lemma 5.5.2 all elliptic curves with a Γ-structure are rigid, so if such a diagram exists, it is unique. Therefore the proof of existence can be carried out locally, so we may and do assume that α stems from an H_1-structure ψ on E, with $\alpha = \Gamma\psi$. By Theorem 5.2.3 there exists a unique cartesian diagram of H_1-trivialized elliptic curves, as in the left square of the above diagram. We already saw the right square is cartesian, so by concatenation of cartesian squares we find that the outer rectangle is cartesian, as desired. □

5.6 Level-N structures

In this section, N denotes a positive integer. We define the notion of a level-N structure on a relative elliptic curve and define a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on the presheaf $[N\text{–str}]$ on $\mathbb{C}\mathbf{Man}$ classifying level-N structures. We show the Weil pairing gives a decomposition $[N\text{–str}] = \coprod_{\zeta\in\mu_N^\times}[N\text{–str}]^\zeta$, where each presheaf $[N\text{–str}]^\zeta$ is isomorphic to $[\Gamma(N)\text{–str}]$, where $\Gamma(N)$ is the principal congruence subgroup of level N defined as

$$\Gamma(N) = \ker\left(\mathrm{can} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right).$$

Let $E \to M$ be an elliptic curve.

Definition 5.6.1 (level-N structure)**.** (1) A *level-N structure* on E/M is an isomorphism of M-groups

$$\psi : (\mathbb{Z}/N\mathbb{Z})^2 \times M \to E[N]. \tag{5.11}$$

Alternatively, a level-N structure on E/M is given by a pair of N-torsion points $(P, Q) \in E[N]\times E[N]$ such that for every $m \in M$ the pair $(P_m, Q_m) \in E_m[N]\times E_m[N]$ is a basis for the $(\mathbb{Z}/N\mathbb{Z})$-module $E_m[N]$.
(2) We denote $[N\text{–str}]_{E/M}$ the sheaf on M that assigns to an open subset $U \subset M$ the set of level- N structures on $E_U := E \times_M U \to U$, with the obvious restriction maps.

Let us write $\underline{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}_M$ for the constant M-group assigned to the abstract group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, so a section γ of $\underline{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}_M$ over U is a locally constant function $\gamma : U \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

Lemma 5.6.2. *The sheaf $[N\text{–str}]_{E/M}$ is a $\underline{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}_M$-torsor on M.*

Proof. To define the action, let $\psi \in [N\text{–str}]_{E/M}(U)$ be a section over some open subset $U \subset M$, i.e. $\psi : (\mathbb{Z}/N\mathbb{Z})^2 \times U \to E_U[N]$ is a level-N structure on E_U/U. The composite

$$\psi\circ\gamma^t : (\mathbb{Z}/N\mathbb{Z})^2 \times U \xrightarrow{\gamma^t\times\mathbb{1}_U} (\mathbb{Z}/N\mathbb{Z})^2 \times U \xrightarrow{\psi} E_U[N]$$

defines one more N-structure on E_U/U. It is straightforward to check this defines an action of $\underline{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}_M$ on $[N\text{–str}]_{E/M}$:

$$\gamma\cdot\psi = \psi\circ\gamma^t, \quad \psi \in [N\text{–str}]_{E/M}(U),\ \gamma \in \underline{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}_M(U). \tag{5.12}$$

Since $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is the full automorphism group of the $(\mathbb{Z}/N\mathbb{Z})$-module $(\mathbb{Z}/N\mathbb{Z})^2$, we have an isomorphism

$$\underline{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}_M \times [N\text{–str}]_{E/M} \to [N\text{–str}]_{E/M} \times [N\text{–str}]_{E/M}\,, \qquad (\gamma,\psi) \mapsto (\gamma\cdot\psi, \psi).$$

Since $E[N]$ is a local system of rank-2 free $\mathbb{Z}/N\mathbb{Z}$-modules, the sheaf $[N\text{–str}]_{E/M}$ admits a section locally on M. We conclude the sheaf $[N\text{–str}]_{E/M}$ is a torsor under $\underline{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}_M$. ☐

Definition 5.6.3 (Weil pairing of a level-N structure)**.** The *Weil pairing* of a level-N structure (P, Q) is defined to be the locally constant $\mu_N^\times$-valued function

$$e_N(Q, P) : M \xrightarrow{(Q,P)} E[N] \times E[N] \xrightarrow{e_N} \mu_N^\times. \tag{5.13}$$

We opted for the 'reversal' of P and Q in this definition, to be consistent with requiring an H_1-structure (τ, σ) to satisfy $\sigma \bullet \tau = 1$. Sending a level-N structure to its Weil pairing defines a morphism of sheafs

$$[N\text{–str}]_{E/M} \to \underline{\mu_N^\times}_M, \tag{5.14}$$

equivariant with respect to the determinant map det: $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to (\mathbb{Z}/N\mathbb{Z})^\times$. For every primitive N-th root of unity $\zeta \in \mu_N^\times$, we write $[N\text{–str}]^\zeta_{E/M}$ for the preimage of $\underline{\{\zeta\}}_M$, i.e. for the subsheaf of $[N\text{–str}]_{E/M}$ consisting of those level-N structures with Weil pairing constantly equal to ζ.

It follows that

$$[N\text{–str}]_{E/M} = \bigsqcup_{\zeta \in \mu_N^\times} [N\text{–str}]^\zeta_{E/M}, \tag{5.15}$$

where the disjoint union of the sheafs $[N\text{–str}]^\zeta_{E/M}$ is by definition the sheafification of the presheaf $U \mapsto \bigsqcup_{\zeta \in \mu_N^\times} [N\text{–str}]^\zeta_{E/M}(U)$ on M.

Lemma 5.6.4. *For each $\zeta \in \mu_N^\times$ there exists an isomorphism*

$$[\Gamma(N)\text{–}str]_{E/M} \cong [N\text{–}str]^\zeta_{E/M}. \tag{5.16}$$

Proof. We first prove this for the primitive N-th root of unity $\zeta_N = \exp(2\pi \mathrm{i}/N)$. Let $\psi : \mathbb{Z}^2 \times M \to \underline{H}_1(E/M)$ be an H_1-structure on E/M. Then ψ induces by tensoring with $\mathbb{Z}/N\mathbb{Z}$ a level-N structure on E/M

$$\psi \otimes \mathbb{1}_{\mathbb{Z}/N\mathbb{Z}} : (\mathbb{Z}/N\mathbb{Z})^2 \cong \mathbb{Z}^2 \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\psi \otimes \mathbb{1}_{\mathbb{Z}/N\mathbb{Z}}} \underline{H}_1(E/M) \otimes (\mathbb{Z}/N\mathbb{Z}) \cong E[N]$$

whose Weil pairing is $\zeta_N := \exp(2\pi \mathrm{i}/N)$, as is clear from Definition 4.5.9.

In fact, this defines a morphism of sheafs on M

$$[H_1\text{–str}]_{E/M} \to [N\text{–str}]^{\zeta_N}_{E/M},$$

equivariant with respect to $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Since $[H_1\text{–str}]_{E/M}$ resp. $[\Gamma\text{–str}]^{\zeta_N}_{E/M}$ is a torsor under $\underline{\mathrm{SL}_2(\mathbb{Z})}_M$ resp. $\underline{\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}$, and $\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$ we conclude that there is an isomorphism

$$\begin{aligned} [\Gamma(N)\text{–str}]_{E/M} := \Gamma(N)\backslash\, [H_1\text{–str}]_{E/M} &\xrightarrow{\sim} [N\text{–str}]^{\zeta_N}_{E/M} \\ \Gamma(N)\psi &\mapsto \psi \otimes (\mathbb{Z}/N\mathbb{Z}). \end{aligned} \tag{5.17}$$

For any commutative ring R, the determinant map det: $\mathrm{GL}_2(R)^\times \to R^\times$ is surjective by virtue of the section

$$\langle\cdot\rangle : R^\times \to \mathrm{GL}_2(R)^\times,\ \langle r\rangle = \begin{pmatrix} r & 0 \\ 0 & 1\end{pmatrix}. \tag{5.18}$$

Hence for $m + N\mathbb{Z} \in \mathbb{Z}/N\mathbb{Z}$ and $\zeta \in \mu_N^\times$ we get an isomorphism

$$\begin{aligned} \langle m\rangle : [N\text{–str}]^\zeta_{E/M} &\to [N\text{–str}]^{\zeta^m}_{E/M}, \\ \psi = (P,Q) &\mapsto \psi \circ \langle m\rangle^t = ([m]_E(P), Q) \end{aligned} \tag{5.19}$$

Now let $\zeta \in \mu_N^\times$ be an arbitrary element of $\mathbb{C}^\times$ having order N. Then there exists unique $r \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $\zeta = \zeta_N^r = \exp(2\pi i r/N)$. Composing (5.32) with (5.19) gives an isomorphism

$$\begin{aligned} [\Gamma(N)\text{–str}]_{E/M} := \Gamma(N)\backslash\, [H_1\text{–str}]_{E/M} &\xrightarrow{\sim} [N\text{–str}]^\zeta_{E/M}, \\ \Gamma(N)\psi &\mapsto (\psi \otimes \mathbb{1}_{\mathbb{Z}/N\mathbb{Z}}) \circ \langle m\rangle, \end{aligned} \tag{5.20}$$

where the map $(\psi \otimes \mathbb{1}_{\mathbb{Z}/N\mathbb{Z}}) \circ \langle r\rangle$ is the composite

$$(\mathbb{Z}/N\mathbb{Z})^2 \times M \xrightarrow{\langle r\rangle} (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\psi\otimes \mathbb{1}_{\mathbb{Z}/N\mathbb{Z}}} \underline{H}_1(E/M) \otimes (\mathbb{Z}/N\mathbb{Z}) \cong E[N]. \qquad \square$$

Next, we define a presheaf $[N\text{–str}]$ on $\mathbb{C}$**Man** classifying relative elliptic curves with a level-N structure. We show that the sheaf $[N\text{–str}]$ on $\mathbb{C}$**Man** admits a fine moduli space.

Let $\tilde{h} : E' \to E$ be a cartesian morphism of elliptic curves covering $h : M' \to M$. Then $\tilde{h}$ gives rise to an isomorphism $E'[N] \cong h^*E[N]$. Therefore a level-N structure $\psi = (\mathbb{Z}/N\mathbb{Z})^2 \times M \xrightarrow{\sim} E[N]$ on E/M pulls back to a level-N structure $h^*\psi : (\mathbb{Z}/N\mathbb{Z})^2 \times M' \xrightarrow{\sim} E'[N]$. Note that if $\psi = (P,Q)$ then $h^*\psi = (h^*P, h^*Q)$, where $h^* : E(M) \to E'(M')$ is given by pulling back sections.

We obtain a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-equivariant morphism of sheaves on M', compatible with the Weil pairings, as shown in the commutative diagram

$$\begin{array}{ccc} h^*([N\text{–str}]_{E/M}) & \xrightarrow{\tilde{h}^*} & [N\text{–str}]_{E'/M'} \\ {\scriptstyle h^*e_N}\downarrow & & \downarrow{\scriptstyle e_N} \\ h^*(\underline{\mu_N^\times}_M) & \xrightarrow{\sim} & \underline{\mu_N^\times}_{M'}. \end{array}$$

Definition 5.6.5. (1) Two elliptic curves with a level-N structure $(E \to M, (P,Q))$ and $(E' \to M, (P',Q')$ are said to be *isomorphic over* M if there exists an isomorphism $\tilde{h} : E' \to E$ of elliptic curves over M such that $\tilde{h}(P') = P$ and $\tilde{h}(Q') = Q$.

(2) We denote $[N\text{–str}](M)$ the set of elliptic curves over M with a level-N structure $(E \to M, (P,Q))$ up to isomorphism over M. Given a holomorphic map $h : M' \to M$, pullback defines restriction maps

$$\begin{aligned} [N\text{–str}](h) : [N\text{–str}](M) &\to [N\text{–str}](M'), \\ (E \to M, \psi) &\mapsto (h^*E \to M', h^*\psi). \end{aligned}$$

This defines a presheaf $[N\text{–str}] : \mathbb{C}\mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ on $\mathbb{C}$**Man**. (3) Let $\zeta \in \mu_N^\times$. We denote $[N\text{–str}]^\zeta$ the subpresheaf of $[N\text{–str}]$ classifying relative elliptic curves with a level-N structure with Weil pairing ζ.

Proposition 5.6.6. *Let $\zeta \in \mu_N^\times$. Then there exists an isomorphism of presheaves on* $\mathbb{C}$**Man**

$$[N\text{-}str]^\zeta \cong [\Gamma(N)\text{-}str]. \tag{5.21}$$

Proof. This follows from Corollary 5.7.4. □

Theorem 5.6.7. *Let $N \geq 3$ and let $\zeta \in \mu_N^\times$. Then sheaf $[N\text{–}str]^\zeta$ admits a fine moduli space $(E_{\Gamma(N)} \to Y(N), (P_\zeta, Q_\zeta))$.*

Proof. It follows from Proposition 5.6.6, Theorem 5.5.4 and Example 5.5.3 that there is an isomorphism $[N\text{–str}]^\zeta \cong [\Gamma(N)\text{–str}] \cong \mathrm{h}_{Y(N)}$. Let us explain what the universal level-N structure (P_ζ, Q_ζ) on $E_{Y(N)} \to Y(N)$ is.

The universal H_1-trivialized elliptic curve $(\mathcal{E} \to \mathfrak{H}, \Psi)$ has a level-N structure (P_N, Q_N) with Weil pairing $\zeta_N = \exp(2\pi\mathrm{i}/N)$ given by

$$(P_N(\tau), Q_N(\tau)) = \left(\frac{\tau}{N}, \frac{1}{N}\right). \tag{5.22}$$

For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the two H_1-trivializations Ψ and $\Psi \circ \gamma^t$ induce the same level-N structure if and only if $\gamma \in \Gamma(N)$. Hence for $\gamma \in \Gamma(N)$, the automorphism $([\gamma]_\mathcal{E}, [\gamma])$ of $\mathcal{E} \to \mathfrak{H}$ preserves the level-N structure (P_N, Q_N), which therefore descends to a level-N structure denoted $(P_{\zeta_N}, Q_{\zeta_N})$ on $\mathcal{E}_{Y(N)} = \Gamma(N)\backslash\mathcal{E} \to Y(N) = \Gamma(N)\backslash\mathfrak{H}$.

For an arbitrary $\zeta \in \mu_N^\times$, let $r \in (\mathbb{Z}/N\mathbb{Z})^\times$ be such that $\zeta = \zeta_N^r$. Then the universal level-N structure with Weil pairing ζ on $\mathcal{E}_{Y(N)} \to Y(N)$ is given by $(P_\zeta, Q_\zeta) := ([r](P_{\zeta_N}), Q_{\zeta_N})$. □

Corollary 5.6.8. *Let $N \geq 3$. Then the disjoint union over all $\zeta \in \mu_N^\times$ of the fine moduli spaces $(E_{\Gamma(N)} \to Y(N), (P_\zeta, Q_\zeta))$ for $[N\text{–}str]^\zeta$ constructed in Theorem 5.6.7 is a fine moduli space for $[N\text{-}str]$ and we have*

$$[N\text{–}str] \cong h_{Y(N)\times\mu_N^\times}. \tag{5.23}$$

Proof. This follows from Theorem 5.6.7 and the fact that the Weil pairing of a level-N structure on a relative elliptic curve E/M is constant on each connected component of M. □

5.7 Points of exact order N

Let $E \to M$ be a relative elliptic curve.

Definition 5.7.1. A *point of exact order* N is a section $P \in E(M)$ such that for every $m \in M$ the element P_m of the group E_m has order N.

Points of exact order N are in bijection with injective homomorphism of M-groups $\underline{\mathbb{Z}/N\mathbb{Z}}_M \to E$, in such a way that a point P corresponds to the homomorphism sending $1 \in \underline{\mathbb{Z}/N\mathbb{Z}}(M)$ to $P \in E(M)$.

We denote $[N\text{–pt}]_{E/M}$ the sheaf on M that assigns to an open subset $U \subset M$ the set of points of exact order N of $E_U := E \times_M U \to U$, with the obvious restriction maps. Note that if $\psi = (P, Q)$ is a level-N structure on $E(M)$, then

$\psi(ae_1 + be_2) = aP + bQ$ with $a, b \in \mathbb{Z}/N\mathbb{Z}$ is a point of exact order N whenever a and b generate the unit ideal of $\mathbb{Z}/N\mathbb{Z}$. In particular, there is a natural transformation of sheaves on M

$$\begin{aligned} [N\text{–str}]_{E/M} &\to [N\text{–pt}]_{E/M}, \\ \psi = (P, Q) &\mapsto \psi(e_2) = Q. \end{aligned} \tag{5.24}$$

Definition 5.7.2. We denote the image of the level-N congruence subgroup $\Gamma_1(N)$ under the canonical map $\pi_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ by

$$\overline{\Gamma_1}(N) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}/N\mathbb{Z} \right\}. \tag{5.25}$$

The map (5.24) is invariant for $\bar{\Gamma}_1(N)$.

Lemma 5.7.3. *Let $\zeta \in \mu_N^\times$. The assignment $(P, Q) \mapsto Q$ yields an isomorphism of sheaves on M*

$$\overline{\Gamma_1}(N)\backslash[N\text{–str}]^{\zeta}_{E/M} \to [N\text{–pt}]_{E/M}. \tag{5.26}$$

Proof. Let $U \subset M$ be an open subset, and let $\psi = (P, Q)$ and $\psi' = (P', Q')$ be two level-N structures on E_U/U with Weil pairing ζ. Since $[N\text{–str}]^{\zeta}_{E/M}$ is an $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$-torsor, there exists a unique $\gamma \in \underline{\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}_M(U)$ such that $\psi' = \gamma \cdot \psi = \psi \circ \gamma^t$. Since $\psi : (\mathbb{Z}/N\mathbb{Z})^2 \times M \to E[N]$ is injective, we have that

$$Q = Q' \iff \psi'(e_2) = \psi(e_2) \iff \gamma^t(e_2) = e_2 \iff \gamma \in \overline{\Gamma_1}(N)$$

This shows we have a well-defined injective morphism (5.26) of sheaves on M. We conclude the proof by showing it induces a surjection on the stalks

Let $m \in M$. It remains to be shown that the map $[N\text{–str}]^{\zeta}_{E_m} \to [N\text{–pt}]_{E_m}$, $(P_m, Q_m) \mapsto Q_m$ is surjective. Let $Q_m \in [N\text{–pt}]_{E_m}$ be a point of E_m of exact order N. Since the Weil pairing $e_N : E_m[N] \times E_m[N] \to \mu_N$ is nondegenerate, there exists $P_m \in E_m[N]$ such that $e_N(Q_m, P_m) = \zeta$. Then $(P_m, Q_m) \in [N\text{–str}]^{\zeta}_{E_m}$ is a level-N structure on E_m with Weil pairing ζ, that maps to Q_m, as desired. ☐

Corollary 5.7.4. *There exists an isomorphism of sheaves on M, to be described in the proof,*

$$[\Gamma_1(N)\text{–str}]_{E/M} \xrightarrow{\sim} [N\text{–pt}]_{E/M}. \tag{5.27}$$

Proof. Choose a primitive N-th root of unity $\zeta \in \mu_N^\times$, for example $\zeta_N = \exp(2\pi \mathrm{i}/N)$. The required isomorphism is the composite of the isomorphisms

$$\Gamma_1(N)\backslash\,[H_1\text{–str}]_{E/M} \cong \overline{\Gamma_1}(N)\backslash \left(\Gamma(N)\backslash\,[H_1\text{–str}]_{E/M}\right), \tag{5.28}$$

$$\stackrel{5.5}{\to}\ \bar{\Gamma}_1(N)\backslash[N\text{–str}]^{\zeta}_{E/M} \stackrel{5.9}{\to} [N\text{–pt}]_{E/M}. \tag{5.29}$$

It ought to be remarked that this composite is independent of the choice of the primitive N-th root of unity ζ. Indeed, the isomorphism 5.5 for ζ_N^m is obtained by precomposing that for ζ_N with the diamond isomorphism

$$\langle m \rangle : [N\text{–str}]^{\zeta_N}_{E/M} \to [N\text{–str}]^{\zeta_N^m}_{E/M}, (P, Q) \mapsto (mP, Q),$$

and the latter does not alter the second point in the level-N structure. ☐

Definition 5.7.5. We define a sheaf $[N\text{–pt}]$ on $\mathbb{C}\mathbf{Man}$ classifying relative elliptic curves with a point of exact order N, in the same way as we defined $[N\text{-str}]$ in Definition 5.6.5.

5.8 Cyclic subgroups of order N

Let $E \to M$ be a relative elliptic curve.

Definition 5.8.1. A *cyclic subgroup of order* N on E/M is a closed subset $G \subset E$ such that $G_m = G \cap E_m$ is a cyclic subgroup of order N of E_m for every $m \in M$.

Example 5.8.2. Let Q be a point of exact order N of E/M. Then

$$(\mathbb{Z}/N\mathbb{Z}) \cdot Q := \bigcup_{i=0}^{N-1} ([i] \circ Q)(M) \tag{5.30}$$

is a cyclic subgroup G of order N of E/M. For every $m \in M$, we have that the fibre $G_m = \bigcup_{i=0}^{N-1} [i]\,(Q_m) \subset E_m$ is equal to the subgroup of order N of E_m generated by the element Q_m.

We denote $[N\text{–grp}]_{E/M}$ the sheaf on M that assigns to an open subset $U \subset M$ the set of cyclic subgroups of order N of $E_U := E \times_M U \to U$, with the obvious restriction maps.

Proposition 5.8.3. *There exists an isomorphism of sheaves on M*

$$[\Gamma_0(N)\text{–str}]_{E/M} \xrightarrow{\sim} [N\text{–grp}]_{E/M}. \tag{5.31}$$

We will give the proof of Proposition 5.8.3 at the end of this section, after having doing some preparations.

Example 5.8.2 defines a morphism of sheaves $[N\text{–pt}]_{E/M} \to [N\text{–grp}]_{E/M}$ given by $Q \mapsto (\mathbb{Z}/N\mathbb{Z}) \cdot Q$. This morphism is invariant for the natural action of $(\mathbb{Z}/N\mathbb{Z})^\times$on $[N\text{–pt}]_{E/M}$, where $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ operates by $Q \mapsto [d](Q)$. In fact, this morphism descends to an isomorphism, as the following lemma asserts.

Lemma 5.8.4. *There exists an isomorphism of sheaves on M*

$$(\mathbb{Z}/N\mathbb{Z})^\times \backslash [N\text{–pt}]_{E/M} \to [N\text{–grp}]_{E/M}. \tag{5.32}$$

Proof. Let $m \in M$ be an arbitrary point. It suffices to show bijectivity of the map on the stalk at m, which is $[N\text{–pt}]_{E_m} \to [N\text{–grp}]_{E_m}$, $Q \mapsto (\mathbb{Z}/N\mathbb{Z}) \cdot Q$. This is the plain statement that any cyclic subgroup of order N of an abelian group A has a generator, which is unique up to taking $(\mathbb{Z}/N\mathbb{Z})^\times$multiples. □

Definition 5.8.5. We denote the image of the level-N congruence subgroup $\Gamma_0(N)$ under the canonical map $\pi_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ by

$$\overline{\Gamma_0}(N) = \left\{ \begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix} : b \in \mathbb{Z}/N\mathbb{Z}, d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}. \tag{5.33}$$

Note that $\overline{\Gamma_1(N)}$ is a normal subgroup of $\overline{\Gamma_0(N)}$. The quotient $\bar{\Gamma}_0(N)/\bar{\Gamma}_1(N)$ acts on the sheaf $\overline{\Gamma_1}(N)\backslash[N\text{–str}]^{\zeta}_{E/M}$. It is straightforward to check that the isomorphism (5.32) of sheaves on M is equivariant with respect to the isomorphism of groups

$$\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\pi_N} \bar{\Gamma}_0(N)/\bar{\Gamma}_1(N) \to (\mathbb{Z}/N\mathbb{Z})^\times,$$
$$\begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix} \bar{\Gamma}_1(N) \mapsto d.$$

Proof of Proposition 5.8.3. The required isomorphism is the composite of the isomorphisms

$$\begin{aligned} \Gamma_0(N)\backslash\, [H_1\text{–str}]_{E/M} &\cong (\overline{\Gamma}_0(N)/\overline{\Gamma}_1(N))\backslash\, [\Gamma_1(N)\text{–str}] \\ &\overset{(5.31)}{\to} (\mathbb{Z}/N\mathbb{Z})^\times\backslash\, [N\text{–pt}] \overset{(5.32)}{\to} [N\text{–grp}]\,. \end{aligned}$$

□

Chapter 6

Néron polygons and generalized elliptic curves

In this chapter we consider degenerating families of elliptic curves, called *generalized elliptic curves*. Roughly speaking, a generalized elliptic curve is a holomorphic map of complex manifolds of relative dimension 1, whose fibres are either smooth elliptic curves or 'semistable' degenerations thereof called *Néron polygons*. In the algebraic case they were introduced by Deligne and Rapoport in their landmark paper [DR73] in order to construct a modular compactification of certain moduli stacks of elliptic curves. In the complex-analytic case we will define these in Section 6.5.

In Section 6.1 for $N \geq 1$ we define the Néron N-gon C_N as a cyclic chain ('polygon') of N projective lines. We show that $C_N^{\mathrm{reg}} \cong \mathbb{C}^* \times (\mathbb{Z}/N\mathbb{Z})$ is a complex Lie group that acts on C_N. In Definition 6.2.4, we define the analogue of the Weil e_N-pairing on the N-torsion $E[N]$ of a complex elliptic curve, for the N-torsion $C^{\mathrm{reg}}[N]$ of the regular locus of the Néron N-gon. Following [Del71][Chapitre IV], in Section 6.4 we will show that the cusps of the modular curves $X(N)$, $X_1(N)$ and $X_0(N)$ parametrize isomorphism classes of Néron polygons with a certain level structure on their N-torsion.

In Chapter 9 we will construct, for suitable congruence subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$, a generalized elliptic curve $\mathcal{D}_\Gamma \to X(\Gamma)$ which compactifies $\mathcal{E}_\Gamma \to Y(\Gamma)$. The fibre of this so-called *Shioda modular surface* $\mathcal{D}_\Gamma$ over a cusp t of $X(\Gamma)$ having width h is a Néron h-gon: $\mathcal{D}_\Gamma|_t \cong C_h$.

If $\tilde{\Gamma} \subset \Gamma$ is a second congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then we will construct a holomorphic extension $\mathcal{D}_{\tilde{\Gamma}} \to \mathcal{D}_\Gamma$ of the natural map $p_{\Gamma,\tilde{\Gamma}} \colon \mathcal{E}_{\tilde{\Gamma}} = \tilde{\Gamma}\backslash\mathcal{E} \to \Gamma\backslash\mathcal{E} = \mathcal{E}_\Gamma$ which covers the natural map $p_{\Gamma,\tilde{\Gamma}} \colon X(\tilde{\Gamma}) = \tilde{\Gamma}\backslash\mathfrak{H}^* \to \Gamma\backslash\mathfrak{H}^* = X(\Gamma)$ in Section 9.4. If $\tilde{s} \in \mathrm{Cusps}(\Gamma)$ has width $\tilde{h}$ and its image $s = p_{\Gamma,\tilde{\Gamma}}(\tilde{s}) \in \mathrm{Cusps}(\Gamma)$ has width h, then the map on the fibres $\mathcal{D}_{\tilde{\Gamma}}|_{\tilde{s}} \to \mathcal{D}_\Gamma|_s$ is a contraction map $u_{h,\tilde{h}} \colon C_{\tilde{h}} \to C_h$ that we will define in Section 6.2 for positive integers h and $\tilde{h}$ with h dividing $\tilde{h}$.

6.1 The Néron polygon C_N

In this section for each positive integer N we will construct the *Néron N-gon* C_N as the union of a $\mathbb{Z}/N\mathbb{Z}$-index family of projective lines, each transversally intersecting both of its neighbors so as to form a cyclic chain or 'polygon'. In Chapter 8 we will encounter a version of this with an infinite $\mathbb{Z}$-indexed family of projective lines, deemed a *Néron chain*. In fact, we can construct the Néron chain in one fell sweep with the Néron polygons.

Definition 6.1.1. Let $H \subset \mathbb{Z}$ be a subgroup, and let $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be its index. We will call the quotient complex analytic space

$$C_N = (\mathbb{CP}^1 \times \mathbb{Z}/H)/((0, i) \sim (\infty, i+1)). \tag{6.1}$$

the *N-sided Néron polygon* or *Néron N-gon* if $N \in \mathbb{Z}_{\geq 0}$ and the *Néron chain* if $N = \infty$.

The irreducible components of C_N are projective lines indexed by $i \in \mathbb{Z}/H$, with the i-th line intersection the $(i+1)$-th transversely. The singular locus C_N^{sing} of C_N consists of these N intersection points. The quotient map $c : \mathbb{P}^1 \times (\mathbb{Z}/N\mathbb{Z}) \to C_N$ is an isomorphism away from C_N^{sing}, so that

$$C_N^{\text{reg}} = \mathbb{C}^* \times (\mathbb{Z}/H). \tag{6.2}$$

We see that there is a natural structure of complex Lie group on C_N^{reg} which in fact extends to a holomorphic action map

$$m : C_N^{\text{reg}} \times C_N \to C_N \tag{6.3}$$

determined by the commutativity of the following diagram

$$\begin{array}{ccc}
\mathbb{C}^\times \times (\mathbb{Z}/H) \times \mathbb{CP}^1 \times (\mathbb{Z}/H) & \xrightarrow{\tilde{m}} & \mathbb{CP}^1 \times (\mathbb{Z}/H) \\
\downarrow {\scriptstyle c \times c} & & \downarrow {\scriptstyle c} \\
C_N^{\text{reg}} \times C_N & \xrightarrow{\quad m \quad} & C_N,
\end{array}$$

where $\tilde{m}$ is given for all $x \in \mathbb{C}^*$, $y \in \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ and $i, j \in \mathbb{Z}/H$ by

$$\tilde{m}((x, i), (y, j)) = (xy, i + j).$$

Let H_1 be a subgroup of H, and let N_1 be its index. Then the Néron N-gon is a quotient of the Néron N_1-gon. To see this, define an action

$$a \colon \mathbb{Z}/H \times C_N \to C_N \tag{6.4}$$

of $\mathbb{Z}/H$ on C_H by biholomorphisms, by requiring commutativity of the diagram

$$\begin{array}{ccc}
(\mathbb{Z}/H\mathbb{Z}) \times (\mathbb{CP}^1 \times (\mathbb{Z}/H\mathbb{Z})) & \xrightarrow{(m,(x,i)) \mapsto (x, i+m)} & \mathbb{CP}^1 \times (\mathbb{Z}/H\mathbb{Z}) \\
\downarrow {\scriptstyle \mathrm{id}_{\mathbb{Z}/H\mathbb{Z}} \times c} & & \downarrow {\scriptstyle c} \\
(\mathbb{Z}/H\mathbb{Z}) \times C_N & \xrightarrow{\quad a \quad} & C_N.
\end{array}$$

Lemma 6.1.2. *For subgroups $H_1 \subset H \subset \mathbb{Z}$, with indices $N_1 = (\mathbb{Z} : H_1)$ and $N = (\mathbb{Z} : H)$, we have a natural isomorphism*

$$C_N \cong (H/H_1) \backslash C_{N_1}. \tag{6.5}$$

Proof. One checks that both are the quotient of $\mathbb{CP}^1 \times \mathbb{Z}/H_1\mathbb{Z}$ by the same equivalence relation. □

6.2 Weil e_N-pairing on C_N

Let $N \in \mathbb{Z}_{\geq 1}$. The N-torsion of C_N^{reg} is the free $\mathbb{Z}/N\mathbb{Z}$-module of rank 2

$$C_N^{\text{reg}}[N] = \mu_N \times \mathbb{Z}/N\mathbb{Z}.$$

The aim of this section is to define a nondgenerate alternating bilinear map

$$e_N : C_N^{\text{reg}}[N] \times C_N^{\text{reg}}[N] \to \mu_N, \tag{6.6}$$

called the *Weil e_N-pairing*, since it is the analogue of the Weil e_N-pairing on the N-torsion of an elliptic curve. We will construct this pairing in a somewhat ad-hoc fashion, using the fact proved in Lemma 6.2.3 that any automorphism of C_N induces an automorphism of the rank-2 free $\mathbb{Z}/N\mathbb{Z}$-module $C_N^{\text{reg}}[N]$ with determinant 1. See [AMRT10][§1.4, Definition (b) preceding Theorem 4.3] for a more intrinsic definition in terms of ample line bundles of degree N on C_N. So let us start with determining the group $\text{Aut}(C_N)$ of automorphisms of C_N which induce a group automorphism of C_N^{reg}.

