The Additive Structure of Scissors Congruence Classes
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1 Introduction

Say you're an early geometer, and you would like to show that two plane figures P, Q have equal area. A reasonable way to do this is to show that they may be decomposed into smaller pieces (where these smaller pieces have disjoint interiors) by way of finitely many cuts, such that the resulting pieces are pairwise congruent figures. Indeed, this is one of the methods that Euclid employs to show that given polygons have equal area [Euc]. If this relationship holds we say that the two figures are *scissors congruent*, or *s.c.* for short.

With the benefits of modern measure theory, we can show that a much wider range of figures have equal area, which we understand as the 2-dimensional Lebesgue measure μ. We can also use it to show that indeed scissors congruence does preserve area. For two polygons P, Q in the plane (considered as sets), the property of having disjoint interiors is equivalent to the fact that their intersection has measure 0. It follows that the area of their union is equal to the sum of their areas, so $\mu(P \cup Q) = \mu(P) + \mu(Q)$. Additionally, the isometries of the plane that induce the congruence relation between polygons are bimeasurable and measure-preserving, so because the measure of area is additive w.r.t. interior disjoint unions, it follows that it is invariant under scissors congruence.

More generally, if instead of considering polygons, we considered a different class of measurable sets, and instead of the group of all isometries we considered a different group of measure-preserving transformations, we can still conclude that scissors congruence must preserve the measure. For example, we could consider the class of polygons whose sides are axis-aligned, i.e. either horizontal or vertical, and as transformations the group of translations in the plane. We combine this data into a 'scissors congruence structure' $(\mathcal{F}, G)$, where $\mathcal{F}$ is a class of figures and G is an $\mathcal{F}$-measurable group of transformations (Definition 2.4.17).

During the early 19'th century it was proved (by several mathematicians independently) that any two polygons in the plane are s.c. (allowing polygonal subdivisions and all isometric transformations) if and only if they have the same area; this is the Wallace-Bolyai-Gerwien Theorem. This left open the question of whether the same could be proved for polyhedra in Euclidean 3-space. This problem was of particular interest to David Hilbert around the turn of the 20'th century, as he was interested in the formal axiomatization of Euclidean geometry. He included it as the third problem in his list presented at the International Congress of Mathematicians in Paris in 1900. Not much later, Hilbert's third problem was the first to be solved. In [Deh01], Hilbert's student Max Dehn showed that, for example, a cube and a regular tetrahedron of the same volume are not scissors congruent, solving the problem in the negative. Dehn was not the first to prove this particular fact, but Hilbert was not aware that it had been proved some years earlier by a mathematician named Ludwik Antoni Birkenmajer [CC18].

Dehn's result would later by improved by Jean-Pierre Sydler in [Syd65]. Børge Jessen provides an excellent account of Dehn's and Sydler's results in [Jes68]. Later, Chih-Han Sah published [Sah79], which contained much of the cutting edge of the subject at the time. For an introduction to some theories of scissors congruence that have developed since then, including an application to algebraic K-theory, see [Zak16].

In this work we investigate the structure of the additive monoid of general s.c. structures. In Section 2 we define all of the necessary machinery to give a powerful and general definition of scissors congruence (Definition 2.4.12). Along the way we give a somewhat non-standard definition of 'polytope' and other similar figures (Section 2.3). The difficulty comes from the fact that for two polyhedra, say, to be scissors congruent we cut them up and paste them back together. If we consider polyhedra to be specific kinds of subsets of $\mathbb{R}^3$ then we run into trouble; if you cut a square in half along some line, which half gets the points that lay on the line? Our definition avoids this issue.

The additive structure of s.c. classes does not always form a commutative monoid in the traditional sense. Consider the set of subfigures of a given square. We cannot add the s.c. class of the square to itself, because there is no room to find two disjoint representatives. Thus the set of s.c. classes forms a *partial commutative monoid* (PCM, Definition 3.1.1), which we call $\mathbf{SC}$. We give some basic properties of these algebraic structures in Section 3.1.

In Section 3.2 we investigate the basic consequences of the definition of addition of s.c. classes. The ordering of subfigure containment can be extended to $\mathbf{SC}$ and this ordering agrees with the natural ordering of $\mathbf{SC}$ as a PCM (Proposition 3.2.9). We place particular emphasis on s.c. structures on $\mathbb{R}^n$ such that $\mathcal{F}$ contains all axis-aligned rectangular prisms, every figure of $\mathcal{F}$ is *Jordan-measurable* (Definition 2.3.9) and bounded, and every transformation of G is an isometry. We call such s.c. structures 'regular Euclidean' (Definition 2.4.20), and they have the property that the Lebesgue measure (or equivalently, the Jordan content) gives a well-defined PCM homomorphism from $\mathbf{SC}$ to the non-negative real numbers.

Proposition (3.2.15). *If $(\mathcal{F}, G)$ is a s.c. structure where the figures of $\mathcal{F}$ are Lebesgue-measurable, then the Lebesgue measure μ induces a homomorphism of partial commutative monoids $\mu\colon \mathbf{SC} \to [0, \infty]$.*

Theorem (3.2.17). *If $(\mathcal{F}, G)$ is a regular Euclidean s.c. structure, then for s.c. classes $\alpha, \beta \in \mathbf{SC}$ we have $\alpha < \beta$ in the natural ordering of $\mathbf{SC}$ if and only if $\mu(\alpha) < \mu(\beta)$.*

Section 3.3 is all about Zylev's theorem. It gives mild sufficient conditions for the addition of $\mathbf{SC}$ to be *cancellative*.

Definition (3.3.2). *An s.c. structure is called* cancellative *if for all figures $P, Q, T \in \mathcal{F}$ such that $P \sqcup T$ is s.c. with $Q \sqcup T$ we have that P is s.c. with Q.*

Definition (3.3.4). *An s.c. structure $(\mathcal{F}, G)$ is called* positive *if no figure $P \in \mathcal{F}$ is s.c. with a strict subfigure of P.*

Definition (3.3.5). *Let $P, Q \in \mathcal{F}$ be figures. Then P is said to be* larger than twice Q *if there is a subfigure $Q' \subset P$ s.c. to Q and whenever $Q_1 \subset P$ is s.c. to Q then there exists a $Q_2 \subset P$ s.c. to Q such that $Q_1 \cap Q_2 = [\emptyset]$.*

Definition (3.3.6). *An s.c. structure $(\mathcal{F}, G)$ is* uniform *if for any two non-empty figures P, Q there is a decomposition $Q = Q_1 \sqcup \cdots \sqcup Q_n$ such that P is larger than twice Q_i for all I.*

Zylev's theorem (3.3.7). *If an s.c. structure $(\mathcal{F}, G)$ is positive and uniform, then it is cancellative.*

Zylev's theorem justifies the use of group-theoretic methods to investigate scissors congruence as is common in the literature. We also show that there is a sense in which there is no 'algebraic proof' of Zylev's theorem (Theorem 3.3.9), which shows that the combinatorics of figures cannot entirely be discarded. Finally we apply Zylev's theorem to show that regular Euclidean structures are always cancellative (Theorem 3.3.15).

In the short Section 3.4 we give a final structural theorem about regular Euclidean s.c. structures: their natural monoids **SC** may be decomposed into two independent components.

Lateral Group Theorem (3.4.5). *If $(\mathcal{F}, G)$ is a regular Euclidean s.c. structure, then there is an abelian group* $\mathrm{Lat}(\mathbf{SC})$*, called the* lateral group *of* **SC***, such that there is an isomorphism*

$$\mathbf{SC} \cong ((0, \infty) \times \mathrm{Lat}(\mathbf{SC})) \cup \{0\},$$

where $(0, \infty)$ is the additive semigroup of positive real numbers.

In Section 3.5 we investigate how all these ideas manifest themselves in the concrete cases of a few regular Euclidean s.c. structures. We give proofs of the Wallace-Bolyai-Gerwien theorem (3.5.3) and Dehn's theorem (3.5.13), which give effective *invariants* of the scissors congruence of polygons and polyhedra, respectively.

Finally, Section 4 poses some interesting questions which further research may answer.

2 Scissors congruence of space figures

2.1 Quotient rings of sets

First we will want to define what we mean by a 'figure'. Taking them to be subsets of $\mathbb{R}^n$ is too narrow, as for instance we do not care whether a triangle includes its boundary line segments. These get left on the cutting room floor during the dissection process, so to speak. We will a priori specify such boundary sets as *null sets*, using the language of *rings of sets*.

2.1.1 Definition. *A* ring of sets *on a set X is a subcollection $\mathcal{R} \subset \mathcal{P}(X)$ that:*

- *Contains $\emptyset$.*
- *Is closed under finite unions, so if $A, B \in \mathcal{R}$, then $A \cup B \in \mathcal{R}$.*
- *Is closed under set difference (or 'relative complement'), so if $A, B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}$.*

An ideal of sets *of $\mathcal{R}$ is a subcollection $\mathcal{N} \subset \mathcal{R}$ such that:*

- *$\emptyset \in \mathcal{N}$.*
- *If $N, M \in \mathcal{N}$, then $N \cup M \in \mathcal{N}$.*
- *If $N \in \mathcal{N}$ and $A \in \mathcal{R}$ are such that $A \subset N$, then $A \in \mathcal{N}$.*

The elements $A \in \mathcal{R}$ will be called figure representatives *and the elements $N \in \mathcal{N}$ will be called* null sets.

The terminology *ring* and *ideal* is somewhat suggestive, and the reader might wonder what they have to do with the concepts from abstract algebra. The first thing to notice is that a power set $\mathcal{P}(X)$ forms a ring if we take addition to be symmetric difference $\triangle$, and multiplication to be intersection $\cap$. This commutative ring is canonically isomorphic to the ring of all functions $f \colon X \to \mathbb{F}_2$ under pointwise operations, where the codomain is the field of two elements. The second thing to notice is that we may derive the symmetric difference and intersection from the operations of union and set difference, and vice versa, as follows:

2.1.2 Lemma. *For all sets A and B, we have*

$$A \triangle B = (A \setminus B) \cup (B \setminus A), \qquad A \cap B = (A \cup B) \triangle (A \triangle B),$$

$$A \cup B = (A \triangle B) \triangle (A \cap B), \qquad A \setminus B = A \triangle (A \cap B). \qquad \square$$

This allows us to conclude that $\mathcal{R}$ is a ring of sets on X if and only if it is a (non-unital) subring of $\mathcal{P}(X)$. An element $A \in \mathcal{R}$ is a subset of some $N \in \mathcal{N}$ precisely if it is of the form $A = B \cap N$ for a $B \in \mathcal{R}$, so the property that an ideal of sets is closed under taking subsets is equivalent to the defining property of an ideal of a commutative ring that it be stable under multiplication with elements of the ring.

We think of the elements of $\mathcal{N}$ as negligible sets, which we declare as not being relevant to a decomposition of some figure. In the case of polygons, these will be the finite unions of line segments and points, as we would like to say that two polygons joined side-to-side are disjoint enough for our purposes.

2.1.3 Definition. *We define the* quotient ring $\mathcal{F} := \mathcal{R}/\mathcal{N}$ *of* $\mathcal{R}$ *modulo* $\mathcal{N}$ *as the set of equivalence classes of the relation* $\sim$ *defined on* $\mathcal{R}$ *as*

$$A \sim B \iff A \triangle N = B \text{ for some } N \in \mathcal{N}.$$

The elements $P \in \mathcal{F}$ *will be called* figures. *If* $A \in \mathcal{R}$ *is a figure representative, let* $[A] \in \mathcal{F}$ *be its equivalence class w.r.t.* $\sim$*, i.e. the figure it represents.*

We can show that $[A] = [B]$ if and only if there are null sets $N, M \in \mathcal{N}$ such that $A \cup N = B \cup M$. Alternatively, $[A] = [B]$ if and only if $A \triangle B \in \mathcal{N}$.

A figure might most accurately be called an 'almost-set', by way of analogy with concepts such as almost-everywhere equality and almost-certain events from measure theory. We will stick to the 'figure' terminology, as it aligns with the classes of objects we want to study. Moreover, we will often call a quotient ring a 'class of figures' on X.

2.1.4 Example. *If* X *is any set, then its power set* $\mathcal{P}(X)$ *is a ring of sets on* X*. The collection* $\{\emptyset\}$ *containing only the empty set is an ideal of* $\mathcal{P}(X)$*. The resulting class of figures we denote by* $\mathcal{P}(X)/\{\emptyset\}$*.*

2.1.5 Notation. *If* $\mathcal{F}$ *is a class of figures, we let* $\mathcal{R}(\mathcal{F})$ *denote its underlying ring of figure representatives and we let* $\mathcal{N}(\mathcal{F})$ *denote its ideal of null sets, so* $\mathcal{F} := \mathcal{R}(\mathcal{F})/\mathcal{N}(\mathcal{F})$*.*

2.1.6 Notation. *We will use the letters* $A, B, C, \ldots$ *for figure representatives, and* $P, Q, R, \ldots$ *for figures.*

An important fact is that given a family $(\mathcal{R}_i)_{i \in I}$ of rings of sets on X, their intersection $\bigcap_i \mathcal{R}_i$ will also be a ring of sets. An analogous thing holds for a family of ideals on some ring $\mathcal{R}$. A consequence is that given any collection of subsets $\mathcal{S} \subset \mathcal{P}(X)$, there is a smallest ring on X containing them, and given any collection of figure representatives $\mathcal{A} \subset \mathcal{R}$ there is a smallest ideal of $\mathcal{R}$ containing them. We call these the ring and ideal *generated by* $\mathcal{S}$ and $\mathcal{A}$ respectively.

Because rings and ideals of sets correspond precisely to specific kinds of algebraic rings, we can leverage the theory of modular arithmetic to extend the algebraic operations of sets to figures. The following brief aside formalizes this.

2.1.7 Definition. *For a natural number* $n \in \mathbb{N}$*, we define an* n-ary slice function *to be a function* $s \colon \{1, \ldots, n\} \to \{0, 1\}$*. An* abstract n-ary Boolean combination *is a set of* n*-ary slice functions. Let* Bool_n *be the collection of all* n*-ary abstract Boolean combinations.*

As the name implies, abstract Boolean combinations represent 'concrete' combinations of n-tuples of sets, as follows.

2.1.8 Definition. *If* $s \colon \{1, \ldots, n\} \to \{0, 1\}$ *is a slice function and* $A_1, \ldots, A_n \subset X$ *are subsets, then their* Boolean slice $f_s(A_1, \ldots, A_n)$ *corresponding to* s *is the unique subset of* X *such that* $x \in f_s(A_1, \ldots, A_n)$ *if and only if* $x \in A_i$ *for every* i *with* $s(i) = 1$ *and* $x \notin A_j$ *for every* j *with* $s(j) = 0$*. This defines a function* $f_s \colon \mathcal{P}(X)^n \to \mathcal{P}(X)$*, given formulaically by*

$$f_s(A_1, \ldots, A_n) = \left(\bigcap_{i \in s^{-1}(1)} A_i \right) \setminus \left(\bigcup_{j \in s^{-1}(0)} A_j \right),$$

where an empty intersection evaluates to X and an empty union evaluates to $\emptyset$.

If $V \in \mathrm{Bool}_n$ is an abstract Boolean combination, its corresponding (concrete) n-ary Boolean combination *is the function $f_V : \mathcal{P}(X)^n \to \mathcal{P}(X)$ given by*

$$f_V(S_1, \ldots, S_n) = \bigcup_{s \in V} f_s(S_1, \ldots, S_n).$$

All common 'algebraic' operations on sets, such as unions, intersections, complements, symmetric differences, etc. are in fact (concrete) Boolean combinations. For example, as in Figure 1, the binary union of two sets $A \cup B$ corresponds to $V = \{s_1, s_2, s_3\} \in \mathrm{Bool}_2$, where

$$\begin{aligned} &s_1(1) = 1, \qquad s_2(1) = 1, \qquad s_3(1) = 0, \\ &s_1(2) = 0, \qquad s_2(2) = 1, \qquad s_3(2) = 1. \end{aligned}$$

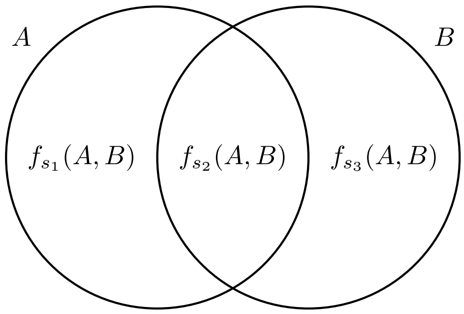

Figure 1: A Venn diagram showing the union of two disks A and B as a disjoint union of Boolean slices corresponding to slice functions s_1, s_2, s_3.

Interpreting this, an element $x \in X$ is contained in $A \cup B$ whenever it is either in A and not in B, in both A and B, or in B and not in A. It is no coincidence that no element of $A \cup B$ satisfies more than one of these cases.

2.1.9 Lemma. *Boolean slices corresponding to distinct slice functions are disjoint.*

Proof. Let $s_1, s_2 : \{1, \ldots, n\} \to \{0, 1\}$ be distinct slice functions and let $A_1, \ldots, A_n$ be sets. There is an $i \in \{1, \ldots, n\}$ such that $s_1(i) \neq s_2(i)$. Without loss of generality, assume that $s_1(i) = 1$ and $s_2(i) = 0$. If $x \in f_{s_1}(A_1, \ldots, A_n)$ then $x \in A_i$, so $x \notin f_{s_2}(A_1, \ldots, A_n)$. Analogously, if $x \in f_{s_2}(A_1, \ldots, A_n)$ then $x \notin A_i$, so $x \notin f_{s_1}(A_1, \ldots, A_n)$. We conclude that $f_{s_1}(A_1, \ldots, A_n) \cap f_{s_2}(A_1, \ldots, A_n) = \emptyset$. Because figures are disjoint if and only if they have disjoint representatives, the same holds for $P_1, \ldots, P_n \in \mathcal{F}$. $\square$

In fact, it is not unreasonable to *define* the collection of 'algebraic set operations' to be exactly the Boolean combinations. This definition can be seen as 'generator-free' or 'unbiased'. No special privilege is given to the binary combinations. The fact that you can build up *all* of the combinations out of just the binary ones is a theorem we can prove about the Boolean combinations.

As a sanity check, observe that indeed there are two nullary Boolean combinations on $\mathcal{P}(X)$: one that returns the empty set $\emptyset$ and one that returns the universe set X. These correspond to the empty set of slice functions and the set that contains only the unique slice function $\emptyset \to \{0, 1\}$ respectively.

It should come as no surprise that the Boolean combinations are closed under composition. That is, if $n, k_1, \ldots, k_n$ are integers, and $V \in \mathrm{Bool}_n$, $V_i \in \mathrm{Bool}_{k_i}$ are Boolean combinations, then the function that maps $(A_{1,1}, \ldots, A_{1,k_1}, \ldots, A_{n,1}, \ldots, A_{n,k_n})$ onto

$$f_V\Big(f_{V_1}(A_{1,1}, \ldots, A_{1,k_1}), \ldots, f_{V_n}(A_{n,1}, \ldots, A_{n,k_n})\Big)$$

is another Boolean combination. In fact, this operation can be abstracted into a set of 'composition' operations on abstract Boolean combinations, with signatures of the form $\mathrm{Bool}_n \times \mathrm{Bool}_{k_1} \times \cdots \times \mathrm{Bool}_{k_n} \to \mathrm{Bool}_{k_1+\cdots+k_n}$. This composition operation makes the sequence $(\mathrm{Bool}_n)_{n\in\mathbb{N}}$ into what's called an 'operad', but as it is not terribly important for our purposes we will not go into further detail.

2.1.10 Definition. *The* zero *n-ary slice function is the unique slice function $z\colon \{1,\dots,n\} \to \{0,1\}$ such that $z(i) = 0$ for all $i \in \{1,\dots,n\}$. A slice function is called* non-zero *if it is not a zero slice function.*

2.1.11 Definition. *A combination $V \in \mathrm{Bool}_n$ is called* bounded *if all of the slice functions it contains are non-zero.*

A zero slice function corresponds to the concrete combination that maps $A_1,\dots,A_n$ onto the complement (in X) of their union. This is an 'unbounded' combination, in the sense that it may map a collection of bounded sets (such as sets that represent polytopes, see Definition 2.3.3) onto an unbounded set. It is not too hard to see that the bounded Boolean combinations are closed under composition. We can use this fact to give a more elegant definition of a ring of sets: it is a subcollection of $\mathcal{P}(X)$ that is closed under all bounded Boolean combinations. The following lemma serves to illustrate that such combinations are automatically well-defined on figures as well.

