Deformation Quantisation
Velten, F.

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Author :	Florian Velten
Student ID :	2124432
Supervisor – LION :	Dr. S. P. Patil
Supervisor – MI :	Dr. B. Mesland

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Florian Velten

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Abstract

This thesis discusses a lesser known formulation of quantum mechanics called *deformation quantisation*. This theory provides a more intuitive way to study quantum systems, in which confusing topics such as the relation between the operator commutator and the classical Poisson bracket and the concept of classical limit become very obvious. After a short introduction, Chapter 2 introduces the mathematical foundation of Hamiltonian classical mechanics, symplectic geometry, and states the Darboux Theorem. Chapter 3 develops the theory of Hochschild cohomology and gives a classification of the cohomology spaces of the algebra of smooth functions on a manifold. In Chapter 4, deformations of algebras are defined and results of Hochschild cohomology are used to prove lemmas about star products, which are smooth deformations of the algebra of smooth functions on a manifold. And Chapter 5 introduces the special case of the *Moyal star product* and shows how it can be used to obtain deformation quantisation. In this chapter, it is also shown that deformation quantisation is completely equivalent to the Hilbert space formalism which is traditionally taught in undergraduate studies, and the simple harmonic oscillator is treated as an example. Finally, Chapter 6 summarises the possible benefits and drawbacks of teaching deformation quantisation instead of the Hilbert space formalism and lists some avenues of further study.

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Chapter 1

Introduction

When undergraduate students follow their first quantum mechanics course, they are likely surprised by the contrast between the Hilbert space formalism and the Newtonian, Lagrangian, and Hamiltonian formulations of classical mechanics. Physical particles are replaced by their wave functions and properties of particles that are easy to understand and visualise, such as position, velocity, momentum, and energy, are replaced by the at first rather opaque notion of operators on a Hilbert space. The student learns about bras and kets, about state vectors, operators, eigenfunctions, and the Schrödinger equation. All in all, they have to get used to a completely new mathematical toolbox, apply it as a new, initially unintuitive description of microscopic reality, and are then told that it all relates back to what they have already learned via the rather vague notion of classical limit. This is, of course, a recipe for confusion. But it does not have to be this way.

Instead of introducing a whole new mathematical theory, there is a way to describe quantum mechanics which only requires a single additional concept while building upon Hamiltonian mechanics: a non-commutative deformation. This is deformation quantisation, a formulation of quantum mechanics based on the momentum phase space that the student already knows. The central object is the *Moyal star product*, which introduces corrections to the classical theory as a power series in the quantum parameter $\hbar$. Its non-commutative and non-local nature admits a very intuitive quantum theory as an extension of the classical.

In this thesis, we will introduce the mathematical background and discuss the basics of deformation quantisation. Chapter 2 treats elementary symplectic geometry, which is the mathematical setting of Hamiltonian mechanics. Chapter 3 introduces Hochschild cohomology, a tool that is needed in Chapter 4, which discusses the theory of algebra deformations. This forms the mathematical basis for deformation quantisation, which we explain and apply to the simple harmonic oscillator in Chapter 5. Finally, Chapter 6 contains a conclusion and possible paths for further study.

To follow the mathematics of Chapters 2, 3, and 4, a decent understanding of smooth manifolds, (co)tangent bundles, and differential forms, as well as basic knowledge of rings and algebras is required. To understand the physics of Chapter 5, experience with both Hamiltonian mechanics and the Hilbert space formulation of quantum mechanics is necessary.

Chapter 2

Symplectic Geometry

We study symplectic geometry because it provides a natural setting for classical mechanics. To see why, we need to ask ourselves what data is needed to describe a physical system. Classically, such a system consists of n particles in three-dimensional space, so we certainly need some way of describing *where* the particles are. But not only that, we also need to know *where they're going*, that is their velocities. Moreover, we need a set of rules that prescribes how these quantities change with time, which should be equivalent to Newton's laws of motion.

Let us make this a bit more mathematical. The time evolution of a particle's position describes the particle's orbit in the ambient space, and its velocity is tangent to this orbit at each point in time. Thus, we very naturally arrive at the study of smooth manifolds and their tangent spaces. In particular, the geometry of the ambient space yields a $3n$-dimensional smooth manifold M called the *configuration space* of the system. Specifying a point in M is specifying the positions of the $3n$ particles. If we also want to include the velocities, we move to the tangent bundle TM, which is the $6n$-dimensional *velocity phase space*.

Now we need to represent Newton's laws in some way. There are several ways to do this, but the one we will be focussing on here is the Hamiltonian approach. Given the system's Lagrangian $L \in \mathcal{C}^\infty(TM)$, which encodes Newton's laws as the Euler-Lagrange equations, a *Legendre transform* is applied to move to the cotangent bundle T^*M, which is the ordinary *(momentum) phase space*. (The Legendre transform is a powerful mathematical tool, whose strength is often overlooked by physicists. For a concise and clear geometric explanation of the Legendre transform, which also touches upon physical applications in classical and statistical mechanics, see [18].) The Hamiltonian approach is completely equivalent to the Lagrangian approach even in the very general setting we are working in here. This is a result of Theorem 20.10 in [4], a series of lecture notes on symplectic geometry which defines and explores the Legendre transform in the setting of manifolds, among other topics.

In moving to the momentum phase space, the Lagrangian turns into the system's Hamiltonian, a smooth function $H \in \mathcal{C}^\infty(T^*M)$ on the cotangent bundle. This function represents the total energy of the system and should generate the flow of time, that is to say determine the time evolution of the system. In particular, we would like to associate to our Hamiltonian H a vector field X_H on T^*M; the phase space orbits of the particles are then the vector field's integral curves.

Moving forward, we follow the reasoning of Dr. Henry Cohn's essay on his website*. Since Newton's laws of motion are linear differential equations and the dynamics should not depend on the total energy of the system, the vector field X_H should depend linearly on $\mathrm{d}H$. Thus, we are looking for a smooth map

$$\Omega_1(T^*M) \to \mathcal{X}(T^*M),$$

or, equivalently, a section of the vector bundle $\mathrm{Hom}(T^*M, TM)$. Now, it turns out to be more convenient to instead look for a section of $\mathrm{Hom}(TM, T^*M)$, a bundle which is isomorphic to $T^*M \otimes T^*M$.

Let ω be a section of $T^*M \otimes T^*M$, then what properties do we require ω to have? Reasoning backwards, we would like ω to associate to X_H the differential $\mathrm{d}H$ of our Hamiltonian. In other words, given another vector field X, we would like to have $\omega(X_H, X) = \mathrm{d}H\,(X)$, so $\mathrm{d}H = \iota_{X_H}\omega$. Now recall that we needed to solve for X_H given $\mathrm{d}H$, so ω should be non-degenerate. This means nothing but that the map

$$\hat{\omega} : \mathcal{X}(T^*M) \to \Omega^1(T^*M) : X \mapsto \iota_X\omega$$

should be an isomorphism.

Next is a more physical requirement, namely conservation of energy. We should require that the total energy of the system is constant, that is that the Hamiltonian is constant along the flow lines of its induced vector field. This yields $0 = X_H(H) = \mathrm{d}H\,(X_H) = \omega(X_H, X_H)$, which implies that ω should be alternating: $\omega \in \Omega^2(T^*M)$.

Finally, just like the total energy, we want the laws of physics to be the same at every point in time. Let θ_t denote the flow of X_H, then we require $\theta_t^*\omega = \omega$ for all t, so $\frac{\mathrm{d}}{\mathrm{d}t}\theta_t^*\omega = 0$. Using the Lie derivative and Cartan's Magic formula, we calculate

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\theta_t^*\omega = \theta_t^*\mathcal{L}_{X_H}\omega &= \theta_t^*(\iota_{X_H}\,\mathrm{d}\omega + \mathrm{d}(\iota_{X_H}\omega)) \\ &= \theta_t^*(\iota_{X_H}(\mathrm{d}\omega) + \mathrm{d}(\mathrm{d}H)) = \theta_t^*(\iota_{X_H}(\mathrm{d}\omega)). \end{aligned}$$

If we require ω to be closed, the above equation is true. Hence, we have a closed non-degenerate 2-form ω; such a differential form is called *symplectic*.

So all in all, classical mechanics can be described by specifying a symplectic form on the cotangent bundle of the configuration space manifold. The question now is, does such an object exist, and if so, is there a canonical, physically motivated choice? As we will see, the answer is "yes".

2.1 Symplectic Manifolds

Let us state the above construction as a definition.

*`https://cohn.mit.edu/symplectic`, accessed at 13:06 on July 14th, 2023.

Definition 2.1: Symplectic Manifold

Let M be a $2n$-dimensional smooth manifold. A *symplectic form* on M is a non-degenerate closed 2-form $\omega \in \Omega^2(M)$. The pair (M, ω) is called a *symplectic manifold*.

Non-degeneracy means that interior multiplication of ω with a vector field defines a linear isomorphism

$$\hat{\omega} : \mathcal{X}(T^*M) \to \Omega^1(T^*M) : X \mapsto \iota_X \omega.$$

Symplectic manifolds have many nice properties, and non-degeneracy has many different definitions, a lot of which are equivalent. For a comprehensive treatment, see Chapter 22 in [15] or the aforementioned lecture notes [4].

The prototypical example of a symplectic manifold is $\mathbf{R}^{2n}$ with

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i\,, \tag{2.1}$$

where $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ are the coordinates on $\mathbf{R}^{2n}$. As it turns out, symplectic manifolds are special in the sense that locally, there always exist coordinates such that the symplectic form looks like the one in equation 2.1.

Theorem 2.2: Darboux's Theorem

Let (M, ω) be a symplectic manifold of dimension $2n$ and let $p \in M$ be a point. There exists a chart (U, φ) around p such that in the coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ induced by (U, φ), we have

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

Such coordinates are called *Darboux coordinates*.

A proof can be found in [15] below Theorem 22.13 and a sketch of a different proof in Problem 22-19 of the same text book.

The Darboux Theorem has an interesting implication, which is very desirable from a physics standpoint: if we can describe a classical system using symplectic geometry, there are always coordinates such that the laws of physics can be described by the same differential equations, irrespective of the system and its state. So let us see if we can find a symplectic model for classical physics.

2.2 The Canonical Symplectic Form on T^*M

Let M be a smooth n-manifold and let $\pi : T^*M \to M$ be the vector bundle projection. Its pullback $\mathrm{d}\pi^*$ maps covector fields on M to covector fields on T^*M.

Definition 2.3: Tautological 1-form

The *tautological 1-form* on T^*M is the 1-form τ which is defined for points $(q,\varphi) \in T^*M$ and vectors $v \in T_{(q,\varphi)}(T^*M)$ by

$$\tau(q,\varphi)(v) = (\mathrm{d}\pi^*(q,\varphi)(\varphi))(v) = \varphi(\mathrm{d}\pi(q,\varphi)(v)).$$

The tautological 1-form first projects a tangent vector $v \in T_{(q,\varphi)}(T^*M)$ onto M, and then applies φ to the resulting vector $\mathrm{d}\pi(q,\varphi)(v) \in T_qM$. Now, we can define a symplectic form on the cotangent bundle.

Definition 2.4: Canonical Symplectic Form

The 2-form $\omega = -\,\mathrm{d}\tau$ is called the *canonical symplectic form* on T^*M.

The facts that τ is smooth and that ω is symplectic are proven in [15] in Proposition 22.11. In the proof, we can also find the coordinate representation of these two objects, which we will highlight here. Let (q^i) denote smooth local coordinates around a point $q \in M$ and let (q^i, φ_i) be the corresponding coordinates around a point $(q,\varphi) \in T^*M$, so that we have $\varphi = \sum_{i=1}^n \varphi_i \,\mathrm{d}q^i$. The natural coordinates on $T^*(T^*M)$ around (q,φ) are now given by the local frame $(\mathrm{d}q^i, \mathrm{d}\varphi_i)$, hence we have $\mathrm{d}\pi^*(\mathrm{d}q^i) = \mathrm{d}q^i$, from which it follows that

$$\tau(q,\varphi) = \mathrm{d}\pi^*(q,\varphi)(\varphi) = \sum_{i=1}^n \varphi_i \,\mathrm{d}q^i.$$

The canonical symplectic form thus reads

$$\omega = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}\varphi_i$$

in local coordinates. In particular, the natural coordinates on T^*M are Darboux coordinates for the canonical symplectic form.

In the next section, we will see why this symplectic form is physically interesting, and thus why it is so convenient that it always exists on the momentum phase space of a configuration space manifold.