Lemma 6.2.1. *Let $N \in \mathbb{Z}_{\geq 1}$. Then there is an isomorphism*

$$\mu_N \times \{\pm 1\} \cong \text{Aut}\,(C_N) \tag{6.7}$$

given for all $\zeta \in \mu_N$, $u \in \{\pm 1\}$, $x \in \mathbb{C}^*$ and $i \in \mathbb{Z}/N\mathbb{Z}$ by

$$(\zeta, u)(x, i) = \left(\left(x\zeta^i\right)^u, ui\right).$$

Proof. In [DR73][Prop. 1.10] the above formula is used to identify $\text{Aut}(C_N)$ with a semi-direct product $\mu_N \rtimes \{\pm 1\}$. In fact, this semi-direct product is (also) a direct product, since the involution $(1, -1)$ of C_N extending inversion on C_N^{sm} is clearly a central element of $\text{Aut}(C_N)$. □

Lemma 6.2.2. *There is an isomorphism of rank-2 free $(\mathbb{Z}/N\mathbb{Z})$-modules*

$$\begin{aligned} \Phi : (\mathbb{Z}/N\mathbb{Z})^2 &\to C_N^{\text{reg}}[N], \\ (1,0) &\mapsto (0,1), \\ (0,1) &\mapsto (\zeta_N, 0)\,. \end{aligned} \tag{6.8}$$

Proof. Let $(x, i) \in \mathbb{C}^* \times (\mathbb{Z}/N\mathbb{Z}) = C_N^{\text{reg}}$. Since $[N]_{C_N^{\text{reg}}}(x, i) = (x^N, Ni) = (x^N, 0)$, we have $(x, i) \in C_N^{\text{reg}}[N]$ if and only if $x^N = 1$, that is, $x \in \mu_N$.

We conclude that $C_N^{\text{reg}} = \mu_N^\times \times (\mathbb{Z}/N\mathbb{Z})$. Since μ_N is a cyclic group of order N generated by ζ_N, it is then clear that Φ is an isomorphism. □

Lemma 6.2.3. *The following map, defined for $i \in \mathbb{Z}/N\mathbb{Z}$ and $u \in \{\pm 1\}$ by*

$$\begin{aligned} \delta : \operatorname{Aut}(C_N) = \mu_N \times \{\pm 1\} &\to \bar{P}_N, \\ (\zeta_N^i, u) &\mapsto u \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{6.9}$$

is an isomorphism such that for every $\alpha \in \operatorname{Aut}(C_N)$ we have

$$\alpha \circ \Phi = \Phi \circ \delta(\alpha)^t. \tag{6.10}$$

The matrix $\omega_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ defines a $(\mathbb{Z}/N\mathbb{Z})$-valued pairing

$$\begin{aligned} (\mathbb{Z}/N\mathbb{Z})^2 \times (\mathbb{Z}/N\mathbb{Z})^2 &\to \mathbb{Z}/N\mathbb{Z}, \\ ((a,b),(c,d)) &\mapsto (a,b)\omega_1(c,d)^t = bd - ac. \end{aligned}$$

Using the isomorphisms $\Phi : (\mathbb{Z}/N\mathbb{Z})^2 \to C_N^{\text{reg}}[N]$ and $\mathbb{Z}/N\mathbb{Z} \to \mu_N$, $a \mapsto \zeta_N^a$ we carry ω_1 over to a μ_N-valued pairing on $e_N : C_N^{\text{reg}}[N] \times C_N^{\text{reg}}[N] \to \mu_N$.

Definition 6.2.4. The *Weil e_N-pairing* on the Néron N-gon C_N is the map

$$\begin{aligned} e_N : C_N^{\text{reg}}[N] \times C_N^{\text{reg}}[N] &\to \mu_N, \\ ((\zeta, i), (\eta, j)) &\mapsto \zeta^j / \eta^i. \end{aligned} \tag{6.11}$$

By construction the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{Z}/N\mathbb{Z})^2 \times (\mathbb{Z}/N\mathbb{Z})^2 & \xrightarrow{\Phi \times \Phi} & C_N^{\text{reg}} \times C_N^{\text{reg}}[N] \\ \downarrow{\scriptstyle \omega_1} & & \downarrow{\scriptstyle e_N} \\ (\mathbb{Z}/N\mathbb{Z}) & \xrightarrow{a \mapsto \zeta_N^a} & \mu_N. \end{array}$$

Lemma 6.2.5. *The Weil e_N-pairing is a natural nondegenerate alternating bilinear pairing on $C_N^{\text{reg}}[N]$, that is, for all $x, y, z \in C_N^{\text{reg}}[N]$ and $\alpha \in \operatorname{Aut}(C_N)$ we have*

- $e_N(x, x) = 1$ (alternating);
- $e_N(x, m(y, z)) = e_N(x, y) e_N(x, z)$ *and* $e_N(m(x, y), z) = e_N(x, z) e_N(y, z)$ (bilinear);
- *if* $e_N(x, w) = 1$ *for every* $w \in C_N^{\text{reg}}[N]$ *then* $x = (1, 1)$ (nondegenerate);
- $e_N(\alpha(x), \alpha(y)) = e_N(x, y)$ (natural).

Proof. Properties (1) and (2) are straightforward from the definition. For property (3), simply note that if $x = (\zeta, i)$, then $e_N(x, (\zeta_N, 0)) = \zeta_N^{-i}$ and $e_N(x, (0, 1)) = \zeta$. To show property (4), it suffices to show that $\operatorname{Aut}(C_N) = \mu_N \times \{\pm 1\}$ acts on $C_N^{\text{reg}}[N]$ via automorphisms of determinant 1. This is clear from Lemma 6.2.3, since visibly we have $\overline{P}_N \subset \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

□

6.3 Contractions

Let h and N be positive integers, with h dividing N. In Lemma 6.3.2 will construct a surjective holomorphic map $u_{h,N}\colon C_N \to C_h$, which restricts to an isomorphism $u_{h,N}^{-1}(C_h^{\mathrm{reg}}) = \mathbb{C}^* \times (\mathbb{Z}/N\mathbb{Z})[h] \to C_N^{\mathrm{reg}} = \mathbb{C}^* \times (\mathbb{Z}/h\mathbb{Z})$ given by $(x, (N/h)i) \mapsto (x, i)$ for all $x \in \mathbb{C}^*$ and $i \in \mathbb{Z}/h\mathbb{Z}$, and contracts all irreducible components of C_N not meeting $\mathbb{C}^* \times (\mathbb{Z}/N\mathbb{Z})[h]$.

Definition 6.3.1 (component group)**.** Given a $\mathbb{C}$-analytic space X, we denote $\pi_0(X)$ the set of connected components of X. There is a natural map $\nu = \nu_X : X \to \pi_0(X)$ sending a point $x \in X$ to the connected component of x.

If G is a complex Lie group, then $\pi_0(G)$ has a unique group structure with respect to which $\nu_G : G \to \pi_0(G)$ is a homomorphism. We call $\pi_0(G)$ the *component group* of G.

Now let $N \in \mathbb{Z}_{\geq 1}$. The regular locus of the Néron N-gon C_N is the complex Lie group $C_N^{\mathrm{reg}} = \mathbb{C}^* \times \mathbb{Z}/N\mathbb{Z}$, which has component group $\pi_0\left(C_N^{\mathrm{reg}}\right) = \mathbb{Z}/N\mathbb{Z}$.

In the remainder of this section we fix a positive divisor h of N. Then we have an isomorphism $[h/N] : (\mathbb{Z}/N\mathbb{Z})[h] = (N/h)\mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/h\mathbb{Z}$ given for all $i \in \mathbb{Z}$ by $[h/N]((N/h)i + N\mathbb{Z}) = i + h\mathbb{Z}$.

Lemma 6.3.2. *Let h be a positive divisor of N. Then there exists a unique holomorphic map*

$$u = u_{h,N} : C_N \to C_h \tag{6.12}$$

that restricts to an isomorphism

$$\begin{array}{ccc} u^{-1}(C_h^{\mathrm{reg}}) & = & \mathbb{C}^* \times (\mathbb{Z}/N\mathbb{Z})[h] \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathbb{C}^*} \times [h/N]} \\ C_h^{\mathrm{reg}} & = & \mathbb{C}^* \times (\mathbb{Z}/h\mathbb{Z}). \end{array} \tag{6.13}$$

The morphism u contracts onto one point each irreducible component of C_N which is the closure of a connected component of C_N^{reg} whose order in $\pi_0\left(C_N^{\mathrm{reg}}\right) = \mathbb{Z}/N\mathbb{Z}$ does not divide h. Moreover, the following diagram commutes:

$$\begin{array}{ccc} u^{-1}(C_h^{\mathrm{reg}}) \times C_N & \longrightarrow & C_N \\ \big\downarrow{\scriptstyle u\times u} & & \big\downarrow{\scriptstyle u} \\ C_h^{\mathrm{reg}} \times C_h & \longrightarrow & C_h. \end{array}$$

Proof. See [DR73][Prop. IV.1.3]. □

Next, we compare the automorphism groups of C_N and C_N via the contraction map $u_{h,N} : C_n \to C_h$.

Lemma 6.3.3. *For every automorphism $\alpha \in \mathrm{Aut}\,(C_N) = \mu_N \times \{\pm 1\}$ there exists a unique automorphism $\beta_{h,N}(\alpha) \in \mathrm{Aut}\,(C_h) = \mu_h \times \{\pm 1\}$ with the property that*

$$u_{h,N} \circ \alpha = \beta_{h,N}(\alpha) \circ u_{h,N}. \tag{6.14}$$

If $\alpha = (\zeta, u) \in \mu_N \times \{\pm 1\}$ *then* $\beta_{N/h}(\alpha) = \left(\zeta^{N/h}, u\right) \in \mu_h \times \{\pm 1\}$, *and this defines a surjective group homomorphism*

$$\beta_{h,N} : \mathrm{Aut}\,(C_N) \to \mathrm{Aut}\,(C_h)\,. \tag{6.15}$$

Proof. We have to show that if $\alpha = (\zeta, u) \in \mu_N \times \{\pm 1\}$ with $\zeta \in \mu_N$ and $u \in \{\pm 1\}$, then $\beta = \left(\zeta^{N/h}, u\right) \in \mu_h \times \{\pm 1\}$ satisfies $\beta \circ u_{h,N} = u_{h,N} \circ \alpha$. It suffices to consider the cases $\alpha = -1$ and $\alpha = \zeta \in \mu_N$. For the first case, note that $u_{h,N}$ intertwines the involutions on C_N and C_h defined by -1.

For the second case, let $\zeta \in \mu_N$, which preserves the irreducible components of C_N, and similarly $\zeta^{N/h}$ preserves the singular points of C_h. This shows that $\beta \circ u_{h,N} = u_{h,N} \circ \alpha$ on each irreducible components of C_N that is contracted onto a singular point of C_h.

Now consider an irreducible component $\mathbb{CP}^1 \times \{(N/h)i + N\mathbb{Z}\}$ of C_N with is mapped isomorphically by $u_{h,N}$ to the irreducible component $\mathbb{CP}^1 \times \{i + h\mathbb{Z}\}$ of C_h via the identity map on $\mathbb{CP}^1$. The action of ζ on $\mathbb{P}^1 \times \{iN/h + N\mathbb{Z}\} \subset C_N$ is via multiplication by $\zeta^{iN/h}$, while the action of $\zeta^{N/h}$ on $\mathbb{P}^1 \times \{i + h\mathbb{Z}\}$ is via multiplication by $\left(\zeta^{N/h}\right)^i = \zeta^{iN/h}$. This shows that $\beta \circ u_{h,N} = u_{h,N} \circ \alpha$ on each irreducible component of C_N that is not contracted by $u_{h,N}$, hence on all of C_N.

Finally, since $[N/h] : \mu_N \to \mu_h$, $\zeta \mapsto \zeta^{N/h}$ is a surjective homomorphism, so is $\beta_{h,N} : \mu_N \times \{\pm 1\} \to \mu_h \times \{\pm 1\}$, $(\zeta, u) \mapsto \left(\zeta^{N/h}, u\right)$. □

6.4 Modular interpretation of $\mathrm{Cusps}(N), \mathrm{Cusps}_1(N)$ and $\mathrm{Cusps}_0(N)$

This section gives a modular interpretation for the sets of cusps of the modular curves $X(N)$, $X_1(N)$ and $X_0(N)$ introduced by Deligne and Rapoport in their seminal paper [DR73].

Definition 6.4.1. Let h and N be positive integers.

(1) A *level-N structure on C_h* is a group isomorphism $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \to C_h^{\mathrm{reg}}[N]$. The set of level-$N$ structures on C_h is denoted $[N\text{–str}]_{C_h}$.

(2) A *point of (exact) order N on C_h* is an element P of the group C_h^{reg} having order N. The set of points of exact order N on C_h is denoted $[N\text{–pt}]_{C_h}$.

(3) A *cyclic subgroup of order N on C_h* is a cyclic subgroup G of the group C_h^{reg} having order N. The set of cyclic subgroups of order N on C_h is denoted $[N\text{–grp}]_{C_h}$.

Example 6.4.2. Let N and h be positive integers.

(1) The *standard level-N structure* on C_N is the isomorphism Φ that we defined in Lemma 6.2.2, given by $\Phi(a, b) = \left(\zeta_N^b, a\right)$ for all $a, b \in \mathbb{Z}/N\mathbb{Z}$.

(2) Suppose that $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \to C_h^{\mathrm{reg}}[N]$ is a level-N structure. Then image $\phi(0, 1) \in C_h^{\mathrm{reg}}[N]$ of the vector $(0, 1) \in (\mathbb{Z}/N\mathbb{Z})^2$ is a point of exact order N of C_h. This defines a forgetful map

$$[N\text{–lvl}]_{C_h} \to [N\text{–pt}]_{C_h}, \\ \phi \mapsto \phi(0, 1).$$

(3) Suppose that $P \in C_h^{\mathrm{reg}}[N]$ is a point of exact order N. Then P generates a a cyclic subgroup $(\mathbb{Z}/N\mathbb{Z}) \cdot P := \{[i](P) : i \in \mathbb{Z}/N\mathbb{Z}\}$ of order N on C_h. This defines a forgetful map

$$[N\text{–pt}]_{C_h} \to [N\text{–grp}]_{C_h},$$
$$P \mapsto (\mathbb{Z}/N\mathbb{Z}) \cdot P.$$

Remark 6.4.3. Let $h \in \mathbb{Z}_{\geq 1}$. From the canonical isomorphism $C_h^{\mathrm{reg}}[N] = \mu_N \times (\mathbb{Z}/h\mathbb{Z})$, we read off that the set $[N\text{–lvl}]_{C_h}$ is nonempty if and only if $N \mid h$. On the other hand, for every value of h there exists on C_h a point $(\zeta_N, 0 + h\mathbb{Z})$ of order N as well as a cyclic subgroup $\mu_N \times \{0 + h\mathbb{Z}\}$ of order N, both contained in the identity component $\mathbb{C}^* \times \{0 + h\mathbb{Z}\} \cong \mathbb{C}^*$ of $C_h^{\mathrm{reg}} = \mathbb{C}^* \times \mathbb{Z}/h\mathbb{Z}$.

Definition 6.4.4. Let $\zeta \in \mu_N^\times$. We say a level-N structure ϕ on C_N has Weil pairing ζ if $e_N(\phi(0,1), \phi(1,0)) = \zeta$. We let $[N\text{–str}]_{C_N}^{\zeta} \subset [N\text{–str}]_{C_N}$ be the subset consisting of level-N structures that have Weil pairing ζ.

The Weil e_N-pairing provides a decomposition $[N\text{–str}]_{C_N} = \bigsqcup_{\eta \in \mu_N^\times} [N\text{–str}]_{C_N}^{\eta}$.

Example 6.4.5. The standard level-N structure Φ on C_N from Example 6.4.2(1) has Weil pairing $e_N(\Phi(0,1), \Phi(1,0)) = e_N\left((\zeta_N, 0), (0,1)\right) = \zeta_N$.

Definition 6.4.6. We define a left action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on $[\mathrm{lvl}-\mathrm{N}]_{C_N}$ by

$$\gamma\phi = \phi \circ \gamma^t.$$

Note that if ϕ has Weil pairing ζ, then $\gamma\phi = \phi \circ \gamma^t$ has Weil pairing $\zeta^{\det(\gamma)}$. Thus the map $[N\text{–str}] \to \mu_N^\times$ sending a level- N structure ϕ to $e_N(\phi(0,1), \phi(1,0))$ is equivariant for $\det : \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to (\mathbb{Z}/N\mathbb{Z})^\times$. In particular, the action of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ preserves the decomposition $[\mathrm{lvl}-\mathrm{N}]_{C_N} = \bigsqcup_{\zeta \in \mu_N^\times} [\mathrm{lvl}-\mathrm{N}]_{C_N}^{\zeta}$.

Lemma 6.4.7. *There exists a commutative diagram in which the horizontal maps are bijective*

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & [N\text{–str}]^{\zeta_N} \\ \downarrow & & \downarrow{\scriptstyle \phi \mapsto \phi(0,1)} \\ \overline{\Gamma}_1(N) \backslash \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & [N\text{–pt}] \\ \downarrow & & \downarrow{\scriptstyle P \mapsto (\mathbb{Z}/N\mathbb{Z}) \cdot P} \\ \overline{\Gamma}_0(N) \backslash \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & [N\text{–grp}]\,. \end{array} \tag{6.16}$$

Proof. The top horizontal map is defined by sending $\gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\gamma\Phi = \Phi \circ \gamma$; it is an isomorphism because $[N\text{–str}]_{C_N}^{\zeta_N}$ is a simply transitive $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$-set.

Since the automorphism group of $(\mathbb{Z}/N\mathbb{Z})^2$ is $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we see that the map $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to [\mathrm{lvl}-\mathrm{N}]_{C_N}$, $\gamma \mapsto \Phi \circ \gamma^t$ is bijective. Since the level-N structure $\gamma\Phi$ has Weil pairing $\zeta_N^{\det(\gamma)}$, it restricts to a bijection $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to [\mathrm{lvl}-\mathrm{N}]_{C_N}^{\zeta_N}$.

The stabilizer of the vector $(0,1) \in (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ for the $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$-action is given by

$$\bar{\Gamma}_1(N) := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}/N\mathbb{Z} \right\} = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) : \gamma^t(1,0) = (1,0) \right\}.$$

Hence for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ we have, using that Φ is injective,

$$\Phi\left(\gamma_1^t(0,1)\right) = \Phi\left(\gamma_2^t(0,1)\right) \Longleftrightarrow \gamma_1^t(0,1) = \gamma_2^t(0,1) \Longleftrightarrow \bar{\Gamma}_1(N)\gamma_1 = \bar{\Gamma}_1(N)\gamma_2.$$

This equivalence says precisely that the top isomorphism $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to [\mathrm{lvl} - \mathrm{N}]_{C_N}^{\zeta_N}$, $\gamma \mapsto \gamma\Phi = \Phi \circ \gamma^t$, induces a well-defined injection $\bar{\Gamma}_1(N)\backslash\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to [\mathrm{pt} - \mathrm{N}]_{C_N}$, $\bar{\Gamma}_1(N)\gamma \mapsto \Phi\left(\gamma^t(0,1)\right)$. It is surjective as well, because the top right map is surjective owing to the nondegeneracy of the Weil e_N-pairing, i.e. any point of exact order N on C_N can be extended to a level-N on C_N structure with prescribed Weil pairing $\zeta \in \mu_N^\times$.

Similarly, the top horizontal isomorphism is seen to descend to a well-defined injective map $\bar{\Gamma}_0(N)\backslash\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to [N\text{–pt}]_{C_N}, \bar{\Gamma}_0(N)\gamma \mapsto \Phi\left(\gamma^t(\{0\} \times (\mathbb{Z}/N\mathbb{Z}))\right)$ because the stabilizer of the subgroup $\{0\} \times \mathbb{Z}/N\mathbb{Z} \subset (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ is given by

$$\bar{\Gamma}_0(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) : \gamma^t(\{0\} \times \mathbb{Z}/N\mathbb{Z}) = \{0\} \times \mathbb{Z}/N\mathbb{Z} \right\}.$$

To check it is surjective, we have to check the bottom right map is surjective, which is clear since by definition a *cyclic* subgroup of order N on C_N^{reg} is generated by an element that has order N. □

There is an evident action of $\mathrm{Aut}(C_h)$ on each of the sets $[\mathrm{lvl} - \mathrm{N}]_{C_h}$, $[\mathrm{pt} - \mathrm{N}]_{C_h}$ and $[\mathrm{grp} - \mathrm{N}]_{C_h}$.

Lemma 6.4.8. *There exists a commutative diagram in which the horizontal maps are bijective*

$$\begin{array}{ccc} \mathrm{Cusps}(N) = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\overline{P}_N & \longrightarrow & \mathrm{Aut}(C_N)\backslash\, [N\text{–}str]_{C_N}^{\zeta_N} \\ \downarrow & & \downarrow {\scriptstyle [\phi] \mapsto [\phi(0,1)]} \\ \mathrm{Cusps}_1(N) = \overline{\Gamma}_1\backslash\, \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\overline{P}_N & \longrightarrow & \mathrm{Aut}(C_N)\backslash\, [N - \mathrm{pt}]_{C_N} \\ \downarrow & & \downarrow {\scriptstyle [P] \mapsto [(\mathbb{Z}/N\mathbb{Z})\cdot P]} \\ \mathrm{Cusps}_0(N) = \overline{\Gamma}_0(N)\backslash\, \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\overline{P}_N & \longrightarrow & \mathrm{Aut}(C_N)\backslash\, [N - \mathrm{grp}]_{C_N}. \end{array} \tag{6.17}$$

Proof. The diagram results from diagram (6.16) whose horizontal bijections intertwine the right action of $\overline{P_N}$ and left action of $\mathrm{Aut}\,(C_N)$ through equation (6.10). □

Theorem 6.4.9. *For each $N \in \mathbb{Z}_{\geq 1}$, we have the following modular interpretation of the set of cusps* $\mathrm{Cusps}(N)$ *of the modular curve $X(N)$:*

$$\mathrm{Cusps}(N) = \{\textit{Néron } N\textit{-gons with a level-}N \textit{ structure with Weil pairing } \zeta_N\}/\cong. \tag{6.18}$$

Proof. The bijection is provided by the top horizontal map in diagram (6.17). □

Finally we give, following [DR73][Chapitre IV, §4], a modular description of the cusps of the modular curves $X_1(N)$ and $X_0(N)$ in terms of so-called ample points of order N resp. ample cyclic subgroups of order N on Néron polygons.

Definition 6.4.10. Consider a Néron polygon C_h for some $h \in \mathbb{Z}_{\geq 1}$.

(1) Let G be a cyclic subgroup of order N on C_h. We say that G is *ample* if it meets each connected component of C_h^{reg}. Equivalently we may demand that $(\mathbb{Z}/N\mathbb{Z}) \cong G \to \pi_0\left(C_h^{\mathrm{reg}}\right) \cong \mathbb{Z}/h\mathbb{Z}$ be a surjective homomorphism. We denote the set of ample points of order N on C_h by $[N\text{–pt}]_{C_h}^{\mathrm{ample}}$.

(2) We say a point P of exact order N on a Néron polygon C_h is *ample* if the cyclic subgroup $(\mathbb{Z}/N\mathbb{Z}) \cdot P$ of order N generated by P is ample in the sense of part (1) of this definition. It is equivalent to ask that the image of P in $\pi_0(C_h^{\mathrm{reg}})$ have exact order N. We denote the set of ample cyclic subgroups of order N on C_h by $[N\text{–grp}]_{C_h}^{\mathrm{ample}}$.

Remark 6.4.11. Let $h \in \mathbb{Z}_{\geq 1}$. There exists an ample point of exact order N (resp. an ample cyclic subgroup of order N) on the Néron polygon C_h if and only if h divides N.

Theorem 6.4.12. *We have the following modular description of the sets* $\mathrm{Cusps}_1(N)$ *resp.* $\mathrm{Cusps}_0(N)$ *of cusps of the modular curves* $X_1(N)$ *resp.* $X_0(N)$*:*

$$\mathrm{Cusps}_1(N) = \{\textit{Néron polygons with an ample point of exact order } N\}/\cong,$$
$$\mathrm{Cusps}_0(N) = \{\textit{Néron polygons with an ample cyclic subgroup of order } N\}/\cong.$$

Proof. Let $P \in [N\text{–pt}]_{C_N}$. Let h be the order of P in $\pi_0(C_N^{\mathrm{reg}}) = \mathbb{Z}/N\mathbb{Z}$. We have that P is ample if and only if $h = N$. If $h < N$ then taking the image of P under the isomorphism of complex Lie groups (6.13) gives an ample point $P_1 := u_{h,N}(P) \in C_h^{\mathrm{reg}}$ of order N on C_h.

Conversely, let h be a positive divisor of N and consider an ample point $P_1 \in C_h^{\mathrm{reg}}$ of order N on C_h. Then P_1 lifts along $u_{h,N}$ to a point P of order N on C_N, whose order in $\pi_0(C_N^{\mathrm{reg}})$ is h. The correspondence $P \leftrightarrow P_1$ sets up a bijection between the set of points of order N on C_N whose order in $\pi_0(C_N^{\mathrm{reg}})$ is h, and the set of ample points of order N on C_h.

Assembling these bijections for all positive divisors h of N gives the bijection

$$[N\text{–pt}]_{C_N} = \bigsqcup_h [N\text{–pt}]_{C_h}^{\mathrm{ample}}.$$

Lemma 6.3.3 asserts that each automorphism of C_h lifts along $u_{h,N}$ to an automorphism of C_N. It follows that the above bijection descends to a bijection

$$\mathrm{Aut}(C_N)\backslash[N\text{–pt}]_{C_N} \cong \bigsqcup_h \mathrm{Aut}(C_h)\backslash[N\text{–pt}]_{C_h}^{\mathrm{ample}}.$$

The composition of the middle horizontal bijection in (6.17) with this bijection gives the desired modular description of $\mathrm{Cusps}_1(N)$.

Moving on to $\mathrm{Cusps}_0(N)$, the same reasoning for cyclic subgroups of order N as was carried out for points of order N yields a bijection

$$\mathrm{Aut}(C_N)\backslash[N\text{–grp}]_{C_N} \cong \bigsqcup_h \mathrm{Aut}(C_h)\backslash[N\text{–grp}]^{\mathrm{ample}}_{C_h}.$$

The composition of the bottom horizontal bijection in (6.17) with this bijection gives the desired modular description of $\mathrm{Cusps}_0(N)$. □

6.5 Generalized elliptic curves

In this section we will define *generalized elliptic curves* where the generalization consists in allowing the singular Néron polygons from Section 6.1 as fibres in addition to smooth elliptic curves. In keeping with the functorial spirit of this thesis, we discuss homomorphisms and pullbacks.

Recall from Theorem 2.2.9 that a flat holomorphic map $f : X \to Y$ of complex spaces is submersive at a point $x \in X$ if and only if the fibre $X_{f(x)} = f^{-1}(f(x))$ is a complex manifold at x. We will write $X^{\mathrm{sm}} = \bigcup_{y\in Y} f^{-1}(y)^{\mathrm{reg}}$ for the open subset of X on which f is a submersion; its complement is an analytic (closed) subset denoted $X^{\mathrm{nsm}} = \bigcup_{y\in Y} f^{-1}(y)^{\mathrm{sing}}$.

Definition 6.5.1. (1) Let M be a complex manifold. A *generalized elliptic curve* over M consists of a proper, flat morphism with reduced fibres $f : E \to M$, where E is a complex space, together with a structure of M-group on E^{sm} that extends to an action of E^{sm} on E over M, such that the fibre of these data over a point $m \in M$ are either an elliptic curve or a Néron polygon C_N for some $N \in \mathbb{Z}_{\geq 1}$.

(2) Let $h : M' \to M$ be a morphism in $\mathbb{C}\mathbf{Man}$, and let $f' : E' \to M'$ be a second generalized elliptic curve. A *homomorphism of generalized elliptic curves covering* h is a holomorphic map $\tilde{h} : E' \to E$ covering h such that $\tilde{h}\,(E'^{\mathrm{sm}}) \subset E^{\mathrm{sm}}$ and the following diagram commutes:

$$\begin{array}{ccc} E'^{\,\mathrm{sm}} \times_{M'} E' & \longrightarrow & E' \\ \big\downarrow{\scriptstyle \tilde{h}\times\tilde{h}} & & \big\downarrow{\scriptstyle \tilde{h}} \\ E^{\mathrm{sm}} \times_M E & \longrightarrow & E. \end{array}$$

We call $\tilde{h}$ *cartesian* if it induces a biholomorphism $E' \xrightarrow{\sim} E \times_M M'$.

Example 6.5.2. (1) Any relative elliptic curve $f : E \to M$ is a generalized elliptic curve.

(2) For every $N \in \mathbb{Z}_{\geq 1}$ and complex manifold M we have that $C_N \times M \to M$ is a generalized elliptic curve, called the *constant N-gon over M*.

Lemma 6.5.3. *Let $h : M' \to M$ be a morphism in $\mathbb{C}\mathbf{Man}$ and let $f : E \to M$ be a generalized elliptic curve. Then the fibre product $f' : E' := E \times_M M' \to M'$ is naturally a generalized elliptic curve over M'.*

Proof. The three properties of a morphism of complex spaces of being proper, of being flat and of having reduced fibres are all stable under pullback. For every point $m' \in M'$ we have that $E'_{m'} \cong E_{h(m)}$. In particular this shows that $E'^{\text{sm}} = E^{\text{sm}} \times_M M'$. The group law $E'^{\text{sm}} \times_{M'} E'^{\text{sm}} \to E'^{\text{sm}}$ and action map $E'^{\text{sm}} \times_{M'} E' \to E'$ are defined as the base change of $E^{\text{sm}} \times_M E^{\text{sm}} \to E^{\text{sm}}$ resp. $E^{\text{sm}} \times_M E \to E$ along $h : M' \to M$. Then clearly the fibres of $f' : E' \to M'$ are elliptic curves or Néron polygons, since the fibres of $f : E \to M$ are such. □

We have defined level-N structures, points of exact order N and cyclic subgroups of order N in Definitions 5.6.1, 5.7.1 resp. 5.8.1 for relative elliptic curves and in Definition 6.4.1 for Néron polygons. We conclude this section by giving the common generalization of these definitions to generalized elliptic curves.

Definition 6.5.4. Let $f : E \to M$ be a generalized elliptic curve and let $N \in \mathbb{Z}_{\geq 1}$.

(1) A *level-N structure* on the generalized elliptic curve E/M is an isomorphism of M-groups $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \times M \to E^{\text{sm}}[N]$.
(2) A *point of exact order* N on E/M is a section $Q : M \to E^{\text{sm}}$ of f such that for each point $m \in M$ the element $Q_m := Q(m)$ of the group $(E_m)^{\text{reg}}$ have order N.
(3) A *cyclic subgroup of order* N on E/M is an M-subgroup $G \subset E^{\text{sm}}$ such that for each point $m \in M$ the subgroup $G_m := G \cap E_m$ of the group $(E_m)^{\text{reg}}$ have order N.

We will say a point Q of exact order N resp. a cyclic subgroup G of order N on a generalized elliptic curve E/M is *ample* if for each point $m \in M$ such that the fibre E_m is a singular Néron polygon, the point Q_m resp. cyclic subgroup G_m is ample in the sense of Definition 6.4.10.

Definition 6.5.5. Let $f : E \to M$ be a generalized elliptic curve and let $N \in \mathbb{Z}_{\geq 1}$.

(1) We will say a cyclic subgroup $G \subset E^{\text{sm}}$ of order N on E/M is *ample* if for every $m \in M$ it intersects each connected component of the fibre $(E^{\text{sm}})_m$.
(2) We will say a point Q of exact order N on E/M is *ample* if the cyclic subgroup $G := (\mathbb{Z}/N\mathbb{Z}) \cdot P = \bigcup_{i=0}^{N-1} [i]_E(P)$ it generates is ample in the sense of part (1) of this definition.