2.1.12 Venn Diagram Lemma. *If $\mathcal{F}$ is a quotient ring of sets on X, then for any bounded $V \in \mathrm{Bool}_n$, the Boolean combination restricts to an operation $(\mathcal{R}(\mathcal{F}))^n \to \mathcal{R}(\mathcal{F})$, and furthermore the operation $f_V\colon \mathcal{F}^n \to \mathcal{F}$ defined as $f_V([A_1],\dots,[A_n]) = [f_V(A_1,\dots,A_n)]$ is well-defined.*

Proof. Let $s \in V$ be a slice function. Recall that the function $f_s\colon (\mathcal{P}(X))^n \to \mathcal{P}(X)$ is given by

$$f_s(A_1,\dots,A_n) = \left(\bigcap_{i\in s^{-1}(1)} A_i\right) \setminus \left(\bigcup_{j\in s^{-1}(0)} A_j\right).$$

Note that if s is not a zero slice function then the intersection will not be vacuous, so the right hand side is guaranteed to be a combination of the A_i using operations under which we know $\mathcal{R}(\mathcal{F})$ to be closed. So, if $A_1,\dots,A_n \in \mathcal{R}(\mathcal{F})$ then $f_s(A_1,\dots,A_n) \in \mathcal{R}(\mathcal{F})$. The set $f_V(A_1,\dots,A_n)$ is simply the union of these sets for all $s \in V$, so it too is an element of $\mathcal{R}(\mathcal{F})$.

We know from the theory of rings that the operations $[A] \triangle [B] = [A \triangle B]$ and $[A] \cap [B] = [A \cap B]$ are well-defined. Because the union and set difference operators may be derived from these two (Lemma 2.1.2), it follows that $f_V([A_1],\dots,[A_n]) = [f_V(A_1,\dots,A_n)]$ is well-defined for all $[A_1],\dots,[A_n] \in \mathcal{F}$. $\square$

The Venn Diagram Lemma serves mostly as an aid for intuition. It shows that we can apply whatever operations we would normally use for sets to figures without worry.

Take note that if P is a figure, it does not in general make sense to ask whether some $x \in X$ is contained in P, as this might not be independent of the representative of P. However, we can still define an inclusion relation among figures.

2.1.13 Definition. *If $P, Q \in \mathcal{F}$ are figures we say that P is* contained *in Q, written $P \subset Q$, if $P \cap Q = P$. In this case we say that P is a* subfigure *of Q.*

It is readily verified that $P \subset Q$ if and only if there are representatives A, B of P and Q respectively such that $A \subset B$ as subsets. This ordering defines a lattice (without greatest element) on $\mathcal{F}$, which is to say that it defines a partial ordering such that any two elements have both a least upper bound and a greatest lower bound, given by the union and intersection respectively. The empty figure $[\emptyset]$, whose representatives are precisely the null sets, is the smallest element of this lattice.

2.1.14 Definition. *We say that figures P and Q are* disjoint *if $P \cap Q = [\emptyset]$. If P is any figure, and $P_1,\dots,P_n$ are figures such that $P = P_1 \cup \cdots \cup P_n$ and P_i and P_j are disjoint whenever $i \neq j$, then we say that the P_i form a* decomposition *of P. If this is the case, we write $P = P_1 \sqcup \cdots \sqcup P_n$.*

In the literature figures that are disjoint are also called 'interior-disjoint' or 'non-overlapping'. Decompositions of figures are the bread and butter of any theory of scissors congruence, and we will see them a lot. The collection of decompositions of a figure P can be partially ordered as follows.

2.1.15 Definition. *Let $P \in \mathcal{F}$ be a figure and let $P = Q_1 \sqcup \cdots \sqcup Q_n$ and $P = R_1 \sqcup \cdots \sqcup R_k$ be decompositions of P. We say that the decomposition consisting of the Q_i* refines *the other if for all $1 \leq i \leq n$ there is some $1 \leq j \leq k$ with $Q_i \subset R_j$.*

The following is easy to verify.

2.1.16 Proposition. *Refinement is a reflexive transitive antisymmetric relation on the set of decompositions of P.* □

In fact, we can show that the set of decompositions is *downward-directed* w.r.t. refinement.

2.1.17 Refinement Lemma. *Any two decompositions of some figure P have a common refinement. Explicitly, if $P = Q_1 \sqcup \cdots \sqcup Q_n = R_1 \sqcup \cdots \sqcup R_k$, then there is some decomposition $P = S_1 \sqcup \cdots \sqcup S_t$ such that each Q_i and R_i may be decomposed into finitely many S_i.*

Proof. Let $t = n \cdot k$, and let us write the S-figures as indexed by pairs (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq k$. Let $S_{i,j} = Q_i \cap R_j$. These figures will form the required decomposition. □

This statement can be made stronger. The set of decompositions is a *lattice* (without a minimal element) under refinement; any non-empty finite set of decompositions has a greatest lower bound and a least upper bound w.r.t. refinement. We will not need this stronger result but it is also not hard to prove.

The proof of the Refinement Lemma is rather simple in our formalism, but we state the result as a proper lemma nonetheless, as it is quite fundamental to any theory of general scissors congruence. Indeed, the different formalisms of abstract scissors congruence given in [Sah79] and [Zak17] satisfy similar properties.

2.2 Generating figures, bases, and convexity

We would like some easy ways of defining classes of figures.

2.2.1 Definition. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are classes of figures on the same set X, we say that $\mathcal{F}_1$ is a* subclass *of $\mathcal{F}_2$ if $\mathcal{R}(\mathcal{F}_1) \subset \mathcal{R}(\mathcal{F}_2)$ and $\mathcal{N}(\mathcal{F}_1) = \mathcal{R}(\mathcal{F}_1) \cap \mathcal{N}(\mathcal{F}_2)$. We write $\mathcal{F}_1 \leq \mathcal{F}_2$.*

Morally speaking, $\mathcal{F}_1$ and $\mathcal{F}_2$ have 'the same null sets'. This relation partially orders the collection of all classes of figures on X.

2.2.2 Proposition. *If $\mathcal{F}$ is a class of figures and $\mathcal{F}' \leq \mathcal{F}$ is a subclass, then there is an injective function $i\colon \mathcal{F}' \to \mathcal{F}$ such that if $V \in \mathrm{Bool}_n$ is any bounded Boolean combination and $P_1, \ldots, P_n \in \mathcal{F}'$ then $i(f_V(P_1, \ldots, P_n)) = f_V(i(P_1), \ldots, i(P_n))$.*

Proof. Consider the inclusion map $\mathcal{R}(\mathcal{F}') \to \mathcal{R}(\mathcal{F})$. Postcomposing it with the quotient map $\mathcal{R}(\mathcal{F}) \to \mathcal{F}$ we get a function $\mathcal{R}(\mathcal{F}') \to \mathcal{F}$. It is constant on the $\sim$-equivalence classes of $\mathcal{R}(\mathcal{F}')$ because it preserves unions (in fact, it preserves all bounded Boolean combinations) and $\mathcal{N}(\mathcal{F}') \subset \mathcal{N}(\mathcal{F})$. Let i be the induced function $\mathcal{F}' \to \mathcal{F}$. All we have left to show is that i is injective. Let $A, B \in \mathcal{R}(\mathcal{F}')$ be such that $i([A]) = i([B])$. Then by definition there is some $N \in \mathcal{N}(\mathcal{F})$ such that $A \triangle B = N$, so $N \in \mathcal{R}(\mathcal{F}')$, so $[A] = [B]$ in $\mathcal{F}'$ as required. □

In other words, we can identify a subclass of $\mathcal{F}$ with a subset of $\mathcal{F}$ without worry.

2.2.3 Definition. *A subset $\mathcal{H} \subset \mathcal{F}$ is called a* generating set *for $\mathcal{F}$ if for every $P \in \mathcal{F}$ there are $H_1, \ldots, H_n \in \mathcal{H}$ and a (bounded) Boolean combination $V \in \mathrm{Bool}_n$ with $f_V(H_1, \ldots, H_n) = P$. We call $\mathcal{H}$ a* basis *for $\mathcal{F}$ if every figure $P \in \mathcal{F}$ may be written as a finite disjoint union of elements of $\mathcal{H}$. Equivalently, if every P decomposes into elements of $\mathcal{H}$.*

Note that if $\mathcal{H}$ is a basis for $\mathcal{F}$, then any collection of figures that contains $\mathcal{H}$ is also a basis.

2.2.4 Proposition. *Given some class of figures $\mathcal{F}$ and a subset $\mathcal{S} \subset \mathcal{R}(\mathcal{F})$, there is a unique smallest subclass $\mathcal{F}_\mathcal{S} \leq \mathcal{F}$ such that $\mathcal{S} \subset \mathcal{R}(\mathcal{F}_\mathcal{S})$, which we call the class* generated by $\mathcal{S}$ *relative to $\mathcal{F}$. The classes $[S]$ of representatives $S \in \mathcal{S}$ will form a generating set for $\mathcal{F}_\mathcal{S}$.* $\square$

This proposition allows us to define one 'large' class of figures first, by fully specifying its ring of representatives and its ideal of null sets, and after that we only need to give a generating collection of representing sets for any smaller class. In section 2.3 we will apply this to defining various classes of n-polytopes. First, some more results on generating sets of figures.

With the help of the Venn Diagram Lemma (Lemma 2.1.12) we can turn any generating set into a basis.

2.2.5 Proposition. *If $\mathcal{H} \subset \mathcal{F}$ is a generating set of figures, then the collection*

$$S(\mathcal{H}) = \bigcup_{n \in \mathbb{N}} S_n(\mathcal{H}),$$

where

$$S_n(\mathcal{H}) = \Big\{ f_s(H_1, \dots, H_n) : s \text{ is a non-zero } n\text{-ary slice function and } H_1, \dots, H_n \in \mathcal{H} \Big\},$$

is a basis for $\mathcal{F}$.

Proof. Recall from Lemma 2.1.9 that Boolean slices corresponding to distinct slice functions are disjoint. Consider some element $P \in \mathcal{F}$. Because $\mathcal{F}$ is generated by $\mathcal{H}$, there will be some $n \in \mathbb{N}$, a $V \in \mathrm{Bool}_n$, and $H_1, \dots, H_n \in \mathcal{H}$ such that $P = f_V(H_1, \dots, H_n)$. By definition we have

$$f_V(H_1, \dots, H_n) = \bigcup_{s \in V} f_s(H_1, \dots, H_n),$$

so because this is a disjoint union and no $s \in V$ is the zero function, it follows that $S(\mathcal{H})$ is a basis for $\mathcal{F}$. $\square$

In general Boolean slices might be very complicated figures (they capture a lot of the complexity of $\mathcal{F}$, after all). In some cases we can make a slightly friendlier basis as follows.

2.2.6 Definition. *We say that a generating set $\mathcal{H} \subset \mathcal{F}$* has local complements *if for any two $P, Q \in \mathcal{H}$ there is a $Q' \in \mathcal{H}$ such that $P \setminus Q = P \cap Q'$.*

2.2.7 Proposition. *Let $C(\mathcal{H}) \subset \mathcal{F}$ denote the collection of all non-empty finite intersections of elements of $\mathcal{H}$. If $\mathcal{H}$ has local complements, then $C(\mathcal{H})$ will be a basis for $\mathcal{F}$.*

Proof. We show that if $\mathcal{H}$ has local complements, then $C(\mathcal{H}) = S(\mathcal{H})$, so $C(\mathcal{H})$ is a basis for $\mathcal{F}$. Clearly $C(\mathcal{H}) \subset S(\mathcal{H})$.

Let $H_1, \dots, H_n \in \mathcal{H}$ be a collection of figures and let $s\colon \{1, \dots, n\} \to \{0, 1\}$ be a non-zero slice function. We know that

$$f_s(H_1, \dots, H_n) = \left(\bigcap_{i \in s^{-1}(1)} H_i \right) \setminus \left(\bigcup_{j \in s^{-1}(0)} H_j \right).$$

The intersection of H_i's on the left is an element of $C(\mathcal{H})$ by definition. Subtracting the finite union of figures on the right is equivalent to repeated subtraction of single figures, so we may write

$$f_s(H_1, \dots, H_n) = ((G \setminus H_{j_1}) \setminus \dots) \setminus H_{j_k},$$

where $G \in C(\mathcal{H})$. To show that $f_s(H_1, \dots, H_n) \in C(\mathcal{H})$ it therefore suffices to show that $G \setminus H \in C(\mathcal{H})$ whenever $G \in C(\mathcal{H})$ and $H \in \mathcal{H}$, and the result will follow inductively. Note that G is an intersection of elements of $\mathcal{H}$, so there is some $G' \in \mathcal{H}$ such that $G = G \cap G'$. Because $\mathcal{H}$ has local complements there is an $H' \in \mathcal{H}$ such that $G' \setminus H = G' \cap H'$, so

$$G \setminus H = (G \cap G') \setminus H = G \cap (G' \setminus H) = G \cap G' \cap H' \in C(\mathcal{H}),$$

which is all we need to conclude that $S(\mathcal{H}) \subset C(\mathcal{H})$, and we are done. $\square$

2.2.8 Definition. *If $\mathcal{F}$ is a class of figures in $\mathbb{R}^n$, we say that a $P \in \mathcal{F}$ is* convex *if it may be represented by an $A \in \mathcal{R}(\mathcal{F})$ that is a convex set, i.e. for all $x, y \in A$ and $t \in [0,1]$ we have $tx + (1-t)y \in A$.*

2.2.9 Corollary. *If $\mathcal{F}$ has a generating set $\mathcal{H}$ of convex figures that has local complements, then the collection of convex elements of $\mathcal{F}$ is a basis.*

Proof. Note that intersections of convex sets are convex, so intersections of convex figures will also be convex. It follows that the basis $C(\mathcal{H})$ consists of convex figures, so the collection of all convex figures will also be a basis. □

Now we give a final way of constructing subclasses of figures from a larger class, after which we will be ready to define polytopes and various other sorts of figures.

2.2.10 Definition. *Let $\mathcal{F}$ be some class of figures on $\mathbb{R}^n$. Then $P \in \mathcal{F}$ is said to be* bounded *if it has some representative A such that A is bounded, i.e. there is some real number $r > 0$ such that for all points $x \in A$ the distance from x to the origin is less than r.*

2.2.11 Proposition. *If $\mathcal{F}$ is a class of figures on $\mathbb{R}^n$, then the set $\mathrm{Bounded}(\mathcal{F})$ of bounded elements of $\mathcal{F}$ is a subclass of $\mathcal{F}$.*

Proof. Clearly the class of figures generated by the bounded figure representatives of $\mathcal{R}(\mathcal{F})$ is a subclass of $\mathcal{F}$. From the fact that unions and differences of bounded sets are bounded it follows that this subclass consists of exactly the bounded figures of $\mathcal{F}$. □

2.3 Polytopes and other examples

We now define our notion of polytopes, and some other classes of figures along the way, in the manner that most naturally fits into our theory. For this section let $n \geq 1$ be some positive integer parameter.

2.3.1 Definition. *We define the class of figures Alg_n as follows. Let its ring of figure representatives be the smallest ring of set on $\mathbb{R}^n$ that contains the solution sets of all polynomial inequalities. That is, let $\mathcal{R}(\mathrm{Alg}_n)$ be generated by sets of the form*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : p(x_1, \ldots, x_n) \leq 0\}$$

for some multivariable real polynomial p. Such a set is called a semi-algebraic set. *Let its ideal of null sets $\mathcal{N}(\mathrm{Alg}_n)$ be generated by the solution sets of non-trivial polynomial equations, i.e. sets of the form*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : p(x_1, \ldots, x_n) = 0\},$$

where p is a non-zero *polynomial. Such sets are also called* algebraic hypersurfaces. *We will call Alg_n the class of* semi-algebraic figures *of $\mathbb{R}^n$.*

2.3.2 Definition. *Let UPol_n be the subclass of Alg_n generated by the solution sets of all linear inequalities. That is, sets of the form*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_1x_1 + \cdots + a_nx_n \leq b\}$$

for real numbers $a_1, \ldots, a_n, b$. Such a set is called a closed half-space. *Recall that by the definition of a subclass we have $\mathcal{N}(\mathrm{UPol}_n) = \mathcal{N}(\mathrm{Alg}_n) \cap \mathcal{R}(\mathrm{UPol}_n)$. We call UPol_n the class of* unbounded n-polytopes.

2.3.3 Definition. *Let $\mathrm{Pol}_n = \mathrm{Bounded}(\mathrm{UPol}_n) \leq \mathrm{UPol}_n$. This is the class of* n-polytopes.

We will call the elements of Pol_1 'line segments', those of Pol_2 'polygons', and those of Pol_3 'polyhedra'. We will also refer to n-polytopes as simply 'polytopes' if our statement does not depend on the dimension. There are some other related classes of figures in $\mathbb{R}^n$ that we can define in much the same way.

2.3.4 Definition. *Let* Circ *be the subclass of* Alg_2 *generated by* $\mathcal{R}(\mathrm{Pol}_2)$ *and sets of the form*

$$\left\{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \leq r\right\}.$$

That is, the circular disks. Its ideal of null sets will be generated by the null polygons and the circles. This gives us the class of circle figures.

2.3.5 Definition. *Let* Orth_n *be the subclass of n-polytopes whose boundaries are axis-aligned. Explicitly,* Orth_n *is the class of bounded elements of the subclass of* UPol_n *generated by sets of the form*

$$\{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_i \leq b\}, \qquad \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_i \geq b\}$$

for some index $1 \leq i \leq n$ and $b \in \mathbb{R}$. This is the class of ortholinear n-figures.

We have $\mathrm{Orth}_n < \mathrm{Pol}_n < \mathrm{UPol}_n < \mathrm{Alg}_n$ for every $n \geq 2$.

2.3.6 Proposition. *Any figure in* UPol_n*,* Pol_n*, or* Orth_n *may be decomposed into convex figures of the same class.*

Proof. Note that each of the classes is generated by a collection of convex figures that has local complements, so the result follows from Corollary 2.2.9. □

To conclude, let us give a pair of large classes.

2.3.7 Definition. *Let* $\mathcal{R}(\mathrm{Borel}_n)$ *be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$, and let* $\mathcal{N}(\mathrm{Borel}_n)$ *be the collection of sets that have n-dimensional Lebesgue measure 0. Then* Borel_n *is the class of* Borel n-figures.

2.3.8 Proposition. *Each of the previously given classes of figures are subclasses of the various* Borel_n*.*

Proof. This follows from the fact that the semi-algebraic subsets of $\mathbb{R}^n$ (i.e. subsets defined by polynomial inequalities) that have Lebesgue measure 0 are precisely those that are contained in some finite union of algebraic hypersurfaces of $\mathbb{R}^n$ (and are therefore precisely the null sets of the various classes of figures). This may be shown using the fact that any semi-algebraic set can be decomposed into a finite number of algebraic sets homeomorphic with open hypercubes, which then necessarily have lower dimension than the ambient space if they are contained in a finite union of algebraic hypersurfaces. Such in-depth discussions on semi-algebraic sets are beyond our scope, and may be found in [BCR98] (in particular, Theorem 2.3.6 on the decomposition of semi-algebraic sets). □

Note that we call the elements of Borel_n 'Borel figures', as opposed to 'Lebesgue figures'. This is because of the measure theoretic fact that any Lebesgue measurable subset of $\mathbb{R}^n$ is the symmetric difference of a Borel set and a subset of $\mathbb{R}^n$ with Lebesgue measure 0, which is then necessarily Lebesgue measurable. Hence, any Borel figure may be represented by an actual Borel set.

2.3.9 Definition. *Let $A \subset \mathbb{R}^n$ be a set. We define the* inner Jordan content *of A to be the greatest lower bound of the n-volume of all finite interior-disjoint unions of closed n-cubes contained in A, and its* outer Jordan content *to be the least upper bound of the n-dimensional volume of all finite interior-disjoint unions of closed n-cubes containing A.*

2.3.10 Definition. *We say that A is* Jordan measurable *if its inner and outer Jordan contents are equal, in which case we call it simply the* Jordan content *of A.*

2.3.11 Definition. *Let* Jordan_n *be the class of figures on $\mathbb{R}^n$ whose figure representatives are the Jordan measurable sets, and whose ideal of null sets is the collection of sets with Jordan content 0.*

2.3.12 Proposition. *We have* $\mathrm{Alg}_n \leq \mathrm{Jordan}_n \leq \mathrm{Borel}_n$ *for all n.*

Proof. We will only sketch the idea of the proof. Citations are supplied where the reader might be interested in further detail. For $A \subset \mathbb{R}^n$ it holds that A is Jordan measurable if and only if it is Lebesgue measurable and its (topological) boundary has Lebesgue measure 0. Another fact is that if A is Jordan measurable, then its Lebesgue measure and Jordan content agree. In particular, sets with Jordan content 0 have Lebesgue measure 0, so $\mathrm{Jordan}_n \leq \mathrm{Borel}_n$ (see [Enc]). Because semi-algebraic sets have boundaries with Lebesgue measure 0 (as in the proof of Proposition 2.3.8, this follows from [BCR98] Theorem 2.3.6) it follows also that $\mathrm{Alg}_n \leq \mathrm{Jordan}_n$. □

2.4 Subfigure correspondences and scissors congruence

In order to define scissors congruence as a relation between figures, we need to be sure that the transformations we allow send figures to figures.

