Universiteit Leiden
The Netherlands

Continuous functions through the lens of the Baire Category Theorem
Bauer, B.P.A.

B.P.A. Bauer

Continuous functions through the lens of the Baire Category Theorem

Bachelor thesis

27 August 2023

Thesis supervisor: dr. M. Roelands

Leiden University
Mathematical Institute

Contents

1 Introduction

When asked to motivate the interest in a certain mathematical field, many a mathematician will bring up the beauty of their subject of study. Though the aesthetic appreciation of mathematics will always be a subjective thing, mathematical beauty is often said to be found in the elegance of a brief proof, the generality of a mathematical tool or the simplicity of an identity that is able to connect multiple mathematical areas. Especially in undergraduate mathematics, mathematical objects are often considered to be beautiful if they are elegant and simple. Particularly in analysis, which at surface seems a muddle of ε-δ definitions and lengthy calculations of derivatives and integrals, students can be relieved to work with smooth, analytical functions, with 'nice' properties and an abundance of theorems to apply to. However, focused on the ideal of the smooth function, it is easy to lose sight of many classes of functions that demonstrate 'weird' behaviour: the kind of functions that are presented as a counterexample to show how things can 'go wrong' in calculus. A collection of such wonderful counterexamples can be found in [GO64]. Though these examples demonstrate the ways our assumptions about continuity and differentiability are not always right, the maturing mathematician can find beauty in these counterexamples and excitement in the seemingly ugly functions.

Two of such 'nice' properties that continuous functions may possess on an interval of their domain are differentiability and monotonicity. In this thesis, we zoom in on the extremes within the space of continuous real-valued functions on $[0, 1]$, equipped with the supremum norm. We will explore the existence of functions that are nowhere differentiable, of functions that are everywhere differentiable and nowhere monotone, and the question of whether each nowhere differentiable function is nowhere monotone as well. Additionally, we will show that every continuous function can be approximated by a nowhere differentiable function, highlighting the prevalence of nowhere differentiability.

The key ingredient for proving these statements will be the Baire Category Theorem, which makes a fundamental topological distinction between two types (or categories) of spaces, stating whether they are in a sense 'full' or 'large', or more 'empty'. The reader may be familiar with this theorem through the proofs of the Open Mapping Theorem and Banach-Steinhaus Theorem, both results about linear functions on Banach spaces (see [Fol99, Theorem 5.10 and Theorem 5.13]).

Though the results in this thesis are not novel, this thesis presents the proofs in a more detailed and insightful way, explaining all concepts at undergraduate level, so that the theorems are understandable to a broader audience. Especially the material in Section 3.3 provides a lot more detail to the proof of the theorem that everywhere differentiable nowhere monotone functions exist, which was originally proven by Clifford Weil [Wei76]. As we delve into these theorems, we also highlight and prove other noteworthy results, such as Vitali's Covering Theorem in Section 3.2 and Tao's Theorem for the uniform convergence of functions whose derivatives converge uniformly in Section 3.3.

This thesis is now divided into the following components. Section 2 lays the required groundwork for the rest of this thesis. Section 2.1 to Section 2.4 contain a recap of topological spaces and measure theory, to ascertain the reader is familiar with the main concepts. Section 2.5 and Section 2.6 introduce the necessary definitions for the Baire Category Theorem and contain a proof of the theorem. Section 3.1 centres around proving the fact that nowhere differentiable functions are dense in the space of real-valued continuous functions on $[0, 1]$. Section 3.2 focuses on the relation between differentiability and monotonicity of continuous functions. The main result of this section is Lebesgue's Theorem for the Differentiability of Monotone Functions, which states that a function that is monotone and continuous on an interval is also differentiable on that interval. We conclude with a proof that there exist continuous functions that are nowhere monotone and everywhere differentiable in Section 3.3. With these results in mind, the reader of this thesis will hopefully have a more nuanced understanding of the counterintuitive behaviour of continuous functions, and be able to appreciate the beauty of both these supposedly unattractive examples and the application of the Baire Category Theorem as a general technique in proofs.

2 Preliminaries

In this thesis, a basic knowledge of topology, measure theory and functional analysis is required. Though it is assumed that the reader is familiar with these topics, this chapter serves as a reference point to understand the different steps in the proofs later on. The first two sections are dedicated to a short summary of properties of different kinds of topological spaces, in particular metric spaces and normed spaces. Next, we present a few specific results from the field of real analysis and measure theory. The proofs of these results can be found in standard literature. The final two sections build up to the Baire Category Theorem, including a proof of the theorem for complete metric spaces and for locally compact Hausdorff spaces.

2.1 Topological spaces

In this section, the main topological concepts and results that are used in this thesis are presented. Though most of this thesis is dedicated to normed vector spaces, the topological framework in this section allows us to give a more general proof of the Baire Category Theorem.

We denote a topological space by $(S, \mathcal{T})$. For a subset $A \subset S$, the set A will be assumed to have the subspace topology $\mathcal{T}_A := \{U \cap A \colon U \in \mathcal{T}\}$ induced by $\mathcal{T}$. To prevent confusion with varying definitions found in the literature, we mean that a set $N \subset S$ is a *neighbourhood* of a point $x \in S$ or a set $A \subset S$ when there exists an open $U \subset S$ such that $U \subset N$ and $x \in U$ or $A \subset U$ respectively. We denote the closure and the interior of a set $A \subset S$ by cl A and int A respectively. In case it could be confusing in which set we consider the closure or interior of A, we introduce a subscript. Hence, if $A \subset U$ for some set $U \subset S$, we can distinguish between $\text{cl}_U A$ and $\text{cl}_S A$. A set A is *dense* in S if cl $A = S$. The following equivalent notions of density will often reappear in this thesis.

Proposition 2.1.1. *Let S be a topological space, and let $A \subset S$. Then the following are equivalent.*

(i) A is dense in S.

(ii) A intersects every non-empty open set $U \subset S$.

(iii) The interior of $S \setminus A$ is empty.

The topological spaces in this thesis have some further structure, which allows us to separate points and sets by open sets. We examine two kinds of restrictions. The first is called a *Hausdorff space*, for which it holds that for $x, y \in S$ with $x \neq y$, there are open sets $A, B \in \mathcal{T}$ with $x \in A$, $y \in B$ such that $A \cap B = \emptyset$. The second one is a stronger property. A space S is *regular* if for $x \in S$ and $F \subset S$ closed, $x \notin F$, there exist open disjoint sets containing x and F respectively.

Another property we will need in this thesis is *compactness*, the property that every open cover of the space has a finite subcover. An *open cover* of a topological space $(S, \mathcal{T})$ is defined as a subset $\mathcal{U} \subset \mathcal{T}$ for which $S = \cup_{U \in \mathcal{U}} U$. Similarly, an open cover of a subset A of S is some $\mathcal{U} \subset \mathcal{T}$ for which $A \subset \cup_{U \in \mathcal{U}} U$. We call $(S, \mathcal{T})$ *compact* if for every open cover $\mathcal{U}$ of S, there exists a finite subset $\mathcal{U}' \subset \mathcal{U}$ such that $S = \cup_{U \in \mathcal{U}'} U$. A subspace $A \subset S$ is compact if for every open cover $\mathcal{U} \subset \mathcal{T}$ with $A \subset \cup_{U \in \mathcal{U}} U$ there is a finite subset $\mathcal{U}' \subset \mathcal{U}$ such that $A \subset \cup_{U \in \mathcal{U}'} U$. Alternatively, a topological space is compact if and only if every family $\mathcal{F}$ of closed subsets with the *finite intersection property* has a non-empty intersection. A family $\mathcal{F}$ has the finite intersection property if every finite subcollection of sets in $\mathcal{F}$ has non-empty intersection. This property follows from the definition of open covers by taking complements.

Between compactness, closed subspaces and Hausdorff spaces a few useful connections exist.

Proposition 2.1.2. *Let $(S, \mathcal{T})$ be a topological space, and $A \subset S$ a subspace.*

(i) If S is compact and A is closed, then A is compact.

(ii) If S is Hausdorff and A is compact, then A is closed.

(iii) If S is compact and Hausdorff, then S is regular.

(iv) If S is compact and Hausdorff, $x \in S$ and U is a neighbourhood of x, then U contains a compact neighbourhood of x.

When working with functions from one topological space to another, it is necessary to have a concept of how much a function preserves the topology of one space to another. A function that preserves this topological structure is called a *homeomorphism*, which is a continuous bijection with continuous inverse. The following identities express to which extent continuous functions preserve topological properties.

Proposition 2.1.3. *Let f be a continuous function mapping $(S, \mathcal{T}_S)$ to $(V, \mathcal{T}_V)$. If $K \subset S$ is compact, then $f(K)$ is compact.*

Proposition 2.1.4. *Let $f\colon S \to V$ be a continuous bijection. If S is compact and V is Hausdorff, then f is a homeomorphism.*

A slightly more relaxed property than compactness is that of a *locally compact* space, which requires every point to have a compact neighbourhood. We are especially interested in locally compact Hausdorff spaces, which are often abbreviated as *LCH spaces*. An example of such a space is $\mathbb{R}^n$ with the topology derived from the Euclidean metric (see next section). A LCH space has the property that it can be embedded in a compact Hausdorff space.

Definition 2.1.5. Let $(S, \mathcal{T})$ be a topological space. Let ∞ denote a point outside of S and set $S_\infty := S \sqcup \{\infty\}$. Topologize S_∞ with $\mathcal{T}_\infty := \mathcal{T} \sqcup \{S_\infty \setminus K : K \subset S \text{ is compact}\}$. The space S_∞ along with the inclusion $f\colon S \to S_\infty$ is called the *Alexandroff extension* of S.

If S is a LCH space that is non-compact, then S_∞ is the *one-point compactification* of S satisfying the following properties. A proof of this construction can be found in [Run05, Theorem 3.3.26].

Theorem 2.1.6 (One-point compactification)**.** *Let $(S, \mathcal{T})$ be a non-compact LCH space. Then there is a compact Hausdorff space $(S_\infty, \mathcal{T}_\infty)$ along with a function $f\colon S \to S_\infty$ such that the following two statements hold.*

(i) f is a homeomorphism onto its image.

(ii) The set $S_\infty \setminus f(X)$ consists of just one point.

The topological space $(S_\infty, \mathcal{T}_\infty)$ is unique up to homeomorphism.

2.2 Metric spaces and the space of continuous functions

This section introduces metric spaces and the space of continuous functions on a compact metric space, the main topic of this thesis. The main property of metric spaces that we are interested in is completeness, which will also be defined below. In addition, we present a proof that the space of continuous functions on a compact Hausdorff space is complete, due to its importance in the rest of this thesis.

Throughout this section, we denote a metric space by (X, d). If the metric is clear from context, we often just write the metric space as X. The space $\mathbb{R}$ equipped with the *Euclidean metric* $d(x, y) := |x - y|$ is a metric space. The Euclidean metric is the standard metric on $\mathbb{R}$, so if no other metric is specified, $\mathbb{R}$ is assumed to have this metric. We denote open balls of radius $r > 0$ around a point $x \in X$ by $B_r(x) := \{y \in X : d(x, y) < r\}$.

In the analysis of metric spaces, the conditions for a sequence to converge are of fundamental interest. Throughout this thesis, a sequence or enumeration of points $\{x_n\}_{n\geq 1}$ in X will be denoted as $\{x_n\}_n$. The same notation is used for a sequence of sets. Every converging sequence is a Cauchy sequence, but the converse does not always hold. If every Cauchy sequence converges, the space is called *complete*. This property can also be understood as the metric space being 'big enough' to contain all the limits of the Cauchy sequences, or to be without 'holes'. As such, $\mathbb{R}$ is a complete metric space, but $\mathbb{Q}$ is not. The convergence of sequences allows for the following alternative characterisation of a set being closed in a space. For the proof of this result, see [Run05, Proposition 2.3.4].

Proposition 2.2.1. *Let (X, d) be a metric space, and let $A \subset X$. Then the closure of A is the set of points in X that are the limit of a sequence in A converging in X.*

The next proposition expresses the relation between closed and complete sets. The proof can be found in [Run05, Proposition 2.4.5].

Proposition 2.2.2. *Suppose (X, d) is a metric space, and let $A \subset X$. Then the following implications hold.*

(i) If A is complete, then it is closed.

(ii) If X is complete, then A is complete if and only if it is closed.

We are not only interested in the convergence of sequences of points in X, but also in the convergence of sequences of functions. Below are two modes of convergence to describe the behaviour of a sequence of functions in regard to a limit function, if it exists.

For $n \geq 1$, let $f_n \colon X \to Y$ be a function between two metric spaces. We say f_n *converges pointwise* to f if for every $x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

We say f_n *converges uniformly* to f if

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0.$$

This supremum might not exist, for example, if $X \subset \mathbb{R}$ is not a bounded set. A classic example of a sequence of functions $\{f_n\}_n$ that converges pointwise but not uniformly is given by $f_n \colon [0, 1] \to \mathbb{R}$, $f_n(x) := x^n$, which converges to

$$f(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

An important property of uniform convergence is that if each f_n is continuous, the *limit function* f is also continuous, while in the case of pointwise convergence, f may fail to be continuous as above. This proof of this property can be found in [Mun14, Theorem 21.6].

Theorem 2.2.3 (Uniform Limit Theorem)**.** *For each $n \geq 1$, let $f_n \colon X \to Y$ be a continuous function between a topological space X and a metric space Y. If $\{f_n\}_n$ converges uniformly to f, then f is continuous.*

Compactness and completeness are closely related through Proposition 2.1.2 and Proposition 2.2.2. In $\mathbb{R}^n$, a specific equivalence holds. For the proof of this theorem, see [Run05, Corollary 2.5.12].

Theorem 2.2.4 (Heine-Borel)**.** *Let $K \subset \mathbb{R}^n$ (with the Euclidean metric). Then K is compact if and only if it is bounded and closed in $\mathbb{R}^n$.*

We can now introduce the main object of study in this thesis. Let M be a compact Hausdorff space. We then write

$$C(M) := \{f \colon M \to \mathbb{R} \colon f \text{ is continuous}\}.$$

The uniform metric is defined as

$$d_\infty(f,g) := \sup_{x\in M} |f(x) - g(x)|\,.$$

To ensure the metric is well-defined, the existence of the supremum is necessary, which is indeed the case. The continuity of the function $h := f - g$ implies, together with Proposition 2.1.3, that $h(M)$ is a compact set in $\mathbb{R}$. By the Heine-Borel Theorem (Theorem 2.2.4), $h(M)$ must be bounded in $\mathbb{R}$. Based on the standard properties of $\mathbb{R}$, we can conclude that d_∞ satisfies all the necessary requirements of a metric.

Theorem 2.2.5. *The space $(C(M), d_\infty)$ is complete.*

Proof. Let $\{f_n\}_n$ be a Cauchy sequence for d_∞ in the space $C(M)$. For abritrary $x \in M$, the sequence $\{f_n(x)\}_n$ is Cauchy in $\mathbb{R}$ with the standard metric, because $d_\infty(f_n, f_m)$ is always an upper bound for $|f_n(x) - f_m(x)|$ for any $n, m \in \mathbb{N}$. Since $\mathbb{R}$ is a complete metric space, for each $x \in M$, the limit $\lim_{n\to\infty} f_n(x)$ exists.

Now, define $f\colon M \to \mathbb{R}$ by $f(x) := \lim_{n\to\infty} f_n(x)$, which is well-defined as each limit exists and is unique. For arbitrary $\eta > 0$, there exists some $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$d_\infty(f_n, f_m) = \sup_{z\in M} |f_n(z) - f_m(z)| < \eta,$$

as the sequence is Cauchy. Letting $m \to \infty$, for any $x \in M$ we find

$$|f_n(x) - f(x)| \leq \eta.$$

To prove that f is continuous, let $x \in M$ and $\varepsilon > 0$ be arbitrary. Because of the property above, there exists some $N \geq 1$ such that $\sup_{z\in M} |f_N(z) - f(z)| < \frac{1}{3}\varepsilon$. Because f_N is continuous, there exists some open neighbourhood U of x such that for $y \in U$, we have $|f_N(x) - f_N(y)| < \frac{1}{3}\varepsilon$. Hence, for any $y \in U$, with the triangle inequality, we find

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\
&\leq 2 \sup_{z\in M} |f(z) - f_N(z)| + |f_N(x) - f_N(y)| \\
&< 2 \cdot \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon \\
&= \varepsilon,
\end{aligned}$$

so f is continuous.

Then $f \in C(M)$, so all that is left is to show that $f_n \to f$ as $n \to \infty$. Since for any arbitrary η, we have some $N \geq 1$ such that $d_\infty(f_n, f) \leq \eta$ if $n \geq N$, this is indeed the case. □

2.2.1 Normed vector spaces

We consider one more specific kind of topological space: the normed vector space. The space of continuous functions $C(M)$ is also a vector space, and we will equip this space with a norm. In this section, we shortly examine a few properties of the normed vector space.

We denote a normed vector space over $\mathbb{R}$ by $(E, \|\cdot\|)$ with the norm $\|\cdot\|\colon E \to \mathbb{R}_{\geq 0}$. The standard norm on $\mathbb{R}$ is the Euclidean norm, given by $\|x\| := |x|$ for $x \in \mathbb{R}$. The space $C(M)$ is a normed vector space with the norm $\|\cdot\|_\infty\colon C(M) \to \mathbb{R}$ defined by

$$\|f\|_\infty := \sup_{x\in M} |f(x)|.$$

This is the standard norm on $C(M)$. We call a vector space E *complete* if it is complete with respect to the metric induced by the norm through setting $d(x, y) := \|x - y\|$ for all $x, y \in E$. By Theorem 2.2.5, the normed vector space $C(M)$ is complete.

In addition, the following result holds. Its proof can be found in [RY08, Corollary 2.19].

Proposition 2.2.6. *If $\|\cdot\|$ is any norm on a finite-dimensional space E over $\mathbb{R}$, then E is a complete metric space.*

For infinite-dimensional normed vector spaces, the associated metric space may not be complete. Normed vector spaces that are complete under the metric associated to the norm are called *Banach spaces*. Finite-dimensional normed vector spaces and $C(M)$ (with M a compact Hausdorff space) are both Banach spaces.

2.3 Theorems from real analysis

The following results will be used throughout this thesis. They apply specifically to functions on $\mathbb{R}$ with the standard norm.

Proposition 2.3.1 (Weierstrass M-test)**.** *Let $\{M_k\}_k$ be a sequence of non-negative real numbers such that $\sum_{k\geq 1} M_k < \infty$. Let $\{g_k\}_k$ be a sequence of real-valued functions on an interval $I \subset \mathbb{R}$. If $|g_k(x)| \leq M_k$ for all $x \in I$, then $\sum_{k\geq 1} g_k$ converges uniformly on I.*

Proof. Let $f_n(x) := \sum_{k=1}^{n} g_k(x)$ for $n \geq 1$. We have to show that this sequence converges uniformly to $f\colon I \to \mathbb{R}$ given by $f(x) := \lim_{n\to\infty} f_n(x)$. As the $\sum_{k\geq 1} M_k$ converges, we have

$$\lim_{n\to\infty} \sum_{k\geq n+1} M_k = 0.$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| &= \lim_{n\to\infty} \sup_{x\in I} \left| \sum_{k\geq n+1} g_k(x) \right| \\
&\leq \lim_{n\to\infty} \sup_{x\in I} \sum_{k\geq n+1} |g_k(x)| \\
&\leq \lim_{n\to\infty} \sum_{k\geq n+1} M_k \\
&= 0.
\end{aligned}$$

This implies that f_n converges uniformly to f. □

This next result is a specific case of the Stone-Weierstrass Theorem. For the proof, see [HS65, Corollary 7.31].

