Formalising the Van Kampen Theorem for Directed Topology

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Formalizing the Van Kampen Theorem for Directed Topology

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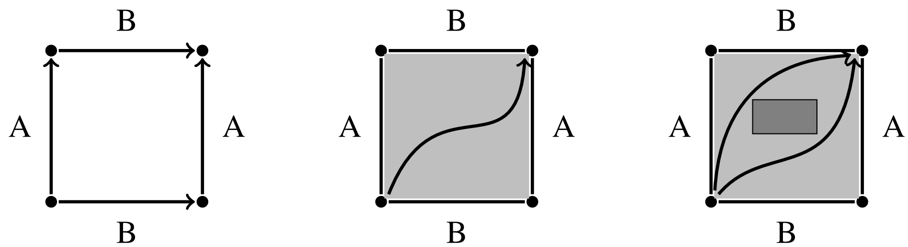
Abstract

Directed topology is a fairly new field of mathematics with applications in concurrency. It extends the concept of a topological space by adding a notion of directedness in which directed paths play a very important role. There are direction preserving maps between directed spaces called directed maps. A special case of these is a directed path homotopy that transforms one directed path into another. Using these deformations, directed paths are partitioned into equivalence classes and a special category, the fundamental category, can be linked to a directed space. In this thesis we will explain these definitions and present a special theorem: a directed version of the Van Kampen Theorem. This theorem allows the calculation of fundamental categories by combining local knowledge about paths. Our main contribution is the formalization of this material using the Lean proof assistant and we show how we have implemented this.

Contents

1 Introduction

Directed topology, an extension of topology, is a field of mathematics that has been developed in order to abstractly analyze structures where paths with directions are present. The original motivation, given by Fajstrup et al. in 1999, stems from its applications in concurrency, a field researching the simultaneous execution of programs [FRG99]. Given two programs A and B, we can execute those sequentially in two ways: either first A or first B. That relatively simple case can be seen in the figure below on the left. If parallel execution is possible, any path from the bottom left to the top right is a valid execution path on the condition that the path never turns backward. After all, once an instruction has been executed it cannot be undone. This situation is already substantially more complicated than the sequential version.



Possible execution paths of two programs A and B under three conditions: sequential (left), simultaneous (middle) and simultaneous with obstacles (right).

In the case of parallel execution, two programs could try to write to the same location at the same time, which could lead to unintended behavior. In order to prevent this, we disallow such execution paths. We would obtain a case like the square on the right with the darker rectangle as an obstacle in it. If an execution path were to pass through that space, both programs would write to the shared location at the same time and we consider that invalid. There are now two truly different paths: the one through the top left and the one through the bottom right. Indeed, these two executions write to the shared location in a different order. Of course, depending on the programs there may be no or many different obstacles. In order to make all of these intuitions precise, the field of directed topology has been developed.

1.1 Directed Topology

There are multiple ways to define the notion of directedness in a topological space. The original approach of Fajstrup et al. uses spaces with a partial ordering or a local partial ordering [FRG06]. We use the somewhat more flexible d-spaces defined by Grandis by letting a directed space be a topological space with a distinguished set of paths [Gra03]. These paths are called directed paths. Between directed spaces there are direction preserving maps called directed maps and together they form the category **dTop**. A directed version of path homotopies can be used to relate directed paths and obtain equivalence classes. These classes are respected by concatenation of paths and with that the fundamental category of a directed space can be defined. It is the directed equivalent of the fundamental groupoid of a topological space [Bro06]. There is also a directed analogue of the fundamental group called the fundamental monoid, but often it cannot distinguish between different forms of directedness. The main theorem this thesis is concerned with is a directed version of the Van Kampen Theorem, originally stated and proven by Grandis [Gra03]. It allows the calculation of the fundamental category of a directed space by combining the fundamental categories of two subspaces. As we will see, the Lebesgue Number Lemma plays an important rule in the proof.

1.2 Contributions

Our main contribution has been the formalization of definitions and theorems relating to directed topology, in particular the Van Kampen Theorem. Formalization is the use of a theorem prover, in our case Lean, in order to write definitions, theorems and proofs in a rigorous manner. A computer is then able to verify the correctness of these statements using type theory or another formal method. The major advantage of formalizing a theorem is that anyone can be fully certain about the correctness of a theorem without needing to sift through its proof in detail. An additional advantage is that the computer is able to aid with proofs by automatically filling in steps or suggesting what theorems to apply. That is why some formalization languages are also called proof assistants. Lastly, the digital nature of formalization lends itself to an efficient form of cooperation by means of file sharing.
We wrote the formalization using the works already present in MathLib. MathLib is a library containing the formalization in Lean of a vast amount of mathematical theory and it is the product of many different collaborators. We have uploaded all of our files in a git repository and it contains a guide on how to get it to run locally. The version of Lean we used is 3.50.3. Resources to learn and understand Lean can be found on the Lean community website.

1.3 Thesis overview

In Section 2, we give a brief overview of the mathematical background on topology and category theory needed to understand the main ideas and concepts of this thesis. Section 3 defines the notion of directed spaces and directed maps and we give a few examples. In Section 4 the definitions and some properties of directed homotopies and directed path homotopies are given. We use those to define relations on the set of directed paths between two points. In Section 5 the equivalence classes of paths under these relations are used to define the fundamental category and the fundamental monoid. The Van Kampen Theorem is stated in Section 6 and an application is given. Finally, in Section 7 we reflect on the ideas presented in this thesis and give some suggestions for further research.
Small excerpts from the Lean formalization can be found throughout the thesis. They can be recognized by the `monospace` font used. The surrounding text explains the ideas of the formalization and references the corresponding file in which the code can be found.

2 Preliminaries

This thesis assumes basic mathematical knowledge. Familiarity with topology and category theory is useful and this chapter will explain some of the necessary background. For further information about these topics, see [Wal14] (topology) or [Lei14] (category theory).

2.1 Topology

The field of topology is concerned with the properties of spaces and shapes "up to deformation". Think of squares, circles, donuts or spheres. The notion of a topological space is central.

Definition 2.1 (Topological space)**.** A topological space is a pair $(X, \mathcal{T})$, consisting of a set X together with a set $\mathcal{T} \subseteq \mathcal{P}(X)$ such that:

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

(2) If $\mathcal{U} \subseteq \mathcal{T}$ is a set of elements of $\mathcal{T}$, then $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

(3) If $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called open sets and $\mathcal{T}$ is called the topology on X. If $\mathcal{T}$ is clear from context, the set X is often referred to as the topological space instead of the pair $(X, \mathcal{T})$.
Informally speaking, if a point x is an element of some open set $U \in \mathcal{T}$ then all points "close" to x are also elements of U. A subset $V \subseteq X$ is called closed if its complement $X \setminus V$ is open.
A continuous map between two topological spaces is then a map that maps points that are close together to points that are close together:

Definition 2.2 (Continuous map)**.** A continuous map between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a map $f : X \to Y$ of sets such that for all $V \in \mathcal{T}_Y$ it holds that $f^{-1}(V) \in \mathcal{T}_X$.

Compositions of continuous maps are again continuous. It is also easy to show that the identity map is always continuous.
One special topological space is the unit interval $[0, 1]$, a subspace of the real number line $\mathbb{R}$. Continuous maps $\gamma : [0, 1] \to X$ with X a topological space are called paths in X. If γ_1 and γ_2 are two paths in X with $\gamma_1(1) = \gamma_2(0)$, then you can concatenate those two paths by first following γ_1 and then γ_2. This new path is denoted $\gamma_1 \odot \gamma_2$ and given by:

$$(\gamma_1 \odot \gamma_2)(t) = \begin{cases} \gamma_1(2t), & t \leq \frac{1}{2}, \\ \gamma_2(2t-1), & \frac{1}{2} < t. \end{cases}$$

These paths turn out to be important invariants of topological spaces and information about them can be captured in the so called fundamental groupoid [Bro06, chapter 6]. Within this thesis we will take a look at the directed variant: the fundamental category.

2.2 Category Theory

Category theory is used to capture abstract similarities between seemingly unrelated mathematical fields. It turns out that many different concepts can be classified as categories:

Definition 2.3 (Category)**.** A category C consists of a collection of objects, $\mathrm{Ob}(C)$, together with sets of morphisms $C(A, B)$ for all $A, B \in \mathrm{Ob}(C)$, a composition map $\circ : C(B, C) \times C(A, B) \to C(A, C)$ and an identity $\mathrm{id}_A \in C(A, A)$ for all $A \in \mathrm{Ob}(C)$ such that:

(1) For any $A, B \in \mathrm{Ob}(C)$ and $f \in C(A, B)$ we have $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

(2) For any $A, B, C, D \in \mathrm{Ob}(C)$ and $f \in C(A, B)$, $g \in C(B, C)$, $h \in C(C, D)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Instead of $A \in \mathrm{Ob}(C)$, we often write $A \in C$. Another way to write $f \in C(A, B)$ is $f : A \to B$, as morphisms often behave like maps.

Example 2.4. The category **Set** has as objects sets and as morphisms between two sets exactly the maps between those two sets. Composition of morphisms is exactly the composition of those maps. Each set has an identity map and composition of maps is associative, so properties 1 and 2 are satisfied.

Example 2.5. **Top** is the category of topological spaces and continuous maps. They are respectively the objects and the morphisms. Identities are continuous and composition is associative, so it is indeed a category.

There are also maps between different categories, called functors.

Definition 2.6 (Functor)**.** Let C and $\mathcal{D}$ be two categories. A functor $F : C \to \mathcal{D}$ consists of a map on objects and a map on morphisms, where it maps a morphism $f : A \to B$ to a morphism $F(f) : F(A) \to F(B)$, such that the following holds:

(1) For any $A \in C$ we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

(2) For any $A, B, C \in C$, $f : A \to B$ and $g : B \to C$ we have $F(g \circ f) = F(g) \circ F(f)$.

In other words, a functor needs to respect identities and composition.

When formulating the Van Kampen Theorem in Section 6.1, we use the term pushout square. This is a specific type of a so-called colimit. There are many ways to define those, but as we only need the pushout square colimit, we will not bother with the generalization.

Definition 2.7 (Pushout square)**.** Let C be a category, $A, B, C, D \in C$ four objects and $i : A \to B$, $j : A \to C$, $f : B \to D$ and $g : C \to D$ four morphisms such that $f \circ i = g \circ j$. We can draw them in a *commutative square*:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "f"'] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

We say that that square is a pushout square if for any object $X \in C$ and pair of morphisms $p : B \to X$ and $q : C \to X$ with $p \circ i = q \circ j$ there is a unique $r : D \to X$ such that $p = r \circ f$ amd $q = r \circ g$.

D is then called the pushout of the square.

3 Directed Spaces

In this section, we will look at the basic structure of a directed space. With directed maps as morphisms, the category of directed spaces **dTop** is obtained.

3.1 Directed Spaces

A directed space is a topological space with a distinguished set of paths, whose elements are called directed paths. This is analogous to the set of open sets in a topological spaces. Similarly, the set of directed paths must satisfy some properties. Firstly, constant paths must be directed. Secondly, if there are two directed paths that connect in an end and start point, the concatenation of those two paths should again be directed. Intuitively, if you can walk from point A to point B and from B to C, then you should be able to walk from A to C by following those paths consecutively. Lastly, it should be possible to walk a part of a directed path at a different speed, as long as the path is not walked backward. This can be captured in the property that monotone subparametrizations of directed paths must also be directed paths.

Definition 3.1 (Directed space)**.** A directed space is a topological space X, together with a set of paths in X, denoted P_X. That set must satisfy the following three properties:

(1) For any point $x \in X$, we have that $0_x \in P_X$, where 0_x is the constant path in x.

(2) For any two paths $\gamma_1, \gamma_2 \in P_X$ with $\gamma_1(1) = \gamma_2(0)$, we have that $\gamma_1 \odot \gamma_2 \in P_X$.