2.4.1 Definition. *Let G be a subgroup of the group of all bijections $g\colon X \to X$. If $\mathcal{F}$ is a class of figures on X, we say that G is* measurable *with respect to $\mathcal{F}$ if for all $g \in G$ we have $g(A) \in \mathcal{R}(\mathcal{F})$ and $g(N) \in \mathcal{N}(\mathcal{F})$ for all $A \in \mathcal{R}(F)$ and $N \in \mathcal{N}(\mathcal{F})$.*

Because G is a group we know that it must also hold that $g^{-1}(A) \in \mathcal{R}(F)$ for all $A \in \mathcal{R}(F)$ (and analogously for null sets). In words, G is $\mathcal{F}$-measurable if all of its transformations $g \in G$ both preserve and reflect figure representatives and null sets.

2.4.2 Proposition. *If G is an $\mathcal{F}$-measurable group of transformations, then the action of G on figures $[A] \in \mathcal{F}$ defined as $g([A]) = [g(A)]$ is well-defined.*

Proof. Let $A, B \in \mathcal{R}(\mathcal{F})$ represent the same figure. We have $A \triangle B \in \mathcal{N}(\mathcal{F})$, so $g(A) \triangle g(B) = g(A \triangle B) \in \mathcal{N}(\mathcal{F})$. □

2.4.3 Proposition. *If a figure P decomposes as $P = P_1 \sqcup \cdots \sqcup P_n$ and $g \in G$ is some transformation, then $g(P)$ decomposes as $g(P) = g(P_1) \sqcup \cdots \sqcup g(P_n)$.*

Proof. Let $A, A_1, \ldots, A_n \in \mathcal{F}$ be figure representatives of $P, P_1, \ldots, P_n$ such that $A = A_1 \cup \cdots \cup A_n$. Such a union always exists, but note that the A_i are not necessarily properly disjoint. Rather, $A_i \cap A_j$ is a null set whenever $i \neq j$. We have

$$\begin{aligned} g(P) = [g(A)] &= [g(A_1 \cup \cdots \cup A_n)] = [g(A_1) \cup \cdots \cup g(A_n)] \\ &= [g(A_1)] \cup \cdots \cup [g(A_n)] = g(P_1) \cup \cdots \cup g(P_n). \end{aligned}$$

Moreover, if $i \neq j$ then

$$g([A_i]) \cap g([A_j]) = [g(A_i) \cap g(A_j)] = [g(A_i \cap A_j)] = [\emptyset].$$

The second-to-last equality holds because g is injective, and the last one holds because $A_i \cap A_j$ is a null set, and this is preserved by g. □

This property of transformations is important; any transformation carries subfigures to subfigures, and their configurations relative to each other are preserved as well. We take this property as a primitive notion of morphism between figures. First, we need a bit of algebraic machinery.

2.4.4 Definition. *Let X be a set. A* small category *$\mathscr{C}$ on X consists of the following data:*

* *For all $x, y \in X$, a set $\mathrm{Hom}(x, y)$ of* morphisms *from x to y,*
* *For all $x \in X$, a distinguished element $1_x \in \mathrm{Hom}(x, x)$,*
* *For all $x, y, z \in X$, an operation $\circ\colon \mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, z)$, called* composition,

which are required to satisfy

* *Each morphism ϕ belongs to a unique $\mathrm{Hom}(x, y)$,*
* *For all $\phi \in \mathrm{Hom}(x, y)$, $1_y \circ \phi = \phi \circ 1_x = \phi$,*

* *For all $x, y, z, w \in X$ and $\phi \in \mathrm{Hom}(x, y)$, $\psi \in \mathrm{Hom}(y, z)$, and $\chi \in \mathrm{Hom}(z, w)$, $(\chi \circ \psi) \circ \phi = \chi \circ (\psi \circ \phi)$.*

A groupoid *$\mathscr{G}$ is a small category that also satisfies the following:*

* *For all $x, y \in X$ and $\phi \in \mathrm{Hom}(x, y)$ there exists a $\psi \in \mathrm{Hom}(y, x)$ such that $\psi \circ \phi = 1_x$ and $\phi \circ \psi = 1_y$. It follows that this ψ is unique and we denote it by ϕ^{-1}.*

2.4.5 Definition. *If $\mathscr{G}$ is a groupoid, we will write $\phi \in \mathscr{G}$ to mean that ϕ is some morphism of $\mathscr{G}$, so there are $x, y \in X$ such that $\phi \in \mathrm{Hom}(x, y)$. Because* Hom*-sets are mutually disjoint, these x and y will be unique, and we will call them the 'domain' and 'codomain' of ϕ respectively. So, if $\phi, \psi \in \mathscr{G}$, then $\psi \circ \phi$ is 'defined' if and only if the domain of ψ equals the codomain of ϕ.*

If it is unclear from the context what category a given Hom-set belongs to, we will write $\mathrm{Hom}_{\mathscr{C}}(x, y)$ and $\mathrm{Hom}_{\mathscr{D}}(x, y)$, etc. to distinguish them.

2.4.6 Definition. *A* subgroupoid *of a groupoid $\mathscr{G}$ on a set X consists of a subset $Y \subset X$ and a groupoid $\mathscr{H}$ on Y such that $\mathrm{Hom}_{\mathscr{H}}(x, y) \subset \mathrm{Hom}_{\mathscr{G}}(x, y)$ for all $x, y \in Y$. A subgroupoid is called* wide *if $Y = X$, and it is called* full *if $\mathrm{Hom}_{\mathscr{H}}(x, y) = \mathrm{Hom}_{\mathscr{G}}(x, y)$ for all $x, y \in Y$.*

Note that a full subgroupoid is uniquely determined by its subset of elements $Y \subset X$. Now we can define the 'data type' of our geometric transformations.

2.4.7 Definition. *Let P be a figure. We let $\mathcal{S}(P)$ be the set of all subfigures of P. A* subfigure correspondence *between figures P and Q is a bijection $\phi\colon \mathcal{S}(P) \to \mathcal{S}(Q)$ that preserves finite disjoint unions, so*

* *$\phi([\emptyset]) = [\emptyset]$, and*
* *$\phi(P_1 \sqcup \cdots \sqcup P_n) = \phi(P_1) \sqcup \cdots \sqcup \phi(P_n)$ for all disjoint $P_1, \ldots, P_n \in \mathcal{S}(P)$.*

2.4.8 Definition. *If we define for any two figures P, Q the set $\mathrm{Hom}(P, Q)$ to consist of all subfigure correspondences between P and Q, and we compose correspondences as functions, then this will give us a groupoid. This is the* groupoid of all subfigure correspondences*, which we denote $\mathscr{A}(\mathcal{F})$, or simply $\mathscr{A}$ if the class of figures is clear from context.*

Proof. We need to show that the composition of two subfigure correspondences is again a subfigure correspondence, and the same for a subfigure correspondence's inverse function. The former is trivially verified simply by writing out the definition. For the latter, let $\phi\colon \mathcal{S}(P) \to \mathcal{S}(Q)$ be a subfigure correspondence, and let $\phi^{-1}\colon \mathcal{S}(Q) \to \mathcal{S}(P)$ be its inverse function. Then

$$\phi^{-1}([\emptyset]) = \phi^{-1}(\phi([\emptyset])) = [\emptyset],$$

and if $Q_1, \ldots, Q_n \in \mathcal{S}(Q)$ are disjoint, then

$$\phi^{-1}(Q_1 \sqcup \cdots \sqcup Q_n) = \phi^{-1}\left(\phi(\phi^{-1}(Q_1)) \sqcup \cdots \sqcup \phi(\phi^{-1}(Q_n))\right)$$

$$= \phi^{-1}\left(\phi\left(\phi^{-1}(Q_1) \sqcup \cdots \sqcup \phi^{-1}(Q_n)\right)\right) = \phi^{-1}(Q_1) \sqcup \cdots \sqcup \phi^{-1}(Q_n),$$

which is all that needed to be shown. □

Note that if $\phi\colon \mathcal{S}(P) \to \mathcal{S}(Q)$ is a subfigure correspondence, then by surjectivity there is some $P' \subset P$ with $\phi(P') = Q$, so $\phi(P) = \phi(P') \sqcup \phi(P \setminus P') = Q \sqcup \phi(P \setminus P')$, so because Q is the largest element of $\mathcal{S}(Q)$ it must be that $\phi(P) = Q$. Proofs of the following propositions are immediate.

2.4.9 Proposition. *Any transformation $g \in G$ induces a subfigure correspondence between P and $g(P)$ given by $R \mapsto g(R)$.* □

2.4.10 Proposition. *If $R \subset P$ is a subfigure and ϕ is a subfigure correspondence between P and Q, then the restriction of ϕ to $\mathcal{S}(R)$ is a subfigure correspondence between R and $\phi(R)$, which we write $\phi|_R$.* □

2.4.11 Definition. *A G-*congruence *between P and Q is a subfigure correspondence $\phi\colon \mathcal{S}(P) \to \mathcal{S}(Q)$ such that there is a transformation $g \in G$ with $\phi(R) = g(R)$ for all $R \subset P$. If such a congruence exists, then P and Q are said to be G-*congruent.

The congruences form a wide subgroupoid of $\mathscr{A}$. We could have just as well defined a 'congruence' between P and Q to be a transformation $g \in G$ such that $g(P) = Q$, but in light of the following definition we prefer to speak in terms of subfigure correspondences.

2.4.12 Definition. *A G-*scissors congruence *between P and Q is a subfigure correspondence $\phi\colon \mathcal{S}(P) \to \mathcal{S}(Q)$ such that there is some decomposition $P = P_1 \sqcup \cdots \sqcup P_n$ such that $\phi|_{P_i}\colon \mathcal{S}(P_i) \to \mathcal{S}(\phi(P_i))$ is a G-congruence for every i. If such a correspondence exists we say that P and Q are G-*scissors congruent *(or G-*s.c. *for short), which we will also denote as $P \simeq_G Q$.*

Commonly in the literature the relation of scissors congruence will be defined slightly differently, but it is no great difficulty to see that they are equivalent.

2.4.13 Proposition. *Two figures P, Q are s.c. if and only if there are decompositions $P = P_1 \sqcup \cdots \sqcup P_n$ and $Q = Q_1 \sqcup \cdots \sqcup Q_n$ such that for all $1 \leq i \leq n$ there is a $g_i \in G$ with $g_i(P_i) = Q_i$.* □

If the group of transformations G is clear from context, we will often speak of simply 'scissors congruence' or 's.c.', and write $P \simeq Q$. A decomposition $P = P_1 \sqcup \cdots \sqcup P_n$ such that $\phi|_{P_i}$ is a G-congruence for every i is said to 'witness' the scissors congruence ϕ.

2.4.14 Proposition. *The G-scissors congruences form a wide subgroupoid of $\mathscr{A}(\mathcal{F})$. We denote this groupoid by $\mathscr{S}_G(\mathcal{F})$, or simply by $\mathscr{S}$ if the other data is clear from context.*

Proof. All we need to show is that compositions and inverses of scissors congruences are themselves scissors congruences. First we treat the inverses. Let $\phi \in \mathscr{S}$ be a scissors congruence from P to Q. Let $P = P_1 \sqcup \cdots \sqcup P_n$ be a decomposition witnessing ϕ. We claim that the decomposition $Q = \phi(P_1) \sqcup \cdots \sqcup \phi(P_n)$ witnesses ϕ^{-1} as a scissors congruence. Let i be an arbitrary index. By definition there is some $g \in G$ such that $\phi|_{P_i}\colon \mathcal{S}(P_i) \to \mathcal{S}(\phi(P_i))$ is given by $R \mapsto g(R)$. Let $g^{-1}\colon X \to X$ be the inverse transformation of g. It is easily verified that $\phi^{-1}|_{\phi(P_i)}\colon \mathcal{S}(\phi(P_i)) \to \mathcal{S}(P_i)$ may be given by $R' \mapsto g^{-1}(R')$. It follows immediately that the inverse function of a scissors congruence is another scissors congruence; we simply invert the mapping piecewise.

Now let $\phi, \psi \in \mathscr{S}$ scissors congruences from P to Q and from Q to R respectively. Let $P = P_1 \sqcup \cdots \sqcup P_n$ and $Q = Q_1 \sqcup \cdots \sqcup Q_k$ witness ϕ and ψ respectively. Then $P = \phi^{-1}(Q_1) \sqcup \cdots \sqcup \phi^{-1}(Q_k)$ will be another decomposition of P. By the Refinement Lemma (Lemma 2.1.17), these two decompositions of P have a common refinement $P = S_1 \sqcup \cdots \sqcup S_\ell$. Let S_* be an arbitrary figure in this decomposition. There are indices i, j such that $S_* \subset P_i \cap \phi^{-1}(Q_j)$ (and if S_* is not empty these indices are unique). Let $g, h \in G$ be such that $\phi|_{P_i}$ and $\psi|_{Q_j}$ are given by $T \mapsto g(T)$ and $T' \mapsto h(T')$ respectively. It follows that $(\psi \circ \phi)|_{S_*}$ may be given by $T \mapsto (h \circ g)(T)$. Thus $P = S_1 \sqcup \cdots \sqcup S_\ell$ witnesses $\psi \circ \phi$ and we are done. □

2.4.15 Corollary. *Scissors congruence is an equivalence relation.* □

It's worth emphasizing in words how scissors congruences are composed; we reflect the decomposition of the intermediate figure along the first scissors congruence and then we take its common refinement with the decomposition of the first figure, after which the relevant congruences on the resulting pieces will be given by the compositions of the congruences that correspond to the original pieces.

2.4.16 Remark. *We define the 'data' of a scissors congruence to be its action on the subfigures of its domain. This is one of the ways of having scissors congruences retain their function-like characteristics in spite of the quotient abstraction that we use to define figures. Another option is to have a scissors congruence between P and Q be an 'almost-function', an equivalence class of partial functions $P \to Q$ that are defined and equal everywhere except on a null set. Because we mostly care about the structure of figures somewhat independent of their ambient space, we opt for the former.*

To define a notion of scissors congruence we need a set X, a class of figures $\mathcal{F}$ on X, and an $\mathcal{F}$-measurable group of transformations G of X. In all of the following examples we will take $X = \mathbb{R}^n$ to be some finite-dimensional Euclidean coordinate space. The dimension of the ambient space will be clear from the name of the class of figures, so in light of this, the following definition.

2.4.17 Definition. *A* scissors congruence structure *(or 's.c. structure') on a set X is a pair $(\mathcal{F}, G)$, where $\mathcal{F}$ is a class of figures on X and G is an $\mathcal{F}$-measurable group of transformations of X.*

2.4.18 Definition. *An s.c. structure $(\mathcal{F}, G)$ is said to be* finer *than another s.c. structure $(\mathcal{F}', G')$ if $\mathcal{F} \leq \mathcal{F}'$ and $G \subset G'$. This relation partially orders the s.c. structures on a given set.*

As is common, if $(\mathcal{F}, G)$ is finer than $(\mathcal{F}', G')$, then $(\mathcal{F}', G')$ is said to be *coarser* than $(\mathcal{F}, G)$. Recall that if $\mathcal{F} \leq \mathcal{F}'$ then we can identify the figures of $\mathcal{F}$ with a subset of $\mathcal{F}'$ as in Proposition 2.2.2. Let $(\mathcal{F}, G)$ be finer than $(\mathcal{F}', G')$ and let $P, Q \in \mathcal{F}$. Then P and Q being $(\mathcal{F}, G)$-scissors congruent implies that their images under the identification of $\mathcal{F}$ with a subset of $\mathcal{F}'$ are $(\mathcal{F}', G')$-scissors congruent. In other words, the former structure's s.c. relation $\simeq$ is 'finer' (i.e. more discerning) than the latter's.

2.4.19 Definition. *Let E_n (the 'Euclidean group' of dimension n) be the smallest group of transformations of $\mathbb{R}^n$ that contains all translations and all linear orthogonal transformations (equivalently, the group of all* isometries*). Let T_n be the group of all* translations *of $\mathbb{R}^n$.*

2.4.20 Definition. *An s.c. structure $(\mathcal{F}, G)$ on $\mathbb{R}^n$ is said to be* Euclidean *if $(\mathcal{F}, G) \leq (\mathrm{Borel}_n, E_n)$. It is called* regular Euclidean *if $(\mathrm{Orth}_n, T_n) \leq (\mathcal{F}, G) \leq (\mathrm{Bounded}(\mathrm{Jordan}_n), E_n)$.*

2.4.21 Definition. *We list the following notable s.c. structures.*

* (Pol_n, E_n), *the* scissors congruence of polytopes.
* (Circ, E_2), *the* scissors congruence of circle figures.
* (Orth_n, T_n), *the* scissors congruence of ortholinear figures.
* (Alg_n, E_n), *the* scissors congruence of semi-algebraic figures.

Of these, only the last is not regular Euclidean (because semi-algebraic figures may be unbounded). The scissors congruence of polytopes will be our primary motivating example.

3 The ordered additive monoid of scissors congruence classes

3.1 Some facts about partial commutative monoids

In the next section we will define an addition operation on scissors congruence classes of figures. To this end, we will first lay out the basics of partial commutative monoids and their relation to partially ordered groups.

3.1.1 Definition. *A* partial commutative monoid (PCM) *is a set M together with a constant $0 \in M$, a subset $D \subset M \times M$, and a binary operation $+\colon D \to M$, called* addition*. The statement that $(a, b) \in D$ we state by saying that '$a+b$ is defined'. For all $a, b, c \in M$ we require the following:*

* *(associativity) $(a + b) + c$ is defined if and only if $a + (b + c)$ is defined, and these expressions are equal if they are.*
* *(unity) $a + 0$ and $0 + a$ are defined and equal to a.*
* *(commutativity) if $a + b$ is defined then so is $b + a$ and they are equal.*

If $a + b$ is defined for all $a, b \in M$, then M is called a (total) commutative monoid.

In context we will not always specify assumptions like '$a+b$ is defined'. For example, an assumption such as 'assume that $a + b = c$' should formally be read as 'assume that $a + b$ is defined and that $a + b = c$'.

To keep things relatively short, we will assume familiarity with basic abstract-algebraic facts and terminology. For example, an *abelian group* is a commutative monoid in which every element has an additive inverse. Additionally, several upcoming propositions are listed without proof as their results are either well-known or easy to derive.

3.1.2 Definition. *In the following definitions, M is an arbitrary PCM.*

* *M is* sharp *if $a + b = 0$ implies $a = b = 0$ for all $a, b \in M$.*
* *M is* positive *if $a + b = a$ implies $b = 0$ for all $a, b \in M$.*
* *M is* cancellative *if $a + c = b + c$ implies $a = b$ for all $a, b, c \in M$.*

3.1.3 Definition. *Any PCM may be equipped with its* natural ordering $\leq$, *which is the relation on M defined as $a \leq b$ if and only if there exists $c \in M$ such that $a + c = b$.*

The natural ordering of a PCM is in general reflexive and transitive but not always antisymmetric, even if M is sharp.

3.1.4 Definition. *A* PCM homomorphism *is a function $f : M \to N$ between PCMs M, N such that $f(0_M) = 0_N$ and $f(a +_M b) = f(a) +_N f(b)$ for all $a, b \in M$ such that $a +_M b$ is defined..*

3.1.5 Definition. *A* monotone map *is a function $f : X \to Y$ between (pre)ordered sets $(X, \leq_X)$, $(Y, \leq_Y)$ such that $x \leq_X y$ implies $f(x) \leq_Y f(y)$ for all $x, y \in X$.*

3.1.6 Proposition. *Any PCM homomorphism $f : M \to N$ is monotone w.r.t. the natural orderings of M and N.* □

3.1.7 Definition. *Let M be a PCM. A* congruence *on M is an equivalence relation $\sim$ such that if $a \sim b$ and $c \sim d$, then $a + c \sim b + d$.*

3.1.8 Proposition. *Given any relation $\sim$ on a PCM M, there is a unique smallest congruence $\sim^*$ on M such that $a \sim b$ implies $a \sim^* b$. This is called the congruence* generated by $\sim$. □

3.1.9 Proposition. *A relation $\sim$ on a PCM M is a congruence if and only if there is a commutative monoid N and a PCM homomorphism $f : M \to N$ such that $a \sim b$ if and only if $f(a) = f(b)$.* □

In fact, we can make a stronger statement; the first of the universal properties we need.