Theorem 2.3.2 (Weierstrass Approximation Theorem)**.** *Let f be a continuous real-valued function on the interval $[a, b] \subset \mathbb{R}$. For any $\varepsilon > 0$, there exists a polynomial p such that $\|f - p\|_\infty < \varepsilon$.*

According to the Intermediate Value Theorem, each continuous function on a closed interval has the intermediate value property, so if its domain contains the interval $[a, b]$, then f will take every value between $f(a)$ and $f(b)$. Darboux's Theorem ensures the derivative of a function also has the intermediate value property. For the proof, see [Ols04].

Theorem 2.3.3 (Darboux's Theorem)**.** *Let I be a closed interval, and let $f\colon I \to \mathbb{R}$ be a differentiable function. If $a, b \in I$ with $a < b$ and if y lies between $f'(a)$ and $f'(b)$ then there exists a number $x \in [a, b]$ such that $f'(x) = y$.*

The subsequent theorem provides the necessary conditions to draw conclusions about the differentiability of the inverse of a differentiable function. The formulation and proof can be found in [Tao22, Theorem 10.4.2].

Theorem 2.3.4 (Inverse Function Theorem)**.** *Let X, Y be subsets of $\mathbb{R}$ and let $f\colon X \to Y$ be an invertible function with inverse $f^{-1}\colon Y \to X$. Suppose x_0 and y_0 are limit points of X and Y respectively such that $f(x_0) = y_0$. If f is differentiable at x_0, f^{-1} is continuous at y_0 and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 and*

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

2.4 Measure theory

Though most of this thesis makes use of concepts from topology and functional analysis, for the results in Section 3.2 and Section 3.3 we rely on a basis of measure theory. The key properties and theorems used in these sections are listed here.

We refer to a measure space as $(X, \mathscr{A}, \mu)$. The main σ-algebras that we use in this thesis are $\mathfrak{P}(\mathbb{Z}_{\geq 1})$, the power set of $\mathbb{Z}_{\geq 1}$, and the Borel σ-algebra on $\mathbb{R}$. Every interval in $\mathbb{R}$ that will be relevant in this thesis is a measurable set for the Borel σ-algebra.

We often rely on the following properties of measures. A proof can be found in [Coh13, Proposition 1.2.2 and Proposition 1.2.4].

Proposition 2.4.1. *Let $(X, \mathscr{A}, \mu)$ be a measure space.*

(i) If $A, B \in \mathscr{A}$ such that $A \subset B$, then $\mu(A) \leq \mu(B)$.

(ii) If $\{A_n\}_n$ is a sequence of sets in $\mathscr{A}$, then $\mu(\cup_{n\geq 1} A_n) \leq \sum_{n\geq 1} \mu(A_n)$.

Recall that a measure is *finite* if $\mu(X) < \infty$ and *σ-finite* if X can be written as a countable union of sets $\{A_n\}_n$, with each $A_n \in \mathscr{A}$ and $\mu(A_n) < \infty$ for each n.

Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two measurable spaces. We say a function $f\colon X \to Y$ is a *measurable function* if for any $E \in \mathscr{B}$, the set $f^{-1}(E) \in \mathscr{A}$. For measurable functions, we denote the integral of f with respect to μ over a set X as $\int_X f(x)\, \mathrm{d}(\mu x)$. A function f is *integrable* if $\int_X |f(x)|\, \mathrm{d}(\mu x) < \infty$.

The two measures that will appear in this thesis are the *counting measure* and *Lebesgue measure* on $\mathbb{R}$. For $(X, \mathscr{A})$ any measurable space, we can define a function $\nu\colon \mathscr{A} \to [0, \infty]$ by setting $\nu(A) := |A|$, so $\nu(A)$ measures the cardinality of a set. Of course, when A is an infinite set, $\nu(A) = \infty$. This measure is called the *counting measure*. Consider $(\mathbb{Z}_{\geq 1}, \mathfrak{P}(\mathbb{Z}_{\geq 1}), \nu)$. Any function $f\colon \mathbb{Z}_{\geq 1} \to \mathbb{R}$ can be seen as a real-valued sequence with entries $a_n := f(n)$. Any sequence $\{a_n\}_n$ is then a measurable function, and the integral over such a sequence equals the sum $\sum_{n\geq 1} a_n$. If this series is absolutely convergent, the sequence is integrable with respect to the counting measure.

The Lebesgue measure λ on $\mathbb{R}$ is the standard way of assigning a value to intervals in $\mathbb{R}$. This value measures the length or size of an interval. It has the following properties.

(i) Every countable set in $\mathbb{R}$ has λ-measure 0.

(ii) Each interval in $\mathbb{R}$ is λ-measurable. If $[a, b] \subset \mathbb{R}$ with $a < b$, then $\lambda([a, b]) := b - a$. The open and half-open intervals with endpoints a and b are assigned the same value.

(iii) Every λ-measurable set in $\mathbb{R}$ can be approximated from above by open λ-measurable sets and from below by compact λ-measurable sets.

The third property means that λ is a *regular* measure. This property is made more precise in the following proposition. Its proof can be found in [Coh13, Proposition 1.4.1].

Proposition 2.4.2. *Let $A \subset \mathbb{R}$ be λ-measurable. Then the following two statements hold.*

(i) $\lambda(A) = \inf\{\lambda(U) \colon U$ *is open and* $A \subset U\}$.

(ii) $\lambda(A) = \sup\{\lambda(K) \colon K$ *is compact and* $K \subset A\}$.

If a property holds for all points in $\mathbb{R}$ except for on a set with Lebesgue measure zero, we say that the property holds *almost everywhere*. If we need to specify a different general measure μ, we say the property holds μ-almost everywhere. In Section 3.2, the Lebesgue measure will be viewed on the set $[0, 1]$, where the same properties as above hold.

Finally, we present four integral theorems that will be used later in this thesis. The first proposition states that integrable functions are almost everywhere finite. For the proof, we refer to [Coh13, Corollary 2.3.14].

Proposition 2.4.3. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be an integrable function with values in $[-\infty, \infty]$. Then $|f(x)| < \infty$ holds μ-almost everywhere for $x \in X$.*

The theorem below allows us to compute the Lebesgue integral with the Riemann integral. For the proof, see [Coh13, Theorem 2.5.4].

Theorem 2.4.4 (Equality of the Riemann and Lebesgue integral)**.** *Let $f \colon [a, b] \to \mathbb{R}$ be a bounded function. Then the following statements hold.*

(i) f is Riemann integrable if and only if it is continuous at almost every point of $[a, b]$.

(ii) If f is Riemann integrable, then f is Lebesgue integrable and the Riemann and Lebesgue integrals of f coincide.

The next theorem will be used to switch the integral and the limit for a sequence of increasing functions. For the proof, see [Coh13, Theorem 2.4.1].

Theorem 2.4.5 (Monotone Convergence Theorem)**.** *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let both the sequence $\{f_n\}_n$ and f be $[0, \infty]$-valued measurable functions on X. Then if for μ-almost every $x \in X$ we have that $f_n(x) \leq f_{n+1}(x)$ for all $n \geq 1$ and $f(x) = \lim_{n\to\infty} f_n(x)$, then*

$$\int_X f(x)\ d(\mu x) = \lim_{n\to\infty} \int_X f_n(x)\ d(\mu x).$$

The final theorem in this section permits us to change the order of integration when evaluating a double integral, for the proof see [Coh13, Proposition 5.2.1]. We write $\mathscr{A} \otimes \mathscr{B}$ for the σ-algebra on $X \times Y$ generated by the collection of measurable rectangles of the form $A \times B$, for $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

Theorem 2.4.6 (Tonelli's Theorem)**.** *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces. Let $f \colon X \times Y \to [0, \infty]$ be $\mathscr{A} \otimes \mathscr{B}$-measurable. Then f satisfies*

$$\int_{X\times Y} f\ d((\mu \times \nu)(x, y)) = \int_X \left(\int_Y f(x, y)\ d(\nu y) \right) d(\mu x) = \int_Y \left(\int_X f(x, y)\ d(\mu x) \right) d(\nu y).$$

2.5 Sets of the first and second category

Finally, a few more basic definitions from topology are needed to state and prove the Baire Category Theorem. Recall that we call a set A dense in some topological space S if its closure is the whole space. We have also seen two equivalent characterisations in Proposition 2.1.1. The standard example of a dense space is $\mathbb{Q} \subset \mathbb{R}$ with the standard metric. The formulation of Proposition 2.2.1 (in the context of metric spaces) means that we can think of the closure of $\mathbb{Q}$ as the set containing all the limit points of $\mathbb{Q}$ to make $\mathbb{R}$ complete. For general topological spaces, this means that every point in the space S is arbitrarily 'close' to a member of A. We now explore the concept of *nowhere dense* and *meagre* sets, whose points, on the other hand, are not clustered in the space. They are in fact 'mostly empty'.

Definition 2.5.1 (Nowhere dense)**.** A set E in a topological space S is *nowhere dense* if its closure contains no neighbourhoods. In other words, E is nowhere dense if for all non-empty, open $U \subset S$, there exists a non-empty open set $V \subset U$ for which $E \cap V = \emptyset$.

A trivial example of a nowhere dense set is any finite set in $\mathbb{R}$ with the standard topology. The empty set is also clearly nowhere dense. In a discrete space, the empty set is the only nowhere dense set. Other examples are $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{R}$ viewed as the horizontal axis of the plane in $\mathbb{R}^2$, both with the standard topology.

Note that dense and nowhere dense are not opposite definitions. If E is not dense, it means that its closure fails to fill some neighbourhood in the space. On the other hand, if E is nowhere dense, the closure of E contains no neighbourhood at all. There are many different definitions of being nowhere dense, some more intuitive than others. It is useful to be able to switch between these for different proofs.

Proposition 2.5.2. *The following are all equivalent definitions of E being nowhere dense in a topological space S.*

(i) For all non-empty open $U \subset S$, there exists a non-empty open $V \subset U$ with $E \cap V = \emptyset$.

(ii) For every non-empty open $U \subset S$, the interior of $U \setminus E$ is non-empty.

(iii) The interior of the closure of E is empty.

(iv) The complement of the closure of E is dense.

Proof. We will start by showing the first statement implies the second. Let $U \subset S$ be an arbitrary non-empty open set. By *(i)*, there exists a non-empty open set $V \subset U$ such that $E \cap V = \emptyset$. Then $V \subset U \setminus E$, so by definition $V \subset \text{int}\,(U \setminus E)$. As V was non-empty, we can conclude the interior of $U \setminus E$ is non-empty.

Now assume the second statement is in place. We prove this implies *(iii)*. Assume *(iii)* does not hold, so the interior of the closure of E is non-empty. Then there exists some non-empty open set $U \subset \text{cl}\,E$. By *(ii)*, int $(U \setminus E)$ is non-empty, so there exists some non-empty open set $V \subset U \setminus E$. This implies that $V \subset \text{cl}\,E \setminus E$. However, E is dense in its closure, so V must intersect E by Proposition 2.1.1, which poses a contradiction with our assumption that *(iii)* does not hold.

Assume now that *(iii)* holds. Suppose $S \setminus \text{cl}\,E$ is not dense. By Proposition 2.1.1, there is some non-empty open $U \subset S$ such that $U \cap (S \setminus \text{cl}\,E) = \emptyset$. Then $U \subset \text{cl}\,E$, but the interior of cl E is empty, so we arrive at a contradiction once again. Hence, *(iii)* implies *(iv)*.

Finally, assume $S \setminus \text{cl}\,E$ is dense. Let $U \subset S$ be any non-empty open set. By Proposition 2.1.1, $V := (S \setminus \text{cl}\,E) \cap U$ is a non-empty open set. Because $E \cap (S \setminus \text{cl}\,E) = \emptyset$, $V \cap E = \emptyset$, so we satisfy the requirements of *(i)*. These implications all together prove the equivalence of the four statements. □

With this definition, we can make a fundamental distinction between classes of sets.

Definition 2.5.3 (Meagre)**.** We call a set of the *first category* if it can be written as a countable union of nowhere dense sets. The term *meagre set* is more commonly used. Sets that cannot be described as such, are called *sets of the second category* (or *non-meagre*). If a set E is meagre, then the set $X \setminus E$ is called *comeagre*.

Since the empty set is nowhere dense, a set of the second category cannot be empty. Therefore, if we want to show a space is non-empty, we can do so by showing it is of the second category.[1] This is just one of the useful consequences of Baire's Theorem.

2.6 The Baire Category Theorem

The Baire Category Theorem, or Baire's Theorem for short, has a few different versions, two of which will be considered in this thesis. Both of these give sufficient conditions for a topological space to be a Baire space, as is defined below. The theorem is named after René-Louis Baire, who proved the result for Euclidean space $\mathbb{R}^n$ in 1899. There are a number of different ways to introduce Baire's Theorem, due to the many different definitions of nowhere dense sets. The main results and proofs in this section are based on [Mun14, Section 48, specifically p. 295-297] and [Con14, Theorem 1.6.1].

Definition 2.6.1 (Baire space)**.** A topological space S is called a *Baire space* if, for every countable collection $\{A_n\}_n$, where each A_n is closed and has empty interior in S, their union $\cup_{n\geq 1} A_n$ also has empty interior in S.

In this definition, each set A_n is nowhere dense. If no countable union of these sets contains an interior point, it can never be equal to the whole space S. Hence, a Baire space cannot be meagre. An equivalent definition that will be used in the proof is the following.

Proposition 2.6.2. *A topological space S is a Baire space if and only if for any countable collection of sets $\{U_n\}_n$, each open and dense in S, their intersection $\cap_{n\geq 1} U_n$ is also dense.*

Proof. For the right implication, suppose S is a Baire space and let $\{U_n\}_n$ be a countable collection of dense, open sets in S. Let $A_n := S \setminus U_n$ for each n. Then A_n is closed and has empty interior in S by Proposition 2.1.1. We have that $S \setminus \cap_{n\geq 1} U_n = \cup_{n\geq 1}(S \setminus U_n) = \cup_{n\geq 1} A_n$, which has empty interior because S is Baire, so $\cap_{n\geq 1} U_n$ is dense in S by Proposition 2.1.1 as well.

It can be checked that the left implication follows in the same way, setting $U_n := S \setminus A_n$ for a collection of closed sets A_n, each with empty interior in S. $\square$

We need one more property of Baire spaces before stating Baire's Theorem.

Proposition 2.6.3. *If Y is a Baire space and $X \subset Y$ is open, then X is a Baire space.*

Proof. We first show that if $X \subset Y$ is an open subspace of Y and $A \subset X$ is nowhere dense in X, then A is also nowhere dense in Y. Indeed, arguing by contradiction, assume that A is not nowhere dense in Y. By Proposition 2.5.2, we have that

$$U := \mathrm{int}_Y(\mathrm{cl}_Y(A)) \neq \emptyset.$$

Because of the definition of the subspace topology, $U \cap X$ is open in X. In addition, $U \cap X \subset \mathrm{cl}_Y(A) \cap X = \mathrm{cl}_X(A)$. However, $\mathrm{int}_X(\mathrm{cl}_X(A)) = \emptyset$ because A is nowhere dense in X, so $\mathrm{cl}_X(A)$ cannot contain an open set. This yields a contradiction, so we must have that A is nowhere dense in Y.

[1] The word 'category' is somewhat controversial. It was introduced by Baire, but many authors have since rejected the term for its lack of meaning to what the concept stands for. Since the word still appears in the Baire Category Theorem, I have used it here. In the remainder of this thesis, the term 'meagre' will be used instead.

Let $\{A_n\}_n$ be a countable collection of closed sets with empty interior in X. For each n, A_n is nowhere dense in X, so by the above, all A_n are nowhere dense in Y. Because Y is a Baire space, we have $\text{int}_Y(\cup_{n\geq 1} A_n) = \emptyset$. If $\text{int}_X(\cup_{n\geq 1} A_n) \neq \emptyset$, then there exists some open non-empty $U \subset \cup_{n\geq 1} A_n$. Then U is also open in Y, by the property of the subspace topology and the fact that X is open in Y. However, this contradicts that the interior of $\cup_{n\geq 1} A_n$ is empty in Y. It then must hold that the interior of $\cup_{n\geq 1} A_n$ is also empty in X, so X is a Baire space. □

We now present a proof of the Baire Category Theorem for two types of topological spaces. The idea of the proof, both for the metric space and for the compact Hausdorff space (which will be extended to a locally compact Hausdorff space), is to take a collection of dense and open sets and an arbitrary non-empty open set A. We then construct a nested sequence in these dense sets that all intersect A, and show the intersection of all these sets with A is non-empty, in order to apply Proposition 2.6.2. This particular formulation of the theorem is one of the most general versions, but the Baire Category Theorem can be further extended to the case of complete pseudometric spaces.

Theorem 2.6.4 (Baire Category Theorem)**.** *Every complete metric space and every locally compact Hausdorff space is a Baire space.*

Proof. For the first part of the theorem, assume (X, d) is a complete metric space. Let $\{U_n\}_n$ be a collection of open and dense sets in X. Let $A \subset X$ be any non-empty open set in X. We show that $\cap_{n\geq 1} U_n$ intersects A.

Because U_1 is dense and open, $A \cap U_1$ is non-empty and open, and there exists an open ball around a point $x_1 \in A \cap U_1$ with radius $r_1 < 1$ such that $\text{cl } B_{r_1}(x_1) \subset A \cap U_1$. For any $n \geq 2$, consider $B_{r_{n-1}}(x_{n-1}) \cap U_n$. This set is open and non-empty. Then there exists some $B_{r_n}(x_n)$ with radius $r_n < \frac{1}{n}$ such that

$$\text{cl } B_{r_n}(x_n) \subset B_{r_{n-1}}(x_{n-1}) \cap U_n \subset A,$$

the latter inclusion following from the fact that $B_{r_1}(x_1) \subset A$ and each subsequent open ball is chosen as a subset of the previous open ball.

As we now have a nested sequence of open balls, we find that for $n > N$

$$\text{cl } B_{r_n}(x_n) \subset B_{r_N}(x_N) \cap U_N \subset A \cap U_N.$$

For arbitrary $\varepsilon > 0$ and $N > \frac{2}{\varepsilon}$, for $n, m \geq N$ it holds that $\text{cl } B_{r_n}(x_n) \subset B_{r_N}(x_N)$ and $\text{cl } B_{r_m}(x_m) \subset B_{r_N}(x_N)$, which implies that $d(x_n, x_m) < \frac{2}{N} < \varepsilon$, so $\{x_n\}_n$ is Cauchy. Since X is complete, $x_n \to x$ for some $x \in X$.

