Passenger's Strategic Behavior In a Transportation Station

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1 Introduction

Indisputably, public transit plays a vital role in transporting passengers worldwide. As urban populations continue to grow, cities face increasing traffic congestion challenges, and transportation network efficiency challenges have become more pressing. Therefore, it is vital to prioritize enhancing managerial and administrative practices, reducing travel times, and improving overall accessibility and mobility. In response to these demands, modern transportation systems have incorporated real-time services, providing customers with real-time information such as departure time and crowd density. With advancements in technology and widespread internet usage, access to this information has become more convenient than ever before. In this study, we consider a stylized stochastic model of a transportation system, employing a game theoretical framework to study the impact of information provision on system performances in the presence of strategic passengers.

The study of queueing systems from a game-theoretic perspective was first conducted by Naor (1969), who investigated the strategic behavior of customers in M/M/1 queues. In this model, customers make decisions based on their knowledge of the queue length, deciding whether to join the line or balk [1]. Building upon Naor's work, Edelson and Hildebrand (1975) further investigated the same queue system without providing any information to the customers [2]. Hassin and Haviv's (2003) and Stidham's (2009) books extensively explore the primary methodologies and numerous findings [3], [4]. Additionally, Hassin's (2016) book offers detailed summaries of over 600 papers focusing on rational queueing, providing a comprehensive overview of the research conducted in this area [5].

Manou, Economou, and Karaesmen (2014) formulated a transportation system model with finite and strategic customer arrivals following a Poisson Process. In this study, the transportation facility, with random capacity, visits the station under a renewal process. The research focuses on examining customers' equilibrium strategies under two scenarios: the observable case, where customers observe the queue length, and the unobservable case, where customers lack such information [6].

Manou et al. (2017) expanded the research to a transportation model where the facilities have infinite capacity. The authors considered a pricing problem where an administrator imposes a fee, and the customers observe their delay information. For three information levels, they explore the effect of the customers' reward, the unit waiting cost and the distribution of the arriving facilities on the customer behavior and the fee imposed [7].

Finally, Logothetis and Economou (2023) studied the effect of information on a transportation system with finite capacity. To determine the ideal level of provided information, they derive the customer equilibrium strategies and compare the equilibrium throughput and social welfare under various combinations of the information offered [8].

This thesis centers on a transportation system with infinite capacity and studies the influence of information when customers make strategic decisions regarding joining or balking. This research includes two distinct scenarios regarding the homogeneity of the customer population. The first scenario assumes customers are homogeneous in their evaluations of reward and waiting costs. In contrast, the second scenario considers customers as inhomogeneous in evaluating their waiting costs.

The structure of this thesis is as follows. Section 2 provides a detailed description of the transportation system, encompassing key assumptions and operational characteristics. It also includes essential definitions and concepts from the existing literature on rational queueing, along with a specific delineation of the information levels utilized. In Section 3, we analyze the customer equilibrium strategies and examine the corresponding system performances across three different information levels, drawing comparisons between them. Furthermore, we contrast the performances observed within the two distinct scenarios. In addition, in Section 4, we describe the equilibrium strategies and system performances for the inhomogeneous scenario. In Section 5, we compare these performances with those of the homogeneous scenario.

2 Model

2.1 Transportation system

In this research project, we consider a transportation system where all customers present at the time of transportation arrival are served simultaneously.
Customers arrive at the platform according to a Poisson process denoted by $\{P(t)\}$ with rate λ. The transportation facility visits the station according to a renewal process $\{M(t)\}$. The typical interarrival time for the process $\{M(t)\}$ is represented by the random variable X, with distribution $F(t)$ and density $f(t)$. We assume that the platform and the facility have infinite capacity, so every customer joining the system will receive service.

2.2 Strategic customers framework

The customers, upon arrival at the station, decide whether to join or to balk, with the objective of maximizing their expected net benefit. We assume that their decisions are strategic, in the sense that individuals' beliefs regarding how others will decide, affect their choices. Thus, the whole situation can be regarded as a game among the customers. The primary objective is to determine the equilibrium strategy of the customers.

Moreover, we make the following assumptions about the customers:

- Customers are aware of both the operational and economic parameters. They know which specific distribution the arriving facilities follow, which values the parameters of the distributions have, and what the reward and cost are.
- Customers are selfish; they only consider personal profit.
- Customers are rational; they can calculate the best decision for them.

We assume the presence of a reward-cost structure in the system, wherein customers are allowed to decide to either join or balk the system upon their arrival. Each customer incurs a cost (c) per unit of time (t) while simultaneously assigning a value or reward (r) to the service provided by the facility.

To describe the equilibrium strategy, we first need to define some concepts. In the resulting game, since we deal with an infinite number of customers, we assume a common set of strategies S and payoff function G. In particular, $G(s|s')$ is the payoff for a tagged customer who selects the strategy s, when everyone else selects the strategy s'. An equilibrium strategy is a strategy $s^e \in S$ that

$$s^e \in \arg_{s \in S} G(s|s^e).$$

In other words, an equilibrium strategy s^e is a strategy chosen by the customers such that no customer has the incentive to denote from this strategy initially [3]. Moreover, a strategy s_1 is strictly dominated by strategy s_2 if $G(s_1|s) < G(s_2|s) \forall s \in S$. Note that if a strategy is a dominant strategy, then it is also an equilibrium strategy [9].

2.3 Information Levels

In this research project, we mainly focus on the effect of information on customers' strategic behavior and the resulting performance of the system. To this end, we investigate the following scenarios regarding the information conveyed to the customers.

- The no-information level ($\emptyset$).The customers are not provided with any additional information.
- The information level "Age" (A). Customers are informed regarding transportation facility, which is the time between the last facility left and the customers' arrival.
- The information level "Residual time" (R). The exact waiting time is known for all customers. This is the time between the arrival of a customer and the next arriving facility.

