Recurrence of Confined Random Walks

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Introduction

There has been much interest in the study of random walks in random environments. Of particular challenge are problems in which the walker affects the environment. In this context, even the most fundamental question of recurrence versus transience remains open. In [DHS14] the authors consider the discrete-time simple random walk on non-decreasing connected graphs, state three natural conjectures about recurrence/transience, and prove partial results for subgraphs of $\mathbb{Z}^d, d \geq 3$, that satisfy a bounded shape condition.

Under this bounded shape condition [DHS14] proposes the three conjectures. The first conjecture states that only the asymptotic growth rate of the domain matters for the transience/recurrence of such simple random walks. The second conjecture says that if a random walk $\{Y_t\}$ on a graph $\mathbb{G}_t \subseteq \mathbb{G}_t^{'}$ is transient, then the random walk $\{Y_t^{'}\}$ on $\mathbb{G}'_t$ is also transient. The third conjecture states that if the simple random walk on a fixed graph $\mathbb{G}_\infty$ of uniformly bounded degrees is recurrent, then the same applies to simple random walks on non-decreasing $\mathbb{G}_t \subseteq \mathbb{G}_\infty$, for any choice of $\mathbb{G}_t \uparrow \mathbb{G}_\infty$.

In [Hua19] random walks on evolving graphs are considered. For sequences of finite graphs increasing monotonically towards a limiting infinite graph, they establish transition probability upper bounds. It is proved that for any increasing sets $\mathbb{D}_t \uparrow \mathbb{Z}^d$, $d \geq 3$, having roughly the shape of a ball $\mathbb{B}_{f(t)} \subseteq \mathbb{D}_t \subseteq \mathbb{B}_{Cf(t)}$, for some $f(t) \uparrow \infty$ and finite constant C, whenever $\int_0^\infty f(t)^{-d} dt < \infty$, the random walk $\{X_t\}$ that takes steps in $\{\mathbb{D}_t\}$ almost surely visits every vertex of $\mathbb{Z}^d$ finitely often. Under additional technical conditions, the converse is also true. In particular, there is a recurrent phase when the domain grows sufficiently slowly, despite the fact that the limiting graph is transient.

[Ami+20] looks at Random Walks in Changing Environments (RWCE), specifically at the conductances between vertices for simple random walks. The following theorem is proved: If $\{X_t, G_t\}$ is a monotone increasing adaptive RWCE on $\mathbb{N}$, bounded above by some recurrent connected graph $G_\infty = (\mathbb{N}, C_\infty)$, then the random walk is recurrent, where adaptive means that the random walk and the evolving graph are dependent.

The last conjecture in [DHS14] and the main results of [Hua19] and [Ami+20] closely relate to the main topic of this bachelor thesis. We consider, evolving domains or graphs that alter at every time step t according to a specific set of rules that may depend on the position of the simple random walk. Whereas [Hua19] looks at simple random walks in dimension $d \geq 3$, we focus on simple random walks in dimension $d = 2$. It is known that the simple random walk on the infinite lattice $\mathbb{Z}^2$ is recurrent. This is easily shown, however, the difficulty comes when we introduce finite domains and we look at boundary points. Since the simple random walk on the infinite lattice, $\mathbb{Z}^2$, is recurrent. It is easy to show that the simple random walk on a bounded domain in $\mathbb{Z}^2$ is also recurrent. But what would happen when we allow for evolving domains? Do these domains need certain criteria to still be recurrent or can we just pick any evolving domain and the recurrence of the simple random walk still follows?

Our hypotheses are that the domains had to have the following properties: At all times, the domain has to be, has to be convex in a certain sense, and has to contain the origin. By looking at a specific type of domain in $\mathbb{Z}^2$, fulfilling the so-called augmented $\star$-property, we gain some insight in the world of evolving domains.

The main question we want to answer: Is the simple random walk on an evolving, connected domain which satisfying the augmented $\star$-property recurrent?

The structure of this bachelor thesis is as follows: In the first chapter we introduce the simple random walk on $\mathbb{Z}^1$ and $\mathbb{Z}^2$ and prove the recurrence. In the second chapter we take a step aside, into the field of electrical networks, and we prove the recurrence of the simple random walk on $\mathbb{Z}^2$ in terms of infinite resistance in a graph to infinity in electrical networks. In the third chapter we look at a simple random walk in a finite subdomain on $\mathbb{Z}^2$ and we introduce techniques, such as Markovian Coupling and the definition of the augmented $\star$-property, to prove the recurrence of a simple random walk. In the fourth chapter we introduce evolving domains and prove that the simple random walk on an evolving, connected, augmented $\star$-property domain is recurrent. We conclude this paper with some open problems.

1 Simple Random Walks

"A drunk man will always find his way home but a drunk bird may get lost forever". This quote refers to the recurrence of the simple random walk in two and three dimensions. Before we get into more detail, we define what a random walk on $\mathbb{Z}^1$ is and provide some basic definitions and notations to establish a basis, which we can extend to higher dimensions.

1.1 Simple Random Walk on $\mathbb{Z}^1$

The first question that comes to mind is: *"What exactly is a random walk?"* We start by considering a simple random walk on $\mathbb{Z}$. We fix an $N \in \mathbb{N}$. The configuration space, all the different steps available, is given by the set of binary sequences of length N, i.e.,

$$\Omega_N = \{\omega = (\omega_1, \cdots, \omega_N) \in \{-1, +1\}^N\},$$

Write

$$X_k(\omega) = \omega_k, \;\; 1 \leq k \leq N, \;\; \omega \in \Omega_N$$

to denote the projection on the k-th component of ω, which is to be thought of as the step of the random walk at time k. The position of the random walk after n steps is given by

$$S_n(\omega) = \sum_{k=1}^{n} X_k(\omega), \quad 1 \leq n \leq N, \quad S_0(\omega) = 0.$$

In this way, for every $\omega \in \Omega_N$ we obtain a trajectory $(S_n)_{n=1}^N$, also called a path. We take the uniform distribution as the probability distribution on Ω_N, i.e.,

$$P_N(A) = |A| 2^{-N}, \quad A \subset \Omega_N.$$

This implies that all binary sequences ω, i.e., all trajectories, have the same probability.

Definition 1 (**Simple Random Walk**). *The sequence of random variables $(S_n)_{n=1}^N$ on the finite probability space (Ω_N, P_N) is called a simple random walk of length N starting at the origin.*

We also make a lot of use of the facht that a simple random walk is a Markov process.

