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Schipper, R.S.D.

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Compactifications of Metric Spaces

Raoul Schipper
s2056410
Supervisor: Mark Roelands
Bachelor Mathematics

Universiteit Leiden
The Netherlands

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Introduction

Compactifications in mathematics date back to the late 19^{th} and early 20^{th} centuries. Maurice Fréchet and Felix Hausdorff both laid the foundational work on compact spaces, the concept that every open cover has a finite subcover. The notion of compactification emerged as an extension of these compact spaces. The motivation behind compactifications was to find ways to extend a given space in a topological sense by adding limit points or 'points at infinity'.

The process of compactification in topology entails changing a given topological space into a compact space, and there exist various methods to achieve this goal. The idea underlying compactifications is the embedding of the original topological space into a compact one. Within the scope of this thesis, our focus lies specifically on metric compactifications, which involve the embedding of metric spaces into compact spaces. Notably, in the case of the real numbers, this process entails the addition of the points ∞ and $-\infty$.

In the first chapter, we introduce some fundamental concepts and results that are necessary for this thesis. Then, we define the compactification of metric spaces in the second chapter, where we will give an example of the real numbers. We will also demonstrate how we can extend isometries to homeomorphisms on metric compactifications. In the final chapter, we consider the metric compactification of the Euclidean d-dimensional space equipped with the p-norm. This compactification is particularly interesting due to the fact that we can explicitly compute the 'points at infinity'. To determine these points, we utilize the fact that the space is metrizable.

1 Preliminaries

In this chapter we will introduce basic definitions, propositions and theorems that will appear throughout this thesis. All vector spaces will be taken over $\mathbb{R}$.

1.1 Topological spaces

Definition 1.1 (Subspace Topology)**.** Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X. Define

$$\mathcal{T}_Y := \{Y \cap U \mid U \in \mathcal{T}\}.$$

We call $\mathcal{T}_Y$ the *subspace topology* on Y.

Definition 1.2 (Neighborhood)**.** Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A *neighborhood* of x is a set N such that there is an open set $U \subset N$ with $x \in U$.

Definition 1.3 (Open Cover)**.** Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. An *open cover* of Y is a subset $S \subset \mathcal{T}$ such that $Y \subset \bigcup_{U \in S} U$.

Definition 1.4 (Compact)**.** Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. Then Y is *compact* if for all open covers S of Y there is a finite subset $S' \subset S$ such that $\bigcup_{U \in S'} U \cap Y = Y$.

Proposition 1.5 (Closed Subset in a Compact Set is Compact)**.** *Let $(X, \mathcal{T})$ be a compact space and let $Y \subset X$ be closed. Then Y is compact.*

Proof. Let $\{F_i\}_{i \in I}$ be an open cover of Y. Since Y^c is open, we know that $\{F_i\}_{i \in I} \cup Y^c$ is an open cover of X. Due to compactness of X there is a finite subset $J \subset I$ such that $\{F_i\}_{i \in J} \cup Y^c$ is an open cover of X. Thus $\{F_i\}_{i \in J}$ is a finite open subcover of Y, hence Y is compact. □

Definition 1.6 (Continuity)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a function. Then f is *continuous* if $f^{-1}(A) \in \mathcal{T}_X$ for all $A \in \mathcal{T}_Y$.

Definition 1.7 (Homeomorphism)**.** A *homeomorphism* between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a continuous bijection $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ with a continuous inverse.

Proposition 1.8. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and let $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous function. If $C \subset X$ is compact then $f(C)$ is also compact.*

Proof. For the proof see [1, Proposition 9.5]. □

Definition 1.9 (Basis)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $B \subset \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every set in $\mathcal{T}$ can be written as a union of elements of B.

Definition 1.10 (Subbasis)**.** Let $(X, \mathcal{T})$ be a topological space. We call the subset $S \subset \mathcal{T}$ a *subbasis* if the set of finite intersections of elements in S is a basis for $\mathcal{T}$.

Definition 1.11 (Product Topology)**.** Let I be an index set and for every $i \in I$ let $(X_i, \mathcal{T}_i)$ be a topological space. The *product topology* on $X := \prod_{i \in I} X_i$ is the topology $\mathcal{T}$ such that the set

$$B := \left\{ \prod_{i \in I} U_i \mid U_i \in \mathcal{T}_i,\ U_i \neq X_i \text{ for finitely many } i \right\}$$

is a basis for $\mathcal{T}$.

Remark 1.12. An equivalent definition uses projections. For every $i \in I$ we define the projection $p_i \colon X \to X_i$. The product topology of X is the set $\mathcal{T}$ such that

$$S := \{p_i^{-1}(U) \mid i \in I, U \in \mathcal{T}_i\}$$

is a subbasis for $\mathcal{T}$.

Definition 1.13 (Hausdorff Space)**.** The topological space $(X, \mathcal{T})$ is a *Hausdorff space* if for all distinct $x, y \in X$ there are disjoint $U, V \in \mathcal{T}$ with $x \in U$ and $y \in V$.

Definition 1.14 (Finite Intersection Property)**.** Let X be a set and let Y be a set of subsets of X. We say that Y has the *finite intersection property* if for all finite sets $Y' \subset Y$ the set $\bigcap_{F \in Y'} F$ is nonempty.

Proposition 1.15 (Compactness Equivalence)**.** *Let $(X, \mathcal{T})$ be a topological space then the following statements are equivalent:*

(i) *$(X, \mathcal{T})$ is compact,*

(ii) *for any set $\{Y_i\}_{i \in I}$ of closed subsets of X with the finite intersection property, the intersection $\bigcap_{i \in I} Y_i$ is nonempty.*

Proof. (i) $\Rightarrow$ (ii):
Let $\{Y_i\}_{i \in I}$ be a set of closed subsets of X with the finite intersection property. Assume that $\bigcap_{i \in I} Y_i = \varnothing$. This means that $X = (\bigcap_{i \in I} Y_i)^c = \bigcup_{i \in I} Y_i^c$, so $\{Y_i^c\}_{i \in I}$ is an open cover of X. There exists a finite set $J \subset I$ such that $\bigcup_{j \in J} Y_j^c = X$. Hence, $\varnothing = (\bigcup_{j \in J} Y_j^c)^c = \bigcap_{j \in J} Y_j$ but this implies that $\{Y_i\}_{i \in I}$ does not have the finite intersection property anymore. By contradiction we find $\bigcap_{i \in I} Y_i \neq \varnothing$.
(i) $\Leftarrow$ (ii):
Let $\{Y_i\}_{i \in I}$ be an open cover of X. Assume it does not have a finite subcover. In other words, for any finite $J \subset I$ we have $\bigcup_{j \in J} Y_j \neq X$. This means that $\varnothing \neq (\bigcup_{j \in J} Y_j)^c = \bigcap_{j \in J} Y_j^c$, hence, $\{Y_i^c\}_{i \in I}$ has the finite intersection property and thus $\varnothing \neq \bigcap_{i \in I} Y_i^c = (\bigcup_{i \in I} Y_i)^c$ but $\bigcup_{i \in I} Y_i = X$. By contradiction we find that $\{Y_i\}_{i \in I}$ has a finite subcover and X is compact. □

1.2 Nets and ordered sets

Definition 1.16 (Preorder)**.** A relation $\leq$ is called a *preorder* of a set X if it has the following properties for all $x, y, z \in X$:

(i) $x \leq x$ (reflexivity),

(ii) if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

Definition 1.17 (Partially Ordered Set)**.** Let X be a set. Then $(X, \leq)$ is called a *partially ordered set* if for all $x, y, z \in X$ the preorder also satisfies:

if $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry).

Definition 1.18 (Directed Set)**.** A *directed set* is a nonempty set X with a preorder $\leq$ and every pair of elements in X has an *upper bound* in X. That is, for all $x, y \in X$ there exists $z \in X$ with $x \leq z$ and $y \leq z$.

Definition 1.19 (Totally Ordered Set)**.** Let $(X, \leq)$ be a partially ordered set. We call X a *totally ordered set* if for all $x, y \in Y$ we have $x \leq y$ or $y \leq x$.

Definition 1.20 (Maximal Element)**.** Let $(X, \leq)$ be a partially ordered set. We say that $m \in X$ is a *maximal element* if for all $x \in X$ with $m \leq x$, it holds that $m = x$.

Theorem 1.21 (Zorn's Lemma)**.** *Let A be an partially ordered set which has the property that all totally ordered subsets of A have an upper bound in A. Then the set A contains at least one maximal element.*

Definition 1.22 (Net)**.** Let X be a set and I a directed set. Then we call $\{x_i\}_{i \in I}$ with $x_i \in X$ a *net* in X.

Definition 1.23 (Net Convergence)**.** A net $\{x_i\}_{i \in I}$ in a topological space $(X, \mathcal{T})$ is said to *converge* to $x \in X$ if for every open neighbourhood $U \in \mathcal{T}$ of x there is an $i_0 \in I$ such that for all $i \geq i_0$ it holds that $x_i \in U$.

In general, continuity is not implied by converging sequences. However, the more general concept of a net does imply continuity. To illustrate this with an example, we will look at a function between topological spaces that is not continuous, but which does preserve converging sequences. For this example we will need the definition of the cocountable topology.

Definition 1.24 (Cocountable Topology)**.** The *cocountable* topology of a set X, denoted by $\mathcal{T}_{cc}$, consists of the empty set and all the cocountable subsets of X, which are those sets whose complement in X are countable.

Lemma 1.25. *Let X be an uncountable set equipped with the cocountable topology. Then every convergent sequence in X is eventually constant.*

Proof. Let $(x_n)_n$ be a sequence converging to $x \in X$. When we define the set $U := X \setminus \{x_n \mid x_n \neq x\}$ it follows that $U^c = \{x_n \mid x_n \neq x\}$ is countable, so U is open in X. Since $x \in U$ there exists an N such that for all $m \geq N$ we have $x_m \in U$. This means that $x_m = x$ for all $m \geq N$. $\square$

Let X be an uncountable set and the denote the discrete topology by $\mathcal{T}_{dis}$. Consider the identity function $id\colon (X, \mathcal{T}_{cc}) \to (X, \mathcal{T}_{dis})$. Suppose $(x_n)_n$ converges to x for $\mathcal{T}_{cc}$, then by Lemma 1.25 we know that the sequence is eventually constant, hence

$$\lim_{n\to\infty} id(x_n) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x = id(x).$$

So we see that $(id(x_n))_n$ converges to $id(x) \in X$ for $\mathcal{T}_{dis}$. However when we take $\{x\} \in \mathcal{T}_{dis}$ we see that $id^{-1}(\{x\}) = \{x\}$, but $\{x\} \notin \mathcal{T}_{cc}$ so id is not continuous. The following definition is from [5, p.281].