2.3 Hamiltonian Vector Fields

Earlier, we encountered the isomorphism

$$\hat{\omega} : \mathcal{X}(T^*M) \to \Omega^1(T^*M) : X \mapsto \iota_X\omega.$$

Since its inverse is well-defined, we can associate to each 1-form a vector field on M. Moreover, we can use it to associate a vector field to each *smooth function* on M:

Definition 2.5: Hamiltonian Vector Field

Let $f \in \mathcal{C}^\infty(M)$. The *Hamtiltonian vector field* associated to f is

$$X_f = \hat{\omega}^{-1}(\mathrm{d}f).$$

It is the vector field X_f such that $\iota_{X_f}\omega = \mathrm{d}f$.

On page 574 in [15], Lee calculates the form of a Hamiltonian vector field X_f in Darboux coordinates:

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial y^i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y^i}.$$

The Hamiltonian vector fields that correspond to the coordinate functions will be of interest later:

$$X_{x^i} = -\frac{\partial}{\partial y^i} \quad \text{and} \quad X_{y^i} = \frac{\partial}{\partial x^i}. \tag{2.2}$$

The combination of Hamiltonian vector fields and the symplectic form induces an important structure on the algebra of smooth functions.

Definition 2.6: Poisson Algebra

Let A be an associative algebra over a field $\mathbf{F}$. A *Poisson bracket* on A is an antisymmetric $\mathbf{F}$-bilinear map $\{.,.\} : A \times A \to A$ that satisfies the following two identities.

$$\begin{aligned} \textbf{Jacobi identity:} &\quad \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \\ \textbf{Leibniz rule:} &\quad \{fg,h\} = f\{g,h\} + \{f,h\}g. \end{aligned}$$

The pair $(A,\{.,.\})$ is called a *Poisson algebra*.

Lemma 2.7

Let $f,g \in \mathcal{C}^\infty(M)$. The pairing $\{f,g\} = \omega(X_f, X_g)$ defines a Poisson bracket on $\mathcal{C}^\infty(M)$ as $\mathbf{R}$-algebra.

Proof. The antisymmetry and bilinearity of the bracket follow immediately from the definitions and the antisymmetry of ω. A proof the Jacobi identity can be found in Proposition 22.19 of [15]. The Leibniz rule follows from properties of the exterior derivative:

$$\mathrm{d}(fg) = \mathrm{d}f \cdot g + f \cdot \mathrm{d}g\,,$$

hence

$$\begin{aligned}\{fg,h\} &= \omega(X_{fg},X_h) = \mathrm{d}(fg)\,(X_h)\\ &= \mathrm{d}f\,(X_h)\cdot g + f\cdot \mathrm{d}g\,(X_h)\\ &= \omega(X_f,X_h)g + f\omega(X_g,X_h)\\ &= \{f,h\}g + f\{g,h\}.\end{aligned}$$

□

Readers who are familiar with Hamiltonian mechanics will probably recognise the notation of the Poisson bracket, and for good reasons: this particular instance of overlapping notation is not accidental. Recall that the natural coordinates on the momentum phase space T^*M, which we will now suggestively denote by (q^i, p^i), are Darboux. It then follows from equation 2.2 that we have

$$X_{q^i} = -\frac{\partial}{\partial p^i} \quad \text{and} \quad X_{p^i} = \frac{\partial}{\partial q^i}.$$

Suppose that we now fix a Hamiltonian function $H \in \mathcal{C}^\infty(M)$. First, note the following identity:

$$\begin{aligned}X_f(g) &= \mathrm{d}g\,(X_f) = \omega(X_g,X_f)\\ &= -\omega(X_f,X_g) = -\,\mathrm{d}f\,(X_g)\\ &= -X_g(f).\end{aligned}$$

Using this, we can let the Hamiltonian vector fields of the canonical coordinates act on H to get

$$\frac{\partial H}{\partial p^i} = -X_{q^i}(H) = X_H(q^i) \quad \text{and} \quad \frac{\partial H}{\partial q^i} = X_{p^i}(H) = -X_H(p^i).$$

But since the phase space orbits corresponding to physical trajectories are the integral curves of X_H, letting X_H act on a function simply corresponds to taking the time derivative of said function, so the above reduces to Hamilton's equations of motion:

$$\frac{\partial H}{\partial p^i} = \frac{\mathrm{d}q^i}{\mathrm{d}t} \quad \text{and} \quad \frac{\partial H}{\partial q^i} = -\frac{\mathrm{d}p^i}{\mathrm{d}t}.$$

In general, the time evolution of a time-independent smooth function f is, accordingly, given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = X_H(f) = \omega(X_f, X_H) = \{f, H\},$$

which is also a well-known equation of Hamiltonian mechanics. This shows that symplectic geometry indeed serves as a mathematical theory that describes classical mechanics.

Chapter 3

Hochschild Cohomology

Before we can build upon symplectic geometry and turn to deformation quantisation, we must discuss an important mathematical tool that will appear here and there in the future. This tool is the cohomology of multilinear maps from Cartesian products of an algebra with itself into itself, which was developed by Gerhard Hochschild in his 1945 paper [13]. Nowadays, Hochschild homology and cohomology are treated mostly in advanced texts on non-commutative geometry such as the lectures notes in [6].

An important theorem of Hochschild (co)homology is the *Hochschild-Konstant-Rosenberg Theorem*, which provides a classification of the Hochschild (co)homology spaces. In practice, this is useful because it allows for the decomposition of a cochain into the sum of an antisymmetric object and a coboundary. We will see a smooth version of this decomposition as well as prove the classification result for the smooth case. The decomposition and a handful of small lemmas will be useful in the next chapter, in which we will discuss deformations of algebras.

3.1 Definitions

The definitions in any theory of cohomology are formally very similar, independently of the specific type. In the following, we will refer to the corresponding definitions of the De Rham cohomology to illustrate this. Note that many terms will have the prefix "co-", which has everything to do with the fact that we are studying *co*homology, instead of homology.

We will first discuss the general version of the theory, generally following [5], and then move towards the special case of smooth functions on a manifold.

Definition 3.1: Algebra

Let $\mathbf{F}$ be a field. An $\mathbf{F}$-*algebra* is an $\mathbf{F}$-vector space A supplied with a bilinear "multiplication" map $\cdot : A \times A \to A$.

Throughout this thesis, all algebras will be associative and unital, that is the multiplication will be associative and will have a unit element.

Definition 3.2: Cochains and the Hochschild Coboundary Operator

Let A be an $\mathbf{F}$-algebra and let $k \geq 0$ be an integer. A *k-cochain* is a k-linear map $C : A^k \to A$, in other words an element of $\mathrm{Hom}_{\mathbf{F}}(A^{\otimes k}, A)$, with the convention $A^0 = A^{\otimes 0} = \mathbf{F}$. The vector space of k-cochains on A is denoted by $C^k(A)$.

The *Hochschild coboundary operator* $\partial : C^k(A) \to C^{k+1}(A)$ is given by

$$(\partial C)(u_0, \ldots, u_k) = u_0 C(u_1, \ldots, u_k) + (-1)^{k+1} C(u_0, \ldots, u_{k-1}) u_k + \sum_{r=1}^{k} (-1)^r C(u_0, \ldots, u_{r-1} \cdot u_r, \ldots, u_k)$$

for integers $k \geq 1$ and by

$$(\partial C)(u_0) = u_0 C(1) - C(1) u_0$$

for $k = 0$.

The cochains play the same role in Hochschild cohomology as the differential forms on a manifold play in De Rham cohomology, and the coboundary operator is comparable to the exterior derivative. Both $\mathbf{F}$-linearity of the coboundary operator and the fact that the image of a p-cochain is a $(p+1)$-cochain are clear from the definition. A proof that $\partial^2 = 0$ is nothing more than expanding the definition and cancelling terms; a short one can be found on pages 1 and 2 of [5].

Now let $k \geq 1$ be an integer. A k-cochain C is called a *coboundary* (compare: exact form) if there exists a $(k-1)$-cochain B such that $C = \partial B$, and it is called a *cocycle* (compare: closed form) if $\partial C = 0$. Note that every coboundary is a cocycle. Thanks to the linearity of ∂, both the coboundaries and the cocycles form vector spaces in each degree.

Definition 3.3: Hochschild Cohomology Spaces

Let $k \geq 0$ be an integer. The vector space of k-cocycles of A is denoted by $Z^k(A)$, and the vector space of k-coboundaries of A is denoted by $B^k(A)$, with the convention $B^0(A) = 0$.

Note that $B^k(A)$ is a subspace of $Z^k(A)$ for every k. The *k-th Hochschild cohomology space* is defined to be the quotient

$$H^k_{Hs}(A) = Z^k(A)/B^k(A).$$

Next, we study the spaces of low degree, which represent interesting objects related to the algebra. Let $k = 0$. Note that the image of a 0-cochain $C : \mathbf{F} \to A$ is completely determined by the image of $1 \in \mathbf{F}$, so $C^0(A) \cong A$. Moreover, since $B^0(A) = 0$, we have $H^0_{Hs}(A) \cong Z^0(A)$. If we write $C(1) = u_1$, then C is a cocycle precisely when $(\partial C)(u_0) = u_0 u_1 - u_1 u_0 = 0$ for all $u_0 \in A$. But this just means that u_1 should commute with all elements of A, hence $H^0_{Hs}(A)$ is just the *centre* of A. In particular, if A is commutative, it follows that $H^0_{Hs}(A) \cong A$.

Let $k = 1$. For $C \in C^1(A)$, the cocycle condition reads

$$(\partial C)(u_0, u_1) = u_0 C(u_1) - C(u_0 u_1) + C(u_0) u_1 = 0.$$

This is equivalent with stating that $C : A \to A$ should be a *derivation* on A, that is a linear map satisfying $C(ab) = aC(b) + C(a)b$ for all $a, b \in A$, so $Z^1(A) \cong \mathrm{Der}(A)$ is the vector space of derivations.

In higher degrees, a classification of the Hochschild cohomology spaces for a specific type of algebra is possible using the Hochschild-Konstant-Rosenberg Theorem.

3.2 Smooth Hochschild Cohomology

Since physicists are only interested in functions that are nice enough, which in practice means smooth, we will now specialise by letting A be the algebra of smooth functions on a manifold. Useful to us will be a smooth version of the Hochschild-Konstant-Rosenberg formula, which states that every cocycle is the sum of its total antisymmetrisation and a coboundary. But in order to study it, let us first specify the smooth version of the cohomology theory.

Definition 3.4: Smooth Hochschild Cohomology

Let M be a smooth manifold and let $A = \mathcal{C}^\infty(M)$ be the $\mathbf{R}$-algebra of real-valued smooth functions on M. A Hochschild k-cocycle on A is called *differential* if it is a differential operator in each of its arguments.

Let us make a quick note on differential operators on manifolds. On $\mathbf{R}^n$, a differential operator is just a $\mathcal{C}^\infty(\mathbf{R}^n)$-linear combination of compositions of the partial derivative operators. In multi-index notation*, a differential operator of order $m \in \mathbf{N}$ (in other words, an m-differential operator) thus takes the form

$$\begin{aligned} D &: \mathcal{C}^\infty(\mathbf{R}^n) \to \mathcal{C}^\infty(\mathbf{R}^n) \\ D(f) &= \sum_{|\alpha| \le m} d_\alpha D^\alpha(f) \end{aligned}$$

for smooth functions $d_\alpha \in \mathcal{C}^\infty(\mathbf{R}^n)$.

How can we extend this definition to manifolds? The answer is probably not surprising: we can pull differential operators on $\mathbf{R}^n$ back via charts and then glue them together. But there is actually a slightly more elegant solution. In essence, one can define *locality* of a linear operator $D : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ by requiring that for all $f \in \mathcal{C}^\infty(M)$, we have $\mathrm{supp}(D(f)) \subseteq \mathrm{supp}(f)$. A local linear operator D is then called *differential* if, for all $p \in M$,

$$\varphi \circ D \circ \varphi^{-1} : \mathbf{R}^n \to \mathbf{R}^n$$

*https://en.wikipedia.org/wiki/Multi-index_notation, accessed at 13:32 on July 14th, 2023 is differential at $\varphi(p)$ for a chart (U, φ) around p. Perhaps also unsurprisingly, differential operators on manifolds are not at all rare: multiplication by smooth functions are differential operators of order 0, vector fields are 1-differential, and *compositions of vector fields* yield differential operators up to arbitrary order. See [17] of for a more in-depth discussion of these short remarks.

To get back to our discussion on smooth Hochschild cohomology, we set it up in exactly the same way as for the general case, but using differential cochains instead of general ones. The result is that we can completely classify the cohomology spaces using the following theorem, after Theorem 13 of [8].