3.1.10 Proposition. *If $\sim$ is a congruence on a PCM M, then there is a PCM $M/{\sim}$, unique up to isomorphism, and a homomorphism $q : M \to M/{\sim}$ with the following universal property. If N is a PCM and $f : M \to N$ is a homomorphism such that $a \sim b$ implies $f(a) = f(b)$, then there is a unique homomorphism $\bar{f} : M/{\sim} \to N$ such that $f = \bar{f} \circ q$.* □

The PCM $M/{\sim}$ is called the *quotient PCM* of M by $\sim$ and the homomorphism q is the *quotient map*. We can state this proposition in another way: there is a *natural bijection* between the set of PCM homomorphisms $M \to N$ that are constant on the $\sim$-congruence classes and the set of PCM homomorphisms $M/{\sim} \to N$. This is a good perspective on universal properties to keep in mind.

3.1.11 Proposition. *If M is a PCM, then there is a total commutative monoid $\overline{M}$, called the* enveloping monoid *of M, together with an injective PCM homomorphism $i : M \to \overline{M}$ such that if N is any total commutative monoid and $f : M \to N$ is a PCM homomorphism, then there is a* unique *PCM homomorphism $\bar{f} : \overline{M} \to N$ such that $f = \bar{f} \circ i$.*

Proof. See [Weh17], Section 2.1. □

The universal property necessitates that $\overline{M}$ is generated as a monoid by the image of M under i. More precisely, every element $a \in \overline{M}$ is of the form $a = \sum_j i(a_j)$ with $a_j \in M$ for all j. We think of $\overline{M}$ as the commutative monoid of *formal sums* of elements of M, where we identify formal sums with the result of that sum in M whenever this is defined.

3.1.12 Proposition. *For all $a, b \in M$ we have $a \leq b$ if and only if $i(a) \leq i(b)$.*

Proof. By Proposition 3.1.6 we have that $a \leq b$ implies $i(a) \leq i(b)$. For the other direction, define the total commutative monoid $M^{\sqcup\infty}$ as the set $M \cup \{\infty\}$ (where $\infty \notin M$) with addition $\oplus$ defined as $a \oplus b = a + b$ whenever $a, b \in M$ are such that $a + b$ is defined, and $a \oplus b = \infty$ if not. Additionally, $e \oplus \infty = \infty$ for all $e \in M^{\sqcup\infty}$. This operation is clearly associative and $0 \in M$ serves as its unit element.

The inclusion function $f\colon M \to M^{\sqcup\infty}$ is a PCM homomorphism. By the universal property of $\overline{M}$, there is a homomorphism $\bar{f}\colon \overline{M} \to M^{\sqcup\infty}$ such that $f = \bar{f} \circ i$. Assume that $a, b \in M$ are such that $a \nleq b$. There is no $c \in M$ such that $a + c = b$, hence there is no $e \in M^{\sqcup\infty}$ such that $f(a) \oplus e = f(b)$. It follows that $i(a) \nleq i(b)$ in $\overline{M}$, because otherwise we can again use Proposition 3.1.6 to derive $f(a) \leq f(b)$ in $M^{\sqcup\infty}$. ☐

This shows that in fact the homomorphism $i\colon M \to \overline{M}$ is an *order embedding*. We can now shamelessly identify the entire structure of M with its image in $\overline{M}$ under i.

3.1.13 Proposition. *If M is sharp, positive, or cancellative, then so is $\overline{M}$.* ☐

3.1.14 Proposition. *If M is a PCM, then there is a group M^{gr}, unique up to isomorphism, called the* Grothendieck group *of M, and a PCM homomorphism* $\mathrm{gr}\colon M \to M^{\mathrm{gr}}$ *such that if G is any abelian group and $f\colon M \to G$ is a PCM homomorphism, then there is a unique group homomorphism $\bar{f}\colon M^{\mathrm{gr}} \to G$ such that $f = \bar{f} \circ \mathrm{gr}$. If M is cancellative, then* gr *is injective.*

Proof. For total commutative monoids this result is well-known, and we will not prove it. We can extend the idea of the Grothendieck group to PCMs rather easily; simply take the Grothendieck group of the enveloping monoid of M. If G is an abelian group, then the enveloping monoid gives us a natural bijection between the PCM homomorphisms $M \to G$ and the homomorphisms $\overline{M} \to G$. Taking the Grothendieck group in turn gives us a bijection to the homomorphisms $\overline{M}^{\mathrm{gr}} \to G$, so if we compose these bijections we get the desired universal property. ☐

We will use the notation 'gr' to refer to both canonical PCM homomorphisms $\mathrm{gr}\colon M \to M^{\mathrm{gr}}$ and $\mathrm{gr}\colon \overline{M} \to M^{\mathrm{gr}}$.

In general M^{gr} will be generated by the elements of $\overline{M}$, in the sense that every element $x \in M^{\mathrm{gr}}$ is of the form $x = \mathrm{gr}(a) - \mathrm{gr}(b)$ for $a, b \in \overline{M}$. If M is cancellative (so $\overline{M}$ is too), however, we can identify $\overline{M}$ with the subset of M^{gr} consisting of the images of elements of $\overline{M}$ under gr. In this case we say that every element of M^{gr} is of the form $a - b$ with $a, b \in \overline{M}$.

3.1.15 Definition. *A* partially ordered abelian group *or* pogroup *is an abelian group G with a partial order $\preceq$ defined on it such that for all $x, y, z \in G$ we have that $x \preceq y$ implies $x + z \preceq y + z$ (this property is called* translation invariance*).*

3.1.16 Definition. *Let G be a pogroup. Its* positive cone *is the set $G_{\succeq 0} := \{x \in G : 0 \preceq x\}$.*

3.1.17 Proposition. *In a pogroup G, we have $x \leq y$ if and only if $y - x \in G_{\succeq 0}$.* ☐

The positive cone of a pogroup is closed under addition. In fact, the set $G_{\succeq 0}$ forms a commutative monoid.

3.1.18 Proposition. *If G is an abelian group and $P \subset G$ is a sharp submonoid, then there is a unique ordering $\preceq$ on G that makes G into a pogroup such that $P = G_{\succeq 0}$.*

Proof. For $x, y \in G$ we define $x \preceq y$ to mean that $y - x \in P$. Reflexivity follows from the fact that $0 \in P$. For transitivity, assume that $x \leq y$ and $y \leq z$. Then $z - x = (y - x) + (z - y) \in P$, so $x \leq z$. For antisymmetry, assume that $x \leq y$ and $y \leq x$. Then $x - y \in P$ and $y - x \in P$, so $0 = (x - y) + (y - x) \in P$. Because P is sharp it follows that $x - y = 0$, so $x = y$. For translation invariance let $x, y, z \in G$ with $x \preceq y$. Then $(y + z) - (x + z) = y - x \in P$, so $x + z \preceq y + z$. The fact that $\preceq$ is the unique such ordering follows from Proposition 3.1.17. ☐

3.1.19 Definition. *If M is a sharp cancellative PCM, then we define $\leq$ on M^{gr} to be the unique partial ordering on M^{gr} such that $M^{\mathrm{gr}}_{\geq 0} = \overline{M}$. We will also call this the* natural ordering *of M^{gr}.*

3.1.20 Proposition. *The ordering $\leq$ on M^{gr} extends the ordering $\leq$ on $\overline{M}$. That is, for $a, b \in \overline{M}$ we have that $a \leq b \in \overline{M}$ if and only if $a \leq b$ in M^{gr}.*

Proof. Recall that we identified $\overline{M}$ with its image in M^{gr} under the universal homomorphism $\mathrm{gr}\colon \overline{M} \to M^{\mathrm{gr}}$. The statement that '$b - a \in \overline{M}$' should technically be read as 'there is a $c \in \overline{M}$ such that $\mathrm{gr}(b) - \mathrm{gr}(a) = \mathrm{gr}(c)$'. Equivalently, $\mathrm{gr}(a) + \mathrm{gr}(c) = \mathrm{gr}(b)$. Because $\overline{M}$ is a total monoid it follows that $\mathrm{gr}(a + c) = \mathrm{gr}(b)$. Because M is cancellative we have that gr is injective, hence $a + c = b$ and we are done. □

This shows that we are free to use the symbol '$\leq$' for both the natural ordering on M and $\overline{M}$ as well as for this ordering on M^{gr} without risking any ambiguity.

3.2 Addition of s.c. classes

Recall from Corollary 2.4.15 that scissors congruence is an equivalence relation. We will study the set of all equivalence classes under scissors congruence.

3.2.1 Definition. *Let $\mathbf{SC}(\mathcal{F}, G)$ or simply by $\mathbf{SC}$ in context denote the set of $\simeq$-equivalence classes of $(\mathcal{F}, G)$. Its elements are called* s.c. classes *and we will denote them by the letters α, β, γ, etc. If $P \in \mathcal{F}$ is a figure, we will denote its s.c. class as $\mathrm{Scis}(P) \in \mathbf{SC}$.*

A natural observation to make is that disjoint unions of scissors congruent figures are scissors congruent, so $P_1 \sqcup Q_1 \simeq P_2 \sqcup Q_2$ if $P_1 \simeq P_2$ and $Q_1 \simeq Q_2$. Recall that in these expressions the symbol $\sqcup$ implicitly states that $P_1 \cap Q_1 = P_2 \cap Q_2 = [\emptyset]$.

3.2.2 Definition. *Let $\alpha, \beta \in \mathbf{SC}$ be s.c. classes. If there are figures $P \in \alpha$ and $Q \in \beta$ such that $P \cap Q = [\emptyset]$, then we define the* sum *of s.c. classes as $\alpha + \beta := \mathrm{Scis}(P \sqcup Q)$.*

Note that the sum of two s.c. classes is not always defined. As a simple example, consider the Euclidean s.c. structure on $\mathbb{R}$ where $\mathcal{R}(\mathcal{F}) = \{\mathbb{R}, \emptyset\}$ and $G = \{\mathrm{id}_{\mathbb{R}}\}$. We cannot find two disjoint representatives of $\mathrm{Scis}([\mathbb{R}])$, because $[\mathbb{R}]$ is its only representing figure.

3.2.3 Proposition. *Let $(\mathcal{F}, G)$ be an s.c. structure. Then $\mathbf{SC}$ is a sharp PCM with $+$ as addition and $0 = \mathrm{Scis}([\emptyset])$. We call $\mathbf{SC}$ the* natural monoid *of $(\mathcal{F}, G)$.* □

3.2.4 Definition. *An s.c. structure $(\mathcal{F}, G)$ is called* accommodating *if for any $\alpha, \beta \in \mathbf{SC}$ the sum $\alpha + \beta$ is defined.*

We will mostly restrict ourselves to studying accommodating s.c. structures, but this is no great loss. First of all, any s.c. structure on $\mathbb{R}^n$ such that all of its figures are bounded and such that its group of transformations contains all translations will be accommodating, as two figures can always be translated far enough away from each other so as to be disjoint.

3.2.5 Proposition. *If $(\mathcal{F}, G)$ is an s.c. structure on $\mathbb{R}^n$ such that $T_n \subset G$ and every figure in $\mathcal{F}$ is bounded, then $(\mathcal{F}, G)$ is accommodating.*

Proof. What we need to show is that if $P, Q \in \mathcal{F}$, then there are figures $P' \simeq P$ and $Q' \simeq Q$ such that $P' \cap Q' = [\emptyset]$. Because P and Q are bounded, there is a real number $r > 0$ such that P and Q both have representatives that are contained in an n-sphere with radius r centered at the origin. Let P' be the figure attained by translating P in some direction by a distance of r and let Q' be the figure attained by translating Q in the opposite direction by the same distance. Then P' and Q' are disjoint figures s.c. to P and Q respectively. □

3.2.6 Corollary. *Regular Euclidean s.c. structures are accommodating.* □

Secondly, we can perform a simple construction to freely extend any s.c. structure to an accommodating one in a way that loses little information about the original space, as in the following proposition.

3.2.7 Definition. *Let X be a set, and let $(\mathcal{F}, G)$ be an s.c. structure on X. We define the* accomodating extension *of $(\mathcal{F}, G)$ to be the s.c. structure $(\overline{\mathcal{F}}, \overline{G})$ on the set $\overline{X} := X \times \mathbb{N}$ as follows.*

We let $\mathcal{R}(\overline{\mathcal{F}})$ be generated by sets of the form $A \times \{n\}$ with $A \in \mathcal{R}(\mathcal{F})$ and $n \in \mathbb{N}$, and analogously for $\mathcal{N}(\overline{\mathcal{F}})$. We let $\overline{G}$ be the direct product of G and the symmetric group on $\mathbb{N}$. More precisely, the transformations $g \in \overline{G}$ are of the form $g(x, n) = (h(x), f(n))$, where $h \in G$ and $f : \mathbb{N} \to \mathbb{N}$ is a bijection. In particular we can take h to be the identity transformation, so the functions that simply permute the copies of X in $\overline{X}$ are transformations.

3.2.8 Proposition. *If $(\mathcal{F}, G)$ is an s.c. structure on X and $(\overline{\mathcal{F}}, \overline{G})$ is its accomodating extension, then $(\overline{\mathcal{F}}, \overline{G})$ is an accomodating s.c. structure and $\mathbf{SC}(\overline{\mathcal{F}}, \overline{G})$ is canonically isomorphic to $\overline{\mathbf{SC}(\mathcal{F}, G)}$, the enveloping monoid of $\mathbf{SC}(\mathcal{F}, G)$.*

Proof. Note that if $A \in \mathcal{R}(\overline{\mathcal{F}})$ is some figure representative, then there will be a finite subset $S \subset \mathbb{N}$ such that all of the n-coordinates of the points of A lie in S. It follows that if $A, B \in \mathcal{R}(\overline{\mathcal{F}})$ are disjoint figure representatives, then we can find some B' congruent to B that lives on a different finite subset of n-coordinates from A, so A and B' are disjoint. This shows that indeed $(\overline{\mathcal{F}}, \overline{G})$ is accommodating.

We can identify $\mathcal{F}$ with a subclass of $\overline{\mathcal{F}}$ by identifying X with the set $X \times \{0\}$. This gives us a PCM homomorphism $f : \mathbf{SC}(\mathcal{F}, G) \to \mathbf{SC}(\overline{\mathcal{F}}, \overline{G})$. Let $i : \mathbf{SC}(\mathcal{F}, G) \to \overline{\mathbf{SC}(\mathcal{F}, G)}$ be the canonical embedding. By the universal property of the enveloping monoid we get a homomorphism from $\bar{f} : \overline{\mathbf{SC}(\mathcal{F}, G)} \to \mathbf{SC}(\overline{\mathcal{F}}, \overline{G})$ such that $f = \bar{f} \circ i$. It is not too hard to see that $\mathbf{SC}(\overline{\mathcal{F}}, \overline{G})$ is generated as a monoid by the image of $\mathbf{SC}(\mathcal{F}, G)$ under f, so $\bar{f}$ is surjective.

The following argument for the injectivity of $\bar{f}$ is essentially a special case of the proof of Proposition 2.2.4 of [Weh17]. Note that figures $[A \times \{0\}]$ and $[B \times \{0\}]$ are $\overline{G}$-congruent if and only if A and B are G-congruent, so the same holds for scissors congruence. It follows that f is injective. Let $\alpha, \beta \in \overline{\mathbf{SC}(\mathcal{F}, G)}$ be such that $\bar{f}(\alpha) = \bar{f}(\beta)$, and let $(\alpha_j)_{j \in J}$, $(\beta_k)_{k \in K}$ be finite sequences such that $\alpha = \sum_j i(\alpha_j)$, $\beta = \sum_k i(\beta_k)$, and $\alpha_j, \beta_k \in \mathbf{SC}(\mathcal{F}, G)$ for all j, k.

We have that $\bar{f}(\alpha) = \sum_j f(\alpha_j) = \sum_k f(\beta_k) = \bar{f}(\beta)$. Let $P \in \overline{\mathcal{F}}$ be such that $\mathrm{Scis}(P) = \bar{f}(\alpha)$. There are finite sequences $(Q_j)_{j \in J}$ and $(R_k)_{k \in K}$ of figures of $\overline{\mathcal{F}}$ such that $\mathrm{Scis}(Q_j) = f(\alpha_j)$ for all $j \in J$, $\mathrm{Scis}(R_k) = f(\beta_k)$ for all $k \in K$, and $P = \bigsqcup_j Q_j = \bigsqcup_k R_k$. By the Refinement Lemma, there are $S_{j,k} \in \overline{\mathcal{F}}$ such that $Q_j = \bigsqcup_k S_{j,k}$ and $R_k = \bigsqcup_j S_{j,k}$ for all j, k. Each $\mathrm{Scis}(Q_j)$ and $\mathrm{Scis}(R_k)$ is in the image of f, so the $\mathrm{Scis}(S_{j,k})$ must also be in the image of f, because they are subfigures of the Q_j and R_k. So there are disjoint figures $S'_{j,k} \in \mathcal{F}$ such that $f(\mathrm{Scis}(S'_{j,k})) = \mathrm{Scis}(S_{j,k})$. We get that

$$f\Big(\mathrm{Scis}\Big(\bigsqcup_k S'_{j,k}\Big)\Big) = f\Big(\sum_k \mathrm{Scis}(S'_{j,k})\Big) = \sum_k f(\mathrm{Scis}(S'_{j,k})) = \mathrm{Scis}(Q_j) = f(\alpha_j),$$

$$f\Big(\mathrm{Scis}\Big(\bigsqcup_j S'_{j,k}\Big)\Big) = f\Big(\sum_j \mathrm{Scis}(S'_{j,k})\Big) = \sum_j f(\mathrm{Scis}(S'_{j,k})) = \mathrm{Scis}(R_k) = f(\beta_k),$$

for all j, k. By injectivity of f we have $\mathrm{Scis}\left(\bigsqcup_k S'_{j,k}\right) = \alpha_j$ and $\mathrm{Scis}\left(\bigsqcup_j S'_{j,k}\right) = \beta_k$, so

$$\alpha = \sum_j i(\alpha_j) = \sum_{j,k} i(\mathrm{Scis}(S'_{j,k})) = \sum_k i(\beta_k) = \beta,$$

so $\bar{f}$ is an isomorphism. $\square$

In other words, the enveloping monoid of the natural monoid of an s.c. structure is again the natural monoid of an s.c. structure. Note that if $(\mathcal{F}, G)$ was already accomodating then its natural monoid is canonically isomorphic to the natural monoid of its accomodating extension; no new information is gained.

3.2.9 Proposition. *Consider the natural ordering of* $\mathbf{SC}$. *We have $\alpha \leq \beta$ if and only if there are figures $P, Q \in \mathcal{F}$ such that* $\mathrm{Scis}(P) = \alpha$, $\mathrm{Scis}(Q) = \beta$, *and $P \subset Q$.*

Proof. Assume that there are such figures P and Q. Defining $\gamma := \mathrm{Scis}(Q \setminus P)$, we get $\alpha + \gamma = \mathrm{Scis}(P) + \mathrm{Scis}(Q \setminus P) = \beta$, so $\alpha \leq \beta$. Now assume that there is a $\gamma \in \mathbf{SC}$ such that $\alpha + \gamma = \beta$. Let P, Q, R be such that $\mathrm{Scis}(P) = \alpha$, $\mathrm{Scis}(Q) = \beta$, $\mathrm{Scis}(R) = \gamma$, and $P \cap R = [\emptyset]$. By assumption there is a scissors congruence ϕ from $P \sqcup R$ to Q. Then $\phi(P) \subset Q$, which is all we need. $\square$

3.2.10 Definition. *Let* IntRay *be the subclass of* UPol_1 *generated by the rays* (n, ∞) *for* $n \in \mathbb{Z}$. *If* T_e *is the infinite cyclic group of transformations consisting of translations by even integers, then* (IntRay, T_e) *is a Euclidean s.c. structure on* $\mathbb{R}$.