(3) For any path $\gamma \in P_X$ and any continuous, monotone map $\varphi : [0,1] \to [0,1]$ we have that $\gamma \circ \varphi \in P_X$.

The elements of P_X are called *directed paths* or *dipaths* .

We will first consider some examples of directed spaces.

Example 3.2 (Directed unit interval)**.** We can equip the unit interval with a rightward direction. This is done by taking $P_{[0,1]} = \{\varphi : [0,1] \to [0,1] \mid \varphi \text{ continuous and monotone}\}$. We will denote this directed space with I. More generally, every (pre)ordered space can be given a set of directed paths this way.

Example 3.3 (Directed unit circle)**.** One of the ways the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ can be made into a directed space is by only allowing paths that go monotonously counterclockwise. Specifically, we take the set of directed paths $P_{S^1} = \{t \mapsto \exp(i\varphi(t)) \mid \varphi : [0,1] \to \mathbb{R} \text{ continuous and monotone}\}$. This directed space will be denoted with S^1_+.

Example 3.4 (Maximal directed space)**.** Any topological space X can be made into a directed space by taking P_X as the set of all paths in X. We will call this the maximal directedness on X. This is also sometimes called the indiscrete or natural directedness.

Example 3.5 (Minimal directed space)**.** Any topological space X can be made into a directed space by taking $P_X = \{0_x \mid x \in X\}$. In other words, only the constant paths are allowed as directed paths. We will call this the minimal directedness on X. This is also sometimes called the discrete directedness.

Example 3.6 (Product of directed spaces)**.** If (X, P_X) and (Y, P_Y) are two directed spaces, then the space $X \times Y$ with the product topology can be made into a directed space by letting $P_{X\times Y} = \{t \mapsto (\gamma_1(t), \gamma_2(t)) \mid \gamma_1 \in P_X \text{ and } \gamma_2 \in P_Y\}$. As we will see in Section 3.2, with this set of directed paths both projection maps will be directed and $(X \times Y, P_{X\times Y})$ becomes a product in the category of directed spaces and directed maps.

Example 3.7 (Induced directed space)**.** Let X be a topological space and (Y, P_Y) a directed space. Let a continuous map $f : X \to Y$ be given. If $\gamma : [0,1] \to X$ is a path in X, then $f \circ \gamma : [0,1] \to Y$ is a path in Y. We can create a direction on X by taking $P_X = \{\gamma \in C([0,1], X) \mid f \circ \gamma \in P_Y\}$. It is not hard to verify that this satisfies all the properties of a directed space. In the special case that X is a subspace of Y and f is the inclusion map, we find that every subspace of a directed space can be given a natural directedness.

We formalized the concept of a directed space by extending the topological_space class. In Lean, a type class is a collection of types and extending a class means adding new types. Other types can be made into an instance of a class by supplying a term for each type. For example, $\mathbb{R}$, the type of real numbers, is an instance of topological_space. A useful property of extending classes is that you can get back any subclasses. In this specific case, we can use to_topological_space to obtain the underlying topological space of a directed space.
In our formalization, we do not explicitly use a set of paths. Rather, being directed is a property of a path itself, analogous to how being open is a property of a set in the topological_space class. Paths in topological spaces have been implemented in Mathlib in the file topology/path_connected.lean. A path has type path x y, where its starting point is x and endpoint is y. This way, concatenation of paths can be done without additional hypotheses. The definition of a directed space can be found in directed_space.lean.

```
class directed_space (α : Type u) extends topological_space α :=
  (is_dipath : ∀ {x y}, path x y → Prop)
  (is_dipath_constant : ∀ (x : α), is_dipath (path.refl x))
  (is_dipath_concat : ∀ {x y z} {γ₁ : path x y} {γ₂ : path y z}, is_dipath γ₁ →
   is_dipath γ₂ → is_dipath (path.trans γ₁ γ₂))
  (is_dipath_reparam : ∀ {x y : α} {γ : path x y} {t₀ t₁ : I} {f : path t₀ t₁},
   monotone f → is_dipath γ → is_dipath (f.map γ.continuous_to_fun))
```

The term is_dipath determines whether a path is a directed path or not. The three other terms are exactly the three properties of a directed space. path.refl x is the constant path in a point x and path.trans is used for the concatenation of paths. Mathlib only has support for reparametrizations of paths (meaning that the endpoints must remain the same), but we want to also allow strict subparametrizations. We do this by interpreting the subparametrization f as a monotone path in $[0, 1]$. Then the path $\gamma \circ f$ can be obtained using path.map, where we interpret γ as a continuous map.
In constructions.lean, different instances of directed spaces can be found: topological spaces with a preorder (Example 3.2), products of directed spaces (Example 3.6) and induced directedness (Example 3.7). For brevity, we introduce a notation for the set of all paths between x and y.

Definition 3.8. If X is a directed space with two points $x, y \in X$, we use the shorthand notation $P_X(x, y)$ for the set $\{\gamma \in P_X \mid \gamma(0) = x \text{ and } \gamma(1) = y\}$.

This can also be seen as a type for our formalization. That is exactly how to interpret the structure dipath, found in dipath.lean:

```
variables {X : Type u} [directed_space X]
structure dipath (x y : X) extends path x y :=
    (dipath_to_path : is_dipath to_path)
```

It extends the path structure and depends on two points x and y in a directed space X. The term dipath_to_path is has type is_dipath to_path. That means that the underlying path it extends must be a directed path. Due to the axioms of a directed space, we can define dipath.refl and dipath.trans analogous to their path-counterparts. On the other hand, path.symm, the reversal of a path, cannot be converted to a directed variant as it is not guaranteed that the reversal of a directed path is directed.
We introduce a notation for a special kind of subpath of a directed path.

Definition 3.9. Let X be a directed space and $\gamma \in P_X$ a directed path. If $n > 0$ and $1 \leq i \leq n$ we will define $\gamma_{i,n} \in P_X$ to be the path from $\gamma(\frac{i-1}{n})$ to $\gamma(\frac{i}{n})$ given by $\gamma_{i,n}(t) = \gamma(\frac{i+t-1}{n})$.

We can now say what it means for a directed path to be covered by a cover of a directed space. This definition will play a big role in proving the Van Kampen Theorem.

Definition 3.10. Let X be a directed space and $\mathcal{U}$ a cover of X. Let $\gamma \in P_X$ be a directed path and $n > 0$ an integer. We say that γ is n-covered (by $\mathcal{U}$) if we have for all $1 \leq i \leq n$ that $\operatorname{Im} \gamma_{i,n} \subseteq U$ for some $U \in \mathcal{U}$. In the special case that $n = 1$, we simply say that γ is covered by U, where $\operatorname{Im} \gamma \subseteq U$.

In `path_cover.lean` we formalize this definition of n-covered in the special case that $\mathcal{U}$ consists of two elements X_0 and X_1:

```
def covered (γ : dipath x₀ x₁) (hX : X₀ ∪ X₁ = univ) : Prop :=
    (range γ ⊆ X₀) ∨ (range γ ⊆ X₁)

def covered_partwise (hX : X₀ ∪ X₁ = set.univ) : Π {x y : X}, dipath x y → ℕ → Prop
| x y γ 0 := covered γ hX
| x y γ (nat.succ n) :=
    covered (split_dipath.first_part_dipath γ
        (inv_I_pos (show 0 < (n.succ + 1), by norm_num))) hX ∧
    covered_partwise (split_dipath.second_part_dipath γ
        (inv_I_lt_one (show 1 < (n.succ + 1), by norm_num))) n
```

Here `covered` corresponds with γ being 1-covered: its image is either contained in X_0 or in X_1. We use this definition to inductively define `covered_partwise`. As it is easier to start at zero in Lean, `covered_partwise hX γ n` corresponds with γ being $(n+1)$-covered. In the case that $n = 0$, we have that `covered_partwise` simply agrees with `covered hX γ`. Otherwise, we use an induction step to define that `covered_partwise hX γ (nat.succ n)` holds if the first part $\gamma_{1,n+2}$ is `covered` and the remainder of γ is `covered_partwise hX γ n`. Note the use of $n+2$ instead of $n+1$ due to the offset between the definitions. The remainder of `path_cover.lean` contains lemmas about conditions for being n-covered.

3.2 Directed Maps

As directed spaces are an extension of topological spaces, directed maps will be extensions of continuous maps. They will need to respect the extra directed structure. If a path in the domain space is given, a path in the codomain space can be obtained by composing the continuous map with the path. If the former is directed, so should be the latter.

Definition 3.11 (Directed map)**.** Let X and Y be two directed spaces. A directed map $f : X \to Y$ is a continuous map on the underlying topological spaces that furthermore satisfies: for any $\gamma \in P_X$, we have that $f \circ \gamma \in P_Y$.

Example 3.12. The map $f : I \to S^1_+$ given by $t \mapsto e^{it}$ is a directed map. Firstly, it is continuous on the underlying topological spaces. Secondly, if $\gamma \in P_I$ is a directed path, that is, continuous and monotone, then $f \circ \gamma$ is given by $t \mapsto \exp(i\gamma(t))$ and that path is by definition of $P_{S^1_+}$ directed.

Any continuous map (on the underlying topological spaces) from a minimally directed space to a directed space is directed. Similarly, any continuous map to a maximally directed space is directed. By construction of the product of directed spaces the continuous projection maps on both coordinates are directed: a directed path in the product space is a pair of directed paths and a projection returns the original directed path. Similarly, if a continuous map $f : X \to Y$ is used to induce a direction on X as in Example 3.7, then f becomes a directed map from X to Y, where X has the induced directedness.
In order to formalize the definition of a directed map in Lean, we first define `directed_map.directed`, which contains exactly the property that a continuous map between two directed spaces maps directed paths to directed paths:

```
def directed {α β : Type*} [directed_space α] [directed_space β] (f : C(α, β)) : Prop
    :=
  ∀ ⦃x y : α⦄ (γ : path x y), is_dipath γ → is_dipath (γ.map f.continuous_to_fun)
```

A directed map is then an extension of the `continuous_map` structure with a proof for being `directed`.

```
structure directed_map (α β : Type*) [directed_space α] [directed_space β] extends
    continuous_map α β :=
  (directed_to_fun : directed_map.directed to_continuous_map)
```

We use the notation $D(\alpha,\beta)$ for the type of directed maps between α and β. Directed paths are also instances of directed maps, because it maps a directed path in I to a monotone subparametrization of itself and we know that that is also directed path. `dipath.lean` contains definitions on how to convert the `dipath` type to the `directed_map` type and the other way around. These are called `to_directed_map` and `of_directed_map` respectively.
Directed spaces and directed maps form a category, which we will denote with **dTop**. There are two functors $Min, Max : \mathbf{Top} \to \mathbf{dTop}$, where Min equips a topological space with the minimal directedness and Max equips a topological space with the maximal directedness. If $U : \mathbf{dTop} \to \mathbf{Top}$ is the forgetful functor that sends a directed space to its underlying topological space, we obtain two adjunctions $Min \dashv U \dashv Max$ [Gra03]. An adjunction is a special relationship between two functors, see [Lei14, chapter 2]. Within **dTop** we find an instance of pushouts as the following lemma shows.