Theorem 3.5: Smooth Hochschild-Konstant-Rosenberg Formula

Let $k \geq 1$ be an integer. For each differential k-cocycle C, there exists a differential $(k-1)$-cochain B such that

$$C = \partial B + C^{-},$$

where

$$C^{-}(u_1, \ldots, u_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) C(u_{\sigma(1)}, \ldots, u_{\sigma(k)})$$

is the necessarily 1-differential total antisymmetrisation of C.

An elementary proof of this theorem in stages can be found as Propositions 2.13 and 2.14 and Theorem 2.15 in [9]. To prove the classification, we need the following two lemmas.

Lemma 3.6

Let $k \geq 0$ be an integer. Every 1-differential k-cochain is a cocycle.

Proof. Let $k \geq 0$ be a integer and let C be a 1-differential k-cochain. The coboundary of C is

$$(\partial C)(u_0, \ldots, u_k) = u_0 C(u_1, \ldots, u_k) + (-1)^{k+1} C(u_0, \ldots, u_{k-1}) u_k + \sum_{r=1}^{k} (-1)^r C(u_0, \ldots, u_{r-1} \cdot u_r, \ldots, u_k).$$

Using the fact that C is 1-differential, hence a derivation in each of its arguments, we can expand the terms in the sum over r to obtain a telescoping sum:

$$(-1)^r C(u_0, \ldots, u_{r-1} \cdot u_r, \ldots, u_k) = (-1)^r \left(u_{r-1} C(u_0, \ldots, u_r, \ldots, u_k) + C(u_0, \ldots, u_{r-1}, \ldots, u_k) u_r \right)$$

for every $1 \leq r \leq k$. By virtue of the commutativity of $\mathcal{C}^\infty(M)$, all terms but the first and last of this summation cancel, leaving just

$$-u_0 C(u_1, \ldots, u_k) + (-1)^k C(u_0, \ldots, u_{k-1}) u_k,$$

which cancels with the two other terms of ∂C. We conclude that $\partial C = 0$. □

Lemma 3.7

Let $k \geq 1$ be an integer. The total antisymmetrisation of every (not necessarily differential) k-coboundary vanishes.

Proof. Let $k \geq 0$ be an integer and let C be a k-cochain. Writing out the total antisymmetrisation of ∂C, we get

$$\begin{aligned}(k+1)! \cdot (\partial C)^-(u_0, \dots, u_k) = \sum_{\sigma \in S_{k+1}} \operatorname{sgn}(\sigma) \Big(& u_{\sigma(0)} C(u_{\sigma(1)}, \dots, u_{\sigma(k)}) \\ & + \sum_{r=1}^{k} (-1)^r C(u_{\sigma(0)}, \dots, u_{\sigma(r-1)} \cdot u_{\sigma(r)}, \dots, u_{\sigma(k)}) \\ & + (-1)^{k+1} C(u_{\sigma(0), \dots, \sigma(k-1)}) u_{\sigma(k)} \Big) .\end{aligned}$$

The terms in this large sum cancel as follows. Fix $\sigma \in S_{k+1}$ and $1 \leq r \leq k$, and define two permutations in cycle notation, namely the transposition $\tau = (r-1, r)$ and the $(k+1)$-cycle $\gamma = (0, 1, \dots, k)$. Then the first summand of the σ-term in the summation over S_{k+1},

$$\operatorname{sgn}(\sigma) u_{\sigma(0)} C(u_{\sigma(1)}, \dots, u_{\sigma(k)}),$$

cancels with the last summand of the $\sigma\gamma$-term,

$$\begin{aligned}&\operatorname{sgn}(\sigma\gamma)(-1)^{k+1} C(u_{\sigma\gamma(0)}, \dots, u_{\sigma\gamma(k-1)}) u_{\sigma\gamma(k)} \\ &= -\operatorname{sgn}(\sigma) C(u_{\sigma(1)}, \dots, u_{\sigma(k)}) u_{\sigma(0)},\end{aligned}$$

since $\operatorname{sgn}(\gamma) = (-1)^k$. Symmetrically, the last summand of the σ-term cancels with the first summand of the $\sigma\gamma^{-1}$-term. And since $\operatorname{sgn}(\tau) = -1$, the r-summand of the σ-term cancels with the r-summand of the $\sigma\tau$-term,

$$\begin{aligned}&\operatorname{sgn}(\sigma\tau)(-1)^r C(u_{\sigma\tau(0)}, \dots, u_{\sigma\tau(r-1)} \cdot u_{\sigma\tau(r)}, \dots, u_{\sigma\tau(k)}) \\ &= -\operatorname{sgn}(\sigma)(-1)^r C(u_{\sigma(0)}, \dots, u_{\sigma(r)} \cdot u_{\sigma(r-1)}, \dots, u_{\sigma(k)}).\end{aligned}$$

This shows that $(\partial C)^- = 0$. □

Corollary 3.8: Classification of $H^k_{Hs}(\mathcal{C}^\infty(M))$

Let M be a smooth manifold and let $k \geq 1$ be an integer. The k-th Hochschild cohomology space of $A = \mathcal{C}^\infty(M)$ is parametrised by the smooth sections of the k-th exterior power of the tangent bundle of M:

$$H^k_{Hs}(A) \cong \Gamma(\Lambda^k TM).$$

Proof. We will show that to each section of $\Lambda^k TM$, we can associate a class in $H^k_{Hs}(A)$, and vice versa. Let $X \in \Gamma(\Lambda^k TM)$, then X is a differential k-linear map $X : A^k \to A$, hence a k-cochain. Since it is also 1-differential in each argument by definition, it is a k-cocycle by Lemma 3.6. Thus, we can simply associate X to its cohomology class $[X] \in H^k_{Hs}(A)$.

Let C be a k-cocycle, then we can write $C = \partial B + C^-$ for a $(k-1)$-cochain B by Theorem 3.5. Since we know by the same theorem that C^- is 1-differential and antisymmetric, we have $C^- \in \Gamma(\Lambda^k TM)$. We would like to associate C^- to $[C] \in H^k_{Hs}(A)$. To show that this is consistent, Let $D \in [C]$, then we have $[C - D] = 0$, hence there exists a $(k-1)$-cochain F such that $C - D = \partial F$. By Lemma 3.7, we then have

$$C^- - D^- = (C-D)^- = (\partial F)^- = 0,$$

so $C^- = D^-$, hence equivalent cocycles have equal total antisymmetrisation, and we can map $[C]$ to C^-. Moreover, it is clear by Theorem 3.5 that C and C^- generate the same cohomology class, so these two associations are each other's inverse, which completes the proof. □

This concludes our discussion of Hochschild cohomology. Using the lemmas of this chapter, we can prove some interesting results in the theory of deformations.

Chapter 4

Deformations

Having introduced the right tools, we are now ready finally to discuss the first half of this thesis's title: *deformation*. In all generality, a deformation of an algebra is a construction of another, larger algebra using formal power series in such a way that it resembles the old one in the lowest degree of the parameter. Along with this construction comes a natural notion of equivalence, as we will see.

Deformations were first introduced by Murray Gerstenhaber in 1964. They turned out to be quite useful in mathematical physics as a tool to describe quantum phenomena using non-commutativity, an application we will see in the next chapter. Aryan Ghobadi's essay [5] contains a fair amount of references detailing the development of the theory, including the original paper by Gerstenhaber.

An important result in the theory of deformations is the existence of a deformation quantisation of an arbitrary Poisson manifold, which was proven by Maxim Kontsevich in 2003 (see [14]). This is good news for physicists, because it implies that any classical system, which can be described using symplectic manifolds, which are special cases of Poisson manifolds, as we have seen in Chapter 2, can be quantised in the way introduced in the next chapter.

We will first develop the theory in all generality, after which we will go into detail on the smooth version of deformations, called star products, for which the algebra to deform is $\mathcal{C}^\infty(M)$ for a manifold M, largely following [8]. This reference omits a lot of details in the proofs of certain lemmas, though, which we fill in here. It also discusses a few more interesting facts regarding star products and their relations to the de Rham cohomology of differential forms on the manifold, which we do not mention here.

Along the way, we introduce two examples of star products on $\mathbf{R}^2$, the second of which, called the *Moyal star product* after theoretical physicist José Moyal, is of utmost importance for physical applications. We end this chapter by constructing a Poisson bracket on $\mathcal{C}^\infty(M)$ given an equivalence class of star products of this algebra, which highlights the relation between star products and symplectic geometry and alludes to the classical limit of quantum mechanics.

4.1 Deformations and Their Equivalences

Recall that given a ring R, we can define the ring of formal power series over R in a formal parameter X as

$$R[[X]] = \left\{ \sum_{k=0}^{\infty} r_k X^k \mid \forall k \in \mathbf{N} : r_k \in R \right\}.$$

The sums here are formal, so there are no convergence problems to be considered. The addition is defined degree by degree in X, and the multiplication is (the natural extension of) polynomial multiplication.

The definitions in this section are adapted from [8].

Definition 4.1: Deformation

Let A be an algebra over a field $\mathbf{F}$. A *deformation* of A is an associative $\mathbf{F}[[\hbar]]$-bilinear product

$$* : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]],$$

which is defined for $a, b \in A$ by

$$a * b = ab + \sum_{k=1}^{\infty} B_k(a, b)\hbar^k,$$

for some $\mathbf{F}$-bilinear $B_k : A \times A \to A$. Where it is convenient, we may write $B_0(a, b) = ab$ for the multiplication on A.

Let us make a few notes about this definition. The new product turns $A[[\hbar]]$ into an $\mathbf{F}$-algebra, whose product is *not* the natural extension of the product of A. In particular, for elements

$$f = \sum_{k=0}^{\infty} f_k \hbar^k \quad \text{and} \quad g = \sum_{k=0}^{\infty} g_k \hbar^k$$

of $A[[\hbar]]$, we have

$$\begin{aligned} f * g &= \sum_{j,k \geq 0} (f_j * g_k)\hbar^{j+k} = \sum_{j,k \geq 0} f_j g_k \hbar^{j+k} + \sum_{j,k \geq 0, m \geq 1} B_m(f_j, g_k)\hbar^{j+k+m} \\ &= \sum_{j,k,m \geq 0} B_m(f_j, g_k)\hbar^{j+k+m}. \end{aligned}$$

The associativity condition places restrictions on the bilinear maps B_k. Namely, for all $a, b, c \in A$, the following two expressions must be equal:

$$\begin{aligned} (a * b) * c &= \sum_{k=0}^{\infty} \hbar^k \sum_{j=0}^{k} B_j(B_{k-j}(a, b), c), \quad \text{and} \\ a * (b * c) &= \sum_{k=0}^{\infty} \hbar^k \sum_{j=0}^{k} B_j(a, B_{k-j}(b, c)). \end{aligned}$$

Moving degree by degree, we see that equality for $k = 0$ is given by the associativity of the product on A since

$$B_0(B_0(a,b),c) = (ab)c = a(bc) = B_0(a, B_0(b,c)),$$

and that for $k \geq 1$, we must require

$$\sum_{j=0}^{k} B_j(B_{k-j}(a,b),c)) = \sum_{j=0}^{k} B_j(a, B_{k-j}(b,c)). \tag{4.1}$$

Moving the $j = 0$ and the $j = k$ terms to the right hand side, this reduces to

$$\begin{aligned}
&\sum_{j=1}^{k-1} B_j(B_{k-j}(a,b),c) - B_j(a, B_{k-j}(b,c)) \\
&\quad = aB_k(b,c) - B_k(ab,c) + B_k(a,bc) - B_k(a,b)c \\
&\quad = (\partial B_k)(a,b,c).
\end{aligned} \tag{4.2}$$

This last equation involves the Hochschild coboundary of the 2-cochain B_k. For the $k = 1$ case, the sum above is empty, so the associativity requirement states that B_1 should be a 2-cocycle. More generally, we conclude that the *first non-zero B_k* is a 2-cocycle.

Next are some remarks of a more intuitive nature. The choice of the symbol $\hbar$ as the formal parameter is motivated twofold. First, the bar distinguishes the symbol from ordinary letters quite well, which is very convenient for legibility. And second, it foreshadows the physical interpretation of $\hbar$ as the reduced Planck constant. This will become clear when we look at actual deformation quantisations and the classical limit, which is traditionally associated with the limit as $\hbar \to 0$ (at least in comparison to other physical quantities with the same dimensions as $\hbar$).