For arbitrary N and $n > N$,

$$x \in \text{cl } B_{r_n}(x_n) \subset B_{r_N}(x_N) \cap U_N \subset A \cap U_N.$$

Since this holds for any N, $x \in A \cap (\cap_{N\geq 1} U_N)$, so $\cap_{N\geq 1} U_N$ is dense. Hence (X, d) is a Baire space.

Now, let $(S, \mathcal{T})$ be a compact Hausdorff space (we will later extend this to the situation where S is LCH). Just as in the metric case, we start with a collection of open and dense sets $\{U_n\}_n$ and show that their intersection has a non-empty intersection with an arbitrary non-empty open set $A \subset S$.

The intersection $A \cap U_1$ is non-empty and open, so there must exist some $x_1 \in A \cap U_1$. Because S is compact Hausdorff, the space is regular according to Proposition 2.1.2, which allows us to separate points and closed sets with open neighbourhoods. The set $S \setminus (A \cap U_1)$ is closed, so there exist disjoint open neighbourhoods $V_1, W_1 \subset S$ such that $x_1 \in V_1$ and $S \setminus (A \cap U_1) \subset W_1$. The fact that S is compact Hausdorff also means that V_1 contains a compact neighbourhood of x_1 by Proposition 2.1.2. As a result, there exists a compact set $K_1 \subset V_1$ such that $x_1 \in Z_1 \subset K_1$ for an open set Z_1. We obtain the inclusions $x_1 \in Z_1 \subset K_1 \subset V_1 \subset A \cap U_1$, using that V_1 and W_1 are disjoint.

Now for $n \geq 2$, consider $A \cap Z_{n-1} \cap U_n$, which is open and non-empty, so it contains at least one point x_n. By regularity of S, we can find disjoint open neighbourhoods V_n, W_n such that $x_n \in V_n$, $S \setminus (A \cap V_{n-1} \cap U_n) \subset W_n$. Then there also exists a compact neighbourhood K_n containing an open neighbourhood Z_n of x_n, so we have that $x_n \in Z_n \subset K_n \subset V_n \subset A \cap Z_{n-1} \cap U_n$. In particular, $K_n \subset Z_{n-1} \subset K_{n-1}$ and $K_n \subset A \cap U_n$.

Proceeding inductively, we find a collection of nested closed sets $\{K_n\}_n$ and for every $n \geq 2$ we have that $x_n \in K_n \subset Z_{n-1} \subset K_{n-1} \subset ... \subset K_1$, with $K_n \subset A \cap (\cap_{k=1}^n U_k)$. Hence, $\{K_n\}_n$ has the finite intersection property. Because S is compact and each K_n is closed, the intersection $\cap_{n \geq 1} K_n$ is non-empty as well. Because $\cap_{n\geq 1} K_n \subset A \cap (\cap_{n \geq 1} U_n)$, this intersection is non-empty, so $\cap_{n \geq 1} U_n$ is dense in S.

Now, assume that $(S, \mathcal{T})$ is locally compact and Hausdorff, and not compact: otherwise, the above procedure would suffice. We can then apply Theorem 2.1.6, so S has a one-point compactification $(S_\infty, \mathcal{T}_\infty)$. The space S_∞ is then compact and Hausdorff, so by the above, a Baire space. It holds that $S \in \mathcal{T}_\infty$, as $S \in \mathcal{T}$ and $\mathcal{T}_\infty = \mathcal{T} \sqcup \{S_\infty \setminus K \colon K \subset S, K \text{ compact}\}$. Hence, S is an open subspace of S_∞, so we can apply Proposition 2.6.3 to conclude that S is a Baire space. □

Remark. Note that the fact that when for every $n \geq 1$ we choose a point $x_n \in B_{r_{n-1}}(x_{n-1}) \cap U_n$ in the metric case of Baire's Theorem, the fact that a choice function exists for the entire sequence of elements depends on the axiom of countable choice, a weaker version of the axiom of choice. It turns out that Baire's Theorem is actually equivalent to the axiom of dependent choice, which is stronger than the axiom of countable choice and weaker than the axiom of choice. This goes beyond the scope of this thesis, but the proof of this statement can be found in [Her06, Theorem 4.106].

3 The differentiability of continuous functions

In this chapter, we delve into two results that follow from the Baire Category Theorem. We explore a specific Banach space: the vector space of real-valued continuous functions on $[0,1]$. We denote this space by $C[0,1]$, which was introduced as $C(M)$ for a general compact Hausdorff space M in Section 2.2. In what follows, the norm $\|\cdot\|_\infty$ on $C(M)$ will be denoted as $\|\cdot\|$, as it is the only norm considered in the rest of this thesis.

To establish a connection between the main theorems in 3.1 and 3.3, Section 3.2 provides a bridging result between these sections. In the proofs of Theorem 3.1.4 and Theorem 3.3.8, Baire's Theorem is applied in a similar way to show functions exist with a certain property. The strategy for these proofs will be to find a suitable space within a complete metric space that does not have the desired property. We will then show that this space is meagre, so it fails to encompass the whole space, which is a Baire space. This is a common thread to follow through the proofs that require quite some calculus trickery.

Though Section 2.5 and Section 2.6 laid the groundwork for a broader context, we narrow our focus to continuous functions on $[0,1]$ in this thesis. For general metric spaces, different analytical methods are needed to define a sensible concept of differentiability. Since we need a compact space, the closed interval $[0,1]$ serves as a convenient example, though the results in this chapter are applicable to any closed interval $[a,b]$ with $a<b$.

Within the proofs, there are multiple times where the differential quotient of a function is calculated by taking a limit. For the boundary points of $[0,1]$, we need to take a one-sided limit instead. While we may omit this remark during the proofs, it is beneficial to keep it in mind.

3.1 Nowhere differentiable continuous functions

The first consequence of Baire's Theorem for continuous functions that we examine is that there exist functions that are continuous but nowhere differentiable. Though this is already a non-intuitive fact, it is even more remarkable that these functions are actually 'typical' in the space of continuous functions, in the sense that the set of nowhere differentiable functions is dense in the set of continuous functions. In this section, we give a proof of this result. However, we prove the existence of nowhere differentiable functions instead by the example of the famous Weierstrass function. This gives more of a sense of what a nowhere differentiable function can look like.

3.1.1 The Weierstrass function

Generally, the Weierstrass function $\widetilde{W}\colon \mathbb{R}\to\mathbb{R}$ is defined for general a and b with a a positive odd integer and $b\in(0,1)$ such that $ab>1+\frac{3}{2}\pi$, and is given by

$$\widetilde{W}(x) := \sum_{n\geq 0} b^n \cos(a^n x\pi).$$

Here, we present the proof that the Weierstrass function given by $W(x) := \sum_{n\geq 0} 5^{-n}\cos(175^n x)$ is continuous and nowhere differentiable. Working with specific a and b makes the proof more insightful than for general a and b, which requires a lot of algebraic manipulation. The proof that $\widetilde{W}$ is continuous and nowhere differentiable, first presented by Karl Weierstrass in 1872, can be found in [Wei95]. Though W does not entirely take the form of $\widetilde{W}$ above (the π is missing in the cos term), by plugging a change of variable given by $x\mapsto x\pi$ into W, we still obtain the Weierstrass function $\widetilde{W}$. Indeed, if $W(x)$ is continuous and nowhere differentiable, then $W(x\pi)$ is as well. It is also easily verified that a and b satisfy the necessary conditions.

We first present a lemma to obtain the appropriate goniometric manipulation.

Lemma 3.1.1. *For all $x, y, z \in \mathbb{R}$ and $K \in \mathbb{Z}_{\geq 1}$, the following statements hold.*

(i) We have $|\cos(x) - \cos(y)| \leq |x - y|$.

(ii) There exists some $p \in \left(z - \frac{\pi}{K}, z + \frac{\pi}{K}\right)$ such that $|\cos(Kz) - \cos(Kp)| \geq 1$.

Proof. For the first statement, let $x, y \in \mathbb{R}$ be given arbitrarily. If $x = y$, the statement holds. Without loss of generality, assume $x < y$. By the Mean Value Theorem, there exists some $z \in (x, y)$ such that

$$|\sin(z)| = \frac{|\cos(x) - \cos(y)|}{|x - y|}.$$

We can rewrite this statement to

$$|\cos(x) - \cos(y)| = |\sin(z)||x - y| \leq |x - y|.$$

For the second statement, let $z \in \mathbb{R}$ and $K \geq 1$ be given. We distinguish between the following cases.

(i) $\cos(Kz) = 0$,

(ii) $0 < \cos(Kz) < 1$ and

(iii) $\cos(Kz) = 1$.

The cases that $-1 < \cos(Kz) < 0$ and $\cos(Kz) = -1$ are symmetric respectively to case (ii) and case (iii).

In case (i), Kz must be of the form $(\frac{1}{2} + n)\pi$ for $n \in \mathbb{Z}$. By multiplying $\left(z - \frac{\pi}{K}, z + \frac{\pi}{K}\right)$ by K, we see that we must find some $Kp \in \left(\left(n - \frac{1}{2}\right)\pi, \left(n + \frac{3}{2}\right)\pi\right)$. Then we can choose $Kp = n\pi$ to find $|\cos(Kz) - \cos(Kp)| = |\pm 1| \geq 1$.

In case (ii), we have that $Kz \in ((-\frac{1}{2} + 2n)\pi, (\frac{1}{2} + 2n)\pi) \setminus \{2n\pi\}$ for some $n \in \mathbb{Z}$. Then if $Kz \in ((-\frac{1}{2} + 2n)\pi, 2n\pi)$, choose $Kp = (2n - 1)\pi$. In case $Kz \in (2n\pi, (\frac{1}{2} + 2n)\pi)$, then choose $Kp = (2n + 1)\pi$. In both cases, $Kp \in (Kz - \pi, Kz + \pi)$, and $\cos(Kp) = -1$. We then find

$$|\cos(Kz) - \cos(Kp)| = |\cos(Kz) + 1| > 1$$

as $\cos(Kz) > 0$.

In case (iii), we have that $Kz = 2n\pi$ for some $n \in \mathbb{Z}$. Choose $Kp = (2n + \frac{1}{2})\pi$, then $Kp \in (Kz - \pi, Kz + \pi)$ and $|\cos(Kz) - \cos(Kp)| = |1 - 0| \geq 1$. □

We also require the following inequality.

Proposition 3.1.2. *Let $a, b, c \in \mathbb{R}$. Then $|a + b + c| \geq |a| - |b| - |c|$.*

Proof. By the triangle inequality, we have that

$$|a| = |a + b + c - b - c| \leq |a + b + c| + |b| + |c|$$

so by substracting $|b|$ and $|c|$, we find the required inequality. □

Theorem 3.1.3. *The function $W : \mathbb{R} \to \mathbb{R}$ defined by*

$$W(x) := \sum_{n \geq 0} 5^{-n} \cos(175^n x)$$

is continuous and not differentiable for any $x \in \mathbb{R}$.

Proof. The easy part of the proof is the continuity of W. Note that $|5^{-n}\cos(175^n x)| \leq 5^{-n}$ for any $n \geq 0$ and value of x. As $\frac{1}{5} < 1$, the sum $\sum_{n\geq 0} 5^{-n}$ is a convergent geometric series, so W converges uniformly by the Weierstrass M-text (Proposition 2.3.1), and W is continuous by Theorem 2.2.3.

To prove W is nowhere differentiable, we show that for any $x \in \mathbb{R}$ and any $M > 0$, there exists a sequence $\{x_n\}_n$ for which $x_n \to x$ as $n \to \infty$, so that there exists some $N > 0$ such that for $n \geq N$ we have

$$\frac{|W(x_n) - W(x)|}{|x_n - x|} \geq M.$$

This implies that the differential quotient of $W(x)$ does not converge, so $W'(x)$ cannot exist.

For $n \geq 1$, choose $x_n \in \left(x - \frac{\pi}{175^n}, x + \frac{\pi}{175^n}\right)$ such that $|\cos(175^n x) - \cos(175^n x_n)| \geq 1$ by applying Lemma 3.1.1. Then $x_n \to x$ as $n \to \infty$. Let N be such that $35^N \geq M \cdot \frac{17\pi}{6}$ and $n \geq N$. Write $W_k(x) := 5^{-k}\cos\bigl(175^k x\bigr)$. Then

$$\begin{aligned} |W(x_n) - W(x)| &= \left|\sum_{k\geq 0} (W_k(x_n) - W_k(x))\right| \\ &\geq |W_n(x_n) - W_n(x)| - \left|\sum_{k=0}^{n-1} (W_k(x_n) - W_k(x))\right| - \left|\sum_{k\geq n+1} (W_k(x_n) - W_k(x))\right|, \end{aligned}$$

using Proposition 3.1.2. For brevity, we write

$$\begin{aligned} |a_n| &:= |W_n(x_n) - W_n(x)| \\ |b_n| &:= \left|\sum_{k=0}^{n-1} (W_k(x_n) - W_k(x))\right| \\ |c_n| &:= \left|\sum_{k\geq n+1} (W_k(x_n) - W_k(x))\right| \end{aligned}$$

We find estimates for each term $|a_n|, |b_n|$ and $|c_n|$ separately.

We observe that

$$|a_n| = 5^{-n}|\cos(175^n x_n) - \cos(175^n x)| \geq 5^{-n},$$

by our choice of x_n.

We also have that

$$\begin{aligned}
|b_n| &= \left|\sum_{k=0}^{n-1} 5^{-k}\left(\cos\left(175^k x_n\right) - \cos\left(175^k x\right)\right)\right| \\
&\leq \sum_{k=0}^{n-1} 5^{-k}\left|\cos\left(175^k x_n\right) - \cos\left(175^k x\right)\right| \\
&\leq \sum_{k=0}^{n-1} 5^{-k}\left|175^k x_n - 175^k x\right| \\
&< \sum_{k=0}^{n-1} 5^{-k} 175^k \frac{\pi}{175^n} \\
&= \frac{\pi}{175^n}\sum_{k=0}^{n-1} 35^k \\
&= \frac{\pi}{175^n}\left(\frac{35^n - 1}{34}\right) \\
&< \frac{\pi}{34}\left(\frac{1}{5}\right)^n \\
&< \frac{1}{34}5^{-n+1},
\end{aligned}$$

applying in order the triangle inequality, the first part of Lemma 3.1.1, the fact that $x_n \in \left(x - \frac{\pi}{175^n}, x + \frac{\pi}{175^n}\right)$ and the formula for geometric series. Finally,

$$\begin{aligned}
|c_n| &= \left|\sum_{k\geq n+1} 5^{-k}\left(\cos\left(175^k x_n\right) - \cos\left(175^k x\right)\right)\right| \\
&\leq \sum_{k\geq n+1} 5^{-k}\left|\cos\left(175^k x_n\right) - \cos\left(175^k x\right)\right| \\
&\leq \sum_{k\geq n+1} 5^{-k}\left(\left|\cos\left(175^k x_n\right)\right| + \left|\cos\left(175^k x\right)\right|\right) \\
&\leq 2\sum_{k\geq n+1} 5^{-k} \\
&= 2\cdot 5^{-n-1}\cdot\frac{5}{4} \\
&= \frac{1}{2}\cdot 5^{-n}.
\end{aligned}$$

Combining all of these estimates, we find

$$|a_n| - |b_n| - |c_n| \geq 5^{-n} - \frac{5}{34}5^{-n} - \frac{1}{2}5^{-n} = \frac{6}{17}5^{-n}.$$

Then it holds that

$$\begin{aligned}
\frac{|W(x_n) - W(x)|}{|x_n - x|} &\geq \frac{6}{17}5^{-n}\cdot\frac{1}{|x_n - x|} \\
&> \frac{6}{17}5^{-n}\cdot\frac{175^n}{\pi} \\
&= \frac{6}{17\pi}35^n \\
&\geq \frac{6}{17\pi}35^N \\
&> M.
\end{aligned}$$

We conclude that W is not differentiable anywhere. $\square$

In Figure 1, a numerical approximation of the graph of the modified Weierstrass function of Theorem 3.1.3 is given. By setting the domain as $[0, 8]$, the oscillating behaviour is well visible.

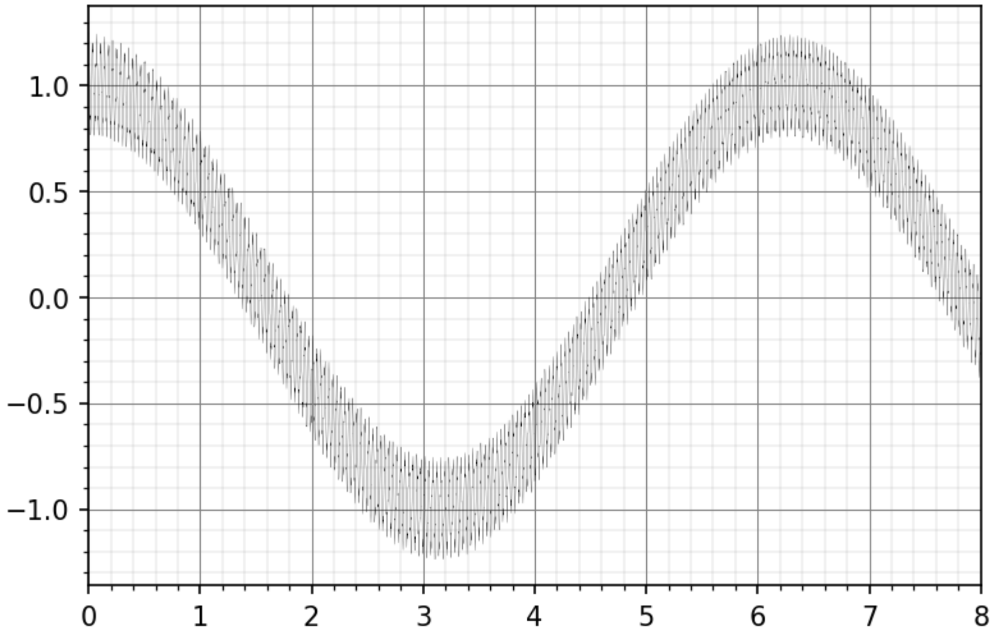

Figure 1: Graph of the modified Weierstrass function for $a = 175$, $b = \frac{1}{5}$, partial sum size $N = 30$ and domain $[0, 8]$.

Zooming in on the image eventually yields a similar view. Figure 2 shows a graph of the same functions as Figure 1, but on the domain $[0, 0.05]$.