Definition 1.26 (Pointwise Convergence Topology)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Define Y^X as the set of all functions from X to Y. Consider $x \in X$ and $U \in \mathcal{T}_Y$ and define

$$\mathcal{U}(x, U) := \{f \in Y^X \mid f(x) \in U\}.$$

The sets $\mathcal{U}(x, U)$ form a subbasis for the topology on Y^X, which is called the *topology of pointwise convergence*.

Remark 1.27. Later in this thesis we will be working with metric spaces such as $Y = \mathbb{R}$. For $f \in \mathbb{R}^X, x \in \mathbb{R}$ and $\epsilon > 0$ we use the following sets:

$$\mathcal{U}(f, x, \epsilon) := \{g \in \mathbb{R}^X \mid |g(x) - f(x)| < \epsilon\}.$$

We can rewrite this set as $\mathcal{U}(x, (f(x) - \epsilon, f(x) + \epsilon))$, since

$$-\epsilon < g(x) - f(x) < \epsilon$$

and

$$f(x) - \epsilon < g(x) < \epsilon + f(x).$$

Hence, $\mathcal{U}(f, x, \epsilon)$ is open in $\mathbb{R}^X$ for all $x \in X$ and for all $\epsilon > 0$.
We can also show for a given $x \in X$ and $f \in \mathbb{R}^X$ that each element $\mathcal{U}(x, U)$ of the subbasis with $f \in \mathcal{U}(x, U)$ contains a set $\mathcal{U}(f, x, \epsilon)$. Let $\mathcal{U}(x, (a, b))$ with $f \in \mathcal{U}(x, (a, b))$. We know that there exists an $\epsilon > 0$ such that

$$\epsilon < \min\{f(x) - a, b - f(x)\}.$$

For $g \in \mathcal{U}(f, x, \epsilon)$ we have

$$-\epsilon < g(x) - f(x) < \epsilon$$

and also

$$a < -\epsilon + f(x) < g(x) < \epsilon + f(x) < b.$$

Hence, $g \in \mathcal{U}(x, (a, b))$. This shows that for a net $\{f_i\}_{i\in I}$ to converge to f in the pointwise convergence topology, $f_i(x)$ needs to converge to $f(x)$ for every $x \in \mathbb{R}$. This is formulated in the following lemma.

Lemma 1.28 (Pointwise Convergence)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and consider functions $f_i\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ indexed by a directed set I. The net $\{f_i\}_{i\in I}$ converges pointwise to f if and only if $\{f_i(x)\}_{i\in I}$ converges to $f(x)$ in Y for all $x \in X$.*

Proof. The proof is a reformulation of [5, Lemma 43.3]. □

Definition 1.29 (Continuity at a point)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ a function. We call f *continuous at a point* $x \in X$ if for every neighborhood U_Y of $f(x)$ there exists a neighborhood U_X of x such that $f(U_X) \subset U_Y$.

Proposition 1.30 (Continuity and Nets)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a function. Then the following statements are equivalent:*

(i) *if the net $\{x_i\}_i$ in X converges to $x \in X$ then the net $\{f(x_i)\}_i$ in Y converges to $f(x) \in Y$,*

(ii) *for every $\mathcal{O} \in \mathcal{T}_Y$ with $f(x) \in \mathcal{O}$ we know that $f^{-1}(\mathcal{O})$ is a neighborhood of x.*

Proof. (i) $\Rightarrow$ (ii):
Let $\mathcal{O} \in \mathcal{T}_Y$ with $f(x) \in \mathcal{O}$. Assume that $f^{-1}(\mathcal{O})$ is not a neighborhood of x. Then for every open neighborhood $U \in \mathcal{T}_X$ of x there in an element x_U such that $x_U \in U \setminus f^{-1}(\mathcal{O})$. Define I as the set of all open neighborhood of x ordered by reverse set inclusion. That is, for all $V_1, V_2 \in I$ with $V_1 \subset V_2$ we have $V_1 \geq V_2$. We see that the net $\{x_U\}_{U\in I}$ converges to x but $\{f(x_U)\}_{U\in I}$ does not converge to $f(x)$.
(i) $\Leftarrow$ (ii):
Let $\{x_i\}_{i\in I}$ be a net in X that converges to $x \in X$ and let U be a neighborhood of $f(x)$. Then $f^{-1}(U)$ is a neighborhood of x and we know that there is a $j_0 \in I$ such that for all $j \geq j_0$ we have $x_j \in f^{-1}(U)$. Hence, for all $j \geq j_0$ we have $f(x_j) \in U$. □

Remark 1.31. A function f is continuous if the aforementioned properties hold for every $x \in X$.

Proposition 1.32. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The product topology and the topology of pointwise convergence coincide on Y^X.*

Proof. We will assume that $\{f_i\}_{i\in I}$ converges to f in the product topology and then prove that the net also converges to f for the topology of pointwise convergence. This means that for any finite set $\{x_1, \ldots, x_m\} \subset X$ and any open neighborhood $U_i \subset X$ of $f(x_i)$ for $1 \leq i \leq m$ there exists a i_0 such that if $j \geq i_0$ then $f_j(x_i) \in U_i$ for all $1 \leq i \leq m$. So, in particular for $\{x\} \subset X$ and each neighborhood U with $f(x) \in U$ there exists an i_0 such that for all $j \geq i_0$ we have $f_j(x) \in U$. This means that $\{f_i\}_{i\in I}$ converges pointwise to f.
Conversely, we assume that the net $\{f_i\}_{i\in I}$ converges pointwise to f and let

$\{x_1, \dots, x_m\} \subset X$. For all $1 \leq k \leq m$ let $U_k \subset Y$ be an open neighborhood such that $f(x_k) \in U_k$. We know that for all $1 \leq k \leq m$ there exists an i_k such that for all $j \geq i_k$ we have $f_j(x_k) \in U_k$. Let i_0 be such that $i_0 \geq i_k$ for all $1 \leq k \leq m$. It follows that $f_j(x_k) \in U_k$ for all $j \geq i_0$. Hence, the net $\{f_i\}_{i \in I}$ converges in the product topology. □

1.3 Metric spaces and norms

Definition 1.33 (Metric space)**.** Let X be a set and define $d \colon X \times X \to \mathbb{R}$. The function d is called a *metric* on the set X if for all $x, y, z \in X$ the following properties hold:

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) = 0 \Leftrightarrow x = y$,

(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a *metric space*.

Definition 1.34 (Isometry)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Then the function $f \colon (X, d_X) \to (Y, d_Y)$ is an *isometry* if for all $x, y \in X$ we have $d_X(x, y) = d_Y(f(x), f(y))$.

Definition 1.35 (Dense)**.** Let (X, d) be a metric space. A set $Y \subset X$ is *dense* in X if for all $x \in X$ and for all $\epsilon > 0$ there exists an $y \in Y$ such that $d(x, y) < \epsilon$.

Definition 1.36 (Separable space)**.** Let (X, d) be a metric space. If there exists a countable subset $Y \subset X$ such that Y is dense in X then (X, d) is called *separable*.

Definition 1.37 (Proper metric space)**.** Let (X, d) be a metric space. If for all $r > 0$ and $x \in X$ the closed ball $\mathcal{B}_r(x) := \{y \in X \mid d(x, y) \leq r\}$ is compact, then (X, d) is called a *proper metric space*.

To clarify, given a metric space (X, d), we will be using the notation $\mathcal{B}_r(x)$ for closed balls, the notation

$$\mathcal{O}_r(x) := \{y \in X \mid d(x, y) < r\}$$

for open balls and $\mathcal{O}$ for general open sets.

Theorem 1.38 (Bolzano–Weierstrass)**.** *Let $(x_n)_n$ be a bounded sequence in $\mathbb{R}^d$ with the Euclidean topology. Then $(x_n)_n$ has a convergent subsequence.*

Proof. For the proof see [6, Theorem 3.4.8]. □

Lemma 1.39. *Every convergent sequence in $\mathbb{R}$ with the Euclidean topology is bounded.*

Proof. Let $(x_n)_n$ be a sequence in $\mathbb{R}$ that converges to x. Then there is an N such that for all $n \geq N$ we get $|x_n - x| < 1$. This means that $|x_n| \leq |x| + 1$ for all $n \geq N$. Define $M := \max\{|x_1|, \ldots, |x_N|, |x| + 1\}$. This is possible since this set is finite. We see that $|x_n| \leq M$ for all n, thus $(x_n)_n$ is bounded. □

Proposition 1.40 (Hölder's Inequality)**.** *Let $x, y \in \mathbb{R}^d$ and let $p > 1$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q}.$$

Proof. For the proof see [9, Theorem 1.3.12]. □

Definition 1.41 (Norm)**.** Let X be a vector space. A norm on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for all $x, y \in X$ and $s \in \mathbb{R}$

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|sx\| = |s| \|x\|$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 1.42 (Norm equivalence)**.** Let X be a vector space. Two norms $\|\cdot\|$ and $\|\cdot\|'$ are *equivalent*, denoted by $\|\cdot\| \sim \|\cdot\|'$, if there exist $A, B \in \mathbb{R}_{>0}$ such that for all $x \in X$

$$A \|x\|' \leq \|x\| \leq B \|x\|'.$$

Proposition 1.43. *Norm equivalence is an equivalence relation.*

Proof. For the proof see [2, Theorem 2.16, Theorem 2.17]. □

For the proof following proposition we used [2, Theorem 2.16].