The ordinary interpretation of the term "deformation" is reflected as follows: a mathematical deformation changes the algebra product on A only in higher orders of some formal parameter. If we "substitute" $\hbar = 0$ in all expressions, the $\mathbf{F}$-algebra $A[[\hbar]]$ with product $*$ simply reduces to A with its predefined product. One can think of the parameter $\hbar$ as "small" (in some non-mathematical sense), which also reflects the classical limit mentioned in the previous paragraph.

Right now, interesting to us are not individual deformations but rather equivalence classes of them. By also extending the linear endomorphisms of A into formal power series, we can construct isomorphisms of $A[[\hbar]]$, which are used to relate two star products.

Definition 4.2: Equivalence of Deformations

Two deformations $*$ and $*'$ on an $\mathbf{F}$-algebra A are called *equivalent* if there exists an $\mathbf{F}[[\hbar]]$-linear map $T : A[[\hbar]] \to A[[\hbar]]$ given by a series

$$T = \mathrm{id}_A + \sum_{k=1}^{\infty} T_k \hbar^k$$

for $\mathbf{F}$-linear coefficients $T_k : A \to A$, such that for all $f, g \in A[[\hbar]]$, we have

$$T(f * g) = T(f) *' T(g).$$

Where it is convenient, we may write $T_0 = \mathrm{id}_A$.

This definition also deserves some remarks. In terms of the T_k the action of T is defined as

$$\begin{aligned} T\left(\sum_{j=0}^{\infty} f_j \hbar^j\right) &= \sum_{j=0}^{\infty} T(f_j)\hbar^j = \sum_{j=0}^{\infty} f_j \hbar^j + \sum_{j\geq 0, k\geq 1} T_k(f_j)\hbar^{j+k} \\ &= \sum_{j,k\geq 0} T_k(f_j)\hbar^{j+k}. \end{aligned}$$

The fact that T is just the identity in zeroth order continues the idea that we are modifying the product on A only in higher degrees. As stated earlier, we may view the map T as an element of the ring of power series over $\mathrm{End}_{\mathbf{F}}(A)$. It is a well-known fact that a power series over a ring is invertible if and only if its degree 0 term is; the inverse can be constructed inductively. Since the identity is clearly an invertible linear endomorphism of A, it follows that T is invertible and thus that the above definition defines a symmetric relation on deformations. It is clear that the relation is also reflexive (simply take $T_k = 0$ for all $k \geq 1$), so let us check transitivity. Note that if

$$T(f * g) = T(f) *' T(g) \quad \text{and} \quad T'(f *' g) = T'(f) *'' T'(g),$$

we also have

$$(T' \circ T)(f * g) = T'(T(f) *' T(g)) = (T' \circ T)(g) *'' (T' \circ T)(g),$$

so we only need to verify that $T' \circ T$ is of the required form. Linearity is clear, and a quick calculation shows that for all $a \in A$, we have

$$(T' \circ T)(a) = \sum_{k=0}^{\infty} \hbar^k \sum_{j=0}^{k} (T'_j \circ T_{k-j})(a).$$

Therefore, we get

$$T' \circ T = \sum_{k=0}^{\infty} \hbar^k \sum_{j=0}^{k} T'_j \circ T_{k-j} = \mathrm{id}_A + \sum_{k=1}^{\infty} \hbar^k \left(T_k + T'_k + \sum_{j=1}^{k-1} T'_j \circ T_{k-j} \right).$$

Since the factor in brackets is clearly a linear endomorphism of A for all k, it follows that $T' \circ T$ is indeed a power series over $\mathrm{End}_{\mathbf{F}}(A)$ with degree 0 term equal to id_A, so the relation is also transitive. Therefore, we have defined an equivalence relation on the deformations of an algebra A, justifying the use of the word "equivalent".

4.2 Star Products

While general deformations of algebras are an interesting and rich topic by themselves (see for example [5]), a physicist is more interested in special, nice cases that allow them to model reality. Let us therefore discuss the case that the coefficients of both the deformations and the equivalences are differential.

Definition 4.3: Star Product

A *star product* on a smooth manifold M is a deformation $*$ on $A = \mathcal{C}^\infty(M)$ with coefficient functions B_k that are differential in each slot.

It may be helpful to have a simple example of a star product at hand.

Example 4.4: A Simple Star Product

Let $M = \mathbf{R}^2$, then we can define a simple star product by

$$f * g = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \partial_1^k f \cdot \partial_2^k g$$

for $f, g \in \mathcal{C}^\infty(\mathbf{R}^2)$ and by $\hbar$-linear extension to the entirety of $\mathcal{C}^\infty(\mathbf{R}^2)[[\hbar]]$. In particular, we have $B_k(f,g) = \frac{1}{k!}\partial_1^k f \cdot \partial_2^k g$.

Proof. It is clear that this definition satisfies all requirements of a star product except for associativity, which we will now check explicitly. For this, recall equation 4.1, which reads

$$\sum_{j=0}^{k} B_j(B_{k-j}(f,g),h) = \sum_{j=0}^{k} B_j(f, B_{k-j}(g,h)).$$

This equation must be satisfied for all $k \geq 0$, where the $k = 0$ case is trivial, as explained earlier. Expanding both sides, we need to check that

$$\begin{aligned} \text{Left} &= \sum_{j=0}^{k} \frac{1}{j!} \partial_1^j \left(\frac{1}{(k-j)!} \partial_1^{k-j} f \cdot \partial_2^{k-j} g \right) \cdot \partial_2^j h \\ &= \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} \sum_{m=0}^{j} \binom{j}{m} \partial_1^{k-j+m} f \cdot \partial_1^{j-m} \partial_2^{k-j} g \cdot \partial_2^j h \end{aligned}$$

is equal to

$$\begin{aligned}\text{Right} &= \sum_{j'=0}^{k} \frac{1}{j'!}\partial_1^{j'} f \cdot \partial_2^{j'} \left(\frac{1}{(k-j')!}\partial_1^{k-j'} g \cdot \partial_2^{k-j'} h \right) \\ &= \sum_{j'=0}^{k} \frac{1}{k!}\binom{k}{j'} \sum_{m'=0}^{j'} \binom{j'}{m'} \partial_1^{j'} f \cdot \partial_2^{m'}\partial_1^{k-j'} g \cdot \partial_2^{k-j'+j'-m'} h.\end{aligned}$$

This may look daunting, but we can see from the mixed partial derivatives acting on g that the (j,m) term in Left will only cancel with the (j',m') term in Right if $j-m = k-j'$ and $k-j = m'$. Thus, we can attempt to substitute $j' = k-j+m$ and $m' = k-j$ into Right. The factor with partial derivatives readily becomes

$$\partial_1^{k-j+m} f \cdot \partial_2^{k-j}\partial_1^{j-m} g \cdot \partial_2^{j} h,$$

which is the same expression found in Left. As for the numerical factors, we have

$$\begin{aligned}\frac{1}{k!}\binom{k}{k-j+m}\binom{k-j+m}{k-j} &= \frac{1}{k!}\frac{k!}{(k-j+m)!(j-m)!}\frac{(k-j+m)!}{(k-j)!m!} \\ &= \frac{1}{(k-j)!(j-m)!m!} \\ &= \frac{1}{k!}\frac{k!}{j!(k-j)!}\frac{j!}{m!(j-m)!} \\ &= \frac{1}{k!}\binom{k}{j}\binom{j}{m},\end{aligned}$$

which shows that they, too, are the same in both Left and Right. This proves that Left = Right, hence we have indeed defined an associative star product. □

Unfortunately, working explicitly with even such an innocent looking star product becomes unwieldy quite quickly. It is thus all the more fortunate that there is an intimate relationship between star products and operator composition, which we will encounter in the next chapter.

Moving on to equivalences, it turns out that we do not need to impose any extra conditions on the coefficients of an equivalence of star products due to Proposition 17 in [8]:

Lemma 4.5

If $*$ and $*'$ are equivalent star products and T is an equivalence between them, then the coefficients of T are differential operators.

The proof uses the following lemma.

Lemma 4.6

Let T be an equivalence of star products with coefficients T_r. For all integers $k \geq 1$, we have

$$(\mathrm{id} + \hbar T_1 + \cdots + \hbar^k T_k)^{-1} \circ T = \mathrm{id} + \hbar^{k+1} T_{k+1} + \mathcal{O}(\hbar^{k+2}).$$

Proof. Fix $k \geq 1$ and define $U = \mathrm{id} + \hbar T_1 + \cdots + \hbar^k T_k$ and $T' = U^{-1} \circ T$, then we aim to calculate the coefficients of T' from the equality

$$T = U \circ T' = \sum_{r=0}^{\infty} \hbar^r \sum_{s=0}^{\min\{r,k\}} T_s \circ T'_{r-s}.$$

In degree 0, we simply have $\mathrm{id} = \mathrm{id}$. In degrees r with $1 \leq r \leq k$, we have

$$T_r = T'_r + T_1 \circ T'_{r-1} + \cdots + T_{r-1} \circ T'_1 + T_r.$$

For $r = 1$, this yields $T_1 = T'_1 + T_1$, hence $T'_1 = 0$. For $r = 2$, we then have

$$T_2 = T'_2 + T_1 \circ T'_1 + T_2 = T'_2 + T_2,$$

from which we conclude $T'_2 = 0$. This continues up to and including $r = k$, yielding $T'_r = 0$ for $1 \leq r \leq k$. Finally, in order $\hbar^{k+1}$, we get

$$T_{k+1} = T'_{k+1} + T_1 \circ T'_k + \cdots + T_k \circ T'_1 = T'_{k+1},$$

which finishes the proof. □

Proof of Lemma 4.5. Write

$$T = \mathrm{id} + \sum_{r=1}^{\infty} \hbar^r T_r.$$

The proof is by induction on the degree of the coefficients of T. For the base case, let k be the smallest positive integer such that $T_k \neq 0$. The statement of equivalence of $*$ and $*'$ via T reads

$$T(f * g) = T(f) *' T(g).$$

Expanding the left side, we get

$$T(f * g) = T\left(\sum_{r=0}^{\infty} B_0(f,g)\hbar^r\right) = \sum_{r,s \geq 0} T_s(B_r(f,g))\hbar^{r+s},$$

and expanding the right side, we get

$$\begin{aligned} T(f) *' T(g) &= \left(\sum_{s=0}^{\infty} T_s(f)\hbar^s\right) *' \left(\sum_{t=0}^{\infty} T_t(g)\hbar^t\right) \\ &= \sum_{r,s,t \geq 0} B'_r(T_s(f), T_t(g))\hbar^{r+s+t}. \end{aligned}$$

Equating these two expressions and moving degree by degree, we get the following. In degree 0, we simply have $fg = fg$. Since $T_r = 0$ for $1 \leq r < k$, degrees r with $1 \leq r < k$ are also simple, namely $B_r(f,g) = B'_r(f,g)$; apparently, the two star products must have equal coefficients up to and including order $k-1$.

In degree k, we have

$$\begin{aligned} B_k(f,g) + T_k(fg) &= B'_k(f,g) + T_k(f)g + fT_k(g) \\ B_k(f,g) - B'_k(f,g) &= fT_k(g) - T_k(fg) + T_k(f)g \\ &= (\partial T_k)(f,g), \end{aligned}$$

where the last equation involves the Hochschild coboundary of T_k, which is apparently a differential 2-cochain. By Theorem 3.5, it therefore equals the sum of its total antisymmetrisation and the boundary of a differential 1-cochain, which we call E, that is $\partial T_k = (\partial T_k)^- + \partial E$. But note that ∂T_k is also clearly a 2-coboundary, hence has vanishing total antisymmetrisation by Lemma 3.7. It follows that $\partial T_k = \partial E$, so $T_k - E$ is a 1-cocycle, in other words a derivation on A. But the derivations on $A = \mathcal{C}^\infty(M)$ are precisely the vector fields, hence $T_k - E$ is a differential operator. Since E is differential as well, we conclude that T_k is a differential operator, proving the base case.

For the induction step, suppose that $T_1, \ldots, T_k$ are all differential and at least one of them is non-zero, and suppose that T_{k+1} is also non-zero. We aim to prove that T_{k+1} is differential. For this, we set U and T' as in the proof of Lemma 4.6 and define the star product $*''$ by

$$f *'' g = U^{-1}(U(f) *' U(g)).$$

Since all coefficients of U and of $*'$ are differential, $*''$ also has differential coefficients, hence is indeed a star product. We also have

$$f *' g = U(U^{-1}(f) *'' U^{-1}(g)).$$

Using this equation, we calculate

$$\begin{aligned} T'(f * g) &= U^{-1}(T(f * g)) = U^{-1}(T(f) *' T(g)) \\ &= (U^{-1} \circ U)((U^{-1} \circ T)(f) *'' (U^{-1} \circ T)(g)) \\ &= T'(f) *'' T'(g), \end{aligned}$$

so T' is an equivalence of star products. Therefore, we can apply the same argument as just now in the base case to show that the first non-zero coefficient of T', which by assumption and by Lemma 4.6 equals T_{k+1}, is differential. This completes the proof. □

Example 4.7: The Moyal Star Product

Another simple, but very important, example of a star product is the *Moyal star product*, which is also defined on $M = \mathbf{R}^2$. To write it down in a concise way, we need to introduce a new notation.