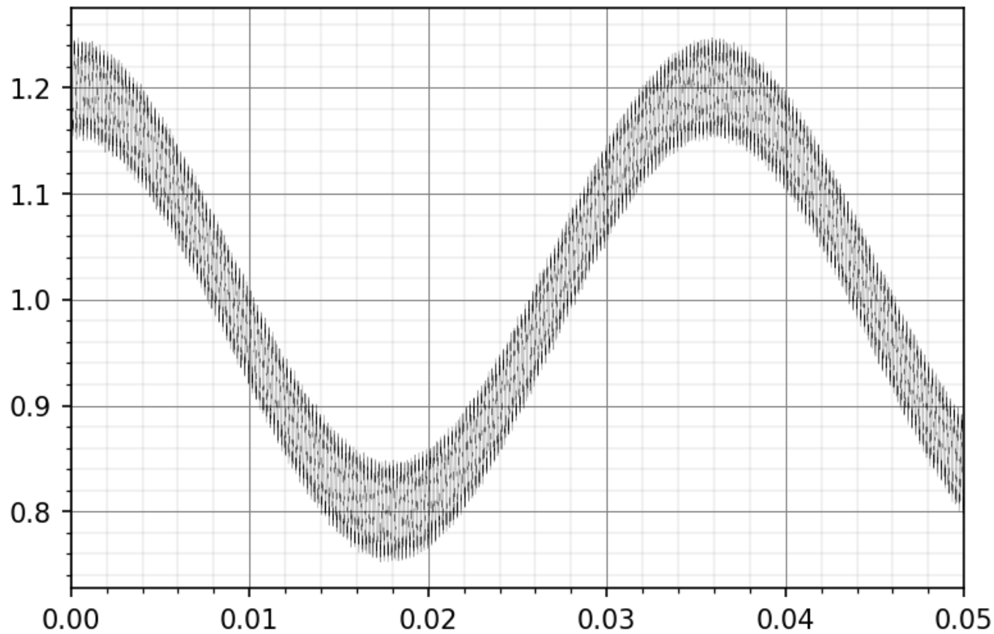

Figure 2: Graph of the modified Weierstrass function for $a = 175$, $b = \frac{1}{5}$ and partial sum size $N = 30$ and domain $[0, 0.05]$.

3.1.2 The density of nowhere differentiable functions

By the example of Theorem 3.1.3, it is clear that nowhere differentiable continuous functions exist. The Weierstrass function for more general a and b yields an entire class of such functions. Baire's Theorem allows us to draw an even stronger conclusion: nowhere differentiable functions are dense in the space of continuous functions.

In the proof below, we show that the space of functions that are differentiable in at least one point x_0 is a subset of the space of functions f that satisfy the equation $|f(x)-f(x_0)| \leq n|x-x_0|$ for some $n \geq 1$ and all $x \in [0,1]$. We establish this space is meagre in $C[0,1]$. We can then conclude that the complement of this set is dense in $C[0,1]$. The proof is based on [Fol99, Exercise 42 on p. 165] and [Whe22].

Theorem 3.1.4 (Density of nowhere differentiable functions)**.** *The space N of nowhere differentiable functions is dense in $C[0,1]$.*

Proof. For $n \geq 1$, let E_n be the set of all $f \in C[0,1]$ for which there exists some x_0 (possibly depending on f) such that for all $x \in [0,1]$,

$$|f(x) - f(x_0)| \leq n|x - x_0|.$$

Let $n \geq 1$ be given. Now we show that E_n is nowhere dense in $C[0,1]$. To this end, we prove that E_n is closed in $C[0,1]$ and has empty interior. By Proposition 2.2.1, we want to show that every convergent sequence in E_n converges in E_n. Let $\{f_k\}_k$ be a sequence in E_n that converges to some $f \in C[0,1]$. We must show that $f \in E_n$.

For arbitrary $\varepsilon > 0$, there exists some $K \in \mathbb{Z}_{\geq 1}$ such that for $k \geq K$, $\|f_k - f\| < \varepsilon$. Because for each $k \geq K$, we have $f_k \in E_n$, there must exist $x_k \in [0,1]$ such that $|f_k(x) - f_k(x_k)| \leq n|x - x_k|$. We can write this as the sequence $\{x_k\}_k$ in $[0,1]$. Because $[0,1]$ is bounded, the sequence is bounded and must have a convergent subsequence $\{x_{k_l}\}_{k_l}$ by the Bolzano-Weierstrass Theorem. We call the limit of this sequence x_0. Take $\{f_{k_l}\}_{k_l}$ as the associated subsequence of $\{f_k\}_k$ to $\{x_{k_l}\}_{k_l}$. Finally, for any $y \in [0,1]$, we have that

$$|f_{k_l}(y) - f(y)| \leq \sup_{x \in [0,1]} |f_{k_l}(x) - f(x)| = \|f_{k_l} - f\| < \varepsilon.$$

Now, set $\beta := \frac{1}{2+2n}\varepsilon$. Then choose L_1 such that for all $k_l \geq L_1$, we have $\|f - f_{k_l}\| < \beta$. Choose L_2 such that for all $k_l \geq L_2$, $|x_0 - x_{k_l}| < \beta$. Set $L := \max\{L_1, L_2\}$. Now, let $k_l \geq L$ be arbitrarily given.

Using these estimates, we find for any $x \in M$ that

$$\begin{aligned}
|f(x) - f(x_0)| &\leq |f(x) - f_{k_l}(x)| + |f_{k_l}(x) - f_{k_l}(x_0)| + |f_{k_l}(x_0) - f(x_0)| \\
&\leq 2\|f - f_{k_l}\| + |f_{k_l}(x) - f_{k_l}(x_0)| \\
&\leq 2\|f - f_{k_l}\| + |f_{k_l}(x) - f_{k_l}(x_{k_l})| + |f_{k_l}(x_{k_l}) - f_{k_l}(x_0)| \\
&\leq 2\|f - f_{k_l}\| + n|x - x_{k_l}| + n|x_{k_l} - x_0| \\
&\leq 2\|f - f_{k_l}\| + n|x - x_0| + n|x_0 - x_{k_l}| + n|x_{k_l} - x_0| \\
&< 2\beta + n|x - x_0| + 2n\beta \\
&= (2 + 2n)\beta + n|x - x_0| \\
&= \varepsilon + n|x - x_0|.
\end{aligned}$$

Since we can find this inequality for arbitrary small ε, it must hold that $|f(x)-f(x_0)| \leq n|x-x_0|$, so $f \in E_n$.

We still need to show that E_n has empty interior. Let $f \in E_n$ and $\varepsilon > 0$. Consider the open ball $B_\varepsilon(f) := \{h \in E_n : \|h - f\| < \varepsilon\}$. By the Weierstrass Approximation Theorem (Theorem 2.3.2), there exists some polynomial $p \in C[0,1]$ with real coefficients such that $\|f - p\| < \frac{1}{2}\varepsilon$. Set $B := \|p'\|$ and $k := \frac{\varepsilon}{2(n+B+1)}$. Then we define $\phi\colon [0,1] \to \mathbb{R}$ by

$$\phi(x) := \begin{cases} (B+n+1)(x-kj) & \text{if } kj \le x < k(j+1), j \in \mathbb{Z} \text{ even} \\ (B+n+1)(k(j+1)-x) & \text{if } kj \le x < k(j+1), j \in \mathbb{Z} \text{ odd.} \end{cases}$$

This is a 'sawtooth' function, which separates $[0,1]$ in parts of length $2k$. In the intervals where $kj \le x < k(j+1)$ for j even, ϕ is increasing, else it is decreasing. We have that

$$\|\phi\| = (B+n+1)(k(j+1) - kj) = (B+n+1)k = \frac{1}{2}\varepsilon$$

and the slope of ϕ is given by its height $\frac{1}{2}\varepsilon$ divided by the length of the interval k, so depending on whether ϕ is increasing or decreasing, the slope is equal to $\pm\frac{\varepsilon}{2k} = \pm(B+n+1)$. Now let $g := p + \phi$. Then $\|f - g\| \le \|f - p\| + \|\phi\| < \varepsilon$, so $g \in B_\varepsilon(f)$. However, by taking the limit of x to any $x_0 \in [0,1]$ for which the limit exists, we find

$$\lim_{x \to x_0} \frac{|g(x_0) - g(x)|}{|x_0 - x|} = |g'(x_0)| = |p'(x_0) \pm (B+n+1)| \ge n+1.$$

If this limit does not exist because g is not differentiable in x_0 (which occurs for a finite number of points because ϕ is piecewise linear), we can take a one-sided limit and arrive at the same inequality. Hence, there exists no x_0 such that $|g(x) - g(x_0)| \le n|x - x_0|$, so $g \notin E_n$. So E_n has empty interior.

Now we show that $C[0,1] \setminus N \subset \cup_{n\ge1} E_n$, meaning that the functions that are differentiable in some point x_0 are a subset of $\cup_{n\ge1} E_n$. Let $f \in C[0,1]$ be a function that is differentiable in some point x_0. Then there exist $K \in \mathbb{N}$ and $\delta > 0$ such that for all $x \in [0,1]$ with $|x - x_0| < \delta$,

$$\frac{|f(x_0) - f(x)|}{|x_0 - x|} \le K,$$

where δ can depend on K. This can be rewritten as $|f(x_0) - f(x)| \le K|x_0 - x|$. We show that there exists some $n \ge 1$ such that $f \in E_n$. For $|x_0 - x| < \delta$, taking $n = K$ suffices. If $|x_0 - x| \ge \delta$, we find the following estimate.

$$\begin{aligned} |f(x) - f(x_0)| &\le |f(x)| + |f(x_0)| \\ &\le 2\|f\| \\ &\le 2\|f\| \frac{|x - x_0|}{\delta}. \end{aligned}$$

Taking $n \ge \max\left\{K, \frac{2\|f\|}{\delta}\right\}$ as a whole number, we find $f \in E_n$, so $f \in \cup_{n\ge1} E_n$.

By Proposition 2.5.2, the set $F_n := C[0,1] \setminus E_n = C[0,1] \setminus \operatorname{cl} E_n$ is dense in $C[0,1]$. The set F_n is open as well as dense. By the Baire Category Theorem (Theorem 2.6.4), $C[0,1]$ is a Baire space, so we can apply Proposition 2.6.2 to find $\cap_{n\ge1} F_n = \cap_{n\ge1} C[0,1] \setminus E_n = C[0,1] \setminus \cup_{n\ge1} E_n$ is also dense in $C[0,1]$.

As $C[0,1] \setminus N \subset \cup_{n\ge1} E_n$, it also holds that $C[0,1] \setminus \cup_{n\ge1} E_n \subset N$. The left hand side is dense in $C[0,1]$, so N must also be dense in $C[0,1]$. $\square$

Figure 3 depicts a sawtooth function ϕ used in the approximation of a function $f \in E_n$ in the proof of Theorem 3.1.4. The image is accessed from [Whe22].

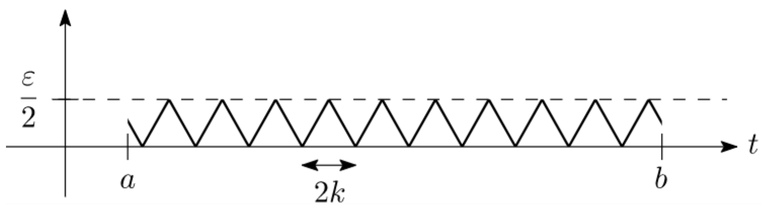

Figure 3: Illustration of the function ϕ on an interval $[a, b]$. Each 'sawtooth' has a height of $\frac{\varepsilon}{2}$ and a width of $2k$.

3.2 Nowhere monotone functions

The Weierstrass function is a nowhere differentiable function that is constantly oscillating, as we can see in Figure 1. In fact, there is no point where the function either increases or decreases. It is useful to have a precise definition of what we mean by *not increasing* and *not decreasing*.

Definition 3.2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We say f is *not increasing* in a point $x \in \mathbb{R}$ if for any neighbourhood N of x, there exists some $t \in N$ such that $(f(t)-f(x))(t-x) < 0$. Similarly, f is *not decreasing* in x if for any neighbourhood N, there exists some $t \in N$ such that $(f(t) - f(x))(t-x) > 0$. We say f is *not monotone in x* if f is neither increasing nor decreasing, and f is *nowhere monotone* if f is not monotone for any $x \in \mathbb{R}$.

From Figure 1, it seems likely that the Weierstrass function is nowhere monotone. In this section, we ascertain that each nowhere differentiable function is in fact nowhere monotone. This statement is the reverse implication of Lebesgue's Theorem for the Differentiability of Monotone Functions. The proof that continuous monotone functions are differentiable almost everywhere was first published in 1904 by Henri Lebesgue in [Leb04]. This section aims to present a concise proof of this theorem. The definitions and proofs in this section are drawn from [HS65, Section 17]. The building blocks of the proof of Lebesgue's Theorem are the concept of Dini derivatives and that of the Vitali cover, which we introduce below. This version of the proof of Lebesgue's Theorem does not require f to be continuous, so we are working in a more general context than on $C[0, 1]$.

Definition 3.2.2 (Dini derivatives)**.** Let $x \in \mathbb{R}$ and $\delta > 0$. If f is a real-valued function defined on $[x, x+\delta)$, we define

$$D_+f(x) := \liminf_{h\downarrow 0} \frac{f(x+h) - f(x)}{h}$$

and

$$D^+f(x) := \limsup_{h\downarrow 0} \frac{f(x+h) - f(x)}{h}.$$

If f is a real-valued function on $(x-\delta, x]$, define

$$D_-f(x) := \liminf_{h\uparrow 0} \frac{f(x+h) - f(x)}{h}$$

and

$$D^-f(x) := \limsup_{h\uparrow 0} \frac{f(x+h) - f(x)}{h}.$$

These numbers can take any value on the extended reals and are known as the *Dini derivatives of f at x*. More specifically, $D_+f(x)$ is the *lower right derivative*, $D^+f(x)$ is the *upper right derivative*, $D_-f(x)$ is the *lower left derivative* and $D^-f(x)$ is the *upper left derivative.*

Remark. It obviously holds that

$$D_+f(x) \le D^+f(x)$$

and

$$D_-f(x) \leq D^-f(x)$$

if the Dini derivatives exist for $x \in \mathbb{R}$.

The Dini derivative is a generalisation of the usual concept of differentiation. Not every function has a derivative, but the Dini derivatives of a function do always exist if f is defined on a half-open interval at x and if we allow all values on the extended reals.

Definition 3.2.3. Let $f\colon [0,1] \to \mathbb{R}$ and $x \in [0,1]$. If $D_+f(x) = D^+f(x)$, then f has a *right derivative at* x and we write this value as $f'_+(x)$. Similarly, we write the left derivative of f at x as $f'_-(x)$ if the upper and lower left derivative of f coincide at x. If $f'_+(x)$ and $f'_-(x)$ exist and are equal, then we say f is *differentiable at* x and we write $f'(x)$ for the common value $f'_+(x) = f'_-(x)$. This number is called the *derivative of* f *at* x. The derivative is allowed to take the value ∞ or $-\infty$.

For a monotone function to be differentiable in a point in its domain, the Dini derivatives need to exist and be equal, and the derivative should be finite. In the proof of Lebesgue's Theorem, we will show that the Lebesgue measure λ of the sets where the upper and lower right derivatives are not equal is zero (and the same construction works for the left derivatives). Then we will show that the set where the derivative is not finite also has Lebesgue measure zero. As the Lebesgue measure is the only measure used in this section, the word Lebesgue will be omitted.

First, we need two other theorems to support the proof.

Theorem 3.2.4. *Let* $f\colon [0,1] \to \mathbb{R}$. *Then there exist only countably many points* $x \in (0,1)$ *where* $f'_-(x)$ *and* $f'_+(x)$ *exist and are not equal (they may be infinite).*

Proof. We define

$$A := \{x \in (0,1)\colon f'_-(x), f'_+(x) \text{ exist and } f'_+(x) < f'_-(x)\}$$

and

$$B := \{x \in (0,1)\colon f'_-(x), f'_+(x) \text{ exist and } f'_+(x) > f'_-(x)\}.$$

For each $x \in A$, there then exists some $r_x \in \mathbb{Q}$ such that $f'_+(x) < r_x < f'_-(x)$. Then we choose $s_x, t_x \in \mathbb{Q}$ such that $0 < s_x < x < t_x < 1$ and

$$\frac{f(y) - f(x)}{y - x} > r_x \quad \text{if } s_x < y < x, \tag{1}$$

while

$$\frac{f(y) - f(x)}{y - x} < r_x \quad \text{if } x < y < t_x. \tag{2}$$

Multiplying (1) and (2) with $y - x$, we obtain

$$f(y) - f(x) < r_x(y - x), \tag{3}$$

the inequality in (1) switching as $y - x < 0$. We define $\phi\colon A \to \mathbb{Q}^3$ by $\phi(x) := (r_x, s_x, t_x)$. Since $\mathbb{Q}^3$ is countable, if we prove that ϕ is injective, A must also be countable. To this end, let $x, y \in A$ with $x \neq y$ such that $\phi(x) = \phi(y)$ be given. Then $(s_x, t_x) = (s_y, t_y)$ and both x and y fall within this interval. By (3), it holds that

$$f(y) - f(x) < r_x(y - x)$$

and

$$f(x) - f(y) < r_y(x - y).$$

Adding both of these inequalities together and using that $r_x = r_y$, we obtain $0 < 0$. Hence, ϕ must be injective, so A is countable. Following the same procedure for B, we obtain that there exist only countably many points where $f'_-(x)$ and $f'_+(x)$ exist and are not equal. $\square$

The other theorem needed is known as Vitali's Covering Theorem. This theorem applies for general sets in $\mathbb{R}^d$ for $d \geq 1$, but we only use the result for $\mathbb{R}$. The theorem roughly states that we can cover any set E in $\mathbb{R}$ up to a set of measure zero with a countable family of pairwise disjoint closed intervals with positive but arbitrarily small lengths. If the size of E is finite, then only finitely many of such intervals are needed to cover E up to a set with an arbitrarily small size. Formally, a Vitali cover is defined as follows.

Definition 3.2.5 (Vitali cover)**.** Let $E \subset \mathbb{R}$. A family $\mathcal{V}$ of closed intervals in $\mathbb{R}$, each with positive length, is called a *Vitali cover of* E if for each $x \in E$ and $\varepsilon > 0$, there exists an interval $I \in \mathcal{V}$ such that $x \in I$ and $\lambda(I) < \varepsilon$.

Theorem 3.2.6 (Vitali's Covering Theorem)**.** *Let $E \subset \mathbb{R}$ and $\mathcal{V}$ any non-empty Vitali cover of E. Then there exists a pairwise disjoint countable family $\{I_n\}_n \subset \mathcal{V}$ such that*

$$\lambda\left(E \setminus \bigcup_{n\geq 1} I_n\right) = 0.$$

In addition, if $\lambda(E) < \infty$, then for each $\varepsilon > 0$, there exists a pairwise disjoint finite family $\{I_1, ..., I_n\} \subset \mathcal{V}$ such that

$$\lambda\left(E \setminus \bigcup_{k=1}^{n} I_k\right) < \varepsilon.$$

Proof. We first prove the case where $\lambda(E) < \infty$. Choose an open set V such that $E \subset V$ and $\lambda(V) < \infty$. Let $\mathcal{V}_0 := \{I \in \mathcal{V} \colon I \subset V\}$. Let $x \in E$ and $\varepsilon > 0$ be arbitrary. Because V is open, there exists some $\alpha > 0$ such that $x \in (x - \alpha, x + \alpha) \subset V$. As $\mathcal{V}$ is a Vitali cover of E, there exists a closed interval $I \in \mathcal{V}$ such that $x \in I$ and $\lambda(I) < \min\{\alpha, \varepsilon\}$. Then $I \subset (x - \alpha, x + \alpha)$, so $I \subset V$ and $I \in \mathcal{V}_0$. Hence, $\mathcal{V}_0$ is a Vitali cover for E.