Proposition 1.44. *Let X be a finite dimensional vector space. Then all norms on X are equivalent.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for X. We note that we can write every $x \in X$ uniquely as $x = \sum_{j=1}^{n} \lambda_j e_j$ for some $\lambda_j \in \mathbb{R}$. Let $\|\cdot\|$ be an arbitrary norm on X and define $\left\|\sum_{j=1}^{n} \lambda_j e_j\right\|'' := (\sum_{j=1}^{n} |\lambda_j|^2)^{\frac{1}{2}}$. Then

$$\begin{aligned}\left\| \sum_{j=1}^{n} \lambda_j e_j \right\| &\leq \sum_{j=1}^{n} \|\lambda_j e_j\| \\ &= \sum_{j=1}^{n} |\lambda_j| \|e_j\|.\end{aligned}$$

Due to Hölder's Inequality with $p = q = \frac{1}{2}$ we get

$$\sum_{j=1}^{n} |\lambda_j| \, \|e_j\| \leq \left(\sum_{j=1}^{n} |\lambda_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|e_j\|^2 \right)^{\frac{1}{2}}$$
$$= A \left\| \sum_{j=1}^{n} \lambda_j e_j \right\|''$$

with $A := (\sum_{j=1}^{n} \|e_j\|^2)^{\frac{1}{2}}$. Hence $\|x\| \leq A \|x\|''$ for all $x \in X$.
Define the function $f \colon \mathbb{R}^d \to \mathbb{R}$ by

$$f(\lambda_1, \ldots, \lambda_n) := \left\| \sum_{j=1}^{n} \lambda_j e_j \right\|.$$

This function is continuous on $\mathbb{R}^d$ with respect to the Euclidean topology. Define

$$S := \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^d \middle| \sum_{j=1}^{n} |\lambda_j|^2 = 1 \right\}.$$

It is clear that S is bounded. Since $\{1\} \subset \mathbb{R}$ is a closed subset and f is continuous, we know that $f^{-1}(\{1\}) = S$ is also closed. By the Heine-Borel Theorem, see [7, Corollary 2.5.12], we know that S is compact and thus by the Extreme Value Theorem [8, 4.5.2] we know that there must be a $(\mu_1, \ldots, \mu_n) \in S$ such that

$$f(\mu_1, \ldots, \mu_n) \leq f(\lambda_1, \ldots, \lambda_n)$$

for all $(\lambda_1, \ldots, \lambda_n) \in S$. Define $a := f(\mu_1, \ldots, \mu_n)$. If $a = 0$ then

$$\left\| \sum_{j=1}^{n} \mu_j e_j \right\| = 0,$$

which would imply that $\sum_{j=1}^{n} \mu_j e_j = 0$. Then $\mu_1 = \cdots = \mu_n = 0$ contradicting the fact that $(\mu_1, \ldots, \mu_n) \in S$. We conclude that $a > 0$.
If $\|x\|'' = \left\| \sum_{j=1}^{n} \lambda_j e_j \right\| = 1$ then the $(\lambda_1, \ldots, \lambda_n) \in S$ and thus $a \leq \|x\|$. Hence, if $x \in X \backslash \{0\}$ and $\left\| \frac{x}{\|x\|''} \right\|'' = 1$ we see that $a \leq \left\| \frac{x}{\|x\|''} \right\|$ and thus $a \|x\|'' \leq \|x\|$. If $x = 0$ it also holds that $a \|x\|'' \leq \|x\|$. We conclude that $a \|x\|'' \leq \|x\| \leq A \|x\|''$ for all $x \in X$ and thus the two norms are equivalent.
Any other norm $\|\cdot\|'$ on X is then also equivalent to $\|\cdot\|''$. Due to Proposition 1.43 we conclude that $\|\cdot\|$ and $\|\cdot\|'$ are also equivalent. Hence, all norms on X are equivalent. □

Definition 1.45. Let $p > 1$ be a real number. We define the function

$$\|\cdot\|_p : \mathbb{R}^d \to \mathbb{R}$$

by

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Proposition 1.46. *The function $\|\cdot\|_p : \mathbb{R}^d \to \mathbb{R}$ is a norm.*

Proof. For any $x \in \mathbb{R}^d$ we see that $\|x\|_p^p$ is a sum of nonnegative numbers. Thus $\|x\|_p \geq 0$. We can clearly see that if $x = 0$ then $\|x\|_p = 0$. If $\|x\|_p = 0$ then $x = 0$ since $\|x\|_p^p$ is the sum of all nonnegative numbers. Thus property (i) of Definition 1.41 holds.
Let $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$. Then we see that

$$\begin{aligned}
\|sx\| &= \left(\sum_{i=1}^{n} |sx_i|^p \right)^{1/p} \\
&= \left(\sum_{i=1}^{n} |s|^p |x_i|^p \right)^{1/p} \\
&= \left(|s|^p \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \\
&= |s| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \\
&= |s| \, \|x\| .
\end{aligned}$$

Thus property (ii) holds.
Let $x, y \in \mathbb{R}^d$ be nonzero. Then

$$\begin{aligned}
|x_i + y_i|^p \leq (|x_i| + |y_i|)^p &= (|x_i| + |y_i|)(|x_i| + |y_i|)^{p-1} \\
&= |x_i|(|x_i| + |y_i|)^{p-1} + |y_i|(|x_i| + |y_i|)^{p-1}.
\end{aligned}$$

and if we take the sum of both sides, we get

$$\sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} |x_i|(|x_i| + |y_i|)^{p-1} + \sum_{i=1}^{n} |y_i|(|x_i| + |y_i|)^{p-1}.$$

The first inequality is because we know that $|x_i + y_i|^p \leq |x_i|^p + |y_i|^p$ due to the fact that $f(x) := x^p$ is a strictly increasing function on $\mathbb{R}_{\geq 0}$. We will now use Hölder's Inequality with q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^{n} |x_i|(|x_i| + |y_i|)^{p-1} \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q} \right)^{1/q}$$

and

$$\sum_{i=1}^{n} |y_i|(|x_i| + |y_i|)^{p-1} \leq \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\right)^{1/q}.$$

Thus we get

$$\sum_{i=1}^{n} |x_i + y_i|^p \leq \left(\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}\right) \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\right)^{1/q}.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$ we also know that $(p-1)q = p$. Hence, we get

$$\sum_{i=1}^{n} |x_i + y_i|^p \leq \left(\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}\right) \left(\sum_{i=1}^{n} |x_i + y_i|^{p}\right)^{1/q}.$$

By dividing both sides by the right factor we get

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

Since $1 - \frac{1}{q} = \frac{1}{p}$ we get

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

We conclude that property (iii) holds. □

From now on we will refer to $\|\cdot\|_p$ as the *p-norm*.

Definition 1.47 (Dual Norm)**.** Let $\|.\|$ be a norm on $\mathbb{R}^d$. For all $x \in \mathbb{R}^n$ define

$$\|x\|^* := \sup\{|x^T y| : \|y\| \leq 1, y \in \mathbb{R}^d\}.$$

We call this the *dual norm* of $\|\cdot\|$.

In general, this is a well-defined norm as we can see in [3, Theorem 4.1]. In particular, we will show below that this defines a dual norm for the p-norm.

Definition 1.48 (Sign Function)**.** Define the function $\mathrm{sgn}\colon \mathbb{R} \to \{-1, 0, 1\}$ by

$$\mathrm{sgn}(x) := \begin{cases} -1 & \text{if}\, x < 0, \\ 0 & \text{if}\, x = 0, \\ 1 & \text{if}\, x > 0. \end{cases}$$

Proposition 1.49 (p-norm and q-norm Duality)**.** *Let $p > 1$. The dual norm of the p-norm is the q-norm with $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Let $p > 1$ and $x, y \in \mathbb{R}^d$ be given such that $\|y\|_p \leq 1$. With Hölder's Inequality we get

$$\left|x^T y\right| = \left|\sum_{i=1}^{n} x_i y_i\right| \leq \sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q \leq \|x\|_q .$$

Now that we know $\|x\|^*$ is at most $\|x\|_q$, it is sufficient to show that there exists a $y \in \mathbb{R}^d$ such that $\|y\|_p \leq 1$ and the sum is equal to the q-norm of x. Assume $x \neq 0$, since for $x = 0$ it is trivial that $\|0\|^* = \|0\|_q$ holds. Define

$$z := (\operatorname{sgn}(x_1)|x_1|^{q-1}, \dots, \operatorname{sgn}(x_n)|x_n|^{q-1})$$

and $y := \frac{z}{\|z\|_p}$. Note that $\|y\|_p = 1$. Then

$$|x^T y| = \frac{\sum_{i=1}^{n} |x_i|^q}{\left(\sum_{i=1}^{n} |x_i|^{(q-1)p}\right)^{1/p}} = \frac{(\|x\|_q)^q}{(\|x\|_q)^{q/p}} = \|x\|_q ,$$

which follows from the fact that $(q-1)p = q$ and $q - \frac{q}{p} = 1$. We conclude that $\|x\|^* = \|x\|_q$ for all $x \in \mathbb{R}^d$. □

Remark 1.50. In general, for a linear function $\phi : \mathbb{R}^n \to \mathbb{R}$ we know that there exist $y_1, \dots, y_n$ such that the following holds

$$\phi(x) \leq |\phi(x)| \overset{(a)}{=} \left|\sum_{k=1}^{n} y_k x_k\right| \leq \sum_{k=1}^{n} |y_k x_k| \overset{(b)}{\leq} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} .$$

Equality (a) holds due to the fact that ϕ is a linear function and every linear function whose codomain is one dimensional can be written in this way. Inequality (b) is due to Hölder's Inequality.

2 Metric Compactification

In this chapter we will be introducing the notion of a metric compactification. The idea is that any given metric space can be embedded into a compact topological space. The closure of this embedded space is compact and will be called the metric compactification. The extra elements that are added can be thought of as 'points at infinity'.

2.1 Defining the metric compactification

Consider an arbitrarily given metric space (X, d). Equip $\mathbb{R}^X$ with the topology of pointwise convergence denoted by $\mathcal{T}_{pw}$. Fix some $x_0 \in X$ and define the function $\Psi\colon (X, d) \to (\mathbb{R}^X, \mathcal{T}_{pw})$ by

$$\Psi(x)(y) := d(y, x) - d(x_0, x). \tag{1}$$

Proposition 2.1. *The function $\Psi\colon (X, d) \to (\mathbb{R}^X, \mathcal{T}_{pw})$ is injective and continuous.*

Proof. For injectivity, let $x_1, x_2 \in X$ and assume that $\Psi(x_1)(y) = \Psi(x_2)(y)$ for all $y \in X$. We get the following equations

$$\begin{aligned} 0 &= \Psi(x_1)(x_1) - \Psi(x_2)(x_1) = d(x_1, x_1) - d(x_0, x_1) - d(x_1, x_2) + d(x_0, x_2) \\ &= -d(x_0, x_1) - d(x_1, x_2) + d(x_0, x_2), \end{aligned}$$

and

$$\begin{aligned} 0 &= \Psi(x_2)(x_2) - \Psi(x_1)(x_2) = d(x_2, x_2) - d(x_0, x_2) - d(x_1, x_2) + d(x_0, x_1) \\ &= -d(x_0, x_2) - d(x_1, x_2) + d(x_0, x_1). \end{aligned}$$

When we add them together we get

$$\begin{aligned} 0 &= -d(x_0, x_1) - d(x_1, x_2) + d(x_0, x_2) - d(x_0, x_2) - d(x_1, x_2) + d(x_0, x_1) \\ &= -2d(x_1, x_2). \end{aligned}$$

This leads to $d(x_1, x_2) = 0$ and $x_1 = x_2$. Thus, Ψ is injective.
To prove continuity, we first remark that for all $z \in X$ we have

$$\begin{aligned} |\Psi(y)(z) - \Psi(x)(z)| &= |d(z, y) - d(x_0, y) - (d(z, x) - d(x_0, x))| \\ &\leq |d(z, y) - d(z, x)| + |d(x_0, x) - d(x_0, y)| \\ &\leq 2d(x, y). \end{aligned}$$

When we have a net $\{x_i\}_{i \in I}$ in X which converges to x, we see that for all $y \in X$ we have

$$\lim_{i \in I} |\Psi(x)(y) - \Psi(x_i)(y)| = 0.$$

So, $\Psi(x_i)$ converges pointwise to $\Psi(x)$ and thus Ψ is continuous. □

The closure of the image will become the compactification of the metric space. To prove that the closure of the image is compact, we need Tychonoff's Theorem. For the proof of Tychonoff's Theorem we will follow the proof in [4, Theorem 5D].