Chapter 7

Toric geometry

Toric geometry is a fruitful source of examples in algebraic geometry, a thesis this thesis hopes to support. A *toric variety* is a $\mathbb{C}$-variety which contains an algebraic torus as an open dense subvariety. One builds a toric variety $F(N, \Sigma)$ from two ingredients: a finite free $\mathbb{Z}$-module N, called the *lattice*, and a set of nicely arranged cones in the finite-dimensional real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, called the *fan*. The torus can be thought of analytically as $N \otimes_{\mathbb{Z}} \mathbb{C}^*$, but will be defined in schematic parlance as $T_N = \operatorname{Spec} \mathbb{C}[M]$, where $M = \operatorname{Hom}(N, \mathbb{Z})$ is the dual lattice of N. The toric variety $F(N, \Sigma)$ is acted upon by T_N, such that the T_N-orbits correspond to the cones in the fan Σ. This *orbit-cone correspondence* is the subject of Section 7.3.

We call a pair (N, Σ) as above a *rational partial polyhedral decomposition*, or RPP decomposition for short. The assignment of a toric variety $F(N, \Sigma)$ to an RPP decomposition is functorial. In fact, Theorem 7.1.8 states that F is an equivalence of categories between the category of RPP decompositions and the category of normal toric varieties and *toric morphisms*.

Section 7.3 discusses fibre products of toric varieties. Its central result is that the fibre product of two toric morphisms is again a toric variety, provided one of the morphisms is flat with reduced fibres.

In Section 7.4 we discuss projective toric morphisms. Section 7.5 provides a major source of examples of these, viz. *star-subdivisions*. Section 7.4 includes a description of (very) ample torus-invariant divisors relative to a toric morphism for technical reasons detailed in Section 10.4.

7.1 Toric varieties from fans

In this section we will give a synopsis of the construction of a toric variety from combinatorial data. We refer to [Ful93] for a comprehensive treatment including proofs.

Definition 7.1.1. (1) A $\mathbb{C}$-*variety* is, for us, an integral separated scheme locally of finite type over $\mathbb{C}$. A *morphism of* $\mathbb{C}$-*varieties* is a morphism of schemes over $\mathbb{C}$. The category of $\mathbb{C}$-varieties is denoted $\mathbb{C}$–**Vrt**. We will write $X \times Y$ for the product of two objects X and Y in $\mathbb{C}$–**Vrt**, even though it is given by the fibre products $X \times_{\operatorname{Spec} \mathbb{C}} Y$ in the category of schemes.

(2) A $\mathbb{C}$-*algebraic group* is a group object in the category of $\mathbb{C}$-varieties. The category of $\mathbb{C}$-algebraic groups and homomorphisms thereof is denoted $\mathbb{C}$–**AlgGrp**.

We now discuss how to construct an algebraic torus T_N from a *lattice* N.

Definition 7.1.2. A *lattice* is a free $\mathbb{Z}$-module of finite rank N. A *homomorphism* between two lattices is an additive map.

We write $\mathbb{G}_m = \mathbb{G}_{m,\mathbb{C}}$ for the multiplicative algebraic group over $\mathbb{C}$, whose $\mathbb{C}$-points are given by $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$. For each integer $k \in \mathbb{Z}$ the map $z \mapsto z^k$ is an endomorphism of $\mathbb{G}_m$, which gives an identification

$$\mathrm{Hom}_{\mathbb{C}\text{–}\mathbf{AlgGrp}}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}.$$

Let N be a lattice. Then its $\mathbb{Z}$-dual $M = N^\vee = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is also a lattice. Write $T_N = \operatorname{Spec} \mathbb{C}[M]$ for the $\mathbb{C}$-algebraic torus attached to N; its $\mathbb{C}$-points are given by $T_N(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. The group of 1-*parameter subgroups of* T_N is

$$\mathrm{Hom}_{\mathbb{C}\text{–}\mathbf{AlgGrp}}(\mathbb{G}_m, T_N) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) = N \tag{7.1}$$

and the *character group* of T_N is given by

$$\mathrm{Hom}_{\mathbb{C}\text{–}\mathbf{AlgGrp}}(T_N, \mathbb{G}_m) = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) = M. \tag{7.2}$$

The assignment $N \mapsto T_N$ gives rise to an equivalence of categories from the category of lattices to the category of algebraic tori over $\mathbb{C}$. In particular, for two lattices N and N' we have

$$\mathrm{Hom}_{\mathbb{C}\text{–}\mathbf{AlgGrp}}(T_N, T_{N'}) = \mathrm{Hom}(N, N'). \tag{7.3}$$

Composition gives a pairing

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}\text{–}\mathbf{AlgGrp}}(T_N, \mathbb{G}_m) \times \mathrm{Hom}_{\mathbb{C}\text{–}\mathbf{AlgGrp}}(\mathbb{G}_m, T_N) &\to \mathrm{Hom}_{\mathbb{C}\text{–}\mathbf{AlgGrp}}(\mathbb{G}_m, \mathbb{G}_m), \\ (g, f) &\mapsto g \circ f\end{aligned} \tag{7.4}$$

which under the above identifications corresponds to the *duality pairing*

$$\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}. \tag{7.5}$$

We now discuss a vast extension of the above construction of algebraic tori, which produces torus embeddings.

Definition 7.1.3. (0) A *torus embedding* $T \subset X$ consists of an algebraic torus T, contained as a Zariski-dense open subset in an algebraic variety X, together with an action $T \times X \to X$ that extends the group law on T, i.e. such that the diagram

$$\begin{array}{ccc} T \times X & \longrightarrow & X \\ \cup & & \cup \\ T \times T & \longrightarrow & T \end{array}$$

commutes.

(1) A *morphism* from one torus embedding $T \subset X$ to a second $T' \subset X'$ is a dominant morphism of algebraic varieties $f : X \to X'$ that restricts to a surjective group homomorphism $f|_T : T \to T'$ and is equivariant with respect to the respective torus actions, in the sense that it renders the following diagram commutative:

$$\begin{array}{ccc} T \times X & \longrightarrow & X \\ {\scriptstyle f|_T \times f}\downarrow & & \downarrow{\scriptstyle f} \\ T' \times X' & \longrightarrow & X'. \end{array}$$

(2) When we say a torus embedding $T \subset X$ has a certain property of $\mathbb{C}$-schemes (i.e. separated, normal, proper, smooth), then it is meant that X has that property.

In toric geometry one constructs a torus embedding starting from certain combinatorial data called a rational partial polyhedral decomposition, or RPP decomposition for short.

For a lattice N, we will call the tensor product $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ the *realification* of N, for it is a real vector space with integral structure N. Given subsets $R \subset \mathbb{R}$ and $\sigma \subset N_{\mathbb{R}}$ we write

$$R \cdot \sigma = \left\{ \sum_{i=1}^{n} r_i v_i : n \in \mathbb{Z}_{\geq 0}, r_i \in R, v_i \in \sigma \text{ for all } 1 \leq i \leq n \right\}.$$

Let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R})$ be the dual vector space, with integral structure M. We have a duality pairing $M_{\mathbb{R}} \otimes_{\mathbb{R}} N_{\mathbb{R}} \to \mathbb{R}$ obtained from the duality pairing $M \otimes N \to \mathbb{Z}$ by the scalar extension $\mathbb{Z} \to \mathbb{R}$. Given $u \in M_{\mathbb{R}}$ we write $u^{\perp} = \{v \in N_{\mathbb{R}} : \langle u, v \rangle = 0\}$; provided $u \neq 0$ this is a hyperplane in $N_{\mathbb{R}}$.

Definition 7.1.4. Let N be a lattice and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ its realification.

(1) A *convex cone* in $N_{\mathbb{R}}$ is a subset $\sigma \subset N_{\mathbb{R}}$ such that $\sigma = \mathbb{R}_{\geq 0} \cdot \sigma$.
(2) A *convex polyhedral cone* is a cone $\mathbb{R}_{\geq 0} \cdot S$ for a finite subset $S \subset N_{\mathbb{R}}$.
(3) A *rational convex polyhedral cone* is a cone $\mathbb{R}_{\geq 0} \cdot S$ for a finite subset $S \subset N$.
(4) The dimension of a cone $\sigma \subset N_{\mathbb{R}}$, denoted $\dim(\sigma)$, is defined to be the dimension of its $\mathbb{R}$-span, i.e. $\dim(\sigma) := \dim(\mathbb{R} \cdot \sigma)$.

For any subset $\sigma \subset N_{\mathbb{R}}$ we define its *dual*

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\} \tag{7.6}$$

which visibly is a convex cone. If σ is a (rational) convex polyhedral cone, then $\sigma^{\vee}$ is again such a cone, and the duality theory of convex analysis states that $(\sigma^{\vee})^{\vee} = \sigma$, see [Ful93][Section 1.2, (9), (1)]. A *face* of a convex polyhedral cone σ is a subset $u^{\perp} \cap \sigma$ for some $u \in \sigma^{\vee}$. If we write $\sigma = \mathbb{R}_{\geq 0} \cdot S$, then the face of σ cut out by u equals $u^{\perp} \cap \sigma = \mathbb{R}_{\geq 0} \cdot T$ with $T = \{v \in S : \langle u, v \rangle = 0\} \subset S$. Therefore a face of a (rational) polyhedral cone is again such a cone. Any intersection of faces of σ is again a face of σ, and a face of a face of σ is a face of σ [Ful93][Section 1.2, statements (3), (4)].

Proposition-Definition 7.1.5. *For a convex polyhedral cone σ, the following are equivalent:*

(1) $\sigma \cap (-\sigma) = \{0\}$*;*
(2) $\mathbb{R} \cdot \sigma^\vee = M_\mathbb{R}$*;*
(3) *there exists* $u \in \sigma^\vee$ *with* $u^\perp \cap \sigma = \{0\}$*.*

A convex polyhedral cone is called strongly convex *if it satisfies any, hence all, of these equivalent conditions.*

Proof. See [Ful93][Section 1.2, (13)] for a proof of these equivalences. □

Let σ be a convex rational polyhedral cone in $N_\mathbb{R}$. Then the monoid $N_\sigma := N \cap \sigma$ is a finitely generated monoid, see [Ful93][§1.2, Proposition 1]. The *unit group* of N_σ is trivial, i.e. one has $N_\sigma^\times := N_\sigma \cap (-N_\sigma) = \{0\}$, if and only if σ is strongly convex, i.e. one has $\sigma \cap (-\sigma) = \{0\}$.

Let σ be a strongly convex rational polyhedral cone. A nonzero $n \in N_\sigma$ is called *indecomposable* if $n = n_1 + n_2$ with $n_1, n_2 \in N_\sigma$ implies that $n_1 = 0$ or $n_2 = 0$. The set of indecomposable elements of N_σ is the smallest set of generators of N_σ.

Definition 7.1.6 (fan)**.** Let N be a lattice. A *fan* in $N_\mathbb{R}$ is a collection Σ of strongly convex rational polyhedral cones in $N_\mathbb{R}$ such that

- if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$;
- if $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a face both of σ and of τ (hence belongs to Σ).

Definition 7.1.7. (0) A *rational partial polyhedral decomposition* or *RPP decomposition* is a pair (N, Σ) consisting of a lattice N and a fan Σ in $N_\mathbb{R}$.

(1) A *morphism* $h : (N, \Sigma) \to (N', \Sigma')$ between RPP decompositions is an additive map $h : N \to N'$ with finite cokernel such that its scalar extension $h_\mathbb{R} : N_\mathbb{R} \to N'_\mathbb{R}$ maps each cone in Σ into some cone of Σ', i.e. for each $\sigma \in \Sigma$ there exists $\sigma' \in \Sigma'$ with $h_\mathbb{R}(\sigma) \subset \sigma'$.

By abuse of notation, if no ambiguity is likely to arise we will drop the subscript from $h_\mathbb{R}$ and simply write $h\colon N_\mathbb{R} \to N'_\mathbb{R}$.

Theorem 7.1.8. *There exists an equivalence of categories from the category of rational partial polyhedral decompositions to the category of normal torus embeddings, written* $(N, \Sigma) \mapsto (T_N \subset F(N, \Sigma))$*, and* $h \mapsto F(h)$*.*

Proof. See [OM78][Theorem 4.1]. □

If Σ is the set of faces of a strongly convex rational polyhedral cone σ in $N_\mathbb{R}$, then we will also write $F(N, \sigma)$ for $F(N, \Sigma)$. Denoting the duality pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ and $M_\sigma = \{\xi \in M : \langle \xi, \sigma \rangle \geq 0\}$, we have $F(N, \Sigma) = \operatorname{Spec} \mathbb{C}\,[M_\sigma]$.

For a general RPP decomposition (N, Σ), the toric variety $F(N, \Sigma)$ is obtained by gluing together the affine toric varieties $F(N, \sigma)$ for the cones $\sigma \in \Sigma$, as we will discuss in the next section.

Definition 7.1.9. A strongly convex rational polyhedral cone σ is called *nonsingular* if there exists a $\mathbb{Z}$-basis B of N and a subset $S \subset B$ such that $\sigma = \mathbb{R}_{\geq 0} \cdot S$. We call σ *singular* if it is not nonsingular.

Theorem 7.1.10. *For an RPP decomposition (N, Σ) we have the following equivalences:*

(1) *$F(N, \Sigma)$ is of finite type over $\mathbb{C}$ if and only if Σ is a finite set;*
(2) *$F(N, \Sigma)$ is smooth over $\mathbb{C}$ if and only if each cone $\sigma \in \Sigma$ is nonsingular;*
(3) *$F(N, \Sigma)$ is proper over $\mathbb{C}$ if and only if $\bigcup_{\sigma\in\Sigma} \sigma = N_{\mathbb{R}}$;*
(4) *$F(N, \Sigma)$ is affine over $\mathbb{C}$ if and only if Σ consists of the faces of a single cone in $N_{\mathbb{R}}$.*

Proof. Parts (1), (2), (3) and (4) are proved as Thm. 4.1, Thm. 4.3, Thm. 4.5 resp. Thm 4.2 in [OM78]. □

Definition 7.1.11. We say a morphism $h : (N, \Sigma) \to (N', \Sigma')$ of RPP decompositions is

(1) *proper* if for every $\sigma' \in \Sigma'$ the set $\{\sigma \in \Sigma : h(\sigma) \subset \sigma'\}$ is finite and its union is $h^{-1}(\sigma')$;
(2) *weakly semistable* if for every $\sigma \in \Sigma$ there exists $\sigma' \in \Sigma'$ with $h(N\cap\sigma) = N'\cap\sigma'$.

Theorem 7.1.12. *For a morphism $h : (N, \Sigma) \to (N', \Sigma')$ of RPP decompositions, we have the following equivalences.*

(1) *$F(h)$ is proper if and only if h is proper;*
(2) *$F(h)$ is flat with reduced fibres if and only if h is weakly semistable;*
(3) *$F(h)$ is birational if and only if $h : N \to N'$ is an isomorphism of $\mathbb{Z}$-modules.*

Proof. Part (1) is classical and can be found for example in [OM78][Theorem 4.4].

Part (2) is more recent and proved in [Mol19][Theorem 2.1.4].

For part (3) note that $F(h)$ is birational if and only if it restricts an isomorphism of algebraic tori $T_N \to T_{N'}$. This happens precisely when $h : N \to N'$ is an isomorphism, because the functor $N_0 \mapsto T_{N_0}$ from lattices to $\mathbb{C}$-algebraic tori is fully faithful by (7.3), hence reflects isomorphisms. □

Notation 7.1.13. Let Σ be a fan in $N_{\mathbb{R}}$, and let k be a positive integer. We will denote $\Sigma_{<k} = \{\sigma \in \Sigma : \dim(\sigma) < k\}$ the subset of Σ consisting of those cones that have dimension less than k. Then $(N, \Sigma_{<k})$ is again an RPP decomposition, and the natural map $F(N, \Sigma_{<k}) \to F(N, \Sigma)$ is an open embedding complementary to a closed subset of codimension at least k. For $l \in \mathbb{Z}_{\geq 0}$ we define $\Sigma_{\leq l} = \Sigma_{<l+1}$, and let $\Sigma(l) = \{\sigma \in \Sigma : \dim(\sigma) = l\}$ be the set of l-dimensional cones of Σ.

7.2 The orbit-cone correspondence

In this section, we show that given an RPP decomposition (N, Σ) the cones in the fan Σ are in bijection with the T_N-orbits of the toric variety $F(N, \Sigma)$.

Theorem 7.2.1 (orbit-cone correspondence)**.** (1) *Let (N, Σ) be an RPP decomposition, and let $T_N \subset F(N, \Sigma)$ be the associated torus embedding. Then there exists a bijection*

$$\begin{aligned} &\Sigma \xrightarrow{\sim} T_N \backslash F(N, \Sigma), \\ &\sigma \mapsto O(\sigma) := \operatorname{Hom}\left(M \cap \sigma^{\perp}, \mathbb{C}^*\right). \end{aligned} \tag{7.7}$$

Let n be the rank of N. We have $\dim F(N,\sigma) = n - \dim(\sigma)$. Moreover, the Zariski-closure of $O(\sigma)$ is

$$V(\sigma) := \overline{O(\sigma)} = \bigcup_{\tau \in \Sigma : \sigma \subset \tau} O(\tau). \tag{7.8}$$

(2) *Let $h : (N, \Sigma) \to (N', \Sigma')$ be a homomorphism of RPP decompositions. Then we have that $F(h)(O(\sigma)) = O(\sigma')$ where σ' is the smallest cone of Σ' containing $h(\sigma)$.*

Proof. For part (1) see [CLS11][Theorem 3.2.6].

Before starting the proof of part (2), we make a couple of observations. Because $F(h) : F(N, \Sigma) \to F(N', \Sigma')$ is equivariant with respect to the surjective homomorphism $T_N \to T_{N'}$ the image under $F(h)$ of a T_N-orbit $O(\sigma_0)$ with $\sigma_0 \in \Sigma$ is a $T_{N'}$-orbit $O(\tau_0)$ for a uniquely determined $\tau_0 \in \Sigma'$.

Now consider two cones $\sigma_1, \sigma_2 \in \Sigma$. From part (1) it follows that $\sigma_1 \subset \sigma_2$ if and only if $O(\sigma_2) \subset \overline{O(\sigma_1)} = V(\sigma_1)$. Write $O(\tau_i) = F(h)(O(\sigma_i))$ with $\tau_i \in \Sigma'$ for $i = 1, 2$. If $O(\sigma_2) \subset \overline{O(\sigma_1)}$ then continuity of the map $F(h)$ implies that $O(\tau_2) = F(h)(O(\sigma_2)) \subset F(h)(\overline{O(\sigma_1)}) \subset \overline{F(h)(O(\sigma_1))} = \overline{O(\tau_1)}$. We conclude that $\sigma_1 \subset \sigma_2$ implies $\tau_1 \subset \tau_2$.

Using part (1) it then follows that for $\sigma_0 \in \Sigma$ and $\tau_0 \in \Sigma'$ we have the implication

$$F(h)(O(\sigma_0)) = O(\tau_0) \implies F(h)(F(N, \sigma_0)) \subset F(N, \tau_0). \tag{7.9}$$

We will now begin the actual proof of part (2). First of all since $h(\sigma) \subset \sigma'$ it is clear that $F(h)(O(\sigma)) \subset F(h)(F(N,\sigma)) \subset F(N', \sigma')$. Now $F(N', \sigma')$ decomposes into the orbits $O(\tau')$ with τ' a face of σ'. We conclude that $F(h)(O(\sigma)) = O(\tau)$ for a certain face τ of σ'. It remains to prove that $\tau = \sigma'$.

The implication (7.9) shows there is a toric morphism $F(h) : F(N, \sigma) \to F(N', \tau)$. By full faithfulness of the equivalence F, it must be induced by a morphism of RPP decompositions $h : (N, \sigma) \to (N', \tau')$. In particular we have $h(\sigma) \subset \tau$. Since $\tau \subset \sigma'$ and σ' is chosen to be the smallest cone in Σ' with $h(\sigma) \subset \sigma'$ we conclude that $\tau' = \sigma'$, as was left to be shown. □

Alternatively, $\sigma \in \Sigma'$ as in part (2) of the Orbit-Cone correspondence may be pinned down as the unique cone of Σ' such that $h(\sigma)$ is contained in σ' and meets the relative interior of σ'.

Remark 7.2.2. Any reduced T_N-stable closed subvariety of the toric variety $F(\Sigma)$ has the shape $\bigcup_{\rho \in P} O(\rho)$ for a unique subset $P \subset \Sigma$ having the property that each cone in Σ that contains a cone in P, belongs itself to P.

7.3 Fibre products of toric varieties

The purpose of this section is to show that the fibre product of two toric varieties $F(N_1, \Sigma_1)$ and $F(N_2, \Sigma_2)$ over a third one $F(N_0, \Sigma_0)$ taken in the category of $\mathbb{C}$-varieties may be described as a toric variety $F(N, \Sigma)$ under mild conditions which are satisfied in the applications we have in mind. We start by showing that the category of RPP decompositions has a final objects and (binary) fibre products.

Theorem 7.3.1. (1) *The RPP decomposition* $(\mathbb{Z}^0, \{\{0\}\})$ *is the final object in the category of RPP decompositions. We have that* $F(\mathbb{Z}^0, \{0\}) = \operatorname{Spec} \mathbb{C}$ *is the final object in the category of* $\mathbb{C}$*-varieties.*

(2) *Let* $h_1 : (N_1, \Sigma_1) \to (N_0, \Sigma_0)$ *and* $h_2 : (N_2, \Sigma_2) \to (N_0, \Sigma_0)$ *be morphisms of RPP decompositions. Then* $(N, \Sigma) = (N_1 \times_{N_0} N_2, \Sigma_1 \otimes_{\Sigma^0} \Sigma_2)$*, where*

$$\Sigma_1 \otimes_{\Sigma_0} \Sigma_2 = \{\sigma_1 \times_{\sigma_0} \sigma_2 : \sigma_i \in \Sigma_i \text{ for all } i \in \{0,1,2\} \text{ and } h_1(\sigma_1) \cup h_2(\sigma_2) \subset \sigma_0\} \tag{7.10}$$

is an RPP decomposition. The projection maps $\pi_i : N \to N_i$ $(i = 1, 2)$ *define morphisms of RPP decompositions* $\pi_i : (N, \Sigma) \to (N_i, \Sigma_i)$*. The fibre product of* h_1 *and* h_2 *in the category of RPP decompositions is given by* (N, Σ) *and the morphisms* π_i*.*

Proof. Part (1) is left to the reader. For part (2) the reader may consult [Mol19][Def. 2.2.1] and the discussion following it. □

Theorem 7.3.2. *Let the notation be as in Theorem 7.3.1. If* h_2 *is weakly semistable, then* π_1 *is weakly semistable and the diagram*

$$\begin{array}{ccc} F(N,\Sigma) & \xrightarrow{\pi_2} & F(N_2,\Sigma_2) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ F(h,\Sigma_1) & \xrightarrow{h_2} & F(N_0,\Sigma_0) \end{array}$$

is cartesian in the category of $\mathbb{C}$*-varieties.*

Proof. See [Mol19][Lemma 2.2.6]. □

Corollary 7.3.3. *Let* $n \in \mathbb{Z}_{\geq 0}$ *and let* $h_i : (N_i, \Sigma_i) \to (N_0, \Sigma_0)$ $(1 \leq i \leq n)$ *be morphisms of RPP decompositions with common target. Then the fibre product* (N, Σ) *of* $\{h_i : 1 \leq i \leq n\}$ *exists in the category of RPP decompositions. If at most one of the morphism* h_i *is not weakly semistable, then* $F(N, \Sigma)$ *is the fibre product of the collection of* $F(N_i, \Sigma_i)$ *over* $F(N_0, \Sigma_0)$ *in the category of* $\mathbb{C}$*-varieties.*

Proof. For $n = 0$ this is the content of Theorem 7.3.1(1). For $n = 1$ the statement is trivial. For $n = 2$ this follows from Theorem 7.3.1(2) and Theorem 7.3.2.

Now let $n \geq 3$ and suppose the statement has been proved for $n-1$. By renumbering, we may assume that $h_1, \ldots, h_{n-1}$ are weakly semistable, and then we find that their fibre product (N', Σ') is weakly semistable over (N_0, Σ_0). Furthermore $F(N', \Sigma')$ is the fibre product of $F(N_i, \Sigma_i)$ for $1 \leq i \leq n-1$ over $F(N_0, \Sigma_0)$. Applying the case of binary fibre products to (N', Σ') and (N_n, Σ_n) completes the induction step. The proof is concluded by induction on n. □

Any category that has binary fibre products and a final object also has binary products. Indeed, the fibre product of two objects over the final object is a product for these two objects.

Corollary 7.3.4. *Let* (N_1, Σ_1) *and* (N_2, Σ_2) *be RPP decompositions. Then we have that* $(N_1 \times N_2, \Sigma_1 \otimes \Sigma_2)$ *with* $\Sigma = \Sigma_1 \otimes \Sigma_2 := \{\sigma_1 \times \sigma_2 : \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}$ *is an RPP decomposition, the projection maps* $\pi_i : N_1 \times N_2 \to N_i$ $(i = 1, 2)$ *define morphisms of RPP decompositions* $\pi_i : (N_1 \times N_2, \Sigma_1 \otimes \Sigma_2) \to (N_i, \Sigma_i)$ $(i = 1, 2)$*, and together these form the product of* N_1 *and* N_2 *in the category of RPP decompositions.*

Moreover, the toric variety $F(N_1 \times N_2, \Sigma_1 \otimes \Sigma_2)$ *together with the morphisms* $F(\pi_i) : F(N_1 \times N_2, \Sigma_1 \otimes \Sigma_2) \to F(N_i, \Sigma_i)$ $(i = 1, 2)$ *form the product of* $F(N_1, \Sigma_1)$ *and* $F(N_2, \Sigma_2)$ *in the category of* $\mathbb{C}$*-varieties.*

Proof. The first statement follows on combining Theorem 7.3.1(1) and (2). For the final statement we note that any morphism to the final object in the category of RPP decompositions is weakly semistable, so that Theorem 7.3.2 may be invoked. □

Remark 7.3.5. We remark in passing that fibre products in toric (or logarithmic) geometry form in general a quite subtle subject, see e.g. [Ogu18] for an extensive treatment.

7.4 Toric projective morphisms

In this section we first recall the definitions of prime, Weil and Cartier divisors. Then we describe the torus-invariant divisors on a toric variety in terms of 1-dimensional cones in the corresponding fan.

Definition 7.4.1. Let X be a normal (irreducible) $\mathbb{C}$-variety X, which is locally of finite type but not necessarily quasi-compact.

(1) A *prime divisor* Z on X is an irreducible closed subvariety of X of codimension 1. Let X be a normal irreducible $\mathbb{C}$-variety. The local ring $\mathcal{O}_{X,Z}$ of X along Z is an integrally closed Noetherian local ring of Krull dimension 1, hence a discrete valuation ring, whose field of fractions is the function field $\mathbb{C}(X)$ of X. We denote the normalized valuation corresponding to $\mathcal{O}_{X,Z}$ by $\mathrm{ord}_Z : \mathbb{C}(X)^\times \to \mathbb{Z}$.

(2) A *Weil divisor* on X is a function $Z \mapsto a_Z$ that assigns to every prime divisor Z on X an integer $a_Z \in \mathbb{Z}$ such that each point in X has an open neighborhood U which meets only finitely many prime divisors Z of X with $a_Z \neq 0$; we denote this function by the formal sum $D = \sum_Z a_Z Z$. The set of Weil divisor on X is a group for componentwise addition $(\sum_Z a_Z Z) + (\sum_Z b_Z Z) = \sum_Z (a_Z + b_Z) Z$, called the *divisor group* of X and denoted $\mathrm{Div}(X)$.

(3) Let $f \in \mathbb{C}(X)^\times$. Then $\mathrm{div}(f) = \sum_Z \mathrm{ord}_Z(f) Z$ defines a divisor on X. A divisor having this shape is called a *principal divisor*. A *Cartier divisor* on X is a Weil divisor $D = \sum_Z a_Z Z$ such that X is covered by open subsets U for which $D|_U$ is a principal divisor on U.

Now let $X = F(N, \Sigma)$ be the (normal irreducible) toric variety attached to an RPP decomposition (N, Σ). The open dense torus T_N in X acts on the variety X via automorphisms, and consequently on the set of prime divisors of X and on the divisor group $\mathrm{Div}(X)$ of X. If $\rho \in \Sigma(1)$, i.e. ρ is a 1-dimensional cone of Σ, then via the Orbit-Cone correspondence we see that $D_\rho := V(\rho)$ is a prime divisor on Z.

Each torus-invariant prime divisor on X has this shape, see [CLS11][§4.1]. The torus-invariant Weil divisors on X are therefore given by $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ where $\Sigma(1) \to \mathbb{Z}$, $\rho \mapsto a_\rho$ is an arbitrary map of sets. We denote by u_ρ the unique primitive vector of a ray $\rho \in \Sigma(1)$.

We now give a criterion for a torus-invariant Weil divisor $D = \sum_Z a_Z Z$ to be Cartier in terms of so-called support functions. We call the set $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subset N_\mathbb{R}$ formed as the union of all cones in Σ, the *support* of Σ.

Definition 7.4.2. A *support function* is a function $\phi : |\Sigma| \to \mathbb{R}$ that agrees on each cone $\sigma \in \Sigma$ with some $\mathbb{R}$-linear function $N_\mathbb{R} \to \mathbb{R}$ and satisfies $\phi(M \cap |\Sigma|) \subset \mathbb{Z}$.

Lemma 7.4.3. *Let $D = \sum_\rho a_\rho \rho$ be a torus-invariant Weil divisor on X. Then D is Cartier if and only if there exists a support function $\phi : |\Sigma| \to \mathbb{R}$ such that $\phi(u_\rho) = -a_\rho$ for every $\rho \in \Sigma(1)$.*

Proof. See [CLS11][Theorem 4.2.12] □

Since each cone $\sigma \in \Sigma$ is the convex hull of the rays $\rho \in \Sigma(1)$ contained in σ, a Cartier divisor D has a unique support function, which we denote ϕ_D.

Definition 7.4.4. Let $f : X \to Y$ be a proper morphism of $\mathbb{C}$-varieties, and let $\mathcal{L}$ be a line bundle on X. We say $\mathcal{L}$ is f-base point free if the adjunction map $f^* f_* \mathcal{L} \to \mathcal{L}$ is surjective. If $\mathcal{L}$ is f-base point free, then we we write $\phi_\mathcal{L} : X \to \mathbb{P}_Y(f_*\mathcal{L})$ for the Y-morphism attached to the epimorphism $f^* f_* \mathcal{L} \to \mathcal{L}$. We say $\mathcal{L}$ is *f-very ample* if $\phi_\mathcal{L}$ is a closed embedding. We say $\mathcal{L}$ is *f-ample* if for every quasi-compact open subset U of Y there exists a positive integer $k > 0$ such that $\mathcal{L}^{\otimes k}|_{f^{-1}(U)}$ is very ample relative to $f : f^{-1}(U) \to U$.

In case $Y = \operatorname{Spec} \mathbb{C}$, we will say that $\mathcal{L}$ is base-point-free (resp. very ample, resp. ample) if it is f-base-point-free (resp. f-very ample, resp. f-ample).

Let (N, Σ) be an RPP decomposition. Let $n = \operatorname{rk}_\mathbb{Z} N = \dim_\mathbb{R} N_\mathbb{R}$. A subset S of $N_\mathbb{R}$ is called *convex* if $tu + (1-t)v \in S$ for all $u, v \in S$ and $t \in [0, 1]$. We say Σ has *full-dimensional convex support* if $|\Sigma|$ is a convex subset of $N_\mathbb{R}$ and each maximal cone $\sigma \in \Sigma$ has dimension $\dim(\sigma) = n$.

Henceforth assume that $|\Sigma|$ has full-dimensional convex support. A *wall* of Σ is a cone $\tau \in \Sigma(n-1)$ such that $\tau = \sigma \cap \sigma'$ for any two distinct $\sigma, \sigma' \in \Sigma(n)$.

Definition 7.4.5. Let S be a nonempty convex subset of $N_\mathbb{R}$. A function $\phi : S \to \mathbb{R}$ is called *convex* if for all $u, v \in S$ and $t \in [0, 1]$ we have the inequality

$$t\phi(u) + (1-t)\phi(v) \leq \phi(tu + (1-t)v). \tag{7.11}$$

If in addition, the above inequality is an equation only if u and v belong to one and the same cone of Σ, then we say ϕ is *strictly convex*.