Definition 2 (**Markov Process**). *A Markov Process is a stochastic process that satisfies the Markov Property, characterized by the "Memorylessness"*

Definition 3 (**Markov Property**). *Let $(\Omega, \mathcal{F}, P)$ be a probability space with totally ordered collections of subsets $(\mathcal{F}_s, s \in I)$ from some index set I; and let (S, S) be a measurable space. A (S, S)-valued stochastic process $X = \{X_t : \Omega \to S\}_{t \in I}$ is said to possess the Markov property if for each $AinS$ and each $s, t \in I$ with $s < t$*

$$P(X_t \in A | \mathcal{F}_s) = P(X_t \in A | X_s).$$

In other words, a stochastic process is a Markov process if the probability of a move only depends on the present and not on the past.

Example 1 (**A fair coin**). *Assume we have a fair coin, look at the line of natural numbers and start on the point 0. If we throw heads, then we move one space to the right, $+1$. If we throw tails, then we move on space to the left, -1. If we toss a coin N times, then we get a set $A \subset \Omega_N$ that consists of N elements, each either $+1$ or -1. If we plot the movement, i.e., the summation of all the steps, then we get a simple random walk on $\mathbb{Z}^1$.*

The next question that comes to mind is: *"Do we always return to our starting point?"* The answer is: *"The answer depends on a multitude of factors."* For example, we have to turn our attention to the dimension we are working in. Returning, with probability 1, to our starting point or any point that we are interested in is called recurrence.

Definition 4 (**Recurrence of Simple Random Walk**). *The random walk $(S_n)_{n \in \mathbb{N}_0}$ is called recurrent when either $P(\sigma_0 < \infty) = 1$, i.e., the probability of the return time being finite equals 1. The state 0 is called recurrent if and only if $\sum_{n=1}^{\infty} P_n(0) = \infty$, where $P_n(0) = \mathbb{P}(S_n = 0)$ is the probability that the random walk returns to the origin at time step n.*

1.2 Recurrence of Simple Random Walk on $\mathbb{Z}^1$

In this section we prove that a simple random walk on $\mathbb{Z}^1$ is recurrent. We start by mentioning Stirling's approximation.

Stirling's Approximation
Stirling's Approximation or Stirling's formula is an approximation for factorial:

$$n! \sim \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n. \tag{1.1}$$

where $\sim$ means that the two quantities are asymptotic, i.e., their ratio tends to 1 as n goes to infinity.

Theorem 5. *The Simple Random Walk on $\mathbb{Z}^1$ is recurrent.*

Proof. First of all we have to acknowledge that we can only return in an even number of steps. If we want to return to the starting point, then we have to take the same number of steps to the right and to the left.

We only look at $P_{2n}(0)$, i.e., the probability of returning to the origin in $2n$ steps. We can find a closed formula. Of the $2n$ steps, only n are free to choose. The other n steps have to be the opposite of the first n steps chosen, since we need to return. So the number of paths we can take is $\binom{2n}{n}$. Since going to the left or to the right has probability $\frac{1}{2}$, we get the following closed formula:

$$\begin{aligned}
P_{2n}(0) &= \binom{2n}{n}\Big(\frac{1}{2}\Big)^n\Big(\frac{1}{2}\Big)^n = \frac{(2n)!}{n!(2n-n)!}\frac{1}{2^{2n}} \\
&\sim \frac{\sqrt{4\pi n}(\frac{2n}{e})^{2n}}{2\pi n(\frac{n}{e})^{2n}}\frac{1}{2^{2n}} \text{ (By Stirling's Approximation)} \\
&= \frac{\sqrt{4\pi n}2^{2n}n^{2n}e^{-2n}}{2\pi n e^{-2n}n^{2n}}\frac{1}{2^{2n}} \\
&= \frac{1}{\sqrt{\pi n}}, \quad n \to \infty.
\end{aligned}$$

So

$$\sum_{n=1}^{\infty} P_n(0) = [1+o(1)]\sum_{n=1}^{\infty}\frac{1}{\sqrt{\pi n}} = \infty.$$

From this we conclude that the one-dimensional simple random walk is recurrent. □

1.3 Recurrence of Simple Random Walk on $\mathbb{Z}^2$

"Can we extend the recurrence of one-dimensional random walk to higher dimensions?" In this section we look at random walks on $\mathbb{Z}^2$ and we prove the recurrence of Simple Random Walk on $\mathbb{Z}^2$. The following theorem is of the utmost importance in understanding simple random walk on $\mathbb{Z}^2$.

Theorem 6 (Decomposition of Simple Random walk on $\mathbb{Z}^2$). *Simple random walk on $\mathbb{Z}^2$ can be decomposed into two independent simple random walks on $\mathbb{Z}$, i.e., $\mathbb{P}^{\mathbb{Z}^2}_{2n}(S_{2n}=0) = [\mathbb{P}^{\mathbb{Z}}_{2n}(S_{2n}=0)]^2$.*

A formal proof is given in [DS84]. The proof depends on the Markov property and on a certain way to decompose the steps of simple random walk on $\mathbb{Z}^2$ into two components. The decomposition of the two-dimensional simple random walk is illustrated in Figure 1.

Let S_n be the position of the two-dimensional simple random walk, and for every n define the positions of the two one-dimensional simple random walks S^1_n and S^2_n by the orthogonal projection of S_n onto the respective diagonals 1 and 2. The steps of a random walk are the differences

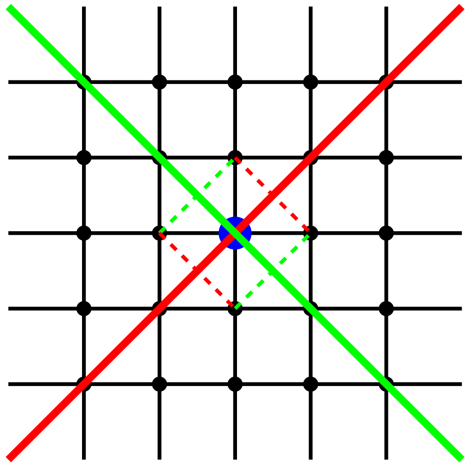


Figure 1: Decomposition of a two-dimensional simple random walk on $\mathbb{Z}^2$ into two simple random walks on Z^1.

between two successive positions. The two-dimensional step X_i can take the values Up, Down, Left and Right. The following table gives the relation between X_i, X_i^1 and X_i^2.

X_i	Up	Down	Left	Right
X_i^1	$+1$	-1	-1	$+1$
X_i^2	$+1$	-1	$+1$	-1

This table illustrates that the distribution of X^1 given X^2 is the same as the marginal distribution of X^1, and thus $\mathbb{P}(X^1 = 1) = \mathbb{P}(X^1 = -1) = \mathbb{P}(X^2 = 1) = \mathbb{P}(X^2 = -1) = \frac{1}{2}$. So in this way two one-dimensional random walks correspond precisely to one two-dimensional random walk and the other way around.