3.2.11 Proposition. *The natural ordering of* $\mathbf{SC}(\mathrm{IntRay}, T_e)$ *is not antisymmetric.*

Proof. In this scissors congruence we have $[(2n, \infty)] \simeq [(2k, \infty)]$ for any $n, k \in \mathbb{Z}$. However, $[(0, \infty)] \not\simeq [(1, \infty)]$. Letting $\alpha = \mathrm{Scis}([(0, \infty)])$, $\beta = \mathrm{Scis}([(1, \infty)])$, $\gamma = \mathrm{Scis}([(1, 2)])$, and $\delta = \mathrm{Scis}([(0, 1)])$, we get $\alpha + \gamma = \beta$ and $\beta + \delta = \alpha$, so $\alpha \leq \beta$ and $\beta \leq \alpha$. Because $\alpha \neq \beta$ we conclude that $\leq$ is not antisymmetric. $\square$

3.2.12 Definition. *If* α, β *are s.c. classes such that* $\alpha \leq \beta$ *and* $\beta \leq \alpha$ *we say that they are* order equivalent. *Conversely, we interpret* $\alpha < \beta$ *to mean that* $\alpha \leq \beta$ *and* $\beta \not\leq \alpha$ *(only if* $\leq$ *is antisymmetric is this equivalent to* $\alpha \leq \beta$ *and* $\alpha \neq \beta$*).*

3.2.13 Examples. *The following examples of s.c. structures and their natural monoids are relatively simple to characterize; we leave the verification to the reader.*

- *If* $\mathcal{F}$ *contains only the empty figure, then* $\mathbf{SC}$ *is the trivial monoid.*
- *Let* IntSeg *be the subclass of* Pol_1 *generated by the integer intervals* $[n, n+1]$ *for* $n \in \mathbb{Z}$. *Let* T_{int} *be the group of integer translations of* $\mathbb{R}$. *The figures of* IntSeg *may be classified up to scissors congruence by how many intervals of length* 1 *they contain. The natural monoid* $\mathbf{SC}$ *is isomorphic to the additive monoid of natural numbers* $\mathbb{N}$ *with its usual ordering.*
- *Consider* (Pol_1, E_1), *the scissors congruence of line segments. Any line segment* $P \in \mathrm{Pol}_1$ *is s.c. to a figure represented by an interval* $[0, \ell] \subset \mathbb{R}$, *where* $\ell \in \mathbb{R}_{\geq 0}$ *is the total length of the intervals that make up* P. *It follows that* $\mathbf{SC}$ *is isomorphic to the additive monoid of non-negative real numbers* $[0, \infty)$ *with its usual ordering.*

If our s.c. structures satisfy the regularity properties of being affine or Euclidean, we can relate the natural monoid of s.c. classes to the real numbers under addition via the usual Lebesgue measure.

3.2.14 Definition. *Let* $\mathbb{R}^n$ *be an arbitrary Euclidean space and let* μ *denote the Lebesgue measure on* $\mathbb{R}^n$. *If* $(\mathcal{F}, G)$ *is an s.c. structure on* $\mathbb{R}^n$ *with* $\mathcal{F} \leq \mathrm{Borel}_n$, *then we may define the Lebesgue measure on figures of* $\mathcal{F}$ *by setting* $\mu([A]) := \mu(A)$. *Moreover, if* $(\mathcal{F}, G)$ *is Euclidean, then we can define* μ *even on s.c. classes of* $\mathbf{SC}$ *by setting* $\mu(\mathrm{Scis}(P)) := \mu(P)$.

3.2.15 Proposition. *The measure functions given in the previous definition are well-defined given the respective assumptions on* $(\mathcal{F}, G)$. *Additionally, if* $P, Q \in \mathcal{F}$ *are disjoint then* $\mu(P \sqcup Q) = \mu(P) + \mu(Q)$, *and if* $\alpha, \beta \in \mathbf{SC}$ *are such that* $\alpha + \beta$ *is defined then* $\mu(\alpha + \beta) = \mu(\alpha) + \mu(\beta)$.

Proof. Assume that $(\mathcal{F}, G)$ is such that $\mathcal{F} \leq \mathrm{Borel}_n$. Every figure representative $A \in \mathcal{R}(\mathcal{F})$ is Lebesgue-measurable, and $\mathcal{N}(\mathcal{F})$ consists of exactly the figure representatives $N \in \mathcal{R}(\mathcal{F})$ such that $\mu(N) = 0$. From this it follows that $\mu([A]) := \mu(A)$ is well-defined. Recall that figures P and Q are disjoint if and only if they have representatives A and B such that A and B are disjoint as sets. It follows from the additivity of the Lebesgue measure that

$$\mu(P \sqcup Q) = \mu(A \cup B) = \mu(A) + \mu(B) = \mu(P) + \mu(Q).$$

If $(\mathcal{F}, G)$ is additionally Euclidean then G is contained in the Euclidean group; every $g \in G$ is an isometry. Isometries preserve Lebesgue measure. Recall that P and Q are scissors congruent (so $\mathrm{Scis}(P) = \mathrm{Scis}(Q)$) if and only if there are decompositions $P = P_1 \sqcup \cdots \sqcup P_n$ and $Q = Q_1 \sqcup \cdots \sqcup Q_n$ and transformations $g_1, \ldots, g_n \in G$ such that $g_i(P_i) = Q_i$ for all i. If this is the case then

$$\mu(P) = \mu(P_1 \sqcup \cdots \sqcup P_n) = \mu(P_1) + \cdots + \mu(P_n) = \mu(Q_1) + \cdots + \mu(Q_n) = \mu(Q),$$

so $\mu(\mathrm{Scis}(P)) := \mu(P)$ is well-defined.

Lastly, if $\alpha, \beta \in \mathbf{SC}$ are such that $\alpha + \beta$ is defined, then by definition there are disjoint figures P, Q with $\mathrm{Scis}(P) = \alpha$ and $\mathrm{Scis}(Q) = \beta$, so

$$\mu(\alpha + \beta) = \mu(P \sqcup Q) = \mu(P) + \mu(Q) = \mu(\alpha) + \mu(\beta). \qquad \square$$

In other words, if $(\mathcal{F}, G)$ is Euclidean then μ will be a PCM homomorphism from $\mathbf{SC}$ to the additive monoid $[0, \infty]$ of extended non-negative real numbers.

3.2.16 Proposition. *If $(\mathcal{F}, G)$ is Euclidean, then for $\alpha, \beta \in \mathbf{SC}$ it holds that if $\alpha < \beta$, then $\mu(\alpha) \leq \mu(\beta)$. This inequality is strict if $\mu(\beta) < \infty$.*

Proof. Let α, β be s.c. classes such that $\alpha < \beta$. By definition there is a $\gamma > 0$ such that $\alpha + \gamma = \beta$, so $\mu(\alpha) + \mu(\gamma) = \mu(\beta)$. It follows that $\mu(\alpha) \leq \mu(\beta)$. If $\mu(\beta) < \infty$, then $\mu(\alpha), \mu(\gamma) < \infty$ as well, so it follows that $\mu(\gamma) > 0$, hence $\mu(\alpha) < \mu(\beta)$. $\square$

Moreover, if $(\mathcal{F}, G)$ is *regular* Euclidean we get a much stronger result.

3.2.17 Theorem. *If $(\mathcal{F}, G)$ is regular Euclidean, then $\alpha, \beta \in \mathbf{SC}$ it holds that $\alpha < \beta$ if and only if $\mu(\alpha) < \mu(\beta)$.*

Proof. Assume that $\mu(\alpha) < \mu(\beta)$. Let $A, B \in \mathcal{R}(\mathcal{F})$ be bounded figure representatives such that $\mathrm{Scis}([A]) = \alpha$ and $\mathrm{Scis}([B]) = \beta$. For $p \in \mathbb{N}$, we will say that an 2^{-p}*-cubelet* is a figure representative in $\mathcal{R}(\mathrm{Orth}_n)$ of the form

$$\left[a_1 \cdot 2^{-p}, (a_1 + 1) \cdot 2^{-p}\right] \times \cdots \times \left[a_n \cdot 2^{-p}, (a_n + 1) \cdot 2^{-p}\right]$$

for $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. That is, the 2^{-p}-cubelets are exactly the closed $2^{-p} \times \cdots \times 2^{-p}$ cubes whose vertex coordinates are integer multiples of 2^{-p}. Note that any two distinct 2^{-p}-cubelets represent disjoint figures, and because G contains all translations, any two (figures represented by) 2^{-p}-cubelets are congruent. For an arbitrary figure representative $D \in \mathcal{R}(\mathcal{F})$, let $C_p^{\mathrm{in}}(D)$ denote the union of all 2^{-p}-cubelets that are contained in D, and let $C_p^{\mathrm{out}}(D)$ denote the union of all 2^{-p}-cubelets that intersect D. Note that for all $p \geq 0$ we have $C_p^{\mathrm{in}}(D) \subset D \subset C_p^{\mathrm{out}}(D)$. Moreover, if $p_1 \leq p_2$, then $C_{p_1}^{\mathrm{in}}(D) \subset C_{p_2}^{\mathrm{in}}(D)$ and $C_{p_1}^{\mathrm{out}}(D) \supset C_{p_2}^{\mathrm{out}}(D)$, because a 2^{-p}-cubelet is the interior-disjoint union of exactly 2^n-many $2^{-(p+1)}$-cubelets. If D is bounded, so are $C_p^{\mathrm{in}}(D)$ and $C_p^{\mathrm{out}}(D)$.

If $D \subset \mathbb{R}^n$ is Jordan measurable and bounded, then

$$\lim_{p \to \infty} \mu\left(C_p^{\mathrm{in}}(D)\right) = \mu(D) = \lim_{p \to \infty} \mu\left(C_p^{\mathrm{out}}(D)\right) < \infty,$$

so because $\mathcal{F} \leq \mathrm{Bounded}(\mathrm{Jordan}_n)$ this applies to our case. Let $\varepsilon > 0$ be a positive real number such that $\varepsilon < (\mu(B) - \mu(A))/2$. It follows from the above equation that there are p_A, p_B such that $\mu(C_{p_A}^{\mathrm{out}}(A)) \leq \mu(A) + \varepsilon$ and $\mu(C_{p_B}^{\mathrm{in}}(B)) \geq \mu(B) - \varepsilon$. Let $p := \max(p_A, p_B)$. We find that

$$\mu\left(C_p^{\mathrm{out}}(A)\right) \leq \mu\left(C_{p_A}^{\mathrm{out}}(A)\right) \leq \mu(A) + \varepsilon < \mu(B) - \varepsilon \leq \mu\left(C_{p_B}^{\mathrm{in}}(B)\right) \leq \mu\left(C_p^{\mathrm{in}}(B)\right).$$

However, we know that $C_p^{\mathrm{out}}(A)$ and $C_p^{\mathrm{in}}(B)$ are simply interior-disjoint unions of uniformly sized cubelets, each of measure 2^{-pn}, so the above inequality shows that $C_p^{\mathrm{in}}(B)$ must simply be made out of strictly *more* cubelets. Because G contains all translations it follows that there is a scissors congruence between $[C_p^{\mathrm{out}}(A)]$ and a strict subfigure of $[C_p^{\mathrm{in}}(B)]$, so

$$\alpha \leq \mathrm{Scis}\left([C_p^{\mathrm{out}}(A)]\right) < \mathrm{Scis}\left([C_p^{\mathrm{in}}(B)]\right) \leq \beta. \qquad \square$$

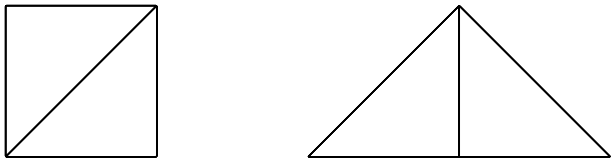

Figure 2: A simple scissors congruence.

3.3 Regularity properties and Zylev's theorem

Let us think again about the early geometer. They have shown that a square is scissors congruent to an isosceles right triangle whose perpendicular sides are the same length as the square's diagonal, from which they conclude that the two figures have the same area content (see Figure 2).

They conceive of another method to show that two figures are equal in area. If we have two figures P, Q and we can find a third figure T disjoint from both such that $P \sqcup T \simeq Q \sqcup T$ then we can also conclude that P and Q have equal area. Indeed, Euclid uses this method to show that two parallelograms of equal height on equal bases have the same area ([Euc], Book 1, Proposition 35). We can also use this technique to prove the Pythagorean theorem, as in Figure 3. Thinking of the area content of a figure as a real number value, this argument makes an appeal to the cancellative property of the additive monoid of non-negative real numbers. When does this same property hold for the natural monoid **SC** of scissors congruence classes? As it turns out, if we impose some fairly weak conditions on our scissors congruence relation we can give a constructive method for turning such a proof of area equality into a scissors congruence.

3.3.1 Definition. *Two figures P and Q are said to be* directly equicomplementable *if there are s.c. figures $T_1 \simeq T_2$ such that $P \sqcup T_1 \simeq Q \sqcup T_2$. They are* equicomplementable *if there is a finite sequence of figures $R_1, \dots, R_n$ such that $R_1 = P$, $R_n = Q$, and R_k is directly equicomplementable with R_{k+1} for all $1 \leq k < n$. In short, equicomplementability is the transitive closure of direct equicomplementability.*

Note that scissors congruent figures are directly equicomplementable (where $T_1 = T_2 = [\emptyset]$), so equicomplementability is a priori a coarser relation.

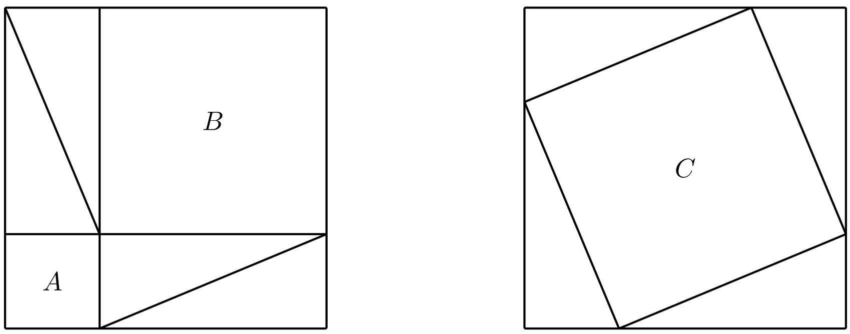

Figure 3: A "visual proof" of the Pythagorean theorem. It shows that the sum of the squares of the right sides of a right triangle $A \sqcup B$ is directly equicomplementable with the square of the hypotenuse C. Note that $A \sqcup B$ and C are directly equicomplementable even if $G = T_2$, the group of translations of the plane.

3.3.2 Definition. *An s.c. structure $(\mathcal{F}, G)$ is called* cancellative *if for all figures P, Q it holds that P and Q are equicomplementable if and only if they are scissors congruent.*

We will want to abstract cancellativity and the following related properties to the level of the natural monoid. We do this now explicitly for cancellativity but we will state the other equivalences without retracing a very similar proof.

3.3.3 Lemma. *An s.c. structure $(\mathcal{F}, G)$ is cancellative if and only if for all $\alpha, \beta, \gamma \in$ **SC** we have that if $\alpha + \gamma = \beta + \gamma$, then $\alpha = \beta$. In other words, $(\mathcal{F}, G)$ is cancellative if and only if* **SC** *is* *cancellative as a PCM.*

Proof. Assume that $(\mathcal{F}, G)$ is cancellative and let $\alpha, \beta, \gamma \in \mathbf{SC}$ be such that $\alpha + \gamma = \beta + \gamma$. Let $S \in \alpha + \gamma$ be arbitrary. By definition there are $P \in \alpha$, $Q \in \beta$, and $R_1, R_2 \in \gamma$ with $S \simeq P \sqcup R_1 \simeq Q \sqcup R_2$. Again by definition we find that P and Q are (directly) equicomplementable, so by cancellativity they are scissors congruent. Hence, $\alpha = \beta$.

Now assume that for all $\alpha, \beta, \gamma \in \mathbf{SC}$ we have that if $\alpha + \gamma = \beta + \gamma$, then $\alpha = \beta$. Let P, Q be equicomplementable figures. Let $R_1, \ldots, R_n$ be such that $P = R_1$, $Q = R_n$, and R_k and R_{k+1} are directly equicomplementable for all k. Let k be arbitrary and let T_1, T_2 be scissors congruent figures such that $R_k \sqcup T_1 \simeq R_{k+1} \sqcup T_2$. Let $\gamma = \mathrm{Scis}(T_1) = \mathrm{Scis}(T_2)$. We have

$$\mathrm{Scis}(R_k) + \gamma = \mathrm{Scis}(R_k \sqcup T_1) = \mathrm{Scis}(R_{k+1} \sqcup T_2) = \mathrm{Scis}(R_{k+1}) + \gamma,$$

so by assumption we have $\mathrm{Scis}(R_k) = \mathrm{Scis}(R_{k+1})$, so $R_k \simeq R_{k+1}$. It follows inductively that P and Q are scissors congruent, so $(\mathcal{F}, G)$ is cancellative. □

We define the following auxiliary properties.

3.3.4 Definition. *An s.c. structure $(\mathcal{F}, G)$ is* positive *if for all $P, Q \in \mathcal{F}$ we have that if $Q \subset P$ and $Q \simeq P$, then $Q = P$. In other words, no figure is s.c. with a strict subfigure.*

Indeed $(\mathcal{F}, G)$ being positive coincides with $\mathbf{SC}(\mathcal{F}, G)$ being positive in the sense of Definition 3.1.2.

3.3.5 Definition. *For figures P and Q, we say that P is* larger than twice Q *if there is a subfigure $Q' \subset P$ s.c. to Q and whenever $Q_1 \subset P$ is s.c. to Q then there exists a $Q_2 \subset P$ s.c. to Q such that $Q_1 \cap Q_2 = [\emptyset]$.*

3.3.6 Definition. *An s.c. structure $(\mathcal{F}, G)$ is* uniform *if for all non-empty figures P, Q there is a decomposition $Q = Q_1 \sqcup \cdots \sqcup Q_n$ such that P is larger than twice Q_i for all I.*

The algebraic equivalent of uniformity is that for all non-zero $\alpha, \beta \in \mathbf{SC}$ there are $\beta_1, \ldots, \beta_n \in \mathbf{SC}$ with $\beta = \beta_1 + \cdots + \beta_n$ such that for all i we have $\alpha \geq \beta_i$ and if γ_i is such that $\beta_i + \gamma_i = \alpha$, then $\beta_i \leq \gamma_i$.

Remark. *Algebraically it might seem sensible to say that α is larger than twice β if $\alpha \geq \beta + \beta$. This is a strictly weaker property than the one we have defined. Consider the additive monoid of cardinal numbers no greater than $\aleph_0$. In the weaker definition we have that $\aleph_0$ is larger than twice itself, because $\aleph_0 + \aleph_0 = \aleph_0$. In the stronger definition it is not, because we have $\aleph_0 + 0 = \aleph_0$, yet $\aleph_0 \not\leq 0$.*

The following theorem originated in [Zyl65].

3.3.7 Zylev's Theorem. *If an s.c. structure is positive and uniform then it is cancellative.*

We will adapt the proof of Zylev's Theorem from [Sah79]. This proof rests upon the following somewhat technical lemma, which holds without any assumptions on the s.c. structure.

3.3.8 Lemma. *If P and Q are figures that are directly equicomplementable in such a way that there are figures $R_1, \ldots, R_t, S_1, \ldots, S_t$ such that $P \sqcup R_1 \sqcup \cdots \sqcup R_t \simeq Q \sqcup S_1 \sqcup \cdots \sqcup S_t$, and $R_i \simeq S_i$ for all i, and such that P is larger than twice each R_i, then P and Q are scissors congruent.*

Proof. Let $P_\dagger$ and $Q_\dagger$ be figures. We will call a tuple $P, Q, R_1, \ldots, R_t, S_1, \ldots, S_t$ such that $P_\dagger \simeq P$, $Q_\dagger \simeq Q$, $P \sqcup R_1 \sqcup \cdots \sqcup R_t \simeq Q \sqcup S_1 \sqcup \cdots \sqcup S_t$, and P is larger than twice each R_i a *complementation of length t* for $P_\dagger$ and $Q_\dagger$. The figures $P \sqcup R_1 \sqcup \cdots \sqcup R_t$ and $Q \sqcup S_1 \sqcup \cdots \sqcup S_t$ we will call the *dominant figures* for this complementation. A complementation of length 0 is just a pair P, Q such that $P_\dagger \simeq P \simeq Q \simeq Q_\dagger$, so it guarantees a scissors congruence between $P_\dagger$ and $Q_\dagger$. The statement of the lemma is exactly that if $P_\dagger$ and $Q_\dagger$ have some complementation of length t, then they are scissors congruent. We will show that if $P_\dagger$ and $Q_\dagger$ have a complementation of length $t > 0$, then there is also a complementation of length $t - 1$, so by induction it will follow that $P_\dagger$ and $Q_\dagger$ are

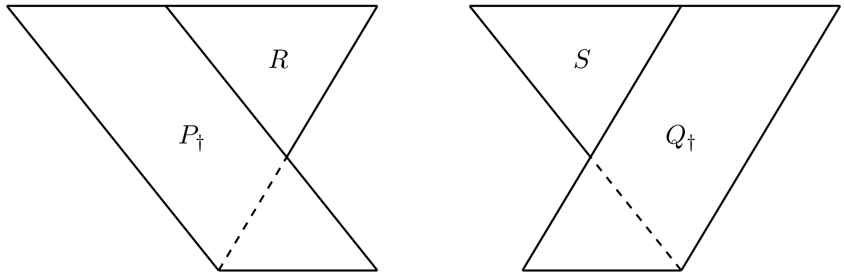

Figure 4: We will demonstrate the algorithm of Lemma 3.3.8 using this example, Proposition 35 from Book 1 of Euclid's Elements [Euc]. The parallellograms $P_\dagger$ and $Q_\dagger$ are on equal bases and of equal heights. The congruent triangles R and S are added to them, resulting in scissors congruent figures $P_\dagger \sqcup R$ and $Q_\dagger \sqcup S$. This is a complementation of length 1, because P and Q are larger than twice R and S. The dashed line indicates where to cut the figures to make a decomposition that witnesses the scissors congruence.

scissors congruent. The following algorithm has 3 steps. The process is illustrated by the Figures 4 through 8.