Lemma 3.13. *Let $X \in$ **dTop** be a directed space and X_1 and X_2 two open subspaces such that $X = X_1 \cup X_2$. Take $X_0 = X_1 \cap X_2$ as the intersection of X_1 and X_2. Let $i_k : X_0 \to X_k$ and $j_k : X_k \to X$, with $k \in \{1, 2\}$ be the inclusion maps. We then get a pushout square in **dTop**:*

$$\begin{tikzcd} X_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"'] & X \end{tikzcd}$$

Proof. Let $Y \in$ **dTop** be another directed space and $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ two directed maps such that $f_1 \circ i_1 = f_2 \circ i_2$. We will now construct an unique directed map $f : X \to Y$ such that $f \circ j_1 = f_1$ and

$f \circ j_2 = f_2$. Note that $f \circ j_k$ as a map is simply the restriction of f to X_k. As X is covered by X_1 and X_2, it follows that f needs to be defined by

$$f(x) = \begin{cases} f_1(x), & x \in X_1, \\ f_2(x), & x \in X_2. \end{cases}$$

This already gives us uniqueness. If $U \subseteq Y$ is open, then $f^{-1}(U) = f_1^{-1}(U) \cup f_2^{-1}(U)$ is open as a union of open subsets and thus f is continuous. Here we use the fact that f_1 and f_2 are continuous and that an open subset of an open subset is open in the entire space. In order to see that f is directed, we need to use the Lebesgue Number Lemma [Mun75, p. 179-180]. We will use it to cut up a path into pieces and then apply the two maps f_1 and f_2 independently and concatenate the results together.
Let $\gamma \in P_X$ be any directed path. We have that $[0,1] = \gamma^{-1}(X) = \gamma^{-1}(X_1) \cup \gamma^{-1}(X_2)$, so $\gamma^{-1}(X_1)$ and $\gamma^{-1}(X_2)$ form an open cover of $[0,1]$. By applying the Lebesgue Number Lemma we can find an integer $n > 0$ such that for all $1 \leq i \leq n$ we have that $[\frac{i}{n}, \frac{i+1}{n}] \subseteq \gamma^{-1}(X_{k_i})$ with k_i either 1 or 2, i.e. γ is n-covered. With a suitable bijective and monotone reparametrization $\varphi : [0,1] \to [0,1]$, we have that

$$\gamma \circ \varphi = \gamma_{1,n} \odot (\gamma_{2,n} \odot \ldots (\gamma_{n-1,n} \odot \gamma_{n,n})).$$

Note that each of the paths $\gamma_{i,n}$ is directed as they are monotone subparametrizations of γ. We obtain:

$$\begin{gathered}(f \circ \gamma) \circ \varphi = f \circ (\gamma \circ \varphi) = f \circ (\gamma_{1,n} \odot (\gamma_{2,n} \odot \ldots (\gamma_{n-1,n} \odot \gamma_{n,n}))) = \\ (f_{k_1} \circ \gamma_{1,n}) \odot ((f_{k_2} \circ \gamma_{2,n}) \odot \ldots ((f_{k_{n-1}} \circ \gamma_{n-1,n}) \odot (f_{k_n} \circ \gamma_{n,n})))\end{gathered}$$

The directedness of the maps f_1 and f_2 tells us that each of the paths $f_{k_i} \circ \gamma_{i,n}$ is directed. By the property of concatenation of directed paths we find that $(f \circ \gamma) \circ \varphi$ is directed. As φ is a bijective monotone reparametrization, so is its inverse. This give us that $f \circ \gamma = (f \circ \gamma) \circ \varphi \circ \varphi^{-1}$ is directed, so f is a directed map. From this, it follows that the above square is indeed a pushout square. ◻

From proof of the lemma, it follows that within the category **Top** a topological space X and a covering of it with two open subsets X_1 and X_2 also form a pushout square. When X_1 and X_2 are both closed, they still form a pushout in **Top** but that is not guaranteed in **dTop**. The latter is shown by the following example.

Example 3.14. Take $X = [0,1]$ (maximally directed), $X_1 = \{0\} \cup \left(\bigcup_{i=0}^{\infty} \left[\frac{1}{2i+2}, \frac{1}{2i+1}\right]\right)$ and $X_2 = \{0\} \cup \left(\bigcup_{i=1}^{\infty} \left[\frac{1}{2i+1}, \frac{1}{2i}\right]\right)$. We have that $X_1 \cap X_2 = \{\frac{1}{n} \mid n \in \mathbb{Z}_{>1}\}$. Let $Y = [0,1]$ with directed paths given by $P_Y = \{0_0\} \cup \{\gamma : [0,1] \to (0,1] \mid \gamma \text{ continuous}\}$. The directed paths in Y are thus paths contained in $(0,1]$ and the constant path in 0. Note that this collection does indeed satisfy the three properties of a directed space.
The point 0 in X_1 is not connected by any paths to any other points, so we claim that there is a directed map $f_1 : X_1 \to Y$ given by $f_1(x) = x$: we have that for any $\gamma \in P_{X_1}$ that $\gamma = 0_0$ or γ is contained in $\left[\frac{1}{2i+1}, \frac{1}{2i+2}\right]$ for some $i \geq 0$ and thus in $(0,1]$. In both cases, $f_1 \circ \gamma$ is directed in Y, making f_1 a directed map. Similarly we have a directed map $f_2 : X_2 \to Y$ given by $f_2(x) = x$.
If $i_1 : X_1 \cap X_2 \to X_1$ and $i_2 : X_1 \cap X_2 \to X_2$ are the inclusion maps, then $(f_1 \circ i_1)(x) = x = (f_2 \circ i_2)(x)$ for all $x \in X_1 \cap X_2$, so $f_1 \circ i_1 = f_2 \circ i_2$ holds. If X were the pushout of X_1 and X_2, there would have to be a unique directed map $f : X \to Y$ such that $f \circ j_1 = f_1$ and $f \circ j_2 = f_2$. Clearly f must be defined as in the proof of Lemma 3.13:

$$f(x) = \begin{cases} f_1(x), & x \in X_1, \\ f_2(x), & x \in X_2, \end{cases} = \begin{cases} x, & x \in X_1, \\ x, & x \in X_2, \end{cases} = x.$$

If $\gamma \in P_X$ is the directed path given by $t \mapsto t$, we would require that $f \circ \gamma \in P_Y$. As $f \circ \gamma$ is neither the constant path 0_0 nor a path in $(0, 1]$, we find a contradiction to the directedness of f. X is therefore not the pushout of X_1 and X_2, even though X_1 and X_2 are both closed.

From this example, it also follows that Max does not preserve colimits, as a pushout is an instance of a colimit.

4 Directed Homotopies

In this section, we will look at directed homotopies and directed path homotopies. Those two realize the idea of deformation, while respecting the directedness of a directed space.

4.1 Homotopies

A directed homotopy is the deformation of one directed map into another.

Definition 4.1 (Directed homotopy)**.** Let X and Y be two directed spaces. A homotopy between two directed maps $f, g : X \to Y$ is a directed map $H : I \times X \to Y$ such that for all $x \in X$ we have that $H(0, x) = f(x)$ and $H(1, x) = g(x)$.

Note that $I \times X$ has the product directedness. We say that H is a directed homotopy from f to g. This order matters, as unlike in the topological case a directed homotopy cannot generally be reversed. In our formalization, we adhere to the method used in defining homotopies between continuous maps in Mathlib, found in `topology/homotopy/basic.lean`. In an analogous manner, the structure extends the `directed_map (I × X) Y` structure and has two extra properties.

```
structure dihomotopy (f₀ f₁ : directed_map X Y) extends D((I × X), Y) :=
  (map_zero_left' : ∀ x, to_fun (0, x) = f₀.to_fun x)
  (map_one_left' : ∀ x, to_fun (1, x) = f₁.to_fun x)
```

As a directed map is always a continuous map on the underlying topological spaces, we can define how to convert a `dihomotopy` to a `homotopy`. Conversely, if we are given a `homotopy` and we know that it is directed, we can obtain a `dihomotopy`.
If $f : X \to Y$ is a directed map, there is an identity homotopy H from f to f, given by $H(t, x) = f(x)$. Also, if G is a directed homotopy from f to g and H a directed homotopy from g to h, we obtain a directed homotopy $G \otimes H$ from f to h given by

$$(G \otimes H)(t, x) = \begin{cases} G(2t, x), & t \leq \frac{1}{2}, \\ H(2t - 1, x), & \frac{1}{2} < t. \end{cases}$$

These two constructions are called `refl` and `trans` in `directed_homotopy.lean`. In both cases the coercion of a `homotopy` into a `dihomotopy` is used, by supplying the proof that both homotopies are directed. Here we use the fact that Mathlib already contains proofs that the constructed maps are indeed homotopies, i.e. continuous and satisfying the two mapping properties.

4.2 Path Homotopies

Definition 4.2 (Directed path homotopy)**.** Let X be a directed space and $x, y \in X$ two points. A directed path homotopy between two directed paths $\gamma_1, \gamma_2 \in P_X(x, y)$ is a directed homotopy $H : I \times I \to X$ from γ_1 to γ_2 such that additionally for all $t \in [0, 1]$ we have that $H(t, 0) = x$ and $H(t, 1) = y$.

In other words, a path homotopy is a homotopy between two paths that keeps both endpoints fixed. Again we say that H is a directed path homotopy from γ_1 to γ_2. Between two paths γ_1 and γ_2 in I with the same endpoints exists a path homotopy under the condition that $\gamma_1(t) \leq \gamma_2(t)$ for all $t \in I$ as the following example shows.

Example 4.3. Let $t_0, t_1 \in I$ be two points and $\gamma_1, \gamma_2 \in P_I(t_0, t_1)$. If $\gamma_1(t) \leq \gamma_2(t)$ for all $t \in I$, there is a directed path homotopy H from γ_1 to γ_2 given by $H(t, s) = (1 - t) \cdot \gamma_1(s) + t \cdot \gamma_2(s)$. It is continuous by continuity of paths, multiplication and addition. It is easy to show that $H(a_0, b_0) \leq H(a_1, b_1)$ if $a_0 \leq a_1$ and $b_0 \leq b_1$. From this, it follows that H is directed, because a directed path in $I \times I$ is exactly a pair of monotone maps $I \to I$ by definition.
Note that H interpolates two paths γ_1 and γ_2. The formalized proof of it being a directed map can be found in the file `interpolate.lean`.

Let $x, y, z \in X$ be three points and $\beta_1, \gamma_1 \in P_X(x, y)$ and $\beta_2, \gamma_2 \in P_X(y, z)$. If there are two directed path homotopies G from β_1 to γ_1 and H from β_2 to γ_2, we can construct a directed path homotopy $G \odot H$ from $\beta_1 \odot \beta_2$ to $\gamma_1 \odot \gamma_2$ given by

$$(G \odot H)(t, s) = \begin{cases} G(t, 2s), & s \leq \frac{1}{2}, \\ H(t, 2s - 1), & \frac{1}{2} < s. \end{cases}$$

Let X be a directed space and $x, y \in X$ two points. If $\gamma_1, \gamma_2 \in P_X(x, y)$ are two directed paths with a path homotopy from γ_1 to γ_2, we will write $\gamma_1 \rightsquigarrow \gamma_2$. This defines a relation on the set $P_X(x, y)$, but that relation is not guaranteed to be an equivalence relation, as it is generally not symmetric. This is due to the fact that the reversal of a directed path may not be directed. The following lemma, which we haven't formalized in Lean, shows this.