Theorem 2.2 (Tychonoff)**.** *The product of compact spaces is compact for the product topology.*

Proof. Let X_i be a compact space for all $i \in I$ and equip $X := \prod_{i \in I} X_i$ with the product topology. Let Y be a set of closed sets of X which has the finite intersection property. If $\bigcap_{A \in Y} A$ is a nonempty set, then X is compact due to Proposition 1.15.
Define the set

$$Y' := \{A \subset \mathcal{P}(X) \mid Y \subset A, A \text{ has the finite intersection property}\}.$$

Then Y' is a partially ordered set by set inclusion. We will show that Y' has a maximal element with the use of Zorn's Lemma. We will first prove that every totally ordered subset in $(Y', \subset)$ has an upper bound. Let $W \subset Y'$ be some totally ordered subset. Consider $\bigcup_{A \in W} A$. To prove that this element has the finite intersection property, we will take elements $b_1, \ldots, b_n \in \bigcup_{A \in W} A$. Then there exists an $B \in W$ with $b_1, \ldots, b_n \in A$ and since S has the finite intersection property we also know that the intersection of all the b_i is nonempty. This means that $\bigcup_{A \in W} A$ has the finite intersection property and that $\bigcup_{A \in W} A \in Y'$. Hence, $\bigcup_{A \in W} A$ is an upper bound for W. Now that we have proved that every totally ordered subset of Y' has an upper bound, by Zorn's Lemma it follows that Y' has a maximal element, which we shall denote by Y_M.
For each projection $p_i \colon Y_M \to X_i$ we denote the image as Y_M^i. To show that the Y_M^i also have the finite intersection property, let $y_1^i, \ldots, y_n^i \in Y_M^i$. We know that for all $1 \leq j \leq n$ we have $y_j^i = p_i(y_j)$ for some $y_j \in Y_M$. Since Y_M has the finite intersection property, we know that there exists some $x \in \bigcap_{j=1}^n y_j$ and thus $p_i(x) \in \bigcap_{j=1}^n y_j^i$. Define the set $C_M^i := \{\overline{A} \mid A \in Y_M^i\}$. Since Y_M^i has the finite intersection property, we know that C_M^i also has the finite intersection property. Since X_i is compact there exists $x_i \in \bigcap_{A \in C_M^i} A$ by Proposition 1.15. Define $x := (x_i)_{i \in I}$ and $C_M := \{\overline{A} \mid A \in Y_M\}$. We will prove that $x \in \bigcap_{A \in C_M} A$ and therefore, also $x \in \bigcap_{A \in Y} A$.
Consider the projection $p_i \colon X \to X_i$. Let $\mathcal{O} \subset X$ be an open set such that $x \in \mathcal{O}$. We know that there exists an element B of the basis of the product topology with $x \in B$ such that $B \subset \mathcal{O}$. This element B can be written as a finite intersection of elements of the subbasis of the product topology. So we know that there are a finite number of $\mathcal{O}_{i_1}, \ldots, \mathcal{O}_{i_n}$ with $\mathcal{O}_{i_j} \subset X_{i_j}$ such that $B = \bigcap_{j=1}^n p_{i_j}^{-1}(\mathcal{O}_{i_j})$ and $x \in \bigcap_{j=1}^n p_{i_j}^{-1}(\mathcal{O}_{i_j}) \subset \mathcal{O}$.
This means that $x_{i_j} \in \mathcal{O}_{i_j}$. Thus the intersection of $\mathcal{O}_{i_j}$ with every set of $Y_M^{i_j}$ is nonempty, since $A \in Y_M^{i_j}$ is dense in $\overline{A} \in C_M^{i_j}$. This also means that the intersection of $p_{i_j}^{-1}(\mathcal{O}_{i_j})$ with every set of Y_M is nonempty, since for $A \in Y_M$ and $A^{i_j} \in Y_M^{i_j}$ we have $A^{i_j} \cap \mathcal{O}_{i_j} = p_{i_j}(p_{i_j}^{-1}(O_{i_j}) \cap A)$. If $p_{i_j}^{-1}(\mathcal{O}_{i_j})$ is empty,

then $A^{i_j} \cap \mathcal{O}_{i_j}$ will also be empty. Since the intersection of finite elements of Y_M is also an element in Y_M we know that $p_{i_j}^{-1}(\mathcal{O}_{i_j})$ also intersects these elements. Hence, $Y_M \cup \{p_{i_j}^{-1}(\mathcal{O}_{i_j})\}$ has the finite intersection property. Thus, $p_{i_j}^{-1}(\mathcal{O}_{i_j}) \in Y_M$ because if $p_{i_j}^{-1}(\mathcal{O}_{i_j}) \notin Y_M$ then $Y_M \subsetneq Y_M \cup \{p_{i_j}^{-1}(\mathcal{O}_{i_j})\}$ and then Y_M would not be a maximal element. For the same reason we know that $\bigcap_{j=1}^{n} p_{i_j}^{-1}(\mathcal{O}_{i_j}) \in Y_M$ and also $\mathcal{O} \in Y_M$. This means that the intersection of $\mathcal{O}$ with every set of Y_M is nonempty. Since $x \in \mathcal{O}$ and $\mathcal{O}$ was arbitrarily chosen open set it follows that $x \in C_M$. We know that $Y \subset Y_M$ and $Y \subset C_M$ hold since all sets in Y are closed. It follows that x is also in the intersection of every set in Y. Thus Y has the finite intersection property and thus X is compact by Proposition 1.15. □

Proposition 2.3. *The closure of the image of* Ψ *is compact in* $(\mathbb{R}^X, \mathcal{T}_{pw})$

Proof. For all $x, y \in X$ we have

$$\begin{aligned} |\Psi(x)(y)| &= |d(y,x) - d(x_0,x)| \\ &\leq d(x_0,y). \end{aligned}$$

When we consider every $\Psi(x) \in \mathbb{R}^X$ as an element of $\prod_{y \in X} \mathbb{R}$ in the sense that a function couples an element of X to an element of $\mathbb{R}$. We see that

$$\Psi(X) \subset \prod_{y \in X} [-d(x_0,y), d(x_0,y)].$$

Every closed and bounded subset of $\mathbb{R}$ is compact, thus every $[-d(x_0,y), d(x_0,y)]$ is also compact. By Tychonoff's Theorem we know that the product of compact sets is also compact, thus $\prod_{y \in X}[-d(x_0,y), d(x_0,y)]$ is a compact and closed set with regard to the product topology. With Proposition 1.32 we see that $\prod_{y \in X}[-d(x_0,y), d(x_0,y)]$ is also a compact and closed set with regard to the pointwise convergence topology. Since $\overline{\Psi(X)}$ is now a closed subset of a compact set it follows by Proposition 1.5 that $\overline{\Psi(X)}$ is also compact. □

Now that we know that the closure of the image of Ψ is compact, the following definition will be well-defined.

Definition 2.4 (Metric Compactification)**.** Let (X, d) be a metric space, let $x_0 \in X$ and $\mathcal{T}_{pw}$ the topology of pointwise convergence on $\mathbb{R}^X$. Define the function $\Psi \colon (X, d) \to (\mathbb{R}^X, \mathcal{T}_{pw})$ as in (1). Then $\overline{\Psi(X)}$ is called the *metric compactification* of X.

One can wonder what happens to the metric compactification of a compact metric space. The following proposition will show that no new elements are added to the metric compactification of an already compact metric space.

Proposition 2.5. *Let* (X, d) *be a compact metric space. Then* $\Psi(X)$ *is compact.*

Proof. Since Ψ is continuous due to Proposition 2.1 and X is compact we know that $\Psi(X)$ must also be compact due to Proposition 1.8. □

The choice of $x_0 \in X$ in Definition 2.4 is irrelevant. We will denote the compactification with basis point $z \in X$ as $\overline{\Psi_z(X)}$ and prove that the metric compactification is unique up to a homeomorphism.