We obtain four combinations of the information levels: AR, A, R, and $\emptyset$. In the level AR, a customer knows the age and the residual time. With the use of age, a customer is informed of the age and the residual time. With the help of the age, a customer will estimate his waiting time. However, in this one, a customer also has the residual time information, which is the exact waiting time. Therefore, in the information level AR, a tagged customer will only use the knowledge of the precise waiting time and not the age, and thus, we have the following remark.

Remark 1. *The information level AR behaves the same as the information level R.*

Thus, in this paper, the set of information levels is

$$I = \{A, R, \emptyset\}. \tag{1}$$

2.4 Utility and Performance

We consider an arbitrarily tagged customer who arrives at the system and observes the information I. The expected utility for this customer, given that the customer population follows a strategy $\mathbf{q} \equiv q^i$ is denoted by $G^I(i|q^i)$, with $I \in \{\emptyset, A, R\}$. The facilities have infinite capacity, ensuring that the probability of a customer receiving service equals 1. To determine the expected utility, we define

$$G^I(i|q^i) = r - cE^I(i|q^i). \tag{2}$$

Here, $E^I(i|q^i)$ represents the conditional expectation of the customers' waiting time until the next facility arrives, given that the other customers follow strategy q^i and information i is provided under the information scenario I.

In addition to evaluating the utility of a specific customer, we can also examine the system's performance. The two key performance measures are equilibrium throughput and social welfare. The *equilibrium throughput* is defined as the number of customers served per time unit [4]. Since all customers joining the platform are guaranteed a seat in the facility, the throughput is equivalent to the number of customers entering the system per time unit.
Social welfare is defined as the total expected net benefit per time unit of the customers in the transportation system. The social objective aims to maximize the total rewards received minus the total waiting cost [3].

2.5 Two scenarios regarding the homogeneity

In this paper, we examine two scenarios regarding the homogeneity of the customers: (1) Customers are homogeneous in their reward and waiting cost, and (2) customers are inhomogeneous in their waiting cost evaluation. In the second scenario, customers do not have identical perceptions o their waiting cost. That is, different customers, have varying valuations regarding the cost associated with waiting. We capture the difference by assuming the cost is a random variable according to the uniform distribution $U(\alpha, \beta)$, with $\alpha < \beta$.

3 Effect of information in customer strategic behavior

In this section, we investigate the influence of information on customer strategic behavior. Specifically, in the subsequent subsections, we derive the customer equilibrium strategies corresponding to each information level as discussed in subsection 2.3. Additionally, we analyze the resulting system dynamics under the equilibrium strategy and calculate the key system performance measures. Furthermore, we present several theoretical and numerical findings to compare the distinct information levels.

3.1 Equilibrium Strategies and Performance Measures

This subsection aims to determine the customer equilibrium strategies and performance measures for each information level.

3.1.1 The no-information level

In the no-information level, a general customer strategy $\mathbf{q}$, is specified by a single joining probability q. As described in section 2.4, the net benefit of a tagged customer who decides to join, knowing the other customers follow a strategy $\mathbf{q}$, is given by

$$G^{\emptyset}(q) = r - cE^{\emptyset}(q).$$

The customers arrive according to the Poisson Process, which means we can apply the PASTA property. This means that a customer who comes at a given time observes the system as an arrival at a random moment. Therefore, it equals the residual waiting time at an arbitrary instant. Thus, by applying the renewal theory, we have $E^{\emptyset}(q) = \frac{E(X^2)}{2E(X)}$. Therefore, the net benefit for a tagged customer in the no-information level is equal to

$$G^{\emptyset}(q) = r - c \cdot \frac{E(X^2)}{2E(X)}. \tag{3}$$

The net benefit does not depend on the other customers' strategy q, so the strategy of a tagged customer is a dominant strategy.
We can use this argument to determine the equilibrium strategy. A customer joins the system when $G^{\emptyset}(q) \geq 0$ holds, and we assume for the uniqueness that a customer balks for $G^{\emptyset}(q) < 0$. We have that

$$G^{\emptyset}(q) \geq 0 \Leftrightarrow r - c\frac{E(X^2)}{2E(X)} \geq 0 \Leftrightarrow \frac{E(X^2)}{2E(X)} \leq \frac{r}{c}. \tag{4}$$

So a customer joins when $\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}$ holds.

In the same way, we find that a customer balks when $\frac{E(X^2)}{2E(X)} > \frac{r}{c}$ holds. Therefore, in the information level $\emptyset$, the equilibrium strategy is equal to

$$q^e = \begin{cases} 1 & \text{if } \frac{E(X^2)}{2E(X)} \leq \frac{r}{c}, \\ 0 & \text{else.} \end{cases} \tag{5}$$

Now we can determine the performance measures. These are the equilibrium throughput and the social welfare. The throughput is the number of customers served per time unit. Everyone who enters the system will be served, which equals the number of customers joining the platform per time unit. The equilibrium strategy is dominant, so the throughput is equal to the equilibrium strategy times the parameter of the Poisson process.