Step 1: We want to have that $P \sqcup R_1 \sqcup \cdots \sqcup R_t = Q \sqcup S_1 \sqcup \cdots \sqcup S_t$ (i.e. we want the dominant figures to be *equal*, rather than merely scissors congruent). By assumption, there is a scissors congruence ϕ from $P \sqcup R_1 \sqcup \cdots \sqcup R_t$ to $Q \sqcup S_1 \sqcup \cdots \sqcup S_t$, so by the definition of subfigure correspondences we get

$$\phi(P) \sqcup \phi(R_1) \sqcup \cdots \sqcup \phi(R_t) = Q \sqcup S_1 \sqcup \cdots \sqcup S_t.$$

If we simply replace P with $\phi(P)$ and each R_i with $\phi(R_i)$ then we get a complementation of length t such that the dominant figures are equal. Let W be the dominant figure.

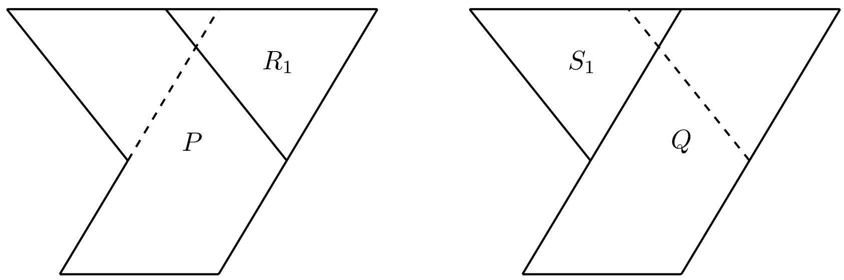

Figure 5: After Step 1 of Lemma 3.3.8, we have a new complementation of length 1 for $P_\dagger$ and $Q_\dagger$, where the dominant figures are equal. We have $P = \phi(P_\dagger)$ and $Q = Q_\dagger$. The dashed lines show that the figures $R_1 = \phi(R)$ and $S_1 = S$ are not disjoint, so we need to apply Step 2.

Step 2: If R_1 and S_1 are disjoint (or if another pair R_i and S_i are disjoint, rename them to R_1 and S_1), then go to Step 3. If not, let $R = R_1 \sqcup \cdots \sqcup R_t$. By assumption P is larger than twice S_1, so there is a figure $T \subset P \setminus S_1$ that is scissors congruent to S_1, so also to R_1. Let ψ be a fixed scissors congruence between S_1 and T. Let $U = S_1 \cap R$. The figure U is the 'overlap' that we need to eliminate in order to reduce the length of the complementation. We have a decomposition $U = U_1 \sqcup \cdots \sqcup U_t$ where $U_i = S_1 \cap R_i$ for all i. Define $U_i' = \psi(U_i)$ for each i. Additionally, define $U' = \psi(U) = U_1' \sqcup \cdots \sqcup U_t'$.

So, we have a new decomposition of the dominant figure $W = P' \sqcup R_1' \sqcup \cdots \sqcup R_t'$ where $P' = (P \setminus U') \sqcup U$ and $R_i' = (R_i \setminus U_i) \sqcup U_i'$. In other words, to make P' from P we subtract a subfigure s.c. to $U = S_1 \cap R = S_1 \setminus P$ and attach it to the R_i, after which we remove U from R and attach it to P. Crucially, we now have that S_1 and R_1' are disjoint, because $S_1 \subset P'$. We now rename P' to P and the R_i' to R_i, as they will form the parts of our new complementation.

Step 3: We know that R_1 and S_1 are disjoint. We also know that they are scissors congruent, so

$$W \setminus R_1 = S_1 \sqcup (W \setminus (R_1 \sqcup S_1)) \simeq R_1 \sqcup (W \setminus (R_1 \sqcup S_1)) = W \setminus S_1.$$

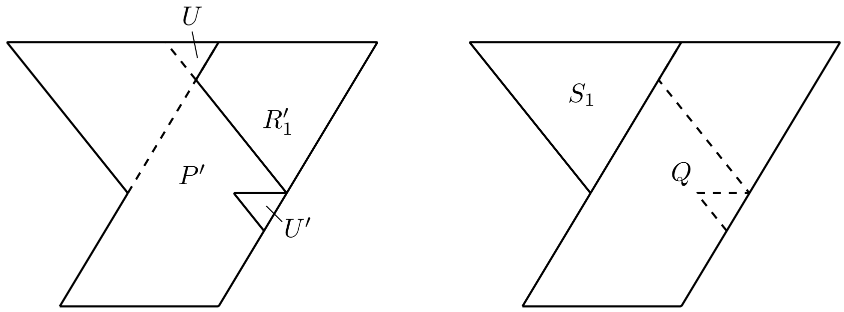

Figure 6: After Step 2 of Lemma 3.3.8, we have a complementation of length 1 for $P_\dagger$ and $Q_\dagger$ where the added figures R_1' and S_1 are disjoint. We can now apply Step 3.

In other words, $P, Q, R_2, \ldots, R_t, S_2, \ldots, S_t$ is a complementation of length $t-1$ for $P_\dagger$ and $Q_\dagger$. This is all that needed to be shown. □

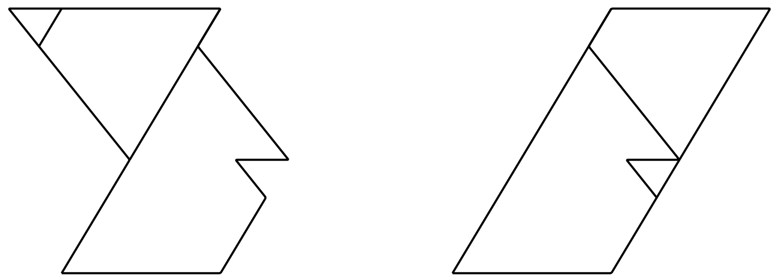

Figure 7: After Step 3 of Lemma 3.3.8, we have a scissors congruence from a figure that is scissors congruent with $P_\dagger$ to $Q_\dagger$.

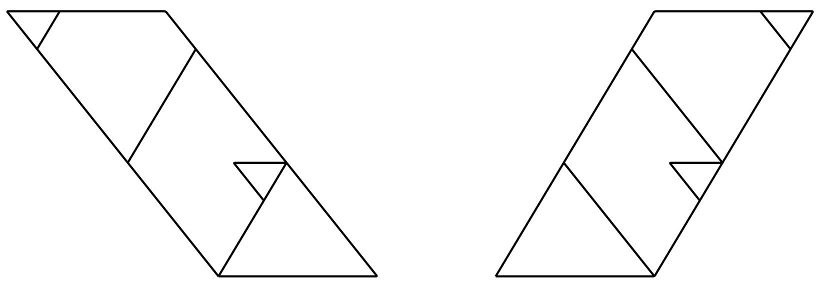

Figure 8: Reflecting back the scissors congruence of Figure 7 to $P_\dagger$, we get a pair of decompositions that witnesses the fact that $P_\dagger \simeq Q_\dagger$. Note that, in the constructed scissors congruence, the small triangle in the central hexagon does not get exchanged with the corresponding small triangle in the other parallellogram. Rather, the central triangle gets swapped with the one in the corner. This is not necessary to make a scissors congruence; it is an artifact of the construction.

Proof. (Zylev's Theorem) Assume $(\mathcal{F}, G)$ is positive and uniform. By transitivity, it is enough to show that if P and Q are directly equicomplementable figures, then they are scissors congruent.

Let P, Q, R, S be figures such that $R \simeq S$ and $P \sqcup R \simeq Q \sqcup S$. If P is empty then R is s.c. with both S and $Q \sqcup S$, so by positivity it must be that Q is empty, so P and Q are scissors congruent. Assume P is nonempty. By uniformity (and taking the common refinement of the decomposition guaranteed by uniformity with a decomposition witnessing the scissors congruence) there are decompositions $R = R_1 \sqcup \cdots \sqcup R_t$ and $S = S_1 \sqcup \cdots \sqcup S_t$ with R_i congruent to S_i for all i, and P larger than twice each R_i. These are precisely the hypotheses for Lemma 3.3.8, so we conclude that P and Q are scissors congruent. □

Note that the proof of Lemma 3.3.8 is rather geometric, in the sense that we manipulate individual figures in such a way as to make them disjoint. This seems to suggest that there 'is no algebraic proof' of Zylev's theorem. This may be formalized in the following theorem, which states that a similar algebraic result is simply *not true*.

3.3.9 Theorem. *There is a sharp commutative monoid M equipped with its natural ordering such that*

- *for all $\alpha, \beta \in M$, if $\alpha + \beta = \alpha$, then $\beta = 0$.*
- *if $\alpha, \beta \in M$ are non-zero, then there are $\beta_1, \dots, \beta_n \in M$ such that $\beta = \beta_1 + \cdots + \beta_n$ and for all i we have $\alpha \geq \beta_i$ and if γ_i is such that $\beta_i + \gamma_i = \alpha$, then $\beta_i \leq \gamma_i$.*
- *there are $\alpha, \beta, \gamma \in M$ such that $\alpha + \gamma = \beta + \gamma$, but $\alpha \neq \beta$.*

Proof. Let $M := \big((0, \infty) \times (0, 1]\big) \cup \{0\}$, with addition defined on nonzero elements as

$$(x, y) + (z, w) := (x + z, \min(1, y + w)).$$

Then M is a sharp commutative monoid.

For positivity, let $\alpha, \beta \in M$ be such that $\alpha + \beta = \alpha$. If $\beta = 0$ then we have shown what we need to, and if $\alpha = 0$ then it follows immediately that $\beta = 0$, so we can assume for the sake of contradiction that both α and β are nonzero. Because they are non-zero they are of the form $\alpha = (x, y)$ and $\beta = (z, w)$. We have

$$(x, y) = (x, y) + (z, w) = (x + z, \min(1, y + w)),$$

so $x = x + z$, which contradicts that $z > 0$. We conclude that $\alpha + \beta = \alpha$ implies $\beta = 0$.

For uniformity, let $\alpha, \beta \in M$ be non-zero. Again, let $\alpha = (x, y)$ and $\beta = (z, w)$. Let $m_1 := 2 \cdot \lceil \frac{z}{x} + 1 \rceil$ and $m_2 := 2 \cdot \lceil \frac{w}{y} + 1 \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function that maps a real number r onto the smallest integer that is not smaller than r. Let $n := \max(m_1, m_2)$ and define

$$\gamma := \left(\frac{z}{n}, \frac{w}{n} \right).$$

We have $\beta = \gamma + \cdots + \gamma$, where this is a sum of n terms. Moreover,

$$\gamma + \gamma = \left(\frac{2z}{n}, \min\left(1, \frac{2w}{n}\right) \right).$$

It follows that

$$\frac{2z}{n} = \frac{2z}{\max(2 \cdot \lceil \frac{z}{x} + 1 \rceil, 2 \cdot \lceil \frac{w}{y} + 1 \rceil)} \leq \frac{z}{\lceil \frac{z}{x} + 1 \rceil} < \frac{z}{\frac{z}{x}} = x,$$

and likewise

$$\min\left(1, \frac{2w}{n}\right) \leq \frac{2w}{n} < y.$$

Let $\delta := \left(x - \frac{z}{n}, y - \frac{w}{n}\right)$. Note that both coordinates are positive and the second is no greater than 1. We have $\gamma + \delta = \alpha$, hence $\gamma \leq \alpha$. Now let $\delta' \in M$ be an arbitrary element such that $\gamma + \delta' = \alpha$. All we need to show is that $\gamma \leq \delta'$. We know for a fact that δ' is non-zero, so let $\delta' = (v, u)$. Necessarily we have $v = x - \frac{z}{n}$. If $u < y - \frac{w}{n}$, then $\gamma + \delta' < \alpha$, so $u \in \left[y - \frac{w}{n}, 1\right]$. We have $\frac{w}{n} \leq y - \frac{w}{n}$, so certainly $\frac{w}{n} \leq u$. We conclude that $\gamma \leq \delta'$, hence α is larger than twice γ as required.

Finally, we have $(1, 1) + (1, 1) = (2, 1) = (1, \frac{1}{2}) + (1, 1)$, but $(1, 1) \neq (1, \frac{1}{2})$, so M is not cancellative. □

3.3.10 Corollary. *There are sharp commutative monoids that are not the natural monoid of some scissors congruence structure.* □

As an aside, this is far from the easiest way to prove Corollary 3.3.10. Any natural monoid of an s.c. structure is what's known as a *refinement monoid* (see [Weh17]); such commutative monoids satisfy the property that if $x_1 + \cdots + x_n = y_1 + \cdots + y_k$, then there exist $z_{i,j}$ such that $x_i = \sum_j z_{i,j}$ and $y_j = \sum_i z_{i,j}$ for all $1 \le i \le n$ and $1 \le j \le k$.

Clearly $(\mathcal{F}, G)$ being positive is necessary for the Zylev Property to hold, as any witness to the failure of positivity also witnesses the failure of cancellativity. Uniformity is not strictly necessary, as one of the example s.c. structures in 3.2.13 (the scissors congruence of integer intervals) is not uniform but its natural monoid is isomorphic to the monoid of natural numbers $\mathbb{N}$, which is cancellative. In fact, we have the following alternative conditions for cancellativity to hold.

3.3.11 Lemma. *If $(\mathcal{F}, G)$ is positive and $\alpha, \beta \in \mathbf{SC}$ are distinct s.c. classes such that there exists a $\gamma \in \mathbf{SC}$ with $\alpha + \gamma = \beta + \gamma$, then α and β are necessarily incomparable.*

Proof. We can prove this algebraically. Assume that α, β, γ be such that α and β are comparable and $\alpha + \gamma = \beta + \gamma$. We may assume without loss of generality that $\beta = \alpha + \delta$. We then have $\alpha + \gamma = \alpha + \delta + \gamma$, so $\alpha + \gamma$ absorbs δ. By positivity it follows that $\delta = 0$, so $\alpha = \beta$. It follows that if α and β distinct, then they must be incomparable. □

3.3.12 Corollary. *If $(\mathcal{F}, G)$ is positive and $\mathbf{SC}$ is linearly ordered by the natural ordering, then it is cancellative.* □

3.3.13 Proposition. *If $(\mathcal{F}, G)$ is positive, then $\le$ is antisymmetric, but the converse does not always hold.*

Proof. If α, β are such that $\alpha \le \beta$ and $\beta \le \alpha$, then there are γ, δ such that $\alpha + \gamma = \beta$ and $\beta + \delta = \alpha$. It follows that $\alpha + \gamma + \delta = \alpha$, so $\gamma + \delta = 0$, and by sharpness it follows that $\gamma = \delta = 0$, so $\alpha = \beta$.

For an example where the converse does not hold, let $\mathcal{R}(\mathcal{F})$ be the smallest σ-algebra on $\mathbb{R}$ that contains the integer intervals $[n, n+1]$ with $n \in \mathbb{Z}$. Let $\mathcal{N}(\mathcal{F})$ be the set of all subsets of $\mathbb{Z}$. Let $S_{\mathbb{Z}}$ denote the symmetric group on $\mathbb{Z}$, i.e. the group of all bijections $\mathbb{Z} \to \mathbb{Z}$. We let G be the group of transformations on $\mathbb{R}$ defined as follows. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote its integer part and d_x its fractional part, so $x = \lfloor x \rfloor + d_x$, where $\lfloor x \rfloor \in \mathbb{Z}$ and $d_x \in [0, 1)$. We let G consist of all transformations g of the form $g(x) = \sigma(\lfloor x \rfloor) + d_x$, where $\sigma \in S_{\mathbb{Z}}$ is some permutation. In short, the elements of G simply permute the half-open integer intervals that make up $\mathbb{R}$.

Much like the natural numbers example from 3.2.13, the figures of $\mathcal{F}$ may be characterized by the cardinality of the set of integer intervals of length 1 they contain. In this case $\mathbf{SC}$ is isomorphic to the monoid of all cardinal numbers smaller than or equal to $\aleph_0$ under addition. Cardinals are partially ordered, but $\aleph_0 + \aleph_0 = \aleph_0$, so $(\mathcal{F}, G)$ is not positive. □

3.3.14 Proposition. *If $(\mathcal{F}, G)$ is accommodating, positive, and $\mathbf{SC}$ has a maximal element, then $\mathbf{SC} = \{0\}$.*

Proof. If some class α is a maximal element of $\mathbf{SC}$, then $\alpha + \alpha \le \alpha$, so by positivity we must have that $\alpha = 0$. However, because every element is comparable to 0, it must be that $\beta \le 0$ holds for every $\beta \in \mathbf{SC}$, so $\mathbf{SC} = \{0\}$. □

One reason that Zylev's theorem is so useful is that it shows that a lot of scissors congruence structures on familiar classes of figures are cancellative.

3.3.15 Theorem. *If $(\mathcal{F}, G)$ is a regular Euclidean s.c. structure on $\mathbb{R}^n$, then it is positive and uniform, hence cancellative.*

Proof. Recall that μ denotes the Lebesgue measure on $\mathbb{R}^n$, and that if $(\mathcal{F}, G)$ is Euclidean then it is well-defined on s.c. classes (Proposition 3.2.15). For positivity, note that $\mu(\alpha) = 0$ if and only if $\alpha = 0$, so we can 'pull back' the positivity of the real numbers to $\mathbf{SC}$. Explicitly, let $\alpha, \beta \in \mathbf{SC}$ be such that $\alpha + \beta = \alpha$. Then

$$\mu(\alpha) = \mu(\alpha + \beta) = \mu(\alpha) + \mu(\beta),$$

so $\mu(\beta) = 0$, hence $\beta = 0$ as required.

For uniformity, let $P, Q \in \mathcal{F}$ be non-empty. Because $(\mathcal{F}, G)$ is regular Euclidean, we know that Q is contained in a finite union of cubes of arbitrarily small size (recall the proof of Theorem 3.2.17). Taking the measure of these cubes to be smaller than $\mu(P)/2$, we find that we can decompose Q into figures $Q_1, \ldots, Q_n$ such that $\mu(Q_i) < \mu(P)/2$ for all i. By Theorem 3.2.17 it follows that in fact if R_i is such that $\mathrm{Scis}(Q_i) + \mathrm{Scis}(R_i) = \mathrm{Scis}(P)$, then $\mu(Q_i) < \mu(R_i)$, so $\mathrm{Scis}(Q_i) < \mathrm{Scis}(R_i)$ as required. □

Consider the following construction, which is used in [Jes68] and [Zak16] to analyze scissors congruence by algebraic means.

3.3.16 Definition. *Let $\mathcal{P}$ be the free abelian group generated by all figures $P \in \mathcal{F}$. Let $\mathcal{E}$ be the subgroup of $\mathcal{P}$ generated by $[\emptyset]$, elements of the form $(P \sqcup Q) - P - Q$ for $P, Q \in \mathcal{F}$ such that $P \cap Q = [\emptyset]$, and elements of the form $P - Q$ if P is congruent to Q. For $P \in \mathcal{F}$ let $[P]$ denote its congruence class in $\mathcal{P}/\mathcal{E}$ (where technically this is the congruence class of the generator of $\mathcal{P}$ corresponding to P).*

3.3.17 Proposition. *Let $(\mathcal{F}, G)$ be an s.c. structure. Then there is an isomorphism from $\mathcal{P}/\mathcal{E}$ to $\mathbf{SC}^{\mathrm{gr}}$ that sends $[P]$ to $\mathrm{gr}(\mathrm{Scis}(P))$ for all $P \in \mathcal{F}$.*

Proof. Let $f_1 \colon \mathcal{F} \to \mathcal{P}/\mathcal{E}$ be given by $f_1(P) = [P]$, and let $g_1 \colon \mathcal{F} \to \mathbf{SC}^{\mathrm{gr}}$ be given by $g_1(P) = \mathrm{gr}(\mathrm{Scis}(P))$. First we will show that if $P \simeq Q$, then $f_1(P) = f_1(Q)$.

First we show that if $P = P_1 \sqcup \cdots \sqcup P_n$ in $\mathcal{F}$, then $[P] = [P_1] + \cdots + [P_n]$. By induction it suffices to show this for the case $P = P_1 \sqcup P_2$. Note that

$$[P] - [P_1] - [P_2] = [(P_1 \sqcup P_2) - P_1 - P_2] = 0,$$

so $[P] = [P_1] + [P_2]$ as required.