Proposition 2.6. *Let $x_0, x_1 \in X$. Then there is a homeomorphism between $(\overline{\Psi_{x_0}(X)}, \mathcal{T}_{pw})$ and $(\overline{\Psi_{x_1}(X)}, \mathcal{T})_{pw}$.*

Proof. Let $\{\Psi_{x_0}(y_i)\}_{i\in I}$ be a net of functions in $\Psi_{x_0}(X)$ that converges to the element $h_{x_0} \in \overline{\Psi_{x_0}(X)}$. For all $x \in X$ we have

$$\begin{aligned}\Psi_{x_1}(y_i)(x) &= d(x, y_i) - d(x_1, y_i) = d(x, y_i) - d(x_0, y_i) + d(x_0, y_i) - d(x_1, y_i)\\ &= d(x, y_i) - d(x_0, y_i) - (d(x_1, y_i) - d(x_0, y_i))\\ &= \Psi_{x_0}(y_i)(x) - \Psi_{x_0}(y_i)(x_1).\end{aligned}$$

Hence

$$\lim_{i\in I} \Psi_{x_1}(y_i)(x) = h_{x_0}(x) - h_{x_0}(x_1). \tag{2}$$

Define $H\colon \overline{\Psi_{x_0}(X)} \to \overline{\Psi_{x_1}(X)}$ by $h \mapsto h - h(x_1)$. Due to (2) we know that H is well-defined.
To prove that H is continuous, let $(h_i)_{i\in I}$ be a net in $\overline{\Psi_{x_0}(X)}$ converges pointwise to $h \in \overline{\Psi_{x_0}(X)}$. For all $x \in X$ we have

$$\begin{aligned}\lim_{i\in I} |H(h_i)(x) - H(h)(x)| &= \lim_{n\to\infty} |h_i(x) - h_i(x_0) - (h(x) - h(x_0))|\\ &= |h(x) - h(x_0) - (h(x) - h(x_0))|\\ &= 0.\end{aligned}$$

Hence, $H(h_i)$ converges pointwise to $H(h)$ and thus H is continuous.
To prove that H is a bijection we define $G\colon \overline{\Psi_{x_1}(X)} \to \overline{\Psi_{x_0}(X)}$ with

$$h \mapsto h + h(x_1).$$

We note that for all $h \in \overline{\Psi_{x_0}(X)}$ we get

$$G \circ H(h) = G(H(h)) = G(h - h(x_1)) = h - h(x_1) + h(x_1) = h$$

and

$$H \circ G(h) = H(G(h)) = H(h + h(x_1)) = h + h(x_1) - h(x_1) = h$$

so G is the inverse of H.
Since H is a continuous bijection from a compact space to a Hausdorff space, we know by [1, Corollary 9.7] that H is a homeomorphism. □

2.2 The metric compactification of $\mathbb{R}$

Let us consider the case $X = \mathbb{R}$ with the Euclidean metric. The metric compactification of $\mathbb{R}$ will have two additional points, denoted by $+\infty$ and $-\infty$. In $\mathbb{R}$ it is sufficient to work with sequences instead of with nets (we will prove this in Proposition 3.6).

Let $x_0 := 0$ and consider a monotone sequence $(x_n)_n$ such that $x_n \to \infty$. Then

$$\begin{aligned}\lim_{n\to\infty} \Psi(x_n)(x) &= \lim_{n\to\infty} (d(x, x_n) - d(0, x_n)) \\ &= \lim_{n\to\infty} (|x - x_n| - |0 - x_n|) \\ &= \lim_{n\to\infty} (-x + x_n - x_n) = -x, \end{aligned} \tag{3}$$

because for large enough n, we have $x_n > x$ and $x_n > 0$. Hence, $\Psi(\infty) = -id$. Likewise for the sequence $(-x_n)_n$ in $\mathbb{R}$ with $-x_n \to -\infty$. For all $x \in \mathbb{R}$ we have

$$\begin{aligned}\lim_{n\to\infty} \Psi(-x_n)(x) &= \lim_{n\to\infty} (d(x, -x_n) - d(0, -x_n)) \\ &= \lim_{n\to\infty} (|x + x_n| - |0 + x_n|) \\ &= \lim_{n\to\infty} (x + x_n - x_n) = x,. \end{aligned} \tag{4}$$

because for large enough n, we have $x+x_n > 0$ and $x_n > 0$. Hence, $\Psi(-\infty) = id$. To see that there are only two elements in the boundary $\overline{\Psi(\mathbb{R})} \setminus \Psi(\mathbb{R})$, let $h \in \overline{\Psi(\mathbb{R})} \setminus \Psi(\mathbb{R})$. Then we know there exists a sequence $(y_n)_n$ in $\mathbb{R}$ such that $\lim_{n\to\infty} \Psi(y_n)(x) = h(x)$ for all $x \in \mathbb{R}$. Then there are two possibilities: $(y_n)_n$ is bounded in $\mathbb{R}$ or $(y_n)_n$ is not bounded in $\mathbb{R}$. In the first case, due to the Bolzano-Weierstrass Theorem 1.38, there is a subsequence (y_{n_k}) which converges to some $z \in \mathbb{R}$. We will show that $\Psi(y_{n_k})$ and $\Psi(y_n)_n$ converge to the same limit. Let $\epsilon > 0$ be given. Then

$$|h(x) - \Psi(z)(x)| \leq |h(x) - \Psi(y_{n_k})(x)| + |\Psi(y_{n_k})(x) - \Psi(z)(x)|.$$

Due to the fact that $(y_{n_k})_k$ is a subsequence of $(y_n)_n$ and because $(y_{n_k})_k$ converges to z, we know that we can choose n_k big enough such that

$$|h(x) - \Psi(y_{n_k})(x)| < \frac{1}{2}\epsilon$$

and

$$|\Psi(y_{n_k})(x) - \Psi(z)(x)| < \frac{1}{2}\epsilon.$$

Thus,

$$|h(x) - \Psi(z)(x)| \leq |h(y) - \Psi(y_{n_k})(x)| + |\Psi(y_{n_k})(x) - \Psi(z)(x)| < \epsilon \tag{5}$$

and $\lim_{k\to\infty} \Psi(y_{n_k})(x) = h(x)$.

For the second case, since the sequence $(y_n)_n$ is not bounded it cannot converge

in $\mathbb{R}$. We know that the sequence cannot converge to both ∞ and $-\infty$, since if we could find a subsequence $(y_{n_k})_k$ that converges to ∞ and another subsequence $(y_{n_l})_l$ that converges to $-\infty$. For $(y_{n_k})_k$ we can use the same method in (5) and with the result of (3) we find that $(y_n)_n$ converges to $-id$. In the same way we can use $(y_{n_l})_l$ and the result of (4) and the method of (5) to show that $(y_n)_n$ converges to id. Since $-id \neq id$ we know that $(y_n)_n$ cannot converge to both ∞ and $-\infty$. Hence, $(y_n)_n$ converges to $-id$ or id.
To show that id and $-id$ are in the boundary of $\overline{\Psi(\mathbb{R})}$ we need to show that we cannot write them as functions of the form

$$\Psi(y)(x) = |x - y| - |y|.$$

Since id and $-id$ are linear functions and there is no y for which $\Psi(y)(x)$ is linear, we know that id and $-id$ must be in the boundary of $\overline{\Psi(\mathbb{R})}$. Hence, we have proved that the boundary contains only two elements ∞ and $-\infty$.

2.3 Extending isometries to homeomorphisms

This paragraph is dedicated to the following theorem in which we extend surjective isometries on the metric space to homeomorphisms on the metric compactification.

Theorem 2.7. *Let $\phi\colon (X,d) \to (X,d)$ be a surjective isometry and define the function $\Phi\colon (\overline{\Psi(X)}, \mathcal{T}_{pw}) \to (\overline{\Psi(X)}, \mathcal{T}_{pw})$ for $h \in \overline{\Psi(X)}$ and $x \in X$ by*

$$\Phi(h)(x) := h(\phi^{-1}(x)) - h(\phi^{-1}(x_0)).$$

Then Φ is a homeomorphism.

Proof. We first have to prove that Φ is well-defined, then we will prove that it is continuous.
Let $h \in \overline{\Psi(X)}$, then we know that there is a converging net $\{y_i\}_{i\in I}$ in X such that $\lim_{i\in I} \Psi(y_i)(x) = h(x)$. Then

$$\begin{aligned}
\Phi(h)(x) &= \lim_{i\in I}(\Psi(y_i)(\phi^{-1}(x)) - \Psi(y_i)(\phi^{-1}(x_0))) \\
&= \lim_{i\in I}(d(\phi^{-1}(x), y_i) - d(y_i, x_0) - (d(\phi^{-1}(x_0), y_i) - d(y_i, x_0))) \\
&= \lim_{i\in I}(d(\phi^{-1}(x), y_i) - d(\phi^{-1}(x_0), y_i)) \\
&= \lim_{i\in I}(d(\phi(\phi^{-1}(x)), \phi(y_i)) - d(\phi(\phi^{-1}(x_0)), \phi(y_i))) \\
&= \lim_{i\in I}(d(x, \phi(y_i)) - d(x_0, \phi(y_i))) \\
&= \lim_{i\in I}(\Psi(\phi(y_i))(x)).
\end{aligned}$$

Since $\Psi(\phi(y_i)) \in \overline{\Psi(X)}$ and the net converges pointwise we also know that $\lim_{i\in I}(\Psi(\phi(y_i)) \in \overline{\Psi(X)}$. Thus Φ is well-defined.

Let $\{h_i\}_{i\in I}$ be a net in $\overline{\Psi(X)}$ that converges pointwise to h. For all $x \in X$ we have

$$\begin{aligned}\lim_{i\in I}(\Phi(h_i)(x)) &= \lim_{i\in I}(h_i(\phi^{-1}(x)) - h_i(\phi^{-1}(x_0)))\\ &= h(\phi^{-1}(x)) - g(\phi^{-1}(x_0))\\ &= \Phi(h)(x).\end{aligned}$$

Since $\Phi(h_i)(x)$ converges pointwise to $\Phi(h)(x)$, we know that Φ is continuous.
To prove that Φ is a bijection, we define $\Theta\colon (\overline{\Psi(X)}, \mathcal{T}_{pw}) \to (\overline{\Psi(X)}, \mathcal{T}_{pw})$ by $\Theta(h)(x) := h(\phi(x)) - h(\phi(x_0))$. We will prove that it is well-defined, continuous, and also that it is the inverse of Φ. We know that there is a converging net $\{y_i\}_{i\in I}$ in X such that

$$\lim_{i\in I}\Psi(y_i)(x) = \lim_{i\in I} d(y_i, x) - d(y_i, x_0) = h(x).$$

Then the following holds

$$\begin{aligned}\Theta(h)(x) &= \lim_{i\in I}(\Psi(y_i)(\phi(x)) - \Psi(y_i)(\phi(x_0)))\\ &= \lim_{i\in I}(d(\phi(x), y_i) - d(y_i, x_0) - (d(\phi(x_0), y_i) - d(y_i, x_0)))\\ &= \lim_{i\in I}(d(\phi(x), y_i) - d(\phi(x_0), y_i))\\ &= \lim_{i\in I}(d(x, \phi^{-1}(y_i)) - d(x_0, \phi^{-1}(y_i)))\\ &= \lim_{i\in I}(\Psi(\phi^{-1}(y_i))(x)).\end{aligned}$$

Since $\Psi(\phi^{-1}(y_i)) \in \overline{\Psi(X)}$ holds and the net converges, it is also clear that $\lim_{i\in I}(\Psi(\phi^{-1}(y_i)) \in \overline{\Psi(X)}$ is true. Thus Θ is well-defined.
Let $h \in \overline{\Psi(X)}$. Then

$$\begin{aligned}\Theta\circ\Phi(h)(x) &= \Theta(h)(\phi^{-1}(x)) - \Theta(h)(\phi^{-1}(x_0))\\ &= h(\phi(\phi^{-1}(x))) - h(\phi(\phi^{-1}(x_0))) - (h(\phi(\phi^{-1}(x_0))) - h(\phi(\phi^{-1}(x_0))))\\ &= h(x) - h(x_0).\end{aligned}$$

We know that there is a net $\{\Psi(y_i)\}_{i\in I}$ such that $\Psi(y_i)(x)$ converges to $h(x)$ for every $x \in X$. Hence, it also holds for $x_0 \in X$. Then

$$\Psi(y_i)(x_0) = d(x_0, y_i) - d(x_0, y_i) = 0,$$

Thus, we know $h(x_0) = 0$ and we conclude $\Theta\circ\Phi(h)(x) = h(x)$. In the same way, we can prove $\Psi\circ\Theta = id$. This means that Θ is the inverse of Φ.
Since Φ is a continuous bijection from a compact space to a Hausdorff space, we know by [1, Corollary 9.7] that Φ is a homeomorphism. □

3 The p-norm on $\mathbb{R}^d$

In this chapter we will be looking at the metric compactification of $\mathbb{R}^d$ with p-norm. We can explicitly express this compactification. To determine this, we will first prove that $\overline{\Psi(\mathbb{R}^d, \|\cdot\|_p)}$ is metrizable. It is then sufficient to work with sequences instead of nets.