Lemma 4.4. *Let X be a directed space and $x, y \in X$. The relation $\rightsquigarrow$ on $P_X(x, y)$ is reflexive and transitive, but it is not always symmetric.*

Proof. The fact that the relation is reflexive and transitive can be obtained from Section 4.1. If $\gamma \in P_X(x, y)$, then the directed homotopy given by $H(t, s) = \gamma(s)$ satisfies the additional conditions of a path homotopy, so $\gamma \rightsquigarrow \gamma$. Similarly if $\gamma_1, \gamma_2, \gamma_3 \in P_X(x, y)$ with G a directed path homotopy from γ_1 to γ_2 and H a directed path homotopy from γ_2 to γ_3, then $G \otimes H$ is a directed homotopy from γ_1 to γ_3. It additionally holds that

$$(G \otimes H)(t, 0) = \begin{cases} G(2t, 0), & t \leq \frac{1}{2}, \\ H(2t - 1, 0), & \frac{1}{2} < t. \end{cases} = \begin{cases} x, & t \leq \frac{1}{2}, \\ x, & \frac{1}{2} < t \end{cases} = x,$$

and analogously that $(G \otimes H)(t, 1) = y$, so it is also a directed path homotopy and we find $\gamma_1 \rightsquigarrow \gamma_3$. A counterexample to symmetry is as follows: let $\gamma_1, \gamma_2 : I \to I$ be the paths given by

$$\gamma_1(t) = \begin{cases} 0, & t \leq \frac{1}{2}, \\ 2t - 1, & \frac{1}{2} < t. \end{cases} \quad \text{and} \quad \gamma_2(t) = \begin{cases} 2t, & t \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < t. \end{cases}$$

We have that $\gamma_1 \rightsquigarrow \gamma_2$, because of Example 4.3. On the other hand, if H were a directed path homotopy from γ_2 to γ_1 we would require that $H(0, \frac{1}{2}) = \gamma_2(\frac{1}{2}) = 1$ and $H(1, \frac{1}{2}) = \gamma_1(\frac{1}{2}) = 0$, contradicting with directedness. □

In order get an equivalence relation on the set of directed paths between two points, we will take the symmetric transitive closure of this relation.

Definition 4.5. Let X be a directed space and $x, y \in X$ two points. We say that two dipaths $\gamma_1, \gamma_2 \in P_X(x, y)$ are equivalent, or $\gamma_1 \simeq \gamma_2$, if there is an integer $n \geq 0$ together with dipaths $\beta_i \in P_X(x, y)$, for each $1 \leq i \leq n$, such that

$$\gamma_1 \rightsquigarrow \beta_1 \leftsquigarrow \ldots \rightsquigarrow \beta_n \leftsquigarrow \gamma_2.$$

This alternating sequence of arrows is also called a zigzag. As $\gamma_2 \leftsquigarrow \gamma_2$ holds for any path γ_2 by reflexivity, we can always assume that there is an odd number of paths in a zigzag between two paths γ_1 and γ_2. By taking $n = 0$, it follows that $\gamma_1 \simeq \gamma_2$ holds if $\gamma_1 \rightsquigarrow \gamma_2$. More precisely, $\simeq$ is the smallest equivalence relation on $P_X(x, y)$ such that that property holds [Lei14, p. 129]. As $\simeq$ is an equivalence relation, we can talk about the equivalence classes of paths, denoted by $[\gamma]$. An important property of these equivalence classes is that they are invariant under maps and path reparametrization.

Lemma 4.6. *Let X, Y be directed spaces and $x, y \in X$. Let $\gamma_1, \gamma_2 \in P_X(x, y)$ and $f : X \to Y$ directed. If $\gamma_1 \simeq \gamma_2$, then $f \circ \gamma_1 \simeq f \circ \gamma_2$.*

Proof. Let $n > 0$ odd and $\beta_i \in P_X(x, y)$ for $1 \leq i \leq n$ such that $\gamma_1 \rightsquigarrow \beta_1 \leftsquigarrow \beta_2 \rightsquigarrow \ldots \rightsquigarrow \beta_n \leftsquigarrow \gamma_2$. If $H : I \times I \to X$ is a directed path homotopy from γ_1 to β_1, then $f \circ H$ is a directed path homotopy from $f \circ \gamma_1$ to $f \circ \beta_1$. We find that $f \circ \gamma_1 \rightsquigarrow f \circ \beta_1$. Repeating this for all other arrows in the zigzag gives us $f \circ \gamma_1 \rightsquigarrow f \circ \beta_1 \leftsquigarrow f \circ \beta_2 \rightsquigarrow \ldots \rightsquigarrow f \circ \beta_n \leftsquigarrow f \circ \gamma_2$, so $f \circ \gamma_1 \simeq f \circ \gamma_2$. □

Lemma 4.7. *Let X be a directed space and $x, y \in X$. Let $\gamma \in P_X(x, y)$ and $\varphi, \varphi' : I \to I$ continuous and monotone with $\varphi(0) = \varphi'(0) = 0$ and $\varphi(1) = \varphi'(1) = 1$. Then $\gamma \circ \varphi \simeq \gamma \circ \varphi'$.*

Proof. As γ is a directed map from I to X, it is enough by Lemma 4.6 to show that $\varphi \simeq \varphi'$. Let $\beta_1 = \varphi \odot 0_1$ and $\beta_2 = 0_0 \odot \varphi'$. Then, by applying Example 4.3 three times, we obtain the zigzag

$$\varphi \rightsquigarrow \beta_1 \leftsquigarrow \beta_2 \rightsquigarrow \varphi'.$$

This shows that $\varphi \simeq \varphi'$, completing the proof. □

In the next section, we will construct the fundamental category of a directed space. For that we need the following four additional equalities.

Lemma 4.8. *Let X, Y be directed spaces and $x, y, z, w \in X$. Let $\beta_1, \gamma_1 \in P_X(x, y)$, $\beta_2, \gamma_2 \in P_X(y, z)$ and $\gamma_3 \in P_X(z, w)$ such that $\beta_1 \simeq \gamma_1$ and $\beta_2 \simeq \gamma_2$. Then the following holds:*

(1) $\beta_1 \odot \beta_2 \simeq \gamma_1 \odot \gamma_2$

(2) $0_x \odot \gamma_1 \simeq \gamma_1$

(3) $\gamma_1 \odot 0_y \simeq \gamma_1$

(4) $(\gamma_1 \odot \gamma_2) \odot \gamma_3 \simeq \gamma_1 \odot (\gamma_2 \odot \gamma_3)$

Proof. Statements 2, 3 and 4 are direct applications of Lemma 4.7 as they are all reparametrizations. We will now show statement 1. Let $n, m > 0$ odd and $p_i, q_j \in P_X(x, y)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that

$$\beta_1 \rightsquigarrow p_1 \leftsquigarrow p_2 \rightsquigarrow \dots \rightsquigarrow p_n \leftsquigarrow \gamma_1,$$
$$\beta_2 \rightsquigarrow q_1 \leftsquigarrow q_2 \rightsquigarrow \dots \rightsquigarrow q_m \leftsquigarrow \gamma_2.$$

Let G be a directed path homotopy from β_1 to p_1 and H be the identity homotopy from β_2 to β_2. Then $G \odot H$ is a directed path homotopy from $\beta_1 \odot \beta_2$ to $p_1 \odot \beta_2$. We obtain a zigzag

$$\beta_1 \odot \beta_2 \rightsquigarrow p_1 \odot \beta_2 \leftsquigarrow p_2 \odot \beta_2 \rightsquigarrow \dots \rightsquigarrow p_n \odot \beta_2 \leftsquigarrow \gamma_1 \odot \beta_2,$$

so $\beta_1 \odot \beta_2 \simeq \gamma_1 \odot \beta_2$. Analogously we obtain a zigzag

$$\gamma_1 \odot \beta_2 \rightsquigarrow \gamma_1 \odot q_1 \leftsquigarrow \gamma_1 \odot q_2 \rightsquigarrow \dots \rightsquigarrow \gamma_1 \odot q_m \leftsquigarrow \gamma_1 \odot \gamma_2.$$

This results in $\gamma_1 \odot \beta_2 \simeq \gamma_1 \odot \gamma_2$ and combining both equivalences gives us $\beta_1 \odot \beta_2 \simeq \gamma_1 \odot \gamma_2$. □

The definition of a directed path homotopy and the three lemmas above have all been been formalized in `directed_path_homotopy.lean`. For the path homotopies, we followed the more general approach from MathLib, where we first defined directed homotopies that satisfy some property P. Thereafter we defined `dihomotopy_rel` as directed homotopies that are fixed on a select subset of points. This is all defined in `directed_homotopy.lean`. A path homotopy is a homotopy that is fixed on both endpoints, that is, on $\{0, 1\} \subseteq I$, so we can define a directed path homotopy as

```
abbreviation dihomotopy (p₀ p₁ : dipath x₀ x₁) :=
    directed_map.dihomotopy_rel p₀.to_directed_map p₁.to_directed_map {0, 1}
```

As a directed homotopy is defined between two directed maps, we need to convert both paths `p₀` and `p₁` to directed maps. The construction $\odot$ is called `hcomp` and $\otimes$ is called `trans`. If $f, g \in D(I, I)$ are two directed maps with $f(t) \leq g(t)$ for all $t \in I$, the definition `dihomotopy.reparam` constructs a homotopy from $\gamma \circ f$ to $\gamma \circ g$. This is done by composing γ and the homotopy obtained from Example 4.3. If H is an homotopy from p to q with $p, q \in P_X(x, y)$, and $f : X \to Y$ is a directed map, then the homotopy from $f \circ p$ to $f \circ q$ given by $f \circ H$ is exactly what `dihomotopy.map` entails.
Now we can define the relations $\rightsquigarrow$ and $\simeq$. These are called `pre_dihomotopic` and `dihomotopic` respectively.

```
    def pre_dihomotopic : Prop := nonempty (dihomotopy p₀ p₁)
    def dihomotopic : Prop := eqv_gen pre_dihomotopic p₀ p₁
```

The term nonempty means exactly that there exists some dihomotopy, which corresponds with our definition of $\rightsquigarrow$. `eqv_gen` gives the smallest equivalence relation generated by a relation, which is exactly what we want. The lemmas `map`, `reparam` and `hcomp` in the `dihomotopic`-namespace now correspond with Lemma 4.6, Lemma 4.7 and the first point of Lemma 4.8 respectively.
This gives us enough tools to construct the so called fundamental category.

5 Fundamental Structures

In this section, we define two structures that contain information about directed paths up to deformation in a directed space: the fundamental category and the fundamental monoid.

5.1 The Fundamental Category

Using the properties found in Section 4.2, we can define a category that captures the information of all paths up to directed deformation in a directed space.

Definition 5.1 (Fundamental Category)**.** Let X be a directed space. The fundamental category of X, denoted by $\vec{\Pi}(X)$, is a category that consists of:

- Objects: points $x \in X$.
- Morphisms: $\vec{\Pi}(X)(x, y) = P_X(x, y)/\simeq$.
- Composition: $[\gamma_2] \circ [\gamma_1] = [\gamma_1 \odot \gamma_2]$.
- Identity: $\mathrm{id}_x = [0_x]$.

Remark. The fact that this category is well defined follows from Lemma 4.8. Due to property 1, composition is well defined. Due to properties 2 and 3, the constant path behaves as an identity and property 4 gives us associativity.

Note that $\vec{\Pi}$ maps objects in **dTop** to objects in **Cat**. It turns out that it can also be defined on morphisms.

Definition 5.2. Let $f : X \to Y$ be a directed map. We define $\vec{\Pi}(f) : \vec{\Pi}(X) \to \vec{\Pi}(Y)$ as the functor:

- On objects: $\vec{\Pi}(f)(x) = f(x)$.
- On morphisms: $\vec{\Pi}(f)([\gamma]) = [f \circ \gamma]$.