Theorem 7.4.6. *Let D be a torus-invariant Cartier divisor on a proper toric variety X. Then the following statements are equivalent:*

(1) *the support function $\phi_D : |\Sigma| \to \mathbb{R}$ is convex;*
(2) *the line bundle $\mathcal{O}_X(D)$ is base-point free.*

Proof. See [CLS11][Theorem 6.1.7]. □

Theorem 7.4.7. *Let D be a torus-invariant Cartier divisor on a proper toric variety X. Then the following statements are equivalent:*

(1) *the support function $\phi_D : |\Sigma| \to \mathbb{R}$ is strictly convex;*

(2) *the line bundle $\mathcal{O}_X(D)$ is ample;*

(3) *the line bundle $\mathcal{O}_X(kD)$ is very ample for all $k \geq \max\{1, n-1\}$.*

Proof. See [CLS11][Lemma 6.1.13 and Theorem 6.1.14]. □

Theorem 7.4.8. *Let $f : X \to X'$ be a proper toric morphism induced by a morphism of RPP decompositions $\phi : (N, \Sigma) \to (N', \Sigma')$. Let D be a torus-invariant Cartier divisor on X. Then we have $|\Sigma| = \phi_{\mathbb{R}}^{-1}(|\Sigma'|)$, and the following equivalences hold.*

(1) *The line bundle $\mathcal{O}_X(D)$ is f-base point free if and only if for each $\sigma' \in \Sigma'$ the restriction of the support function ϕ_D to $\phi_{\mathbb{R}}^{-1}(\sigma')$ is convex.*

(2) *The line bundle $\mathcal{O}_X(D)$ is f-ample if and only if for each $\sigma' \in \Sigma'$ the restriction of the support function ϕ_D to $\phi_{\mathbb{R}}^{-1}(\sigma')$ is strictly convex.*

Proof. We have already shown in Theorem 7.1.10(3) that f is proper if and only if $|\Sigma| = \phi_{\mathbb{R}}^{-1}(|\Sigma'|)$.
(1) This follows from Theorem 7.4.6, relative base-point-freeness being Zariski-local on the base and the fact that for every affine Noetherian scheme $U = \operatorname{Spec} A$ and coherent $\mathcal{O}_U$-module $\mathcal{F}$ the map $\Gamma(U, \mathcal{F}) \otimes_A \mathcal{O}_U \to \mathcal{F}$ is an epimorphism.
(2) See [CLS11][Theorem 7.2.11]. □

Theorem 7.4.9. *Let $f : X \to Y$ be a toric morphism. Then f is projective if and only if f is proper and there exists an f-ample torus-invariant Cartier divisor D on X.*

Proof. See [CLS11][Theorem 7.2.12]. □

7.5 Star-subdivisions

There is an elegant geometric way of producing projective toric morphisms by performing a star-subdivision on a fan.

A vector v in a lattice N, embedded as always in its realification $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, is called *primitive* if $\{\alpha \in \mathbb{R} : \alpha v \in N\} = \mathbb{Z}$.

Definition 7.5.1. Let (N, Σ) be an RPP decomposition, and $v \in N$ be a primitive vector. We define the *star-subdivision* of Σ at v to be the set consisting of the following cones in $N_{\mathbb{R}}$:

- those cones in Σ that do not contain v;
- for a cone $\sigma \in \Sigma$ containing v and a face τ of σ that does *not* contain v, the cone spanned by τ and v.

We will denote this set by $\Sigma^*(v)$. Thus, the star-subdivision of Σ at v is given by

$$\Sigma^*(v) = \{\tau \in \Sigma \mid v \notin \tau\} \sqcup \{\tau + \mathbb{R}_{\geq 0} \cdot v \mid v \notin \tau \in \Sigma \text{ and } \tau \cup \{v\} \subset \sigma \in \Sigma\}\,. \quad (7.12)$$

Geometrically one obtains $\Sigma^*(v)$ from Σ by decomposing each cone in Σ that contains v into smaller cones with $\mathbb{R}_{\geq 0} \cdot v$ as extremal ray. It is immediate from the definition that $\{\tau \in \Sigma : v \notin \tau\} = \{\tau \in \Sigma^*(v) : v \notin \tau\}$, so formation of the star-subdivision of Σ at v alters only the set of cones containing v.

Definition 7.5.2. A fan Σ' in $N_{\mathbb{R}}$ is called a *refinement* of Σ if each cone in Σ' is contained in a cone of Σ, and each cone in Σ is the union of cones in Σ'.

It follows from Theorem 7.1.12(1,3) that a refinement induces a proper birational morphism $F(\mathrm{id}_N) : F(N, \Sigma') \to F(N, \Sigma)$.

Lemma 7.5.3. *Let (N, Σ) and v be as in Definition 7.5.1. Then:*

(1) *the star-subdivision $\Sigma^*(v)$ of Σ at v is a refinement of Σ;*

(2) *the induced toric morphism $F(N, \Sigma^*(v)) \to F(N, \Sigma)$ is birational and projective.*

Proof. (1) See [CLS11][Lemma 11.1.3].

(2) The morphism $F(N, \Sigma^*(v)) \to F(N, \Sigma)$ is birational by Theorem 7.1.12(3). It is projective by [CLS11][Lemma 11.1.6] □

Let V be a set of primitive vectors in $N \cap |\Sigma|$ such that each cone $\sigma \in \Sigma$ contains at most one $v \in V$. Since forming the star-subdivision of Σ at a vector v_0 only alters the cones in Σ that contain v_0, we can perform the star-subdivision of Σ simultaneously at all vectors $v \in V$ to obtain a refinement $\Sigma^*(V)$ of V. This fan $\Sigma^*(V)$ consists of the following cones in $N_{\mathbb{R}}$:

- those cones in Σ that do not contain any $v \in V$;
- for a cone $\sigma \in \Sigma$ containing some $v \in V$ and a face τ of σ that does *not* contain v, the cone spanned by τ and v.

In a formula, the refinement $\Sigma^*(V)$ of Σ is the disjoint union

$$\Sigma^*(V) = \{\tau \in \Sigma \mid \tau \cap V = \emptyset\} \sqcup \bigsqcup_{v \in V} \{\tau + \mathbb{R}_{\geq 0} \cdot v \mid v \notin \tau \in \Sigma \text{ and } \tau \cup \{v\} \subset \sigma \in \Sigma\}\,.$$

Remark 7.5.4. The assertions in Lemma 7.5.3 remain true if we replace v by V, i.e. $\Sigma^*(V)$ is a refinement of Σ and $F(\mathrm{id}_N) : F(N, \Sigma^*(V)) \to F(N, \Sigma)$ is a birational projective toric morphism.

Chapter 8

The Tate curve

Let $h \geq 1$ be an integer. In this chapter we will describe a construction due to Tate of a generalized elliptic curve $\mathrm{Tate}_h \to \Delta$ over the unit disk with a single singular fibre at 0 that is a Néron h-gon. This is the local model at a cusp of width h for the compactification of the universal elliptic curve $\mathcal{E}_\Gamma \to Y(\Gamma)$ to a generalized elliptic curve $\mathcal{D}_\Gamma \to X(\Gamma)$, to be constructed in Chapter 9.

We start by constructing a toric variety $\mathcal{G}$ over the affine line $\mathcal{G}^0 = \mathrm{Spec}\mathbb{C}[q]$ with an action of $\mathbb{Z}$ and a multiplication map $\mathcal{G}^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} \to \mathcal{G}$. Then we use the analytification functor from Section 2.6 to obtain a complex manifold over Δ. It inherits an action of $\mathbb{Z}$ over Δ, which will be shown to be proper and free. This allows us to take the quotient for the action of the subgroup $h\mathbb{Z} \subset \mathbb{Z}$ for every $h \in \mathbb{Z}_{\geq 1}$. This quotient is the desired generalized elliptic curve Tate_h, called the *h-sided Tate curve.*

8.1 The toric variety $\mathcal{G}$

Let $N^1 = \mathbb{Z}^{\{0,1\}}$ and let $M^1 = \mathrm{Hom}\,(N^1, \mathbb{Z})$ be its $\mathbb{Z}$-dual. Consider the collection of cones in $N^1_{\mathbb{R}}$

$$\Sigma^1 = \{0\} \cup \{l_i : i \in \mathbb{Z}\} \cup \{\sigma_j : j \in \frac{1}{2} + \mathbb{Z}\}$$

given by

$$l_i = \mathbb{R}_{\geq 0} \cdot (1, i), \quad \sigma_j = \mathbb{R}_{\geq 0} \cdot (1, j - \frac{1}{2}) + \mathbb{R}_{\geq 0} \cdot (1, j + \frac{1}{2})$$

(here and in the rest of this section, i denotes an integer and j a non-integral half-integer). Then l_i is the face that $\sigma_{i-\frac{1}{2}}$ and $\sigma_{i+\frac{1}{2}}$ have in common, while σ_j is the closed region bounded by $l_{j-\frac{1}{2}}$ and $l_{j+\frac{1}{2}}$. Further, $\sigma_j \cap \sigma_{j'} = \{0\}$ if $|j - j'| \geq 2$ and $l_i \cap l_{i'} = \{0\}$ whenever $|i - i'| \geq 1$. Thus the conditions in Definition 7.1.6(1) for Σ^1 to be a fan in $N^1_{\mathbb{R}}$ are fullfilled.

Definition 8.1.1. We let $\mathcal{G}$ be toric variety attached to the RPP decomposition (N^1, Σ^1).

We will now describe $\mathcal{G}$ in quite a bit of detail. It is not difficult to see that

$$M^1_{l_i} = \mathbb{Z}_{\geq 0} \cdot \{(-i,1),(i,-1),(1,0)\},$$
$$M^1_{\sigma_j} = \mathbb{Z}_{\geq 0} \cdot \{(\frac{1}{2}-j,1),(j+\frac{1}{2},-1)\}.$$

We write $\chi^m \in \mathbb{C}[M^1]$ for the element $m \in M^1$ viewed as a character of the torus $T_{N^1} = \operatorname{Spec}\mathbb{C}[M^1]$, and put

$$x_i = \chi^{(-i,1)}, \quad y_i = \chi^{(i,-1)}, \quad q = \chi^{(1,0)} \in \mathbb{C}[M^1]. \tag{8.1}$$

The affine open subsets $F(N^1,\sigma) = \operatorname{Spec}\mathbb{C}[M^1_\sigma] \subset F(N^1,\Sigma^1)$ for the nonzero cones $\sigma \in \Sigma^1$ are given by

$$F(N^1,\sigma_j) = \operatorname{Spec}\mathbb{C}[x_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \cong \mathbb{A}^2_{\mathbb{C}},$$
$$F(N^1,\sigma_{i-\frac{1}{2}}) \cap X_{\sigma_{i+\frac{1}{2}}} = X_{l_i} = \operatorname{Spec}\mathbb{C}[q,x_i,y_i]/(x_iy_i-1) \cong \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{G}_{\mathbb{C}}.$$

The gluing of $F(N^1,\sigma_{i-\frac{1}{2}}) = \operatorname{Spec}\mathbb{C}[x_{i-1},y_i]$ and $F(N^1,\sigma_{i+\frac{1}{2}}) = \operatorname{Spec}\mathbb{C}[x_i,y_{i+1}]$ along $F(N^1,l_i) = \operatorname{Spec}\mathbb{C}[q,x_i,y_i]/(x_iy_i-1)$ is described as follows: we have that $F(N^1,l_i) \subset F(N^1,\sigma_{i-\frac{1}{2}})$ is the open subset where y_i is invertible, whereas $F(N^1,l_i)$ is the open subset of $F(N^1,\sigma_{i+\frac{1}{2}})$ where x_i is invertible, and the coordinates are related by:

$$\begin{aligned} x_{i-1} &= x_i q, \\ y_{i+1} &= y_i q. \end{aligned} \tag{8.2}$$

Consider the fan (N^0,Σ^0) where $N^0 = \mathbb{Z}^{\{0\}}$ and $\Sigma^0 = \{\{0\},\tau^0\}$ consists of the two faces of the strongly convex rational polyhedral cone

$$\tau^0 = \mathbb{R}_{\geq 0} \cdot 1 \subset \mathbb{R} = (N^0)_{\mathbb{R}}.$$

Let $M^0 = \operatorname{Hom}(N^0,\mathbb{Z}) = \mathbb{Z}^{\{0\}}$ be the $\mathbb{Z}$-dual of N^0. The monoid $(M^0)_{\tau^0} = \mathbb{Z}_{\geq 0} \cdot 1$ is free with generator 1, so that $F(N^0,\Sigma^0) = \operatorname{Spec}\mathbb{C}[q]$ is an affine line over $\mathbb{C}$, whose affine coordinate we again denote $q = \chi^1$.

Lemma 8.1.2. *The homomorphism of lattices*

$$\begin{aligned} \pi : N^1 = \mathbb{Z}^{\{0,1\}} &\to N^0 = \mathbb{Z}^{\{0\}}, \\ (a_0,a_1) &\mapsto a_0, \end{aligned} \tag{8.3}$$

defines a weakly semistable morphism of RPP decompositions $(N^1,\Sigma^1) \to (N^0,\Sigma^0)$.

Proof. We need only show that for every cone $\sigma \in \Sigma^1$ there is a cone $\tau \in \Sigma^0$ with $\pi(\sigma \cap N^1) = \tau \cap N^0$. This is the case since $h(l_i \cap N^1) = \tau^0 \cap N^0 = h(\sigma_j \cap N^1)$ for all $i \in \mathbb{Z}$ and $j \in \frac{1}{2}+\mathbb{Z}$, and trivially $h(\{0\} \cap N^1) = \{0\} \cap N^0$. □

Now by functoriality π induces a toric morphism

$$f = F(\pi) : \mathcal{G} = F(N^1,\Sigma^1) \to \mathcal{G}^0 = F(N^0,\Sigma^0), \tag{8.4}$$

which extends the map $\operatorname{Spec}\mathbb{C}[\pi^*] : T_{N^1} = \operatorname{Spec}\mathbb{C}[M^1] \to T_{N^0} = \operatorname{Spec}\mathbb{C}[M^0]$ of algebraic tori obtained from $\pi^\vee : M^0 \to M^1$. The fact that $q := \chi^1 \in \mathbb{C}[M^0]$ gets pulled back to $q := \chi^{(1,0)} \in \mathbb{C}[M^1]$ justifies calling both characters q. Since $x_{j-\frac{1}{2}}y_{j+\frac{1}{2}} = \chi^{(\frac{1}{2}-j,1)}\chi^{(\frac{1}{2}+j,-1)} = \chi^{(1,0)} = q$ we have the coordinate description

$$F(\pi) : F(N^1,\sigma_j) = \operatorname{Spec}\mathbb{C}[x_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \to F(N^0,\tau^0) = \operatorname{Spec}\mathbb{C}[q], \\ x_{j-\frac{1}{2}}y_{j+\frac{1}{2}} \leftarrow\!\shortmid q. \tag{8.5}$$

Lemma 8.1.3. *The open subset where the morphism $f = F(\pi) : \mathcal{G} \to \mathcal{G}^0$ is smooth is given by*

$$\mathcal{G}^{\mathrm{sm}} = F(N^1, \Sigma^1_{\leq 1}) = \bigcup_{i\in\mathbb{Z}} F(N^1, l_i). \tag{8.6}$$

Proof. Since smoothness of a morphism is local on the source, we may restrict our attention to the affine open subset $F(N^1,\sigma_j)$ corresponding to a maximal cone σ_j of the fan Σ^1. There the morphism $F(N^1,\sigma_j) = \operatorname{Spec}\mathbb{C}[x_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \to \mathcal{G} = \operatorname{Spec}\mathbb{C}[q]$ is given by $q \mapsto x_{j-\frac{1}{2}}y_{j+\frac{1}{2}}$ according to equation (8.6). It follows from the Jacobi criterion that the morphism is smooth but at the point $O(\sigma_j) = V(x_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$. Hence $\mathcal{G}^{\mathrm{sm}} \cap F(N^1,\sigma_j) = F(N^1, l_{j-\frac{1}{2}}) \cup F(N^1, l_{j+\frac{1}{2}})$, as desired. □

Given a scheme X and a global regular function $g \in \mathcal{O}_X(X)$, we denote $V(g)$ the closed subscheme of X where g vanishes, and $D(g)$ its open complement. In this notation, the origin of the affine line $\mathcal{G}^0 = \operatorname{Spec}\mathbb{C}[q]$ is given by $V(q)$, and its complement is denoted $D(q)$.

We will now examine the fibre $\mathcal{G} \times_{\mathcal{G}^0} V(q)$ of $f = F(\pi)$ over the origin $V(q) \subset \mathcal{G}^0$.

Lemma 8.1.4. *The fibre $\mathcal{G} \times_{\mathcal{G}^0} V(q)$ is the union of a $\mathbb{Z}$-indexed family of projective lines $\{V(l_i) : i \in \mathbb{Z}\}$, with $V(l_{j-\frac{1}{2}})$ and $V(l_{j+\frac{1}{2}})$ intersecting transversally in $V(\sigma_j)$.*

Proof. We will make frequent use of the Orbit-Cone correspondence Theorem 7.2.1. Since π maps each nonzero cone of Σ^1 surjectively onto $\tau^0 \in \Sigma^0$, we have that $f^{-1}(D(q)) = O(\{0\})$. Since $V(l_i) = V(\sigma_{i-\frac{1}{2}}) \cup O(l_i) \cup V(\sigma_{i+\frac{1}{2}})$, it follows that $\mathcal{G} \times_{\mathcal{G}^0} V(q) = \bigcup_{i\in\mathbb{Z}} V(l_i)$.

Let $j \in \frac{1}{2} + \mathbb{Z}$. Locally on the affine open subset $F(N^1,\sigma_j)$ we have that

$$F(N^1,\sigma_j) \cap V(q) = V(x_{j-\frac{1}{2}}) \cup V(y_{j+\frac{1}{2}}) \subset \operatorname{Spec}\mathbb{C}[x_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] = F(N^1,\sigma_j) \tag{8.7}$$

is given by the union of the two coordinate axes $V(x_{j-\frac{1}{2}}) = F(N^1,\sigma_j) \cap V(l_{j+\frac{1}{2}})$ and $V(x_{j+\frac{1}{2}}) = F(N^1,\sigma_j) \cap V(l_{j-\frac{1}{2}})$. For each $i \in \mathbb{Z}_{\geq 1}$ we see that the orbit $O(l_i) = F(N^1, l_i) \cap V(q) = \operatorname{Spec}\mathbb{C}[x_i, y_i]/(x_i y_i - 1)$ is a projective line with two points removed, the origins $V(\sigma_{i-\frac{1}{2}})$ and $V(\sigma_{i+\frac{1}{2}})$ of $F(N^1,\sigma_{i-\frac{1}{2}})$ resp. $F(N^1,\sigma_{i+\frac{1}{2}})$. □

8.2 Fibre power of Tate curve

For every integer $k \geq 1$ we let $\mathcal{G}^k$ be the k-th fibre power of the morphism of $\mathbb{C}$-varieties $\mathcal{G} \to \mathcal{G}^0$, i.e. the fibre product of k copies of $\mathcal{G}$ over $\mathcal{G}^0$

$$\mathcal{G}^k := \mathcal{G} \times_{\mathcal{G}^0} \mathcal{G} \times_{\mathcal{G}^0} \cdots \times_{\mathcal{G}^0} \mathcal{G}. \tag{8.8}$$

The aim of this section is to show that $\mathcal{G}^k$ is the toric variety attached to a suitable fan. Our motivation is twofold. First, we would like to define a multiplication map $m : \mathcal{G}^{\text{sm}} \times_{\mathcal{G}} \mathcal{G}^1 \to \mathcal{G}$, whose domain is an open subset of $\mathcal{G}^2$. Secondly, a toric desingularization of $\mathcal{G}^k$ will be involved in the construction of Kuga-Sato varieties in Chapter 10.

Recall the toric morphism $f = F(\pi) : \mathcal{G}^1 := F(N^1, \Sigma^1) \to \mathcal{G}^0 := F(N^0, \Sigma^0)$ attached to the morphism of RPP decompositions $\pi : (N^1, \Sigma^1) \to (N^0, \Sigma)$ we defined in Lemma 8.1.2. From Theorem 7.3.2 and Lemma 8.1.2 we see that $\mathcal{G}^k$ is the toric variety attached to the k-th fibre power (N^k, Σ^k) of the weakly semi-stable morphism of RPP decompositions $\pi : (N^1, \Sigma^1) \to (N^0, \Sigma^0)$. It is natural to view the k-th fibre power $N^k = N^1 \times_{N^0} N^1 \times_{N^0} \cdots \times_{N^0} N^1$ as the free $\mathbb{Z}$-module $N^k = \mathbb{Z}^{\{0,1,\dots,k\}}$ of rank $k+1$, such that the projection onto the m-th factor ($m \in \{1, 2, \dots, k\}$) is $\pi_m : N^k \to N^1, (a_0, a_1, \dots, a_k) \mapsto (a_0, a_m)$ and the projection onto N^0 is $\pi : N^k \to N^0, (a_0, a_1, \dots, a_k) \mapsto a_0$. We will describe the cones in the fan Σ^k in $N^k_{\mathbb{R}} = \mathbb{R}^{\{0,1,\dots,k\}}$ based on an alternative description of those in Σ^1, using that these cones are determined by their intersection with $\pi^{-1}(1)$, where $1 \in \mathbb{R} = N^0_{\mathbb{R}}$ is the generator for the monoid $N^0 \cap \tau^0 = \mathbb{Z}_{\geq 0}$.

For $k \in \mathbb{Z}_{\geq 1}$, let $H^k = \{1\} \times \mathbb{R}^k \subset N^k_{\mathbb{R}}$ be the preimage of $1 \in \mathbb{R} = N^0_{\mathbb{R}}$ under π. Let us also write $H^k_{\mathbb{Z}} = H^k \cap N^k = \{1\} \times \mathbb{Z}^k \subset N^k_{\mathbb{R}}$. Note that the fibre H^k above 1 in the k-th fibre power $N^k_{\mathbb{R}}$ is given by the usual k-th power H^1 of the fibre above 1 in $N^1_{\mathbb{R}}$. Via the isomorphism $\mathbb{R}^k \cong \{1\} \times \mathbb{R}^k$ we consider $\mathbb{R}^k$ to be embedded in $N^k_{\mathbb{R}}$ as the affine subspace H^k. It will be practical not to reflect this embedding in our notation, so whenever we write $(x_1, x_2, \dots, x_k) \in \mathbb{R}^k$ the point $(1, x_1, \dots, x_k) \in H^k$ is to be understood. Let $B^k = [0,1]^k$ be the unit k-dimensional hypercube, or unit k-cube for short, situated in $\mathbb{R}^k = H^k$. Let $\tau^k = \mathbb{R}_{\geq 0} \cdot B^k$ be the cone over $B^k \subset H^k$. In this notation, each maximal cone in Σ^1 is a $\mathbb{Z}$-translate of the cone $\tau^1 = \mathbb{R}_{\geq 0} \cdot B^1$ over the unit interval $B^1 \subset H^1$. Consequently, the maximal cones in Σ^k are $\mathbb{Z}^k$-translates of the cone $\tau^k = \mathbb{R}_{\geq 0} \cdot B^k$ over the unit k-cube $B^k \subset H^k$. Let us set

$$Z^k = F(N^k, \tau^k) \subset F(N^k, \Sigma^k).$$

Thus the affine toric varieties attached to the maximal cones in Σ^k form an affine open cover $\{a(\underline{n})(Z^k) : \underline{n} \in \mathbb{Z}^k\}$ of $F(N^k, \Sigma^k)$, on which $\mathbb{Z}^k$ acts simply transitively. Since the study of singularities is a local affair, we will focus on the study and resolution of the singularities of Z^k first. The desingularization of Z^k furnishes one for each of its $\mathbb{Z}^k$-translates as well, and these automatically patch together to give the desired desingularization of the whole $F(N^k, \Sigma^k) = \bigcup_{\underline{n} \in \mathbb{Z}^k} (a(\underline{n})(Z^k))$.

Lemma 8.2.1. *The toric variety $\mathcal{G}^k$ is nonsingular if and only if $k = 1$.*

Proof. Since $\mathcal{G}^k$ is covered by the $\mathbb{Z}^k$-translates of $F(N^k, \tau^k)$, smoothness of $\mathcal{G}^k$ is equivalent to that of $Z^k = F(N^k, \tau^k)$, for which Theorem 7.1.10(2) gives a necessary and sufficient criterion.

The set of fundamental generators of τ^k consists of the 2^k vectors $(1, b_1, b_2, \dots, b_k)$ with $b_m \in \{0, 1\}$:

$$\tau^k = \sum_{(b_m)_{m=1}^k \in \{0,1\}^k} \mathbb{R}_{\geq 0} \cdot (1, b_1, \dots, b_k). \tag{8.9}$$

If $k = 1$ then each maximal cone $\mathbb{R}_{\geq 0} \cdot [i, i+1] = \mathbb{R}_{\geq 0} \cdot \{(1,i),(1,i+1)\}$ is nonsingular since $\{(1,i),(1,i+1)\}$ is a $\mathbb{Z}$-basis of $N^1 = \mathbb{Z}^{\{0,1\}}$.

If $k \geq 2$ then $2^k > k+1$, whence the number of extremal rays of τ^k exceeds the dimension of τ^k and the cone τ^k is singular. □

8.3 The multiplication $m : \mathcal{G}^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} \to \mathcal{G}$

In this section we construct using toric machinery a toric morphism $\mathcal{G}^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} \to \mathcal{G}$ which restricts to a structure of $\mathcal{G}^0$-group scheme on the smooth locus $\mathcal{G}^{\mathrm{sm}}$ and defines an action of $\mathcal{G}^{\mathrm{sm}}$ on $\mathcal{G}$ over $\mathcal{G}^0$.

Let $D(q) = \operatorname{Spec} \mathbb{C}[q, q^{-1}] = F(N^0, \{0\}) \subset F(N^0, \Sigma^0) = \mathcal{G}^0$ be the complement of the origin. Since the only $\sigma \in \Sigma^1$ such that $\pi(\sigma) = \{0\} \subset N^0$ is $\sigma = \{0\}$, we have $F(\pi)^{-1}(F(N^0,\{0\})) = F(N^1,\{0\})$. It follows that there is an isomorphism of $D(q)$-schemes

$$\mathcal{G} \times_{\mathcal{G}^0} D(q) \cong \operatorname{Spec} \mathbb{C}[q, q^{-1}, x_0, x_0^{-1}] \cong \mathbb{G}_{\mathrm{m},D(q)}, \tag{8.10}$$

where $\mathbb{G}_{\mathrm{m},D(q)}$ is the multiplicative group scheme over $D(q)$. Via this isomorphism we transport the group law on $\mathbb{G}_{\mathrm{m},D(q)}$ to $\mathcal{G} \times_{\mathcal{G}^0} D(q)$, and the resulting morphism of $D(q)$-schemes is denoted

$$m : (\mathcal{G} \times_{\mathcal{G}^0} D(q)) \times_{D(q)} (\mathcal{G} \times_{\mathcal{G}^0} D(q)) \to \mathcal{G} \times_{\mathcal{G}^0} D(q). \tag{8.11}$$

Lemma 8.3.1. *We have that $m = F(\mu)$, where $\mu : (N^2, \{0\}) \to (N^1, \{0\})$ is the morphism of RPP decompositions given by the homomorphism*

$$\begin{aligned} \mu : N^2 &\to N^1, \\ (a_0, a_1, a_2) &\mapsto (a_0, a_1 + a_2). \end{aligned} \tag{8.12}$$

Proof. First note that $\pi \circ \mu = \pi$, so $F(\mu)$ is a morphism over $D(q)$. Let us write $\mu^* : M^1 = \mathbb{Z}^{\{0,1\}} \to M^2 = \mathbb{Z}^{\{0,1,2\}}$ for the homomorphism dual to μ. We have that $\mu^*(\chi^{(0,1)}) = \chi^{(0,1,1)} = \chi^{(0,1,0)}\chi^{(0,0,1)}$, so μ pulls back the 'fibre coordinate' $\chi^{(0,1)}$ on $\mathcal{G}^1$ to the product of the 'fibre coordinates' $\chi^{(0,1,0)}$ and $\chi^{(0,0,1)}$ on $\mathcal{G}^2$, as desired. □

Corollary 8.3.2. *The multiplication map m in (8.11) extends to a multiplication map*

$$m : \mathcal{G}^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} \to \mathcal{G}. \tag{8.13}$$

Proof. In Lemma 8.1.3 we have shown that $\mathcal{G}^{1,\mathrm{sm}} = F(N^1, \Sigma^1_{\leq 1})$, which implies that $\mathcal{G}^{1,\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} = F(N^2, \Sigma^1_{\leq 1} \otimes_{\Sigma^0} \Sigma^1)$. In the above lemma we showed that m is the toric morphism attached to the homomorphism $\mu : N^2 \to N^1$, $(a_0, a_1, a_2) \mapsto (a_0, a_1 + a_2)$. Thus our task is to show that μ sends each cone in the fan $\Sigma^1_{\leq 1} \otimes_{\Sigma^0} \Sigma^1$ into a cone of the fan Σ^1. It suffices to check this for maximal cones, which have the shape $l_i \times_{\tau^0} \sigma_j$ for certain $i \in \mathbb{Z}$ and $j \in \frac{1}{2} + \mathbb{Z}$. Recalling that $l_i = \mathbb{R}_{\geq 0} \cdot (1, i)$ and $\sigma_j = \mathbb{R}_{\geq 0} \cdot \{(1, j - \frac{1}{2}), (1, j + \frac{1}{2})\}$, we indeed have that

$$\begin{aligned}\mu(l_i \times_{\tau^0} \sigma_j) &= \mu(\mathbb{R}_{\geq 0} \cdot \{(1, i, j - \frac{1}{2}), (1, i, j + \frac{1}{2})\}) \\ &= \mathbb{R}_{\geq 0} \cdot \{(1, i + j - \frac{1}{2}), (1, i + j + \frac{1}{2})\} = \sigma_{i+j}.\end{aligned}$$

□

8.4 The action of $\mathbb{Z}$ on $\mathcal{G}$

In this section, we define an action of $\mathbb{Z}$ on $\mathcal{G}$ via toric automorphisms. In the next section we will construct Tate curves as quotients for this action. We note in passing that this action extends to one of $\Gamma^1 := \mathbb{Z} \rtimes \mu_2$ on $\mathcal{G}$, and in general, that $\Gamma^k = \mathbb{Z}^k \rtimes \mu_2^k \rtimes S_k$ is the group of toric automorphism of $\mathcal{G}^k$, as we will see in (10.6).

Let $m \in \mathbb{Z}$. The $\mathbb{Z}$-automorphism of N^1 given by the shear map

$$\begin{aligned}\alpha(m) : N^1 &\to N^1, \\ (a_0, a_1) &\mapsto (a_0, ma_0 + a_1),\end{aligned} \tag{8.14}$$

induces an $\mathbb{R}$-automorphism of $N^1_{\mathbb{R}} = \mathbb{R}^{\{0,1\}}$ that maps the cone σ_j onto the cone σ_{j+m}. Therefore $\alpha(m)$ is an automorphism of the RPP decomposition (N^1, Σ^1). Visibly $\alpha(m) \circ \alpha(m') = \alpha(m + m')$ for all $m, m' \in \mathbb{Z}$, so that $\alpha : \mathbb{Z} \to \mathrm{Aut}((N^1, \Sigma^1))$ is a homomorphism. Thus we have defined an action of $\mathbb{Z}$ on the RPP decomposition (N^1, Σ^1), which via the equivalence of categories F corresponds to an action of $\mathbb{Z}$ on the toric variety $\mathcal{G} = F(N^1, \Sigma^1)$,

$$\begin{aligned}a : \mathbb{Z} \\ \to \mathrm{Aut}(\mathcal{G}), \\ m \\ \mapsto F(\alpha(m)).\end{aligned}$$

We will encode the homomorphism a in a morphism of $\mathbb{C}$-schemes

$$a : \mathbb{Z} \times \mathcal{G} \to \mathcal{G}. \tag{8.15}$$

The toric automorphism $a(m)$ of $F(N^1, \Sigma^1)$ restricts to isomorphisms

$$a(m) : F(N^1, \sigma_j) \xrightarrow{\sim} F(N^1, \sigma_{j+m}),$$

which are in coordinates given by

$$\begin{aligned}x_{i+m} = \chi^{(i+m,-1)} &\mapsto \chi^{(i,-1)} = x_i, \\ y_{i+m} = \chi^{(-i-m,1)} &\mapsto \chi^{(-i,1)} = y_i.\end{aligned}$$

Further, $a(m)$ stabilizes the open subset $\mathcal{G}^{\mathrm{sm}} = F(N^1, \Sigma^1_{\leq 1})$ on which $f : \mathcal{G} \to \mathcal{G}^0$ is smooth.