Let $I_1 \in \mathcal{V}_0$. If $E \subset I_1$, it would hold that $\lambda(E \setminus I_1) = 0$ so then the construction would be complete. If this is not the case, we proceed by induction. Suppose $I_1, I_2, ..., I_n \in \mathcal{V}_0$ have been selected and are pairwise disjoint. If $E \subset \cup_{k=1}^n I_k$, then the construction is complete. If this does not hold, we write

$$A_n := \cup_{k=1}^n I_k, \quad U_n := V \setminus A_n.$$

The set A_n is closed as it is a finite union of closed sets, so U_n is open and $U_n \cap E \neq \emptyset$ as $E \not\subset A_n$. We then define

$$\delta_n := \sup\{\lambda(I) \colon I \in \mathcal{V}_0, I \subset U_n\}. \tag{1}$$

Each $\lambda(I)$ is bounded by $\lambda(V) < \infty$, so the supremum exists. The supremum has the property that for any $\eta > 0$, there exists some $y_\eta \in \{\lambda(I) \colon I \in \mathcal{V}_0, I \subset U_n\}$ such that $y_\eta > \delta_n - \eta$, so if we set $\eta := \frac{1}{2}\delta_n$, we can find some $I_{n+1} \in \mathcal{V}_0$ such that $I_{n+1} \subset U_n$ and $y_\eta := \lambda(I_{n+1}) > \frac{1}{2}\delta_n$. If E is not contained in the union of a finite number of I_n, we end up with an infinite sequence $\{I_n\}_n$ of pairwise disjoint elements of $\mathcal{V}_0$ (assuming the axiom of dependent choice). Let $A := \cup_{n\geq 1} I_n$. We then must show that $\lambda(E \setminus A) = 0$.

For $n \geq 1$, let J_n be the closed interval with the same midpoint as I_n and such that

$$\lambda(J_n) = 5\lambda(I_n).$$

We then have

$$\lambda\left(\bigcup_{n\geq 1} J_n\right) \leq \sum_{n\geq 1} \lambda(J_n) = 5\sum_{n\geq 1} \lambda(I_n) = 5\lambda(A) \leq 5\lambda(V) < \infty, \tag{2}$$

using that all I_n are pairwise disjoint. Because $\sum_{n\geq 1}\lambda(J_n)$ is a converging series consisting of non-negative terms, for any $\zeta > 0$ there exists some $K \geq 1$ such that for $k \geq K$, $\sum_{n\geq k}\lambda(J_n) < \zeta$. Hence, $\lim_{k\to\infty}\sum_{n\geq k}\lambda(J_n) = 0$, so as

$$0 \leq \lambda\left(\bigcup_{n\geq k} J_n\right) \leq \sum_{n\geq k}\lambda(J_n),$$

by the Squeeze Theorem, it holds that

$$\lim_{k\to\infty}\lambda\left(\bigcup_{n\geq k} J_n\right) = 0.$$

If we can prove $E \setminus A \subset \cup_{n\geq k} J_n$ for all $k \geq 1$, we have that $\lambda(E \setminus A) = 0$. Let $k \geq 1$ and $x \in E \setminus A$ be given. Then we have $x \in E \setminus A_k \subset U_k$, and U_k is open, so we can find some $\beta > 0$ such that $(x - \beta, x + \beta) \subset U_k$ and some $I \in \mathcal{V}_0$ such that $x \in I$ and $\lambda(I) < \beta$. Hence, $I \subset U_k$. For any $n \geq 1$,

$$0 < \delta_n < 2\lambda(I_{n+1}),$$

and since we have that $\lambda(I_n) \to 0$ as $n \to \infty$ by (2), $\delta_n \to 0$ as $n \to \infty$ as well by the Squeeze Theorem. So there exists some $n \geq 1$ such that $\delta_n < \lambda(I)$. Let q be the smallest number for which this occurs. We then have that $I \not\subset U_q$ by (1). Furthermore, for any $n \geq 1$, we have $\delta_{n+1} \leq \delta_n$, since $A_n \subset A_{n+1}$ implies that $U_{n+1} \subset U_n$, which in turn implies that $\{\lambda(I)\colon I \in \mathcal{V}_0, I \subset U_{n+1}\} \subset \{\lambda(I)\colon I \in \mathcal{V}_0, I \subset U_n\}$ and δ_{n+1} and δ_n are the suprema of these sets respectively. We have shown that $I \subset U_k$, from which it follows that

$$\delta_q < \lambda(I) \leq \delta_k,$$

so $k < q$ as $\{\delta_n\}_n$ is a decreasing sequence.

Because $I \not\subset U_q$, it holds that $I \cap \cup_{n=1}^{q} I_n \neq \emptyset$. In addition, $I \subset U_{q-1}$, because q is the smallest n for which $\delta_n < \lambda(I)$. This implies that $I \cap \cup_{n=1}^{q-1} I_n = \emptyset$. From this, it follows that

$$I \cap I_q \neq \emptyset \tag{3}$$

and because $I \subset U_{q-1}$, we also have that

$$\lambda(I) \leq \delta_{q-1} < 2\lambda(I_q). \tag{4}$$

Since $\lambda(J_q) = 5\lambda(I_q)$, (3) and (4) together with the fact that $k < q$ imply that

$$I \subset J_q \subset \cup_{n\geq k} J_n. \tag{5}$$

We conclude that $x \in \cup_{n\geq k} J_n$, so $E \setminus A \subset \cup_{n\geq k} J_n$ and we find $\lambda(E \setminus A) = 0$.

We still need to show that we can cover E up to a set of arbitrarily small size. For this, let $\varepsilon > 0$ be given and choose k large enough that

$$\sum_{n\geq k+1}\lambda(I_n) < \varepsilon.$$

Then

$$E \setminus A_k \subset (E \setminus A) \cup \left(\bigcup_{n\geq k+1} I_n\right),$$

so

$$\lambda\left(E\setminus\bigcup_{n=1}^{k}I_n\right)=\lambda(E\setminus A_k)\leq 0+\lambda\left(\bigcup_{n\geq k+1}I_n\right)<\varepsilon.$$

Finally, we show the statement holds for $\lambda(E)=\infty$. For each $n\in\mathbb{Z}$, let $E_n:=E\cap(n,n+1)$, so $E=\cup_{n=-\infty}^{\infty}E_n\sqcup\mathbb{Z}$. Let $\mathcal{V}_n:=\{I\in\mathcal{V}\colon I\subset(n,n+1)\}$. Then $\mathcal{V}_n$ is a Vitali cover for E_n (through the same argument as at the beginning of the proof) and $\lambda(E_n)<\infty$, so we can find a countable pairwise disjoint family $\mathcal{I}_n\subset\mathcal{V}_n$ such that $\lambda(E_n\setminus\cup_{I\in\mathcal{I}_n}I)=0$ for each $n\in\mathbb{Z}$ by the first part of the proof. Let $\mathcal{I}:=\cup_{n=-\infty}^{\infty}\mathcal{I}_n$. Then $\mathcal{I}$ is a countable pairwise disjoint subcollection of $\mathcal{V}$ and

$$E\setminus\left(\bigcup_{I\in\mathcal{I}}I\right)\subset\mathbb{Z}\cup\left(\bigcup_{n=-\infty}^{\infty}E_n\setminus\left(\bigcup_{I\in\mathcal{I}_n}I\right)\right),$$

since if $x\in E\setminus(\cup_{I\in\mathcal{I}}I)$, then either

$$x\in\mathbb{Z}\setminus\left(\bigcup_{I\in\mathcal{I}}I\right)=\mathbb{Z}$$

because all $I\in\mathcal{I}$ are disjoint from $\mathbb{Z}$, or

$$x\in\left(\bigcup_{n=-\infty}^{\infty}E_n\right)\setminus\bigcup_{I\in\mathcal{I}}I=\left(\bigcup_{n=-\infty}^{\infty}E_n\right)\setminus\left(\bigcup_{k=-\infty}^{\infty}\bigcup_{I\in\mathcal{I}_k}I\right)=\bigcup_{n=-\infty}^{\infty}E_n\setminus\left(\bigcup_{I\in\mathcal{I}_n}I\right),$$

because $E_n\setminus\cup_{I\in\mathcal{I}_k}I=E_n$ if $n\neq k$.

Hence,

$$\lambda\left(E\setminus\left(\bigcup_{I\in\mathcal{I}}I\right)\right)\leq\lambda(\mathbb{Z})+\sum_{n=-\infty}^{\infty}\lambda\left(E_n\setminus(\cup_{I\in\mathcal{I}_n}I)\right)=0.$$

□

Figure 4 gives a visual example to illustrate the inclusions of Equation (5). For a given interval I, the blue intervals show the smallest possible size I_q and furthest distance from I, given that the intersection of I and I_q is non-empty. The orange and green interval then depict the possible places for the interval J_q, showing that this always includes I.

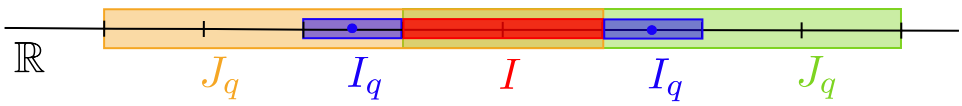

Figure 4: Illustration of Equation (5).

Theorem 3.2.7 (Lebesgue's Theorem for the Differentiability of Monotone Functions)**.** *Let $f\colon [0,1] \to \mathbb{R}$ be a monotone function. Then f is differentiable almost everywhere on $[0,1]$.*

Proof. Suppose f is non-decreasing (otherwise, consider $-f$). Let

$$E := \{x \in [0,1)\colon D_+f(x) < D^+f(x)\}.$$

For every pair $u, v \in \mathbb{Q}_{>0}$ with $u < v$, let

$$E_{u,v} := \{x \in E\colon D_+f(x) < u < v < D^+f(x)\}.$$

Then $E = \cup_{u,v\in\mathbb{Q},0<u<v} E_{u,v}$. Because this union is countable, we can reduce the problem to showing that $\lambda(E_{u,v}) = 0$ for all $0 < u < v$ in $\mathbb{Q}$, since

$$\lambda(E) = \lambda\left(\bigcup_{\substack{u,v\in\mathbb{Q}\\0<u<v}} E_{u,v}\right) \leq \sum_{\substack{u,v\in\mathbb{Q}\\0<u<v}} \lambda(E_{u,v}).$$

We assume the contrary: there exist some $u, v \in \mathbb{Q}_{>0}$ with $u < v$ such that $\alpha := \lambda(E_{u,v}) > 0$. Let ε be such that

$$0 < \varepsilon < \frac{\alpha(v-u)}{u+2v}.$$

Now, by regularity of λ (Proposition 2.4.2), we can choose an open set $U \supset E_{u,v}$ such that $\lambda(U) < \alpha + \varepsilon$. For each $x \in E_{u,v}$, we show there exist arbitrarily small $h > 0$ such that $[x, x+h] \subset U \cap [0,1]$ for which

$$f(x+h) - f(x) < uh. \tag{1}$$

Let $x \in E_{u,v}$ be given. Let $\eta > 0$ such that $D_+f(x) + 2\eta < u$. Because

$$D_+f(x) = \lim_{\delta\downarrow 0}\left(\inf_{0<h<\delta} \frac{f(x+h)-f(x)}{h}\right),$$

there exists some $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$,

$$\inf_{0<h<\delta} \frac{f(x+h)-f(x)}{h} - D_+f(x) \leq \left|\inf_{0<h<\delta} \frac{f(x+h)-f(x)}{h} - D_+f(x)\right| < \eta.$$

Furthermore, for a given $0 < \delta \leq \delta_0$, there exists some $0 < h_0 < \delta$ such that

$$\frac{f(x+h_0)-f(x)}{h_0} - \inf_{0<h<\delta} \frac{f(x+h)-f(x)}{h} \leq \left|\frac{f(x+h_0)-f(x)}{h_0} - \inf_{0<h<\delta} \frac{f(x+h)-f(x)}{h}\right| < \eta.$$

Combining these estimates together, we find

$$D_+f(x) + \eta > \inf_{0<h<\delta} \frac{f(x+h)-f(x)}{h} > \frac{f(x+h_0)-f(x)}{h_0} - \eta.$$

As $u > D_+f(x) + 2\eta$, we have

$$u > \frac{f(x+h_0)-f(x)}{h_0}$$

so $f(x+h_0) - f(x) < uh_0$. Because we chose $0 < h_0 < \delta$ for any $0 < \delta \leq \delta_0$, the number h_0 is arbitrarily small.

Let

$$\mathcal{V} := \{[x, x+h] \subset U \cap [0,1]\colon h > 0, f(x+h) - f(x) < uh\}.$$

Then $\mathcal{V}$ is a Vitali cover of $E_{u,v}$. Hence, by the second part of Theorem 3.2.6, there exists a finite, pairwise disjoint subfamily $\{[x_i, x_i + h_i]\}_{i=1}^m$ of $\mathcal{V}$ such that

$$\lambda\left(E_{u,v} \setminus \bigcup_{i=1}^m [x_i, x_i + h_i]\right) < \varepsilon.$$

Let $V := \cup_{i=1}^m (x_i, x_i + h_i)$. Consequently,

$$\begin{aligned}\lambda(E_{u,v} \setminus V) &= \lambda(E_{u,v} \setminus \cup_{i=1}^m [x_i, x_i + h_i]) + \lambda(\cup_{i=1}^m \{x_i\} \cup \{x_i + h_i\}) \\ &= \lambda(E_{u,v} \setminus \cup_{i=1}^m [x_i, x_i + h_i]) \\ &< \varepsilon. \end{aligned} \tag{2}$$

As $V \subset U$, we have

$$\sum_{i=1}^m h_i = \lambda(V) \leq \lambda(U) < \alpha + \varepsilon,$$

so with (1) it follows that

$$\sum_{i=1}^m (f(x_i + h_i) - f(x_i)) < u \sum_{i=1}^m h_i < u(\alpha + \varepsilon). \tag{3}$$

For all $y \in E_{u,v} \cap V$, there exist arbitrarily small $k > 0$ such that $[y, y + k] \subset V$ and

$$f(y + k) - f(y) > vk, \tag{4}$$

which follows by applying the the same steps as for (1) to the fact that for all $y \in E_{u,v}$, it holds that $D^+ f(y) > v$. We can again apply Theorem 3.2.6 to the Vitali cover of all closed intervals $[y, y + k]$ to find a finite, pairwise disjoint subfamily $\{[y_j, y_j + k_j]\}_{j=1}^n$ such that

$$\lambda\left((E_{u,v} \cap V) \setminus \bigcup_{j=1}^n [y_i, y_i + k_i]\right) < \varepsilon.$$

Combining this inequality with (2) yields

$$\begin{aligned}\alpha &= \lambda(E_{u,v}) \\ &= \lambda(E_{u,v} \setminus V) + \lambda(E_{u,v} \cap V) \\ &< \varepsilon + \varepsilon + \lambda\left((E_{u,v} \cap V) \cap \left(\bigcup_{j=1}^n [y_j, y_j + k_j]\right)\right) \\ &\leq 2\varepsilon + \sum_{j=1}^n k_j. \end{aligned} \tag{5}$$

Applying (5) and (4) in order, we deduce

$$v(\alpha - 2\varepsilon) < v \sum_{j=1}^n k_j < \sum_{j=1}^n (f(y_j + k_j) - f(y_j)). \tag{6}$$

Because $\cup_{j=1}^n [y_j, y_j + k_j] \subset V \subset \cup_{i=1}^m [x_i, x_i + h_i]$ and f is non-decreasing,

$$\sum_{j=1}^n (f(y_j + k_j) - f(y_j)) \leq \sum_{i=1}^m (f(x_i + h_i) - f(x_i)). \tag{7}$$

Finally, by combining (6), (7) and (3) we find that

$$v(\alpha - 2\varepsilon) < u(\alpha + \varepsilon).$$

However, by our assumption of ε, $v(\alpha-2\varepsilon) > u(\alpha+\varepsilon)$. So we arrive at a contradiction. Therefore $\lambda(E_{u,v}) = 0$ and $\lambda(E) = 0$. This means that $f_+(x)$ exists almost everywhere on $[0, 1]$. The same process can be followed to prove that $f'_-(x)$ exists almost everywhere on $[0, 1]$, by defining a set E on $(0, 1]$ instead. Applying Theorem 3.2.4 yields that f'_+ and f'_- are equal almost everywhere, so f' exists almost everywhere on $[0, 1]$ (the end points 0 and 1 also have measure 0).

As a final step, we have to show that the set $F := \{x \in (0, 1) : f'(x) = \infty\}$ has measure zero. To this end, let $\beta > 0$ be given. For each $x \in F$, there exist arbitrarily small $h > 0$ such that $[x, x + h] \subset (0, 1)$ and

$$f(x + h) - f(x) > \beta h. \tag{8}$$

These intervals form a Vitali cover, so by the the first part of Theorem 3.2.6, there exists a countable pairwise disjoint subfamily $\{[x_n, x_n + h_n]\}_n$ of these intervals such that

$$\lambda(F \setminus \cup_{n\geq1}[x_n, x_n + h_n]) = 0.$$

We then have that

$$\begin{aligned}\lambda(F) &= \lambda(F \setminus \cup_{n\geq1}[x_n, x_n + h_n]) + \lambda(F \cap (\cup_{n\geq1}[x_n, x_n + h_n]))\\ &\leq 0 + \lambda(\cup_{n\geq1}[x_n, x_n + h_n])\\ &= \sum_{n\geq1} h_n. \end{aligned}\tag{9}$$

Equations (8) and (9) together with the fact that the intervals $[x_n, x_n + h_n]$ are pairwise disjoint imply that

$$\beta\lambda(F) \leq \beta \sum_{n\geq1} h_n < \sum_{n\geq1}(f(x_n + h_n) - f(x_n)) \leq f(1) - f(0).$$

Since this inequality holds for all $\beta > 0$, it follows that $\lambda(F) = 0$. So f has a finite derivative almost everywhere. □

3.3 Everywhere differentiable and nowhere monotone functions

In the previous section, we derived from Lebesgue's Theorem that a nowhere differentiable function is nowhere monotone. A natural question to ask is whether the converse holds: is a nowhere monotone function also nowhere differentiable? The answer is no, as there actually do exist functions that are nowhere monotone and everywhere differentiable. It is even harder to intuitively think of such functions than it is to think of examples of nowhere differentiable functions, as these properties seem contradictory. The property of being nowhere monotone means a function is very "rough", while the property of being everywhere differentiable is mostly thought of as "smooth". Instinctively, if a function f is differentiable in a point x, we would say if $f'(x) > 0$, f is increasing and if $f'(x) < 0$, f is decreasing. It seems that if f is neither decreasing nor increasing in x, we have $f'(x) = 0$. How can a differentiable function then be nowhere monotone without being a constant function (which is again monotone)?

A challenge to this naive assumption is that a function being increasing or decreasing is a property of its neighbourhood, not of one point. Functions that are everywhere differentiable but nowhere monotone are tellingly referred to as *differentiable monsters* in [Che21]. Lemma 3.3.1 and Corollary 3.3.2 are retrieved from this article. These differentiable monsters have the property that their derivative vanishes on a dense set, as we will see below. Here, we use the abbreviation $[g > 0] := \{x \in [0, 1] : g(x) > 0\}$. The sets $[g < 0]$ and $[g = 0]$ are defined similarly.