To calculate the throughput, we use the renewal reward theorem. A renewal cycle in this case is the time interval between two successive visits of the transportation. In particular, we set $R_k = N_{q^e}(X_k)$ to be the reward in the k renewal cycle, where $N_{q^e}(t)$ denotes the number of customers in the system in $[0, t]$. That is, the customers who are served during a renewal cycle. Thus the total reward until time t would be $R(t) = \sum_{k=1}^{M(t)} R_k$. Since, (R_k, X_k) is a i.i.d sequence of random variables, the renewal reward theorem states that

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E(N_{q^e}(X))}{E(X)}$$

It is easy to see that $N_{q^e}(t)$ is Poisson process with rate λq^e, and thus, we obtain immediately that

$$TH^{\emptyset} = \lambda q^e = \begin{cases} 0 & \text{when } \frac{E(X^2)}{2E(X)} > \frac{r}{c}, \\ \lambda & \text{when } \frac{E(X^2)}{2E(X)} \leq \frac{r}{c}. \end{cases} \tag{6}$$

The expected social benefit per time unit is expressed by

$$SW^{\emptyset} = \lambda q^e G^{\emptyset}(q^e) = \lambda q^e (r - cE(q^e)).$$

Thus, we have that

$$SW^{\emptyset} = \begin{cases} 0 & \text{when } \frac{E(X^2)}{2E(X)} > \frac{r}{c}, \\ \lambda(r - c\frac{E(X^2)}{2E(X)}) & \text{when } \frac{E(X^2)}{2E(X)} \leq \frac{r}{c}. \end{cases} \tag{7}$$

3.1.2 The information level Age

In this subsection, we assume that the arriving customers are informed about the elapsed time a from the last transportation visit. Then, a joining strategy is a function $\mathbf{q} = \{q(a), a \geq 0\}$, where $q(a)$ is the joining probability for a customer that arrives at the station a time units after the last visit of the transportation facility. In this case, the expected net benefit for a tagged customer that arrives and is informed that the age of the process $M(t)$ is a and the rest of the customers use a general strategy $\mathbf{q}$, is given by

$$G^A(a|\mathbf{q}) = r - c \cdot E^A(a|\mathbf{q}). \tag{8}$$

The conditional mean forward recurrence time at the arrival instant of the tagged customer, given that he finds age $A = a$ is the mean residual life function $m(a) = E(X - a|X > a)$. Thus, we have that

Remark 2. *The mean residual life function is*

$$m(a) = \int_a^{\infty} \frac{1 - F(u)}{1 - F(a)} du. \tag{9}$$

The net benefit is independent of $\mathbf{q}$, so a dominant equilibrium strategy exists. Consider that a customer joins when $G^A(a|\mathbf{q}) \geq 0$. Then we have

$$q^e(a) = \begin{cases} 1 & \text{if } m(a) \leq \frac{r}{c}, \\ 0 & \text{else.} \end{cases} \tag{10}$$

We analyze the performance measures. As described in section 3.1.1, the throughput can be calculated by the equilibrium strategy times the parameter λ. However, information is added this time, so we have to integrate over the age. Therefore, for the information level age, the throughput is equal to

$$TH^A = \lambda \int_0^{\infty} q^e(a) \frac{1 - F(a)}{E(X)} da. \tag{11}$$

Again, by integrating over a, we calculate the net benefit per time unit. The social welfare is then specified as

$$SW^A = \lambda \int_0^{\infty} q^e(a)(r - c \cdot m(a)) \frac{1 - F(a)}{E(X)} da. \tag{12}$$

3.1.3 The information level Residual time

We assume in this subsection that the customers are provided with information regarding the exact arrival time of the next transportation facility. That, is, each customer knows his exact waiting time. Similarly to the previous subsection, a general customer strategy is a function $\mathbf{q} = \{q(w), w \geq 0\}$, where $q(w)$ denotes the joining probability when the waiting time is exactly w times units. Therefore, the expected time a customer has to wait equals w. Thus, the net benefit of a tagged customer is specified as

$$G^R(w|\mathbf{q}) = r - c \cdot w. \tag{13}$$

The net benefit is independent of q; therefore, a unique dominant joining strategy exists. Using the argument that a customer joins when $G^R(w|\mathbf{q})$ holds, the equilibrium strategy is specified as

$$q^e(w) = \begin{cases} 1 & \text{if } w \leq \frac{r}{c}, \\ 0 & \text{else.} \end{cases} \tag{14}$$

We integrate over w, and the equilibrium strategy equals 1 when $w \leq \frac{r}{c}$. Therefore, in the information level R, the throughput and social welfare are specified, respectively, as

$$TH^R = \lambda \int_0^\infty q^e(w) \frac{1 - F(w)}{E(X)} dw. \tag{15}$$

$$TH^R = \lambda \int_0^{\frac{r}{c}} \frac{1 - F(w)}{E(X)} dw. \tag{16}$$

$$SW^R = \lambda \int_0^\infty q^e(w)(r - c \cdot w) \frac{1 - F(w)}{E(X)} dw. \tag{17}$$

$$SW^R = \lambda \int_0^{\frac{r}{c}} (r - c \cdot w) \frac{1 - F(w)}{E(X)} dw. \tag{18}$$

3.2 Theoretical Comparisons of the Information Levels

In this section, we compare the various information cases using the derived formulas from the equilibrium throughput and social welfare.

3.2.1 Comparison of $\emptyset$ and A

First, we compare the throughput of the no-information level and the information level of the age as functions of the arrival rate λ. To this end, we have the following result.

Theorem 1 (Comparison of $TH^\emptyset$ and TH^A)**.** *It holds that*

$$\begin{cases} TH^A \geq TH^\emptyset \; for \; \frac{E(X^2)}{2E(X)} > \frac{r}{c}, \\ TH^A \leq TH^\emptyset \; for \; \frac{E(X^2)}{2E(X)} \leq \frac{r}{c}. \end{cases} \tag{19}$$

Proof. Let $\frac{E(X^2)}{2E(X)} > \frac{r}{c}$. We have

$$TH^\emptyset = \lambda \mathbb{1}\{\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}\} = 0.$$

Clearly, we have $TH^A \geq 0$. So we have $TH^A \geq TH^\emptyset$.
Now, let $\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}$. We have

$$TH^\emptyset = \lambda \mathbb{1}\{\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}\} = \lambda.$$

Let $J = \{a : m(a) \le \frac{r}{c}\}$. We have

$$\begin{aligned} TH^A &= \lambda \int_0^\infty \mathbb{1}\{m(a) \le \frac{r}{c}\} dF_A(a) \\ &= \lambda \int_J dF_A(a) \\ &\le \lambda \int_0^\infty dF_A(a) \\ &= \lambda = TH^\emptyset. \end{aligned}$$

So for $\frac{E(X^2)}{2E(X)} \le \frac{r}{c}$ we have $TH^A \le TH^\emptyset$. □

According to Theorem 1, the information level A yields better results regarding the throughput when $\frac{E(X^2)}{2E(X)} > \frac{r}{c}$.