Now we show that if $P \simeq Q$, then $f_1(P) = [P] = [Q] = f_1(Q)$. Let the decompositions $P = P_1 \sqcup \cdots \sqcup P_n$ and $Q = Q_1 \sqcup \cdots \sqcup Q_n$ be such that P_i is congruent to Q_i for all i. Then $[P_i] = [Q_i]$ for all i by definition, so

$$[P] = [P_1] + \cdots + [P_n] = [Q_1] + \cdots + [Q_n] = [Q].$$

So if $P \simeq Q$, then $f_1(P) = f_1(Q)$. It follows that this defines a function $f_2 \colon \mathbf{SC} \to \mathcal{P}/\mathcal{E}$ such that $f_2(\mathrm{Scis}(P)) = [P]$.

If $P, Q \in \mathcal{F}$ are disjoint, then

$$f_2(\mathrm{Scis}(P) + \mathrm{Scis}(Q)) = f(\mathrm{Scis}(P \sqcup Q)) = [P \sqcup Q] = [P] + [Q] = f_2(\mathrm{Scis}(P)) + f_2(\mathrm{Scis}(Q)),$$

so together with the fact that $f_2(0) = 0$ we find that f_2 is a PCM homomorphism $\mathbf{SC} \to \mathcal{P}/\mathcal{E}$. Because $\mathcal{P}/\mathcal{E}$ is a group, this in turn defines a homomorphism $f_3 \colon \mathbf{SC}^{\mathrm{gr}} \to \mathcal{P}/\mathcal{E}$ such that $f_3(\mathrm{gr}(\mathrm{Scis}(P))) = [P]$. This will be our isomorphism.

Now we show that g_1 defines its inverse. The group $\mathcal{P}$ is the free abelian group generated by the elements of $\mathcal{F}$, so its universal property gives us a homomorphism $g_2 \colon \mathcal{P} \to \mathbf{SC}^{\mathrm{gr}}$ such that $g_2(P) = \mathrm{gr}(\mathrm{Scis}(P))$. Now we show that if $E \in \mathcal{E}$, then $g_2(E) = 0$. If $E = [\emptyset]$, then $g_2(E) = \mathrm{gr}(\mathrm{Scis}([\emptyset])) = 0$ so we are done. If there are disjoint $P, Q \in \mathcal{F}$ such that $E = (P \cup Q) - P - Q$, then

$$g_2(E) = \mathrm{gr}(\mathrm{Scis}(P \sqcup Q)) - \Big(\mathrm{gr}(\mathrm{Scis}(P)) + \mathrm{gr}(\mathrm{Scis}(Q))\Big) = 0.$$

We find that g_2 maps the generators of $\mathcal{E}$ to 0, hence $\mathcal{E}$ is contained in the kernel of g_2. It follows that g_2 defines a homomorphism $g_3 \colon \mathcal{P}/\mathcal{E} \to \mathbf{SC}^{\mathrm{gr}}$ such that $g_3([P]) = \mathrm{gr}(\mathrm{Scis}(P))$.

For all $P \in \mathcal{F}$ we have $g_3(f_3(\mathrm{gr}(\mathrm{Scis}(P)))) = \mathrm{gr}(\mathrm{Scis}(P))$ and $f_3(g_3([P])) = [P]$, so because $\mathbf{SC}^{\mathrm{gr}}$ and $\mathcal{P}/\mathcal{E}$ are generated as groups by the $\mathrm{gr}(\mathrm{Scis}(P))$ and $[P]$ respectively it follows that f_3 and g_3 are inverse isomorphisms. □

3.3.18 Corollary. *If $(\mathcal{F}, G)$ is cancellative, then for all $P, Q \in \mathcal{F}$ we have $P \simeq Q$ if and only if $[P] = [Q]$ in $\mathcal{P}/\mathcal{E}$.*

Proof. Recall from Proposition 3.1.14 that $\mathbf{SC}$ embeds into $\mathbf{SC}^{\mathrm{gr}}$ if $\mathbf{SC}$ is cancellative. By the isomorphism between $\mathcal{P}/\mathcal{E}$ and $\mathbf{SC}^{\mathrm{gr}}$ the corollary follows. □

3.4 Linear volume components and lateral groups

Recall Theorem 3.2.17; for s.c. classes $\alpha, \beta \in \mathbf{SC}$ in a regular Euclidean s.c. structure we have $\alpha < \beta$ if and only if $\mu(\alpha) < \mu(\beta)$, where μ is the Lebesgue measure. This means that the natural ordering of $\mathbf{SC}$ is 'almost linear' in the following sense.

3.4.1 Proposition. *If $(\mathcal{F}, G)$ is regular Euclidean, then for all $\alpha, \beta \in \mathbf{SC}$ the following are equivalent:*

* *$\mu(\alpha) = \mu(\beta)$.*
* *α and β are equal or incomparable.*
* *for all $\gamma \in \mathbf{SC}$ we have $\gamma < \alpha$ if and only if $\gamma < \beta$.*
* *for all $\gamma \in \mathbf{SC}$ we have $\alpha < \gamma$ if and only if $\beta < \gamma$.*

We call this equivalence the laterality *property of regular Euclidean s.c. structures.*

Proof. Assume that $(\mathcal{F}, G)$ is regular Euclidean. We will only show the equivalence of the first three items. The fourth will follow by turning all of the $<$-signs in the proof around. Because $\alpha < \beta$ if and only if $\mu(\alpha) < \mu(\beta)$, clearly the first two items are equivalent.

Let $\alpha, \beta, \gamma \in \mathbf{SC}$ be such that $\mu(\alpha) = \mu(\beta)$ and $\gamma < \alpha$ It follows that $\mu(\gamma) < \mu(\alpha) = \mu(\beta)$, so $\gamma < \beta$ as required.

Now assume that $\alpha, \beta \in \mathbf{SC}$ are such that for all $\gamma \in \mathbf{SC}$ we have $\gamma < \alpha$ if and only if $\gamma < \beta$. If either $\alpha < \beta$ or $\beta < \alpha$ holds, then by assumption it follows that either $\alpha < \alpha$ or $\beta < \beta$, which is a contradiction. It follows that we must have $\mu(\alpha) = \mu(\beta)$. □

In this rest of this section, $(\mathcal{F}, G)$ will denote an arbitrary regular Euclidean s.c. structure (and as usual $\mathbf{SC}$ will denote its natural monoid).

3.4.2 Definition. *If X is an ordered set, let $\lessgtr$ denote the relation on X such that $x \lessgtr y$ if and only if neither $x < y$ nor $y < x$ holds.*

Clearly for $\alpha, \beta \in \mathbf{SC}$ we have $\alpha \lessgtr \beta$ if and only if $\mu(\alpha) = \mu(\beta)$.

3.4.3 Proposition. *The relation $\lessgtr$ is a congruence of commutative monoids on $\mathbf{SC}$. Moreover, μ gives an isomorphism between $\mathbf{SC}/\lessgtr$ and the additive monoid $[0, \infty)$.*

Proof. Recall from Proposition 3.2.15 that $\mu\colon \mathbf{SC} \to [0, \infty)$ is a monoid homomorphism (because every $P \in \mathcal{F}$ is bounded we know that no figure has infinite measure). By Proposition 3.1.9 it follows that $\lessgtr$ is a congruence. It also follows by the First Isomorphism Theorem that $\mathbf{SC}/\lessgtr$ is isomorphic under μ to the image in $[0, \infty)$ of $\mathbf{SC}$ under μ. However, this image must equal $[0, \infty)$ as $\mathcal{F}$ contains all rectilinear figures. □

Laterality allows us to prove a structure theorem about $\mathbf{SC}$. Note that, by the universal property of the Grothendieck group, we can extend $\mu\colon \mathbf{SC} \to [0, \infty)$ to a group homomorphism $\mu\colon \mathbf{SC}^{\mathrm{gr}} \to \mathbb{R}$. This gives us the following short exact sequence of abelian groups,

$$0 \longrightarrow \ker(\mu) \longrightarrow \mathbf{SC}^{\mathrm{gr}} \xrightarrow{\mu} \mathbb{R} \longrightarrow 0,$$

where $\ker(\mu)$ is the kernel of the group homomorphism μ, i.e. the subgroup $\{x \in \mathbf{SC}^{\mathrm{gr}} : x \lessgtr 0\} \subset \mathbf{SC}^{\mathrm{gr}}$. We can in fact guarantee that this short exact sequence splits for all regular Euclidean s.c. structures.

3.4.4 Proposition. *There is a subgroup* $\mathbf{J} \subset \mathbf{SC}^{\mathrm{gr}}$ *such that* $\mu|_{\mathbf{J}} \colon \mathbf{J} \to \mathbb{R}$ *is an isomorphism.*

Proof. Let $\mathbf{J}$ be the subgroup of $\mathbf{SC}^{\mathrm{gr}}$ generated by the s.c. classes of the form

$$\beta_r := \mathrm{Scis}\left([[0,r] \times [0,1]^{n-1}]\right)$$

where $r \in \mathbb{R}_{\geq 0}$ is a non-negative real number. Because $\mathrm{Orth}_n \leq \mathcal{F}$, these are indeed elements of $\mathbf{SC}$. All we need to show is that if $r_1, r_2 \geq 0$, then $\beta_{r_1} + \beta_{r_2} = \beta_{r_1+r_2}$, but this follows immediately from the fact that G contains all translations. We have $\mu(\beta_r) = r$, so μ gives an isomorphism between $\mathbf{J}$ and $\mathbb{R}$. □

3.4.5 Lateral Group Theorem. *If* $(\mathcal{F}, G)$ *is regular Euclidean, then there is an abelian group* $\mathrm{Lat}(\mathbf{SC})$*, called the* lateral group *of* $\mathbf{SC}$*, such that there is an isomorphism*

$$\mathbf{SC} \cong ((0,\infty) \times \mathrm{Lat}(\mathbf{SC})) \cup \{0\}.$$

This is called the lateral decomposition *of* $\mathbf{SC}$*.*

Proof. This group can of course be given as $\mathrm{Lat}(\mathbf{SC}) := \ker(\mu) \subset \mathbf{SC}^{\mathrm{gr}}$. From the previous proposition it follows that $\mathbf{SC}^{\mathrm{gr}}$ is isomorphic to $\mathbb{R} \times \ker(\mu)$, namely as the internal direct sum $\mathbf{SC}^{\mathrm{gr}} = \mathbf{J} \oplus \ker(\mu)$. We can write any $x \in \mathbf{SC}^{\mathrm{gr}}$ uniquely as $x = v_x + \ell_x$, where $v_x \in \mathbf{J}$ and $\ell_x \in \ker(\mu)$ (the 'v' stands for 'volume'). It follows for $x, y \in \mathbf{SC}^{\mathrm{gr}}$ that $x < y$ if and only if $v_x < v_y$. By definition for $x \in \mathbf{SC}^{\mathrm{gr}}$ we have $x \in \mathbf{SC}$ if and only if $0 \leq x$, so this is equivalent to $x = 0$ or $0 < v_x$. In other words, $\mathbf{SC} \subset \mathbf{SC}^{\mathrm{gr}}$ consists exactly of 0 and the elements of the form $v + \ell \in \mathbf{SC}^{\mathrm{gr}}$ with $v \in \mathbf{J}_{>0}$ and $\ell \in \ker(\mu)$ arbitrary. We conclude that $\mathbf{SC} \cong ((0,\infty) \times \ker(\mu)) \cup \{0\}$ as claimed. □

The s.c. structures (Pol_n, E_n), (Orth_n, T_n), and (Circ, E_2) are all regular Euclidean, so their natural monoids each have a lateral decomposition.

3.5 The monoids of the scissors congruence of polygons, ortholinear polygons, circle figures, and polyhedra

Let us now calculate a few facts about the various s.c. structures that we care about. In some cases this amounts to *classifying* scissors congruence, in the sense that we give effective necessary or sufficient conditions. The proofs of this section will not be quite as formal as the ones in the rest of this work, as it would simply take up too much space and would also not make the results any clearer. We make several appeals to 'well-known' properties of polytopes and we make no attempt to rigorously calculate our geometric manipulations. As an example, we do not prove the following result.

3.5.1 Definition. *An* n-simplex *is an* n*-polytope represented by the convex hull of* $n+1$ *affinely independent points.*

3.5.2 Proposition. *The* n*-simplices form a basis for* Pol_n*.* □

For the formal details of the theory of polytopes, [Grü03] is an excellent resource.

Euclid showed in Book 1 of the Elements [Euc] (Proposition 45, specifically) that any polygon in the plane has the same area as a rectangle where one of the sides has length 1. Euclid's notion of 'equal area' is a priori coarser than (Pol_n, E_n)-scissors congruence. He also uses the idea that if $P \sqcup T$ has the same area as $Q \sqcup T$, then so do P and Q. By Zylev's theorem this is no stronger than scissors congruence. Finally, he uses that if $P \sqcup P'$ has the same area as $Q \sqcup Q'$, where P is congruent to P' and Q is congruent to Q', then P and Q have equal area. In other words, halves of equal areas are equal. For polyhedra it turns out that this being-halves-of-equal-figures condition also implies that $P \simeq Q$ (this follows indirectly from Dehn and Sydler's theorems, 3.5.13 and 3.5.15; see also the discussion on the vector space structure of $\mathrm{Lat}(\mathbf{SC}(\mathrm{Pol}_3, E_3))$ in Section 4), but it is not clear whether this follows from some more general principle like Zylev's theorem does.

In any case, for polygons there is a constructive proof that any polygon is s.c. to a rectangle where one of the sides has length 1.

3.5.3 Wallace-Bolyai-Gerwien Theorem. *The area function* μ *gives an isomorphism between* $\mathbf{SC}(\mathrm{Pol}_2, E_2)$ *and* $[0, \infty)$.

Proof. Let $P \in \mathrm{Pol}_2$ be an arbitrary polygon. We will show that P is s.c. to a rectangle of the form $[0, r] \times [0, 1]$, from which it follows that the lateral group of $\mathbf{SC}$ is trivial, i.e. area is a complete invariant of scissors congruence.

First we apply Proposition 3.5.2 to decompose P into triangles.

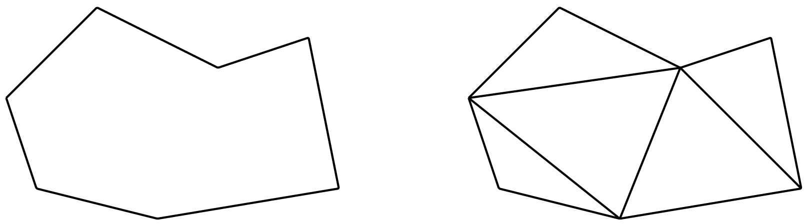

Figure 9: Triangulating a polygon.

We treat every triangle separately. An arbitrary triangle is s.c. to a rectangle as shown by the following figure.

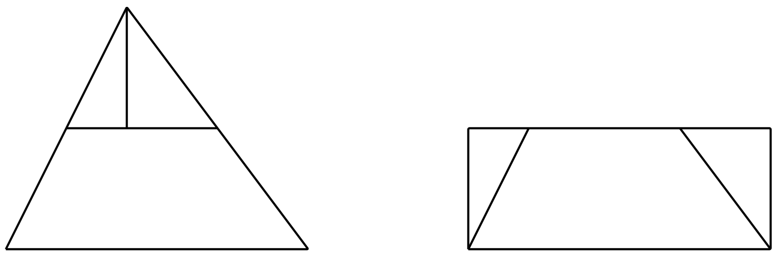

Figure 10: Turning triangles into rectangles.

Note that only rotations by 180° are necessary. We now turn the rectangle into one such that one of the sides lengths is in $[1, 2)$. We can do this by cutting the rectangle in half and stacking one half on top of the other. This halves one of the side lengths and doubles the other using only translations.

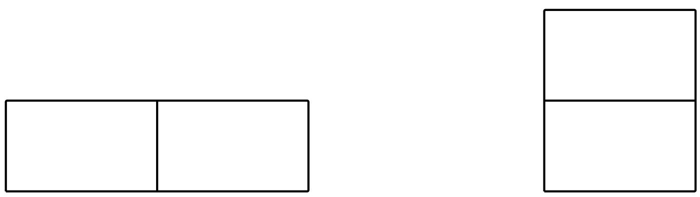

Figure 11: Getting rectangles to the right format.

Finally, we can turn a rectangle where one of the sides has length $\ell \in [1, 2)$ into one with a side length of 1 as in the following figure.

Now we are left with several rectangles, one for each of the triangles we decomposed P into. Using E_2-transformations we can place all of these rectangles next to each other and 'paste' them together to form a rectangle of the form $[0, r] \times [0, 1]$. The real number r must necessarily equal the area $\mu(P)$, which proves the theorem. $\square$

It seems that the only time in the previous proof where we need the full transformation group E_2 is at the end where we reorient the rectangles to paste them all together. However, even here it turns

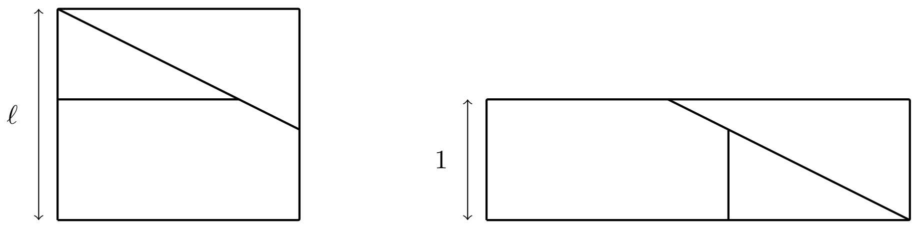

Figure 12: Normalizing the side lengths of a rectangle.

out not to be necessary. It is a cute geometry exercise to show that any two rectangles of the same area are (Pol_2, T_2)-scissors congruent, no matter their relative orientations. It follows that if G is the subgroup of E_2 generated by the translations and a rotation by 180°, then $\mathrm{Lat}(\mathbf{SC}(\mathrm{Pol}_2, G))$ is trivial.

The nervous reader might see the Wallace-Bolyai-Gerwien theorem and fear that $\mathbf{SC}(\mathrm{Pol}_n, E_n)$ is isomorphic to $[0, \infty)$ for all $n \geq 1$. This would mean that all of our work for Zylev's theorem and lateral groups was for nothing (at least as far as polytopes are concerned). Let me reassure this reader; it already fails for $n = 3$ (Corollary 3.5.14). Before tackling this result let us first consider two other cases: the scissors congruence of (Circ, E_2) and that of (Orth_n, T_n).

For the proofs of the coming propositions we use the following helpful concept from [Sch09].

3.5.4 Definition. *If P is a figure of one of the classes* Pol_n, Orth_n, *or* Circ, *we say that* $P = P_1 \sqcup \cdots \sqcup P_n$ *is a* clean decomposition *of P if for all distinct $i, j \in \{1, \ldots, n\}$ we have that P_i and P_j either don't share a boundary or they share exactly one lower-dimensional face (where in the case of* Circ *this may be a circular arc).*

3.5.5 Proposition. *Any decomposition of P may be refined by a clean decomposition of P.*

Proof. Let $\mathcal{F} \in \{\mathrm{Pol}_n, \mathrm{Orth}_n, \mathrm{Circ} : n \geq 1\}$. We define a set of figures $\mathcal{H}$. If $\mathcal{F} = \mathrm{Pol}_n$, let $\mathcal{H}$ be the set of half-spaces in UPol_n. If $\mathcal{F} = \mathrm{Orth}_n$, let $\mathcal{H}$ be the set of axis-aligned half-spaces. If $\mathcal{F} = \mathrm{Circ}$, let $\mathcal{H}$ be the set of half-planes together with the set of disks. These sets $\mathcal{H}$ are not technically generating sets in our definition of the term, because their elements are not themselves figures of $\mathcal{F}$ (because they are not bounded). Nevertheless, they do generate $\mathcal{F}$ in the sense that for every $P \in \mathcal{F}$ there are $H_1, \ldots, H_n \in \mathcal{H}$ and a $V \in \mathrm{Bool}_n$ such that $P = f_V(H_1, \ldots, H_n)$.