Lemma 3.3.1. *Let* $f\colon [0,1] \to \mathbb{R}$ *be a differentiable function on* $[0,1]$. *Then* f *is nowhere monotone on* $[0,1]$ *if and only if both* $[f' > 0]$ *and* $[f' < 0]$ *are dense.*

Proof. First, assume f is nowhere monotone. Suppose $[f' > 0]$ or $[f' < 0]$ is not dense. We assume $[f' > 0]$ is not dense. Then there exists a non-empty, open set U such that $[f' > 0] \cap U$ is empty. Then there exist $x \in U$ and $r > 0$ such that $(x - r, x + r) \subset U$. Let $t \in (x - r, x)$ be given. By the Mean Value Theorem, there exists some $\zeta \in (t, x)$ such that

$$0 \geq f'(\zeta) = \frac{f(x) - f(t)}{x - t},$$

so as $t < x$, $f(x) \leq f(t)$. Hence, we have found a neighbourhood for which f is decreasing, which contradicts the assumption that f is nowhere monotone. The case that $[f' < 0]$ is not dense follows similarly, where instead we find a neighbourhood V where f is increasing.

For the converse, assume $[f' > 0]$ and $[f' < 0]$ are both dense and f is not nowhere monotone. Without loss of generality, assume f is increasing on some interval $(a, b) \subset [0, 1]$. Let $x \in [f' < 0] \cap (a, b)$ be given and let N be any neighbourhood of x. Then N contains some open set $U \cap (a, b)$ which contains x. Then as

$$0 > f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

we can find h sufficiently small and nonzero so that $x + h \in U \cap (a, b)$ and

$$\frac{f(x + h) - f(x)}{h} < 0.$$

Hence, we found an interval in (a, b) where f is not increasing, a contradiction. □

We can apply this lemma to show f' vanishes on a dense set.

Corollary 3.3.2. *Let* $f\colon [0,1] \to \mathbb{R}$ *be a differentiable function on* $[0,1]$. *If* f *is nowhere monotone on* $[0,1]$, *then* $[f' = 0]$ *is dense.*

Proof. We assume f is nowhere monotone on $[0, 1]$. Let $U \subset [0, 1]$ be a non-empty open interval. By Lemma 3.3.1 both $[f' > 0]$ and $[f' < 0]$ are dense, so there exist $x \in U \cap [f' > 0]$ and $y \in U \cap [f' < 0]$. Without loss of generality, assume $x < y$. Then $[x, y] \subset U$ and $f'(x) > 0$ and $f'(y) < 0$. The value 0 lies between $f'(x)$ and $f'(y)$, so by Darboux's Theorem (Theorem 2.3.3), there exists a $z \in [x, y]$ such that $f'(z) = 0$. Then $z \in U \cap [f' = 0]$, so $[f' = 0]$ is dense. □

This characterisation gives us some intuition for the kind of functions we are looking for. The study of these functions goes back over a century, with the first example of such a differentiable monster given by Alfred Köpcke in 1887 in [Köp87]. Köpcke's construction is very complicated, as were some other attempts at examples that followed. For an overview of the study of these functions, see [Bru83, Section 2]. Instead of constructing differentiable monsters explicitly, their existence can actually be proven with the Baire Category Theorem, as shown by Clifford Weil in [Wei76] in 1976. Weil's proof is very concise, though he leaves out a number of non-trivial steps. This section is dedicated to presenting Weil's proof including those details, so that the proof is understandable to students of an undergraduate level.

3.3.1 Pompeiu derivatives

In order to prove that differentiable monsters exist, Weil focuses on the space of derivatives of these functions. As the above discussion demonstrates, we will look for a set of derivates that vanish on a dense set and are non-zero elsewhere. Finding such a function in the first place is not a trivial matter. One type of function that satisfies this property is called a Pompeiu derivative, named after Dimitrie Pompeiu who constructed them in 1905. His work is published in [Pom07]. The class of Pompeiu derivatives challenges the assumption that derivatives of continuous functions are continuous. We first discuss the construction of these Pompeiu derivatives and discuss some of their properties as they were proved by Pompeiu, which we will later need in Weil's proof. We start by constructing a bijective function F, whose tangent line will be vertical on a dense set of points in its domain. By taking the inverse of F, we construct a function whose derivative is zero on a dense set of points, but non-zero on the other points in its domain.

Lemma 3.3.3. *Let $\{a_n\}_n$ be a sequence of strictly positive real numbers with $\sum_{n\geq 0} a_n < \infty$. Let $\{q_n\}_n$ be some ordering of the rationals in $[0,1]$. For $n \geq 1$, define $F_n\colon [0,1] \to \mathbb{R}$ as*

$$F_n(x) := a_n \sqrt[3]{x - q_n}$$

and define $F\colon [0,1] \to \mathbb{R}$ as

$$F(x) := a_0 + \sum_{n\geq 1} F_n(x).$$

Then the function F is a homeomorphism onto $[0,1]$ up to rescaling and translating F with a constant.

Proof. As each a_n is positive and the cube root is a strictly increasing function, F is strictly increasing as well. To show F is a homeomorphism, we first prove that F is continuous and bijective if we modify F by rescaling.

To show the continuity of F, note that for any $n \geq 1$ and any $x \in [0,1]$

$$|F_n(x)| = a_n |\sqrt[3]{x - q_n}| \leq a_n,$$

as $|x - q_n| \leq 1$. Since the series $\sum_{n\geq 1} a_n$ converges by construction, by the Weierstrass M-test (Proposition 2.3.1) we can conclude that $\{\sum_{k=1}^{n} F_k(x)\}_n$ converges absolutely and uniformly on $[0,1]$ to $F - a_0$. Adding a_0, Theorem 2.2.3 yields that F is continuous through the continuity of each of the functions F_n (since the cube root is a continuous function on $[0,1]$).

The proof that F is injective is immediate by it being strictly increasing. We have

$$F(0) = a_0 - \sum_{n\geq 1} a_n \sqrt[3]{q_n},$$

so setting $a_0 := \sum_{n\geq 1} a_n \sqrt[3]{q_n}$, we obtain $F(0) = 0$. Then $F(x) > 0$ for any $x \in (0,1]$. Multiply F by the factor $\frac{1}{F(1)}$. We still denote the rescaled function as F. Now $F(1) = 1$. Since F is continuous, by the Intermediate Value Theorem, F maps $[0,1]$ to $[0,1]$, which yields that F is bijective.

As $[0,1]$ is compact and $\mathbb{R}$ is Hausdorff with respect to the standard norm and F is bijective and continuous, by Proposition 2.1.4, F is a homeomorphism. $\square$

In the following results of this section, F is to be understood as the homeomorphism defined in Lemma 3.3.3. We first establish that the series defined below is absolutely convergent. Then we examine if and on which sets this series is the derivative of F.

Lemma 3.3.4. *The series*

$$f(x) := \frac{1}{3}\sum_{n\geq 1}\frac{a_n}{(x-q_n)^{\frac{2}{3}}},$$

converges almost everywhere.

Proof. Define $f(n,x)\colon \mathbb{Z}_{\geq 1}\times[0,1]\to[0,\infty]$ as

$$(n,x)\mapsto \frac{a_n}{(x-q_n)^{\frac{2}{3}}}.$$

Consider the power set σ-algebra on $\mathbb{Z}_{\geq 1}$ and the Borel σ-algebra on $[0,\infty]$. We verify that the inverse images of f under $\{\infty\}$ and an open interval $(a,b)\subset[0,1]$ with $a<b$, are elements of the product σ-algebra generated by $\mathfrak{P}(\mathbb{Z}_{\geq 1})$ and $\mathcal{B}[0,1]$. We write

$$\begin{aligned} f^{-1}(\{\infty\}) &= \{(n,x)\in\mathbb{Z}_{\geq 1}\times[0,1]\colon f(n,x)=\infty\} \\ &= \{(n,q_n)\colon n\in\mathbb{Z}_{\geq 1}\} \\ &= \cup_{n\geq 1}\{(n,q_n)\}. \end{aligned}$$

Then it holds that $\{(n,q_n)\}\in\mathfrak{P}(\mathbb{Z}_{\geq 1})\times\mathcal{B}[0,1]$ for each $n\geq 1$, since singleton sets are subsets of both $\mathfrak{P}(\mathbb{Z}_{\geq 1})$ and $\mathcal{B}[0,1]$, so this set is also included in the product of these σ-algebras. Furthermore, countable unions of elements of a σ-algebra are again elements of the σ-algebra, so $f^{-1}(\{\infty\})$ is a measurable set.

Set

$$\alpha_n := \left(\frac{a_n}{a}\right)^{\frac{3}{2}} \quad\text{and}\quad \beta_n := \left(\frac{a_n}{b}\right)^{\frac{3}{2}}.$$

We then have that

$$\begin{aligned} f^{-1}((a,b)) &= \{(n,x)\in\mathbb{Z}_{\geq 1}\times[0,1]\colon f(n,x)\in(a,b)\} \\ &= \left\{(n,x)\in\mathbb{Z}_{\geq 1}\times[0,1]\colon a<\frac{a_n}{(x-q_n)^{\frac{2}{3}}}<b\right\} \\ &= \left\{(n,x)\in\mathbb{Z}_{\geq 1}\times[0,1]\colon \left(\frac{a}{a_n}\right)^3<\frac{1}{(x-q_n)^2}<\left(\frac{b}{a_n}\right)^3\right\} \\ &= \{(n,x)\in\mathbb{Z}_{\geq 1}\times[0,1]\colon \beta_n<|x-q_n|<\alpha_n\} \\ &= \cup_{n\geq 1}\left(\{n\}\times\left(B_{\alpha_n}(q_n)\setminus \mathrm{cl}\ B_{\beta_n}(q_n)\right)\cap[0,1]\right). \end{aligned}$$

Each $\{n\}$ is measurable in the powerset of $\mathbb{Z}_{\geq 1}$. The open ball around q_n of radius α_n and the closure of the ball around q_n with radius β_n are both measurable (in $\mathcal{B}[0,1]$), so their difference is as well. It describes precisely those $x\in[0,1]$ with a distance of at least β_n to q_n and less than α_n. Finally, we need to intersect this set with $[0,1]$ to ensure the interval is still a subset of $[0,1]$, and the intersection with a measurable set is again measurable. The product of these sets is then measurable for the product σ-algebra, and the countable union over these sets is then measurable as well.

Since we can write

$$(a,\infty)=\bigcup_{\substack{n\geq 1\\ n>a}}(a,n),$$

the preimage of f of each measurable set of $\mathcal{B}[0,\infty]$ can be described with unions, intersections and complements of the previous sets (of course $f^{-1}(\{0\})=\emptyset$), so we find f is measurable.

Let

$$g(x) := \sum_{n\geq 1} f(n,x) = \lim_{n\to\infty} \sum_{k=1}^{n} f(k,x).$$

By Proposition 2.4.3, it holds that if $\int_{[0,1]} g(x)\ \mathrm{d}(\lambda x) < \infty$, then $g(x) < \infty$ for almost all $x \in [0,1]$, where λ is the Lebesgue measure on $[0,1]$. The absolute value sign is omitted as g is a non-negative function. As both $(\mathbb{Z}_{\geq 1}, \mathfrak{P}(\mathbb{Z}_{\geq 1}), \nu)$ and $([0,1], \mathcal{B}[0,1], \lambda)$ are σ-finite measure spaces and f is a non-negative function, we apply Tonelli's Theorem (Theorem 2.4.6) to obtain

$$\int_{[0,1]} g(x)\ \mathrm{d}(\lambda x) = \int_{[0,1]} \sum_{n\geq 1} f(n,x)\ \mathrm{d}(\lambda x) = \sum_{n\geq 1} \int_{[0,1]} f(n,x)\ \mathrm{d}(\lambda x), \tag{1}$$

where we use that $\int_{[1,\infty)} f(n,x)\ \mathrm{d}(\nu n) = \sum_{n\geq 1} f(n,x)$, with ν the counting measure. Now, for sufficiently large k and n, define

$$g_{k,n}(x) := \begin{cases} f(n,x) \cdot \mathbb{1}_{[0,1]\setminus\left(q_n - \frac{1}{k}, q_n + \frac{1}{k}\right)} & \text{if } q_n \neq 0, 1 \\ f(n,x) \cdot \mathbb{1}_{\left[\frac{1}{k}, 1\right]} & \text{if } q_n = 0 \\ f(n,x) \cdot \mathbb{1}_{\left[0, 1-\frac{1}{k}\right]} & \text{if } q_n = 1, \end{cases}$$

then $\{g_{k,n}\}_k$ is an increasing sequence with

$$\lim_{k\to\infty} g_{k,n}(x) = f(n,x) \cdot \mathbb{1}_{[0,1]\setminus\{q_n\}}$$

for any $x \in [0,1]$. In addition, for each sufficiently large k and n, we have that $g_{k,n}$ is a bounded, continuous function on the set $[0, q_n - \frac{1}{k}] \cup [q_n + \frac{1}{k}, 1]$. By Theorem 2.4.4, this implies that the Riemann and Lebesgue integrals of $g_{k,n}$ coincide. First, suppose $q_n = 0$. With the Monotone Convergence Theorem (Theorem 2.4.5) we find

$$\begin{aligned}
\int_{(0,1]} f(n,x)\ \mathrm{d}(\lambda x) &= \int_{(0,1]} \lim_{k\to\infty} g_{k,n}(x)\ \mathrm{d}(\lambda x) \\
&= \lim_{k\to\infty} \int_{(0,1]} g_{k,n}(x)\ \mathrm{d}(\lambda x) \\
&= \lim_{k\to\infty} \int_{\left[\frac{1}{k},1\right]} f(n,x)\ \mathrm{d}(\lambda x) \\
&= \lim_{k\to\infty} \int_{\frac{1}{k}}^{1} f(n,x)\ \mathrm{d}x \\
&= \lim_{k\to\infty} 3a_n - 3a_n \left(\frac{1}{k}\right)^{\frac{1}{3}} \\
&= 3a_n.
\end{aligned}$$

Following the same steps, for $q_n = 1$ we have

$$\begin{aligned}
\int_{[0,1)} f(n,x)\ \mathrm{d}(\lambda x) &= \lim_{k\to\infty} \int_{0}^{1-\frac{1}{k}} f(n,x)\ \mathrm{d}x \\
&= \lim_{k\to\infty} 3a_n \left(-\frac{1}{k}\right)^{\frac{1}{3}} + 3a_n \\
&= 3a_n.
\end{aligned}$$

Finally, suppose $q_n \neq 0, 1$. Similarly, we then find

$$\begin{aligned}
\int_{[0,1]\setminus\{q_n\}} f(n,x)\ \mathrm{d}(\lambda x) &= \int_{[0,1]\setminus\{q_n\}} \lim_{k\to\infty} g_{k,n}(x)\ \mathrm{d}(\lambda x) \\
&= \lim_{k\to\infty} \int_{[0,1]\setminus\{q_n\}} g_{k,n}(x)\ \mathrm{d}(\lambda x) \\
&= \lim_{k\to\infty} \int_{[0,1]\setminus\left(q_n-\frac{1}{k},q_n+\frac{1}{k}\right)} f(n,x)\ \mathrm{d}(\lambda x) \\
&= \lim_{k\to\infty} \int_0^{q_n-\frac{1}{k}} f(n,x)\ \mathrm{d}x + \int_{q_n+\frac{1}{k}}^1 f(n,x)\ \mathrm{d}x \\
&= \lim_{k\to\infty} -6a_n\left(\frac{1}{k}\right)^{\frac{1}{3}} + 3a_n\left((1-q_n)^{\frac{1}{3}} + q_n^{\frac{1}{3}}\right) \\
&= 3a_n\left((1-q_n)^{\frac{1}{3}} + q_n^{\frac{1}{3}}\right).
\end{aligned}$$

We also have that

$$\int_{[0,1]\setminus\{q_n\}} f(n,x)\ \mathrm{d}(\lambda x) = \int_{[0,1]} f(n,x)\ \mathrm{d}(\lambda x),$$

because the Lebesgue integral over a set of measure 0 equals 0. Combining these calculations together with (1), we find

$$\begin{aligned}
\int_{[0,1]} g\ \mathrm{d}(\lambda x) &= \sum_{n\geq 1} 3a_n\left((1-q_n)^{\frac{1}{3}} + q_n^{\frac{1}{3}}\right) \\
&\leq 6\sum_{n\geq 1} a_n,
\end{aligned}$$

if $q_n \neq 0, 1$, and

$$\begin{aligned}
\int_{[0,1]} g\ \mathrm{d}(\lambda x) &= \sum_{n\geq 1} 3a_n \\
&\leq 6\sum_{n\geq 1} a_n,
\end{aligned}$$

in case $q_n = 0, 1$. In all cases, the integral is finite as the series $\sum_{n\geq 1} a_n$ converges. We find that $\int_{[0,1]} g(x)\ \mathrm{d}(\lambda x)$ is finite, so as a result, the set $\{x \in [0,1] \colon g(x) = \infty\}$ has measure zero. We have that $f = \frac{1}{3}g$, so we conclude that f converges almost everywhere. □

In the remainder of Section 3.3.1, f refers to the series defined in Lemma 3.3.4.

Lemma 3.3.5. *Define the set $A := \{x \in [0,1] \colon f(x)$ converges on $\mathbb{R}\}$. Then for $x \in A$, it follows that F' exists and coincides with f. On this set, $f(x) > 0$. For $x \in [0,1] \setminus A$, the function F has a vertical tangent line at x, or*

$$\frac{F(x+h) - F(x)}{h} \to \infty$$

as $h \to 0$.

Proof. Due to Lemma 3.3.4, A is non-empty. For $x \in A$, it is clear that $f(x) > 0$ as each of the terms is positive (for $x \neq q_n$). We now show that for $x \in A$, F' exists and coincides with f on A.

We can write out the differential quotient of F in the point x for $h \neq 0$ as

$$\frac{F(x+h)-F(x)}{h} = \sum_{n\geq 1} a_n \frac{\sqrt[3]{x+h-q_n} - \sqrt[3]{x-q_n}}{h}.$$

Substituting $P_n := x+h-q_n$ and $Q_n := x - q_n$, we find

$$\begin{aligned}
\frac{P_n^{\frac{1}{3}} - Q_n^{\frac{1}{3}}}{h} &= \frac{P_n^{\frac{1}{3}} - Q_n^{\frac{1}{3}}}{P_n - Q_n} \\
&= \frac{P_n^{\frac{1}{3}} - Q_n^{\frac{1}{3}}}{(P_n^{\frac{1}{3}} - Q_n^{\frac{1}{3}})(P_n^{\frac{2}{3}} + P_n^{\frac{1}{3}} Q_n^{\frac{1}{3}} + Q_n^{\frac{2}{3}})} \\
&= \frac{1}{P_n^{\frac{2}{3}} + P_n^{\frac{1}{3}} Q_n^{\frac{1}{3}} + Q_n^{\frac{2}{3}}},
\end{aligned}$$

so

$$\frac{F(x+h)-F(x)}{h} = \sum_{n\geq 1} \frac{a_n}{P_n^{\frac{2}{3}} + P_n^{\frac{1}{3}} Q_n^{\frac{1}{3}} + Q_n^{\frac{2}{3}}} =: \phi_\infty(h). \tag{1}$$

We want to prove that $\phi_\infty(h)$ converges for h small enough (x is fixed), so that the function is well-defined, and then we take the limit of h to zero.