3.1 $\overline{\Psi(\mathbb{R}^d, \|\cdot\|_p)}$ is metrizable

We will be using the fact that $(\mathbb{R}^d, \|\cdot\|_p)$ is a proper metric space.

Proposition 3.1. *Every compact metric space is separable.*

Proof. Let (X, d) be a compact metric space. Let $\mathcal{O}_r(x)$ be the open ball around x with radius r. For all $n \in \mathbb{N}$ we define $C_n := \{\mathcal{O}_{\frac{1}{n}}(x) \mid x \in X\}$. Since for every n, this is an open cover of X, and X is compact, there is a finite open subcover $F_n \subset C_n$. Define $Y := \bigcup_{n\in\mathbb{N}} F_n$. We only want the center of each set so we define

$$Z := \{x \mid \mathcal{O}_{\frac{1}{n}}(x) \in Y\}.$$

Since Y is a countable union of countable sets, Y itself is countable and thus Z is also countable. Let $x \in X$ and $\epsilon > 0$. Then there exists an m such that $\frac{1}{m} < \epsilon$. Since F_m is a finite open cover of X we know that there exists a $y \in Z$ such that $x \in \mathcal{O}_{\frac{1}{m}}(y)$. Thus $d(x, y) < \frac{1}{m} < \epsilon$ and we conclude that Z is dense in X. Thus (X, d) is separable. □

Proposition 3.2. *Let $p > 1$ in $\mathbb{R}$. Then the unit ball in $(\mathbb{R}^d, \|\cdot\|_p)$ is compact.*

Proof. We can see that the unit ball

$$\mathcal{B}_1^2 := \{x \in \mathbb{R}^d \mid \|x\|_2 \leq 1\}$$

is bounded with respect to the Euclidean norm. Consider the identity function $f\colon (\mathbb{R}^n, \|\cdot\|_2) \to \mathbb{R}$ by $x \mapsto \|x\|_2$. Since f is continuous and the set $[0, 1]$ is closed we know that $f^{-1}([0, 1]) = \mathcal{B}_1^2$ is also closed. With the Heine–Borel Theorem we conclude that $\mathcal{B}_1^2$ is compact with respect to the Euclidean norm.
Let $p > 1$ be a real number. Consider the function $id\colon (\mathbb{R}^d, \|\cdot\|_p) \to (\mathbb{R}^d, \|\cdot\|_2)$. We will prove that this is a homeomorphism. Let $(x_n)_n$ be a sequence that converges to x with respect to $\|\cdot\|_p$ and let $(y_n)_n$ be a sequence that converges to y with respect to $\|\cdot\|_2$. Due to Proposition 1.44 we know that there exist A and B such that $A\|x\|_p \leq \|x\|_2 \leq B\|x\|_p$ for all $x \in \mathbb{R}^d$. It follows that $(x_n)_n$ converges to x with respect to $\|\cdot\|_p$ if and only if $(y_n)_n$ converges to y with respect to $\|\cdot\|_2$. From this we conclude that id is a homeomorphism.
For all $x \in \mathbb{R}^d$ with $\|x\|_p \leq 1$ we have

$$\|x\|_2 \leq B\|x\|_p \leq B.$$

We know that $\mathcal{B}_1^p \subset \mathcal{B}_B^2$ and $\mathcal{B}_1^p$ is closed for $\|\cdot\|_2$, so $\mathcal{B}_1^p$ is compact for $\|\cdot\|_2$ by Heine-Borel. Since id is a homeomorphism, $\mathcal{B}_1^p$ is compact for $\|\cdot\|_p$. Since we have chosen p randomly, we conclude that the unit ball $\mathcal{B}_1^p$ is compact in $(\mathbb{R}^d, \|\cdot\|_p)$ for every real $p > 1$. □

Proposition 3.3. *Every proper metric space is separable.*

Proof. Let (X, d) be a proper metric space and let $x \in X$. Since for every n the ball $\mathcal{B}_n(x)$ is compact and by Proposition 3.1 the set is also separable, we can define $\mathcal{D}_n(x)$ as the countable dense subset of $\mathcal{B}_n(x)$. Then the set

$$\bigcup_{n=1}^{\infty} \mathcal{D}_n(x)$$

is a countable set, because it is a countable union of countable sets and it lays dense in X. Hence, X is separable. □

With these results we can introduce the notion of metrizability.

Definition 3.4 (Metrizable)**.** Let $(X, \mathcal{T})$ be a topological space. If there exists a metric d on X such that the topology induced by d is the same as $\mathcal{T}$ then $(X, \mathcal{T})$ is called *metrizable*.

The following lemma is used in the proof that $\overline{\Psi(X)}$ is metrizable.

Lemma 3.5. *Let $h \in \overline{\Psi(X)}$. Then for all $y, z \in X$ we have*

$$|h(y) - h(z)| \leq d(y, z).$$

Proof. Let $\{x_i\}_{i \in I}$ be a net in X such that $\lim_{i \in I} \Psi(x_i)(y) = h(y)$ for all $y \in X$. Then

$$\begin{aligned}
|h(y) - h(z)| &= \lim_{i \in I} |h(x_i)(y) - h(x_i)(z)| \\
&= \lim_{i \in I} |d(x_i, y) - d(x_0, x_i) - (d(x_i, z) - d(x_0, x_i))| \\
&= \lim_{i \in I} |d(x_i, y) - d(x_i, z)| \leq d(y, z).
\end{aligned}$$

□

Proposition 3.6. *Let (X, d) be a proper metric space. Then $\overline{\Psi(X)}$ is metrizable.*

Proof. By Proposition 3.3 we know that (X, d) is separable so we know that there is a countable set $Y := \{x_1, x_2, \ldots\} \subset X$ that is dense in X. For all elements $g, h \in \overline{\Psi(X)}$ we define the metric

$$\rho(g, h) := \sum_{k=1}^{\infty} 2^{-k} \min\{1, |g(x_k) - h(x_k)|\}$$

with $x_k \in Y$. We first note that this function is well-defined because

$$\sum_{k=1}^{\infty} 2^{-k} \min\{1, |g(x_k) - h(x_k)|\} \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

To prove that ρ is a metric we note that

$$2^{-k} \min\{1, |g(x_k) - h(x_k)|\} \geq 0$$

and

$$\rho(g, h) \geq 0.$$

Thus property (i) from Definition 1.33 holds.
To prove property (ii) we note

$$\begin{aligned}
\rho(g, h) &= \sum_{k=1}^{\infty} 2^{-k} \min\{1, |g(x_k) - h(x_k)|\} \\
&= \sum_{k=1}^{\infty} 2^{-k} \min\{1, |h(x_k) - g(x_k)|\} \\
&= \rho(h, g).
\end{aligned}$$

For property (iii), assume that $\rho(g, h) = 0$. Then $\min\{1, |g(x_k) - h(x_k)|\} = 0$ holds and thus $|g(x_k) - h(x_k)| = 0$ which means that $g(x_k) = h(x_k)$ for every $x_k \in Y$. Let $x \in X$ and $\epsilon > 0$. There exists an x_k such that $|g(x) - g(x_k)| < \frac{1}{2}\epsilon$ and $|h(x) - h(x_k)| < \frac{1}{2}\epsilon$ by Lemma 3.5. Then we know that

$$\begin{aligned}
|g(x) - h(x)| &\leq |g(x) - g(x_k)| + |g(x_k) - h(x_k)| + |h(x_k) - h(x)| \\
&\leq \frac{1}{2}\epsilon + 0 + \frac{1}{2}\epsilon = \epsilon
\end{aligned}$$

We conclude that $g(x) = h(x)$ for all $x \in X$.
Assume $g = h$. Then $\min\{1, |g(x_k) - h(x_k)|\} = 0$ holds and thus

$$\begin{aligned}
\rho(g, h) &= \sum_{k=1}^{\infty} 2^{-k} \min\{1, |g(x_k) - h(x_k)|\} \\
&= \sum_{k=1}^{\infty} 2^{-k} \cdot 0 = 0
\end{aligned}$$

We conclude that $\rho(g, h) = 0$ if and only if $g = h$.