A partial derivative marked by a superscript L or R, for example ∂_1^R, can only act on functions that are to the left or right of it, respectively. For example, if we write $f\partial_2^R\partial_1^L g$, we mean $\partial_1 f \cdot \partial_2 g$.

The Moyal product is now defined for smooth functions $f, g \in \mathcal{C}^\infty(\mathbf{R}^2)$ as

$$f * g = f \exp\left(\hbar\left(\partial_1^L\partial_2^R - \partial_1^R\partial_2^L\right)\right) g = f\left(\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}(\partial_1^L\partial_2^R - \partial_1^R\partial_2^L)^k\right) g.$$

This star product is equivalent with the one from Example 4.4 via the equivalence

$$T(f) = \exp(\hbar\partial_1\partial_2)f = \sum_{k=0}^{\infty}\frac{\hbar^k}{k!}\partial_1^k\partial_2^k f.$$

We do not prove the details here since the calculations are quite involved. It will, however, become clear in the next chapter that the Moyal star product is indeed associative, and a proof of the equivalence can be found in [16].

Star products are interesting because they relate to Poisson structures on the manifold. This was first established rigorously by Maxim Kontsevich in 2003 [14]: he proved that the set of equivalence classes of star products on a given manifold is in bijection with equivalence classes of so-called *Poisson deformations*. In particular, it allows one to assign an equivalence class of star products to a given Poisson manifold in a natural way. Here, we will not go into detail on Poisson deformations (for details, see [8] and [14]), but we will illustrate how to obtain a Poisson structure from a given equivalence class of star products.

Suppose that we are given a star product $*$ on a manifold. The key insight is to consider the antisymmetric part B_1^- of the first coefficient B_1.

Lemma 4.8

If $*$ and $*'$ are equivalent star products, then $B_1^- = B_1'^-$.

Proof. Let T be an equivalence between $*$ and $*'$, then we first calculate the coefficient B_1' in terms of B_1 and T_1. If we note that for all $f, g \in \mathcal{C}^\infty(M)$, we have

$$f * g = T^{-1}(T(f) *' T(g)),$$

we can calculate the right hand side in terms of the coefficients T_k of T and the coefficients $\tilde{T}_k$ of T^{-1} as follows:

$$\begin{aligned} T(f) *' T(g) &= \left(\sum_{i=0}^{\infty} T_i(f)\hbar^i\right) *' \left(\sum_{j=0}^{\infty} T_j(g)\hbar^j\right) \\ &= \sum_{i,j,k\geq 0} B'_k(T_i(f), T_j(g))\hbar^{i+j+k}, \end{aligned}$$

hence

$$T^{-1}(T(f) *' T(g)) = \sum_{i,j,k,m\geq 0} \tilde{T}_m(B'_k(T_i(f), T_j(g)))\hbar^{i+j+k+m}.$$

While this may look intimidating, the lowest degrees are easily resolved. Moreover, to continue, we only need the coefficient $\tilde{T}_1$. To find it, we use that we have

$$f = T^{-1}(T(f)) = T^{-1}\left(\sum_{k=0}^{\infty} T_k(f)\hbar^k\right) = \sum_{j,k\geq 0} \tilde{T}_j(T_k(f))\hbar^{j+k}$$

for all $f \in \mathcal{C}^\infty(M)$. In degree 1, this reduces to $\tilde{T}_1(f) + T_1(f) = 0$, hence $\tilde{T}_1 = -T_1$.

Going back to the intimidating expression, we calculate in first degree:

$$\begin{aligned} B_1(f,g) &= \tilde{T}_1(fg) + B'_1(fg) + T_1(f)g + fT_1(g), \quad \text{or} \\ B_1(f,g) - B'_1(f,g) &= fT_1(g) - T_1(fg) + T_1(f)g \quad [= (\partial T_1)(f,g)]\,. \end{aligned}$$

Notice that the expression on the right in the last line is symmetric in f and g, hence we conclude $B_1^- - B_1'^- = (B_1 - B'_1)^- = 0$, which is what we set out to prove. □

So we see that the antisymmetric part of the first star product coefficient is an equivalence class invariant. In fact, the equivalence class of $*$ will always contain an element whose first coefficient is antisymmetric. To see this, note that it follows from equation 4.2 that B_1 is a 2-cocycle, hence Theorem 3.5 implies that there exists a 1-coboundary C such that $B_1 = B_1^- + \partial C$. The equivalent star product $*'$ given by $f *' g = T(T^{-1}(f) * T^{-1}(g))$ for $T(f) = f + \hbar C(f)$ then has

$$B'_1 = B_1 - \partial C = B_1^-$$

by the last displayed equation in the proof of the previous lemma.

The next two results follow from the associativity requirement on the coefficients of $*$, which we have calculated to be equation 4.2:

$$(\partial B_k)(a,b,c) = \sum_{j=1}^{k-1} B_j(B_{k-j}(a,b),c) - B_j(a, B_{k-j}(b,c)).$$

For $k = 1$, we find

$$aB_1(b,c) - B_1(ab,c) + B_1(a,bc) - B_1(a,b)c = 0, \tag{4.3}$$

and for $k = 2$, it reads

$$(\partial B_2)(a,b,c) = B_1(B_1(a,b),c) - B_1(a, B_1(b,c)). \tag{4.4}$$

Lemma 4.9

The antisymmetric part B_1^- is a derivation in both arguments.

Proof. This can be shown using equation 4.3 with a few well-chosen substitutions. In the following, the terms that are substituted to obtain the next line are highlighted in a different colour. By definition, we have

$$B_1^-(f,g) = \frac{1}{2}(B_1(f,g) - B_1(g,f)),$$

hence

$$\begin{aligned} 2B_1^-(fg,h) &= B_1(fg,h) - B_1(h,fg) \\ &= fB_1(g,h) + B_1(f,gh) - hB_1(f,g) - B_1(h,fg) \\ &= fB_1(g,h) + B_1(f,gh) - B_1(h,f)g - B_1(hf,g) \\ &= fB_1(g,h) - B_1(h,f)g + B_1(f,h)g - fB_1(h,g) \\ &= f(B_1(g,h) - B_1(h,g)) + (B_1(f,h) - B_1(h,f))g \\ &= 2fB_1^-(g,h) + 2B_1^-(f,h)g. \end{aligned}$$

This shows that B_1^- is a derivation in the first slot; it follows by antisymmetry that it is also a derivation in the second slot. □

Lemma 4.10

The antisymmetric part B_1^- satisfies the Jacobi identity, that is we have

$$B_1^-(f,B_1^-(g,h)) + B_1^-(g,B_1^-(h,f)) + B_1^-(h,B_1^-(f,g)) = 0.$$

Proof. Expanding the six occurrences of B_1^- to expose B_1, we obtain twelve terms. These twelve terms can be paired in such a way that we can substitute equation 4.4 six times. The result is the following expression:

$$\begin{aligned} &B_1^-(f,B_1^-(g,h)) + B_1^-(g,B_1^-(h,f)) + B_1^-(h,B_1^-(f,g)) \\ &\qquad = -\frac{1}{4}[(\partial B_2)(f,g,h) - (\partial B_2)(f,h,g) + (\partial B_2)(g,h,f) \\ &\qquad\qquad -(\partial B_2)(g,f,h) + (\partial B_2)(h,f,g) - (\partial B_2)(h,g,f)] \\ &\qquad = -\frac{3!}{4}(\partial B_2)^-(f,g,h). \end{aligned}$$

But Lemma 3.7 tells us that the total antisymmetrisation of a coboundary vanishes, hence the above must equal 0. □

We have now shown that the antisymmetric bilinear map

$$B_1^- : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

satisfies both the Leibniz rule and the Jacobi identity, hence it is a Poisson bracket on $\mathcal{C}^\infty(M)$ cf. Definition 2.6. A natural convention is to take $\{f,g\} = 2B_1^-(f,g)$ so that we get

$$[f,g]_* \stackrel{\text{def}}{=} f * g - g * f = \hbar\{f,g\} + \mathcal{O}(\hbar^2), \tag{4.5}$$

an identity that links the Poisson bracket to the so-called star commutator. We will see this relation in the next chapter as part of our discussion of the classical limit in deformation quantisation.

Chapter 5

Deformation Quantisation

When undergraduates study quantum mechanics for the first time, they are treated to a range of both new concepts and new mathematics. The simplest quantum system is that of a single particle in one dimension with phase space $\mathbf{R}^2$ and space of observables $\mathcal{C}^\infty(\mathbf{R}^2)$. To quantise this system, we introduce the Hilbert space $\mathcal{H} = L^2(\mathbf{R})$ of square-integrable functions with values in the complex numbers. States are defined as elements in $\mathcal{H}$ and observables as self-adjoint operators on $\mathcal{H}$. Next, the Schrödinger equation and the Hamiltonian operator $\hat{H}$ are introduced, but this is already the first point of possible confusion. To obtain $\hat{H}$ from H, students are told to "just replace" the position and momentum variables by their operators, which seemingly come out of nowhere. This *canonical quantisation scheme* is just something to accept.

A bit later, the time evolution of an observable $\hat{F}$ is given as

$$\frac{\mathrm{d}\hat{F}}{\mathrm{d}t} = \frac{\partial \hat{F}}{\partial t} + \frac{1}{i\hbar}[\hat{F}, \hat{H}],$$

which is the second point of possible confusion. This equation, it is said, reduces to its classical analogue,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\},$$

in the *classical limit*, which seems to imply that the commutator of operators corresponds to the Poisson bracket:

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}[\hat{F}, \hat{H}] = \{f, H\}.$$

But this expression makes no sense because on there is an operator on $\mathcal{H}$ on the left and a smooth function on $\mathbf{R}^2$ on the right!

There is a way, however, to make sense of this mess. The solution is another formulation of quantum mechanics, called *deformation quantisation*. The groundwork for this theory was laid by Hermann Weyl and Eugene Wigner in the early 20th century, but was not expanded to completeness until the end of the Second World War by Hilbrand Groenewold and José Moyal. Perhaps this is why the Hilbert space formalism is more prevalent nowadays — it was developed first.

Deformation quantisation builds directly upon the Hamiltonian formulation of classical mechanics. In fact, the observables are still just smooth functions on phase space. There are only two new mathematical objects a student of the theory will have to grasp: the *Moyal star product* and the fact that states are not any longer represented by points in phase space, but rather by so-called *Wigner functions*. It turns out, though, that this relatively simple theory is completely equivalent to the Hilbert space formalism. Moreover, the limit from above can be interpreted literally in deformation quantisation, which makes it a great theory to teach quantum mechanics from a pedagogical standpoint, at least conceptually.

However, a drawback of the formalism is that the calculations are quite lengthy and tedious, as we will see, and might even be above any level of mathematical fluency that can reasonably be expected from an undergraduate student of physics. Nonetheless, deformation quantisation is a very useful theory with its own specialised applications. For a comprehensive summary of the theory's history and applications, see [3].

5.1 A New Formalism

In order to draw a more complete parallel between deformation quantisation and the Hilbert space formalism in the next section, let us first recall the density operator that is used at a more advanced level of study to differentiation between pure and mixed quantum states. Given an ensemble of pure states $|\psi_i\rangle$ which occur with probabilities p_i, it is defined as the self-adjoint operator

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

so that the expectation value of an observable $\hat{F}$ is now given by

$$\langle\hat{F}\rangle = \sum_i p_i \langle\psi_i|\hat{F}|\psi_i\rangle = \mathrm{Tr}\left(\hat{F}\hat{\rho}\right).$$

As stated above, the setting of deformation quantisation is still smooth functions on phase space, but we must now allow them to take on complex values. The classical observables are still only the real-valued functions — introducing complex values simply gives us more functions to work with. Moreover, we will only consider smooth functions that can be expanded as a power series in $\hbar$. For those interested in a more rigorous mathematical treatment, see Chapter 13 in [10].