The following results states that the information level A is always preferable regarding the social welfare.

Theorem 2 (Comparison of $SW^\emptyset$ and SW^A)**.** *It holds that*

$$SW^A \ge SW^\emptyset. \tag{20}$$

Proof. Let $\frac{E(X^2)}{2E(X)} > \frac{r}{c}$. We have

$$SW^\emptyset = \lambda 1\{\frac{E(X^2)}{2E(X)} \le \frac{r}{c}\}(r - c\frac{E(X^2)}{2E(X)}) = 0.$$

The social welfare can not be less than zero, so we have $SW^A \ge SW^\emptyset$.
Now, let $\frac{E(X^2)}{2E(X)} \le \frac{r}{c}$. We have

$$SW^\emptyset = \lambda 1\{\frac{E(X^2)}{2E(X)} \le \frac{r}{c}\}(r - c\frac{E(X^2)}{2E(X)}) = \lambda(r - c\frac{E(X^2)}{2E(X)}).$$

We have

$$\begin{aligned} SW^A &= \lambda \int_0^\infty \{m(a) \le \frac{r}{c}\}(r - c \cdot m(a)) dF_A(a) \\ &= \lambda \int_J (r - cm(a)) dF_A(a), \end{aligned}$$

where $J = \{a : m(a) \le \frac{r}{c}\}$. Now, since

$$\begin{aligned} \int_0^\infty m(a) dF_A(a) &= \int_0^\infty \int_a^\infty \frac{1 - F(x)}{1 - F(a)} dx dF_A(a) \\ &= \int_0^\infty \int_0^x \frac{1 - F(x)}{1 - F(a)} dF_A(a) dx \\ &= \int_0^\infty (1 - F(x)) \int_0^x \frac{f_A(a)}{1 - F(a)} da dx \\ &= \int_0^\infty \frac{1 - F(x)}{E(X)} da \int_0^x dx \\ &= \int_0^\infty x \frac{1 - F(x)}{E(X)} dx = E(A) = \frac{E(X^2)}{2E(X)}, \end{aligned}$$

we get that

$$\begin{aligned}
SW^{\emptyset} &= \lambda \int_0^{\infty} (r - c \cdot m(a)) dF_A(a) \\
&= \lambda \int_{J'} (r - c \cdot m(a)) dF_A(a) + \lambda \int_J (r - c \cdot m(a)) dF_A(a) \\
&\leq \lambda \int_J (r - c \cdot m(a)) dF_A(a) = SW^A.
\end{aligned}$$

The last inequality holds since in J' we have that $r - cm(a) < 0$. □

3.2.2 Comparison of $\emptyset$ and R

We now compare the information level R with the no-information level. We have the following general result.

Theorem 3 (Comparison of $TH^{\emptyset}$ and TH^R.)**.** *It holds that*

$$\begin{cases} TH^R \geq TH^{\emptyset} \text{ for } \frac{E(X^2)}{2E(X)} > \frac{r}{c}, \\ TH^R \leq TH^{\emptyset} \text{ for } \frac{E(X^2)}{2E(X)} \leq \frac{r}{c}. \end{cases} \tag{21}$$

Proof. Let $\frac{E(X^2)}{2E(X)} > \frac{r}{c}$. We have

$$TH^{\emptyset} = \lambda \mathbb{1}\{\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}\} = 0.$$

Clearly, we have $TH^R \geq 0$. So we have $TH^R \geq TH^{\emptyset}$.
Now, let $\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}$. We have

$$TH^{\emptyset} = \lambda \mathbb{1}\{\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}\} = \lambda.$$

Let $J = \{w : w \leq \frac{r}{c}\}$. We have

$$\begin{aligned}
TH^R &= \lambda \int_0^{\frac{r}{c}} \frac{1 - F(a)}{E(X)} dw \\
&= \lambda \int_0^{\frac{r}{c}} dF_R(w) \\
&= \lambda \int_J dF_R(w) \\
&\leq \lambda \int_0^{\infty} dF_R(w) \\
&= \lambda = TH^{\emptyset}.
\end{aligned}$$

So for $\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}$ we have $TH^R \leq TH^{\emptyset}$. □

We now compare the social welfare of the no-information level and the information level of the residual time. With the following result, we establish that the information level R is always better than the no-information level regarding the social welfare.

Theorem 4 (Comparison of $SW^{\emptyset}$ and SW^R)**.** *It holds that*

$$SW^R \geq SW^{\emptyset} \tag{22}$$

Proof. Let $\frac{E(X^2)}{2E(X)} > \frac{r}{c}$. We have

$$SW^{\emptyset} = \lambda 1\{\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}\}(r - c\frac{E(X^2)}{2E(X)}) = 0.$$

Since the social welfare can not be negative, we have $SW^R \geq SW^{\emptyset}$.
Now, let $\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}$. We have

$$SW^{\emptyset} = \lambda 1\{\frac{E(X^2)}{2E(X)} \leq \frac{r}{c}\}(r - c\frac{E(X^2)}{2E(X)}) = \lambda(r - c\frac{E(X^2)}{2E(X)}).$$

Note that

$$\begin{aligned} SW^R &= \lambda \int_0^{\infty} \{w \leq \frac{r}{c}\}(r - c \cdot w)dF_R(w) \\ &= \lambda \int_J (r - cw)dF_R(w) \\ &\geq \int (r - c \cdot w)dF_R(w) = SW^{\emptyset}. \end{aligned}$$

So, $SW^R \leq SW^{\emptyset}$ holds for $\frac{E(X^2)}{2E(X)}$. □

3.2.3 Comparison of A and R for specific cases

In this last subsection, we compare the age and residual time information levels for some special cases regarding the intervisit distribution. Our initial focus is on the case of constant intervisit times. This case constitutes the most natural case for real-life applications, assuming a reliable transportation facility that visits the station regularly. In this case, the facilities visit the station every X time units, which implies that the residual time is equal to

$$r = X - a. \tag{23}$$

So, for this scenario, the information level A provides all customers with their residual times. Therefore, the throughput and social welfare are the same for the age and residual time when the facilities arrive at constant intervals.