For property (iv), let $g, h, j \in \overline{\Psi(X)}$. Then

$$\begin{aligned}
\rho(g,h) &= \sum_{k=1}^{\infty} 2^{-k} \min\{1, |g(x_k) - h(x_k)|\} \\
&= \sum_{k=1}^{\infty} 2^{-k} \min\{1, |g(x_k) - j(x_k) + j(x_k) - h(x_k)|\} \\
&\leq \sum_{k=1}^{\infty} 2^{-k} \min\{1, |g(x_k) - j(x_k)| + |j(x_k) - h(x_k)|\} \\
&\leq \sum_{k=1}^{\infty} 2^{-k} (\min\{1, |g(x_k) - j(x_k)|\} + \min\{1, |j(x_k) - h(x_k)|\}) \\
&= \sum_{k=1}^{\infty} 2^{-k} \min\{1, |g(x_k) - j(x_k)|\} + \sum_{k=1}^{\infty} 2^{-k} \min\{1, |j(x_k) - h(x_k)|\} \\
&= \rho(g,j) + \rho(j,h).
\end{aligned}$$

We conclude that ρ is a metric.
Let $f \in \mathcal{U}(x, U)$. Due to Remark 1.27 we know that there exists and $\epsilon > 0$ such that $\mathcal{U}(f, x, \epsilon) \subset \mathcal{U}(x, U)$. Then there exists an $x_m \in Y$ such that $d(x, x_m) < \frac{1}{3}\epsilon$. Define

$$\mathcal{O}_{\frac{2^{-m}}{3}\epsilon}(f) := \{g \in \overline{\Psi(X)} \mid \rho(f,g) < \frac{2^{-m}}{3}\epsilon\}$$

and let $g \in \mathcal{O}_{\frac{2^{-m}}{3}\epsilon}(f)$. Without loss of generality we can assume that $0 < \epsilon < 3$. Then

$$\rho(f,g) = \sum_{k=1}^{\infty} 2^{-k} \min\{1, |f(x_k) - g(x_k)|\} \geq 2^{-m} \min\{1, |f(x_k) - g(x_k)|\}.$$

Hence, we get

$$\begin{aligned}
2^{-m} \min\{1, |f(x_m) - g(x_m)|\} &< \frac{2^{-m}}{3}\epsilon \\
\min\{1, |f(x_m) - g(x_m)|\} &< \frac{1}{3}\epsilon.
\end{aligned}$$

Since $\frac{1}{3}\epsilon < 1$ we know that $|f(x_m) - g(x_m)| < \frac{1}{3}\epsilon$. Then

$$\begin{aligned}
|f(x) - g(x)| &\leq |f(x) - f(x_m)| + |f(x_m) - g(x_m)| + |g(x_m) - g(x)| \\
&< 2d(x, x_m) + \frac{1}{3}\epsilon < \epsilon
\end{aligned}$$

due to Lemma 3.5 and because $d(x, x_m) < \frac{1}{3}$. It follows that $g \in \mathcal{U}(f, x, \epsilon)$ and thus $\mathcal{O}_{\frac{2^{-m}}{3}\epsilon}(f) \subset \mathcal{U}(f, x, \epsilon)$. We conclude $\mathcal{T}_{pw} \subset \mathcal{T}_\rho$.
Conversely, let $\mathcal{O}_\epsilon(f)$ for $f \in \overline{\Psi(X)}$ and $\epsilon > 0$. There exists an $m \in \mathbb{N}$ such that

$\sum_{k=m}^{\infty} 2^{-k} < \frac{1}{2}\epsilon$. For all $n \leq m$ we define $\mathcal{U} := \bigcap_{n=1}^{m} \mathcal{U}(f, x_n, \frac{1}{2}\epsilon)$. We note that this is a finite intersection and that $\mathcal{U}$ is open with respect to the pointwise topology. Let $g \in \mathcal{U}$, then the following holds

$$\begin{aligned}
\sum_{k=1}^{\infty} 2^{-k} \min\{1, |f(x_k) - g(x_k)|\} &= \sum_{k=m}^{\infty} 2^{-k} \min\{1, |f(x_k) - g(x_k)|\} \\
&\quad + \sum_{k=1}^{m-1} 2^{-k} \min\{1, |f(x_k) - g(x_k)|\} \\
&< \frac{1}{2}\epsilon + \sum_{k=1}^{m-1} 2^{-k} \min\{1, |f(x_k) - g(x_k)|\} \\
&< \frac{1}{2}\epsilon + \sum_{k=1}^{m-1} 2^{-k} \cdot \frac{1}{2}\epsilon < \epsilon
\end{aligned}$$

We conclude that $h \in \mathcal{O}_\epsilon(g)$ and $\mathcal{U} \subset \mathcal{O}_\epsilon(g)$. Thus $\mathcal{T}_\rho \subset \mathcal{T}_{pw}$. Hence, $\mathcal{T}_{pw} = \mathcal{T}_\rho$ and $\overline{\Psi(X)}$ is metrizable. □

The only thing that remains for us to do, in order to show that the metric compactification of $(\mathbb{R}^d, \|\cdot\|_p)$ is metrizable, is to demonstrate that $(\mathbb{R}^d, \|\cdot\|_p)$ is a proper metric space.

Proposition 3.7. *Let d and p be positive integers. Then $(\mathbb{R}^d, \|\cdot\|_p)$ is a proper metric space.*

Proof. Let $r > 0$ be a real number and $x \in \mathbb{R}^d$. Define the translation

$$T_x \colon (\mathbb{R}^d, \|\cdot\|_p) \to (\mathbb{R}^d, \|\cdot\|_p)$$

with $T_x(y) = y + x$ for all $y \in \mathbb{R}^d$ and the translation

$$T_r \colon (\mathbb{R}^d, \|\cdot\|_p) \to (\mathbb{R}^d, \|\cdot\|_p)$$

with $T_r(y) = ry$ for all $y \in \mathbb{R}^d$. It is clear that both T_x and T_r are homeomorphisms. We know that the unit ball $\mathcal{B}_1^p$ is compact with regard to the p-norm for every real $p > 1$ by Proposition 3.2. Since T_x and T_r are homeomorphisms and $T_r \circ T_x$ is also a homeomorphism, we know that $T_r(T_x(\mathcal{B}_1^p))$ is compact with respect to the p-norm. We conclude that $(\mathbb{R}^d, \|\cdot\|_p)$ is a proper metric space. □

Proposition 3.8. *Let $p > 1$ be a real number then $\overline{\Psi(\mathbb{R}^d, \|\cdot\|_p)}$ is metrizable.*

Proof. Due to Proposition 3.7, $(\mathbb{R}^d, \|\cdot\|_p)$ is a proper metric space and due to Proposition 3.6 we conclude that $\overline{\Psi(\mathbb{R}^d, \|\cdot\|_p)}$ is metrizable. □

3.2 Computing the boundary $\overline{\Psi(\mathbb{R}^d, \|\cdot\|_p)} \setminus \Psi(\mathbb{R}^d, \|\cdot\|_p)$

We have proved that the metric compactification of $(\mathbb{R}^d, \|\cdot\|_p)$ is metrizable, so from now on we can use sequences instead of nets. Before we move on to compute the boundary we need the following notion and result.

Definition 3.9 (Norming Functional)**.** Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and let

$$\phi \colon \mathbb{R}^d \to \mathbb{R}$$

be a function. If $\phi(x) = \|x\|$ for some $x \in \mathbb{R}^d$, then we call ϕ a *norming functional* of x.

We need the following notation. For $p \in \mathbb{R}$ define the set

$$\mathcal{S}_p := \{x \in (\mathbb{R}^d, \|\cdot\|_p) \mid \|x\|_p = 1\}.$$

Proposition 3.10. *Let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Define the function*

$$\mathcal{J} \colon \mathcal{S}_p \to \mathcal{S}_q$$

by

$$\sum_{k=1}^{n} x_k e_k \mapsto \sum_{k=1}^{n} \operatorname{sng}(x_k)|x_k|^{p-1} e_k.$$

Then $\mathcal{J}$ is an isometric isomorphism with the property that every element gets mapped to a norming functional of itself.

Proof. The first thing we need to prove is that the function is well-defined. Let $x \in \mathcal{S}_p$ then the following holds

$$\left\| \sum_{k=1}^{n} \operatorname{sgn}(x_k)|x_k|^{p-1} e_k \right\|_q = \left(\sum_{k=1}^{n} |\operatorname{sgn}(x_k)|x_k|^{p-1}|^q \right)^{\frac{1}{q}} = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{q}} = 1.$$

Hence, $\mathcal{J}$ is well-defined.
For $x_k \neq 0$ we have $\operatorname{sgn}(x_k) = \frac{x_k}{|x_k|}$. Hence, it is not difficult to verify that $\mathcal{J}(x) = \sum_{k=1}^{n} x_k |x_k|^{p-2} e_k$ also holds for $x_k = 0$. For a sequence $(x_n)_n$ in $\mathbb{R}$ which converges to x we have $\lim_{n\to\infty} x_n |x_n|^{p-2} = x|x|^{p-2}$. Thus, $x_k |x_k|^{p-2} e_k$ is continuous and $\mathcal{J}(x) = \sum_{k=1}^{n} x_k |x_k|^{p-2} e_k$ is also continuous.
To prove that $\mathcal{J}$ is an isometric isomorphism we will give an inverse function. Define $\mathcal{J}' \colon \mathcal{S}_q \to \mathcal{S}_p$ by

$$\sum_{k=1}^{n} x_k e_k \mapsto \sum_{k=1}^{n} \operatorname{sgn}(x_k)|x_k|^{q-1} e_k.$$

We will first prove that this function is well-defined. Let $x \in \mathcal{S}_q$ then the following holds

$$\left\| \sum_{k=1}^{n} \operatorname{sgn}(x_k)|x_k|^{q-1} e_k \right\|_p = \left(\sum_{k=1}^{n} |\operatorname{sgn}(x_k)|x_k|^{q-1}|^p \right)^{\frac{1}{q}} = \left(\sum_{k=1}^{n} |x_k|^q \right)^{\frac{1}{q}} = 1.$$

Let $x \in \mathcal{S}_q$ then the following holds

$$\begin{aligned}
(\mathcal{J}' \circ \mathcal{J})(x) &= \mathcal{J}'(\mathcal{J}(x)) \\
&= \mathcal{J}'(\sum_{k=1}^{n} \operatorname{sgn}(x_k)|x_k|^{q-1} e_k) \\
&= \sum_{k=1}^{n} \operatorname{sgn}(\operatorname{sgn}(x_k)|x_k|^{q-1})|\operatorname{sgn}(x_k)|x_k|^{q-1}|^{p-1} e_k \\
&= \sum_{k=1}^{n} \operatorname{sgn}(x_k)|x_k|^{(q-1)(p-1)} e_k \\
&= \sum_{k=1}^{n} \operatorname{sgn}(x_k)|x_k| e_k \\
&= \sum_{k=1}^{n} x_k e_k = x
\end{aligned}$$

since $1 = (q-1)(p-1)$. In the same manner, we can show that $(\mathcal{J}' \circ \mathcal{J}) = id$. Hence, $\mathcal{J}'$ is the inverse of $\mathcal{J}$.
We will now prove that $\mathcal{J}$ sends every element to its own norming functional. Due to Remark 1.50, it suffices to show the following

$$\begin{aligned}
\mathcal{J}(x)(x) &= \sum_{k=1}^{n} x_k \operatorname{sgn}(x_k)|x_k|^{p-1} \\
&= \sum_{k=1}^{n} |x_k||x_k|^{p-1} \\
&= \sum_{k=1}^{n} |x_k|^p = 1 = \|x\|_p \, .
\end{aligned}$$

□

We need the following proposition before we can compute the boundary of $\Psi(\mathbb{R}^d, \|\cdot\|_p)$.