The first new concept, the *Moyal star product*, was introduced in a simplified form in the previous chapter. Here, we define it as the star product $*$ on the $\mathbf{C}$-algebra $\mathcal{C}^\infty(\mathbf{R}^2, \mathbf{C})$ given by

$$f * g = f e^{\frac{i\hbar}{2}(\partial_q^L \partial_p^R - \partial_p^L \partial_q^R)} g.$$

It is not very difficult, but certainly quite a tedious exercise to show that this expression indeed defines a star product cf. Definition 4.3. In fact, every detail save for associativity should be clear, and the associativity follows quite simply from the *Weyl-Wigner correspondence*, which we will discuss in the next section.

An immediately noteworthy property of the Moyal star product is its *non-locality*, as is evident from the fact that it takes partial derivatives of all orders of the input functions. As a result, to calculate the value of $f * g$ at a single point in phase space, it is no longer enough to know f and g in a small neighbourhood, since knowing all partial derivatives of a function is equivalent to knowing the function on the entire input space. This reflects the inherent non-locality of quantum mechanics.

Writing down the new time evolution equation for observables is a very simple task. Where in the classical case we have a Hamiltonian $H \in \mathcal{C}^\infty(\mathbf{R}^2)$ that describes the time evolution of another observable f by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\},$$

we simply take *the same* Hamiltonian and write down

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{1}{i\hbar}[f, H]_* = \frac{\partial f}{\partial t} + \frac{1}{i\hbar}(f * H - H * f) \tag{5.1}$$

in our new formalism. Note that dividing by $\hbar$ is not an issue here even if we were working with formal power series in $\hbar$ because the $*$-commutator of two functions will always cancel any terms of degree 0 in $\hbar$.

The second new concept is that states of a system are represented by so-called *Wigner functions*, which we will here denote by the letter W. The time evolution of a Wigner function is in complete analogue to the von Neumann equation for the time evolution of the density operator:

$$i\hbar\frac{\partial W}{\partial t} = -[W, H]_*.$$

This can be proven quite simply using the Weyl correspondence of the next section.

Using the Wigner function of a state, the expectation value of an observable f can be calculated as an integral over phase space, namely

$$\int \mathrm{d}q\, \mathrm{d}p\, f(q, p)W(q, p).$$

This suggests that we may think of the Wigner functions as probability distributions, but there is one caveat: W may take on negative values. It turns out, however, that this property is far from a liability, because it has as a corollary the uncertainty principle, as we will see two sections hence.

All in all, we can see that deformation quantisation has everything we need for a physical theory: a representation of states and observables, time evolution equations for both, and a way to determine the values of observables in a given state. Moreover, all of these three ingredients are completely analogous to the Hilbert space formalism, while still being based firmly on the phase space of a system and introducing only two new mathematical objects. Let us now discuss why the two formulations of quantum mechanics are not only analogous, but in fact equivalent.

5.2 The Weyl-Wigner Correspondence

To make the correspondence between deformation quantisation and the Hilbert space formalism explicit, we will construct a bijective map Q between functions on phase space (with canonical coordinates q, p) and operators on Hilbert space (with the position and momentum operators denoted by $\hat{q}$ and $\hat{p}$ respectively), namely the *Weyl quantisation*. It is based on the idea that since there are multiple natural choices for an operator corresponding to the monomial $q^k p^m$, for example $\hat{q}^k \hat{p}^m$, $\hat{p}^m \hat{q}^k$, $\hat{q}\hat{p}\hat{q}^{k-1}\hat{p}^{m-1}$, or any other monomial in $\hat{q}$ and $\hat{p}$ with the right degrees, why not pick a combination of each of them? In particular, we pick the total antisymmetrisation of the monomial:

$$Q(q^k p^m) = \frac{1}{(k+m)!} \sum_{\sigma \in S_{k+m}} \sigma(\hat{q}, \dots, \hat{q}, \hat{p}, \dots, \hat{p}),$$

where an element $\sigma \in S_n$ is interpreted as a function that permutes its n input operators.

Lemma 5.1

For all non-negative integers n and complex numbers $a_1, b_1, \dots, a_n, b_n$, we have

$$Q((a_1 q + b_1 p) \dots (a_n q + b_n p)) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(a_1 \hat{q} + b_1 \hat{p}, \dots, a_n \hat{q} + b_n \hat{p}).$$

In particular, we have

$$Q((aq + bp)^n) = (a\hat{q} + b\hat{p})^n$$

for all integers $n \geq 1$ and $a, b \in \mathbf{C}$.

The Weyl quantisation is, in fact, the unique linear map from polynomials in q and p into linear operators acting on $\mathcal{C}_c^\infty(\mathbf{R}, \mathbf{C})$ (smooth functions $\mathbf{R} \to \mathbf{C}$ with compact support) that satisfies the rather nice property in the second displayed equation of this lemma, according to Proposition 13.3 of [10].

Proof of Lemma 5.1. The left hand side of the first displayed equation expands as

$$\begin{aligned} &Q((a_1 q + b_1 p) \dots (a_n q + b_n p)) \\ &= \sum_{\sigma \in S_n} \sum_{\text{indices}} \frac{1}{n!} a_{i_1} \dots a_{i_k} b_{j_1} \dots b_{j_{n-k}} \sigma(\hat{q}, \dots, \hat{q}, \hat{p}, \dots, \hat{p}), \end{aligned}$$

where the sum over indices is such that $i_1 < \dots < i_k$ and $j_1 < \dots < j_{n-k}$ and ranges over $0 \leq k \leq n$. Similarly, the right hand side is expanded by first summing over the permutations in S_n and then over indices. Since the term determined by a specific σ and a specific set of indices in the sum above appears in the expansion of the right hand side simply by picking the same σ and the same set of indices, and since both sides are finite sums, they must be equal by the pigeon hole principle.

The special case simply follows by picking

$$a_1 = \cdots = a_n = a \quad \text{and} \quad b_1 = \cdots = b_n = b.$$

□

Quantisation procedure for polynomials in hand, we would now like to expand it to all smooth functions. This can easily be done using the Fourier transform (at least for sufficiently nice functions, which is all we are interested in anyway):

$$\tilde{f}(\sigma, \tau) = \frac{1}{2\pi} \int \mathrm{d}q \, \mathrm{d}p \, f(q, p) e^{-i(\sigma q + \tau p)}.$$

If we have

$$Q\left(e^{i(\sigma q + \tau p)}\right) = e^{i(\sigma \hat{q} + \tau \hat{p})}, \tag{5.2}$$

then we can define

$$Q(f) = \frac{1}{2\pi} \int \mathrm{d}\sigma \, \mathrm{d}\tau \, \tilde{f}(\sigma, \tau) e^{i(\sigma \hat{q} + \tau \hat{p})}, \tag{5.3}$$

that is we first Fourier transform f in the ordinary way, and on the way back we replace the coordinates q and p by the operators $\hat{q}$ and $\hat{p}$. There are some mathematical details to be ironed out before equation 5.2 can be used because the operators $\hat{q}$ and $\hat{p}$ are unbounded; one way to do this is found in Section 13.3 of [10].

The *Weyl transform* in equation 5.3 has three very nice properties. The first is that it is invertible using the *Wigner transform*: if $\hat{F}$ is the operator corresponding to f, we have

$$f(q, p) = Q^{-1}(\hat{F}) = \frac{1}{2\pi} \int \mathrm{d}y \, e^{ipy} \left\langle q - \frac{\hbar}{2} y \middle| \hat{F} \middle| q + \frac{\hbar}{2} y \right\rangle \tag{5.4}$$

in terms of position eigenvectors. The second is that through this Weyl-Wigner correspondence of phase space functions and Hilbert space operators, the Moyal star product corresponds with operator composition. And the third is that real-valued functions on phase space map to self-adjoint operators on Hilbert space, that is observables in our new formalism map to observables in the old one. These three properties, of which a mathematically rigorous treatment can again be found in Section 13.3 of [10], yield the equivalence between deformation quantisation and the Hilbert space formalism.

Let us illustrate the relation between the Moyal star product and operator composition with a relatively simple calculation, adapted from Section 1.4 of [11]. First, note a special case of the *Campbell-Baker-Hausdorff formula*, which states that for two operators A and B that each commute with their commutator $[A, B]$ (such as is the case for linear combinations of $\hat{q}$ and $\hat{p}$), we have

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

Taking $A = \sigma_1\hat{q} + \tau_1\hat{p}$ and $B = \sigma_2\hat{q} + \tau_2\hat{p}$, it is an easy calculation to find

$$\begin{aligned} e^{i(\sigma_1\hat{q}+\tau_1\hat{p})}e^{i(\sigma_2\hat{q}+\tau_2\hat{p})} &= e^{i((\sigma_1+\sigma_2)\hat{q}+(\tau_1+\tau_2)\hat{p})}e^{-\frac{1}{2}(\sigma_1\tau_2-\sigma_2\tau_1)[\hat{q},\hat{p}]} \\ &= e^{i((\sigma_1+\sigma_2)\hat{q}+(\tau_1+\tau_2)\hat{p})}e^{-i\frac{\hbar}{2}(\sigma_1\tau_2-\sigma_2\tau_1)}. \end{aligned}$$

Now suppose that we have two functions f and g and the two corresponding operators $\hat{F} = Q(f)$ and $\hat{G} = Q(g)$. Then using the Weyl transform (equation 5.3), we calculate

$$\begin{aligned} \hat{F}\hat{G} &= \frac{1}{(2\pi)^4}\int \mathrm{d}\sigma_1\,\mathrm{d}\tau_1\,\mathrm{d}\sigma_2\,\mathrm{d}\tau_2\, e^{i(\sigma_1\hat{q}+\tau_1\hat{p})}e^{i(\sigma_2\hat{q}+\tau_2\hat{p})}\tilde{f}(\sigma_1,\tau_1)\tilde{g}(\sigma_2,\tau_2) \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}\sigma_1\,\mathrm{d}\tau_1\,\mathrm{d}\sigma_2\,\mathrm{d}\tau_2\, e^{i((\sigma_1+\sigma_2)\hat{q}+(\tau_1+\tau_2)\hat{p})}e^{-i\frac{\hbar}{2}(\sigma_1\tau_2-\sigma_2\tau_1)}\tilde{f}(\sigma_1,\tau_1)\tilde{g}(\sigma_2,\tau_2). \end{aligned}$$

The first exponential factor suggests that we might easily be able to write this operator as the Weyl transform of some function. So let us substitute $\sigma = \sigma_1 + \sigma_2$, $\sigma' = (\sigma_1 - \sigma_2)/2$, and similarly for τ. The determinant of this transformation is -1, so the integration measure does not change. Moreover, we have

$$\begin{aligned} \sigma_1\tau_2 - \sigma_2\tau_1 &= \left(\frac{\sigma}{2}+\sigma'\right)\left(\frac{\tau}{2}-\tau'\right) - \left(\frac{\sigma}{2}-\sigma'\right)\left(\frac{\tau}{2}+\tau'\right) \\ &= \left(\frac{\sigma\tau}{4}-\frac{\sigma\tau'}{2}+\frac{\sigma'\tau}{2}-\sigma'\tau'\right) - \left(\frac{\sigma\tau}{4}+\frac{\sigma\tau'}{2}-\frac{\sigma'\tau}{2}-\sigma'\tau'\right) \\ &= -\sigma\tau' + \sigma'\tau, \end{aligned}$$

so that the substitution yields

$$\begin{aligned} \hat{F}\hat{G} = \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma\,\mathrm{d}\tau\, e^{i(\sigma\hat{q}+\tau\hat{p})}&\left[\frac{1}{(2\pi)^2}\int \mathrm{d}\sigma'\,\mathrm{d}\tau'\, e^{i\frac{\hbar}{2}(\sigma\tau'-\sigma'\tau)}\right. \\ &\left.\cdot\,\tilde{f}\left(\frac{\sigma}{2}+\sigma',\frac{\tau}{2}+\tau'\right)\tilde{g}\left(\frac{\sigma}{2}-\sigma',\frac{\tau}{2}-\tau'\right)\right]. \end{aligned} \tag{5.5}$$

To see what the function in square brackets is, let us Fourier transform the star product of f and g. Since the Fourier transform from (q, p)-space into (σ, τ)-space turns partial derivatives into factors ($\mathcal{F}(\partial_q) = i\sigma$ and $\mathcal{F}(\partial_p) = i\tau$) and turns products into convolutions according to

$$\begin{aligned} \mathcal{F}(fg)(\omega) &= 2\pi\int \mathrm{d}\alpha\,\tilde{f}(\alpha)\tilde{g}(\omega-\alpha) \\ &= 2\pi\int \mathrm{d}\alpha'\,\tilde{f}\left(\frac{\omega}{2}+\alpha'\right)\tilde{g}\left(\frac{\omega}{2}-\alpha'\right), \end{aligned}$$

we get

$$\begin{aligned} \mathcal{F}(\partial_q^L\partial_p^R - \partial_q^R\partial_p^L) &= i^2\left(\left(\frac{\sigma}{2}+\sigma'\right)\left(\frac{\tau}{2}-\tau'\right) - \left(\frac{\sigma}{2}-\sigma'\right)\left(\frac{\tau}{2}+\tau'\right)\right) \\ &= -(\sigma'\tau - \sigma\tau') = \sigma\tau' - \sigma'\tau \end{aligned}$$

within the convolution integral, hence

$$\mathcal{F}(f * g)(\sigma,\tau) = \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma'\,\mathrm{d}\tau'\,e^{i\frac{\hbar}{2}(\sigma\tau'-\sigma'\tau)} \cdot \tilde{f}\left(\frac{\sigma}{2}+\sigma',\frac{\tau}{2}+\tau'\right)\tilde{g}\left(\frac{\sigma}{2}-\sigma',\frac{\tau}{2}-\tau'\right),$$

which is precisely the expression in square brackets in equation 5.5. It follows that

$$\hat{F}\hat{G} = \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma\,\mathrm{d}\tau\,e^{i(\sigma\hat{q}+\tau\hat{p})}\mathcal{F}(f*g)(\sigma,\tau) = Q(f*g),$$

so $Q(f)Q(g) = Q(f*g)$, which is what wanted to show.