Next, we consider the system where facilities arrive according to the exponential distribution with parameter $\mu > 0$. The mean residual life function is equal to its mean, i.e., $m(a) = \frac{1}{\mu}$ [10].

First, we compare the throughput of information level age and that of the residual time. The following proposition summarizes the result:

Theorem 5. *Let $F(t)$ be the exponential distribution with rate μ. We have that*

$$\begin{cases} TH^A \leq TH^R \text{ for } \frac{1}{\mu} > \frac{r}{c}, \\ TH^A > TH^R \text{ for } \frac{1}{\mu} \leq \frac{r}{c}. \end{cases} \tag{24}$$

Proof. Let $\frac{1}{\mu} > \frac{r}{c}$ and F the exponential distribution with rate μ. The mean residual life function of the exponential distribution is equal to $\frac{1}{\mu}$. Therefore, we have

$$TH^A = \lambda \int_0^{\infty} \mathbb{1}\{\frac{1}{\mu} \leq \frac{r}{c}\}\frac{(1 - F(a))}{E(X)}da = 0 \leq TH^R.$$

Let $\frac{1}{\mu} \leq \frac{r}{c}$ and F the exponential distribution. We have

$$\begin{aligned} TH^A &= \lambda \int_0^\infty \mathbb{1}\{\frac{1}{\mu} \leq \frac{r}{c}\} \frac{(1-F(a))}{E(X)} da \\ &= \lambda \int_0^\infty k e^{-ka} da \\ &= \lambda k \frac{1}{k} \\ &= \lambda. \end{aligned}$$

For the residual time, we have

$$\begin{aligned} TH^R &= \lambda \int_0^\infty \mathbb{1}\{w \leq \frac{r}{c}\} \frac{(1-F(w))}{E(X)} dw \\ &= \lambda \int_0^{\frac{r}{c}} \mu e^{-\mu w} dw \\ &= \lambda \mu \frac{0!}{\mu} (1 - e^{-\frac{\mu r}{c}} \sum_{i=0}^{0} \frac{(\mu r)^i}{i!}) \\ &= \lambda (1 - e^{-\frac{\mu r}{c}}). \end{aligned}$$

Then we have

$$\begin{aligned} TH^A - TH^R &= \lambda - \lambda(1 - e^{-\frac{\mu r}{c}}) \\ &= \lambda e^{-\frac{\mu r}{c}} \\ &> 0. \end{aligned}$$

So, for $\frac{1}{\mu} \leq \frac{r}{c}$, we have $TH^A > TH^R$. □

Additionally, we can compare the social welfare between the age and the residual time information levels in the exponential distribution scenario. The following Theorem summarizes the result:

Proposition 1. *Let $F(t)$ be the exponential distribution with rate μ. We have*

$$\begin{cases} SW^A \leq SW^R \text{ for } \frac{1}{\mu} > \frac{r}{c}, \\ SW^A > SW^R \text{ for } \frac{1}{\mu} \leq \frac{r}{c}. \end{cases} \tag{25}$$

Proof. Let $\frac{1}{\mu} > \frac{r}{c}$. Then we have $SW^A = 0 \leq SW^R$.
Let $\frac{1}{\mu} \leq \frac{r}{c}$. Then we have

$$SW^A = \lambda \int_0^\infty (r - c\frac{1}{\mu})) \frac{1-F(a)}{E(X)} da.$$

The distribution F is the exponential distribution, so we have

$$\begin{aligned} SW^A &= \lambda \int_0^\infty (r - c\frac{1}{\mu}) \mu e^{-\mu a} da \\ &= \lambda (r \int_0^\infty \mu e^{-\mu a} da - c \int_0^\infty e^{-\mu a} da) \\ &= \lambda (r - \frac{c}{\mu}). \end{aligned}$$

For the residual time, we get $SW^R = \lambda \int_0^{\frac{r}{c}} (r - cw)\mu e^{-\mu w} dw$. We have

$$\begin{aligned} SW^R &= \lambda \int_0^{\frac{r}{c}} (r - cw)\mu e^{-\mu w} dw \\ &= \lambda(r\mu \int_0^{\frac{r}{c}} e^{-\mu w} dw - c\mu \int_0^{\frac{r}{c}} we^{-\mu w} dw) \\ &= \lambda(r - \frac{c}{\mu} + \frac{c}{\mu} e^{\frac{-\mu r}{c}}). \end{aligned}$$

Combining the formulas of SW^A and SW^R we get

$$\begin{aligned} SW^A - SW^R &= \lambda(r - \frac{c}{\mu}) - \lambda(r - \frac{c}{\mu} + \frac{c}{\mu} e^{\frac{-\mu r}{c}}) \\ &= \lambda \frac{c}{\mu} e^{\frac{-\mu r}{c}} \end{aligned}$$

We have $\lambda > 0$ and $\mu, c, r > 0$. Therefore we have the inequality $SW^A - SW^R = \lambda(\frac{c}{\mu} e^{\frac{-\mu r}{c}}) > 0$ which implies that $SW^A > SW^R$ holds for $\frac{1}{\mu} \leq \frac{r}{c}$. □