Theorem 7 (Recurrence of Simple Random Walk on $\mathbb{Z}^2$). *Simple random walk on $\mathbb{Z}^2$ is recurrent.*

Proof. First of all we need an even number of steps to get back to the origin: if we take k steps to the left, then we also need to take k steps to the right in order to return. The same goes for up and down. Hence $P_{2n+1}^{(0)} = 0$ for all $n \in \mathbb{N}$.

By using Theorem 6 we know that the $2n$ step probability of returning to the origin is the same as the product of two one-dimensional $2n$ step probabilities of returning to the origin, i.e.,

$$P_{2n}(0) = \mathbb{P}(S_{2n} = 0) = \mathbb{P}(S_{2n}^1 = 0)\mathbb{P}(S_{2n}^2 = 0).$$

From Theorem 5 we know that $\mathbb{P}(S_{2n}^1 = 0) \sim \frac{1}{\sqrt{\pi n}}$. If we substitute this into the equation, we see that $\mathbb{P}(S_{2n} = 0) \sim (\frac{1}{\sqrt{\pi n}})^2 = \frac{1}{\pi n}$. Taking the infinite sum, we get

$$\sum_{n=1}^{\infty} \mathbb{P}(S_{2n} = 0) = \sum_{n=1}^{\infty} \frac{1}{\pi n} = \infty.$$

Hence by Definition 4 we see that simple random walk on $\mathbb{Z}^2$ is recurrent. □

In this chapter we have defined what a random walk is and we proved the recurrence of the one- and two-dimensional simple random walk. Our goal in the next chapters is to prove recurrence for evolving domains. In the next chapter we relate random walks to electrical networks and try to prove recurrence via comparison with effective resistance.

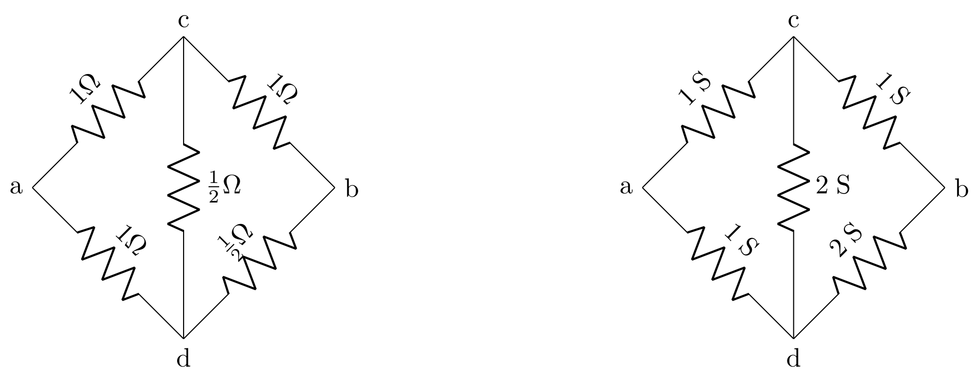
2 Random walks and Electrical Networks

In this chapter we look at electrical networks and how they are connected to random walks. We state and prove Dirichlet's principle and use this to give a different proof of the recurrence of the random walk on $\mathbb{Z}^2$.

2.1 Currents and flows

Our random walks are defined on $\mathbb{Z}^2$ such that each step between neighbouring vertices has the same probability, apart from two boundary vertices **a** and **b**. This can also be seen as an electrical network where each edge between two points carries a unit resistor, (1Ω), and a different resistance at the boundary points. and the boundary vertices are connected to a battery.

In the following example we see two figures. In Figure 2a we see an electrical network with different resistances. In Figure 2b, we see the same network, however, the resistances have been converted to conductances.



(a) A connected graph G with four vertices. This graph can also be seen as a network where the edges are replaced by resistors.

(b) The same graph G as in the left figure, with the alteration that the resistances are converted to conductances in Siemens(S).

Figure 2: The conversion of electrical networks with resistances to conductances.

We define a random walk on the network G to be a Markov chain with transition matrix P and conductance C given by

$$P_{xy} = \frac{C_{xy}}{C_x} \quad \text{with} \quad C_x = \sum_y C_{xy}.$$

For our example in Figure 2b, $C_a = 2, C_b = 3, C_c = 4, C_d = 5$ and the transition matrix P for the associated random walk is

$$P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 \end{pmatrix}.$$

When all the conductances of a network are the same, the associated random walk on the graph G of the network has the property that from each point there is an equal probability of moving to each of the connected points.

Suppose that we have a network of resistors assigned to the edges of a connected graph. Choose two vertices a and b and put a one-volt battery across these vertices establishing a voltage $V_a = 1, V_b = 0$. We are interested in the voltages V_x and the currents i_{xy} in the circuit, as well as giving a probabilistic interpretation to these quantities. We begin with the probabilistic interpretation of the voltage. We will interpret the voltage as a hitting probability.

Interpretation of voltage
When a unit voltage is applied between a and b, making $V_a = 1, V_b = 0$, the voltage V_x at any vertex x represents the probability that a walker starting from x will return to a before reaching b.

Since we have established what the voltage represents, we turn to the interpretation of the current.

Interpretation of current
When a unit current flows into a and out of b, the current i_{xy} flowing through the edge connecting x to y is equal to the expected net number of times that a walker, starting at a and walking until he reaches b, will move along the edge from x to y. These currents are proportional to the currents that arise when a unit voltage is applied between a and b, the constant of proportionality being the effective resistance of the network.

If we impose a voltage V between vertices a and b, then a voltage $V_a = V$ is established at a and $V_b = 0$, and a current $i_a = \sum_x i_{ax}$ will flow into the circuit from the outside source. The amount of current that flows depends upon the overall resistance of the circuit. We define the effective resistance R_{EFF} between a and b by

$$R_{EFF} = \frac{V_a}{i_a}, \quad C_{EFF} = \frac{1}{R_{EFF}} = \frac{i_a}{V_a}.$$

The reciprocal quantity C_{EFF} is also known as the effective conductance. We can interpret the effective conductance probabilistically as an escape probability, where the escape probability, is the probability, starting at a, that the walk reaches b before returning to a.

When we impose a voltage between a and b, voltages V_x are established at the vertices and currents i_{xy} flow through the resistors. We give a characterization of the currents in terms of a quantity called energy dissipation. When a current i_{xy} flows through a resistor, the energy dissipated is $i_{xy}^2 R_{xy}$, which is the product of the current i_{xy} and the voltage $V_{xy} = i_{xy} R_{xy}$. The total energy dissipation in the circuit is defined in the following way:

Definition 8 (**Total energy dissipation** [DS84])**.**

$$E = \frac{1}{2} \sum_{x,y} i_{xy}^2 R_{xy} = \frac{1}{2} \sum_{x,y} i_{xy}^2 (V_x - V_y).$$

The factor $\frac{1}{2}$ is necessary since each edge is counted twice. If a source establishes voltages V_a and V_b at a and b, then the energy supplied is $(V_a - V_b)i_a$, where $i_a = \sum_x i_{ax}$. By conservation of energy, we expect this to be equal to the energy dissipated.