5.3 Properties of Wigner Functions

Given the incredibly nice properties of the Weyl-Wigner correspondence, we should expect that the density operator of the Hilbert space formalism, which completely describes the state it represents, should Wigner transform into the Wigner function of the same state. And this is precisely true: if a state has density operator $\hat{\rho}$, its Wigner function is

$$W(q,p) = \frac{1}{2\pi}\int \mathrm{d}y\,e^{ipy}\left\langle q-\frac{\hbar}{2}y\right|\hat{\rho}\left|q+\frac{\hbar}{2}y\right\rangle. \tag{5.6}$$

In a pure state, that is for $\hat{\rho} = |\psi\rangle\langle\psi|$ for some wave function ψ, we get

$$W(q,p) = \frac{1}{2\pi}\int \mathrm{d}y\,e^{ipy}\psi\left(q-\frac{\hbar}{2}y\right)\psi^*\left(q+\frac{\hbar}{2}y\right),$$

since $\langle q-\frac{\hbar}{2}|\psi\rangle = \psi(q-\frac{\hbar}{2}y)$.

Wigner functions are real-valued, which is easy to check by calculating the complex conjugate of equation 5.6, but also follows from the fact that a density operator is self-adjoint, conform our discussion of the properties of the Weyl-Wigner correspondence. They are also normalised to integrate to 1, in accordance with the quasi-probability distribution interpretation:

$$\begin{aligned}\int \mathrm{d}p\,\mathrm{d}q\,W(q,p) &= \frac{1}{2\pi}\int \mathrm{d}q\,\mathrm{d}p\,\mathrm{d}y\,e^{ipy}\left\langle q-\frac{\hbar}{2}y\right|\hat{\rho}\left|q+\frac{\hbar}{2}y\right\rangle \\ &= \int \mathrm{d}q\,\mathrm{d}y\,\delta(y)\left\langle q-\frac{\hbar}{2}y\right|\hat{\rho}\left|q+\frac{\hbar}{2}y\right\rangle \\ &= \int \mathrm{d}y\,\langle q|\hat{\rho}|q\rangle \\ &= \mathrm{Tr}(\hat{\rho}) \\ &= 1.\end{aligned}$$

To find the time evolution of a Wigner function, we first derive the von Neumann equation for the time evolution of a density operator defined as

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

Recall that the pure states $|\psi_i\rangle$ satisfy the Schrödinger equation $\hat{H}|\psi_i\rangle = i\hbar\partial_t|\psi_i\rangle$, which we can use to write

$$\begin{aligned}
[\hat{H}, \hat{\rho}] &= \sum_i p_i \left(\hat{H}|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle\langle\psi_i|\hat{H}\right) \\
&= \sum_i p_i \left(\hat{H}|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle[\hat{H}|\psi_i\rangle]^\dagger\right) \\
&= \sum_i p_i \left(i\hbar\partial_t|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle[i\hbar\partial_t|\psi_i\rangle]^\dagger\right) \\
&= \sum_i p_i \left(i\hbar\partial_t|\psi_i\rangle\langle\psi_i| + |\psi_i\rangle[i\hbar\partial_t\langle\psi_i|]\right) \\
&= i\hbar\partial_t \sum_i p_i|\psi_i\rangle\langle\psi_i| \\
&= i\hbar\partial_t\hat{\rho}.
\end{aligned}$$

This immediately yields

$$\partial_t W = \frac{1}{i\hbar}[H, W]_* \tag{5.7}$$

as the time evolution equation of a Wigner function.

Taking the trace of an operator $\hat{F}$ correspond to integrating its Wigner transform f. This is a simple calculation:

$$\begin{aligned}
\int \mathrm{d}q\, \mathrm{d}p\, f(q,p) &= \frac{1}{2\pi}\int \mathrm{d}q\, \mathrm{d}p\, \mathrm{d}y\, e^{ipy} \left\langle q - \frac{\hbar}{2}y \right| \hat{F} \left| q + \frac{\hbar}{2}y \right\rangle \\
&= \int \mathrm{d}q\, \mathrm{d}y\, \delta(y) \left\langle q - \frac{\hbar}{2}y \right| \hat{F} \left| q + \frac{\hbar}{2}y \right\rangle \\
&= \int \mathrm{d}q\, \langle q|\hat{F}|q\rangle \\
&= \mathrm{Tr}(\hat{F}).
\end{aligned} \tag{5.8}$$

Next, let us prove a nice property of the Moyal star product, called the *Lone Star Lemma*:

$$\int \mathrm{d}q\, \mathrm{d}p\, f * g = \int \mathrm{d}q\, \mathrm{d}p\, fg = \int \mathrm{d}q\, \mathrm{d}p\, g * f$$

for all functions f and g. To do this, we use an integral representation of the star product, which is derived in the appendix of [12]:

$$\begin{aligned}
f * g = \frac{1}{(\pi\hbar)^2} &\int \mathrm{d}q_1\, \mathrm{d}p_1\, \mathrm{d}q_2\, \mathrm{d}p_2\, f(q_1, p_1) g(q_2, p_2) \\
&\cdot \exp\left(\frac{2}{i\hbar}(p(q_1 - q_2) + q(p_2 - p_1) + (q_2 p_1 - q_1 p_2)\right).
\end{aligned}$$

Integrating this expression over q and p first, we get

$$\begin{aligned}\int \mathrm{d}q\,\mathrm{d}p\, f * g &= \int \mathrm{d}q_1\,\mathrm{d}p_1\,\mathrm{d}q_2\,\mathrm{d}p_2\, f(q_1, p_1)g(q_2, p_2) \\ &\quad \cdot \delta(q_1 - q_2)\delta(p_2 - p_1)\exp\left(\frac{2}{i\hbar}(q_2 p_1 - q_1 p_2)\right) \\ &= \int \mathrm{d}q_1\,\mathrm{d}p_1\, f(q_1, p_1)g(q_1, p_1)\exp\left(\frac{2}{i\hbar}(q_1 p_1 - q_1 p_1)\right) \\ &= \int \mathrm{d}q\,\mathrm{d}p\, fg,\end{aligned}$$

which is precisely what we wanted.

An important corollary of the Lone Star Lemma is that the expectation value of observable f can be calculated by integrating the product fW. To see this, note that if $Q(f) = \hat{F}$, we can use equation 5.8 to get

$$\begin{aligned}\langle \hat{F} \rangle = \mathrm{Tr}(\hat{F}\hat{\rho}) &= \int \mathrm{d}q\,\mathrm{d}p\, Q^{-1}(\hat{F}\hat{\rho}) \\ &= \int \mathrm{d}q\,\mathrm{d}p\, Q^{-1}(\hat{F}) * Q^{-1}(\hat{\rho}) \\ &= \int \mathrm{d}q\,\mathrm{d}p\, f * W \\ &= \int \mathrm{d}q\,\mathrm{d}p\, fW.\end{aligned}$$

In particular, the energy of a state is given by

$$E = \langle \hat{H} \rangle = \int \mathrm{d}q\,\mathrm{d}p\, HW = \int \mathrm{d}q\,\mathrm{d}p\, H * W.$$

For static pure states, we get a more general result. In particular, static pure state Wigner functions are precisely the real-valued *stargenfunctions* of the Hamiltonian, that is the real-valued phase space functions functions W such that

$$H * W = W * H = EW$$

for some real number E, which is, unsurprisingly, the energy of the state. A proof of this can be found as Lemma 1 and 2 in [2]. As a result, static pure state Wigner functions are "$*$-orthogonal" in that

$$W_i * W_j = \frac{1}{2\pi\hbar}\delta_{ij} W_i$$

if the static pure states W_i and W_j have different energies, which is Corollary 1 of [2].

The uncertainty principle, $\sigma_q \sigma_p \geq \frac{\hbar}{2}$, can also be directly derived from properties of pure state Wigner functions, the Lone Star Lemma playing an integral role; a derivation can be found in and Appendix in [3], for example.

The classical limit has a very simple form in deformation quantisation. Where undergraduates are told that the classical limit somehow corresponds to the quantum parameter $\hbar$ becoming negligible, we can take this quite literally in deformation quantisation: simply take the limit $\hbar \to 0$. This has two important effects.

First, the Moyal star product reduces to its degree 0 term, which is just multiplication, and second, the star-commutator $\frac{1}{i\hbar}[f,g]_*$ reduces to the Poisson bracket $\{f,g\}$, similarly to equation 4.5. As a result, the two time evolution equations, one for observables (5.1) and one for Wigner functions (5.7), reduce to their classical analogues:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{1}{i\hbar}[f,H]_* \quad \text{becomes} \quad \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f,H\},$$

and

$$\frac{\partial W}{\partial t} = \frac{1}{i\hbar}[H,W]_* \quad \text{becomes} \quad \frac{\partial W}{\partial t} = \{H,W\}.$$

This last equation is Liouville's Theorem for the *phase space distribution function**, which provides a way to consider multiple states in a classical system at once, analogously to the density operator representing multiple pure states in a quantum system.

To conclude this section, let us make a short note on more particles in multiple dimensions. In the case of n particles in three dimensions, the Moyal product has a simple extension,

$$f * g = f e^{i\frac{\hbar}{2}\sum_{i=0}^{3n}(\partial^L_{q_i}\partial^R_{p_i} - \partial^R_{q_i}\partial^L_{p_i})} g,$$

and so do the Wigner and Weyl transforms, namely

$$Q(f) = \frac{1}{2\pi}\int \mathrm{d}\vec{\sigma}\,\mathrm{d}\vec{\tau}\,\tilde{f}(\vec{\sigma},\vec{\tau})e^{i(\vec{\sigma}\hat{q}+\vec{\tau}\hat{p})},$$

where the Fourier transform of $f(\vec{q},\vec{p})$ is the standard multi-dimensional one, and

$$Q^{-1}(\hat{F}) = \frac{1}{2\pi}\int \mathrm{d}\vec{y}\, e^{i\vec{p}\cdot\vec{y}} \left\langle \vec{q} - \frac{\hbar}{2}\vec{y} \right| \hat{F} \left| \vec{q} + \frac{\hbar}{2}\vec{y} \right\rangle.$$

As a result, all other previous statements also have simple generalisations to higher dimensions.