3.3 Numerical Comparisons of the Information Levels

3.3.1 Erlang distribution

We consider the Erlang distribution with parameters $k \in \mathbb{N}_{\geq 1}$ and $\mu \in (0, \infty)$. It is the distribution of a sum of k independent exponential variables, all with mean $\frac{1}{\mu}$. Moreover, it is the distribution of time until the kth arrival of the Poisson Process with rate μ. Therefore, when the facilities arrive according to an Erlang distribution, we could see the situation as facilities that arrive according to a Poisson process where the customers can only enter the kth facility.
We will show the behavior of the performance measures with three different graphs. We have analytically compared information levels A and R. However, we did not find general proof for all distribution functions. Therefore, we will perform some numerical experiments to plot the equilibrium throughput and social welfare behavior for the two information levels and compare them.
The mean residual life of the Erlang distribution $F(x) = 1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-\mu x} (\mu x)^n$ is given by

$$m(a) = \frac{\mu^{k-1} a^k e^{-ka}}{(k-1)!(1 - F(a))} + \frac{k}{\mu} - a, \tag{26}$$

with $k \in \{1, 2, 3, ...\}$ and $\mu \in (0, \infty)$, see for example [10].
We did some numerical experiments for different values of k.

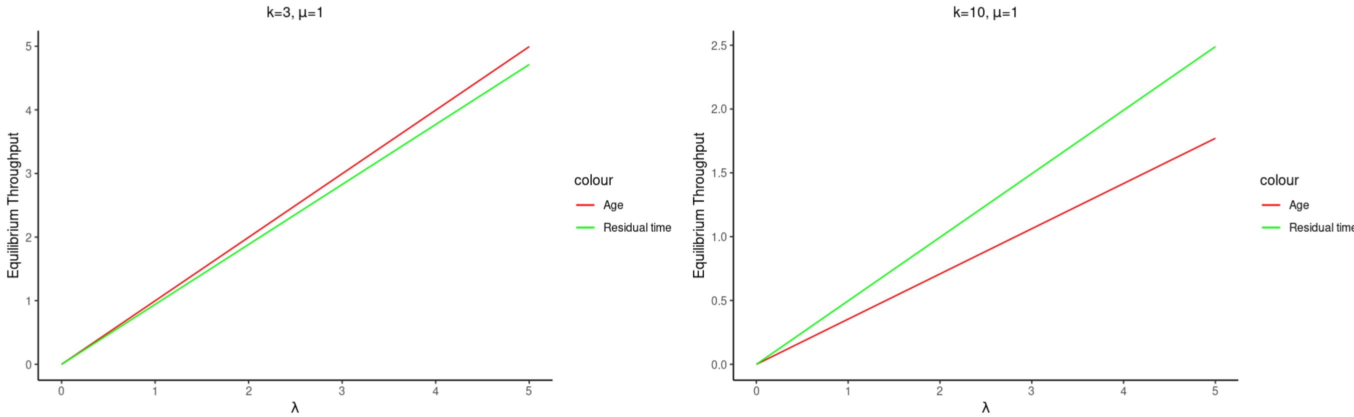

Figure 1: Equilibrium throughput with respect to λ for $r = 5, c = 1, \mu = 1$ and $k = \{3, 10\}$. The facilities arrive according to the Erlang Distribution.

Note that the equilibrium throughput of the age is greater than that of the residual time for $k = 3$. Plotting the social welfare for $k = 10$, we see that this time the throughput of the residual time is greater than that of the age.

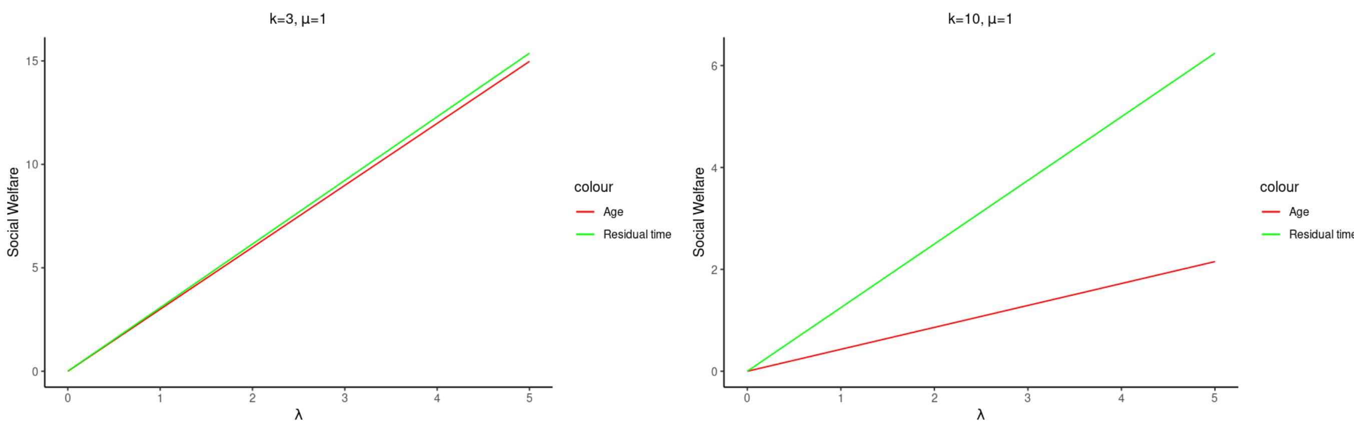

Figure 2: Social welfare with respect to λ for $r = 5, c = 1, \mu = 1$ and $k = \{3, 10\}$. The facilities arrive according to the Erlang Distribution.

For social welfare, both graphs have higher social welfare when the customers know the residual time of the system. By performing many numerical experiments, we have seen that the social welfare for the information level R is always greater than that of information level A.

4 Analysis of the Inhomogeneous scenario

In this section, we use the inhomogeneous scenario. Here, different customers do not have identical perceptions of their waiting costs, which we capture by considering the cost c as a random variable according to a uniform distribution $U(\alpha, \beta)$ with $\alpha < \beta$. For the rest of the section, we only consider the no-information scenario and drop any notation associated with information levels.