Lemma 8.4.1. *The action of $\mathbb{Z}$ on $\mathcal{G} = F(N^1, \Sigma^1)$ is compatible with the multiplication map $m : \mathcal{G}^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} \to \mathcal{G}$ and the projection $f : \mathcal{G} \to \mathcal{G}^0$, i.e. the following diagrams in the category of $\mathbb{C}$-schemes commute:*

$$\begin{array}{ccc} \mathbb{Z} \times \mathcal{G} & \xrightarrow{\quad a \quad} & \mathcal{G} \\ & {\scriptstyle f \circ \mathrm{pr}_2} \searrow \quad \swarrow {\scriptstyle f} & \\ & \mathcal{G}^0, & \end{array} \qquad \begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} \times \mathcal{G} \times_{\mathcal{G}^0} \mathcal{G} & \xrightarrow{a \times a} & \mathcal{G} \times_{\mathcal{G}^0} \mathcal{G} \\ {\scriptstyle m \times m} \downarrow & & \downarrow {\scriptstyle m} \\ \mathbb{Z} \times \mathcal{G} & \xrightarrow{\quad a \quad} & \mathcal{G}. \end{array}$$

Proof. The action of $\mathbb{Z}$ on $N^1 = \mathbb{Z}^{\{0,1\}}$ is compatible with $\pi : N^1 \to N_0, (a_0, a_1) \mapsto a_1$, so the action of $\mathbb{Z}$ on $\mathcal{G}$ is compatible with $f = F(\pi) : \mathcal{G} \to \mathcal{G}^0$.

For $m_1, m_2 \in \mathbb{Z}$ it holds that $\alpha(m_1 + m_2) \circ \mu = \mu \circ (\alpha(m_1) \times \alpha(m_2))$; in fact for any $(a_0, a_1, a_2) \in N^2$ one calculates that

$$\begin{aligned} \alpha(m_1 + m_2)(\mu(a_0, a_1, a_2)) &= \alpha(m_1 + m_2)(a_0, a_1 + a_2) = (a_0, (m_1 + m_2)a_0 + a_1 + a_2) \\ &= \mu((a_0, m_1 a_0 + a_1), (a_0, m_2 a_0 + a_2)) = \mu(\alpha(m_1)(a_0, a_1), \alpha(m_2)(a_0, a_2)). \end{aligned}$$

This shows that $a(m_1 + m_2) \circ \mu = \mu \circ (\alpha(m_1) \times \alpha(m_2))$, as desired. □

8.5 The generalized elliptic curve $\mathrm{Tate}_h = (h\mathbb{Z})\backslash\mathcal{G}|_\Delta$

We now pass from the algebraic to the analytic category. The main result of this section is the construction, for every positive integer h, of a generalized elliptic curve Tate_h over the open unit disk Δ whose single singular fibre is a Néron h-gon at the origin. The h-sided Tate curve Tate_h will serve in Chapter 9 to compactify the universal elliptic curve at a cusp of width h.

Theorem 8.5.1. *Let $h \in \mathbb{Z}_{>0}$. Then there exists a generalized elliptic curve over the open unit disk $f_h : \mathrm{Tate}_h \to \Delta$ with a level-h structure, such that*

(1) *the fibre of Tate_h over $q \in \Delta^*$ is the elliptic curve $f_h^{-1}(q) \cong \mathbb{C}^\times / q^{h\mathbb{Z}}$ with level-h structure (q, ζ_N);*
(2) *the fibre of Tate_h over $0 \in \Delta$ is the Néron h-gon $f_h^{-1}(0) \cong C_h$ with the standard level-h structure $((0, 1), (\zeta_N, 0))$ defined in Lemma 6.2.2.*

Recall that in Section 2.6 we defined an analytification functor $(\cdot)^{\mathrm{an}}$ from the category of reduced separated schemes locally of finite type over $\mathbb{C}$ to the category of (reduced Hausdorff) complex-analytic spaces.

The morphism $f : \mathcal{G} \to \mathcal{G}^0$ of nonsingular toric varieties induces a holomorphic map of complex manifolds

$$f^{\mathrm{an}} : \mathcal{G}^{\mathrm{an}} \to \mathcal{G}^{0,\mathrm{an}}.$$

Since $\mathcal{G}^0 = \operatorname{Spec} \mathbb{C}[q]$, we have that $\mathcal{G}^{0,\mathrm{an}} = \mathbb{C}$, with coordinate called q. Define the *open punctured unit disk* resp. *open unit disk* by

$$\Delta^* = \{q \in \mathbb{C} : 0 < |q| < 1\}, \quad \Delta = \{q \in \mathbb{C} : |q| < 1\}. \tag{8.16}$$

Let us further write

$$\mathcal{G}|_{\Delta^*} = \mathcal{G}^{\mathrm{an}} \times_{\mathbb{C}} \Delta^*, \quad \mathcal{G}|_{\Delta} = \mathcal{G}^{\mathrm{an}} \times_{\mathbb{C}} \Delta. \tag{8.17}$$

as well as $\mathcal{G}|_{\{q\}} = \mathcal{G}^{\mathrm{an}} \times_{\mathbb{C}} \{q\}$ for every $q \in \Delta$. Since by Lemma 2.6.4(6) a morphism of smooth $\mathbb{C}$-varieties $g : X \to Y$ is smooth if and only if $g^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a holomorphic submersion we have $(\mathcal{G}^{\mathrm{sm}})^{\mathrm{an}} = (\mathcal{G}^{\mathrm{an}})^{\mathrm{sm}}$, where $\mathcal{G}^{\mathrm{sm}}$ is the open subset where $f : \mathcal{G} \to \mathcal{G}^0$ is smooth, and $(\mathcal{G}^{\mathrm{an}})^{\mathrm{sm}}$ is the open subset where $f^{\mathrm{an}} : \mathcal{G}^{\mathrm{an}} \to \mathcal{G}^{0,\mathrm{an}}$ is submersive. By functoriality the algebraic multiplication map $m : \mathcal{G}^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} \to \mathcal{G}$ and action map $a : \mathbb{Z} \times \mathcal{G} \to \mathcal{G}$ induce analytic maps

$$m^{\mathrm{an}} : \mathcal{G}|_{\Delta}^{\mathrm{sm}} \times_{\Delta} \mathcal{G}|_{\Delta} \to \mathcal{G}|_{\Delta}, \quad a^{\mathrm{an}} : \mathbb{Z} \times \mathcal{G}|_{\Delta} \to \mathcal{G}|_{\Delta}. \tag{8.18}$$

Proposition 8.5.2. *There exists an isomorphism of Δ^*-groups*

$$\mathcal{G}|_{\Delta^*} \cong \mathbb{C}^* \times \Delta^*. \tag{8.19}$$

Via this isomorphism we have for all $m \in \mathbb{Z}$ and $(x_0, q) \in \mathbb{C}^* \times \Delta^*$,

$$a(m)(x_0, q) = (q^m x_0, q). \tag{8.20}$$

Proof. The construction of m in Section 8.3 established that $\mathcal{G} \times_{\mathcal{G}^0} D(q) \cong \mathbb{G}_{m,D(q)}$ as group schemes over $D(q)$. The action of $\mathbb{Z}$ on $\mathcal{G} \times_{\mathcal{G}^0} D(q) = \operatorname{Spec} \mathbb{C}[x_0, x_0^{-1}, q, q^{-1}]$ is given in these coordinates by $a(m) : (x_0, q) \mapsto (x_{-m}, q) = (q^m x_0, q)$. We conclude the proof by applying $(\cdot)^{\mathrm{an}}$ and restricting to $\Delta^* \subset D(q)^{\mathrm{an}} = \mathbb{C}^*$. □

In Section 6.1 we defined the complex-analytic space

$$C_\infty = (\mathbb{P}^1 \times \mathbb{Z})/((0, i) \sim (\infty, i+1)) \tag{8.21}$$

together with a multiplication map (6.3) and a $\mathbb{Z}$-action (6.4):

$$m : C_\infty^{\mathrm{reg}} \times C_\infty \to C_\infty, \quad a : \mathbb{Z} \times \mathbb{C}_\infty \to C_\infty. \tag{8.22}$$

Proposition 8.5.3. *There is an isomorphism*

$$\mathcal{G}|_{\{0\}} \cong C_\infty \tag{8.23}$$

via which m^{an} and a^{an} in (8.3) correspond to m resp. a in (8.4).

Proof. In Lemma 8.1.4 we saw that $\mathcal{G} \times_{\mathcal{G}^0, f} V(q) = \bigcup_{i \in \mathbb{Z}} V(l_i)$ where $V(l_i) \cong \mathbb{P}^1_{\mathbb{C}}$ is the closure of the orbit $O(l_i) = \mathbb{C}[x_i, x_i^{-1}]$. Thus we have a biholomorphism $x_i : O(l_i)^{\mathrm{an}} \xrightarrow{\sim} \mathbb{C}^*$ extending to an isomorphism $\mathbb{CP}^1 \times \{i\} \to V(l_i)^{\mathrm{an}}$ by sending $0 \mapsto V(\sigma_{i+\frac{1}{2}})$ and $\infty \mapsto V(\sigma_{i-\frac{1}{2}})$. They combine to give a surjection $\mathbb{CP}^1 \times \mathbb{Z} \to \mathcal{G}|_{\{0\}}$ whose fibre above $O(\sigma_j)$ is $\{(0, j - \frac{1}{2}), (\infty, j + \frac{1}{2})\}$. It descends to an isomorphism $C_\infty \to \mathcal{G}|_{\{0\}}$, since $V(l_{j-\frac{1}{2}})$ and $V(l_{j+\frac{1}{2}})$ intersect transversally in the point $O(\sigma_j)$.

The verifications that the algebraically and analytically defined multiplication and action maps agree is clear from the descriptions we have provided of them. □

Lemma 8.5.4. *Let $h \in \mathbb{Z}_{\geq 1}$.*

(1) *The action of* $h\mathbb{Z}$ *on* $\mathcal{G}|_\Delta$ *is free and proper.*
(2) *The quotient map* $c_h := \mathcal{G}|_\Delta \to \mathrm{Tate}_h := (h\mathbb{Z})\backslash\mathcal{G}|_\Delta$ *is an* $h\mathbb{Z}$*-covering space and a local analytic isomorphism to a complex manifold.*
(3) *The morphism* $f^{an}\colon \mathcal{G}|_\Delta \to \Delta$ *is* $(h\mathbb{Z})$*-invariant and descends to an analytic map* $f_h\colon \mathrm{Tate}_h \to \Delta$*.*
(4) *The morphism of* $\mathbb{C}$*-analytic spaces* f_h *is flat.*
(5) *The continuous map* f_h *is proper.*
(6) *The multiplication* $m^{\mathrm{an}}\colon \mathcal{G}^{\mathrm{sm}}|_\Delta \times_\Delta \mathcal{G}|_\Delta \to \mathcal{G}|_\Delta$ *descends to an analytic map* $m_h\colon \mathrm{Tate}_h^{\mathrm{sm}} \times_\Delta \mathrm{Tate}_h \to \mathrm{Tate}_h$*.*
(7) *The map* m *restricts to a commutative* Δ*-group structure on* $\mathrm{Tate}_h^{\mathrm{sm}}$ *and defines an action of* $\mathrm{Tate}_h^{\mathrm{sm}}$ *on* Tate_h *over* Δ*.*
(8) *The fibre of* f_h *above a point* $q \in \Delta^*$ *is* $f_h^{-1}(q) \cong \mathbb{C}^\times/q^{h\mathbb{Z}}$*, while the fibre above* $0 \in \Delta$ *is* $f_h^{-1}(0) \cong C_h$*.*
(9) *The pair* $(f_h\colon \mathrm{Tate}_h \to \Delta, m_h)$ *is a generalized elliptic curve over* Δ*.*

Proof. (1) It suffices to show that $\mathbb{Z}$ acts freely and properly on $\mathcal{G}|_\Delta$. We define a continuous $\mathbb{Z}$-equivariant map $t : \mathcal{G}|_\Delta \to \mathbb{R}$, with $\mathbb{Z}$ acting on $\mathbb{R}$ by translation. For each $i \in \mathbb{Z}$, consider the closed subset of $\mathcal{G}|_\Delta$ given by the following half-open polydisk in $F(N^1, \sigma_{i+\frac{1}{2}})^{\mathrm{an}} \cap \mathcal{G}|_\Delta$ coordinatised by (x_i, y_{i+1}) :

$$F_i = \{(x_i, y_{i+1}) \in \mathbb{C}^2 : |x_i| \le 1, |y_{i+1}| \le 1, |x_i y_{i+1}| < 1\} \subset \mathcal{G}|_\Delta. \tag{8.24}$$

On the interior of F_i let t be given by

$$t := i + \frac{2}{\pi}\arctan\Big(\frac{1-|x_i|}{1-|y_{i+1}|}\Big) \quad \text{if } |x_i|, |y_{i+1}| < 1, \tag{8.25}$$

while setting $t \equiv i$ on $F_i \cap \{x_i = 1\}$ and $t \equiv i+1$ on $F_{i+1} \cap \{y_{i+1} = 1\}$. Since $\lim_{u\downarrow 0} \frac{2}{\pi}\arctan(u) = 0$ and $\lim_{u\to\infty} \frac{2}{\pi}\arctan(u) = 1$, one sees that the restriction of t to each F_i with $i \in \mathbb{Z}$ is continuous. Since $\{F_i : i \in \mathbb{Z}\}$ is a locally finite cover of $\mathcal{G}|_\Delta$ by closed subsets, with $F_{i-1} \cap F_i = F_{i-1} \cap \{y_i = 1\} = F_i \cap \{x_i = 1\}$ for $i \in \mathbb{Z}$ being the only nonempty intersections, it follows that $t : \mathcal{G}|_\Delta \to \mathbb{R}$ is continuous. Note that by construction $F_i = t^{-1}([i, i+1])$. Using (8.2) we see that t is $\mathbb{Z}$-equivariant. Since the $\mathbb{Z}$-action on $\mathbb{R}$ is free and proper, by Lemma 2.3.3 the $\mathbb{Z}$-action on $\mathcal{G}|_\Delta$ is free and proper.

(2) In part (1) we demonstrated that the group $h\mathbb{Z}$ acts freely and properly on the complex manifold $\mathcal{G}|_\Delta$ via biholomorphisms. By Theorem 2.3.5 there exists a categorical quotient $\mathrm{Tate}_h := (h\mathbb{Z})\backslash\mathcal{G}|_\Delta$ in the category of complex manifolds, and the the quotient map $c_h : \mathcal{G}|_\Delta \to \mathrm{Tate}_h$ is an $(h\mathbb{Z})$-covering space and a local analytic isomorphism.

(3) Since the map $f : \mathcal{G} \to \mathcal{G}^0$ is $(h\mathbb{Z})$-invariant by Lemma 8.4.1 the same is true for $f^{\mathrm{an}} : \mathcal{G}|_\Delta \to \Delta$. The universal property of the quotient map $c_h : \mathcal{G}|_\Delta \to \mathrm{Tate}_h$ yields a holomorphic map $f_h : \mathrm{Tate}_h \to \Delta$.

(4) Since $f : \mathcal{G} \to \mathcal{G}^0$ is a flat morphism of $\mathbb{C}$-varieties, and the analytification functor $(\cdot)^{\mathrm{an}}$ preserves flatness by Lemma 2.6.4(4), we find that $f : \mathcal{G}|_\Delta \to \Delta$ is flat. Since $c_h : \mathcal{G}|_\Delta \to \mathrm{Tate}_h$ is a local analytic isomorphism and $f = f_h \circ c_h$, it follows that $f_h : \mathrm{Tate}_h \to \Delta$ is flat as well.

(5) Let $K \subset \Delta$ be a compact subset. It is to be shown that $f_h^{-1}(K) \subset \mathrm{Tate}_h$ is also compact. Consider the closure $\overline{F_i}$ of F_i in $\mathcal{G}^{\mathrm{an}}$, which is

$$\overline{F_i} = \{(x_i, y_{i+1}) \in \mathbb{C}^2 : |x_i| \leq 1, |y_{i+1}| \leq 1\} \subset F(N^1, \sigma_{i+\frac{1}{2}})^{\mathrm{an}}. \tag{8.26}$$

Since $\bar{F_i}$ is a bounded closed subset of $\mathbb{C}^2$, it is compact. Thus the restriction $f^{\mathrm{an}}|_{\overline{F_i}} : \overline{F_i} \to \bar{\Delta} = \{q \in \mathbb{C} : |q| \leq 1\}$ is proper. Further, we have $(f^{\mathrm{an}})^{-1}(\Delta) \cap \bar{F_j} = F_i$ inside $\mathcal{G}^{\mathrm{an}}$.

We deduce that $(f^{\mathrm{an}})^{-1}(K) \cap F_i = (f^{\mathrm{an}})^{-1}(K) \cap \overline{F_i}$ is compact for every $i \in \mathbb{Z}$.

For every $m \in \mathbb{Z}$, the map $a(m)^{\mathrm{an}}$ is an isomorphism of $F_i = t^{-1}([i, i+1])$ onto $F_{i+m} = t^{-1}([i+m, i+m+1])$. Therefore we have that $c_h(\bigcup_{i=0}^{h-1} F_i) = \mathrm{Tate}_h$. Since $f_h \circ c_h = f^{\mathrm{an}}$, we find that

$$f_h^{-1}(K) = f_h^{-1}(K) \cap c_h(\bigcup_{i=0}^{h-1} F_i) = c_h((f^{\mathrm{an}})^{-1}(K) \cap \bigcup_{i=0}^{h-1} F_i) = c_h(\bigcup_{i=0}^{h-1}((f^{\mathrm{an}})^{-1}(K) \cap F_i)) \tag{8.27}$$

is the image of a finite union of compact sets under the continuous map c_h. We conclude that $f_h^{-1}(K)$ is compact, as desired.

(6) The product of the $h\mathbb{Z}$-covering space $\mathcal{G}|_\Delta \to \mathrm{Tate}_h$ with itself is an $(h\mathbb{Z} \times h\mathbb{Z})$-covering space $\mathcal{G}|_\Delta \times \mathcal{G}|_\Delta \to \mathrm{Tate}_h \times \mathrm{Tate}_h$, as asserted in Lemma 2.3.4(i). Restricting it to the subspace $\mathrm{Tate}_h^{\mathrm{sm}} \times_\Delta \mathrm{Tate}_h$ of $\mathrm{Tate}_h \times \mathrm{Tate}_h$ shows the natural map $\mathcal{G}|_\Delta^{\mathrm{sm}} \times_\Delta \mathcal{G}|_\Delta \to \mathrm{Tate}_h^{\mathrm{sm}} \times_\Delta \mathrm{Tate}_h$ is an $(h\mathbb{Z} \times h\mathbb{Z})$-covering map (cf. Lemma 2.3.4(ii)). Since the map $m^{\mathrm{an}} : \mathcal{G}|_\Delta^{\mathrm{sm}} \times_\Delta \mathcal{G}|_\Delta \to \mathcal{G}|_\Delta$ is equivariant with respect to the homomorphism $+ : (h\mathbb{Z}) \times (h\mathbb{Z}) \to (h\mathbb{Z})$, it follows that there is an induced analytic map $m_h : \mathrm{Tate}_h^{\mathrm{sm}} \times_\Delta \mathrm{Tate}_h \to \mathrm{Tate}_h$.

(7) Note first that $\mathcal{G}^{\mathrm{sm}}|_\Delta \to \Delta$ Denote $e : \mathcal{G}^0 \to \mathcal{G}^{\mathrm{sm}}$ the identity section and $i : \mathcal{G}^{\mathrm{sm}} \to \mathcal{G}^{\mathrm{sm}}$ the inversion map of the $\mathcal{G}^0$-group scheme $\mathcal{G}^{\mathrm{sm}}$. Applying $(\cdot)^{\mathrm{an}}$ and restricting to Δ gives holomorphic map $e^{\mathrm{an}} : \Delta \to \mathcal{G}|_\Delta$ and $i^{\mathrm{an}} : \mathcal{G}|_\Delta^{\mathrm{sm}} \to \mathcal{G}|_\Delta^{\mathrm{sm}}$. Then $e_h = c_h \circ e^{\mathrm{an}} : \Delta \to \mathrm{Tate}_h$ is a holomorphic section of f_h, while i^{an} descends as before to a biholomorphism $i_h : \mathrm{Tate}_h \to \mathrm{Tate}_h$ over Δ. The identity, inverse, commutativity and associativity axioms that $(\mathrm{Tate}_h^{\mathrm{sm}}, m_h, e_h, i_h)$ be a commutative Δ-group follow by functoriality from the corresponding axioms that $(\mathcal{G}, m, e, i)$ be a group scheme over $\mathcal{G}^0$. In a similar way the axioms that $m_h : \mathrm{Tate}_h^{\mathrm{sm}} \times_\Delta \mathrm{Tate}_h \to \mathrm{Tate}_h$ be an action over Δ follow from the axioms that $m : \mathcal{G}^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} \to \mathcal{G}$ be an action over $\mathcal{G}^0$.

(8) By Remark 2.3.6 we see for every $q \in \Delta$ that $(h\mathbb{Z}) \backslash (f^{\mathrm{an}})^{-1}(q) \cong f_h^{-1}(q)$. If $q \in \Delta^*$ then by Proposition 8.5.2 we have $(f^{\mathrm{an}})^{-1}(q) \cong \mathbb{C}^*$ with $m \in h\mathbb{Z}$ acting through multiplication by q^m. It follows that $f_h^{-1}(q) = \mathbb{C}^*/q^{h\mathbb{Z}}$. If $q = 0$ then by Proposition 8.5.3 we have that $(f^{\mathrm{an}})^{-1}(0) = C_\infty$ with $h\mathbb{Z}$ acting via the action (6.4). By Lemma 6.1.2 we find that $f_h^{-1}(0) = (h\mathbb{Z}) \backslash C_\infty = C_h$.

(9) By parts (2) we have that Tate_h is a complex manifold. By parts (4) and (5) the map $f_h : \mathcal{G}|_\Delta \to \Delta$ is proper and flat. Part (7) gives the required group law on $\mathcal{G}|_\Delta^{\mathrm{sm}}$ and the action of $\mathcal{G}|_\Delta^{\mathrm{sm}}$ on $\mathcal{G}|_\Delta$. Finally, by part (8) for every $q \in \Delta$ the fibre $f_h^{-1}(q)$ is either an elliptic curve or a Néron polygon C_n for some $n \in \mathbb{Z}_{\geq 1}$. □

Proof of Theorem 8.5.1. To finish the proof of Theorem 8.5.1, we are left to construct a level-h structure

$$\Phi_h : (\mathbb{Z}/h\mathbb{Z})^2 \times \Delta \xrightarrow{\sim} \mathrm{Tate}_h^{\mathrm{sm}}[h] \tag{8.28}$$

on the generalized elliptic curve $f_h \colon \mathrm{Tate}_h \to \Delta$. Consider the injective homomorphism of group complex manifolds over Δ

$$\tilde{\Phi}_h : \mathbb{Z} \times \mathbb{Z}/h\mathbb{Z} \times \Delta \to \mathcal{G}|_\Delta, \tag{8.29}$$

sending (m, n, q) to the point of $F(N^1, \sigma_{m+\frac{1}{2}})^{\mathrm{an}}) = \{(x_m, y_{m+1}) \in \mathbb{C}^2\}$ with coordinates $(x_m, y_{m+1}) = (\zeta_h^n, q\zeta_h^{-n})$.

For the action of $\mathbb{Z}$ on $\mathbb{Z} \times (\mathbb{Z}/h\mathbb{Z}) \times \Delta$ via the first coordinate, the injective homomorphism $\tilde{\Phi}_h$ is equivariant, and therefore it descends to an injective homomorphism

$$\Phi_h : (\mathbb{Z}/h\mathbb{Z})^2 \times \Delta \to \mathrm{Tate}_{h h}^{\mathrm{sm}}. \tag{8.30}$$

A cardinality argument on the fibres shows that Φ_h is an isomorphism onto $\mathrm{Tate}_h^{\mathrm{sm}}[h]$, which is a local system of rank-2 free $\mathbb{Z}/h\mathbb{Z}$-modules over Δ.

Finally we delineate what the map Φ_h looks like on the fibre above a point $q \in \Delta$, that is, describe the level-h structure $(\Phi_h)_q : (\mathbb{Z}/h\mathbb{Z})^2 \to \mathrm{Tate}_h^{\mathrm{sm}} \times_\Delta \{q\}$.

First let $q \in \Delta^*$. Then we have $f_h^{-1}(q)[h] \cong \mathbb{C}^*/q^{h\mathbb{Z}}[h] = \mu_h q^{\mathbb{Z}}/q^{h\mathbb{Z}}$. Since we have $x_m = q^{-m}x_0$ by (8.2), the map $(\Phi_h)_q : (\mathbb{Z}/h\mathbb{Z})^2 \to \mathbb{C}/q^{h\mathbb{Z}}$ is given by $(m, n) \mapsto q^m\zeta_h^n$.

Now suppose that $q = 0 \in \Delta$. We have that $C_h^{\mathrm{reg}}[h] = \mu_h \times (\mathbb{Z}/h\mathbb{Z})$ and $(\Phi_h)_0 : (\mathbb{Z}/h\mathbb{Z})^2 \to \mu_h \times (\mathbb{Z}/h\mathbb{Z})$ is given by $(m, n) \mapsto (\zeta_h^n, m)$. This is the standard level-h structure on C_h defined in Lemma 6.2.2. We conclude that Φ_h is a level-h structure on $f_h : \mathrm{Tate}_h \to \Delta$, which is given on the fibres as in Theorem 8.5.1. This concludes the proof. □

Remark 8.5.5. The 1-sided Tate curve $\mathrm{Tate}_1 \to \Delta$ can also be defined as a complex submanifold of the projective plane $\mathbb{CP}^2 \times \Delta$. In terms of homogeneous coordinates $(X : Y : Z)$ on the fibres, Tate_1 is given by the Weierstrass equation

$$\mathrm{Tate}_1 : Y^2Z + XYZ = X^3 + a_4(q)XZ^2 + a_6(q)Z^3 \tag{8.31}$$

where $a_4, a_6 : \Delta \to \mathbb{C}$ are given by the following power series around 0 with integer coefficients

$$a_4(q) = -\sum_{n=1}^{\infty} 5n^3 \frac{q^n}{1-q^n} \qquad = -\sum_{k=1}^{\infty} 5\sigma_3(k)q^k,$$

$$a_6(q) = -\sum_{n=1}^{\infty} \frac{5n^3 + 7n^5}{12} \cdot \frac{q^n}{1-q^n} \qquad = -\sum_{k=1}^{\infty} \frac{5\sigma_3(k) + 7\sigma_5(k)}{12} q^k,$$

and for positive integers m and n we set

$$\sigma_m(n) = \sum_{d|n, d>0} d^m$$

to be the sum of the m-th powers of the positive divisors of n. (We note that $(5n^3+7n^5)/12$ is an integer for every $n \in \mathbb{Z}$, because $2^2 3 \mid (n-1)n^2(n+1) = n^4 - n^2$, implies $5n^3 + 7n^5 \equiv -5n(n^4 - n^2) \equiv 0 \bmod 12$.) A derivation of these formulae may be found in [DR73][Section 8] or [Kat73][Section A1.2].

8.6 Morphisms between Tate curves

In order to show in Chapter 9 that the construction of the Shioda modular surface $\mathcal{D}_\Gamma$ is functorial in the congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, we will define a holomorphic map $w_{h,\tilde{h}} : \mathrm{Tate}_{\tilde{h}} \to \mathrm{Tate}_h$ covering $p_{\tilde{h}/h} : \Delta \to \Delta$, $z \mapsto z^{\tilde{h}/h}$ for every pair of positive integers h and $\tilde{h}$ with h dividing $\tilde{h}$. It restricts on the fibres over 0 to the contraction map $u_{h,\tilde{h}}\colon \mathrm{Tate}_{\tilde{h}} \to \mathrm{Tate}_h$ from Lemma 6.3.2, as we show in Proposition 8.6.2.

Suppose that $\tilde{\Gamma}$ and Γ are congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with $\tilde{\Gamma} \subset \Gamma$. We saw in Theorem 3.5.1(2) there is a natural holomorphic map $p_{\tilde{\Gamma},\Gamma} : X(\tilde{\Gamma}) \to X(\Gamma_1)$. If $\tilde{s} \in \mathrm{Cusps}(\tilde{\Gamma})$ has width, say, $\tilde{h}$, and its image $s \in \mathrm{Cusps}(\Gamma)$ has width h, then h divides $\tilde{h}$ and in suitable charts (3.12) the map $p_{\tilde{\Gamma},\Gamma}$ is given by $p_e(z) = z^e$, where $e = \tilde{h}/h$, as we showed in Theorem 3.5.6.

So let h and e be positive integers, and set $\tilde{h} = he$. Let $q_{\tilde{h}} \in \Delta^*$ and set $q_h = p_e(q_{\tilde{h}})$. Then $q_{\tilde{h}}^{\tilde{h}} = q_h^h$, so there is a natural isomorphism of elliptic curves

$$\mathrm{Tate}_{\tilde{h}}\,|_{\{q_{\tilde{h}}\}} \cong \mathbb{C}^*/q_{\tilde{h}\mathbb{Z}}^{\tilde{h}} = \mathbb{C}^*/q_h^{h\mathbb{Z}} = \mathrm{Tate}_h\,|_{\{q_h\}}. \tag{8.32}$$

Furthermore, we have defined a contraction map in Lemma 6.3.2

$$\mathrm{Tate}_{\tilde{h}}\,|_{\{0\}} = C_{\tilde{h}} \xrightarrow{u_{h,\tilde{h}}} C_h = \mathrm{Tate}_h\,|_{\{0\}}. \tag{8.33}$$

We are going to show that these maps are the fibres of a holomorphic map

$$w_{\tilde{h},h} : \mathrm{Tate}_{\tilde{h}} \to \mathrm{Tate}_h \text{ covering } p_e : \Delta \to \Delta. \tag{8.34}$$

Proposition 8.6.1. *There exists a holomorphic map $w_{h,\tilde{h}} : \mathrm{Tate}_{\tilde{h}} \to \mathrm{Tate}_h$ fitting into a commutative diagram*

$$\begin{array}{ccc} \mathrm{Tate}_{\tilde{h}} & \xrightarrow{w_{h,\tilde{h}}} & \mathrm{Tate}_h \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{p_{\tilde{h}/h}} & \Delta, \end{array} \qquad \begin{array}{ccc} w_{h,\tilde{h}}^{-1}(\mathrm{Tate}_h^{\mathrm{sm}}) \times_\Delta \mathrm{Tate}_{\tilde{h}} & \xrightarrow{m} & \mathrm{Tate}_{\tilde{h}} \\ {\scriptstyle w_{h,\tilde{h}} \times w_{h,\tilde{h}}}\downarrow & & \downarrow{\scriptstyle w_{h,\tilde{h}}} \\ \mathrm{Tate}_h^{\mathrm{sm}} \times_\Delta \mathrm{Tate}_h & \xrightarrow{m} & \mathrm{Tate}_h. \end{array}$$

Proof. Let e be a positive integer. The endomorphism of the RPP decomposition (N^0, Σ^0) given by $[e] : N^0 \to N^0, a_0 \mapsto ea_0$ induces on the open unit disk $\Delta \subset \mathbb{C} = F(N^0, \Sigma^0)^{\mathrm{an}}$ the e-th powering map $p_e : \Delta \to \Delta, z \mapsto z^e$. This is a surjective holomorphic map whose ramification index at 0 is e, and which is unramified outside of 0.

The endomorphism of the RPP decomposition (N^1, Σ^1) given by

$$\begin{aligned} \rho_e : N^1 = \mathbb{Z}^{\{0,1\}} &\to N^1 = \mathbb{Z}^{\{0,1\}}, \\ (a_0, a_1) &\mapsto (ea_0, a_1), \end{aligned} \tag{8.35}$$

has the following properties:

- ρ_e is a lift of $[e]$ along the projection $\pi : N^1 \to N^0, (a_0, a_1) \mapsto a_0$, i.e. $\pi \circ \rho_e = [e] \circ \pi$;

- ρ_e is equivariant with respect to the group homomorphism $[1/e] : e\mathbb{Z} \to \mathbb{Z}$, that is we have $\rho_e(\alpha(ke)(n))\alpha(k)(\rho_e(n))$ for all $n \in N^1$ and $k \in \mathbb{Z}$;
- ρ_e is compatible with $\mu : N^1 \times_{N_0} N^1 \to N^1$, i.e. for every $(n_1, n_2) \in N^1 \times_{N^0} N^1$ we have $\rho_e(\mu(n_1, n_2)) = \mu(\rho_e(n_1), \rho_e(n_2))$.