It is well behaved on morphisms, because of Lemma 4.6. It is easy to verify that $\vec{\Pi}(f)$ respects composition and identity and it is thus a functor.
In our formalization, we follow the construction of the fundamental groupoid in MathLib found in `algebraic_topology/fundamental_groupoid/basic.lean` closely. The implementation is found in the file `fundamental_category.lean`. The MathLib version has some auxiliary definitions for a reparametrization that show that the two paths $(\gamma_1 \odot \gamma_2) \odot \gamma_3$ and $\gamma_1 \odot (\gamma_2 \odot \gamma_3)$ are equal with relation to $\simeq$ for compatible paths γ_1, γ_2 and γ_3. In order to use these in our directed world, we just need to show that this reparametrization is monotone. This is enough to then define the fundamental category.

```
def fundamental_category (X : Type u) := X

...

instance : category_theory.category (fundamental_category X) :=
{
    hom := λ x y, dipath.dihomotopic.quotient x y,
    id := λ x, ⟦ dipath.refl x ⟧,
    comp := λ x y z, dipath.dihomotopic.quotient.comp,
    id_comp' := λ x y f, quotient.induction_on f
    (λ a, show ⟦ (dipath.refl x).trans a ⟧ = ⟦ a ⟧,
          from quotient.sound (eqv_gen.rel _ _ ⟨dipath.dihomotopy.refl_trans a⟩)),
    comp_id' := /- Proof omitted -/,
    assoc' := /- Proof omitted -/,
}
```

The first definition makes sure that objects of the fundamental category are terms of type X. We then show that `fundamental_category` is an instance of a category by defining the morphisms (`hom`), identities (`id`) and composition (`comp`). The morphisms between two objects `x` and `y` are given by `dipath.dihomotopic.quotient x y`. This is the quotient of `dipath x y` under the `dihomotopic` relation and its definition can be found in `directed_path_homotopy.lean`. The identity on `x` is then the equivalence class (denoted by $[\![\ ]\!]$) of the constant path in `x`. The composition of the equivalence classes of two compatible paths is defined in `dipath.dihomotopic.quotient.comp` as the equivalence class of the concatenation of the two paths.
Proof that the fundamental category is indeed a category are given by `id_comp'`, `comp_id'` and `assoc'`. The first one, `id_comp'`, requires us to show that `(dipath.refl x).trans a` and `a` are two dihomotopic paths. For this, we use `dipath.dihomotopy.refl_trans a`, which is an explicit directed path homotopy from the path `(dipath.refl x).trans a` to `a`. Its existence shows that the two paths are `pre_dihomotopic` and they are thus in the same equivalence class.
The file also contains the definition of the $\vec{\Pi}$-functor from `dTop` to `category_theory.Cat`. Analogous to the undirected MathLib implementation, we use the notation `d`π for this functor.

5.2 The Fundamental Monoid

Definition 5.3 (Monoid)**.** A monoid is triple $(M, \cdot, e)$ consisting of a set M, an associative binary operation $\cdot : M \times M \to M$ and an element $e \in M$ such that for all $g \in M$ it holds that $e \cdot g = g \cdot e = g$.

Example 5.4. The triples $(\mathbb{N}, +, 0)$ and $(\mathbb{N}, \cdot, 1)$ are both monoids. As they both do not have an inverse for each element, they are not groups.

Example 5.5. If C is a locally small category and $x \in C$ and object, then the triplet $(C(x, x), \circ, \mathrm{id}_x)$ is a monoid. This follows directly from the definition of a category. The elements in $C(x, x)$ are called endomorphisms of x.

This last example can be used to define the fundamental monoid of a point in a directed space.

Definition 5.6 (Fundamental monoid)**.** Let X be a directed space and $x \in X$ a point. Then the fundamental monoid of X in x, denoted by $\vec{\pi}(X, x)$ is given by $(C(x, x), \circ, \mathrm{id}_x)$.

Example 5.7. Let $x \in I$ be a point. Let $\gamma \in P_I(x, x)$. Then γ is monotone and $\gamma(0) = \gamma(1) = x$. It follows that $\gamma(t) = x$ for all $t \in I$, so $\gamma = 0_x$. From this, we can conclude that the only morphism in $C(x, x)$ is the identity, so $\vec{\pi}(I, x)$ is the trivial monoid.

Example 5.8. We have that $\vec{\pi}(S^1_+, 1) \cong (\mathbb{N}, +, 0)$. In Section 6.3 we will support this claim by calculating a fundamental monoid in a finite version of the directed unit circle.

Whether we give the unit interval the minimal directedness, the rightward directedness or maximal directedness, the fundamental monoid at any point is the trivial monoid. Their fundamental categories, on the other hand, are able to distinguish the differences in directedness. There are respectively zero, one and two morphisms between two different objects. We see that the fundamental monoid loses information that is contained in the fundamental category.

6 The Van Kampen Theorem

In this section, we will state and prove the Van Kampen Theorem. We follow the proof of Grandis and work out some of the details that were omitted. In Section 6.2 we show how we have formalized this proof by comparing the proof to the Lean code. We conclude with an application of the Van Kampen Theorem in Section 6.3 .

6.1 The Van Kampen Theorem

Before we state and prove the theorem, we will define the notion of being covered for directed homotopies.

Definition 6.1. Let X be a directed space and $\mathcal{U}$ a cover of X. Let $H : I \times I \to X$ be a directed homotopy and $n, m > 0$ two integers. We say that H is (n, m)-covered (by $\mathcal{U}$) if for all $1 \leq i \leq n$ and $1 \leq j \leq m$ the image of $\left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{m}, \frac{j}{m}\right]$ under H is contained in some $U \in \mathcal{U}$.

Once again, by the Lebesgue Number Lemma, for any homotopy H and open cover $\mathcal{U}$ of X, there are $n, m > 0$ such that H is (n, m)-covered by $\mathcal{U}$.

Theorem 6.2 (Van Kampen Theorem)**.** *Let X be a directed space and X_1 and X_2 two open subspaces such that $X = X_1 \cup X_2$ and let $X_0 = X_1 \cap X_2$. Let $i_k : X_0 \to X_i$ and $j_k : X_k \to X$ be the inclusion maps, $k \in \{1, 2\}$. Then we obtain a pushout square in **Cat**:*

$$\begin{tikzcd}
\vec{\Pi}(X_0) \arrow[r, "\vec{\Pi}(i_1)"] \arrow[d, "\vec{\Pi}(i_2)"'] & \vec{\Pi}(X_1) \arrow[d, "\vec{\Pi}(j_1)"] \\
\vec{\Pi}(X_2) \arrow[r, "\vec{\Pi}(j_2)"'] & \vec{\Pi}(X)
\end{tikzcd}$$

Proof. As $j_1 \circ i_1 = j_2 \circ i_2$ and $\vec{\Pi}$ is a functor, the square is commutative. It remains to show it satisfies the property of a pushout square. Let C be any category and $F_1 : \vec{\Pi}(X_1) \to C$ and $F_2 : \vec{\Pi}(X_2) \to C$ be two functors such that $F_1 \circ \vec{\Pi}(i_1) = F_2 \circ \vec{\Pi}(i_2)$. We will explicitly construct a functor F such that $F \circ \vec{\Pi}(j_1) = F_1$ and $F \circ \vec{\Pi}(j_2) = F_2$. The construction will show that this functor is necessarily unique with this property.
Step 1) The objects of $\vec{\Pi}(X)$ are exactly the points of X. If an object $x \in \vec{\Pi}(X)$ is also contained in $\vec{\Pi}(X_1)$, it holds that $F(x) = F(j_1(x)) = (F \circ \vec{\Pi}(j_1))(x)$. The desired condition $F \circ \vec{\Pi}(j_1) = F_1$ then requires us to define $F(x) = F_1(x)$. A similar argument gives us that if $x \in \vec{\Pi}(X_2)$ then $F(x) = F_2(x)$. As X_1 and X_2 cover X, we have that for all $x \in \vec{\Pi}(X)$

$$F(x) = \begin{cases} F_1(x), & x \in X_1, \\ F_2(x), & x \in X_2. \end{cases}$$

By the property that $F_1 \circ \vec{\Pi}(i_1) = F_2 \circ \vec{\Pi}(i_2)$ this is well defined, so we know how F must behave on objects.

Step 2) Let $[\gamma] : x \to y$ be a morphism in $\vec{\Pi}(X)$. Then there is a $n > 0$ such that γ is n-covered by the open cover $\{X_1, X_2\}$, with $\gamma_{i,n}$ covered by X_{k_i}, $k_i \in \{1, 2\}$. One important thing to note is that $\gamma_{i,n}$ can be both seen as a path in X and as a path in X_{k_i} by restricting its codomain. This matters when we talk about $[\gamma_{i,n}]$, as it could be a morphism in $\vec{\Pi}(X)$ and in $\vec{\Pi}(X_{k_i})$. Within this proof will always consider it as a morphism in $\vec{\Pi}(X_{k_i})$ and use $[j_{k_n} \circ \gamma_{i,n}]$ for the morphism in $\vec{\Pi}(X)$. Note that we have that $[\gamma] = [j_{k_n} \circ \gamma_{n,n}] \circ \ldots \circ [j_{k_1} \circ \gamma_{1,n}]$ in $\vec{\Pi}(X)$, as γ is equal to $\gamma_{1,n} \odot (\gamma_{2,n} \odot \ldots (\gamma_{n-1,n} \odot \gamma_{n,n}))$ up to reparametrization. From context, it should always be clear which one we reference. Because we want F to be a functor and thus to respect composition, we find that necessarily

$$\begin{aligned} F[\gamma] &= F([j_{k_n} \circ \gamma_{n,n}] \circ \ldots \circ [j_{k_1} \circ \gamma_{1,n}]) \\ &= F[j_{k_n} \circ \gamma_{n,n}] \circ \ldots \circ F[j_{k_1} \circ \gamma_{1,n}] \\ &= F\left(\vec{\Pi}(j_{k_n})[\gamma_{n,n}]\right) \circ \ldots \circ F\left(\vec{\Pi}(j_{k_1})[\gamma_{1,n}]\right) \\ &= (F \circ \vec{\Pi}(j_{k_n}))[\gamma_{n,n}] \circ \ldots \circ (F \circ \vec{\Pi}(j_{k_1}))[\gamma_{1,n}] \\ &= F_{k_n}[\gamma_{n,n}] \circ \ldots \circ F_{k_1}[\gamma_{1,n}]. \end{aligned}$$

As multiple choices were made, we need to make sure that F is well defined this way. We do this by defining a map $F' : P_X \to C$ by

$$F'(\gamma) = F_{k_n}[\gamma_{n,n}] \circ \ldots \circ F_{k_1}[\gamma_{1,n}],$$

where γ is n-covered with $\gamma_{i,n}$ covered by X_{k_i}. Firstly, we will show that this map is well defined. Secondly, we show that F' respects equivalence classes. From this it follows that F is well defined, as it is simply F' descended on equivalence classes.

Step 3) We first need to make sure that F' does not depend on any choices of k_i. In the case that $\gamma_{i,n}$ is covered by both X_1 and X_2, the value of k_i can be either 1 or 2. The condition that $F_1 \circ \vec{\Pi}(i_1) = F_2 \circ \vec{\Pi}(i_2)$ assures us that both options give us the same value.