We define a flow j from a to b to be an assignment of numbers j_{xy} to pairs xy such that:

Definition 9 (**Flow** [DS84])**.**

1. $j_{xy} = -j_{yx}$.
2. $\sum_y j_{xy} = 0$ *if* $x \neq a, b$.
3. $i_{xy} = 0$ *if* x *and* y *are not adjacent.*

We denote by $j_x = \sum_y j_{xy}$ the flow into x from the outside. We generalize the conservation of energy in the following way:

Definition 10 (**Conservation of energy** [DS84])**.** *Let w be any function defined on the vertices of the graph, and j a flow from a to b. Then*

$$(w_a - w_b) j_a = \frac{1}{2} \sum_{x,y} (w_x - w_y) j_{xy}.$$

Definition 11 (Kirchoff's current law). *Kirchoff's current law says that the algebraic sum of all currents entering and exiting a vertex must equal zero.*

Definition 12 (Thomson's principle). *If i is the unit flow from a to b determined by Kirchoff's law, then the energy dissipation $\frac{1}{2}\sum_{xy} i_{xy}^2 R_{xy}$ minimizes the energy dissipation current $\frac{1}{2}\sum_{xy} j_{xy}^2 R_{xy}$ among all unit flows j from a to b.*

2.2 Rayleigh's Monotonicity Law

In this section we study Rayleigh's monotonicity Law. This law from electrical network theory is an important tool in our study of random walks. Consider a random walk on streets as in Figure 3.

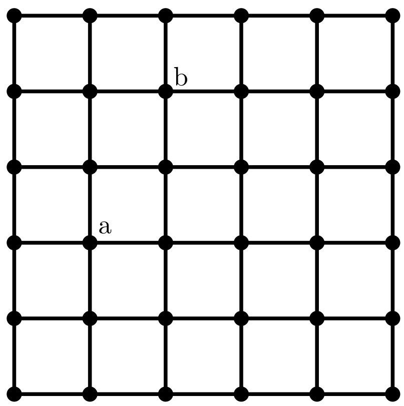


Figure 3: A grid where every edge between two vertices represents a resistor. We want to move from vertex a to vertex b.

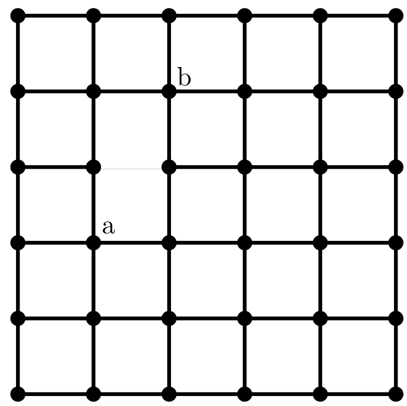


Figure 4: The same grid as in figure 3. This time one unit resistors is replaced by an infinite resistor.

Let P_{escape} be the probability that a walker starting from a reaches b before returning to a. Assign to each edge a unit resistance and maintain a voltage of one volt between a and b. A current i_a will flow into the circuit and we showed that $P_{escape} = \frac{i_a}{C_a}$. Now suppose that one of the streets that is not connected to a is blocked as illustrated in Figure 4.

We show that the probability of escaping to b from a is decreased compared to Figure 3. Consider this problem in terms of our network. Blocking a street corresponds to replacing a unit resistor by an infinite resistor. This has the effect of increasing the effective resistance R_{EFF} of the circuit between a and b. When we put a unit voltage between a and b, less current will flow into the circuit:

$$P_{escape} = \frac{i_a}{4} = \frac{1}{4R_{EFF}}.$$

Thus we need only show that when we increase the resistance in one part of a circuit, the effective resistance increases. This is where Rayleigh's Monotonicity Law comes in.

Theorem 13 (Rayleigh's Monotonicity's Law). *If the resistances of a circuit are increased, the effective resistance R_{EFF} between any two vertices can only increase. If they are decreased, it can only decrease.*

Proof. Let i be the unit current flow from a to b with resistors R_{xy}. Let j be the unit current flow from a to b with resistors $\bar{R}_{xy}$ with $\bar{R}_{xy} \geq R_{xy}$. Then

$$\bar{R}_{EFF} = \frac{1}{2}\sum_{x,y} j_{xy}^2 \bar{R}_{xy} \geq \frac{1}{2}\sum_{x,y} j_{xy}^2 R_{xy}.$$

Since j is a unit flow from a to b, Thomson's principle tells us that the energy dissipation, calculated with resistors R_{xy}, is larger than that for the true currents determined by these resistors, i.e.,

$$\frac{1}{2}\sum_{x,y} j_{xy}^2 R_{xy} \geq \frac{1}{2}\sum_{x,y} i_{xy}^2 R_{xy} = R_{EFF}.$$

Thus, $\bar{R}_{EFF} \geq R_{EFF}$. □

The proof for the case of decreasing resistances is analogous to the proof above.

2.3 Dirichlet Principle

For a large graph it can become complex to analyze the entire graph. Therefore, instead of drawing the entire graph, we can look at the incidence matrix, in which we look at which vertex is connected to which other vertices.

Definition 14 (**Unoriented Incidence Matrix**). *The unoriented incidence matrix of the undirected graph is a $n \times m$ matrix B, where n and m are the numbers of vertices and edges, respectively, such that*

$$B_{i,j} = \begin{cases} 1 & \textit{if vertex } v_i \textit{ is incident with edge } e_j, \\ 0 & \textit{otherwise.} \end{cases}$$

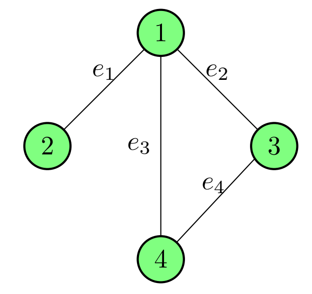


Figure 5: A simply connected graph G.

	e_1	e_2	e_3	e_4
1	1	1	1	0
2	1	0	0	0
3	0	1	0	1
4	0	0	1	1

Table 1: The table representing the unoriented incidence matrix of the simply connected graph G.

We have seen what the incidence matrix is. We can extend the idea to weighted graphs. Instead of indicating that some vertices are connected to some of the edges, now write the weights of the edges if a vertex is incident to an edge, i.e.