Therefore the toric endomorphism $r_e = F(\rho_e)$ of $\mathcal{G}$ associated with ρ_e has the following properties:

(1) r_e is a lift of p_e along the projection $f = F(\pi) : \mathcal{G} \to \mathcal{G}^0$, i.e. $f \circ r_e = p_e \circ f$;
(2) r_e is equivariant with respect to the homomorphism $[1/e] : e\mathbb{Z} \to \mathbb{Z}$;
(3) there exists a commutative diagram

$$\begin{array}{ccc} r_e^{-1}(\mathcal{G}^{\mathrm{sm}}) \times_{\mathcal{G}^0} \mathcal{G} & \xrightarrow{m} & \mathcal{G} \\ \downarrow{\scriptstyle r_e \times r_e} & & \downarrow{\scriptstyle r_e} \\ \mathcal{G}^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G} & \xrightarrow{m} & \mathcal{G}, \end{array}$$

Recall that the h-sided Tate curve $\mathrm{Tate}_h = h\mathbb{Z}\backslash(\mathcal{G}|_\Delta)$ is the quotient of $\mathcal{G}|_\Delta$ by the free and proper action of $h\mathbb{Z}$. By property (2) the toric morphism r_e descends to a holomorphic map $w_{h,\tilde{h}} \colon \mathrm{Tate}_{\tilde{h}} \to \mathrm{Tate}_h$. Property (1) resp. (3) yield commutativity of the left resp. right diagram in Proposition 8.6.1. □

Proposition 8.6.2. *The morphism of analytic spaces over Δ corresponding to the left diagram in Proposition 8.6.1*

$$(w_{h,\tilde{h}}, f) : \mathrm{Tate}_{\tilde{h}} \to \mathrm{Tate}_h \times_{\Delta, p_e} \Delta \tag{8.36}$$

is a desingularization. The induced map on fibres over $0 \in \Delta$ is the contraction map $u_{h,\tilde{h}} : C_{\tilde{h}} \to C_h$ from Section 6.3, provided we identify $\mathrm{Tate}_h \times_\Delta \{0\} = C_h$ as in Theorem 8.5.1.

Proof. Since the morphism $\pi \colon (N^1, \Sigma^1) \to (N^0, \Sigma^0)$ of RPP decompositions is weakly semistable, the fibre product $\mathcal{G} \times_{\mathcal{G}^0, r_e} \mathcal{G}^0$ is the toric variety attached to the fibre product of the corresponding RPP decomposition. We identify its lattice as

$$\begin{aligned} N^1 &\xrightarrow{\sim} N^1 \times_{\pi, N^0, [e]} N^0, \\ (a_0, a_1) &\mapsto ((ea_0, a_1), a_0), \end{aligned} \tag{8.37}$$

and note that the corresponding fan in $N^1_\mathbb{R}$ is

$$\Sigma^{1,e} = \{0\} \cup \{l_{ei} : i \in \mathbb{Z}\} \cup \{\sigma_{j,e} : j \in \frac{1}{2} + \mathbb{Z}\}, \tag{8.38}$$

where $l_{ei} = \mathbb{R}_{\geq 0} \cdot (1, ei)$ and $\sigma_{i+\frac{1}{2},e}$ is the closed region bounded by l_{ei} and $l_{e(i+1)}$ for every $i \in \mathbb{Z}$. The homomorphism $(\rho_e, \pi) : N^1 \to N^1 \times_{\pi, N^0, [e]} N^0$ then becomes the identity on N^1. Thus the toric morphism

$$\psi : F(N^1, \Sigma^1) = \mathcal{G} \to \mathcal{G} \times_{\mathcal{G}^0, r_e} \mathcal{G}^0 \cong F(\Sigma^{1,e}) \tag{8.39}$$

is given by the morphism of RPP decompositions $\mathrm{id} \colon (N^1, \Sigma^1) \to (N^1, \Sigma^{1,e})$. It is the refinement obtained by forming for $m = 1, 2, \ldots, e-1$ subsequent star-subdividision at the sets $V_m = \{(1, m + ei) : i \in \mathbb{Z}\}$.

If $e = 1$, then $\Sigma^{1,1} = \Sigma^1$ is a nonsingular fan and ψ is an isomorphism, as a desingularization of a nonsingular complex space should be. If $e > 1$, then all 2-dimensional cones $\sigma_{j,e}$ in $\Sigma^{1,e}$ are singular, so the singular locus of $F(N^1, \Sigma^{1,e})$ is the set $F(N^1, \Sigma^{1,e})^{\text{sing}} = \bigcup_{j \in \frac{1}{2}+\mathbb{Z}} O(\sigma_{j,e})$. Because a refinement never alters cones of dimension ≤ 1, it follows that $F(N^1, \Sigma^1) \to F(N^1, \Sigma^{1,e})$ restricts to an isomorphism over $F(N^1, \Sigma^{1,e}_{\leq 1}) = F(N^1, \Sigma^{1,e})^{\text{reg}}$, hence ψ is a desingularization.

For every integer k we have that ψ intertwines the action of $ke \in e\mathbb{Z}$ on $\mathcal{G}$ with the pullback of the action of $e \in \mathbb{Z}$ on $\mathcal{G} \times_{\mathcal{G}^0, F([e])} \mathcal{G}^0$, so $\psi|_\Delta : \mathcal{G}|_\Delta \to \mathcal{G}|_\Delta \times_{\Delta, p_e} \Delta$ descends to the desingularization

$$(w_{h,\tilde{h}}, f) : \text{Tate}_{\tilde{h}} = (\tilde{h}\mathbb{Z})\backslash\mathcal{G} \to (h\mathbb{Z})\backslash(\mathcal{G} \times_{\mathcal{G}^0, F([e])} \mathcal{G}^0)|_\Delta = \text{Tate}_h \times_{\Delta, p_e} \Delta. \qquad (8.40)$$

Here we used that taking quotients by proper and free actions commutes with pullback. □

Remark 8.6.3. One calculates that

$$M^1_{\sigma_{j,e}} = \mathbb{Z}_{\geq 0} \cdot \{\chi^{(e(\frac{1}{2}-j),1)}, \chi^{(e(j+\frac{1}{2}),-1)}, \chi^{(1,0)}\}, \qquad (8.41)$$

so that

$$F(N^1, \sigma_{j,e}) = \text{Spec}\, \mathbb{C}[x_{e(j-\frac{1}{2})}, y_{e(j+\frac{1}{2})}, q]/(x_{e(j-\frac{1}{2})} y_{e(j+\frac{1}{2})} - q^e). \qquad (8.42)$$

Therefore the pullback $\text{Tate}_h \times_{\Delta, p_e} \Delta$ of the h-sided Tate curve along $z \mapsto z^e$ has h surface singularities of type A_{e-1}, located at the h singular points of its fibre C_h above 0.

Chapter 9

Shioda modular surfaces

For torsion-free Γ satisfying a regularity condition at the cusps, we show that the universal elliptic curve with a Γ-structure $(\mathcal{E}_\Gamma, \alpha) \to Y(\Gamma)$ from Section 5.3 admits a compactification in Theorem 9.1.1. The resulting object $\mathcal{D}_\Gamma \to X(\Gamma)$ is what Deligne and Rapoport coined a *generalized elliptic curve*, see Section 6.5.

For $N \geq 3$ and $\zeta \in \mu_N^\times$ we show that the level-N structure (P_ζ, Q_ζ) with Weil pairing ζ on $\mathcal{E}_{\Gamma(N)}/Y(N)$ from Theorem 5.6.7 has a unique extension to a level-N structure on $\mathcal{D}_{\Gamma(N)}/X(N)$. We show that each isomorphism class of level-N structures on C_N with Weil pairing ζ occurs in this extension at precisely one cusp of $X(N)$. For $N \geq 5$ the analogous statement concerning $\Gamma_1(N)$ and points of exact order N holds as well.

In Section 9.4 we show that an inclusion $\tilde{\Gamma} \subset \Gamma$ of congruence subgroups satisfying the hypothesis of Theorem 9.1.1 induces a holomorphic map $\tilde{p}_{\Gamma,\tilde{\Gamma}} : \mathcal{D}_{\tilde{\Gamma}} \to \mathcal{D}_\Gamma$ covering $p_{\Gamma,\tilde{\Gamma}} : X(\tilde{\Gamma}) \to X(\Gamma)$. Finally in Theorem 9.5.2 we show that the compact 2-dimensional complex manifolds $\mathcal{D}_\Gamma$ are projective-algebraic in the sense of Definition 2.7.1.

9.1 The compactification $\mathcal{D}_\Gamma \to X(\Gamma)$ of $\mathcal{E}_\Gamma \to Y(\Gamma)$

In this section we carry out the construction of the Shioda modular surfaces $\mathcal{D}_\Gamma$.

Theorem 9.1.1. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a torsion-free congruence subgroup such that all cusps of Γ are regular. Then there exists a generalized elliptic curve $\mathcal{D}_\Gamma \to X(\Gamma)$ extending $\mathcal{E}_\Gamma \to Y(\Gamma)$ whose fibre over a cusp $t \in \mathrm{Cusps}(\Gamma)$ is a Néron h_t-gon, where h_t is the width of t.*

Recall from Equation (3.11) that there is a biholomorphism

$$\begin{aligned} e_h : P_h^+ \backslash \mathfrak{H} &\to \Delta^* \\ \tau &\mapsto \exp(2\pi i \tau / h). \end{aligned}$$

If we let $\tau \in \mathfrak{H}$ and set $q_h = e_h(\tau)$, there is an isomorphism of elliptic curves

$$\begin{aligned} e : \mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) &\to \mathbb{C}^*/q_h^{h\mathbb{Z}} = (\mathrm{Tate}_h)_{q_h} \\ z &\mapsto \exp(2\pi i z). \end{aligned} \tag{9.1}$$

Via this isomorphism, the level-h structures $(\Psi_h)_\tau = (\tau/h, 1/h)$ on $\mathcal{E}_\tau$ and $(\Phi_h)_{q_h}$ on $(\mathrm{Tate}_h)_{q_h} = (q_h, \zeta_h)$ correspond to one other, since $e(\tau/h) = e_h(\tau) = q_h$ and $e(1/h) = e_h(1) = \zeta_h$.

In fact, these isomorphism glue to an isomorphism $\widetilde{e_h}$ of relative elliptic curves that lifts e_h.

Lemma 9.1.2. *There exists an isomorphism of elliptic curves with a level-h structure*

$$\begin{array}{ccc} P_h^+\backslash(\mathcal{E},\Psi_h) & \xrightarrow{\widetilde{e_h}} & (\mathrm{Tate}_h,\Phi_h)|_{\Delta^*} \\ \downarrow & & \downarrow \\ P_h^+\backslash\mathfrak{H} & \xrightarrow{e_h} & \Delta^*. \end{array}$$

Proof. The characters $x = x_0$ and q of the algebraic torus T_{N^1} give rise to an isomorphism $\mathcal{G}|_{\Delta^*} = \{(x,q) : x \in \mathbb{C}^*, q \in \Delta^*\} \cong \mathbb{C}^* \times \Delta^*$. As in Proposition 4.2.4(2), we constructed $\mathcal{E} = (\mathbb{C}\times\mathfrak{H})/L$ as the categorical quotient for the fibral equivalence relation defined by the lattice L in (5.1). The homomorphism $(e, e_h) : \mathbb{C}\times\mathfrak{H} \to \mathbb{C}^* \times \Delta^*$ therefore descends to a cartesian homomorphism of elliptic curves

$$\mathcal{E} = (\mathbb{C}\times\mathfrak{H})/L \to (h\mathbb{Z})\backslash\mathcal{G}|_{\Delta^*} = \mathrm{Tate}_h|_{\Delta^*}$$

covering $e_h : \mathfrak{H} \to \Delta^*$. On fibres it induces the isomorphism (9.1), whence it respects the level-h structures Ψ_h and Φ_h. Using Lemma 5.3.2 this homomorphism is seen to be P_h^+-invariant, so it descends to the desired isomorphism $\widetilde{e}_h : P_h^+\backslash\mathcal{E} \to \mathrm{Tate}_h|_{\Delta^*}$. ☐

Proof of Theorem 9.1.1. Let R be a set of representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/P$. We define $\mathcal{D}_\Gamma \to X(\Gamma)$ to be the pushout of the following diagram, to be explained below, in the category of generalized elliptic curves in which we set $s = [\gamma](\infty)$ and let $h = h_{\Gamma s}$ be the width of the cusp $\Gamma s \in \mathrm{Cusps}(\Gamma)$

$$\begin{array}{ccccccc} \bigsqcup_{\gamma\in R}\mathrm{Tate}_h|_{V_h^*} & \xleftarrow{\sqcup\widetilde{e_h}} & \bigsqcup_{\gamma\in R}P_h^+\backslash\mathcal{E}|_{U_\infty} & \xrightarrow{\sqcup[\gamma]_{\mathcal{E}}} & \bigsqcup_{\gamma\in R}\Gamma_s\backslash\mathcal{E}|_{U_s} & \longrightarrow & \mathcal{E}_\Gamma \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigsqcup_{\gamma\in R}V_h^* & \xleftarrow{\sqcup e_h} & \bigsqcup_{\gamma\in R}P_h^+\backslash U_\infty & \xrightarrow{\sqcup[\gamma]} & \bigsqcup_{\gamma\in R}\Gamma_s\backslash U_s & \longrightarrow & Y(\Gamma). \end{array}$$

The left square is obtained from the isomorphism defined in Lemma 9.1.2 and the inclusion $V_h \subset V_h^*$. The middle square is the disjoint union over $\gamma \in R$ of the isomorphism $[\gamma]_\mathcal{E} : \mathcal{E}|_{U_\infty} \to \mathcal{E}|_{U_s}$ covering $[\gamma] : U_\infty \to U_s$ which is equivariant for the isomorphism $P_h^+ \to \Gamma_s$, $\delta \mapsto \gamma\delta\gamma^{-1}$. The right square arises from our viewing of $\Gamma_s\backslash U_s$ as an open subset of $Y(\Gamma)$. We see that all squares are cartesian homomorphisms of generalized elliptic curves, which renders possible the formation of the pushout $\mathcal{D}_\Gamma \to X(\Gamma)$. ☐

Remark 9.1.3. In the proof of Theorem 9.1.1 we made a choice of a set R of double coset representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/P$. However, the resulting generalized elliptic curve $\mathcal{D}_\Gamma \to X(\Gamma)$ does not depend on this choice of R. Let us briefly explain why this is the case.

Let $\tilde{R}$ be different choice of double coset representatives for $\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})/P$. Theorem 3.5.6 provides an isomorphism between the bottom rows of the diagram (9.1) for R and for $\tilde{R}$. It suffices to show that it lifts to an isomorphism of the top rows as well. This has been done in Lemma 5.3.2 for all but the left most column. Thus it remains to show that for each $\delta \in P$ and $h \in \mathbb{Z}_{\geq 1}$ there is a lift of the automorphism $[\delta]_h$ of Δ defined in Equation (3.14), to an automorphism $[\delta]_{\mathrm{Tate}_h}$ of Tate_h. Since $P = \{\pm 1\} \times P_1^+$, it suffices to consider the case a) that $\delta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and b) that $\delta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

In case a) we have that $[\delta]_h = \mathrm{id}_{\Delta^*}$, though $[\delta]_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ is given by inversion. Therefore we let $[\delta]_{\mathrm{Tate}_h}$ be given by the inversion $i : \mathrm{Tate}_h \to \mathrm{Tate}_h$ induced by the involution $\iota : N^1 \to N^1$, $(a_0, a_1) \mapsto (a_0, -a_1)$.

In case b) we have that $[\delta]_h(z) = \zeta_h z$, which stems from the toric operation of $\zeta_h \in \mathbb{C}^* = (T_{N^0})^{\mathrm{an}}$ on $\Delta \subset \mathbb{C} = \mathcal{G}^{0,\mathrm{an}}$. There is a homomorphic section $s : T_{N^0} \to T_{N^1}$ of the homomorphism $f : T_{N^1} \to T_{N^0}$, induced by the section $\sigma : N^0 \to N^1$, $a_0 \mapsto (a_0, 0)$ of $\pi : N^1 \to N^0$. This implies that the operation of $s(\zeta_h) \in (T_{N^1})^{\mathrm{an}}$ on Tate_h lifts the action of $\zeta_h^n \in (T_{N^0})^{\mathrm{an}}$ on Δ, as desired.

9.2 The level-N structure Ψ_N on $\mathcal{D}_{\Gamma(N)}$

Let $N \geq 3$ be an integer and set $\zeta_N = \exp(2\pi \mathrm{i}/N) \in \mathbb{C} \setminus \mathbb{R}$. By Theorem 5.6.7 there exists a universal elliptic curve $(\mathcal{E}_{\Gamma(N)}, (P_{\zeta_N}, Q_{\zeta_N})) \to Y(N)$ with a level-N structure having Weil pairing ζ_N. It provides a bijection between $|Y(N)|$ and the set of isomorphism classes of complex elliptic curves with a level-N structure with Weil pairing ζ_N.

We now show that this level-N structure $\Psi_N : (\mathbb{Z}/N\mathbb{Z})^2 \times Y(N) \to \mathcal{E}_{\Gamma(N)}$ extends to an ample level-N structure $\Psi_N : (\mathbb{Z}/N\mathbb{Z})^2 \times X(N) \to \mathcal{D}_{\Gamma(N)}$, such that the generalized elliptic curve $(\mathcal{D}_{\Gamma(N)}, \Psi_N) \to X(N)$ with an ample level-N structure provides a bijection between $|X(N)|$ and the set of isomorphism classes of complex generalized elliptic curves with an ample level-N structure.

Proposition 9.2.1. *Let $N \geq 3$. Then there exists an ample level-N structure Ψ_N on the generalized elliptic curve $\mathcal{D}_{\Gamma(N)} \to X(N)$ with Weil pairing ζ_N such that the following map is bijective:*

$$|X(N)| \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{complex generalized elliptic curves with an ample level-}N \\ \textit{structure having Weil pairing } \zeta_N \end{array} \right\} / \cong,$$
$$m \mapsto (\mathcal{D}_{\Gamma(N)}, \Psi_N)_m. \tag{9.2}$$

Proof. The congruence subgroup $\Gamma(N)$ satisfies the hypotheses of Theorem 9.1.1 by Proposition 3.2.2 and Proposition 3.4.8. The theorem constructs the generalized elliptic curve $\mathcal{D}_{\Gamma(N)} \to X(N)$ as the pushout of the diagram (9.1). Since all cusps of $\Gamma(N)$ have width N, the singular fibres of $\mathcal{D}_{\Gamma(N)}$ are Néron N-gons. To construct a level-N structure Ψ_N on $\mathcal{D}_{\Gamma(N)}$, we enhance this diagram to the following diagram of generalized elliptic curves with a level-N structure:

$$\begin{array}{ccccccc}
\bigsqcup_\gamma(\mathrm{Tate}_N, \Phi_N \circ \gamma^t)|_{V_h^*} & \leftarrow & \bigsqcup_\gamma P_N^+\backslash(\mathcal{E}, \Psi_N \circ \gamma^t)|_{U_\infty} & \rightarrow & \bigsqcup_\gamma \Gamma(N)_s\backslash(\mathcal{E}, \Psi_N)|U_s & \rightarrow & (\mathcal{E}_\Gamma, \Psi_N) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigsqcup_\gamma V_h^* & \longleftarrow & \bigsqcup_\gamma P_h^+ \setminus U_\infty & \longrightarrow & \bigsqcup_\gamma \Gamma(N)_s\backslash U_s & \longrightarrow & Y(N).
\end{array}$$

By Lemma 9.1.2 and (5.5) the left resp. middle squares respect the level-N structures. This constructs a level-N structure Ψ_N on $\mathcal{D}_{\Gamma(N)}/X(N)$, which is an extension of the level-N structure Ψ_N on $\mathcal{E}_{\Gamma(N)}/Y(N)$. Therefore the map (9.2) is given on $Y(N)$ by Theorem 3.6.2(1).

Since the level-N structure on the fibre over the cusp $\Gamma(N)s$ with $s = [\gamma](\infty)$ is $\Phi_N \circ \gamma^t$, with Φ_N the standard level$-N$ structure on C_N from Example 6.4.2(1), the map (9.2) is given on $\mathrm{Cusps}(N)$ by equation (6.18).

Now since we have $X(N) = Y(N) \sqcup \mathrm{Cusps}(N)$, and a complex generalized elliptic curve with an ample level-N structure is either a smooth elliptic curve or else a Néron N-gon, it follows that (9.2) is a bijection. □

9.3 The point Q_N of exact order N on $\mathcal{D}_{\Gamma_1(N)}$

For an integer $N \geq 5$, Proposition 3.2.2 (2) asserts that the universal elliptic curve $(\mathcal{E}_{\Gamma_1(N)}, Q_N) \to Y_1(N)$ with a point of exact order N provides a bijection between $|Y_1(N)|$ and isomorphism classes of complex elliptic curves with a point of exact order N. We show that the point $Q_N : Y(N) \to \mathcal{E}_{\Gamma_1(N)}$ of exact order N extends to an ample point $X(N) \to \mathcal{D}_{\Gamma(N)}$ of exact order N, such that the generalized elliptic curve $(\mathcal{D}_{\Gamma(N)}, Q_N) \to X(N)$ with a point of exact order N provides a bijection between $|X_1(N)|$ and isomorphism classes of complex generalized elliptic curves with an ample point of exact order N.

Lemma 9.3.1. *Let $N \geq 5$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, let $s = [\gamma](\infty) \in \mathbb{P}^1(\mathbb{Q})$, and let $t = \Gamma s \in \mathrm{Cusps}(\Gamma)$. Let $Q = \Phi(\gamma^t(0,1)) = (\zeta_N^d, c)$ be the point of exact order N corresponding to t via Lemma 6.4.8. Then the following integers are equal:*

(1) $N/\gcd(c, N)$;

(2) *the order of Q in $\pi_0(C_N^{\mathrm{reg}})$;*

(3) *the width $h_t = h_{\gamma,\Gamma}$ of the cusp t.*

Proof. The first two integers are equal since the order of $c + N\mathbb{Z}$ in $\mathbb{Z}/N\mathbb{Z}$ is $N/\gcd(c, N)$. Denote their common value by h.

Write $\overline{\gamma} = \pi_N(\gamma) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for the reduction of γ modulo N. Consider $\alpha \in \mathrm{Aut}(C_N)$, corresponding under eq. (6.9) to $\overline{\delta} = \pi_N(\delta)$ for some $\delta \in P$. Then Lemma 6.2.3 gives that

$$\alpha(Q) = Q \iff \overline{\Gamma}_1(N)\gamma = \overline{\Gamma}_1(N)\overline{\gamma}\delta \iff \overline{\delta} \in \overline{\gamma}^{-1}\overline{\Gamma}_1(N)\overline{\gamma} \iff \delta \in \gamma^{-1}\Gamma_1(N)\gamma.$$

Thus we find that

$$\#\{\alpha \in \mathrm{Aut}(C_N) : \alpha(Q) = Q\} = (P \cap \gamma^{-1}\Gamma_1(N)\gamma : \Gamma(N)) = h_{\Gamma(N)s}/h_{\Gamma_1(N)s} = N/h_t.$$

On the other hand, for $\zeta \in \mu_N$, viewed as automorphism $\alpha \in \mathrm{Aut}(C_N)$ we have that

$$\alpha(Q) = Q \iff (\zeta^c \zeta_N^d, c) = (\zeta_N^d, c) \iff c \in \mu_{\gcd(c,N)}.$$

We conclude that

$$\gcd(c, N) = \#\{\alpha \in \mathrm{Aut}(C_N) : \alpha(Q) = Q\} = N/h_t.$$

This shows that $h_t = N/\gcd(c, N)$ as desired. □

Proposition 9.3.2. *Let $N \geq 5$. Then there exists an ample point Q_N of exact order N on the generalized elliptic curve $\mathcal{D}_{\Gamma_1(N)} \to X_1(N)$ such that the following map is bijective:*

$$\begin{aligned} |X_1(N)| &\xrightarrow{\sim} \left\{ \begin{matrix} \textit{complex generalized elliptic curves with} \\ \textit{an ample point of order } N \end{matrix} \right\} / \cong, \\ m &\mapsto (\mathcal{D}_{\Gamma(N)}, Q_N)_m. \end{aligned} \tag{9.3}$$

Proof. The proof of this proposition is analogous to Proposition 9.2.1, taking account of Lemma 9.4.1. □

9.4 Morphisms between Shioda modular surfaces

We show in Proposition 9.4.2 that there is a natural map $\mathcal{D}_{\tilde{\Gamma}} \to \mathcal{D}_\Gamma$ between the Shioda modular surface constructed in Theorem 9.1.1 attached to an inclusion $\tilde{\Gamma} \subset \Gamma$.

Consider congruence subroups $\tilde{\Gamma}$ and Γ of $\mathrm{SL}_2(\mathbb{Z})$ such that $\tilde{\Gamma} \subset \Gamma$. In Theorem 3.5.6 we have shown there is a natural holomorphic map

$$p_{\tilde{\Gamma},\Gamma} : X(\tilde{\Gamma}) := \tilde{\Gamma}\backslash\mathfrak{H}^* \to \Gamma\backslash\mathfrak{H}^* =: X(\Gamma).$$

If Γ is torsion-free, then $\tilde{\Gamma}$ is torsion-free as well. There is a natural cartesian homomorphism of elliptic curves

$$\tilde{p}_{\Gamma,\tilde{\Gamma}} : \mathcal{E}_{\tilde{\Gamma}} := \tilde{\Gamma}\backslash\mathcal{E} \to \Gamma\backslash\mathcal{E} =: \mathcal{E}_\Gamma \quad \text{covering} \quad p_{\tilde{\Gamma},\Gamma} : Y(\tilde{\Gamma}) \to Y(\Gamma). \tag{9.4}$$

Furthermore, if all cusps of Γ are regular, the same holds true for $\tilde{\Gamma}$, as the following lemma shows.

Lemma 9.4.1. *Let $\tilde{\Gamma} \subset \Gamma$ be congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Let $\tilde{t} \in \mathrm{Cusps}(\tilde{\Gamma})$, and set $t = p_{\Gamma,\tilde{\Gamma}}(\tilde{t}) \in \mathrm{Cusps}(\Gamma)$. If t is regular and $-1 \notin \Gamma$, then $\tilde{t}$ is regular.*

Proof. Choose $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tilde{t} = \tilde{\Gamma}[\gamma](\infty)$; then we find that $t = \Gamma[\gamma](\infty)$. We have seen in Theorem 3.5.6 that $P_{\tilde{t}} := P_{\gamma,\tilde{\Gamma}} \subset P_{\gamma,\Gamma} =: P_t$. If t is regular and $-1 \notin \Gamma$, then $P_t = P_h^+ := \{\begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$, where h is the width of t. If

$e = (P_t : P_{\tilde{t}}) \in \mathbb{Z}_{\geq 1}$ is the index of $P_{\tilde{t}}$ in P_t, then since P_h^+ is infinite cyclic we have $P_{\tilde{t}} = P_{eh}^+ = \{\begin{pmatrix} 1 & neh \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$. We conclude that $\tilde{t}$ is a regular cusp of $\tilde{\Gamma}$ of width eh. We note also that $-1 \notin \Gamma$ implies $-1 \notin \tilde{\Gamma}$. □

Proposition 9.4.2. *Let $\tilde{\Gamma} \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be congruence subgroups. Assume that Γ is torsion-free and has only regular cusps. Then the map (9.4) extends to a holomorphic map*

$$\tilde{p}_{\tilde{\Gamma},\Gamma} : \mathcal{D}_{\tilde{\Gamma}} \to \mathcal{D}_\Gamma \text{ covering } p_{\tilde{\Gamma},\Gamma} : X(\tilde{\Gamma}) \to X(\Gamma)$$

Proof. We adopt the notation of Theorem 3.5.6, choosing sets $\tilde{R}$ resp. R of double coset representatives for $\tilde{\Gamma}\backslash \mathrm{SL}_2(\mathbb{Z})/P$ resp. $\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})/P$ in such a way that $\tilde{R} \subset \Gamma R$, i.e. so that $\delta(\beta) = 1$ for all $\beta \in \tilde{R}$. We contend that the following diagram covering (3.17) commutes, in which $s = [\gamma](\infty)$, $h = h_{\Gamma s}$, $\tilde{s} = [\beta](\infty)$ and $\tilde{h} = h_{\tilde{\Gamma}\tilde{s}}$.

$$\begin{array}{ccccccc}
\coprod_{\beta\in\tilde{R}} \mathrm{Tate}_{\tilde{h}}|_{V_{\tilde{h}}} & \longleftarrow & \coprod_{\beta\in\tilde{R}} P_{\tilde{h}}^+\backslash\mathcal{E}|_{U_\infty} & \xrightarrow{\coprod[\beta]_\mathcal{E}} & \coprod_{\beta\in\tilde{R}} \tilde{\Gamma}_{\tilde{s}}\backslash\mathcal{E}|_{U_{\tilde{s}}} & \longrightarrow & \mathcal{E}_{\tilde{\Gamma}} \\
\downarrow{\scriptstyle \coprod w_{h,\tilde{h}}} & & \downarrow{\scriptstyle [1]_\mathcal{E}} & & \downarrow{\scriptstyle \coprod[\alpha(\beta)]_\mathcal{E}} & & \downarrow{\scriptstyle \tilde{p}_{\Gamma,\tilde{\Gamma}}} \\
\coprod_{\gamma\in R} \mathrm{Tate}_{h}|_{V_{h}} & \longleftarrow & \coprod_{\gamma\in R} P_{h}^+\backslash\mathcal{E}|_{U_\infty} & \xrightarrow{\coprod[\gamma]_\mathcal{E}} & \coprod_{\gamma\in R} \Gamma_{s}\backslash\mathcal{E}|_{U_{s}} & \longrightarrow & \mathcal{E}_{\Gamma}.
\end{array}$$

The middle square commutes since $\alpha(\beta)\beta = \gamma(\beta)$ implies that $[\alpha(\beta)]_\mathcal{E} \circ [\beta]_\mathcal{E} = [\gamma(\beta)]_\mathcal{E}$. The right square commutes because $\alpha(\beta) \in \Gamma$ and the quotient map $p_\Gamma : \mathcal{E} \to \mathcal{E}_\Gamma$ is Γ-invariant. It remains to show the left square is commutative. We first recall how the horizontal isomorphisms are defined.

Let $k \in \mathbb{Z}_{\geq 1}$. We have a biholomorphism $e_k : P_k^+\backslash\mathfrak{H} \to \Delta^*, \tau \mapsto \exp(2\pi i\tau/k)$ which maps $U_\infty = \{\tau \in \mathbb{C} : \Im\tau > 0\}$ onto $V_k = \{q \in \mathbb{C} : 0 < |q| < \exp(-2\pi/k)\}$. Recall that $\mathcal{E} = V/L$ has been constructed as the quotient of the holomorphic line bundle $V = \mathbb{C} \times \mathfrak{H}$ by the lattice $L = \{(m\tau + n, \tau) : \tau \in \mathfrak{H}, (m, n) \in \mathbb{Z}^2\}$, while $Tate_k|_{\Delta^*}$ has been constructed as the quotient of $\mathcal{G}|_{\Delta^*} = \Delta^* \times \mathbb{C}^*$ by the action of $k\mathbb{Z}$. Now we have a diagram with horizontal isomorphisms and vertical analytic covering maps

$$\begin{array}{ccc}
P_k^+\backslash V/(\mathbb{Z} \times \mathfrak{H}) & \xrightarrow{(e_k, e_1)} & \Delta^* \times \mathbb{C}^* \\
\downarrow & & \downarrow{\scriptstyle c_k} \\
P_k^+\backslash\mathfrak{H} & \xrightarrow{\widetilde{e_k}} & \mathrm{Tate}_k|_\Delta.
\end{array}$$

It thus suffices to show the following diagram commutes,

$$\begin{array}{ccc}
P_{\tilde{h}}^+\backslash V/(\mathbb{Z} \times \mathfrak{H}) & \xrightarrow{(e_{\tilde{h}}, e_1)} & \Delta^* \times \mathbb{C}^* \\
\downarrow & & \downarrow{\scriptstyle p_{\tilde{h}/h} \times [1]} \\
P_h\backslash V/(\mathbb{Z} \times \mathfrak{H}) & \xrightarrow{(e_h, e_1)} & \Delta^* \times \mathbb{C}^*,
\end{array}$$

which is clear because $(p_{\tilde{h}/h} \circ e_{\tilde{h}}) = e_h$ by (3.16) with $\delta = 1$. □

9.5 The Shioda modular surface $\mathcal{D}_\Gamma$ is projective-algebraic

We prove that the Shioda modular surfaces $\mathcal{D}_\Gamma$ constructed in Theorem 9.1.1 are projective-algebraic. The essential point is that the fibration $f : \mathcal{D}_\Gamma \to X(\Gamma)$ onto the projective-algebraic curve $X(\Gamma)$ admits a section.