Let $P \in \mathcal{F}$ and consider an arbitrary decomposition $P = P_1 \sqcup \cdots \sqcup P_n$. Let $H_1, \ldots, H_k \in \mathcal{H}$ be such that there are $V_1, \ldots, V_n \in \mathrm{Bool}_k$ such that $P_i = f_{V_i}(H_1, \ldots, H_k)$ for all i. That is, $H_1, \ldots, H_k$ together may be combined to define all of P_i. Let $V = V_1 \cup \cdots \cup V_n$. This is another abstract k-ary Boolean combination, and moreover we have $f_V(H_1, \ldots, H_k) = P$. Then P may be decomposed into the set $\{f_s(H_1, \ldots, H_k) : s \in V\}$ of Boolean slices. This decomposition refines $P_1, \ldots, P_n$ and this is always a clean decomposition. □

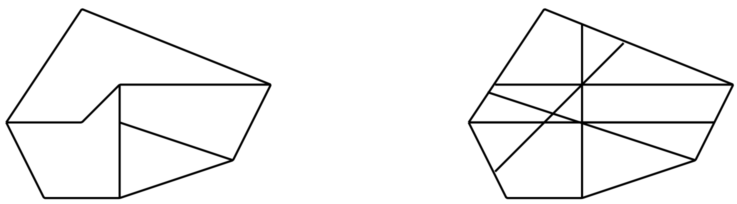

Figure 13: The left decomposition is not a clean decomposition of the polygon, but the construction of Proposition 3.5.5 allows us to refine it into a clean decomposition simply by 'extending' all of the lines used to define the figures of the first decomposition.

3.5.6 Definition. *For a circle figure $P \in$* Circ, *define its* curvature invariant *to be the following function. Let $\mathbb{R}^{[\mathbb{R}]}$ denote the free real vector space on $\mathbb{R}$. That is, for every real number $x \in \mathbb{R}$ we have a linearly independent basis element $[x] \in \mathbb{R}^{[\mathbb{R}]}$. Let C denote the quotient of this space by* *the subspace generated by vectors of the form* $[x] - [-x]$ *for* $x \in \mathbb{R}$. *Define the curvature invariant* $\gamma \colon \mathrm{Circ} \to C$ *as*

$$\gamma(P) = \sum_{\textit{edges } e} \ell_e \cdot [\kappa_e],$$

where $\ell_e \in \mathbb{R}$ *denotes the length of the edge* e *and* $\kappa_e \in \mathbb{R}$ *denotes its* signed curvature. *That is, if the interior of* P *is on the concave side of* e *then* κ_e *is the reciprocal of the radius of the circle that* e *is an arc of (and* $\kappa_e = 0$ *if* e *is a straight line segment), and if the interior of* P *is on the convex side, then* κ_e *is the negative of this value instead.*

3.5.7 Proposition. *The map* γ *defines a homomorphism* $\mathbf{SC}(\mathrm{Circ}, E_2) \to C$.

Proof. It suffices to show that $\gamma([\emptyset]) = 0$, that γ is preserved by congruence, and that if $P = P_1 \sqcup \cdots \sqcup P_n$ then $\gamma(P) = \gamma(P_1) + \cdots + \gamma(P_n)$. The first two are obvious.

By Proposition 3.5.5 it is only necessary to show that γ is additive w.r.t. clean decompositions. Let $P \in \mathrm{Circ}$ and assume that $P = P_1 \sqcup \cdots \sqcup P_n$ is a clean decomposition. Let a *flag* be a pair (e, P_i) where e is an edge of P_i. Note that for flags (e_1, P_i) and (e_2, P_j) we have by the definition of a clean decomposition that either $e_1 = e_2$ or that e_1 and e_2 do not overlap on a curve of positive length. Then

$$\gamma(P_1) + \cdots + \gamma(P_n) = \sum_{\text{flags } (e,\, P_i)} \ell_e \cdot [\kappa_e].$$

Call this quantity S. A flag (e, P_i) lies either in the interior of P or it is contained in some edge of P. In the first case, there will be exactly one other $j \in \{1, \ldots, n\}$ such that (e, P_j) is a flag. The curvatures of e as an edge of P_i and an edge of P_j will be additive inverses, so the contribution of (e, P_i) and S will be 0. In the second case, for some edge $e^\dagger$ of P consider the set of all flags (e, P_i) such that e is contained in $e^\dagger$. None of the edges of these flags overlap (because otherwise they would be in the interior of P), so it must be the case that together they link together to form $e^\dagger$. It follows that their curvatures are all the same and their lengths sum to the length of $e^\dagger$, so we find that $\gamma(P) = \gamma(P_1) + \cdots + \gamma(P_n)$ as desired. □

3.5.8 Corollary. *The lateral group* $\mathrm{Lat}(\mathbf{SC}(\mathrm{Circ}, E_2))$ *is non-trivial.*

Proof. An example of a scissors congruence problem we can formulate in Circ is the problem of the *lune of Hippocrates*.

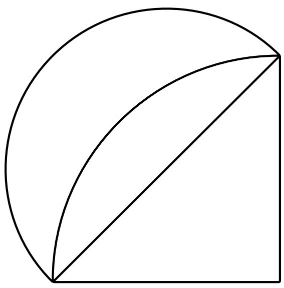

Figure 14: The lune of Hippocrates.

The curves in Figure 14 consist of either straight line segments or circular arcs, so the regions that they delineate represent figures of Circ. It can be shown that if the straight sides of the above triangle are orthogonal radii of the inner circular arc and the hypotenuse is the diameter of the outer arc, then the interiors of the triangle and the lune between the arcs have equal area. However, because the arcs have different radii, the two figures clearly have distinct curvature invariants, so we conclude that area is not the only component of $\mathbf{SC}(\mathrm{Circ}, E_2)$; its lateral group is non-trivial. □

The following scissors congruent invariant is explored in detail in [Spa04].

3.5.9 Definition. *For an axis-aligned rectangular prism $R \in \mathrm{Orth}_n$ (that is, a figure represented by a set of the form $[x_1, z_1] \times \cdots \times [x_n, z_n]$), let its* aspect invariant *be the value $a(R) := w_R^1 \otimes \cdots \otimes w_R^n \in \mathbb{R} \otimes_{\mathbb{Z}} \overset{n}{\cdots} \otimes_{\mathbb{Z}} \mathbb{R}$, where w_R^i is width of the prism R in the i'th coordinate. For an arbitrary ortholinear figure $P \in \mathrm{Orth}_n$, we let $a(P)$ be the sum of the values $a(R_i)$, where $P = R_1 \sqcup \cdots \sqcup R_n$ is a decomposition into axis-aligned rectangular prisms.*

3.5.10 Proposition. *The aspect invariant gives a well-defined homomorphism from $\mathbf{SC}(\mathrm{Orth}_n, T_n)$ to $\mathbb{R} \otimes_{\mathbb{Z}} \overset{n}{\cdots} \otimes_{\mathbb{Z}} \mathbb{R}$.*

Proof. For ease of notation we restrict ourselves to proving the case $n = 2$. The proof is exactly analogous for all other dimensions.

First we give an argument for why $a(R)$ is well-defined when $R \in \mathrm{Orth}_2$ is an axis-aligned rectangle. Consider any decomposition $R = R_1 \sqcup \cdots \sqcup R_m$ where the R_i are also axis-aligned rectangles. We apply Proposition 3.5.5 to refine this into a clean decomposition $R = R'_1 \sqcup \cdots \sqcup R'_t$. As illustrated by Figure 15, such decompositions have a particularly nice form; they can be constructed by first decomposing R horizontally into rectangles of the same height, and then decomposing those rectangles vertically into smaller rectangles of the same width. If a rectangle S is decomposed horizontally into k rectangles $S_1, \ldots, S_k$ of the same height, as

$$\Big[[x_1, x_2] \times [y_1, y_2]\Big] = \Big[[x_1, c_1] \times [y_1, y_2]\Big] \sqcup \cdots \sqcup \Big[[c_{k-1}, x_2] \times [y_1, y_2]\Big],$$

then by definition of the tensor product we have $a(S) = a(S_1) + \cdots + a(S_k)$. The same holds if it is decomposed vertically. It follows that $a(R) = a(R'_1) + \cdots + a(R'_t)$, and because each of the original R_i is also cleanly decomposed into the R'_j it follows that a is also additive for these decompositions, so $a(R) = a(R_1) + \cdots + a(R_m)$ as required.

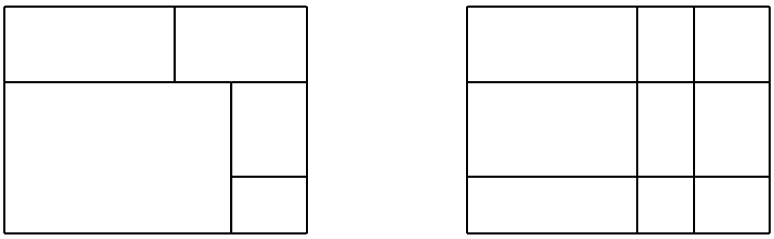

Figure 15: For an axis-aligned rectangle R in Orth_2, any clean decomposition is such that it can be viewed as first dividing R vertically into rectangles and then dividing those horizontally into smaller rectangles, or vice versa.

Well-definedness for arbtirary $P \in \mathrm{Orth}_2$ follows from the fact that any two decompositions of P into rectangles (into $(R_i)_i$ and $(S_j)_j$, say) have a clean decomposition $P = T_1 \sqcup \cdots \sqcup T_t$ as a common refinement. Restricting this decomposition to the R_i and S_j also gives a clean decomposition of those figures, so

$$\sum_i a(R_i) = \sum_k a(T_k) = \sum_j a(S_j),$$

from which we can see that aspect invariant of Definition 3.5.10 is well-defined for ortholinear polygons $P \in \mathrm{Orth}_2$.

The value $a(P)$ is clearly invariant under translations and it maps the empty figure onto 0. We also already showed that it is additive w.r.t. clean decompositions, so it gives a monoid homomorphism $a \colon \mathbf{SC}(\mathrm{Orth}_2, T_2) \to \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$. $\square$

3.5.11 Corollary. *The lateral group $\mathrm{Lat}(\mathbf{SC}(\mathrm{Orth}_n, T_n))$ is non-trivial for $n \geq 2$.*

Now let us give an invariant for (Pol_3, E_3)-scissors congruence.

3.5.12 Definition. *For a polyhedron $P \in \mathrm{Pol}_3$, let its* Dehn invariant *be the value $\Delta(P) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$ given by*

$$\Delta(P) = \sum_{\textit{edges } e} \ell_e \otimes \alpha_e,$$

where ℓ_e is the length of the edge e and α_e is its dihedral angle.

3.5.13 Dehn's Theorem. *The Dehn invariant defines a homomorphism $\mathbf{SC}(\mathrm{Pol}_3, E_3) \to \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$.*

Proof. We make a final appeal to the fact that we only need to show that Δ is additive w.r.t. clean decompositions of some figure P.

Let $P = P_1 \sqcup \cdots \sqcup P_n$ be a clean decomposition. Consider the set of flags (e, P_i) where e is an edge of P_i. Now define

$$S = \Delta(P_1) + \cdots + \Delta(P_n) = \sum_{\text{flags } (e, P_i)} \ell_e \otimes \alpha_e.$$

We can divide the flags (e, P_i) into three disjoint types:

* Type 1: e lies in the interior of P.
* Type 2: e lies on the boundary of P but not within an edge of P.
* Type 3: e is contained in an edge of P.

Then $S = S_1 + S_2 + S_3$ where S_i is the sum of the values $\ell_e \otimes \alpha_e$ taken over the flags of type i. Let (e, P_i) be a flag of type 1 and consider the set of all flags (e', P_j) such that $e' = e$. Because e lies in the interior of P it must be the case that the dihedral angles of the e' sum to exactly 2π (and their lengths must equal ℓ_e). It follows that $S_1 = 0$. By the same argument we find that if (e, P_i) is of type 2 then the dihedral angles of the flags whose edges agree with e must sum to exactly π, hence $S_2 = 0$.

It remains to show that $S_3 = \Delta(P)$. Consider a single edge $e^\dagger$ of P and define E to be the set of all flags (e, P_i) such that e is contained in $e^\dagger$. Let $S_{e^\dagger}$ be the sum over E of the values $\ell_e \otimes \alpha_e$. Note that S_3 is the sum of the $S_{e^\dagger}$ over all edges of P. By the nature of a clean decomposition we know that for $(e_1, P_i), (e_2, P_j) \in E$ we have that either $e_1 = e_2$ or that they do not overlap on a segment of positive length. We can partition E into equivalence classes under the relation $\sim$ defined as $(e_1, P_i) \sim (e_2, P_j)$ whenever $e_1 = e_2$. Then $S_{e^\dagger} = T_1 + \cdots + T_k$ where k is the number of $\sim$-equivalence classes of E. In a $\sim$-equivalence class we know that all flags must have the same length, and their dihedral angles must sum to $\alpha_{e^\dagger}$. It follows that every T_i is of the form $\ell_{e_i} \otimes \alpha_{e^\dagger}$, where e_i is any element of the i'th equivalence class. It follows that

$$S_{e^\dagger} = \ell_{e_1} \otimes \alpha_{e^\dagger} + \cdots + \ell_{e_k} \otimes \alpha_{e^\dagger} = \ell_{e^\dagger} \otimes \alpha_{e^\dagger}.$$

We conclude that the sum of all $S_{e^\dagger}$ must equal $\Delta(P)$, hence $\Delta(P_1) + \cdots + \Delta(P_n) = \Delta(P)$ as required. □

3.5.14 Corollary. *The lateral group* $\mathrm{Lat}(\mathbf{SC}(\mathrm{Pol}_3, E_3))$ *is non-trivial. In other words, volume is not a complete s.c. invariant for polyhedra.*

Proof. Consider a cube C and a regular tetrahedron T of equal (positive) volume. Let ℓ be the length of the edges of T. We have $\Delta(C) = 0$ and $\Delta(T) = 6\ell \otimes \arccos(1/3)$. It suffices to show that the value $\arccos(1/3)$ is $\mathbb{Q}$-linearly independent of π. Assume for the sake of contradiction that the angle $\arccos(1/3)$ is a rational multiple of π. Consider the complex number $z = \frac{1}{3} + i\frac{2}{3}\sqrt{2}$. There is some smallest $n \in \mathbb{Z}_{>0}$ such that $z^n = 1$ (i.e. z is a root of unity). The field extension $\mathbb{Q}(z)$ is quadratic, so $\phi(n) = 2$, where $\phi(n)$ is the totient function that counts the number of integers in the range from 1 to n that are coprime with n. The only n for which $\phi(n) = 2$ are 3, 4, and 6. None of the values z^3, z^4, and z^6 are equal to 1, so we have found our contradiction. We conclude that $\arccos(1/3)$ is not a rational multiple of π, so $\Delta(T) \neq 0$. □

It turns out that volume and the Dehn invariant are a *complete set of invariants* for (Pol_3, E_3)-scissors congruence. The following was shown by Sydler in [Syd65].

3.5.15 Sydler's Theorem. *Volume and the Dehn invariant are the only two* (Pol_3, E_3)*-scissors congruence invariants. That is,* $\mathrm{Lat}(\mathbf{SC}(\mathrm{Pol}_3, E_3))$ *is isomorphic under* Δ *to its image in* $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$*.* $\square$

The proof is fairly technical, and it would be distracting to recount it here. For an English account of the theorem, see [Jes68]. There is more to be said about the Dehn invariant. Unlike the aspect and curvature invariants, it is not surjective onto the given codomain. The interested reader can find a clear and detailed exposition of such facts in [Dup01].

4 Further research

We have given a taste of the various relationships between the properties of an abstract scissors congruence relation, and how these properties (such as cancellativity and the idea of a lateral group) manifest themselves in the concrete examples of Section 3.5. There are many more things to be said about the subject.

Something I couldn't quite make work in the generality that I would have liked is the idea of a lateral *vector space*. One can notice that the images of the curvature, aspect, and Dehn invariants are real vector spaces in a natural way. Because any dilation of $\mathbb{R}^n$ is Borel_n-measurable, it is also the case that any class of figures $\mathcal{F} \leq \mathrm{Borel}_n$ admits a real scalar multiplication; $\lambda \cdot [A]$ is the figure represented by A dilated by a value of λ. This multiplication cannot make $\mathbf{SC}^{\mathrm{gr}}$ into a real vector space, however. Consider a cube Q in Pol_3. We have $\mathrm{Scis}(2 \cdot Q) = 8\,\mathrm{Scis}(Q)$. But this is not the end of the world! As it turns out, in (Pol_3, E_3) this scalar multiplication naturally restricts to a scalar multiplication on $\mathrm{Lat}(\mathbf{SC})$ which does in fact make the lateral group into a real vector space. The following argument is from [Jes68]. Figure 16 illustrates the idea.

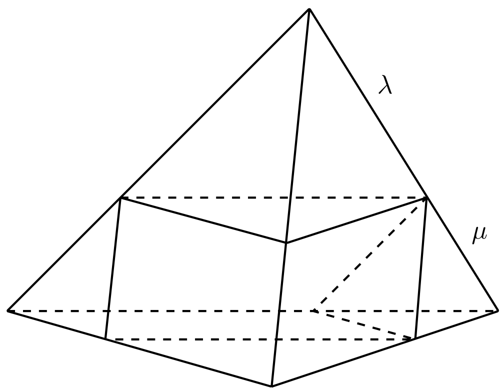

Figure 16: A decomposition of a dilated tetrahedron into two dilated tetrahedra and two triangular prisms. This witnesses the identity $(\lambda + \mu)x = \lambda x + \mu x$ for the scalar multiplication structure on the lateral group $\mathrm{Lat}(\mathbf{SC}(\mathrm{Pol}_3, E_3))$

An arbitrary tetrahedron dilated by a value of $\lambda + \mu$ may be decomposed into two tetrahedra, dilated by λ and μ respectively, and two triangular prisms. Because the tetrahedra form a basis for Pol_3 it follows that *any* polyhedron dilated by $\lambda + \mu$ decomposes into similar polyhedra dilated by λ and μ and some number of prisms. It is not difficult to show using the Wallace-Bolyai-Gerwien theorem that the s.c. classes of prisms are all contained in the subgroup $\mathbf{J}$ of Proposition 3.4.4. So 'modulo prisms' this scalar multiplication does make a vector space out of $\mathbf{SC}(\mathrm{Pol}_3, E_3)/\mathbf{J}$. The theorems of Dehn and Sydler together show that $\mathbf{J}$ is exactly the kernel of the Dehn invariant, so this shows that a priori we could have expected the image of the Dehn invariant to be a real vector space under the natural scalar multiplication. For what other Euclidean s.c. structures can this argument be adapted?

Another question concerns topology. We have an ordered commutative monoid $\mathbf{SC}$, and if the s.c.

structure in question is uniform (or even a weaker variant of the property) then for all $\alpha, \beta \in \mathbf{SC}_{>0}$ there is some γ with $\gamma < \alpha$ and $\gamma < \beta$. This directedness property reminds of the basis of open balls of a metric space. For $P \in \mathcal{F}$ and $\alpha \in \mathbf{SC}_{>0}$, define

$$B_\alpha(P) = \Big\{ Q \in \mathcal{F} : \mathrm{Scis}(P \triangle Q) < \alpha \Big\}.$$

If $R \in B_\alpha(P) \cap B_\beta(Q)$, then there is a $\gamma \in \mathbf{SC}_{>0}$ such that $B_\gamma(R) \subset B_\alpha(P) \cap B_\beta(Q)$, so these 'open balls' do in fact form a basis for a topology on $\mathcal{F}$. If $\mathcal{F}$ is regular Euclidean, then this topology evidently agrees with the one induced by the metric on $\mathcal{F}$ given by $d(P, Q) = \mu(P \triangle Q)$, where μ is the Lebesgue measure, but an advantage is that this 'scissors congruence topology' may be defined in much greater generality. If we fix a particular kind of figure, like a triangle, then we can parametrize the space of all triangles using real numbers. This makes the set of triangles into a quotient space of $\mathbb{R}^6$. Do such topologies agree with the subspace topology of the s.c. topology on Pol_2? Under what circumstances does the function $(P, Q) \mapsto \mathrm{Scis}(P \triangle Q)$ make $\mathcal{F}$ into a complete 'metric' space?

As a final question, consider the idea of morphisms. A 'scissors congruence space' in our formalism consists of three pieces of data: a carrier set X, a class of figures $\mathcal{F}$ on X, and an $\mathcal{F}$-measurable group of transformations of X. When should two such spaces be considered isomorphic? More generally, is there a good notion of structure-preserving map between such spaces? In this work we have spotlighted the 's.c. structure' $(\mathcal{F}, G)$ over the underlying set X, which we think of as mostly a bookkeeping device for defining $\mathcal{F}$ and G. The internal structure of $\mathcal{F}$ really only consists of its ordering. On the other hand, we only need G for its action of creating subfigure correspondences between the elements of $\mathcal{F}$. Both of these things can be wrapped up into one neat package by having an s.c. structure be a *small category* with some extra structure. The objects of this category would be the elements of $\mathcal{F}$, and a morphism $P \to Q$ would consist of a congruence between P and a *subfigure* of Q. What other assumptions do we need to make a theory of scissors congruence of such categories? Functors seem like a natural candidate for morphisms of s.c. structures, but what other properties do they need to satisfy?

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