Definition 15 (**Unoriented Weighted Incidence Matrix**). *The unoriented incidence matrix of the undirected graph is a $n \times m$ matrix B, where n and m are the numbers of vertices and edges, respectively, such that*

$$B_{i,j} = \begin{cases} W_j & \textit{if vertex } v_i \textit{is incident with edge } e_j, \\ 0 & \textit{otherwise.} \end{cases}$$

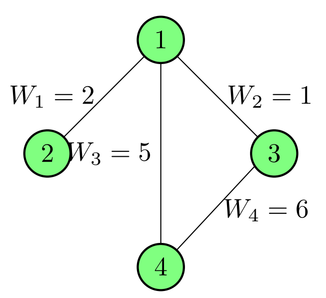


Figure 6: The simply connected weighted graph G'.

	e_1	e_2	e_3	e_4
1	2	1	5	0
2	2	0	0	0
3	0	1	0	6
4	0	0	5	6

Table 2: The table representing the weighted incidence matrix of the graph G'.

Instead of drawing the graph we can also give the Laplacian of the graph. The Laplacian of the graph is a matrix representation of the graph such that on the diagonal we get the degree of each vertex and if vertex i is incident to node j, then we give that a value, corresponding to the amount of edges that are between those nodes.

Definition 16 (**Laplacian**)**.** *Given a simple graph G with n vertices $v_1, \cdots, v_n$, its $n \times m$ Laplacian matrix L is defined element-wise as*

$$L_{i,j} = \begin{cases} deg(v_i), & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

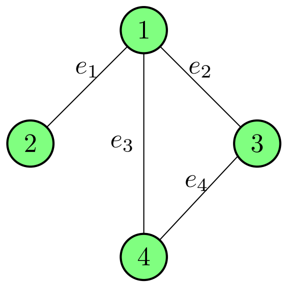


Figure 7: The simple connected graph G.

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

Figure 8: The Laplacian matrix of the simply connected graph G.

Definition 17 (**Weighted Laplacian**)**.** *Given a simple weighted graph G with n vertices $v_1, \ldots, v_n$, its weighted Laplacian matrix L is defined element-wise as*

$$L_{i,j} = \begin{cases} W_i, & \text{if } i = j \\ W_{i,j}, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

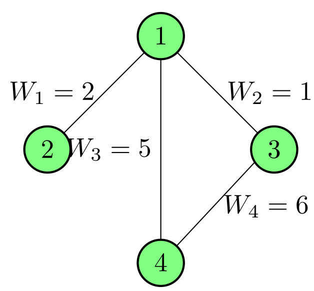


Figure 9: The simply connected weighted graph G'.

$$\begin{pmatrix} 8 & 2 & 5 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 7 & 6 \\ 5 & 0 & 6 & 11 \end{pmatrix}$$

Figure 10: The matrix representing the weighted Laplacian of the graph G'.

Right now we have all the ingredients to start the proof of the Dirichlet Principle. Before we get into that, you might ask yourself, "*What exactly is the Dirichlet Principle?*".

Theorem 18 (**Dirichtlet's Principle**)**.** *The dissipation energy $\mathcal{E}(\vec{x})$ is minimized by the voltage at the vertices satisfying Ohm's Law and Kirchoff's Current Law at the interior vertices.*

The following figure represents an electrical network, where we have placed a voltage meter across the red and blue nodes. In this figure we see the incidence matrix, the conductance matrix and the weighted Laplacian. What we want to find is the vector $\vec{x}$ such that we minimize the dissipation energy $\mathcal{E}(\vec{x})$. $\mathcal{E}(\vec{x})$ is a scalar function of the potentials at the nodes.

We write $\vec{x}$ to denote the vector of potentials at the nodes. A is our incidence matrix, C is our conductance matrix. We define A^TCA to be the weighted Laplacian and $\mathcal{E}(x) = x^T(A^TCA)x$. We observe that this is just a quadratic function and so we can write it in terms of P, Q and R.

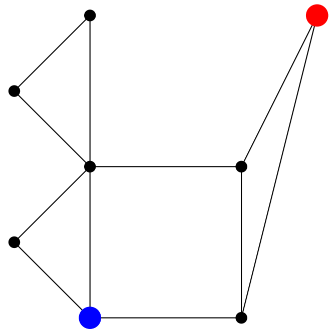


Figure 11: An electrical network. The red dot represents the positive vertex and the blue dot the negative vertex.

Proof. The first part handles the question: *"What is the solution that satisfies Kirchoff's current law here?"*

$$\vec{f} = L\vec{x}$$

is the key equation that relates the divergence of the currents of the vertices at f to the voltage potentials, where L is the Laplacian.

Let us write this in block form and explain what we are doing here. We split the vector f into three components. The plus vertex, the minus vertex and the interior vertices. These are, respectively, equal to the values $C_{eff}, -C_{eff}$ and 0 due to Kirchoff's Current Law. On the right-hand side we have the Laplacian matrix which we can write in terms of P, Q and R. Finally, we have the vector $(\vec{e}, \vec{x})^T$, which is an alternative way of writing the potentials of the plus and minus vertex in one vector $\vec{e}$ and all the potentials of the interior nodes as a vector $\vec{x}$:

$$\vec{f} = L\vec{x},$$
$$\begin{pmatrix} C_{eff} \\ -C_{eff} \\ \vec{0} \end{pmatrix} = \begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix} \begin{pmatrix} \vec{\hat{e}} \\ \vec{\hat{x}} \end{pmatrix}, \quad \vec{\hat{e}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In other words, the potential at the plus vertex C_{eff} is 1, the potential at the minus vertex $-C_{eff}$ is 0. The current into the circuit out of the plus vertex is C_{eff} and the current out of the minus vertex is $-C_{eff}$. and the current in and out of the all the other vertices corresponds to the third vector component 0 (KCL).

Let us write $\vec{f}$ as: $\vec{\hat{f}} = \begin{pmatrix} C_{eff} \\ -C_{eff} \end{pmatrix}$ and let us expand this to determine on the one hand the currents for the plus and minus vertices and on the other hand the currents for the interior vertices:

$$\begin{pmatrix} \vec{\hat{f}} \\ \vec{0} \end{pmatrix} = \begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix} \begin{pmatrix} \vec{\hat{e}} \\ \vec{\hat{x}} \end{pmatrix} \implies \vec{\hat{f}} = P\vec{\hat{e}} + Q^T\vec{\hat{x}} \quad , \quad \vec{0} = Q\vec{\hat{e}} + R\vec{\hat{x}}.$$

Now we have created two expressions, one for the currents for the positive and negative vertices and one for the currents for the interior nodes.