Theorem 9.5.1. *Let X be a connected compact complex manifold of dimension 2. Suppose there exists a surjective holomorphic map $f : X \to Y$ to a Riemann surface Y which admits a holomorphic section $s : Y \to X$. Then X is a projective-algebraic complex surface.*

Proof. By [BHPVdV04][Theorem 6.2] it suffices to exhibit a divisor D on X with positive self intersection number $D^2 > 0$. Let $C = s(Y)$ be the image of the section s, and let G be a general fibre of Y. Since $G^2 = 0$ and $C \cdot G > 0$, for all $n > -C^2/(C \cdot G)$ we have

$$(C + nG)^2 = C^2 + n(C \cdot G) + n^2G^2 = C^2 + n(C \cdot G) > 0.$$

Thus for n sufficiently large, $D = C + nG$ is the desired divisor with $D^2 > 0$. □

Theorem 9.5.2. *Let Γ be a torsion-free congruence subgroup of Γ such that all cusps of Γ are regular. Then the Shioda modular surface $\mathcal{D}_\Gamma$ attached to Γ is projective-algebraic.*

Proof. We need only show the hypotheses in Theorem 9.5.1 are met. We denote the identity section of the generalized elliptic curve $f : \mathcal{D}_\Gamma \to X(\Gamma)$ by $e : X(\Gamma) \to \mathcal{D}_\Gamma$. Since f is proper and $X(\Gamma)$ is compact, we have that $\mathcal{D}_\Gamma$ is compact. Each fibre $f^{-1}(m) \subset \mathcal{D}_\Gamma$ is connected, being a smooth elliptic curve or a Néron polygon, and meets the image $e(X(\Gamma)) \subset \mathcal{D}_\Gamma$ of the identity section, which is connected since $X(\Gamma)$ is connected. This implies that $\mathcal{D}_\Gamma$ is connected. Finally it is clear that $\mathcal{D}_\Gamma$ is a 2-dimensional complex manifold. Thus we conclude by Theorem 9.5.1. □

Chapter 10

Kuga-Sato varieties

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a torsion-free congruence subgroup such that all cusps of Γ are regular. In Chapter 9 we constructed a generalized elliptic curve $\mathcal{D}_\Gamma \to X(\Gamma)$ which is a smooth compactification of the universal elliptic curve with a Γ-structure $(\mathcal{E}_\Gamma \to Y(\Gamma), \Gamma\Psi)$ such that the fibre over $t \in \mathrm{Cusps}(\Gamma)$ is a Néron h_t-gon, where h_t is the width of the cusps t.

Since $\mathcal{E}_\Gamma \to Y(\Gamma)$ is a holomorphic submersion, by Lemma 2.2.11 the k-th fibre power $\mathcal{E}_\Gamma^k := \mathcal{E}_\Gamma \times_{Y(\Gamma)} \mathcal{E}_\Gamma \times_{Y(\Gamma)} \cdots \times_{Y(\Gamma)} \mathcal{E}_\Gamma$ of $\mathcal{E}$ over $Y(\Gamma)$ exists as a holomorphic manifold, and the structure of elliptic curve over $Y(\Gamma)$ on $\mathcal{E}_\Gamma$ defines on $\mathcal{E}_\Gamma^k$ a structure of complex torus of relative dimension k over $Y(\Gamma)$ by Example 4.1.5.

One might hope to construct a smooth compactification of $\mathcal{E}_\Gamma^k \to Y(\Gamma)$ by taking the k-th fibre power of $\mathcal{D}_\Gamma \to X(\Gamma)$. However, for $k \geq 2$ this will not be a complex manifold, but a complex-analytic space that is not a manifold precisely at the points $(x_1, \ldots, x_k) \in \mathcal{D}_\Gamma^k$ where $x_j \in \mathcal{D}_\Gamma^{\mathrm{nsm}}$ for at least two indices j, see Lemma 8.2.1. To remedy the situation, we will go through the same steps as in the construction of $\mathcal{D}_\Gamma$.

(1) Construct a projective desingularisation of $\mathcal{G}^k = (\mathcal{G}_{/\mathrm{Spec}\ \mathbb{C}[q]})^k$ as a sequence of blowups

$$\mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k\langle k-2\rangle \to \ldots \to \mathcal{G}^k\langle 1\rangle \to \mathcal{G}^k\langle 0\rangle = \mathcal{G}^k$$

centered at nonsingular closed subvarieties of $\mathcal{G}^k\langle l\rangle^{\mathrm{nsm}}$.

(2) Construct an action of $\mathbb{Z}^k$ on $\mathcal{G}^k\langle l\rangle$ over $\mathrm{Spec}\ \mathbb{C}[q]$, such that for all $0 \leq l < k-1$ the blowdown morphism $\mathcal{G}^k\langle l+1\rangle \to \mathcal{G}^k\langle l\rangle$ is $\mathbb{Z}^k$-equivariant.

(3) Show that the resulting action of $\mathbb{Z}^k$ on the $(k+1)$-dimensional holomorphic manifold $\mathcal{G}^k\langle k-1\rangle|_\Delta = (\mathcal{G}^k\langle k-1\rangle)^{\mathrm{an}} \times_{\mathbb{C}} \Delta$ is free and proper, and construct the quotient

$$\mathrm{KS}_h^k = (h\mathbb{Z})^k \backslash \mathcal{G}^k\langle k-1\rangle|_\Delta.$$

(4) At each cusp of $X(\Gamma)$, say of width h, we glue in a copy of KS_h^k to $\mathcal{E}_\Gamma^k$. The result of this procedure is a flat, proper, connected holomorphic map $\mathcal{KS}_\Gamma^k \to X(\Gamma)$ whose restriction to the open modular curve is $\mathcal{E}_\Gamma^k \to Y(\Gamma)$.

Furthermore, we will show in Lemma 10.2.4 that the group law $\mathcal{E}_\Gamma^k \times_{Y(\Gamma)} \mathcal{E}_\Gamma^k \to \mathcal{E}_\Gamma^k$ extends to a map $\mathcal{KS}_\Gamma^{k,\mathrm{sm}} \times_{X(\Gamma)} \mathcal{KS}_\Gamma^k \to \mathcal{KS}_\Gamma^k$ with the same properties as the extension $\mathcal{D}_\Gamma^{\mathrm{sm}} \times_{X(\Gamma)} \mathcal{D}_\Gamma \to \mathcal{D}_\Gamma$ of the group law $\mathcal{E}_\Gamma \times_{Y(\Gamma)} \mathcal{E}_\Gamma \to \mathcal{E}_\Gamma$. Moreover, if $\tilde{\Gamma}$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with $\tilde{\Gamma} \subset \Gamma$, we will show in Section Section 10.3 that there is a map $\mathcal{KS}^k_{\tilde{\Gamma}} \to \mathcal{KS}^k_{\Gamma}$ extending the k-th power $p^k_{\Gamma,\tilde{\Gamma}} \colon \mathcal{E}^k_{\tilde{\Gamma}} \to \mathcal{E}^k_{\Gamma}$ of the natural map $p_{\Gamma,\tilde{\Gamma}} \colon \mathcal{E}_{\tilde{\Gamma}} \to \mathcal{E}_{\Gamma}$. We conclude this thesis by showing that the $(k+1)$-dimensional complex manifold $\mathcal{KS}^k_{\Gamma}$ is projective-algebraic in Section 10.4.

10.1 A projective desingularisation of $\mathcal{G}^k$

The aim of this section is to construct a projective desingularization

$$g : \mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k, \tag{10.1}$$

of $\mathcal{G}^k$, which we defined in Section 8.2 as the product of k copies of $\mathcal{G}$ over $\mathcal{G}^0$. There are several equivalent ways of defining g.

Originally Deligne constructed g as a single blowup centered at a cleverly chosen nonreduced closed subscheme of $\mathcal{G}^k$ in his famous paper [Del71][Lemme 5.5]. In this paper he shows that Ramanujan-Petersson's conjecture on the size of the Fourier coefficients of certain modular forms is implied by the Weil conjectures, which he would later also prove.

Scholl in his notes [Sch97][Ch. 7] provides an alternative description of g as a sequence of blowups whose centers are *nonsingular* subvarieties. He first desingularizes members of an affine open cover of $\mathcal{G}^k$, each isomorphic to say Z^k, and then patches these desingularizations together. Then he goes on to explain that the blowup of Z^k can be rooted in toric geometry, where it corresponds to a decomposition of a (cone over a) cube. Based on this latter toric perspective we will give a full combinatorial proof of the existence of a projective desingularization of $\mathcal{G}^k$, without having to resort to explicit calculation with affine charts. Rather it is intrinsically global and does not require patching. However, we refer to [Sch97][Section 7.1] for the proof that this desingularization is a sequence of blowups centered at nonsingular subvarieties.

Let us recall the description of $\mathcal{G}^k$ furnished in Section 8.2 as the toric variety $F(N^k, \Sigma^k)$ attached to a certain RPP decomposition (N^k, Σ^k). We define the lattice $N^k = \mathbb{Z}^{\{0,1,\dots,k\}}$ in $N^k_{\mathbb{R}} = \mathbb{R}^{\{0,1,\dots,k\}}$. Let $\pi : N^k_{\mathbb{R}} \to N^0_{\mathbb{R}}$, $(a_0, a_1, \dots, a_k) \mapsto a_0$ be the projection onto the zeroth coordinate, and consider the affine hyperplane $H^k = \pi^{-1}(1) \subset N^k_{\mathbb{R}}$. Via the isomorphism $\mathbb{R}^k \cong \{1\} \times \mathbb{R}^k = H^k$ we view $\mathbb{R}^k$ as embedded in $N^k_{\mathbb{R}}$. The maximal cones in the fan Σ^k are cones $\sigma_{\underline{i}} = \mathbb{R}_{\geq 0} \cdot B_{\underline{i}}$ over k-dimensional unit-volume hypercubes in $\mathbb{R}^k$ with integral endpoints

$$B_{\underline{i}} = [i_1, i_1+1] \times [i_2, i_2+1] \times \cdots \times [i_k, i_k+1] \subset \mathbb{R}^k, \quad \underline{i} = (i_1, i_2, \dots, i_k) \in \mathbb{Z}^k. \tag{10.2}$$

Just as in Section 8.4, there is an action $\alpha : \mathbb{Z}^k \to \mathrm{Aut}(N^k, \Sigma^k)$ of $\mathbb{Z}^k$ on the RPP decomposition (N^k, Σ^k) given for $\underline{m} = (m_1, m_2, \dots, m_k) \in \mathbb{Z}^k$ by

$$\begin{aligned} \alpha(\underline{m}) : N^k &\to N^k, \\ (a_0, a_1, \dots, a_k) &\mapsto (a_0, m_1 a_0 + a_1, m_2 a_0 + a_2, \dots, m_k a_0 + a_k). \end{aligned} \tag{10.3}$$

Via the equivalence of categories F this corresponds to an action $a : \mathbb{Z}^k \to \mathrm{Aut}(\mathcal{G}^k)$ of $\mathbb{Z}^k$ on $\mathcal{G}^k$ via toric automorphism over $\mathcal{G}^0$. Since $a(\underline{m})(\sigma_{\underline{i}}) = \sigma_{\underline{m}+\underline{i}}$ for $\underline{m}, \underline{i} \in \mathbb{Z}^k$, we see that $\mathbb{Z}^k$ acts simply transitively on the set of maximal cones in Σ^k

$$\Sigma^k(k+1) = \{\sigma_{\underline{i}} : \underline{i} \in \mathbb{Z}^k\}.$$

Consequently, $\mathbb{Z}^k$ acts simply transitively on the members of the affine open cover $\{F(N^k, \sigma_{\underline{i}}) : \underline{i} \in \mathbb{Z}^k\}$ of $\mathcal{G}^k$. To introduce the idea behind the projective desingularization of $\mathcal{G}^k$, let us first construct a projective desingularization for the member $F(N^k, \sigma_{\underline{0}})$ of the affine open cover.

To ease notation, write $B^k = B^k_{\underline{0}} = [0,1]^k$ for the unit k-cube. Let us denote $\upsilon^k = \sigma_{\underline{0}} = \mathbb{R}_{\geq 0} \cdot B^k$ the cone in $N^k_{\mathbb{R}}$ spanned by $\{1\} \times [0,1]^k$. Write Υ^k for the subfan of Σ^k consisting of those cones contained in υ^k, i.e. for the set of faces of υ^k. Finally, we put

$$Z^k := F\left(N^k, \Upsilon^k\right) = \sigma_{\underline{0}}.$$

The set of *vertices* of B^k is $\{0,1\}^k \subset [0,1]^k$. Let $0 \leq l \leq k$. A *codimension-l face* of B^k is the preimage of one of the 2^l vertices of B^l under one of the $\binom{k}{l}$ coordinate projections $\mathbb{R}^k \to \mathbb{R}^l$; thus B^k has $2^l\binom{k}{l}$ faces of codimension l. A *facet* is a face of codimension 1.

More generally, each face θ of B^k spans a cone

$$\sigma_\theta = \mathbb{R}_{\geq 0} \cdot \theta$$

and this sets up a bijection between the faces of B^k and the nonzero faces of υ^k:

$$\Upsilon^k = \{0\} \cup \left\{\sigma_\theta : \theta \text{ is a face of } B^k\right\}.$$

Since υ^k is the unique $(k+1)$-dimensional cone in Υ^k, we have that $O\left(\upsilon^k\right)$ is the unique 0-dimensional T_{N^k}-orbit in Z^k. Thus the point O^k such that $O(\upsilon^k) = \{O^k\}$ is the unique point of Z^k that is fixed under the torus action of T_{N^k}.

We denote the *barycenter* of a positive-dimensional face θ of B^k by $z(\theta)$. Because θ is not a vertex, $2z(\theta)$ is a primitive vector in N^k. For $0 \leq l \leq k-2$ we define the following set of primitive vectors contained in $N \cap |\Upsilon^k|$:

$$V^k_l = \left\{2z(\theta) : \theta \text{ is a codimension-}l\text{ face of } B^k\right\}.$$

Now form a sequence of star-subdivisions

$$\Upsilon^k\langle 0\rangle = \Upsilon^k \text{ and } \Upsilon^k\langle l+1\rangle = \Upsilon^k\langle l\rangle^*\left(V^k_l\right) \text{ for } 0 \leq j < k-1$$

and the corresponding sequence of toric morphisms

$$Z^k\langle k-1\rangle \to Z^k\langle k-2\rangle \to \ldots \to Z^k\langle 1\rangle \to Z^k\langle 0\rangle = Z^k. \qquad (10.4)$$

Description of the fan $\Upsilon^k\langle l\rangle$. An *$l$-flag of faces of* B^k is defined to be a descending sequence of faces of B^k

$$\theta_{\leq l} = \left(B^k = \theta_0 \supset \theta_1 \supset \ldots \supset \theta_l\right)$$

where θ_m has codimension m in B^k. The maximal, i.e. $(k+1)$-dimensional, cones of the fan $\Upsilon^k\langle l\rangle$ correspond bijectively to the l-flags of faces of B^k, by attaching to the l-flag $\theta_{\le l}$ the cone

$$\sigma_{\theta_{\le l}} := \mathbb{R}_{\ge 0} \cdot \{z(\theta_0), z(\theta_1), \dots, z(\theta_{l-1})\} + \mathbb{R}_{\ge 0} \cdot \theta_l.$$

Geometrically, $\Upsilon^k\langle 1\rangle$ is the refinement of Υ^k obtained by decomposing B^k into cones with apex $z(B^k)$ and base one of the $2k$ facets θ_1 of B^k. To obtain $\Upsilon^k\langle 2\rangle$ from $\Upsilon^k\langle 1\rangle$, we further subdivide each facet θ_1 of B^k, which is a $(k-1)$-cube, into $2(k-1)$ pieces corresponding to a facet of θ_2. Then iteratively we subdivide the codimension-j faces of B^k, to obtain the refinement $\Upsilon^k\langle j+1\rangle \to \Upsilon^k\langle j\rangle$.

Since the number of facets of an m-dimensional cube is $2m$, the number of l-flags of faces of B^k is $\prod_{m=k-l+1}^{k}(2m) = 2^l k!/(k-l)!$. We conclude that $Z^k\langle l\rangle$ is the union of the $2^l k!/(k-l)!$ affine toric varieties $F\left(N^k, \sigma_{\theta_{\le l}}\right)$ for each l-flag of faces $\theta_{\le l}$.

Scholl in [Sch97] gives a proof of the following theorem using explicit calculations with affine patches of blowups .

Theorem 10.1.1. *For every $0 \le l < k-1$, the morphism $Z\langle l+1\rangle \to Z^k\langle l\rangle$ in (10.4) is the blowup of $Z^k\langle l\rangle$ at a nonsingular T_{N^k}-stable closed subvariety $F^k\langle l\rangle$ of $Z^k\langle l\rangle^{\mathrm{sing}}$. There exists a T_{N^k}-equivariant isomorphism of pairs*

$$\left(F\left(N^k, \sigma_{\theta_{\le l}}\right), F\left(N^k, \sigma_{\theta_{\le l}}\right) \cap F^k\langle l\rangle\right) \cong \left(\mathbb{A}^l_{\mathbb{C}} \times Z^{k-l}, \mathbb{A}^l_{\mathbb{C}} \times \left\{O^{k-l}\right\}\right)$$

where O^m is the unique fixed point of Z^m under the torus action.

In particular, the composite $Z^k\langle k-1\rangle \to Z^k$ in (10.1) is a T_{N^k}-equivariant projective desingularization of Z^k.

Proof. See [Sch97][Thm. 7.1.2.2] □

Remark 10.1.2. In accordance with Remark 7.2.2 we have the following descriptions. For each $l \in \{0, 1, 2, \dots, k-2\}$ we have

$$F^k\langle l\rangle = \bigcup \left\{V\left(\sigma_{\theta_l}\right) : \theta_l \text{ is a codimension-l face of } B^k\right\},$$
$$Z^k\langle l\rangle^{\mathrm{sing}} = \bigcup \left\{V\left(\sigma_{\theta_{k-2}}\right) : \theta_{k-2} \text{ is a 2-dimensional face of } B^k\right\},$$

while $Z^k\langle k-1\rangle^{\mathrm{sing}} = \emptyset$. Note that indeed $F^k\langle l\rangle \subset Z^k\langle l\rangle^{\mathrm{sing}}$ since $l \le k-2$.

Remark 10.1.3. For each $0 \le l \le k-1$, denote $g_l : Z^k\langle k-1\rangle \to Z^k\langle l\rangle$ the toric morphism induced by the refinement $(N^k, \Upsilon^k\langle k-1\rangle \to (N^k, \Upsilon^k\langle l\rangle)$. Then the inverse image of the singular locus of $Z^k\langle l\rangle$ under g_l is the divisor

$$g_l^{-1}\left(Z^k\langle l\rangle^{\mathrm{sing}}\right) = \bigcup_{m=l}^{k-2} \bigcup \left\{V\left(\mathbb{R}_{\ge 0} \cdot z\left(\theta_m\right)\right) : \theta_m \text{ is a codimension-}m\text{ face of } B^k\right\}.$$

We now give an alternative proof that $Z^k\langle k-1\rangle \to Z^k$ is a projective desingularization based on the combinatorial geometry of the fans involved.

Lemma 10.1.4. *The toric variety $Z^k\langle k-1\rangle$ is smooth.*

Proof. Our proof is based on a volume calculation with the Haar measure on the euclidean space $N_{\mathbb{R}}^k = \mathbb{R}^{\{0,1,\ldots,k\}}$ normalized so that $\operatorname{vol}\left([0,1]^{\{0,1,\ldots,k\}}\right) = 1$.

A $(k+1)$-tuple $(v_m)_{m=1}^k$ of vectors in N^k is a $\mathbb{Z}$-basis for N^k if and only if it has determinant $|\det(v_0, v_1, \ldots, v_k)| = 1$. Consider the $(k+1)$-simplex spanned by these vectors

$$\langle v_0, v_1, \ldots, v_k \rangle := \left\{ \sum_{m=0}^{k} t_m v_i : 0 \leq t_m, \sum_{m=0}^{k} t_m \leq 1 \right\},$$

which has volume

$$\operatorname{vol}\langle v_0, v_1, \ldots, v_k \rangle = \frac{1}{(k+1)!} \cdot |\det(v_0, v_1, \ldots, v_k)|.$$

Let $R = \left\{tv : t \in [0,1], v \in B^k\right\}$ be the closed convex body in $N_{\mathbb{R}}^k$ that is the cone with apex 0 and base B^k. The refinement $\Upsilon^k\langle k-1\rangle \to \Upsilon^k$ divides R into $2^{k-1}k!$ equal-volume $(k+1)$-simplices

$$R \cap \sigma_{\theta_{\leq k-1}} = \langle z(\theta_0), z(\theta_1), \ldots, z(\theta_{k-2}), v_{k-1}, v_k \rangle,$$

one for each $(k-1)$-flag $\theta_{\leq k-1}$ of faces of B^k, where v_{k-1} and v_k are the end points of the line segment θ_{k-1}. Since one has $\operatorname{vol}(R) = \int_0^1 t^k dt = (k+1)^{-1}$, we find that $\operatorname{vol}\left(R \cap \sigma_{\theta_{\leq k-1}}\right) = \left(2^{\bar{k}-1}(k+1)!\right)^{-1}$.

Now consider an arbitrary $(k-1)$-flag $\theta_{\leq k-1}$ of faces of B^k. Write $v_i = 2z(\theta_i)$ for $0 \leq i < k-1$ and $\{v_{k-1}, v_k\} = \partial\theta_{k-1}$, so that $\{v_0, v_1, \ldots, v_k\}$ is the set of fundamental generators for the cone $\sigma_{\theta_{\leq k-1}}$. We now find that

$$\begin{aligned} |\det(v_0, v_1, \ldots, v_{k-2}, v_{k-1}, v_k)| &= \\ |\det(2z(\theta_0), 2z(\theta_1), \ldots, 2z(\theta_{k-2}), v_{k-1}, v_k)| &= \\ 2^{k-1}|\det(z(\theta_0), z(\theta_1), \ldots, z(\theta_{k-2}), v_{k-1}, v_k)| &= \\ 2^{k-1}(k+1)! \operatorname{vol}\langle z(\theta_0), z(\theta_1), \ldots, z(\theta_{k-2}), v_{k-1}, v_k\rangle &= 1. \end{aligned}$$

This shows that each maximal cone in $\Upsilon^k\langle k-1\rangle$ is nonsingular. We conclude using Theorem 7.1.10(3) that $Z^k\langle k-1\rangle = F\left(N^k, \Upsilon^k\langle k-1\rangle\right)$ is a smooth toric variety. □

Proposition 10.1.5. *The morphism $Z^k\langle k-1\rangle \to Z^k$ is a projective desingularization of Z^k.*

Proof. In view of Lemma 7.5.3(2) that toric morphisms induced by star-subdivisions are projective, and the fact that compositions of projective morphisms are projective, the morphism $Z^k\langle k-1\rangle \to Z^k$ is projective. By Lemma 10.1.4 we have that $Z^k\langle k-1\rangle$ is a nonsingular $\mathbb{C}$-variety. It remains to be shown that $g : Z^k\langle k-1\rangle \to Z^k$ restricts to an isomorphism $g^{-1}(Z^{k,\mathrm{sm}}) \to Z^{k,\mathrm{sm}}$.

Similar to Lemma 8.2.1 we see that $Z^{k,\mathrm{sm}} = F(N^k, \Upsilon_{\leq 2}^k)$. Our star-subdivisions are centered at primitive vectors that lie in the relative interior of a cone of dimension greater than 2, so do not alter cones of dimension at most 2. Hence we see by induction on $0 \leq l \leq k-1$ that $\{\sigma \in \Sigma^k\langle l\rangle : \sigma \subset |\Sigma_{\leq 2}^k|\} = \Sigma_{\leq 2}^k$. Since by the Orbit-Cone correspondence we have $g^{-1}(F(N^k, \Sigma_{\leq 2}^k)) = F(N^k, \{\sigma \in \Sigma^k\langle k-1\rangle : \sigma \subset |\Sigma_{\leq 2}^k|\})$, we conclude that there is an isomorphism $g^{-1}(Z^{k,\mathrm{sm}}) \to Z^{k,\mathrm{sm}}$. □

Having resolved the singularities of $\mathcal{G}^k$ locally on the affine open $F(N^k, \sigma_{\underline{0}})$, we are now in a position to construct a global desingularization of $\mathcal{G}^k$. The action of $\mathbb{Z}^k$ on $\mathcal{G}^k$ gives for each $\underline{i} \in \mathbb{Z}^k$ an isomorphism $a(\underline{i})\colon Z^k := F(N^k, \sigma_{\underline{0}}) \to F(N^k, \sigma_{\underline{i}})$, hence a desingularization $Z^k_{\underline{i}} \to F(N^k, \sigma_{\underline{i}})$. Scholl shows in [Sch97][§7.2.5] that the desingularizations $Z^k_{\underline{i}}$ of the $\sigma_{\underline{i}}$ patch together to give one for $\mathcal{G}^k = \bigcup_{\underline{i}\in\mathbb{Z}^k} \sigma_{\underline{i}}$. However, there is a more intrinsic way of seeing this. We simply repeat the procedure via which we constructed the nonsingular refinement $\Upsilon^k\langle k-1\rangle \to \Upsilon^k$, but now consider faces of all k-dimensional cubes $B_{\underline{i}}$, rather than just of $B_{\underline{0}}$. Thus, we set

$$W^k_l = \left\{2z(\theta) : \theta \text{ is a codimension-}l \text{ face of some } B^k_{\underline{i}}\right\},$$

form the sequence of star-subdivisions

$$\Sigma^k\langle 0\rangle = \Sigma^k \text{ and } \Sigma^k\langle j+1\rangle = \Sigma^k\langle j\rangle^*\left(V^k_{k-j}\right) \text{ for } 0 \le j < k-1$$

and the corresponding sequence of toric morphisms

$$\mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k\langle k-2\rangle \to \ldots \to \mathcal{G}^k\langle 1\rangle \to \mathcal{G}^k\langle 0\rangle = \mathcal{G}^k. \tag{10.5}$$

Since the action of Z^k on N^k preserves the fan $\Sigma^k\langle l\rangle$ for every $0 \le l \le k-1$, we see there is an action of $\mathbb{Z}^k$ on $\mathcal{G}^k\langle l\rangle$ such that the morphism $\mathcal{G}^k\langle l\rangle \to \mathcal{G}^k\langle l'\rangle$ is $\mathbb{Z}^k$-invariant for all $0 \le l' \le l \le k-1$.

Scholium 10.1.6. Let $0 \le l \le k-1$. One can describe the full group of toric automorphisms of $\mathcal{G}^k\langle l\rangle$ over $\mathcal{G}^0$ as follows. We define the group

$$\Gamma^k = \mathbb{Z}^k \rtimes \mu_2^k \rtimes S_k,$$

where μ_2^k acts on $\mathbb{Z}^k$ via multiplication, and S_k acts on $\mathbb{Z}^k$ and μ_2^k via permutation. There is a natural action of Γ^k on $H^k_{\mathbb{Z}} := H^k \cap N^k \cong \mathbb{Z}^k$ via integral affine transformations, which restricts to the action of $\mathbb{Z}^k$ by translation, of μ_2^k by multiplication and of S_k by permutation. This action corresponds to an injective homomorphism $\alpha : \Gamma^k \to \mathrm{Aff}\left(H^k_{\mathbb{Z}}\right) = \mathrm{Aut}\left(N^k/N^0\right)$. Since $\mu_2^k \rtimes S^k$ is the group of isometries of the k-cube $[-1,1]^k$, any integral affine transformation $\phi \in \mathrm{Aff}\left(H^k_{\mathbb{Z}}\right)$ which induces a permutation of the set of cubes $\{B_{\underline{i}} : \underline{i} \in \mathbb{Z}^k\}$ belongs to the image of α. By considering primitive vectors in rays of the fan $\Sigma^k\langle l\rangle$, one can show there is an isomorphism

$$\alpha : \Gamma^k \xrightarrow{\sim} \mathrm{Aut}\left(\left(N^k, \Sigma^k\langle l\rangle\right)\right). \tag{10.6}$$

Corollary 10.1.7. *We have that $g : \mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k$ is a projective desingularization of $\mathcal{G}^k$.*

Proof. In Proposition 10.1.5 we have proved that $Z^k\langle k-1\rangle$ is a nonsingular $\mathbb{C}$-variety and that $g : g^{-1}(Z^{k,\mathrm{sm}}) \to Z^{k,\mathrm{sm}}$ is an isomorphism. Because $\mathcal{G}^k$ is covered by $\mathbb{Z}^k$-translates of $Z^k = F(N^k, \sigma_{\underline{0}})$ and $g : \mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k$ is $\mathbb{Z}^k$-equivariant, it follows that $\mathcal{G}^k\langle k-1\rangle$ is smooth and that $g : g^{-1}(\mathcal{G}^{k,\mathrm{sm}}) \to \mathcal{G}^{k,\mathrm{sm}}$ is an isomorphism.

The same argument involving star-subdivisions as in Proposition 10.1.5 shows that $g : \mathcal{G}^k\langle l\rangle \to \mathcal{G}^k\langle l'\rangle$ is a projective morphism for all integers $0 \le l' \le l \le k-1$. Thus the composition $g : \mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k$ of the sequence of projective morphisms starring in (10.5) is projective as well. □

In Section 10.4 we will give an independent proof of the projectivity of the morphism $g : \mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k$ by constructing an explicit g-very ample line bundle on $\mathcal{G}^k$. This line bundle will be used to show that the Kuga-Sato manifolds constructed in the next section are projective-algebraic.

10.2 The compactification $\mathcal{KS}_\Gamma^k$ of $\mathcal{E}_\Gamma^k$

Consider a universal elliptic curve $\mathcal{E}_\Gamma \to Y(\Gamma)$ constructed in Theorem 5.5.4. For each integer $k \geq 2$, the k-th fibre power $\mathcal{E}_\Gamma^k = \mathcal{E}_\Gamma \times_{Y(\Gamma)} \mathcal{E}_\Gamma \times_{Y(\Gamma)} \cdots \times_{Y(\Gamma)} \mathcal{E}_\Gamma$ is a complex torus of relative dimension k over $Y(\Gamma)$. The following theorem shows it has a smooth compactification $\mathcal{KS}_\Gamma^k \to X(\Gamma)$, which we will call the *k-th Kuga-Sato variety attached to* Γ.

Theorem 10.2.1. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a torsion-free congruence subgroup such that all cusps of Γ are regular. Let $k \geq 1$ be an integer. Then there exists a quadruple $(f : \mathcal{KS}_\Gamma^k \to X(\Gamma), m, e, i)$ consisting of*

- *a compact complex manifold $\mathcal{KS}_\Gamma^k$,*
- *a proper holomorphic map $f : \mathcal{KS}_\Gamma^k \to X(\Gamma)$,*
- *a multiplication $m : \mathcal{KS}_\Gamma^{k,\mathrm{sm}} \times_{X(\Gamma)} \mathcal{KS}_\Gamma^k \to \mathcal{KS}_\Gamma^k$,*
- *an identity section $e : X(\Gamma) \to \mathcal{KS}_\Gamma^k$,*
- *an inversion map $i : \mathcal{KS}_\Gamma^k \to \mathcal{KS}_\Gamma^k$,*

having the following four properties:

(1) *the restriction of this quintuple to the open modular curve $Y(\Gamma) \subset X(\Gamma)$ is the $Y(\Gamma)$-complex torus $(f : \mathcal{E}_\Gamma^k \to Y(\Gamma), m, i, e)$ of relative dimension k;*
(2) *the triple (m, i, e) defines a structure of commutative $X(\Gamma)$-group on $\mathcal{KS}_\Gamma^{k,\mathrm{sm}}$;*
(3) *the map m defines an action of $\mathcal{KS}_\Gamma^{k,\mathrm{sm}}$ on $\mathcal{KS}_\Gamma^k$ over $X(\Gamma)$;*
(4) *the maps m and i are compatible in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{KS}_\Gamma^{k,\mathrm{sm}} \times_{X(\Gamma)} \mathrm{KS}_\Gamma^k & \xrightarrow{m} & \mathrm{KS}_\Gamma^k \\ {\scriptstyle i\times i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathrm{KS}_\Gamma^{k,\mathrm{sm}} \times_{X(\Gamma)} \mathrm{KS}_\Gamma^k & \xrightarrow{m} & \mathrm{KS}_\Gamma^k. \end{array}$$

We form the Δ-complex manifold $\mathcal{G}^k\langle k-1\rangle|_\Delta := \mathcal{G}^k\langle k-1\rangle^{\mathrm{an}} \times_{\mathcal{G}^{0,\mathrm{an}}} \Delta$ by applying the analytification functor $(\cdot)^{\mathrm{an}}$ and restricting to $\Delta \subset \mathbb{C} = \mathcal{G}^{0,\mathrm{an}}$. From the action $a : \mathbb{Z}^k \to \mathrm{Aut}(\mathcal{G}^k)$ of $\mathbb{Z}^k$ on $\mathcal{G}^k$ over $\mathcal{G}^0$ defined in (10.3), we obtain an action of $\mathbb{Z}^k$ on $\mathcal{G}^k\langle k-1\rangle|_\Delta$. One deduces from Lemma 8.5.4(i) that the action of $\mathbb{Z}^k$ on $\mathcal{G}^k|_\Delta$ is free and proper, hence that on $\mathcal{G}^k\langle k-1\rangle|_\Delta$ is as well by Lemma 2.3.3.