4.1 Equilibrium strategies and performance meausures

In this subsection, we formulate the equilibrium strategies and performance measures. The customers' strategy can be described as a joining probability q.
As discussed in section 3.1.1, the net benefit can be expressed by:

$$G(q) = r - c\frac{E(X^2)}{2E(X)}. \tag{27}$$

A tagged customer will join the system if his expected net benefit is positive. Solving for c, we get that

$$G(q) \geq 0 \Leftrightarrow r - c\frac{E(X^2)}{2E(X)} \geq 0 \Leftrightarrow c \leq 2r\frac{E(X)}{E(X^2)}. \tag{28}$$

Condition (28) can also be interpreted as the joining threshold regarding the waiting cost, which we denote by $\tilde{c}$. We have

$$\tilde{c} = 2r\frac{E(X)}{E(X^2)}. \tag{29}$$

That said, any customer who has $c > 2r\frac{E(X)}{E(X^2)}$ will balk. Therefore, the percentag of joining customers would be $\mathbb{P}(c \leq \tilde{c}) = F_c(\tilde{c})$, where F_c is the uniform distribution. This is equal to the joining probability q. We then have that

$$q^e = \begin{cases} 0 & \text{when } 2r\frac{E(X)}{E(X^2)} < \alpha \\ 1 & \text{when } 2r\frac{E(X)}{E(X^2)} > \beta \\ \frac{2r\frac{E(X)}{E(X^2)} - \alpha}{\beta - \alpha} & \text{when } \alpha \leq 2r\frac{E(X)}{E(X^2)} \leq \beta. \end{cases} \tag{30}$$

As in the homogeneous scenario, the throughput is calculated by

$$TH = \lambda q^e E(q). \tag{31}$$

Therefore, the throughput is equal to

$$TH = \begin{cases} 0 & \text{for } 2r\frac{E(X)}{E(X^2)} < \alpha, \\ \lambda & \text{for } 2r\frac{E(X)}{E(X^2)} > \beta, \\ \lambda\frac{2r\frac{E(X)}{E(X^2)} - \alpha}{\beta - \alpha} & \text{for } \alpha \leq 2r\frac{E(X)}{E(X^2)} \leq \beta. \end{cases} \tag{32}$$

Moreover, the social welfare is equal to

$$SW = \lambda q^e G(q^e) = \lambda q(r - cE(q^e)).$$

However, we must integrate over all possible $c \sim U(\alpha, \beta)$. We have

$$SW = \begin{cases} 0 & \text{for } r\frac{2E(X)}{E(X^2)} < \alpha, \\ \int_\alpha^\beta \lambda(r - c\frac{E(X^2)}{2E(X)})f(c)dc & \text{for } r\frac{2E(X)}{E(X^2)} > \beta, \\ \int_\alpha^\beta \lambda\frac{r\frac{2E(X)}{E(X^2)} - \alpha}{\beta - \alpha}(r - c\frac{E(X^2)}{2E(X)})f(c)dc & \text{for } \alpha \leq 2r\frac{E(X)}{E(X^2)} \leq \beta. \end{cases} \tag{33}$$

Moreover, for $q^e < 1$, the expected net benefit of a customer equals zero. Therefore, in this situation, social welfare is also equal to zero. We can rewrite equation (33) as

$$SW = \begin{cases} 0 & \text{for } r\frac{2E(X)}{E(X^2)} \leq \beta, \\ \int_{\alpha}^{\beta} \lambda(r - c\frac{E(X^2)}{2E(X)})f(c)dc & \text{for } r\frac{2E(X)}{E(X^2)} > \beta. \end{cases} \tag{34}$$

5 Analytical Comparison of the Homogeneous and Inhomogeneous Scenario

In this section, we will compare the performance measures of the homogeneous and inhomogeneous scenarios. We will denote the throughput and social welfare of the homogeneous scenario by TH_{Hom} and SW_{Hom} respectively, and the systems' performances of the inhomogeneous scenario by TH_{Inh} and SW_{Inh}. For the rest of the section, we only consider the no-information level scenario and therefore drop any notation associated with information levels.

5.1 Equilibrium Throughput

We compare the equilibrium throughput of the homogeneous and inhomogeneous scenarios. This results in the following theorem.

Theorem 6 (Comparison of TH_{Hom} and TH_{Inh})**.** *It holds that*

$$\begin{cases} TH_{Hom} = TH_{Inh} & \textit{for } \frac{E(X^2)}{2E(X)} < \frac{r}{\beta} < \frac{r}{c}, \\ TH_{Hom} < TH_{Inh} & \textit{for } \frac{r}{\beta} < \frac{E(X^2)}{2E(X)} \leq \frac{r}{c}, \\ TH_{Hom} \leq TH_{Inh} & \textit{for } \frac{r}{\beta} < \frac{r}{c} < \frac{E(X^2)}{2E(X)}. \end{cases} \tag{35}$$

Proof. Case 1: Let $\frac{E(X)^2}{2E(X)} < \frac{r}{\beta} < \frac{r}{c}$. Then, the equilibrium joining strategies are $q^e_{Hom} = 1 = q^e_{Inh}$. Then the equilibrium throughputs are

$$TH_{Hom} = \lambda = TH_{Inh}.$$

Case 2: Let $\frac{r}{\beta} < \frac{E(X)^2}{2E(X)} \leq \frac{r}{c}$. Then we have $q^e_{Hom} = 1$ and $q^e_{Inh} = [0, 1)$. We have

$$TH_{Hom} = \lambda, \quad TH_{Inh} = \lambda \frac{2r\frac{E(X)}{E(X^2)} - \alpha}{\beta - \alpha}.$$