The R is non-singular (determinant $\neq 0$) and, in fact, positive definite, since all the values on the diagonal of L are > 0. This implies that $\vec{\hat{x}} = -R^{-1}Q\vec{\hat{e}}$ must be the solution satisfying Kirchoff's current law at the interior vertices and Ohm's law at the edges. Thus apparently $(\vec{\hat{e}}, \vec{\hat{x}})$ minimizes the dissipation energy, as we show next.

We consider the dissipation function and expand it out:

$$\begin{aligned}
\mathcal{E}(\vec{x}) &= \begin{pmatrix} \vec{e}^T & \hat{\vec{x}}^T \end{pmatrix} \begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix} \begin{pmatrix} \vec{e} \\ \hat{\vec{x}} \end{pmatrix} \\
&= \begin{pmatrix} \vec{e}^T & \hat{\vec{x}}^T \end{pmatrix} \begin{pmatrix} P\vec{e} + Q^T\hat{\vec{x}} \\ Q\vec{e} + R\hat{\vec{x}} \end{pmatrix} \\
&= \vec{e}^T P\vec{e} + \vec{e}^T Q^T \hat{\vec{x}}^T + \hat{\vec{x}}^T Q\vec{e} + \hat{\vec{x}}^T R\hat{\vec{x}} \\
&= \color{red}{\vec{e}^T P\vec{e}} + \hat{\vec{x}}^T R\hat{\vec{x}} + 2\hat{\vec{x}}^T Q\vec{e} \\
&= \hat{\vec{x}}^T R\hat{\vec{x}} + 2\hat{\vec{x}}^T Q\vec{e} + C
\end{aligned}$$

The term highlighted in red is a term that does not depend on the potentials at the interior nodes, and so we can see that as a constant C. Let us consider the following expression:

$$\begin{aligned}
(\hat{\vec{x}} + R^{-1}Q\vec{e})^T R(\hat{\vec{x}} + R^{-1}Q\vec{e}) &= \hat{\vec{x}}^T R\hat{\vec{x}} + (R^{-1}Q\vec{e})^T R\hat{\vec{x}} + \hat{\vec{x}}^T RR^{-1}Q\vec{e} + C \\
&= \hat{\vec{x}}^T R\hat{\vec{x}} + 2\hat{\vec{x}}^T Q\vec{e} + C
\end{aligned}$$

We observe that this expression is equal to the expression satisfying Kirchoff's current law. Thus instead of writing the dissipation energy as $\mathcal{E}(\vec{x}) = \vec{x}^T(A^TCA)\vec{x}$ we can write the dissipation function as

$$\begin{aligned}
\mathcal{E}(\vec{x}) &= (\hat{\vec{x}} + R^{-1}Q\vec{e})^T R(\hat{\vec{x}} + R^{-1}Q\vec{e}) + \mathrm{C} \\
&= \hat{\vec{X}}^T R\hat{\vec{X}} + \mathrm{C}
\end{aligned}$$

As we already know, R is a positive definite matrix and thus the minimum occurs when $\hat{\vec{X}}$ is zero, which means that

$$\min(\mathcal{E}) \iff \hat{\vec{X}} = 0$$

$$\begin{aligned}
\hat{\vec{X}} &= 0 \\
\hat{\vec{X}} &= \hat{\vec{x}} + R^{-1}Q\vec{e} = 0 \\
\hat{\vec{X}} = 0 &\iff \hat{\vec{x}} = -R^{-1}Q\vec{e}
\end{aligned}$$

which is exactly the solution that satisfies Kirchoff's Current Law. Hence the dissipation energy $\mathcal{E}(\vec{x})$ is minimized by the voltage at the vertices satisfying Ohm's Law and Kirchoff's Current Law at the interior vertices. □

You might ask yourself now: *"How we can we relate this back to random walks?", "What is the purpose of this?"* Translating the random walk to the electrical network gives us a new way to characterize simple random walks and provides us with another way to prove recurrence or transience.

To determine $p^{(r)}_{\text{escape}}$ electrically, we simply ground all the points of $S^{(r)}$, the points with the same ℓ^1-norm, maintain **0** at one volt, and measure the current $i^{(r)}$ flowing into the circuit. From paragraph 3.4 in [DS84], we know that

$$p^{(r)}_{\text{escape}} = \frac{i^{(r)}}{2d},$$

where d is the dimension of the lattice. Since the voltage being applied is 1, $i^{(r)}$ is just the effective conductance between **0** and $S^{(r)}$, i.e.,

$$i^{(r)} = \frac{1}{R^{(r)}_{EFF}},$$

where $R_{EFF}^{(r)}$ is the effective resistance from $\mathbf{0}$ to $S^{(r)}$. Thus

$$p_{\text{escape}}^{(r)} = \frac{1}{2dR_{\text{EFF}}^{(r)}}.$$

Define R_{EFF}, the effective resistance from zero to infinity, to be

$$R_{\text{EFF}} = \lim_{r\to\infty} R_{\text{EFF}}^{(r)}.$$

This limit exists since $R_{\text{EFF}}^{(r)}$ is an increasing function of r by Rayleigh's Monotonicity's Law, Theorem 13. Hence

$$p_{\text{escape}} = \frac{1}{2dR_{EFF}}.$$

Of course, R_{EFF} may be infinite. In fact, this will be the case if and only if $p_{\text{escape}} = 0$. Thus, the walk is recurrent if and only if the resistance to infinity is infinite.

2.4 Recurrence in Electrical Networks

The random walk is recurrent if and only if the resistance to infinity is infinite or, in other words, if the escape probability, $P_{\text{escape}} = 0$.

We first look at the one dimensional case. Since an infinite line of resistors obviously has infinite resistance, it follows that simple random walk on $\mathbb{Z}^1$ is recurrent.

We are asked to decide whether a d-dimensional lattice has infinite resistance to infinity. The difficulty is that the d-dimensional lattice $\mathbb{Z}^d$ lacks the rotational symmetry of the Euclidean space in which it sits. To see how this lack of symmetry complicates electrical problems, we determine, by solving the appropriate discrete Dirichlet Problem, the voltages for a one volt battery attached between 0 and the vertices of $S^{(3)}$ in $\mathbb{Z}^2$. The resulting voltages are:

			0			
		0	.091	0		
	0	.182	.364	.182	0	
0	.091	.364	1	.364	.091	0
	0	.182	.364	.182	0	
		0	.091	0		
			0			

Figure 12: The voltages at the different distances $S^{(i)}$ calculated by the help of the Dirichlet's Principle.

The voltages at the points of distance $S^{(1)}$ are equal, but the voltages at points of $S^{(2)}$ are not. This means that the resistance from $\mathbf{0}$ to $S^{(3)}$ cannot be written simply as the sum of the resistances from $\mathbf{0}$ to $S^{(1)}$, $S^{(1)}$ to $S^{(2)}$ and $S^{(2)}$ to $S^{(3)}$. Thus, there is a lack of rotational symmetry.