Definition 10.2.2. Let $k, h \in \mathbb{Z}_{\geq 1}$. We define $\mathrm{KS}_h^k = (h\mathbb{Z})^k \backslash \mathcal{G}^k\langle k-1\rangle|_\Delta$ to be the quotient of $\mathcal{G}^k\langle k-1\rangle|_\Delta$ by the free and proper action of $(h\mathbb{Z})^k$.

Lemma 10.2.3. *For all integers $k, h \geq 1$, there exists a biholomorphism $\tilde{e}_h$ covering e_h as in the diagram below:*

$$\begin{array}{ccc} P_h^+\backslash\mathcal{E}^k & \xrightarrow{\tilde{e}_h} & \mathrm{KS}_h^k|_{\Delta^*} \\ \downarrow & & \downarrow \\ P_h^+\backslash\mathfrak{H} & \xrightarrow{e_h} & \Delta^*. \end{array} \tag{10.7}$$

Proof. Since the desingularization $\mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k$ restricts to an isomorphism over $D(q) \subset \mathcal{G}^0$, we have that

$$\mathrm{KS}_h^k|_{\Delta^*} \cong (\mathrm{Tate}_h|_{\Delta^*})^k.$$

Now in the case that $k = 1$ the isomorphism has been constructed in Lemma 9.1.2. The cases that $k \geq 2$ follow by taking fibre powers of this isomorphism. □

Lemma 10.2.4. *Let $k, h \in \mathbb{Z}_{\geq 1}$. There exists a triple (m_h, e_h, i_h) consisting of*

- *a multiplication map $m_h : \mathrm{KS}_h^{k,\mathrm{sm}} \times_\Delta \mathrm{KS}_h^k \to \mathrm{KS}_h^k$ over Δ;*
- *an identity section $e_h : \Delta \to \mathrm{KS}_h^{k,\mathrm{sm}}$ to $f : \mathrm{KS}_h^k \to \Delta$;*
- *an inversion map $i_h : \mathrm{KS}_h^k \to \mathrm{KS}_h^k$ over Δ,*

such that

1. *the restriction of this triple to the puncture unit disk Δ^* makes (10.7) an isomorphism of complex tori of relative dimension k;*
2. *the triple (m, i, e) defines a structure of commutative Δ-group on KS_h^k;*
3. *the map m defines an action of $\mathrm{KS}_h^{k,\mathrm{sm}}$ on KS_h^k over Δ;*
4. *the maps m and i are compatible in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{KS}_h^{k,\mathrm{sm}} \times_\Delta \mathrm{KS}_h^k & \xrightarrow{m} & \mathrm{KS}_h^k \\ \downarrow{\scriptstyle i\times i} & & \downarrow{\scriptstyle i} \\ \mathrm{KS}_h^{k,\mathrm{sm}} \times_\Delta \mathrm{KS}_h^k & \xrightarrow{m} & \mathrm{KS}_h^k. \end{array}$$

Proof. We only sketch this proof, since we have treated the case $k = 1$ in full detail in Section 8.3 and Section 8.5 and the general case $k > 1$ is very similar. Again we denote $g : \mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k$ the projective desingularization constructed in Corollary 10.1.7.

First we observe that $\mathcal{G}^k\langle k-1\rangle^{\mathrm{sm}} = g^{-1}(\mathcal{G}^{k,\mathrm{sm}}) \cong \mathcal{G}^{k,\mathrm{sm}} = F(N^k, \Sigma_{\leq 1}^k)$. The maximal cones in the fan

$$\Sigma_{\leq 1}^k = \{0\} \cup \{\mathbb{R}_{\geq 0} \cdot (1, \underline{i}) : \underline{i} \in \mathbb{Z}^k\}$$

are the rays spanned by the elements in the set $H_\mathbb{Z}^k := H^k \cap N^k = \{1\} \times \mathbb{Z}^k$. Generalizing Lemma 8.3.1 we define the $\mathbb{Z}$-linear map

$$\mu : N^k \times_{N^0} N^k \to N^k,$$
$$(a_0, a_1, a_2, \ldots, a_k, a'_1, a'_2, \ldots, a'_k) \mapsto (a_0, a_1 + a'_1, a_2 + a'_2, \ldots, a_k + a'_k),$$

which is equivariant for $+ : \mathbb{Z}^k \times \mathbb{Z}^k \to \mathbb{Z}^k$. It defines a morphism of RPP decompositions $\mu : (N^k \times_{N^0} N^k, \Sigma_{\leq 1}^k \otimes_{\Sigma^0} \Sigma^k\langle k-1\rangle) \to (N^k, \Sigma^k\langle k-1\rangle)$ hence a toric morphism

$$m : \mathcal{G}^k\langle k-1\rangle^{\mathrm{sm}} \times_{\mathcal{G}^0} \mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k\langle k-1\rangle. \tag{10.8}$$

Similarly there is a morphism of RPP decompositions $\epsilon : (N^0, \Sigma^0) \to (N^k, \Sigma_{\leq 1}^k)$ given by the following section of π:

$$\epsilon : N^0 \to N^k,$$
$$a_0 \mapsto (a_0, 0, \ldots, 0),$$

which defines a section to $f : \mathcal{G}^k \to \mathcal{G}^0$:

$$e : \mathcal{G}^0 \to \mathcal{G}^k\langle k-1\rangle^{\mathrm{sm}}. \tag{10.9}$$

In the same vein there exists an automorphism ι of order 2 of the RPP decomposition $(N^k, \Sigma^k\langle k-1\rangle)$ given by

$$\begin{gathered}\iota : N^k \to N^k,\\ (a_0, a_1, a_2, \dots, a_k) \mapsto (a_0, -a_1, -a_2, \dots, -a_k),\end{gathered}$$

which defines a $\mathbb{Z}^k$-equivariant involutive automorphism i of $\mathcal{G}^k$ over $\mathcal{G}^0$.

It follows now that m^{an}, i^{an} and e^{an} on $\mathcal{G}^k\langle k-1\rangle|_\Delta$ descends to maps m_h, i_h and e_h on KS_h^k as desired. We omit the verification that they verify properties (1)-(4), to avoid repeating the arguments given in Lemma 8.5.4. □

Proof of Theorem 10.2.1. Let R be a set of representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/P$. We construct the triple $\left(f : \mathcal{KS}_\Gamma^k \to X(\Gamma), m, e, i\right)$ as the pushout of the following diagram to be explained below, in which we set $s = [\gamma](\infty)$ and $h = h_{\Gamma s}$,

$$\begin{array}{ccccccc}
\coprod_\gamma \mathrm{KS}_h^k|_{V_h} & \longleftarrow & \coprod_\gamma P_h^+\backslash\mathcal{E}^k|_{U_\infty} & \longrightarrow & \coprod_\gamma \Gamma_s\backslash\mathcal{E}^k|U_s & \longrightarrow & \mathcal{E}_\Gamma^k \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\coprod_\gamma V_h & \longleftarrow & \coprod_\gamma P_h^+ \setminus U_\infty & \longrightarrow & \coprod_\gamma \Gamma_s\backslash U_s & \longrightarrow & Y(\Gamma).
\end{array}$$

The left square is obtained from the isomorphism Proposition 10.3.2 and the inclusion $V_h \subset V_h^*$. The middle square is the disjoint union over $\gamma \in R$ of the isomorphism $([\gamma]\mathcal{E})^k : \left(\mathcal{E}|_{U_\infty}\right)^k \to \left(\mathcal{E}|_{U_s}\right)^k$ covering $[\gamma] : U_\infty \to U_s$ which is equivariant for the isomorphism $P_h^+ \to \Gamma_s, \delta \mapsto \gamma\delta\gamma^{-1}$. The right square arises from our viewing of $\Gamma_s\backslash U_s$ as an open subset of $Y(\Gamma)$. We see that all squares are cartesian, which renders possible the formation of the pushout $\left(f : \mathcal{KS}_\Gamma^k \to X(\Gamma), m, e, i\right)$.

The verification that the triple (f, m, e, i) has the stated properties is local on the base $X(\Gamma)$, and follows from Lemma 10.2.4. By construction, the restriction of (f, m, e) to $Y(\Gamma)$ is the k-th fibre power $\mathcal{E}_\Gamma^k \to Y(\Gamma)$ of the universal elliptic curve $\mathcal{E}_\Gamma \to Y(\Gamma)$. □

Remark 10.2.5. The morphism $\mathcal{KS}_\Gamma^k \to \mathcal{D}_\Gamma^k$ of $\mathbb{C}$-analytic spaces over $X(\Gamma)$ induces an isomorphism between the open subsets where their structure morphism to $X(\Gamma)$ is submersive:

$$\mathcal{KS}_\Gamma^{k,\mathrm{sm}} \cong \left(\mathcal{D}_\Gamma^{\mathrm{sm}}\right)^k.$$

In particular, for every positive integer N we have that

$$\mathcal{KS}_\Gamma^{k,\mathrm{sm}}[N] = \left(\mathcal{D}_\Gamma^{\mathrm{sm}}[N]\right)^k$$

is a local system of rank- $2k$ free $(\mathbb{Z}/N\mathbb{Z})$-modules.

Examples 10.2.6. (1) Let $N \geq 3$. Then the congruence subgroup $\Gamma(N)$ satisfies the hypotheses of Theorem 10.2.1, which constructs the Kuga-Sato variety $\mathcal{KS}^k_{\Gamma(N)}$. It inherits a level- N structure from the the level- N structure Ψ on $\mathcal{D}_{\Gamma(N)}$ constructed in Section 9.2 In fact, the isomorphism $\Psi : (\mathbb{Z}/N\mathbb{Z})^2 \times X(N) \xrightarrow{\sim} \mathcal{D}^{\text{sm}}_{\Gamma(N)}[N]$, $(a_1, a_2, m) \mapsto \Psi_m(a_1, a_2)$, induces an isomorphism

$$\Psi^k : (\mathbb{Z}/N\mathbb{Z})^{2k} \times X(N) \to \left(\mathcal{D}^{\text{sm}}_{\Gamma(N)}[N]\right)^k \cong \mathcal{KS}^{k,\text{sm}}_{\Gamma(N)}[N],$$
$$(a_1, a_2, \ldots, a_{2k-1}, a_{2k}, m) \mapsto (\Psi_m(a_1, a_2), \Psi_m(a_3, a_4), \ldots, \Psi_m(a_{2k-1}, a_{2k})).$$

Thus $\mathcal{KS}^{k,\text{ sm}}_{\Gamma(N)}[N]$ is a globally trivial local system of rank- $2k$ free $(\mathbb{Z}/N\mathbb{Z})$-modules.
(2) Let $N \geq 5$. Then the congruence subgroup $\Gamma_1(N)$ satisfies the hypotheses of Theorem 10.2.1, which constructs the Kuga-Sato variety $\mathcal{KS}^k_{\Gamma_1(N)}$. Recall from Section 9.3 that there is an ample point $Q : X_1(N) \to \mathcal{D}^{\text{sm}}_{\Gamma_1(N)}$ of exact order N. That is, Q is a section of $f : \mathcal{D}^{\text{sm}}_{\Gamma_1(N)}[N] \to X_1(N)$ that induces an injective homomorphism of group holomorphic manifolds over $X_1(N)$

$$(\mathbb{Z}/N\mathbb{Z}) \times X_1(N) \to \mathcal{D}^{\text{sm}}_{\Gamma_1(N)}[N],$$
$$(i, m) \mapsto [i](P_m),$$

whose image meets every connected component of every fibre of $f : \mathcal{D}^{\text{sm}}_{\Gamma_1(N)} \to X_1(N)$. Consequently, $f : \mathcal{KS}^{k,\text{sm}}_{\Gamma_1(N)}[N] \to X_1(N)$ has a section

$$(Q, Q, \ldots, Q) : X_1(N) \to \left(\mathcal{D}^{\text{sm}}_{\Gamma_1(N)}[N]\right)^k \cong \mathcal{KS}^{k,\text{sm}}_{\Gamma_1(N)}[N],$$

that induces an injective homomorphism of group holomorphic manifolds over $X_1(N)$

$$(\mathbb{Z}/N\mathbb{Z})^k \times X_1(N) \to \left(\mathcal{D}^{\text{sm}}_{\Gamma_1(N)}[N]\right)^k \cong \mathcal{KS}^{k,\text{sm}}_{\Gamma_1(N)}[N],$$
$$(i_1, i_2, \ldots, i_k, m) \mapsto ([i_1](P_m), [i_2](P_m), \ldots, [i_k](P_m)),$$

whose image meets each connected component of each fibre of $f : \mathcal{KS}^{k,\text{sm}}_{\Gamma_1(N)} \to X_1(N)$.

10.3 Morphisms between Kuga–Sato varieties

We constructed in Section 9.4 a holomorphic map $\mathcal{D}_{\tilde{\Gamma}} \to \mathcal{D}_{\Gamma}$ of the Shioda modular surfaces attached to an inclusion $\tilde{\Gamma} \subset \Gamma$ of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. In this section we prove as Theorem 10.3.4 that for every $k \in \mathbb{Z}_{\geq 2}$ there exists a holomorphic map $\mathcal{KS}^k_{\tilde{\Gamma}} \to \mathcal{KS}^k_{\Gamma}$ of the Kuga–Sato varieties as well. This will be clear from an alternative description of the maximal cones in the fan $\Sigma^k\langle k-1\rangle$, which will also be used in Section 10.4, which we now provide.

Notation 10.3.1. Let $x_0 \in \mathbb{R}_{\geq 0}$, and let $x \in \mathbb{R}$. Define $\operatorname{dist}(x, x_0\mathbb{Z}) = \min_{i \in \mathbb{Z}} |x - x_0 i|$. This is a piecewise affine function on $\mathbb{R}$. The maximal intervals on which $\operatorname{dist}(\cdot, x_0\mathbb{Z})$ is linear have the shape $x_0[j - \frac{1}{4}, j + \frac{1}{4}]$ for some $j \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$, and on such an interval we have

$$\operatorname{dist}(x, x_0\mathbb{Z}) = \operatorname{sgn}(\lceil j \rfloor - j)(x - jx_0), \quad x \in [j - \frac{1}{4}, j + \frac{1}{4}].$$

Here for $j \in \mathbb{Z}\backslash\frac{1}{2}\mathbb{Z}$ we denote $\lceil j \rfloor$ the unique integer $i \in \mathbb{Z}$ such that $|j-i| = \operatorname{dist}(j, \mathbb{Z})$. If $j \in \mathbb{R} \setminus \{0\}$ we let $\operatorname{sgn}(j) = 1$ if $j > 0$ and $\operatorname{sgn}(j) = -1$ if $j < 0$.

We let S_k be the group of permutations of $\{1, 2, \ldots, k\}$. We consider pairs $(\tau, \underline{j})$ where $\tau : \{1, 2, \ldots, l\} \to \{1, 2, \ldots, k\}$ is an injection and $\underline{j} \in (\frac{1}{4} + \frac{1}{2}\mathbb{Z})^l \times (\frac{1}{2} + \mathbb{Z})^{k-l}$ is a k-tuple with $j_i \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ for all $1 \le i \le l$ and $j_i \in \frac{1}{2} + \mathbb{Z}$ for all $l < i \le k$. To the pair $(\tau, \underline{j})$ we associate the subset of $(k+1)$-tuples $(x_0, x_1, \ldots, x_k) \in N_{\mathbb{R}}^k$ such that the distances $\operatorname{dist}(x_i, x_0) : 1 \le i \le k$ for $i \in \{1, 2, \ldots, k\}$ are ordered according to

$$\begin{aligned} \operatorname{dist}(x_{\tau(1)}, x_0\mathbb{Z}) &\le \operatorname{dist}(x_{\tau(2)}, x_0\mathbb{Z}) &&\le \operatorname{dist}(x_{\tau(2)}, x_0\mathbb{Z}) \le \cdots \le \operatorname{dist}(x_{\tau(l-1)}, x_0\mathbb{Z}) \\ &\le \operatorname{dist}(x_{\tau(l)}, x_0\mathbb{Z}) &&\le \min\{\operatorname{dist}(x_i, x_0\mathbb{Z}) : i \notin \operatorname{image}(\tau)\} \end{aligned} \tag{10.10}$$

(in case $l = 0$ this condition is to be interpreted as being void) and the individual coordinates are subject to the following system of inequalities

$$\begin{cases} x_0 \ge 0, \\ x_{\tau(i)} \in x_0[j_i - \frac{1}{4}, j_i + \frac{1}{4}] \text{ for all } 1 \le i \le l, \\ x_{\tau(i)} \in x_0[j_i - \frac{1}{2}, j_i + \frac{1}{2}] \text{ for all } l < i \le k. \end{cases} \tag{10.11}$$

Proposition 10.3.2. *The full-dimensional cones in $\Sigma^k\langle l\rangle$ are given by $\sigma_{\tau,\underline{j}}$ for a uniquely determined pair $(\tau, \underline{j})$.*

Proof. One may show this by induction on l using the definition of star-subdivisions. □

Corollary 10.3.3. *Let $k \in \mathbb{Z}_{\ge 2}$. Then the union of the k-dimensional cones in $\Sigma^k\langle k-1\rangle$ is the union of the affine hyperplanes $\{x_i = mx_0\}$, $\{x_i + x_j = mx_0\}$ and $\{x_i - x_j = mx_0\}$ for all $m \in \mathbb{Z}$ and $1 \le i < j \le k$, i.e.*

$$\bigcup \Sigma^k(k) = \bigcup_{m\in\mathbb{Z}} \bigcup_{1\le i<j\le k} \{x_i = mx_0\} \cup \{x_i + x_j = mx_0\} \cup \{x_i - x_j = mx_0\}. \tag{10.12}$$

Having established a new description of the fan $\Sigma^k\langle k-1\rangle$ through the boundaries of the maximal cones, we will now explain how to construct a holomorphic map $\mathrm{KS}^k_{\tilde{\Gamma}} \to \mathrm{KS}^k_{\Gamma}$. The procedure of doing so is entirely similar to what we did in Section 8.6 and Section 9.4, so in order to avoid repeating ourselves we only point out the differences.

Theorem 10.3.4. *Let $k \in \mathbb{Z}_{\ge 2}$ and let $\tilde{\Gamma} \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be congruence subgroups. Assume that Γ is torsion-free and has only regular cusps. Then there exists a holomorphic map*

$$\tilde{p}_{\tilde{\Gamma},\Gamma} \colon \mathrm{KS}^k_{\tilde{\Gamma}} \to \mathrm{KS}^k_{\Gamma} \text{ covering } p_{\tilde{\Gamma},\Gamma} \colon X(\tilde{\Gamma}) \to X(\Gamma).$$

Proof. Let h and e be positive integers and set $\tilde{h} = h$. The proof of Proposition 9.4.2 may be repeated line by line except for the construction of a holomorphic map $\mathrm{KS}^k_{\tilde{h}} \to \mathrm{KS}^k_h$ covering $p_e : \Delta \to \Delta$. As in Proposition 8.6.2 it suffices to construct a morphism $\mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k\langle k-1\rangle \times_{\mathcal{G}^0, r_e} \mathcal{G}^0$, where $r_e = F([e])$ with $[e] : N^0 \to N^0$, $a_0 \mapsto ea_0$.

Since $\pi : N^k \to N^0$ is a weakly semistable morphism of RPP decompositions, the latter fibre products is also toric variety, whose lattice admits an isomorphism $(\rho_e, \pi) : N^k \to N^k \times_{N^0,[e]} N^0$, where $\rho_e : N^k = \mathbb{Z}^{\{0,1,\dots,k\}} \to N^k = \mathbb{Z}^{\{0,1,\dots,k\}}$, $(a_0, a_1, \dots, a_k) \mapsto (ea_0, a_1, \dots, a_k)$. Its fan is given by $\Sigma^{k,e}\langle k-1\rangle$, where for every $0 \le l \le k-1$ we let $\Sigma^{k,e}\langle l\rangle = \{\rho_e^{-1}(\sigma) : \sigma \in \Sigma^k\langle l\rangle\}$. In terms of their intersections with $H^k = \{1\} \times \mathbb{R}^k \cong \mathbb{R}^k$, one obtains $\Sigma^{k,e}$ from Σ^k by performing scalar multiplication with a factor e. We have that the maximal cones in the fan $\Sigma^{k,e} = \Sigma^{k,e}\langle 0\rangle$ are cones over $(e\mathbb{Z})^k$-translate of the cube $[0,e]^k \subset H^k$:

$$\sigma_{\underline{i},e} = \mathbb{R}_{\ge 0} \cdot ([ei_1, e(i_1+1)] \times [ei_2, e(i_2+1)] \times \cdots \times [ei_k, e(i_k+1)]) \quad \underline{i} \in \mathbb{Z}^k, \tag{10.13}$$

and $\Sigma^{k,e}\langle l\rangle$ is obtained from $\Sigma^{k,e}$ by performing star-subdivisions similar to how $\Sigma^k\langle l\rangle$ was obtained from Σ^k (but working with cubes that are e times larger). The existence of a toric morphism is now reduced to the following lemma. □

Lemma 10.3.5. *For all $e \in \mathbb{Z}_{\ge 1}$ and $k \in \mathbb{Z}_{\ge 2}$ we have that $\Sigma^{k,e}\langle k-1\rangle$ is a refinement of $\Sigma^k\langle k-1\rangle$.*

Proof. Let $\sigma \in \Sigma^k\langle k-1\rangle(k+1)$ be a maximal cone in $\Sigma^k\langle k-1\rangle$. We need only show that σ is contained in a maximal cone of $\Sigma^{k,e}\langle k-1\rangle$. If this were not the case, then σ would 'stick out' of a cone in $\Sigma^{k,e}\langle k-1\rangle(k+1)$, more precisely, the boundary of σ would not be contained in $\Sigma^{k,e}\langle k-1\rangle(k)$, the union of the boundaries of the cones in $\Sigma^{k,e}\langle k-1\rangle(k+1)$. However it follows from Corollary 10.3.3 that

$$\bigcup \Sigma^{k,e}\langle k-1\rangle(k) \subset \bigcup \Sigma^k\langle k-1\rangle(k),$$

since $\bigcup \Sigma^{k,e}\langle k-1\rangle(k)$ is (10.12) but the first union taking place only over $m \in e\mathbb{Z}$. The boundary of the cone σ is the union of the k-dimensional faces of σ, hence contained in $\Sigma^k\langle k-1\rangle(k)$. Thus we obtain a contradiction. □

10.4 Projectivity

Let $0 \le l \le k-1$ be integers. Let $0 \le m < l$ and let θ_m be a codimension-m face of some $\mathbb{Z}^k$-translate of B^k. Then we write D_{θ_m} for the prime divisor on $\mathcal{G}^k\langle l\rangle$ corresponding to the ray $\mathbb{R}_{\ge 0} \cdot z(\theta_m)$. Although it is important to keep track of the fact that D_{θ_m} is a prime divisor on $\mathcal{G}^k\langle l\rangle$, we have chosen not include the l in the notation. This is justified by the observation that if $l \le l' \le k-1$, then the prime divisor D_{θ_m} on $\mathcal{G}^k\langle l'\rangle$ is the strict transform of the prime divisor D_{θ_m} on $\mathcal{G}^k\langle l\rangle$, and conversely the prime divisor D_{θ_m} on $\mathcal{G}^k\langle l\rangle$ is the image of the prime divisor D_{θ_m} on $\mathcal{G}^k\langle l'\rangle$.

Lemma 10.4.1. *Let $k \ge 2$ and let $0 \le l \le k-1$. Denote $g^k\langle l\rangle : \mathcal{G}^k\langle l\rangle \to \mathcal{G}^k$ the proper toric morphism induced by the refinement $(N^k, \Sigma^k\langle l\rangle) \to (N^k, \Sigma^k)$. Consider the function $\phi_D : |\Sigma^k\langle l\rangle| \to \mathbb{R}$ given by*

$$\phi_D(\underline{x}) := 2 \min_{\tau} \sum_{i=1}^{l} (l-i+1)\,\mathrm{dist}(x_{\tau(i)}, x_0\mathbb{Z}),$$

where τ ranges over all injective maps $\{1, 2, \ldots, l\} \to \{1, 2, \ldots, k\}$. Then ϕ_D is the support function of a torus-invariant $\mathbb{Z}^k$-invariant $g^k\langle l\rangle$-ample Cartier divisor D. We have $D = \sum_{m=0}^{l-1} \sum_{\theta_m} \binom{l-m+1}{2} D_{\theta_m}$ on $\mathcal{G}^k\langle l\rangle$, where θ_m ranges over codimension-m faces of some $\mathbb{Z}^k$-translate of B^k.

Proof. Recall Notation 10.3.1, in particular the description of the maximal cones $\sigma_{\tau,\underline{j}}$ given in Proposition 10.3.2. Consider the linear function

$$\phi_{\tau,\underline{j}}(\underline{x}) = 2\sum_{i=1}^{l}(l-i+1)\,\mathrm{sgn}(\lceil j_i\rfloor - j_i)(x_{\tau(i)} - x_0\lceil j_i\rfloor), \quad \underline{x} \in \sigma_{\tau,\underline{j}}.$$

We claim that $\phi_D(\underline{x}) = \phi_{\tau,\underline{j}}(\underline{x})$ if $\underline{x} \in \sigma_{\tau,\underline{j}}$. Indeed, in view of (10.3.1) the second inequality of (10.11) gives that $\mathrm{dist}(x_{\tau(i)}, x_0\mathbb{Z}) = \mathrm{sgn}(\lceil j_i\rfloor - j_i)(x_{\tau(i)} - x_0\lceil j_i\rfloor)$ for $1 \le i \le l$. In view of the rearrangement inequality, (10.10) shows that τ attains the minimum in (10.4.1).

We will now prove that ϕ_D is convex on each maximal cone σ of Σ^k. We have that $\sigma = \mathbb{R}_{\ge 0} \cdot ([h_1, h_1+1], \ldots [h_k, h_k+1])$ for some $(h_i)_{i=1}^k \in \mathbb{Z}^k$. On each interval $[h_i, h_{i+1}]$ the function $t \mapsto \mathrm{dist}(t, x_0\mathbb{Z})$ is convex. As a consequence, we have that $\mathrm{dist}(x_i, x_0\mathbb{Z}) = \min(x_i - h_i x_0, (h_i+1)x_0 - x_i)$. It follows that

$$\phi_D(\underline{x}) = \min_{\sigma_{\tau,\underline{j}} \subset \sigma} \phi_{\tau,\underline{j}}(\underline{x}),$$

where (τ, j) ranges over all pairs as above such that $\sigma_{\tau,\underline{j}} \subset \sigma$. Since any linear function is convex, and the minimum of a finite set of convex functions is convex, we conclude that ϕ_D is convex on σ.

To prove that ϕ_D is strictly convex on σ, it is necessary and sufficient that ϕ_D is given on distinct cones $\sigma_{\tau,\underline{j}}$ refining σ by distinct linear functions $\phi_{\tau,\underline{j}}$. Let $(\tau, \underline{j})$ and $(\tau', \underline{j}')$ be two pairs as above. If $\phi_{\tau,\underline{j}} = \phi_{\tau',\underline{j}'}$, then comparing the coefficients at x_i in (10.4) shows that $\tau = \tau'$ and $\underline{j} - \underline{j}' \in \mathbb{Z}^k$. If in addition $\sigma_{\tau,\underline{j}} \cup \sigma_{\tau',\underline{j}'} \subset \sigma$, then $\underline{j}, \underline{j}' \in \prod_{i=1}^k \{h_i + \frac{1}{4}, h_i + \frac{3}{4}\}$. We conclude that $(\tau, \underline{j}) = (\tau', \underline{j}')$ as desired. □

Theorem 10.4.2. *The Kuga–Sato variety $\mathcal{E}_\Gamma^k$ is a projective algebraic $(k+1)$-dimensional complex manifold.*

Proof. It follows from Lemma 10.4.1 that $\mathcal{L} := \mathcal{O}(kD)$ is a $\mathbb{Z}^k$-invariant torus-invariant divisor which is very ample relative to $g\colon \mathcal{G}^k\langle k-1\rangle \to \mathcal{G}^k$. Since the support of kD does not meet T_{N^k}, the restriction of $\mathcal{L}$ to T_{N^k} is trivial. By Lemma 2.7.9 we see that $\mathcal{L}^{\mathrm{an}}$ is a very ample divisor relative to $g^{\mathrm{an}} : \mathcal{G}^k\langle k-1\rangle^{\mathrm{an}} \to \mathcal{G}^{k,\mathrm{an}}$, trivialized over Δ^*. Since it is invariant with respect to the $\mathbb{Z}^k$-action on $\mathcal{G}^k\langle k-1\rangle^{\mathrm{an}}$, for every $h \in \mathbb{Z}_{\ge 1}$ it descends to a very ample divisor $\mathcal{L}_h$ on $KS_h^k \to \mathrm{Tate}_h^k$, again with a trivialization over Δ^*.

By construction $\mathcal{KS}_\Gamma^k \to \mathcal{D}_\Gamma^k$ is the pushout of the diagram

$$\begin{array}{ccccccc}
\bigsqcup_\gamma \mathrm{KS}_h^k|_{V_h} & \longleftarrow & \bigsqcup_\gamma P_h^+\backslash\mathcal{E}^k|_{U_\infty} & \longrightarrow & \bigsqcup_\gamma \Gamma_s\backslash\mathcal{E}^k|U_s & \longrightarrow & \mathcal{E}_\Gamma^k \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigsqcup_\gamma \mathrm{Tate}_h^k|_{V_h} & \longleftarrow & \bigsqcup_\gamma P_h^+ \setminus \mathcal{E}^k|_{U_\infty} & \longrightarrow & \bigsqcup_\gamma \Gamma_s\backslash\mathcal{E}^k|_{U_s} & \longrightarrow & \mathcal{E}_\Gamma^k.
\end{array}$$

Trivially, the trivial line bundle $\mathcal{O}$ on $\mathcal{E}_\Gamma^k$ is very ample relative to the identity morphism. For every cusps s of Γ, say of width h, we use the trivialization of $\mathcal{L}_h$ over Δ^* to glue it to the trivial bundle on $\mathcal{E}_\Gamma^k$, yielding a line bundle $\mathcal{L}_0$ on KS_Γ^k. Since very amplitude of a line bundle relative to a morphism is local on the target of the morphism, it follows that $\mathcal{L}_0$ is very ample relative to $g : \mathcal{KS}_\Gamma^k \to \mathcal{D}_\Gamma^k$.

We proved in Theorem 9.5.2 that the Shioda modular surface $\mathcal{D}_\Gamma$ is projective-algebraic. Using Lemma 2.7.4 we deduce that $\mathcal{D}_\Gamma^k = \mathcal{D}_\Gamma \times_{X(\Gamma)} \cdots \times_{X(\Gamma)} \mathcal{D}_\Gamma$ is a projective-algebraic manifold. Since $\mathcal{L}_0$ is very ample relative to $g : \mathcal{KS}_\Gamma^k \to \mathcal{D}_\Gamma^k$ by Corollary 2.7.8, we conclude that $\mathcal{KS}_\Gamma^k$ is also projective-algebraic. □

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