Step 4) The second choice we made is that of n. It is possible that γ is also m-covered for another integer $m > 0$, with $\gamma_{j,m}$ being contained in X_{p_j}. We want to show that

$$F_{k_n}[\gamma_{n,n}] \circ \ldots \circ F_{k_1}[\gamma_{1,n}] = F_{p_m}[\gamma_{m,m}] \circ \ldots \circ F_{p_1}[\gamma_{1,m}].$$

If we refine the partition of γ in n pieces into a partition of mn pieces, that partition will surely also be partwise covered. Let $l_i \in \{1, 2\}$ for all $1 \le i \le mn$ such that $\gamma_{i,mn}$ is covered by X_{l_i}. We now claim that for all $1 \le i \le n$ it holds that $F_{k_i}[\gamma_{i,n}] = F_{l_{mi}}[\gamma_{mi,mn}] \circ \ldots \circ F_{l_{m(i-1)+1}}[\gamma_{m(i-1)+1,mn}]$. As $\gamma_{m(i-1)+k,mn}$ with $1 \le k \le n$ is a part of $\gamma_{i,n}$, we may assume that $l_{m(i-1)+1} = k_i$. This is because F_1 and F_2 agree on $X_1 \cap X_2$. As F_{k_i} is a functor, the claim now follows because functors respect composition and because $\gamma_{i,n}$ is exactly the concatenation of all the smaller paths up to reparametrization. By a similar claim for $F_{p_j}[\gamma_{j,m}]$ we find:

$$\begin{aligned} F_{k_n}[\gamma_{n,n}] \circ \ldots \circ F_{k_1}[\gamma_{1,n}] &= (F_{l_{mn}}[\gamma_{mn,mn}] \circ \ldots \circ F_{l_{m(n-1)+1}}[\gamma_{m(n-1)+1,mn}]) \circ \ldots \circ (F_{l_m}[\gamma_{m,mn}] \circ \ldots \circ F_{l_1}[\gamma_{1,mn}]) \\ &= F_{l_{mn}}[\gamma_{mn,mn}] \circ \ldots \circ F_{l_1}[\gamma_{1,mn}] \\ &= (F_{l_{mn}}[\gamma_{mn,mn}] \circ \ldots \circ F_{l_{(m-1)n+1}}[\gamma_{(m-1)n+1,mn}]) \circ \ldots \circ (F_{l_n}[\gamma_{n,mn}] \circ \ldots \circ F_{l_1}[\gamma_{1,mn}]) \\ &= F_{p_m}[\gamma_{m,m}] \circ \ldots \circ F_{p_1}[\gamma_{1,m}] \end{aligned}$$

We conclude that the definition is independent of the value of n. This makes F' well defined.

Step 5) Before we verify that F' is independent of the choice of representative γ, we will first show that F' satisfies two properties:

$$\forall x \in \vec{\Pi}(X) : F'(0_x) = \mathrm{id}_{F(x)}. \tag{1}$$

$$\forall \gamma \in P_X(x,y), \delta \in P_X(y,z) : F'(\gamma \odot \delta) = F'(\delta) \circ F'(\gamma). \tag{2}$$

Let $x \in \vec{\Pi}(X)$ be given. If $x \in X_1$, then 0_x is already contained in X_1 and so by definition of F' we find $F'(0_x) = F_1[0_x] = \mathrm{id}_{F_1(x)} = \mathrm{id}_{F(x)}$. Otherwise it holds that $x \in X_2$, so $F'(0_x) = F_2[0_x] = \mathrm{id}_{F_2(x)} = \mathrm{id}_{F(x)}$. This proves Eq. (1).
Let $\gamma \in P_X(x,y), \delta \in P_X(y,z)$ be two paths in $\vec{\Pi}(X)$. We can then find a n such that both γ and δ are n-covered, with $\gamma_{i,n}$ contained in X_{k_i} and $\delta_{i,n}$ contained in X_{p_i}. It is then true that $\gamma \odot \delta$ is $2n$-covered as it holds that

$$(\gamma \odot \delta)_{i,2n} = \begin{cases} \gamma_{i,n}, & i \leq n, \\ \delta_{i-n,n}, & i > n. \end{cases}$$

We find:

$$F'(\delta \odot \gamma) = F_{p_n}[(\delta \odot \gamma)_{2n,2n}] \circ \ldots \circ F_{p_1}[(\delta \odot \gamma)_{n+1,2n}] \circ F_{k_n}[(\delta \odot \gamma)_{n,2n}] \circ \ldots \circ F_{k_1}[(\delta \odot \gamma)_{1,2n}] = \\ (F_{p_n}[\delta_{n,n}] \circ \ldots \circ F_{p_1}[\delta_{1,n}]) \circ (F_{k_n}[\gamma_{n,n}] \circ \ldots \circ F_{k_1}[\gamma_{1,n}]) = F'(\delta) \circ F'(\gamma).$$

This shows that Eq. (2) holds.
Step 6) We will now show that F' respects equivalence classes. Then it can descend to the quotient and it follows that F is well defined. If $[\gamma] = [\delta]$ in $\vec{\Pi}(X)$ with δ another path from x to y, we want that

$$F'(\gamma) = F'(\delta). \tag{3}$$

Because of the way the equivalence class is defined, it is enough to show this for γ and δ such that $\gamma \rightsquigarrow \delta$. Let in that case a directed path homotopy H from γ to δ be given. We take $n, m > 0$ such that H is (n,m)-covered by $\{X_1, X_2\}$. Firstly assume that $n > 1$. Restricting H to the rectangle $\left[0, \frac{1}{n}\right] \times [0,1]$ gives us a directed path homotopy H_1 from γ to the directed path η given by $\eta(t) = H\left(\frac{1}{n}, t\right)$. By restricting H to the rectangle $\left[\frac{1}{n}, 1\right] \times [0,1]$ we get a directed path homotopy H_2 from η to δ. It is clear that H_1 is $(1,m)$-covered and that H_2 is $(n-1,m)$-covered. By applying induction, we can conclude that it is enough to show that Eq. (3) holds for $(1,m)$-covered directed path homotopies, as we would obtain that $F'(\gamma) = F'(\eta) = F'(\delta)$. Here the second equality follows from induction and the first equality from the base case.
Step 7) We will prove the case where H is $(1,m)$-covered by showing a more general statement:
Let H be any directed homotopy – not necessarily a path homotopy – from one path $\gamma \in P_X(x,y)$ to another path $\delta \in P_X(x',y')$ that is $(1,m)$-covered, $m > 0$. Let η_0 be the path given by $\eta_0(t) = H(t,0)$ and η_1 be given by $\eta_1(t) = H(t,1)$. Then $F'(\eta_0 \odot \delta) = F'(\gamma \odot \eta_1)$. We do this by induction on m.
In the case that $m = 1$, we have a homotopy contained in X_1 or X_2. Without loss of generality, we can assume it is contained in X_1. Let Γ_1 be the directed homotopy given by $\Gamma_1(t,s) = \eta_0(\min(t,s))$ from 0_x to η_0. Let Γ_2 be the directed homotopy given by $\Gamma_2(t,s) = \eta_1(\max(t,s))$ from η_1 to $0_{y'}$. We then can construct a directed path homotopy from $(0_x \odot \gamma) \odot \eta_1$ to $(\eta_0 \odot \delta) \odot 0_{y'}$ given by $(\Gamma_1 \odot H) \odot \Gamma_2$:

$$
\begin{tikzcd}
x \arrow[r, "\eta_0"] & x' \arrow[r, "\delta"] & y' \arrow[r, "0_{y'}"] & y' \\
x \arrow[u, "0_x"] \arrow[r, "0_x"'] \arrow[ur, phantom, "\Gamma_1"] & x \arrow[u, "\eta_0"] \arrow[r, "\gamma"'] \arrow[ur, phantom, "H"] & y \arrow[u, "\eta_1"'] \arrow[r, "\eta_1"'] \arrow[ur, phantom, "\Gamma_2"] & y' \arrow[u, "0_{y'}"']
\end{tikzcd}
$$

It is a directed path homotopy because $\Gamma_1(t,0) = \eta_0(\min(t,0)) = \eta_0(0) = x$ and $\Gamma_2(t,1) = \eta_1(\max(t,1)) = \eta_1(1) = y'$ for all $t \in I$. As η_0, η_1 and H are all contained in X_1, so is this directed path homotopy. We find that $[\gamma \odot \eta_1] = [\eta_0 \odot \delta]$ in $\vec{\Pi}(X_1)$. This gives us that $F'(\gamma \odot \eta_1) = F_1[\gamma \odot \eta_1] = F_1[\eta_0 \odot \delta] = F'(\eta_0 \odot \delta)$. Let now $m > 1$ and assume the statement holds for $(1, m-1)$-covered homotopies. We can restrict H to $[0,1] \times \left[0, \frac{m-1}{m}\right]$ to obtain a $(1, m-1)$-covered homotopy H_1 and we can restrict H to $[0,1] \times \left[\frac{m-1}{m}, 1\right]$ to obtain a $(1,1)$-covered homotopy H_2:

$$
\begin{tikzcd}
x' \arrow[r, "\delta_1"] & \delta(\frac{m-1}{m}) \arrow[r, "\delta_2"] & y' \\
x \arrow[u, "\eta_0"] \arrow[r, "\gamma_1"'] \arrow[ur, phantom, "H_1"] & \gamma(\frac{m-1}{m}) \arrow[u, "\eta'"'] \arrow[r, "\gamma_2"'] \arrow[ur, phantom, "H_2"] & y \arrow[u, "\eta_1"']
\end{tikzcd}
$$

Note that $F'(\gamma) = F'(\gamma_2) \circ F'(\gamma_1)$, because γ_1 is $(m-1)$-covered, γ_2 is 1-covered and γ is m-covered. Similarly it holds that $F'(\delta) = F'(\delta_2) \circ F'(\delta_1)$. We find:

$$
\begin{aligned}
F'(\gamma \odot \eta_1) &= F'(\eta_1) \circ F'(\gamma) \\
&= F'(\eta_1) \circ (F'(\gamma_2) \circ F'(\gamma_1)) \\
&= (F'(\eta_1) \circ F'(\gamma_2)) \circ F'(\gamma_1) \\
&= (F'(\delta_2) \circ F'(\eta')) \circ F'(\gamma_1) && \text{(Case } m = 1\text{)} \\
&= F'(\delta_2) \circ (F'(\eta') \circ F'(\gamma_1)) \\
&= F'(\delta_2) \circ (F'(\delta_1) \circ F'(\eta_0)) && \text{(Induction Hypothesis)} \\
&= (F'(\delta_2) \circ F'(\delta_1)) \circ F'(\eta_0) \\
&= F'(\delta) \circ F'(\eta_0) \\
&= F'(\eta_0 \odot \delta).
\end{aligned}
$$

This proves the statement. From the statement we find that Eq. (3) holds:

$$
F'(\delta) = F'(\delta) \circ \mathrm{id}_x = F'(\delta) \circ F'(0_x) = F'(0_x \odot \delta) = F'(\gamma \odot 0_y) = F'(0_y) \circ F'(\gamma) = \mathrm{id}_x \circ F'(\gamma) = F'(\gamma).
$$

Here the fourth equality follows from the statement. We conclude that F is well defined.

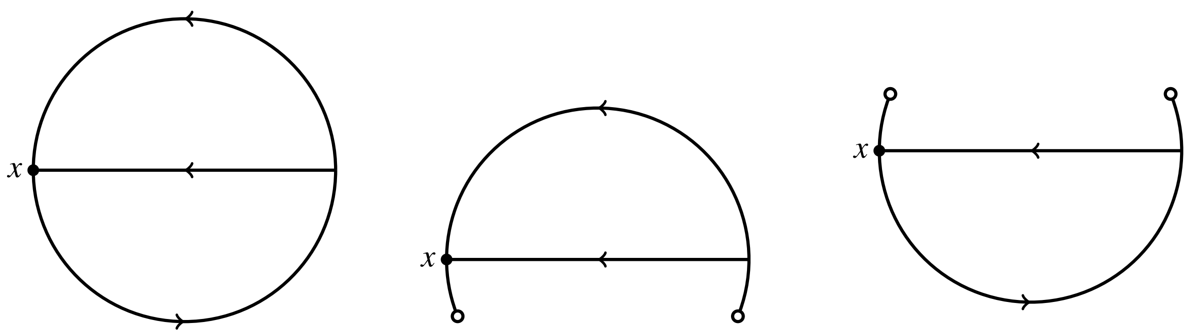
Step 8) As we have that $F[\gamma] = F'(\gamma)$, it is immediate that F is a functor by Eq. (1) and Eq. (2). The equalities $F \circ \vec{\Pi}(j_1) = F_1$ and $F \circ \vec{\Pi}(j_2) = F_2$ are by construction true: if γ is covered by X_1, then $\gamma_{1,1}$ is as well, so $(F \circ \vec{\Pi}(j_1))[\gamma] = F[\gamma] = F'(\gamma) = F_1[\gamma_{1,1}] = F_1[\gamma]$. Here the second $[\gamma]$ is a morphism in $\vec{\Pi}(X)$ and the others are in $\vec{\Pi}(X_1)$. We conclude that the commutative square is indeed a pushout. □

We see that a space X being covered by two open subspaces X_1 and X_2 is a sufficient condition for the conclusion of this theorem to hold. The essence of the proof is that any directed path homotopy can be covered by a grid of rectangles such that it maps each rectangle into either X_1 or X_2. This directly holds as a consequence of the Lebesgue Number Lemma if X_1 and X_2 are open. Without much effort, that covering property can be shown to be also true if $X = X_1^\circ \cup X_2^\circ$, where S° is the topological interior of a subset $S \subseteq X$. Therefore, the condition can be relaxed to the case where X_1 and X_2 are not necessarily open and $X = X_1^\circ \cup X_2^\circ$, where X_1° is the topological interior of X_1. That is how the theorem is stated in the original work of Grandis [Gra03, p. 306].