The method for solving this is to modify the two-dimensional resistor network by "shorting" certain sets of vertices together to get a network where we can readily see that the resistance is infinite.

Short-circuiting involves connecting a given set of nodes with perfectly conducting wires, so that the current can pass freely between them.

Definition 19 (Shorting Law). *Short-circuiting certain sets of vertices together can only decrease the effective resistance of the network between two given vertices.*

When the dimension is 2, we apply the short-circuiting as follows: Short together all vertices on the boundary of the squares around the origin, as shown below:

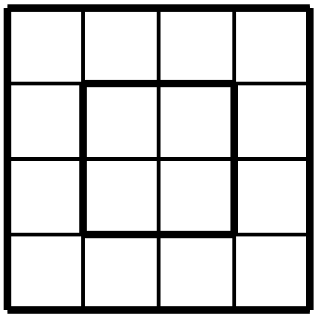


Figure 13: A network on $\mathbb{Z}^2$, the thick lines are the set of vertices that are short circuited.

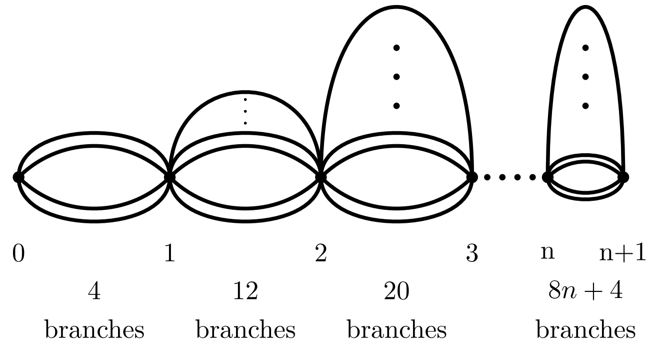


Figure 14: The representation of the short circuited grid in Figure13

If we continue this method with a larger grid, $S^{(n+1)}$ and we short circuit these edges in the same way as in Figure 13 and 14, then we can extrapolate a formula for the number of edges short-circuited to $8n + 4$.

Since the resistors in Figure 14 are in parallel, the resistance between every $S^{(n)}$ is equivalent to one resistor with a resistance of the reciprocal value. Thus, we get that the resistance of the entire network is equal to

$$\sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty$$

and since the resistance of a network to infinity is infinite, we have shown that the simple random walk in two dimensions is recurrent.

3 Finite domains and boundary points

So far we have mostly talked about infinite domains, so you might ask yourself: *"What happens on a finite connected domain?"* and especially: *"What happens at the boundary points?"*

Before we talk about boundary points, we need to specify what sort of random walk we are observing. In this case we look at a *confined simple random walk.*

Definition 20 (**Confined Simple Random Walk**). *A confined simple random walk is a simple random walk which is bounded by domain. The confined simple random walk can only move inside this domain.*

At a boundary point, the confined simple random walk cannot make all four steps. Depending on what boundary point we have, the random walk can make μ different steps, where $\mu \in \{1, 2, 3\}$. The question that remains is: *"What happens to the probability of the remaining steps?* We uniformly distribute the probability of the non-valid step over the valid steps the walk can make, i.e., $P(X_i(\omega) = \frac{1}{4} + \frac{1}{4}\frac{4-\mu}{\mu}$. An example of the probability distribution is shown in Figure 15.

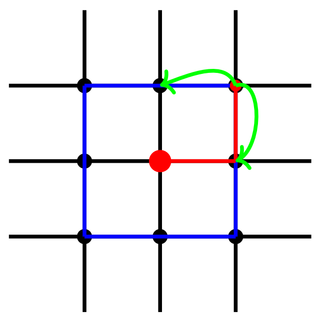


Figure 15: Simple random walk on a finite domain reaching a boundary point. The two green arrows represent the steps the random walk can possibly take. The probability of taking one of the possible steps is increased from $\frac{1}{4}$ to $\frac{1}{2}$.

3.1 Simple Random Walk on a domain in $\mathbb{Z}^2$

It may seem trivial that, because the simple random walk on $\mathbb{Z}^2$ is recurrent, the simple random walk on any domain in $\mathbb{Z}^2$ is also recurrent. In this chapter we introduce a metric ℓ^1 on $\mathbb{Z}^2$ and establish a property that is sharper than recurrence for a class of domains that have the so-called **$\star$-property**. We will define what it means to have the **$\star$-property**, introduce a coupling of two simple random walks, and show that the simple random walk on a domain with the **$\star$-property** is stochastically closer to the origin in ℓ^1-distance than the simple random walk on $\mathbb{Z}^2$.

Definition 21 (**The ℓ^1-norm on $\mathbb{Z}^2$**). *We define the ℓ^1-norm of x to be the following: $||x||_1 = |x_1| + |x_2|$. This norm is also known as the Manhattan norm.*

We have defined a metric, the ℓ^1-norm. *"What do we want to do with this metric?"* We want to talk about two random walks, where one of the random walks is closer to the origin compared to the other random walk. For this we can use the notion of statistical distance and stochastic ordering.

Definition 22 (**Statistical Distance**). *[202a] A statistical distance quantifies the distance between two statistical objects, these two random variables, or two probability distributions, or two samples, or the distance between an individual sample point and a population sample points.*

A stochastic order quantifies the concept of one random variable being "larger" than another. These are usually partial orders, so that one random variable **A** may be neither stochastically larger than, smaller than or equal to another random variable **B**.

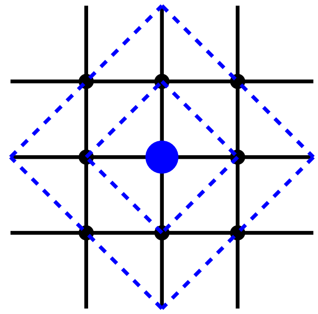


Figure 16: Sets of vertices with the same ℓ^1-distances to the origin.

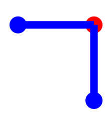
Definition 23 (**Stochastic ordering**). *[202b] A real random variable $\boldsymbol{A}$ is less than a random variable $\boldsymbol{B}$ in stochastic order written $\boldsymbol{A} \preceq \boldsymbol{B}$ if*

$$P(\boldsymbol{A} > x) \leq P(\boldsymbol{B} > x) \text{ for all } x \in (-\infty, \infty),$$

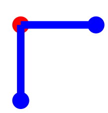
we want to show that one random walk is stochastically closer to the origin than another random walk.