Therefore, we have $TH_{Hom} < TH_{Inh}$.
Case 3: Let $\frac{r}{\beta} < \frac{r}{c} < \frac{E(X)^2}{2E(X)}$. In this case we have $q^e_{Hom} = 0$. However, we have $q^e_{Inh} = [0, 1)$. Therefore, we have

$$TH_{Hom} \leq TH_{Inh}.$$

□

5.2 Social Welfare

We rewrite equation (34) to compare the social welfare of the inhomogeneous and homogeneous scenarios. We have

$$SW_{Inh} = \begin{cases} 0 & \text{for } \frac{E(X^2)}{2E(X)} > \frac{r}{\beta}, \\ \int_\alpha^\beta \lambda(r - c\frac{E(X^2)}{2E(X)})f(c)dc & \text{for } \frac{E(X^2)}{2E(X)} < \frac{r}{\beta}. \end{cases}$$

For $\frac{E(X^2)}{2E(X)} < \frac{r}{\beta}$, we have

$$\begin{aligned} SW_{Inh} &= \int_\alpha^\beta \lambda r f(c)dc - \int_\alpha^\beta \lambda c \frac{E(X^2)}{2E(X)} f(c)dc \\ &= \lambda r - cE(c)\frac{E(X^2)}{2E(X)} \\ &= \lambda(r - c\frac{\alpha + \beta}{2}\frac{E(X^2)}{2E(X)}). \end{aligned}$$

Analysing the social welfare, we obtain the following result.

Theorem 7 (Comparison of SW_{Hom} and SW_{Inh})**.** *It holds that*

$$\begin{cases} SW_{Hom} > SW_{Inh} \Leftrightarrow E(c) > c & \textit{for } \frac{E(X^2)}{2E(X)} < \frac{r}{\beta} < \frac{r}{c}, \\ SW_{Hom} > SW_{Inh} & \textit{for } \frac{r}{\beta} < \frac{E(X^2)}{2E(X)} \leq \frac{r}{c}, \\ SW_{Hom} = SW_{Inh} = 0 & \textit{for } \frac{r}{\beta} < \frac{r}{c} < \frac{E(X^2)}{2E(X)}. \end{cases} \tag{36}$$

Proof. Case 1: Let $\frac{E(X)^2}{2E(X)} < \frac{r}{\beta} < \frac{r}{c}$. Then, the equilibrium joining strategies are $q^e_{Hom} = 1 = q^e_{Inh}$. We have

$$\begin{aligned} SW_{Hom} - SW_{Inh} &= \lambda(r - c\frac{E(X^2)}{2E(X)}) - \lambda(r - \frac{\alpha+\beta}{2}\frac{E(X^2)}{2E(X)} \\ &= \lambda r - \lambda r - c\lambda\frac{E(X^2)}{2E(X)} + \frac{\alpha+\beta}{2}\lambda\frac{E(X^2)}{2E(X)} \\ &= \lambda\frac{E(X^2)}{2E(X)}(\frac{\alpha+\beta}{2} - c) \\ &\geq 0 \Leftrightarrow \frac{\alpha+\beta}{2} > c \Leftrightarrow E(c) > c \end{aligned}$$

So, we have

$$SW_{Hom} > SW_{Inh} \Leftrightarrow E(c) > c.$$

Case 2: Let $\frac{r}{\beta} < \frac{E(X)^2}{2E(X)} \leq \frac{r}{c}$. Then $q^e_{Hom} = 1$ and $q^e_{Inh} = [0, 1)$ holds. We have

$$SW_{Hom} = \lambda(r - c\frac{E(X^2)}{2E(X)}), \quad SW_{Inh} = 0.$$

It is clear that $SW_{Hom} > SW_{Inh}$ holds.
Case 3: Let $\frac{r}{\beta} < \frac{r}{c} < \frac{E(X)^2}{2E(X)}$. In this case we have q^e_{Hom} and $q^e_{Inh} = [0, 1)$. Therefore it holds that

$$SW_{Hom} = 0 = SW_{Inh}.$$

□

6 Discussion

In this research project, our main focus is to investigate the influence of information on passenger strategic behavior within a transportation station. The transportation facility arrives according to a renewal process and has infinite capacity. We compare three distinct levels of information by defining the equilibrium strategies for each information level and analyzing the system's performance under equilibrium.

In particular, we focused on comparing the performance measure of the no-information level with those of the residual time and age. Our analysis revealed that a greater throughput depends on $\frac{E(X^2)}{2E(X)}$ and the ratio between the reward and cost. The social welfare of the age and residual time is always better than that of the no-information level. That means that the total reward of the customers is higher when they have additional information. However, for comparing the information levels A and R, we could only establish proof for the exponential distribution $F(t)$.

In addition, we conducted numerical experiments to compare the age and residual time performance when a facility arrives according to the Erlang distribution. It turns out that social welfare is always higher when customers know their residual time compared to when they have the knowledge of the system's age.

Furthermore, we analyzed customers who capture their costs differently. The equilibrium throughput of the inhomogeneous scenario turns out to be equal to or greater than in the homogeneous scenario. For the social welfare, this is not always true.

Of course, the study needs to be complemented by further considerations. We could consider the information level where customers notice the number of people waiting on the platform, which indicates their waiting time. Arriving customers will see this, which is a logical step to investigate further. Moreover, we could implement the chance of a delay of the next facility.

Additionally, exploring alternative probability distributions that accurately capture the cost associated with waiting is crucial. Although this research project's random selection of cost values is a preliminary approach, practical scenarios often demand a more tailored and realistic representation. Identifying a distribution that accurately reflects individuals' strong preference for shorter waiting times poses a challenging yet essential task for future investigations in this field.

In summary, our model emphasizes the significance of providing some form of information to enhance the overall social welfare of the system. The equilibrium throughput depends on the average waiting time and how customers perceive the value of rewards and costs, which information level proves most advantageous.

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