Write

$$\alpha_n := \sqrt[3]{\frac{P_n}{Q_n}},$$

and

$$\phi_n(h) := \frac{a_n}{(1+\alpha_n+\alpha_n^2) Q_n^{\frac{2}{3}}},$$

then

$$\phi_\infty(h) = \sum_{n\geq 1} \phi_n(h).$$

The polynomial $1+\alpha_n+\alpha_n^2$ has a minimum for $\alpha_n = -\frac{1}{2}$ with value $\frac{3}{4}$, so

$$\frac{1}{1+\alpha_n+\alpha_n^2} \leq \frac{4}{3}$$

for all values of h. Define

$$M_n := \frac{4a_n}{3Q_n^{\frac{2}{3}}} \geq 0.$$

Given that f converges,

$$\sum_{n\geq 1} M_n = 4f(x)$$

must converge as well. Because for each $n \geq 1$ and all values of h,

$$|\phi_n(h)| \leq M_n,$$

the Weierstrass M-test (Proposition 2.3.1) yields that $\sum_{n\geq 1} \phi_n(h)$ converges uniformly on $\mathbb{R}$. By Theorem 2.2.3, we then have that $\phi_\infty(h) = \sum_{n\geq 1} \phi_n(h)$ is continuous.

Now taking $h \to 0$, $P_n \to Q_n$ so

$$\frac{1}{P_n^{\frac{2}{3}} + P_n^{\frac{1}{3}} Q_n^{\frac{1}{3}} + Q_n^{\frac{2}{3}}} \to \frac{1}{3Q_n^{\frac{2}{3}}}.$$

Since $\phi_\infty(h)$ is continuous, we can conclude that

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \phi_\infty(h) = \phi_\infty(0) = f(x),$$

so for $x \in A$, $F'(x)$ exists and equals $f(x)$.

Having described the behaviour of the derivative of F on A, we will turn to its behaviour outside of A. We show that if $x \in [0,1] \setminus A$, the graph of F has a vertical tangent line at x, or

$$\frac{F(x+h) - F(x)}{h} \to \infty$$

as $h \to 0$.

Keeping (1) in mind, we now define for a given $x \in [0,1] \setminus A$ with $x \neq q_n$

$$\Phi_m(h) := \sum_{n=1}^{m} \frac{a_n}{P_n^{\frac{2}{3}} + P_n^{\frac{1}{3}} Q_n^{\frac{1}{3}} + Q_n^{\frac{2}{3}}},$$

so $\phi_\infty(h) = \lim_{m\to\infty} \Phi_m(h)$. Note that all the terms in the partial sums $\Phi_m(h)$ of $\phi_\infty(h)$ are positive. If we can show that for every $M > 0$, we can find some $m \geq 1$ and $\varepsilon > 0$ such that for $|h| < \varepsilon$, $\Phi_m(h) > M$, then it holds that

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \lim_{m\to\infty} \Phi_m(h) = \infty.$$

On $[0,1] \setminus A$, the series $f(x)$ diverges by definition. Hence, if $x \neq q_n$, for any M we can find some m such that

$$f_m(x) := \frac{1}{3} \sum_{n=1}^{m} \frac{a_n}{(x - q_n)^{\frac{2}{3}}} > M.$$

As we have seen before, $\Phi_m(0) = f_m(x)$. Since Φ_m is a finite sum of continuous functions in $h = 0$, Φ_m is also continuous in $h = 0$. Because of this continuity, we can find an interval $(-\varepsilon, \varepsilon)$ where $\Phi_m(h) > M$ as well. Hence, we find that $\Phi_m(h) \to \infty$ as $m \to \infty$ and $h \to 0$.

Now, we consider the case that $x = q_n$ for $n \geq 1$. Then

$$\begin{aligned} \frac{F(q_n + h) - F(q_n)}{h} &= \frac{1}{h} \sum_{n\geq 1} a_n \sqrt[3]{h} \\ &= h^{-\frac{2}{3}} \sum_{n\geq 1} a_n, \end{aligned}$$

which diverges to ∞ as $h \to 0$. □

Combining all of the previous results, we are able to state the theorem that describes Pompeiu's derivative.

Theorem 3.3.6 (Pompeiu's derivative)**.** *The function F has a continuous inverse F^{-1} with bounded derivative everywhere. On $F([0,1] \setminus A)$, F^{-1} has a horizontal tangent line. In other words, there exists a function in $C[0,1]$ with a bounded derivative which vanishes on a dense set but is nonzero on its domain.*

Proof. Because F is a homeomorphism, it has a continuous inverse $F^{-1}\colon [0,1] \to [0,1]$. With the knowledge of the behaviour of the derivative of F, we are able to describe the behaviour of the derivative of F^{-1}.

Since F is differentiable on A, F^{-1} is continuous and $F'(x) = f(x) > 0$ for $x \in A$, we can apply the Inverse Function Theorem (Theorem 2.3.4) for $x \in A$ to find

$$(F^{-1})'(x) = \frac{1}{F'(F^{-1}(x))} = 3\left(\sum_{n\geq 1} \frac{a_n}{(F^{-1}(x) - q_n)^{\frac{2}{3}}}\right)^{-1}.$$

We now prove that on $F([0,1] \setminus A)$, the derivative of F^{-1} exists and equals zero, or that the graph of F^{-1} has a horizontal tangent line. Let $x \in [0,1] \setminus A$. For some sufficiently small h, we can write

$$F(x) + h = F(x + k(h)) \tag{1}$$

for some unique function k depending on h, since F is invertible. This can be rewritten as $k(h) = F^{-1}(F(x) + h) - x$. With this identity, we can write the differential quotient of F^{-1} in $F(x)$ as

$$\frac{F^{-1}(F(x) + h) - F^{-1}(F(x))}{h} = \frac{F^{-1}(F(x + k(h))) - x}{h} = \frac{k(h)}{h}.$$

By continuity of F^{-1}, we have

$$\lim_{h\to 0} k(h) = \lim_{h\to 0} F^{-1}(F(x) + h) - x = 0,$$

so using (1), we find

$$\lim_{h\to 0} \frac{h}{k(h)} = \lim_{h\to 0} \frac{F(x + k(h)) - F(x)}{k(h)} = \lim_{k(h)\to 0} \frac{F(x + k(h)) - F(x)}{k(h)} = \infty,$$

as the graph of F has a vertical tangent line for $x \in [0,1] \setminus A$. This means that $k(h)$ must have the same sign as h eventually. So we can conclude that

$$\lim_{h\to 0} \frac{k(h)}{h} = 0,$$

and $(F^{-1})'(x) = 0$ on $F([0,1] \setminus A)$.

It is easy to see that $\{q_n \colon n \geq 1\} \subset [0,1] \setminus A$, since $(x - q_n)^2 = 0$ if $x = q_n$, so $f(x)$ is not well-defined and does not converge. Since the rationals in $[0,1]$ are dense, the set $[0,1] \setminus A$ is also dense. Because F is a homeomorphism, $F([0,1] \setminus A)$ is also dense, so we find $(F^{-1})'$ is zero on a dense subset of $[0,1]$.

Finally, we show the derivative of F^{-1} is bounded on $[0,1]$. We saw that $(F^{-1})'$ vanishes on $F([0,1] \setminus A)$, so we only consider the set $F(A)$. For $x \in F(A)$ and any $n \geq 1$, we have $1 \geq \sqrt[3]{F^{-1}(x) - q_n} > 0$, so

$$\frac{a_n}{(F^{-1}(x) - q_n)^{\frac{2}{3}}} \geq a_n$$

which implies that

$$\sum_{n\geq 1} \frac{a_n}{(F^{-1}(x) - q_n)^{\frac{2}{3}}} \geq \sum_{n\geq 1} a_n$$

and we find

$$(F^{-1})'(x) = 3\left(\sum_{n\geq 1} \frac{a_n}{(F^{-1}(x) - q_n)^{\frac{2}{3}}}\right)^{-1} \leq 3\left(\sum_{n\geq 1} a_n\right)^{-1}.$$

The final estimate is a finite positive number, so $(F^{-1})'$ is bounded. $\square$

Remark. Note that F^{-1} itself is not an example of a function that is everywhere differentiable and nowhere monotone, because it is the inverse of the function F, which is strictly increasing. It is easy to see that the inverse of a strictly increasing function is strictly increasing itself. Indeed, for $x, y \in [0,1]$, we have $x < y$ if and only if $F(x) < F(y)$. Setting $a := F(x)$ and $b := F(y)$, we find $F^{-1}(a) = x < y = F^{-1}(b)$ if and only if $a < b$.

The Pompeiu derivative is a non-negative function, but by multiplying with a negative scalar, one can find non-positive functions that are zero on a dense set and non-zero outside of that set.

An example of a Pompeiu derivative is shown in Figure 5, retrieved from [Mir21]. Here, the role of F in Lemma 3.3.3 is played by

$$F_N(x) := \sum_{\substack{1 \le m < N \\ \gcd(m,N)=1}} (1.8)^{2-N} \sqrt[3]{x - \frac{m}{N}},$$

and setting $F(x) := \lim_{N\to\infty} F_N(x)$. In this example, F is not rescaled to be a bijection, but for a picture with a finite sum this does not mater. This notation for F_N allows the rational numbers in the set $Q_N := \{\frac{m}{N} : 1 \le m < N, \gcd(m,N) = 1\}$ to be distributed evenly throughout the interval $[0,1]$, so that for any $N \geq 2$, the set $Q_{N-1} \subset Q_N$. The geometric series test shows $\sum_{n\geq 1}(1.8)^{2-n}$ converges.

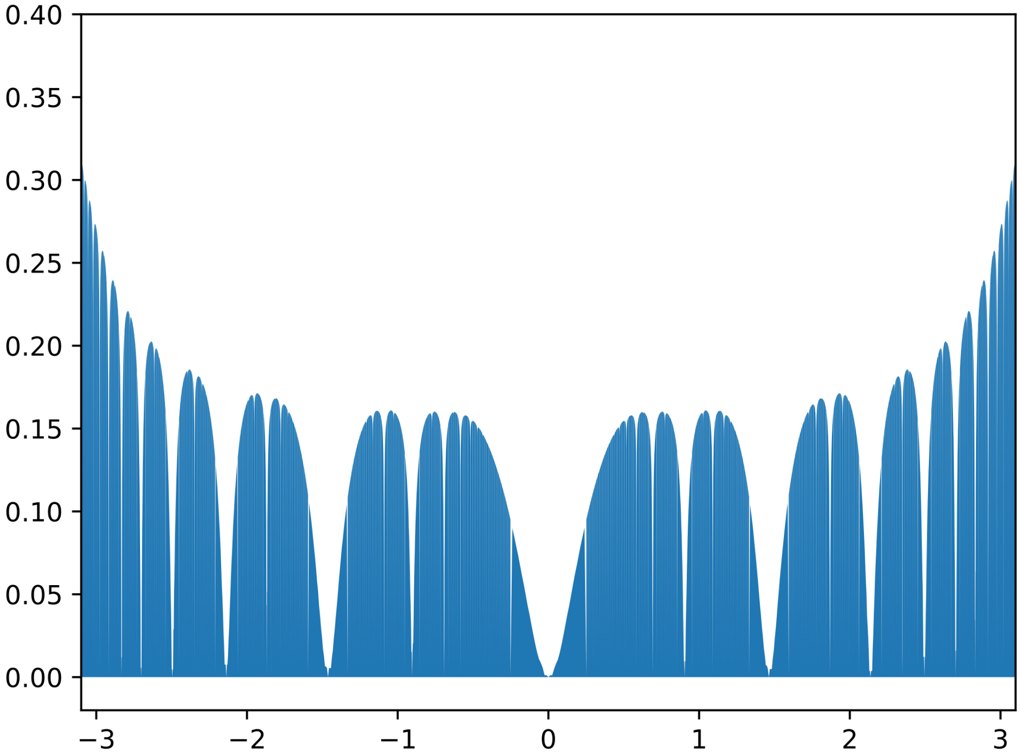

Figure 5: Graph of a Pompeiu function for $a_n := (1.8)^{2-n}$ and partial sum size $N = 50$.

3.3.2 The existence of everywhere differentiable and nowhere monotone functions

Before we can dive into Weil's Theorem, we need one important analysis result. This theorem is based on [Tao23, Theorem 3.7.1].

Theorem 3.3.7 (Uniform limit of a derivative)**.** *For $n \geq 1$, define $F_n \colon [0,1] \to \mathbb{R}$ to be differentiable functions with derivatives $f_n \colon [0,1] \to \mathbb{R}$. Suppose that the derivatives f_n converge* *uniformly to f. In addition, suppose there exists a point $x_0 \in [0,1]$ such that $\lim_{n\to\infty} F_n(x_0)$ exists. Then the functions F_n converge uniformly to a differentiable function F such that $F' = f$.*

Proof. This proof consists of three steps.

Step 1. We first prove a useful fact. Suppose that for any $\eta > 0$ and $n, m \geq 1$, $\|f_n - f_m\| \leq \eta$. Then for all $x \in [0,1]$,

$$|(F_n(x) - F_m(x)) - (F_n(x_0) - F_m(x_0))| \leq \eta |x - x_0|. \tag{1}$$

In order to prove this fact, we apply the Mean Value Theorem on some $x \neq x_0$ and the function $F_n - F_m$. By this theorem, there exists some $\zeta \in (\min(x_0, x), \max(x_0, x))$ such that

$$\begin{aligned}
\frac{|(F_n - F_m)(x) - (F_n - F_m)(x_0)|}{|x - x_0|} &= \left| \frac{(F_n - F_m)(x) - (F_n - F_m)(x_0)}{x - x_0} \right| \\
&= |(F_n - F_m)'(\zeta)| \\
&= |(f_n - f_m)(\zeta)|,
\end{aligned}$$

using the linearity of the differential operator. Assuming $\|f_n - f_m\| \leq \eta$ for some $\eta > 0$ and sufficiently large n, m, this yields

$$\begin{aligned}
\frac{|(F_n(x) - F_m(x)) - (F_n(x_0) - F_m(x_0))|}{|x - x_0|} &\leq \|f_n - f_m\| \\
&\leq \eta.
\end{aligned}$$

By multiplying both sides by $|x - x_0|$, (1) is proven for $x \neq x_0$. If $x = x_0$, both sides of (1) are equal to 0, so in this case the statement holds as well.

Step 2. We show that F_n converges uniformly to a function that we call F. Let $\varepsilon > 0$ be given. The sequence $\{f_n\}_n$ converges uniformly, so it is also a Cauchy sequence. Then for $\eta := \frac{1}{2}\varepsilon$ we can find some $N_1 \geq 1$ such that for $n, m \geq N_1$, $\|f_n - f_m\| < \eta$. In addition, because $\{F_n(x_0)\}_n$ is Cauchy, there also exists some N_2 such that for $n, m \geq N_2$, we have $|F_n(x_0) - F_m(x_0)| < \eta$. Set N as the maximum of N_1 and N_2, and let $n, m \geq N$ be given. We apply (1) for arbitrary $x \in [0,1]$. If $x = x_0$, $|F_n(x) - F_m(x)| < \eta < \varepsilon$, so assume $x \neq x_0$. Then

$$\begin{aligned}
|F_n(x) - F_m(x)| &\leq |(F_n(x) - F_m(x)) - (F_n(x_0) - F_m(x_0))| + |F_n(x_0) - F_m(x_0)| \\
&< \eta |x - x_0| + \eta \\
&\leq \varepsilon,
\end{aligned}$$

since $x, x_0 \in [0,1]$. We find that $\{F_n\}_n$ is a Cauchy sequence for the supremum norm. By Theorem 2.2.5, $(C[0,1], \|\cdot\|)$ is complete, so $F_n \to F$ as $n \to \infty$ for some function F in the supremum norm.

Step 3. Finally, we need to show that F' exists and equals f. Let $\varepsilon > 0$ and $x \in [0,1]$ be given. In the same way as we proved (1), we can apply the Mean Value Theorem to $F_n - F_m$ for n, m large enough that $\|f_n - f_m\| < \frac{1}{3}\varepsilon$ and $y \neq x$, which gives

$$\left| \frac{F_n(y) - F_n(x)}{y - x} - \frac{F_m(y) - F_m(x)}{y - x} \right| < \frac{1}{3}\varepsilon.$$

Letting $m \to \infty$, this yields

$$\left| \frac{F_n(y) - F_n(x)}{y - x} - \frac{F(y) - F(x)}{y - x} \right| \leq \frac{1}{3}\varepsilon.$$

Defining

$$\xi_n(y) := \frac{F_n(y) - F_n(x)}{y - x} - \frac{F(y) - F(x)}{y - x}$$

and

$$\xi(y) := \frac{F(y) - F(x)}{y - x},$$

note that this implies that $\xi_n(y) \to \xi(y)$ as $n \to \infty$ uniformly on $[0,1] \setminus \{x\}$. Because $f_n \to f$, we can also estimate $|f_n(x) - f(x)| \leq \|f_n - f\| < \frac{1}{3}\varepsilon$ for large enough n.

Taking n large enough for the above two estimates, there exists some $\delta > 0$ such that if $|y-x| < \delta$,

$$\left| \frac{F_n(y) - F_n(x)}{y - x} - f_n(x) \right| < \frac{1}{3}\varepsilon$$

since $F_n' = f_n$ on $[0,1]$. Hence, for $0 < |y - x| < \delta$,

$$\begin{aligned}
\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| &\leq \left| \frac{F(y) - F(x)}{y - x} - f_n(x) \right| + |f_n(x) - f(x)| \\
&< \left| \frac{F(y) - F(x)}{y - x} - f_n(x) \right| + \frac{1}{3}\varepsilon \\
&\leq \frac{1}{3}\varepsilon + \left| \frac{F(y) - F(x)}{y - x} - \frac{F_n(y) - F_n(x)}{y - x} \right| + \left| \frac{F_n(y) - F_n(x)}{y - x} - f_n(x) \right| \\
&< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon \\
&= \varepsilon
\end{aligned}$$

We conclude that F is differentiable and $F' = f$. □

Theorem 3.3.8 (Weil). *There exist functions in $C[0,1]$ that are both everywhere differentiable and nowhere monotone.*

Proof. Define

$$D := \{f \in B[0,1] \colon f = F' \text{ for some } F \in C[0,1]\}$$

to be the set of bounded derivatives of functions $F \in C[0,1]$. Just as with $C[0,1]$, we equip D with the supremum norm (this is well-defined because each $f \in D$ is bounded). We first show that D is a complete metric space.