It is however not necessary that *every* directed path homotopy can be covered. The fundamental category of I is a pushout of the fundamental categories $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, but there are directed path homotopies that are not covered by rectangles. Take, for example, the interpolation homotopy (see Example 4.3) between $0_0 \odot \gamma$ and $\gamma \odot 0_1$, with γ the identity map of I. The key is that it is still possible to find *some* homotopy between these two paths that is (n, m)-covered.

The original Van Kampen Theorem, stated by Egbert van Kampen, was concerned with the fundamental group of a space [VK33]. The fundamental group is the undirected version of the fundamental monoid. The version of the theorem for fundamental groups only requires the additional condition that each of X, X_1, X_2 and $X_1 \cap X_2$ is path connected. The fundamental monoid of a directed space in a point, however, is not guaranteed to form a pushout under the same conditions as the following example shows.

Example 6.3. We take X to be the directed unit circle, together with the horizontal diameter directed leftward. Take X_1 to be the top semicircle together with the diameter and X_2 to be the bottom semicircle together with the diameter. Expand them both a little bit to make sure that they are open subspaces.



The three spaces X, X_1 and X_2, from left to right.

In X_1, and therefore also in $X_1 \cap X_2$, the only path from x to x is the constant path, so it follows that $\vec{\pi}(X_1, x) \cong 0$ and $\vec{\pi}(X_1 \cap X_2, x) \cong 0$. Endomorphisms of x in X_2 behave like endomorphisms of 1 in S^1_+, so $\vec{\pi}(X_2, x) \cong (\mathbb{N}, +, 0)$. If $\vec{\pi}(X, x)$ were the pushout, we would find that $\vec{\pi}(X, x) \cong (\mathbb{N}, +, 0)$. This is false, as $\vec{\pi}(X, x) \cong (\mathbb{N}, +, 0) * (\mathbb{N}, +, 0)$ — the free product of $\mathbb{N}$ with itself. We will support this claim in Section 6.3.

With the right conditions, a Van Kampen type theorem for the fundamental monoid holds. One such condition can be extracted from the work of Bubenik [Bub09]. Let X_1 and X_2 be two open subspaces

covering the directed space X with base point x, both containing x. It is now sufficient that for any endomorphism $[\gamma] : x \to x$ we can split it as $[\gamma] = [\gamma_n] \circ \ldots \circ [\gamma_1]$ with $[\gamma_i] : x \to x$ and γ_i contained in either X_1 or X_2.
In Example 6.3 this condition is not fulfilled as one counterclockwise loop along the circle is first contained in X_2 and then in X_1. There is no way to factor that loop into endomorphisms of x in such a way that each morphism is contained in $\vec{\Pi}(X_1)$ or $\vec{\Pi}(X_2)$.

6.2 Formalization

In the formalization of Theorem 6.2 we follow the constructive nature of its proof. It can be found in `directed_van_kampen.lean`. We have the following global variables, corresponding with the assumptions of the Van Kampen Theorem:

```
variables {X : dTop.{u}} {X₁ X₂ : set X}
variables (hX : X₁ ∪ X₂ = set.univ)
variables (X₁_open : is_open X₁) (X₂_open : is_open X₂)
```

The statement `hX` is exactly that X_1 and X_2 cover X. We do not write it as `X₁ ∪ X₂ = X`, as that would not be type correct: the left side has type `set X` and the right side has type `dTop.{u}`. The term `set.univ` does have type `set X` and is defined to be the set containing all elements of X.
Like in the proof, we introduce a category C and two functors $F_1 : \vec{\Pi}(X_1) \to C$ and $F_2 : \vec{\Pi}(X_2) \to C$. Using these we are going to explicitly construct a functor from $\vec{\Pi}(X)$ to C and show that it is unique. We will use that to prove that we indeed have a pushout square.

```
variables {C : category_theory.Cat.{u u}} (F₁ : (dπₓ (dTop.of X₁) ⟶ C))
    (F₂ : (dπₓ (dTop.of X₂) ⟶ C)) (h_comm : (dπₘ i₁) ⋙ F₁ = ((dπₘ i₂) ⋙ F₂))
```

Here `i₁` and `i₂` are the inclusion maps as in the statement of the Van Kampen Theorem. They are obtained by `dTop.directed_subset_hom`, defined in `dTop.lean`. This defines the inclusion morphism $X_0 \to X_1$ in **dTop** in the case that $X_0 \subseteq X_1 \subseteq X$. Here $X_0 = X_1 \cap X_2$. We first define the functor on objects (**Step 1**).

```
def functor_obj (x : dπₓ X) : C :=
    or.by_cases ((set.mem_union x X₁ X₂).mp (filter.mem_top.mpr hX x)) (λ hx,
  F₁.obj ⟨x, hx⟩) (λ hx, F₂.obj ⟨x, hx⟩)
```

Here we take an object x from the fundamental category of X and return an object from the codomain C. We use `filter.mem_top.mpr hX x` to show that $x \in X_1 \cup X_2$. From this, we use `set.mem_union` to obtain $x \in X_1$ or $x \in X_2$ and we can split by those cases to apply either F_1 or F_2. We abbreviate `functor_obj hX F₁ F₂` to `F_obj` in our formalization to maintain clarity. After this definition, there are two lemmas that prove that if $x \in X_k$, then $F_k(x) = F(x)$ for $k \in \{1, 2\}$.
In the proof of Theorem 6.2, F' is first defined and it is then shown to be a valid definition. Within our Lean formalization, we have to do these two parts in the reverse order. Once we have shown the construction is well-defined, we can define F' in our formalization. That is why **Step 2** will be completed later.
We use the definitions of `covered` and `covered_partwise`, shown in Section 3, to define the mapping of morphisms inductively (**Step 3**):

```
def functor_map_of_covered {γ : dipath x y} (hγ : covered γ hX) :
    F_obj x ⟶ F_obj y :=
    or.by_cases hγ
        (λ hγ, functor_map_aux_part_one hX h_comm hγ)
        (λ hγ, functor_map_aux_part_two hX h_comm hγ)
```

```
def functor_map_of_covered_partwise {n : ℕ} : Π {x y : X} {γ : dipath x y} (hγ :
covered_partwise hX γ n), F_obj x ⟶ F_obj y :=
    nat.rec_on n
      (λ x y γ hγ, F₀ hγ)
      (λ n ih x y γ hγ, (F₀ hγ.1) ≫ (ih hγ.2))
```

In `functor_map_of_covered` we define what to do with a path γ that is 1-covered, i.e. map it to $F_1([\gamma])$ or $F_2([\gamma])$ depending on whether γ is covered by X_1 or X_2. `functor_map_aux_part_one` specifies what $F_1([\gamma])$ should be. Again, we use a shorter notation F_0 for `functor_map_of_covered hX h_comm`. We can then define `functor_map__of_covered_partwise` for a n-covered path inductively. If $n = 1$, we can apply F_0 by definition. Otherwise, we are able apply F_0 to the first part of the path, which is 1-covered, and then `functor_map_of_covered_partwise` to the second part, which is $(n-1)$-covered. We abbreviate `functor_map_of_covered_partwise hX h_comm` to F_n.
Note that n is still an input for the definition, so we need to show that this definition is independent of the choice for n. This is captured in the lemma `functor_map_of_covered_partwise_unique` (**Step 4**).

```
lemma functor_map_of_covered_partwise_unique {n m : ℕ} {γ : dipath x y}
(hγ_n : covered_partwise hX γ n) (hγ_m : covered_partwise hX γ m) :
Fₙ hγ_n = Fₙ hγ_m :=
    /- Proof omitted -/
```

This lemma makes use of the following lemma that shows that the image remains the same if we refine the partition of γ, so when we use a nk-covering instead of a n-covering.

```
lemma functor_map_aux_of_covered_partwise_refine {n : ℕ} (k : ℕ) :
  Π {x y : X} {γ : dipath x y} (hγ_n : covered_partwise hX γ n),
  Fₙ hγ_n = Fₙ (covered_partwise_refine hX n k hγ_n) :=
    /- Proof omitted -/
```

Now we know that the image is independent of n, and because a $n > 0$ exists such that γ is n-covered (shown in `has_subpaths`), we can choose one such n and we obtain the following formalization of F', completing **Step 2**.

```
def functor_map_aux (γ : dipath x y) : F_obj x ⟶ F_obj y :=
  Fₙ (classical.some_spec (has_subpaths hX X₁_open X₂_open γ))
```

We have now formalized the F' from the proof of the Van Kampen Theorem and we first show that Eq. (1) and Eq. (2) from Theorem 6.2 hold (**Step 5**).

```
lemma functor_map_aux_refl {x : X} :
```

```
  F_map_aux (dipath.refl x) = 𝟙 (F_obj x) :=
    /- Proof omitted -/

lemma functor_map_aux_trans {x y z : X} (γ₁ : dipath x y) (γ₂ : dipath y z) :
  F_map_aux (γ₁.trans γ₂) = F_map_aux γ₁ ≫ F_map_aux γ₂ :=
    /- Proof omitted -/
```

We arrive at **Step 6** and want to show it is invariant under the `dihomotopic` relation. To do this we need to show the claim from the proof: if we have a directed homotopy H from f to g that is $(1, m)$-covered, then $F'[H(\_, 1)] \circ F'[f] = F'[g] \circ F'[H(\_, 0)]$ (**Step 7**).

```
lemma functor_map_aux_of_homotopic_dimaps {m : ℕ} :
  Π  {f g : D(I, X)} {H : directed_map.dihomotopy f g}
    (hcov : directed_map.dihomotopy.covered_partwise H hX 0 m),
  F_map_aux (dipath.of_directed_map f) ≫ F_map_aux (H.eval_at_right 1) =
F_map_aux (H.eval_at_right 0) ≫ F_map_aux (dipath.of_directed_map g) :=
    /- Proof omitted -/
```

By using induction once again, we end up with the lemma showing us that the choice of representative does not matter.

```
lemma functor_map_aux_of_dihomotopic (γ γ' : dipath x y) (h : γ.dihomotopic γ') :
    F_map_aux γ = F_map_aux γ' :=
    /- Proof omitted -/
```

We can now finally define the behavior on morphisms to obtain a functor by using the universal mapping property of quotients.

```
def functor_map {x y : dπₓ X} (γ : x ⟶ y) : F_obj x ⟶ F_obj y :=
    quotient.lift_on γ F_map_aux
        (functor_map_aux_of_dihomotopic hX X₁_open X₂_open h_comm)

... /- Lemmas about identities and compositions -/

def functor : (dπₓ X) ⟶ C := {
    obj := F_obj,
    map := λ x y, F_map,
    map_id' := λ x, functor_map_id hX X₁_open X₂_open h_comm x,
    map_comp' := λ x y z γ₁ γ₂, functor_map_comp hX X₁_open X₂_open h_comm γ₁ γ₂
}
```

Here `F_map` is an abbreviation for `functor_map hX X₁_open X₂_open h_comm` and `Functor` is analogously abbreviated to simply `F`. Finally, we get to **Step 8**. The remaining lemmas show that $F \circ \vec{\Pi}(j_k) = F_k$ for $k = 1$ and $k = 2$, and that F is the unique functor with this property.

```
lemma functor_comp_left : (dπₘ j₁) ⋙ F = F₁ := /- Proof omitted -/
lemma functor_comp_right : (dπₘ j₂) ⋙ F = F₂ := /- Proof omitted -/
```

```
lemma functor_uniq (F' : (dπₓ X) ⟶ C) (h₁ : (dπₘ j₁) ≫ F' = F₁)
    (h₂ : (dπₘ j₂) ≫ F' = F₂) :
    F' = F := /- Proof omitted -/
```

The Van Kampen Theorem is stated as

```
theorem directed_van_kampen {hX₁ : is_open X₁} {hX₂ : is_open X₂}
    {hX : X₁ ∪ X₂ = set.univ} :
    is_pushout (dπₘ i₁) (dπₘ i₂) (dπₘ j₁) (dπₘ j₂) :=
    /- Proof omitted -/
```

The type `is_pushout` is defined in Lean in terms of colimits and cocones. Those two category theoretical terms are used in the generalization of the definition of a pushout. In `pushout_alternative.lean` we show that our definition of a pushout, as stated in Definition 2.7, implies `is_pushout`. The theorem `directed_van_kampen` now follows easily from the lemmas we proved.