5.4 The Harmonic Oscillator

Let us now treat the most ubiquitous physical system in our new formalism. Recall that a classical example of a harmonic oscillator is a simple mass on a spring, with Hamiltonian $H = p^2/2m + \frac{1}{2}kq^2$. It has as solutions sinusoids with frequency $\omega = \sqrt{k/m}$, which is why the Hamiltonian is often rewritten as

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

*`https://en.wikipedia.org/wiki/Liouville%27s_theorem_(Hamiltonian)`, accessed at 14:30 on July 14th, 2023

Weyl transforming H results in the standard harmonic oscillator Hamiltonian operator by virtue of Lemma 5.1:

$$\hat{H} = Q(H) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2.$$

The most elegant way to treat this system in the Hilbert space formalism is by introducing *ladder operators* $\hat{a}_\pm$, as Griffiths and Schroeter do in Section 2.3.1 of their classic textbook [7]. The ladder operators are given by

$$\hat{a}_\pm = \frac{1}{2\hbar m\omega}(m\omega\hat{q} \mp i\hat{p}),$$

which have commutator $[\hat{a}_-, \hat{a}_+] = 1$ and with which the Hamiltonian can be factored as

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right).$$

But since deformation quantisation is completely equivalent to the Hilbert space formalism, we should be able to use the exact same procedure to find the Wigner functions of the system. So let us define the *ladder functions* as

$$a_\pm(q,p) = Q^{-1}(\hat{a}_\pm) = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega q \mp ip).$$

Note that they are linear in both q and p so that second and higher order derivatives vanish. This makes it exceedingly simple to calculate their star product:

$$\begin{aligned}
a_\pm * a_\mp &= a_\pm \cdot a_\mp + i\frac{\hbar}{2}(\partial_q a_\pm \partial_p a_\mp - \partial_p a_\pm \partial_q a_\mp)\\
&= \frac{1}{2\hbar m\omega}(m^2\omega^2q^2 + p^2) + i\frac{\hbar}{2}\left(\frac{\pm im\omega}{2\hbar m\omega} - \frac{\mp im\omega}{2\hbar m\omega}\right)\\
&= \frac{H}{\hbar\omega} \mp \frac{1}{2},
\end{aligned}$$

which immediately shows that we indeed have

$$[a_-, a_+]_* = 1 \quad \text{and} \quad H = \hbar\omega\left(a_- * a_+ - \frac{1}{2}\right),$$

as expected.

Now let us try to find the ground state Wigner function $W_0(q,p)$ using the identity $a_- * W_0 = 0$. To do so, we first note that for smooth functions f, we have

$$e^{a\partial_x}f(x) = f(x+a).$$

This allows us to express the Moyal star product in a more convenient form for this problem, namely

$$f * g = f e^{i\frac{\hbar}{2}(\partial_q^L\partial_p^R - \partial_q^R\partial_p^L)} g = f\left(q + i\frac{\hbar}{2}\partial_p^R, q - i\frac{\hbar}{2}\partial_q^R\right) g(q,p).$$

Now we can write

$$a_- * W_0 = \frac{1}{\sqrt{2\hbar m\omega}} \left(m\omega \left(q + i\frac{\hbar}{2}\partial_p^R \right) + i \left(p - i\frac{\hbar}{2}\partial_q^R \right) \right) W_0(q,p) = 0.$$

The real and imaginary parts of this differential equation for W_0 can be considered separately. In particular, for the real part, we have

$$\left(m\omega q + \frac{\hbar}{2}\partial_q \right) W_0(q,p) = 0,$$

which implies that we have $W_0(q,p) = A(p)e^{-\frac{m\omega}{\hbar}q^2}$ for some function $A(p)$. Substituting into the imaginary part yields

$$\left(\frac{m\omega\hbar}{2}\partial_p + p \right) A(p) = 0,$$

hence $A(p) = Ce^{-\frac{p^2}{m\omega\hbar}}$ for some constant C. To obtain the constant, we note that Wigner functions should be normalised, that is integrate to 1 over phase space. Thus, we get

$$\begin{aligned} \frac{1}{C} &= \int \mathrm{d}q\, \mathrm{d}p \exp\left(-\frac{p^2}{m\omega\hbar} - \frac{m\omega}{\hbar}q^2 \right) \\ &= \sqrt{\pi m\omega\hbar}\sqrt{\frac{\pi\hbar}{m\omega}} \\ &= \pi\hbar \end{aligned}$$

using the standard Gaussian integral $\int \mathrm{d}x\, e^{-ax^2} = \sqrt{\pi/a}$, from which conclude that

$$W_0(q,p) = \frac{1}{\pi\hbar} \exp\left(-\frac{p^2}{m\omega\hbar} - \frac{m\omega}{\hbar}q^2 \right) = \frac{1}{\pi\hbar} \exp\left(-\frac{2H}{\hbar\omega} \right).$$

How do we know this is correct? Well, we should obtain the same result by Wigner transforming the ground state wave function, which is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

according to equation 2.60 in [7]. Substituting this into the Wigner transform (5.4), we get

$$\begin{aligned} Q^{-1}(\psi_0) &= \frac{1}{2\pi} \int \mathrm{d}y\, e^{ipy} \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{2\hbar} \left(\left(q + \frac{\hbar}{2}y \right)^2 + \left(q - \frac{\hbar}{2}y \right)^2 \right) \right) \\ &= \frac{1}{2\pi}\sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega}{\hbar}q^2} \int \mathrm{d}y \exp\left(-\frac{m\omega\hbar}{4}y^2 + ipy \right). \end{aligned}$$

The exponential before the integral in the last line is already what we want, which is great sign. As for the integral, we can substitute $u = \frac{1}{2}\sqrt{m\omega\hbar} \cdot y - ip/\sqrt{m\omega\hbar}$ to obtain

$$\begin{aligned}
Q^{-1}(\psi_0) &= \frac{1}{2\pi}\sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega}{\hbar}q^2} \int \frac{2\,\mathrm{d}u}{\sqrt{m\omega\hbar}} e^{-u^2-\frac{p^2}{m\omega\hbar}} \\
&= \frac{1}{2\pi}\sqrt{\frac{m\omega}{\pi\hbar}} \frac{2}{\sqrt{m\omega\hbar}} \sqrt{\pi} \exp\left(-\frac{p^2}{m\omega\hbar} - \frac{m\omega}{\hbar}q^2\right) \\
&= \frac{1}{\pi\hbar} \exp\left(-\frac{p^2}{m\omega\hbar} - \frac{m\omega}{\hbar}q^2\right) \\
&= W_0(q,p),
\end{aligned}$$

so our calculations were correct.

The energy of the ground state should be $\frac{1}{2}\hbar\omega$, so let us check if we can find this value if we calculate the energy E_0 of the state W_0. Since the energy is the expectation value of the Hamiltonian, we have

$$\begin{aligned}
E_0 &= \int \mathrm{d}q\,\mathrm{d}p\, H(q,p) W_0(q,p) \\
&= \frac{1}{\pi\hbar} \int \mathrm{d}q\,\mathrm{d}p \left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2\right) \exp\left(-\frac{p^2}{m\omega\hbar} - \frac{m\omega}{\hbar}q^2\right).
\end{aligned}$$

Here, we can use another Gaussian-type integral, which can be obtained by integrating by parts the standard one: $\int \mathrm{d}x\, x^2 e^{-ax^2} = \sqrt{\pi/16a^3}$. This yields

$$\begin{aligned}
E_0 &= \frac{1}{\pi\hbar}\left(\frac{1}{2m}\sqrt{\frac{\pi m^3\hbar^3\omega^3}{4}}\sqrt{\frac{\pi\hbar}{m\omega}} + \frac{1}{2}m\omega^2\sqrt{\frac{\pi\hbar^3}{4m^3\omega^3}}\sqrt{\pi m\hbar\omega}\right) \\
&= \frac{1}{2}\hbar\omega,
\end{aligned}$$

as expected.

Moving forward, Griffiths and Schroeter show that if ψ is an eigenfunction of $\hat{H}$, so are $\hat{a}_\pm\psi$. Using the same procedure but instead working with the properties of $a_\pm$ with respect to $*$, one can show that if W is a function obeying $H * W = EW$, then so are $a_\pm * W$. However, at this point we must be careful to not immediately conclude that $a_\pm * W$ are Wigner functions, because the stargenfunction equation consists of two parts: we must also have $W * H = EW$.

But using the same argument *twice*, one can show that if W is a stargenfunction of H, then so are

$$a_\pm * W * a_\mp. \tag{5.9}$$

We could also have arrived at this conclusion using the fact that Wigner functions correspond to density operators, because the density operator of an energy eigenstate $|\psi\rangle$ also requires a two-sided application of the ladder operators: the new state is

$$\hat{a}_\pm|\psi\rangle\langle\psi|(\hat{a}_\pm)^\dagger = \hat{a}_\pm\hat{\rho}\hat{a}_\mp.$$

It is then easy to see that the n-th excited state has Wigner function

$$W_n = \frac{1}{n!} a_+^{*n} * W_0 * a_-^{*n},$$

where f^{*n} of course means taking the star product of f with itself n times.

Lastly, note that just as in the Hilbert space formalism, states with different energies are orthogonal. This follows from the properties of Wigner functions, but can also be seen using the Weyl-Wigner correspondence. Namely, if $m \neq n$, then

$$W_n * W_m = Q^{-1}(|\psi_n\rangle\langle\psi_n|\psi_m\rangle\langle\psi_m|) = Q^{-1}(0) = 0.$$

This concludes our discussion of the harmonic oscillator in deformation quantisation.

Chapter 6

Conclusion and Outlook

In this thesis, we introduced the mathematical theory behind the Hamiltonian formulation of classical mechanics, explored deformations of the algebra of functions on phase space, and applied what we learned to formulate a new, independent theory of quantum mechanics. We went into detail on symplectic geometry, Hochschild cohomology, and deformation theory, all of which are rich mathematical topics of their own beyond what is necessary for physical applications. Lastly, we learned about the quantum theory of deformation quantisation, a formulation of quantum mechanics that works on a system's phase space, which is independent of, yet completely equivalent to, the Hilbert space formalism and has its own specialised applications.

Deformation quantisation has another benefit of a more pedagogical nature. As we have seen, it is only a small step to introduce just a single new parameter, $\hbar$, and a single new (non-local, non-commutative) operation, the Moyal star product, onto the objects we were already working with, functions on phase space. In fact, the evolution equations of both states and observables change only marginally. Moreover, it is also a very small step to move back to Hamiltonian classical mechanics by taking the classical limit $\hbar \to 0$ quite literally. This contrasts drastically with the relatively large step that the Hilbert space formalism requires by introducing a new concept of states and observables and a completely new time evolution equation.

Unfortunately, the conceptual simplicity comes with a price tag. In particular, calculations involving the Moyal star product and Wigner functions become tedious and lengthy quite quickly. To illustrate this, the reader is encouraged to try to calculate the Wigner function of the first excited state of the harmonic oscillator, for example using equation 5.9. In the author's experience, calculations in the Hilbert space formalism were never this complicated in his undergraduate courses. Additionally, to fully understand how deformation quantisation generalises Hamiltonian mechanics, it is necessary to have at least seen phase space distribution functions, which are currently not part of the undergraduate studies of classical mechanics at the author's university. Nonetheless, deformation quantisation is an elegant formulation of quantum mechanics that can provide a large increase in conceptual understanding to anyone who struggles with the weird properties of the microscopic world.

6.1 Where to Go From Here

Apart from exploring in more detail the mathematical theories of symplectic geometry, Hochschild cohomology, and algebra deformations and the physical theory of deformation quantisation, there are two directions for further study that the author has explored a little as part of the writing process of this thesis. On the one hand, a mathematician may take a deformed algebra as a stepping stone to study *non-commutative geometry* in general. This involves taking an arbitrary algebra in place of the algebra of smooth functions on a manifold and defining a differential structure in the form of a *differential graded Lie algebra*, which is a generalisation of the differential forms on a manifold.

A theoretical physicist may then define, for this non-commutative geometric structure, analogous concepts to (pseudo-)Riemannian metrics, torsion, and curvature and use these in an attempt to take steps towards a theory of quantum gravity. The allure in this approach lies in the *Quantum Spacetime Hypothesis*, which states that spacetime is better modelled by non-commutative geometry, in contrast with the standard assumption that spacetime should be modelled on a continuous space such as $\mathbf{R}^4$. For more information on non-commutative geometry and its application to theoretical physics, we refer to Edwin Beggs's and Shahn Majid's textbook titled "Quantum Riemannian Geometry" [1].

On the other hand, one may use the ideas of deformation quantisation to formulate non-commutative field theories by replacing ordinary powers of a field with star product powers. At first glance, it may seem that this will only complicate the theory and the search for solutions to the field equations, but it turns out that simple non-commutative field theories allow for the construction of simple soliton solutions, that is solutions that look like local excitations of the field. Moreover, the Weyl-Wigner correspondence yields two different, yet totally equivalent, approaches to treat the same theory, which may complement each other. For details on non-commutative field theory and its relation to String Theory, we refer to the Komaba Lectures by Jeffrey Harvey [11].

In conclusion, deformation quantisation is just the tip of the iceberg when it comes to using non-commutative theories to model quantum effects. One may hope that the study of this subject will eventually lead to a discovery of a theory of quantum gravity, but only time will tell when and how we find a way to reconcile relativity with the quantum world.

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