Let $\{f_n\}_n$ be a Cauchy sequence in D. For each $f_n \in D$, there exists some $F_n \in C[0,1]$ such that $f_n = F_n'$. Just as in Theorem 3.3.7, we use the pointwise convergence of f_n to define a limit function f, which exactly as before is shown to converge uniformly to f. Now, we still need to show f is the bounded derivative of some function F. For each n, define $G_n(x) := F_n(x) - F_n(0)$. Each G_n is a function from $[0,1]$ to $\mathbb{R}$ with the property that $G_n(0) = 0$, so $\{G_n(0)\}_n$ converges to 0. In addition, $G_n'(x) = F_n'(x) = f_n(x)$. Taking $x_0 = 0$, we can apply Theorem 3.3.7 for f_n and G_n to find that the functions G_n converge uniformly to G with the property that $G' = f$. To show f is bounded, take any $\varepsilon > 0$. Because $f_n \to f$, there exists some $N \geq 1$ such that $\|f_N - f\| < \varepsilon$. Because each $f_n \in D$, f_N is bounded by some value B, so

$$\|f\| \leq \|f - f_N\| + \|f_N\| \leq \varepsilon + B,$$

so f is bounded. Hence $f \in D$, so each Cauchy sequence in D converges in D as well, showing that D is a complete metric space.

For each $f \in D$, we now define the set $Z(f) := \{x \in [0,1] \colon f(x) = 0\}$ and the space

$$D_0 := \{f \in D \colon Z(f) \text{ is dense in } [0,1]\}.$$

We will show D_0 is complete with respect to the supremum norm as well. In order to do so, we first prove that for $g \in D_0$, $Z(g)$ is a countable intersection of open sets.

Define, for each $n \geq 1$,

$$g_n(x) := \frac{G(x+\frac{1}{n}) - G(x)}{\frac{1}{n}},$$

where $G' = g$. Then for any x, $\lim_{n\to\infty} g_n(x) = G'(x) = g(x)$, which implies $\lim_{n\to\infty} |g_n(x)| = |g(x)|$. If $x \in Z(g)$, then by definition $|g(x)| = 0$. Because g is the limit of g_n, we can rewrite this as $0 = \liminf_{n\to\infty} |g_n(x)| = \lim_{n\to\infty}(\inf_{m\geq n} |g_m(x)|)$. We prove this is equivalent to the statement

$$\text{for all } k > 0 \text{ and } n \geq 1, \text{there exists } m \geq n \text{ such that } |g_m(x)| < \frac{1}{k}. \tag{1}$$

For the right implication, assume the converse holds, so there exist k, n such that for all $m \geq n$, $|g_m(x)| \geq \frac{1}{k}$. This implies that $\inf_{m\geq n} |g_m(x)| \geq \frac{1}{k}$. Using our assumption, we find

$$0 = \liminf_{n\to\infty} |g_n(x)| = \sup_{n\geq 1} \left(\inf_{m\geq n} |g_m(x)| \right) \geq \frac{1}{k},$$

which is a contradiction.

For the left implication, we assume (1) and suppose $\liminf_{n\to\infty} |g_n(x)| =: b > 0$. By definition of the infimum, for some $0 < \varepsilon < b$, there exists some $N \geq 1$ such that for all $n \geq N$, $|g_n(x)| > b - \varepsilon$. Let $k > \frac{1}{b-\varepsilon}$ a whole number and $n \geq N$. By (1) there exists some $m \geq n$ such that $|g_m(x)| < \frac{1}{k} < b - \varepsilon$, but this contradicts the assumption that for all $m \geq n$, $|g_m(x)| > b - \varepsilon$. Hence, $\liminf_{n\to\infty} |g_n(x)| = 0$ is equivalent to (1).

In conclusion, $x \in Z(g)$ is equivalent to $\liminf_{n\to\infty} |g_n(x)| = 0$, which in set-theoretic terms can be expressed as

$$x \in \bigcap_{k\geq 1} \bigcap_{n\geq 1} \bigcup_{m\geq n} \left\{ y \colon |g_m(y)| < \frac{1}{k} \right\}.$$

Hence,

$$Z(g) = \bigcap_{k\geq 1} \bigcap_{n\geq 1} \bigcup_{m\geq n} \left\{ y \colon |g_m(y)| < \frac{1}{k} \right\}.$$

Each set $\{y \colon |g_m(y)| < \frac{1}{k}\}$ is open in $[0, 1]$, so $Z(g)$ is a countable intersection of open sets.

Now let $\{f_n\}_n$ be a Cauchy sequence in D_0. This converges to some $f \in D$. We then need to prove that $Z(f)$ is dense in $[0, 1]$. Write $Z := \cap_{n\geq 1} Z(f_n)$. Because each $Z(f_n)$ is dense and a countable intersection of open sets, Z is again a countable intersection of dense open sets. As $[0, 1]$ is a Baire space, with the Baire Category Theorem we find Z is dense in $[0, 1]$. If $x \in Z$, then for all $n \geq 1$, $x \in Z(f_n)$, so equivalently $0 = \lim_{n\to\infty} |f_n(x)| = |f(x)|$, so $x \in Z(f)$. We find $Z \subset Z(f)$, so $Z(f)$ must also be dense in $[0, 1]$. We conclude $f \in D_0$.

Before we get to the essential part of the proof, we have two things left to prove: we show that D_0 is closed under addition, and that D_0 contains more than just the zero function.

The first part is fairly straightforward. Let $f, g \in D_0$ be given. Because $f + g$ is bounded (taking the sum of the bounds of f and g) and taking the derivative is linear, $f + g \in D$. In addition, $Z(f)$ and $Z(g)$ are dense in $[0, 1]$. Because $Z(f)$ and $Z(g)$ are countable intersections of open, dense sets in $[0, 1]$ and $[0, 1]$ is Baire, $Z(f) \cap Z(g)$ is again a countable intersection of open, dense sets. As a result, $Z(f) \cap Z(g)$ must be dense in $[0, 1]$ as well. Now, if $x \in Z(f) \cap Z(g)$, we have $f(x) = 0 = g(x)$. Then $f(x) + g(x) = 0$, so $x \in Z(f + g)$. We have that $Z(f) \cap Z(g) \subset Z(f + g)$, and because the former is dense, the latter must be as well. Hence, $f + g \in D_0$.

By Theorem 3.3.6, there exists a non-zero function $f\colon [0,1] \to \mathbb{R}$ such that $F' = f$ on its domain, f is bounded and $Z(f)$ is dense in $[0,1]$, so $f \in D_0$. Hence, D_0 contains a Pompeiu derivative.

As we have seen in the proof that nowhere differentiable functions exist (Theorem 3.1.4), the idea of the proof is to define a subset E with the property that functions are somewhere monotone and show this particular set is meagre in D_0. All the work we have done so far is to apply the Baire Category Theorem on D_0, which is a Baire space, so E cannot equal the whole space. Its complement is dense in D_0 (just as in the proof of Theorem 3.1.4). A suitable definition is

$$E := \{f \in D_0 \colon \text{ there is an interval where } f \text{ does not switch sign}\}.$$

Let $\{I_n\}_n$ be some ordering of all the closed intervals with rational endpoints contained in $[0,1]$. Because of the density of the rationals in the reals, it suffices to show that f does not switch sign on any interval I_n in this collection (here, we assume I_n is not a singleton set). Write

$$E_n := \{f \in D_0 \colon f(x) \geq 0 \text{ for } x \in I_n\}$$

and

$$F_n := \{f \in D_0 \colon f(x) \leq 0 \text{ for } x \in I_n\},$$

then

$$E = \bigcup_{n \geq 1} (E_n \cup F_n).$$

We show that, for some $n \geq 1$, E_n is nowhere dense by showing it is closed and its interior is empty. The proof for F_n follows in the same way.

Let $\{f_n\}_n$ be a sequence in E_n converging uniformly to some $f \in D_0$. Suppose $f \notin E_n$. Then there exists some $x_0 \in I_n$ for which $f(x_0) < 0$. As $f_n \to f$, we can choose $\varepsilon := -f(x_0)$, then there exists some N for which if $n \geq N$, $|f_n(x_0) - f(x_0)| \leq \|f_n - f\| < \varepsilon$. Since $f(x_0) < 0$, this implies $f_n(x_0) < 0$, which contradicts $f_n \in E_n$. So E_n is closed.

To show E_n has empty interior, let $f \in E_n$ and $\varepsilon > 0$. Since f vanishes on a dense set in $[0,1]$ and I_n contains an open set, there exists some $x_0 \in I_n$ such that $f(x_0) = 0$ and $x_0 > 0$. By Theorem 3.3.6, there exists a Pompeiu derivative $h \in D_0$ such that $h(y_0) \neq 0$ for some $y_0 \in (0,1)$. Furthermore, h is bounded, non-negative, and there exists some $H \in C[0,1]$ such that $H' = h$. For fixed $s > 0$ and $t \leq 1$, we take $a > 0$ large enough such that

$$x_a := \frac{y_0}{2x_0 a} < t$$

and

$$y_a := \frac{y_0^2}{4x_0^2 a} < s.$$

Now, define $v\colon [0, x_a] \to [0, y_a]$ by

$$v(x) := -ax^2 + \frac{y_0}{x_0}x,$$

then clearly we have that $v(0) = 0$ and $v(x_a) = y_a < s$. Furthermore, the function v is strictly increasing with $v'(0) = \frac{y_0}{x_0}$ and $v'(x_a) = 0$. We also define $w\colon [x_a, t] \to [y_a, s]$ by

$$w(x) := \frac{s - y_a}{(t - x_a)^2}(x - x_a)^2 + y_a.$$

We then have that $w(x_a) = y_a$, $w(t) = s$ and $w'(x_a) = 0$. The function w is also strictly increasing. From this, it follows that

$$q(x) := \begin{cases} v(x) & \text{if } 0 \leq x \leq x_a \\ w(x) & \text{if } x_a < x \leq t \end{cases}$$

is a differentiable, increasing function. In addition, q is a bijection from $[0,t]$ onto $[0,s]$ so that $q'(0) = \frac{y_0}{x_0}$. Finally, we set $s := 1 - y_0$ and $t := 1 - x_0$ and define

$$g(x) := \begin{cases} \frac{y_0}{x_0} x & \text{if } 0 \leq x \leq x_0 \\ q(x - x_0) + y_0 & \text{if } x_0 < x \leq 1. \end{cases}$$

Then g is an increasing bijection from $[0,1]$ onto $[0,1]$ such that $g(x_0) = y_0$. Furthermore, g is differentiable and g' is bounded, being constant on $[0, x_0]$ and equal to q' on $(x_0, 1]$.

Then we find that $(Hg)'(x) = h(g(x))g'(x)$. It holds that this function is zero whenever $h(x) = 0$, as $g(0) = 0$ but $g(x) \neq 0$ for x nonzero, and $g'(x) > 0$ for all $x \in [0,1] \setminus \{x_a\}$. In particular, $(Hg)'(x_0) = h(y_0)g'(x_0) \neq 0$. Hence, $(Hg)' \in D_0$.

Because $(Hg)'$ is a bounded and non-negative function, there exists some $B > 0$ such that $\|(Hg)'\| < B$. Define $\tilde{h} := -\frac{\varepsilon}{B}(Hg)'$, then $\tilde{h}(x_0) < 0$ and

$$\|\tilde{h}\| = \varepsilon B^{-1} \|(Hg)'\| < \varepsilon.$$

Now define $p := f + \tilde{h}$, then $\|f - p\| < \varepsilon$ but $p \notin E_n$, as $p(x_0) = f(x_0) + \tilde{h}(x_0) < 0$. So $B_\varepsilon(f)$ is not a subset of E_n, so E_n is nowhere dense.

Proceeding similarly for F_n, we find that E is a countable of union of nowhere dense sets. Hence, E is a meagre set. Since D_0 is a complete metric space, by the Baire Category Theorem (Theorem 2.6.4), E cannot be the entire space. Its residual $D_0 \setminus E$ must therefore be non-empty. Then there exists some $f \in D_0 \setminus E$ and some $F \in C[0,1]$ such that $F' = f$. If F would be monotone on some interval, its derivative f would not switch sign, which cannot occur as $f \in D_0 \setminus E$. Therefore, F must be a function that is everywhere differentiable and nowhere monotone. □

Corollary 3.3.2 motivates the construction of the space D_0. The following lemma demonstrates that this prevents the derivatives f from being continuous.

Lemma 3.3.9. *Let $I \subset [0,1]$ be a dense set and $f \in C[0,1]$ be such that $f(x) = 0$ for all $x \in I$. Then $f \equiv 0$.*

Proof. Assume there is some $y \in [0,1] \setminus I$ such that $f(y) \neq 0$. By continuity of f, for $\varepsilon = \frac{1}{2}|f(y)|$ there exists some $\delta > 0$ such that for $x \in (y - \delta, y + \delta)$, we have $|f(x) - f(y)| < \varepsilon$. Since I is dense, there exists some $x \in I \cap (y - \delta, y + \delta)$ such that

$$|f(x) - f(y)| = |f(y)| < \varepsilon = \frac{1}{2}|f(y)|,$$

a contradiction. □

We also need to require $f \in D$ to be bounded to define the supremum norm on D, because there are functions $F \in C[0,1]$ whose derivatives fail to be bounded on some interval.

Example 3.3.10. Consider the function $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) := \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

taken from [GO64, Example 6 on p. 37]. Its derivative is

$$f(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Let $M \in \mathbb{Z}_{\geq 1}$ arbitrary. Choose a whole number $N \geq 1$ such that $N > \frac{M^2}{16}$ and $x := \sqrt{\frac{1}{4N\pi}}$. Then

$$\begin{aligned}
|f(x)| &= \left| 2\sqrt{\frac{1}{4N\pi}} \sin(4N\pi) - \sqrt{16N\pi} \cos(4N\pi) \right| \\
&= \left| -\sqrt{16N\pi} \right| \\
&= 4\sqrt{N\pi} \\
&> 4\sqrt{\frac{1}{16}M^2} \\
&= M,
\end{aligned}$$

so f is not bounded.

4 Concluding remarks

In this thesis, our aim was to illustrate the power of the Baire Category Theorem by showing two remarkable facts about the space of continuous functions. Firstly, we demonstrated that any continuous function can be approximated by a nowhere differentiable function. We then demonstrated that every nowhere differentiable function is nowhere monotone. Finally, we established that there exist functions that are everywhere differentiable and nowhere monotone. In summary, we arrived at the following inclusions.

$$\{f \in C[0,1] \colon f \text{ is nowhere differentiable}\} \subset \{f \in C[0,1] \colon f \text{ is nowhere monotone}\} \overset{\text{dense}}{\subset} C[0,1].$$

The existence of functions with these properties serves as a challenge to our intuition of continuous functions. It seems especially strange that not only functions exist that are nowhere monotone and differentiable in a specific point or on a countable set, but actually are everywhere differentiable. This only shows that as mathematicians, we should not expect everything to be beautifully simple, but rather see the beauty in the unappealing.

References

[Bru83] Andrew Bruckner. "Some new simple proofs of old difficult theorems". In: *Real Analysis Exchange* 9.1 (1983), pp. 63–78.

[Che21] Pan Cheng-Han. "Nowhere-monotone differentiable functions and bounded variation". In: *Journal of Mathematical Analysis and Applications* 494.2 (2021), pp. 1–14. DOI: `https://doi.org/10.1016/j.jmaa.2020.124618`.

[Coh13] Donald L. Cohn. *Measure Theory*. 2nd ed. Boston: Birkhäuser, 2013.

[Con14] John B. Conway. *A Course in Point Set Topology*. Ed. by Kenneth A. Ribet and Sheldon Axler. Berlin: Springer, 2014.

[Fol99] Gerald B. Folland. *Real Analysis. Modern Techniques and Their Applications*. 2nd ed. New York: John Wiley & Sons, Inc., 1999.

[GO64] Bernard R. Gelbaum and John M.H. Olmsted. *Counterexamples in Analysis*. San Francisco: Holden-Day, 1964.

[Her06] Horst Herrlich. *Axiom of Choice*. Heidelberg: Springer-Verlag Berlin, 2006.

[HS65] Edwin Hewitt and Karl Stromberg. *Real and Abstract Analysis. A modern treatment of the theory of functions of a real variable*. Heidelberg: Springer-Verlag Berlin, 1965.

[Köp87] Alfred Köpcke. "Ueber Differentiirbarkeit und Anschaulichkeit der stetigen Functionen". In: *Mathematische Annalen* 29.9 (1887), pp. 123–140. DOI: `https://doi.org/10.1007/BF01445174`.

[Leb04] Henri Lebesgue. *Leçons sur l'intégration et la recherche des fonctions primitives, professées au Collège de France par Henri Lebesgue*. Paris: Gauthier-Villars, 1904.

[Mir21] Arami Mira. *File:Pompeiu derivative.svg*. 2021. URL: `https://commons.wikimedia.org/wiki/File:Pompeiu_derivative.svg`. Consulted: 13-07-2023.

[Mun14] James Munkres. *Topology*. 2nd ed. Harlow: Pearson Education Limited, 2014.

[Ols04] Lars Olsen. "A New Proof of Darboux's Theorem". In: *The American Mathematical Monthly* 111.8 (2004), pp. 713–715.

[Pom07] Dimitrie Pompeiu. "Sur les fonctions dérivées". In: *Mathematische Annalen* 63.3 (1907), pp. 326–332. DOI: `https://doi.org/10.1007/BF01449201`.

[Run05] Volker Runde. *A Taste of Topology*. Ed. by Kenneth A. Ribet and Sheldon Axler. Berlin: Springer, 2005.

[RY08] Bryan P. Rynne and Martin A. Youngson. *Linear Functional Analysis*. 2nd ed. London: Springer, 2008.

[Tao22] Terence Tao. *Analysis I*. 4th ed. Singapore: Springer, 2022.

[Tao23] Terence Tao. *Analysis II*. 4th ed. Gateway East, Singapore: Springer, 2023.

[Wei76] Clifford Weil. “On Nowhere Monotone Functions”. In: *Proceedings of the American Mathematical Society* 56.1 (1976), pp. 388–389.

[Wei95] Karl Weierstrass. “Über continuirliche Functionen eines reellen Arguments, die für keinen Werth des letzeren einen bestimmten Differentialquotienten besitzen”. In: *Mathematische Werke von Karl Weierstrass*. Ed. by Königlich Preussichen Akademie der Wissenschaften. Vol. 2. Berlin: Mayer & Mueller, 1895, pp. 71–74.

[Whe22] Miles Wheeler. *Nowhere Differentiable Functions. MA30252: Advanced Real Analysis*. 2022. URL: `https://people.bath.ac.uk/mw2319/ma30252/sec-diff.html`. Consulted: 30-06-2023.

A Python code

Weierstrass function

```
"""
Weierstrass function
"""

# required modules
import numpy as np
import matplotlib.pyplot as plt

# stepsize: the higher the more points are calculated
Step = 10000

# xpoints for range x axis
startx=0
stopx=8
xpoints = np.linspace(startx,stopx,Step)

# Weierstrassfunction without pi in cos term, Npar is up to N partial sum
c = 175
d = 1/5
def weierstrass(x,Npar):
    we=0
    for n in range(0,Npar):
        z=np.float64(c**n*x)
        we=we+np.cos(z)*d**n
    return we

# using different values for NPar, a and b can cause overflow
# plots
f = plt.figure()
f.set_dpi(150)
plt.plot(xpoints,weierstrass(xpoints, 30), linewidth=0.1, color="black")
plt.xlim(startx,stopx)
plt.grid(color='grey', which='major', linestyle='-', linewidth=0.5)
plt.grid(color='grey', which='minor', linestyle='-', linewidth=0.1)
plt.minorticks_on()
plt.show()
```