6.3 Applications

Topological spaces often have uncountably many points and so their fundamental categories have uncountably many objects making them hard to reason about. It is possible to reduce a fundamental category to a subset of objects that captures the essence of the directedness. This makes the fundamental category easier to work with and with the right selection of points, a Van Kampen Theorem still holds. This requires some more theory [Bub09].
Another way to keep it simple is by looking at spaces with finitely many points. The first space X we will look at is the so called *discrete* or *finite* unit circle.

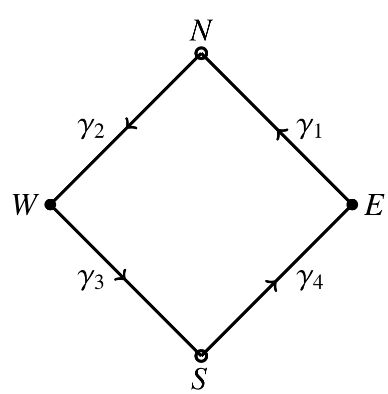
Example 6.4. The discrete unit circle is a finite version of S^1 and consists of four points $X = \{N, E, S, W\}$. E and W are the right and left point of the unit circle and N and S represent the upper and lower arc of the circle. The topology is given by

$$\mathcal{T}_X = \{\emptyset, \{N\}, \{S\}, \{N, S\}, \{N, S, E\}, \{N, S, W\}, \{N, E, S, W\}\}$$

Let now four paths be given by

$$\gamma_1(t) = \begin{cases} E, & t = 0, \\ N, & 0 < t. \end{cases}, \gamma_2(t) = \begin{cases} N, & t < 1, \\ W, & t = 1. \end{cases}, \gamma_3(t) = \begin{cases} W, & t = 0, \\ S, & 0 < t. \end{cases}, \gamma_4(t) = \begin{cases} S, & t < 1, \\ E, & t = 1. \end{cases},$$

Schematically we can represent the space and these paths as:



We now make X directed by setting P_X equal to the set containing any monotone subparametrization of any concatenation of the four paths. Note that any path in X is now considered directed if and only if it visits points in X in a counterclockwise order. This way we mimic the directedness of S^1_+, as each directed path in that space also runs counterclockwise.
We can split X into two open subspaces $X_1 = \{N, S, E\}$ and $X_2 = \{N, S, W\}$. The only directed paths up to monotone reparametrization in X_1 are γ_2, γ_3, $\gamma_2 \odot \gamma_3$ and the constant paths. Its fundamental category has thus three objects with their three identities and three additional morphisms : $[\gamma_2]$, $[\gamma_3]$ and $[\gamma_3] \circ [\gamma_2]$. By symmetry, the fundamental category X_2 has also three objects and three non-trivial morphisms. The intersection of X_1 and X_2 is $\{N, S\}$. Any path from N to S must go through W and any path from S to N must go through E, so the only possible paths in $X_1 \cap X_2$ are the constant (trivial) paths.
As X_1 and X_2 are open, the condition of Theorem 6.2 is satisfied. It follows that X is a pushout of X_1 and X_2.
We now take the category C containing four points $\{n, e, s, w\}$ and whose morphisms are freely generated by the four morphisms $p_1 : e \to n$, $p_2 : n \to w$, $p_3 : w \to s$ and $p_4 : s \to e$. Freely generated means that morphisms in C are exactly compositions of these four morphisms (or identities) and two morphisms are the same if and only if their sequences of these base morphisms are the same up to identities. We claim that $\vec{\Pi}(X)$ is isomorphic to this category C. We do this by showing that C is also the pushout of $\vec{\Pi}(X_1)$ and $\vec{\Pi}(X_2)$. As a pushout is unique up to isomorphism [Lei14], the claim will follows.
We take two maps $j_1 : \vec{\Pi}(X_1) \to C$ and $j_2 : \vec{\Pi}(X_2) \to C$. j_1 is given on the objects by $j_1(N) = n$, $j_1(S) = s$ and $j_1(E) = e$. The non-trivial morphisms are mapped as $j_1([\gamma_1]) = p_1$, $j_1([\gamma_4]) = p_4$ and $j_1([\gamma_4 \odot \gamma_1]) = p_1 \circ p_4$. Similarly j_2 is defined. If it clear that j_1 and j_2 agree on $\vec{\Pi}(X_1 \cap X_2)$, so we have a commutative square.
Let $\mathcal{D}$ be another category and $F_1 : \vec{\Pi}(X_1) \to \mathcal{D}$ and $F_2 : \vec{\Pi}(X_2) \to \mathcal{D}$ two functors that agree on $\vec{\Pi}(X_1 \cap X_2)$. We then define $F : C \to \mathcal{D}$ as $F(n) = F_1(N)$, $F(e) = F_1(E)$, $F(s) = F_1(S)$ and $F(w) = F_2(W)$. On morphisms, we have that necessarily $F(p_1) = F_1([\gamma_1])$, $F(p_2) = F_2([\gamma_2])$, $F(p_3) = F_2([\gamma_3])$ and $F(p_4) = F_1([\gamma_4])$. As the morphisms in C are freely generated by p_1, p_2, p_3 and p_4 this defines F uniquely and it holds that $F \circ j_k = F_k$ for $k \in \{1, 2\}$, so C is indeed a pushout of $\vec{\Pi}(X_1)$ and $\vec{\Pi}(X_2)$. From this it follows that the morphisms in $\vec{\Pi}(X)$ are also freely generated by the four morphisms $[\gamma_1], [\gamma_2], [\gamma_3]$ and $[\gamma_4]$.
We can now ask the question: what do endomorphisms of E in $\vec{\Pi}(X)$ look like? They are of the form $([\gamma_4] \circ [\gamma_3] \circ [\gamma_2] \circ [\gamma_1])^n$ with $n \geq 0$ and each one is different. We now obtain an explicit isomorphism from $\vec{\pi}(X, E)$ to $(\mathbb{N}, +, 0)$ given by $([\gamma_4] \circ [\gamma_3] \circ [\gamma_2] \circ [\gamma_1])^n \mapsto n$. This supports Example 5.8, where we claimed that $\vec{\pi}(S^1_+, 1) \cong (\mathbb{N}, +, 0)$.

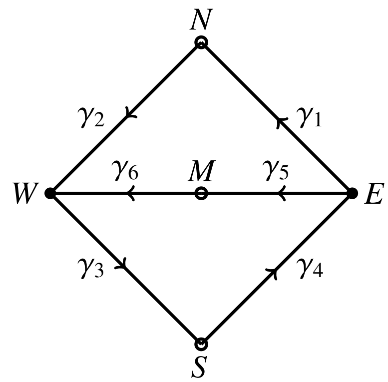
Example 6.5. We will now look at a finite version of the space considered in Example 6.3. This space will be similar to the discrete unit circle we just considered. We take $Y = \{N, E, S, W, M\}$ with the topology given by

$$\mathcal{T}_Y = \{\emptyset, \{N\}, \{M\}, \{S\}, \{N, M\}, \{N, S\}, \{M, S\}, \{N, M, S\}, \{N, M, S, E\}, \{N, M, S, W\}, Y\}.$$

In this case N, M and S all represent open intervals. Note that the discrete unit circle is a subspace of Y in a natural way. We take the four paths $\gamma_1, ..., \gamma_4$ equal to those in the previous example and take two additional paths

$$\gamma_5(t) = \begin{cases} E, & t = 0, \\ M, & 0 < t. \end{cases}, \gamma_6(t) = \begin{cases} M, & t < 1, \\ W, & t = 1. \end{cases},$$

We can represent this space and these paths as:



We take the open subspaces $Y_1 = \{N, M, S, E\}$ and $Y_2 = \{N, M, S, W\}$. The non-trivial morphisms in $\vec{\Pi}(Y_1)$ are $[\gamma_1], [\gamma_4], [\gamma_5], [\gamma_5] \circ [\gamma_4]$ and $[\gamma_1] \circ [\gamma_5]$. In $\vec{\Pi}(Y_2)$ they are $[\gamma_2], [\gamma_3], [\gamma_6], [\gamma_3] \circ [\gamma_2]$ and $[\gamma_3] \circ [\gamma_6]$. $\vec{\Pi}(Y_1 \cap Y_2)$ only consists of three points N, M and S together with their identities. Just like in the previous example, Theorem 6.2 tells us that the morphisms in $\vec{\Pi}(Y)$ are freely generated by the six morphisms $[\gamma_i]$, $1 \leq i \leq 6$.

This time we are interested in the non-trivial endomorphisms of W. If we start in point W, first the morphism $[\gamma_4] \circ [\gamma_3]$ must be followed to reach point E. From there we have a choice to return to W: either $[\gamma_2] \circ [\gamma_1]$ or $[\gamma_6] \circ [\gamma_5]$. We find that endomorphisms of W are sequences of two loops $[\gamma_2] \circ [\gamma_1] \circ [\gamma_4] \circ [\gamma_3]$ and $[\gamma_6] \circ [\gamma_5] \circ [\gamma_4] \circ [\gamma_3]$. This structure is isomorphic to $\mathbb{N} * \mathbb{N}$, agreeing with Example 6.3.

7 Conclusion and Further Research

In this thesis, we presented important concepts from directed topology. These allowed us to state and prove a directed version of the Van Kampen Theorem. If its simple conditions are satisfied, we calculate the fundamental category of a directed space using the fundamental categories of subspaces. Within this thesis, we showcased how we formalized that theorem and the theory leading up to it using the Lean proof assistant.

Our formalization can be extended to other concepts and theorems from directed topology. For example, Bubenik's approach of restricting the fundamental category to a (finite) full subcategory is a clear extension of the theory we have formalized in Lean.

At the moment, MathLib does not have a version of the Van Kampen Theorem for groupoids, originally proven by Brown in 1968 [Bro68, Bro06]. As Grandis based his proof of the directed version on Brown's version, our formalization can conversely be adapted to suit the undirected case. The undirected version could also be shown to be a corollary of the directed version. This is because the fundamental category of a topological space equipped with the maximal directedness coincides with the fundamental groupoid of the topological space.

It might also prove interesting to further investigate sufficient and necessary preconditions for the Van Kampen Theorem, as stated in Section 6.1. As the proof of the Van Kampen Theorem suggests, we want to be able to cover both paths and homotopies, but whether that is truly necessary is something that future research will have to tell.

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