Proposition 3.11. *Let $(X, \|\cdot\|)$ be a normed space. Let $x \in X$ and $\lambda > 0$. Then $\phi \in X^*$ is a norming functional of the vector x if and only if ϕ is a norming functional of the vector λx.*

Proof. Assume that $\phi(x) = \|x\|$. Then the following holds

$$\phi(\lambda x) = \lambda\phi(x) = \lambda \|x\| = \|\lambda x\| .$$

Hence, ϕ also norms λx.
Assume that $\phi(\lambda x) = \|\lambda x\|$. Then the following holds

$$\lambda\phi(x) = \phi(\lambda x) = \|\lambda x\| = \lambda \|x\| .$$

Thus we know that

$$\begin{aligned} \lambda\phi(x) &= \lambda \|x\| \\ \phi(x) &= \|x\| . \end{aligned}$$

Hence, ϕ also norms x. □

We will now state the main result of this section, wherein we explicitly determine both the elements on the boundary of $\overline{\Psi(\mathbb{R}^d, \|\cdot\|_p)}$ and the internal functions.

Theorem 3.12. *Every element in $\overline{\Psi(\mathbb{R}^d, \|\cdot\|_p)} \setminus \Psi(\mathbb{R}^d, \|\cdot\|_p)$ is of the form $\mathcal{J}(x)$ for an $x \in \mathbb{R}^d$.*

Proof. Let $p > 1$ be a real number and define $x_0 := 0$. Let $h \in \overline{\Psi(\mathbb{R}^d, \|\cdot\|_p)}$. Then there exists a sequence $(x_n)_n$ such that $\Psi(x_n) \to h$ as $n \to \infty$ pointwise. There are two possibilities: the sequence is bounded or the sequence is unbounded. In the first case we know by the Bolzano–Weierstrass theorem that there exists a convergent subsequence $(x_{n_k})_k$ such that $x_{n_k} \to x$ as $k \to \infty$ for some $x \in \mathbb{R}^d$. Let $\epsilon > 0$. Then the following holds for $y \in \mathbb{R}^d$

$$\begin{aligned} |h(y) - \Psi(x_n)(y)| &\leq |h(y) - \Psi(x_{n_k})(y) + \Psi(x_{n_k})(y) - \Psi(x_n)(y)| \\ &\leq |h(y) - \Psi(x_{n_k})(y)| + |\Psi(x_{n_k})(y) - \Psi(x_n)(y)|. \end{aligned}$$

Since $\Psi(x_n)$ converges pointwise to h we also know that for all $\epsilon > 0$ there exists an N such that for all $k \geq N$ we have

$$|h(y) - \Psi(x_{n_k})(y)| < \epsilon.$$

Thus, $\Psi(x_n)$ converges to h and we know that $\Psi(x) = h$ with $h \in \Psi(\mathbb{R}^d, \|\cdot\|_p)$. In the second case, if $(x_n)_n$ is unbounded then there is a subsequence $(x_{n_k})_k$ such that $\|x_{n_k}\|_p \to \infty$ for $k \to \infty$. Define $y_k := x_{n_k}$. Since $\mathcal{S}_p$ is compact, we know that $\frac{y_k}{\|y_k\|_p}$ has a convergent subsequence $\frac{y_{k_i}}{\|y_{k_i}\|_p}$ that converges to some

$y \in S_p$. Let $z \in \mathbb{R}^d$ then the followings holds

$$\begin{aligned}
\left\| \frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} - \frac{y_{k_i}}{\|y_{k_i}\|_p} \right\|_p &= \left\| \frac{\|y_{k_i}\|_p (y_{k_i} - z) - \|y_{k_i} - z\|_p y_{k_i}}{\|y_{k_i} - z\|_p \|y_{k_i}\|_p} \right\|_p \\
&= \left\| \frac{\|y_{k_i}\|_p y_{k_i} - \|y_{k_i}\|_p z - \|y_{k_i} - z\|_p y_{k_i}}{\|y_{k_i} - z\|_p \|y_{k_i}\|_p} \right\|_p \\
&= \left\| \frac{(\|y_{k_i}\|_p - \|y_{k_i} - z\|_p) y_{k_i} - \|y_{k_i}\|_p z}{\|y_{k_i} - z\|_p \|y_{k_i}\|_p} \right\|_p \\
&\leq \frac{(\|y_{k_i}\|_p - \|y_{k_i} - z\|_p) \|y_{k_i}\|_p + \|y_{k_i}\| \|z\|_p}{\|y_{k_i} - z\|_p \|y_{k_i}\|_p} \\
&\leq \frac{\|y_{k_i} - y_{k_i} - z\|_p \|y_{k_i}\|_p + \|y_{k_i}\|_p \|z\|_p}{\|y_{k_i} - z\|_p \|y_{k_i}\|_p} \\
&= \frac{2 \|z\|_p \|y_{k_i}\|_p}{\|y_{k_i} - z\|_p \|y_{k_i}\|_p} \\
&= \frac{2 \|z\|_p}{\|y_{k_i} - z\|_p}.
\end{aligned}$$

Note that

$$\|y_{k_i} - z\|_p \geq \|y_{k_i}\|_p - \|z\|_p$$

and thus

$$\frac{2 \|z\|_p}{\|y_{k_i} - z\|_p} \leq \frac{2 \|z\|_p}{\|y_{k_i}\|_p - \|z\|_p}.$$

Since $\|y_{k_i}\|_p - \|z\|_p \to \infty$ for $i \to \infty$, it follows that

$$\lim_{i \to \infty} \frac{2 \|z\|_p}{\|y_{k_i} - z\|_p} \leq \lim_{i \to \infty} \frac{2 \|z\|_p}{\|y_{k_i}\|_p - \|z\|_p} = 0.$$

Hence,

$$\lim_{i \to \infty} \frac{2 \|z\|_p}{\|y_{k_i} - z\|_p} = 0$$

and thus

$$\lim_{i \to \infty} \left\| \frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} - \frac{y_{k_i}}{\|y_{k_i}\|_p} \right\|_p = 0.$$

Let $\epsilon > 0$. Then we know that

$$\begin{aligned}\left\| \frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} - y \right\|_p &= \left\| \frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} - \frac{y_{k_i}}{\|y_{k_i}\|_p} + \frac{y_{k_i}}{\|y_{k_i}\|_p} - y \right\|_p \\ &\leq \left\| \frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} - \frac{y_{k_i}}{\|y_{k_i}\|_p} \right\|_p + \left\| \frac{y_{k_i}}{\|y_{k_i}\|_p} - y \right\|_p .\end{aligned}$$

There exists an N such that for all $j \geq N$ we have

$$\left\| \frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} - \frac{y_{k_i}}{\|y_{k_i}\|_p} \right\|_p + \left\| \frac{y_{k_i}}{\|y_{k_i}\|_p} - y \right\|_p < \epsilon$$

Hence, $\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p}$ converges to y just like $\frac{y_{k_i}}{\|y_{k_i}\|_p}$ as $i \to \infty$.
Due to Proposition 3.10 and since $\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p}, \frac{y_{k_i}}{\|y_{k_i}\|_p} \in \mathcal{S}_p$ we know that $\mathcal{J}$ norms these elements.
With Proposition 3.11 we get the following result

$$\mathcal{J}\left(\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} \right)(y_{k_i} - z) = \|y_{k_i} - z\|_p .$$

Due to Remark 1.50, we know that for $w_1, w_2 \in \mathcal{S}_p$ we have

$$\mathcal{J}(w_1)(w_2) \leq |\mathcal{J}(w_1)(w_2)| \leq \|\mathcal{J}(w_1)\|_q \|w_2\|_p .$$

Since $\|\mathcal{J}(w_1)\|_q = 1$ we conclude

$$\mathcal{J}(w_1)(w_2) \leq \|w_2\|_p . \tag{6}$$

Thus we have the following inequality

$$\|y_{k_i} - z\|_p - \|y_{k_i}\|_p \leq \mathcal{J}\left(\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} \right)(y_{k_i} - z) - \mathcal{J}\left(\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} \right)(y_{k_i}).$$

Due to linearity of $\mathcal{J}\left(\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} \right)$ we know that

$$\begin{aligned}\mathcal{J}\left(\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} \right)(y_{k_i} - z) - \mathcal{J}\left(\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} \right)(y_{k_i}) &= \mathcal{J}\left(\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} \right)(-z) \\ &= -\mathcal{J}\left(\frac{y_{k_i} - z}{\|y_{k_i} - z\|_p} \right)(z).\end{aligned}$$

Again, due to Proposition 3.11 the function $\mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p} \right)$ norms y_{k_i}. Thus the following holds

$$\mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p} \right)(y_{k_i}) = \|y_{k_i}\|_p .$$

Once more, with (6) we deduce

$$\mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p}\right)(y_{k_i}-z)-\mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p}\right)(y_{k_i})\leq \|y_{k_i}-z\|_p-\|y_{k_i}\|_p\,.$$

Due to linearity of $\mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p}\right)$ we know that

$$\begin{aligned}\mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p}\right)(y_{k_i}-z)-\mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p}\right)(y_{k_i}) &= \mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p}\right)(-z)\\ &= -\mathcal{J}\left(\frac{y_{k_i}}{\|y_{k_i}\|_p}\right)(z).\end{aligned}$$

Since both $\frac{y_{k_i}-z}{\|y_{k_i}-z\|_p}$ and $\frac{y_{k_i}}{\|y_{k_i}\|_p}$ converge to y for $i\to\infty$, we conclude

$$-\mathcal{J}(y)(z)\leq \lim_{i\to\infty}\|y_{k_i}-z\|_p-\|y_{k_i}\|_p\leq -\mathcal{J}(y)(z).$$

With the Squeeze Theorem [8, Theorem 2.2.26.8] we see that $h(z)=-\mathcal{J}(y)(z)$. We know that $-\mathcal{J}(y)$ is a linear function and every function $h\in\Psi(\mathbb{R}^d,\|\cdot\|_p)$ is of the form $h(x)=\Psi(y)(x)=\|x-y\|_p-\|y\|_p$ for some y. Since there is no $y\in\mathbb{R}^d$ for which $\|x-y\|_p-\|y\|_p$ is a linear fuction, we conclude that every element in the boundary of $\overline{\Psi(\mathbb{R}^d,\|\cdot\|_p)}$ is of the form $-\mathcal{J}(y)$. $\square$

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