A confined simple random walk is of course bounded inside a domain. The immediate question that comes to mind is *"What kind of domains?"* In this paper we look at connected domains that fulfill the $\star$-property.

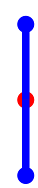
Definition 24 (**The $\star$-property**). *A bounded domain D has the $\star$-property if for every boundary point, i.e., every point in ∂D, there are at least as many steps that lead into the domain as steps that lead you out of the domain.*



(a) A boundary with two steps that leads into the domain and two steps that lead out of the domain.



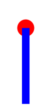
(b) A boundary with two steps that leads into the domain and two steps that lead out of the domain.



(c) A boundary with three steps that leads into the domain and one step that leads out of the domain

(d) A boundary with one step that leads into the domain and three steps that lead out of the domain.

(e) A boundary with one step that leads into the domain and three steps that lead out of the domain.



(f) A boundary with one step that leads into the domain and three steps that lead out of the domain.

Figure 17: Examples of boundary points on $\mathbb{Z}^2$. Figure 5(a),5(b),5(c) satisfy the $\star$-property on $\mathbb{Z}^2$. Figures 5(d),5(e), 5(f) are called tadpoles, which do not fulfill the $\star$-property.

In our goal of proving the recurrence of finite domains we have reached the point where we are looking at the ℓ^1-metric and the $\star$-property of domains and instead of recurrence we are looking at something stronger, namely one random walk being stochastically closer to the origin than the other random walk. From those points, the following theorem arises:

Theorem 25. *A confined simple random walk with a domain that conforms to the $\star$-property is stochastically closer in ℓ^1-norm than the non-confined simple random walk.*

Before we can prove this theorem we need to connect the two random walks. For this we use a technique called Markovian coupling.

3.2 Markovian coupling

In this section we define what a coupling is. We couple two simple random walks on the positive quadrant of $\mathbb{Z}^2$. Under the coupling we show that the random walks are stochastically ordered with respect to the ℓ^1-metric, when the domain of the confined random walk has the $\star$-property, in definition 27.

Definition 26 (**Markovian Coupling**)**.** *Let X_1 and X_2 be two random variables defined on probability spaces (Ω_1, F_1, P_1) and (Ω_2, F_2, P_2). A coupling of X_1 and X_2 is any probability space (Ω, F, P) on which there are two random variables Y_1 and Y_2, such that Y_1 has the same distribution as X_1 while Y_2 has the same distribution as X_2.*

In other words, a coupling of two random variables X and $X^{'}$ taking values in $(E, \mathcal{E})$ is any pair of random variables $(\widehat{X}, \widehat{X^{'}})$ taking values in $(E \times E, \mathcal{E} \bigotimes \mathcal{E})$ such that the marginals have the same distribution as X and $X^{'}$. Since we are working with Markov chains and are only interested in the location at time t, the coupling that we willbe constructing is known as a Markovian coupling [Hol]. In our coupling, the non-confined simple random walk first makes a random step and the confined simple random walk either copies that step when that step keeps it in the domain, or otherwise makes a random step over the allowed steps on its own.

Construction

Let us consider two simple random walks on the positive quadrant of $\mathbb{Z}^2$. $(\widehat{S}_n)_{n=0}^{\infty}$ is the simple random walk on $\mathbb{Z}^2$ and $(\widehat{S}^{'}_n)_{n=0}^{\infty}$ is the simple random walk on a confined connected domain $D \subseteq \mathbb{Z}^2$, containing the origin. We are going to couple these two simple random walks in the following way. Let $\widehat{S}^{'}_n$ follow the same step as $\widehat{S}_n$ when possible. If $\widehat{S}_n$ makes a step that is not possible for $\widehat{S}^{'}_n$, then we split the probabilities over the possible steps that $\widehat{S}^{'}_n$ can make, when $\widehat{S}_n$ increases its ℓ^1 distance. If $\widehat{S}_n$ makes a step that is not possible for $\widehat{S}^{'}_n$ and $\widehat{S}_n$ decreases its ℓ^1 distance, then $\widehat{S}^{'}_n$ also makes a move that decreases its ℓ^1 distance.

So, we first look at the non-confined simple random walk, and then determine what step the confined simple random walk, will or can make. The simple random walks are defined as $\widehat{S}_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $\widehat{S}^{'}_n(\omega) = \sum_{k=1}^{n} X^{'}_k(\omega)$. The coupling is:

$$
\begin{cases}
X^{'}_i(\omega) = X_i(\omega) & \text{if } \widehat{S}^{'}_i(\omega) = \sum_{k=1}^{i-1} X^{'}_k(\omega) + X_i(\omega) \in D. \\
 & \\
X^{'}_i(\omega) = \omega^{'}_i & \text{if } \widehat{S}^{'}_i(\omega) = \sum_{k=1}^{i-1} X^{'}_k(\omega) + X_i(\omega) \notin D. \\
 & \text{If } ||(\omega_1, \cdots, \omega_i)||_1 > ||(\omega_1, \cdots \omega_{i-1})||_1, \\
 & \text{then take step } \omega^{'}_i \text{ with probability } P(X_i(\omega^{'}_i)) = \frac{1}{4} + \frac{1}{4}\frac{4-\mu}{\mu}. \\
 & \text{and if } ||(\omega_1, \cdots, \omega_i)||_1 < ||(\omega_1 \cdots \omega_{i-1})||_1, \\
 & \text{then take step } \omega^{'}_1 \text{ such that } ||(\omega_1, \cdots, \omega^{'}_i)||_1 < ||(\omega_1 \cdots \omega_{i-1})||_1.
\end{cases}
$$

Before we continue, observe that under this coupling we can extend the $\star$-property with tadpoles pointing away from the origin.

In figure 5(d), 5(e) and 5(f), we see that we do not fulfill the $\star$-property. We have three steps to go out of the domain and one step to stay in the domain. However, under the coupling we see that if the non-confined random walk decreases its ℓ^1-distance, then the confined random walk also decreases its ℓ^1-distance. Thus, tadpoles are allowed, however, they needs to be placed in such a way that if the ramdom walk starts on that tadpole, it increases the ℓ^1 distance. An example is shown in Figure 18.

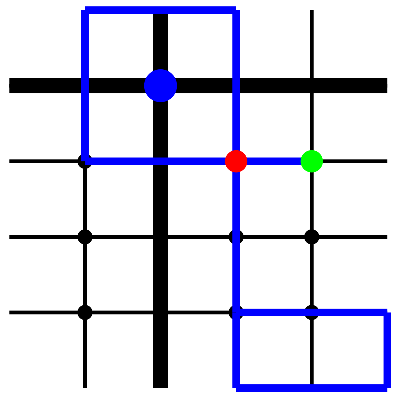


Figure 18: A domain with a tadpole placed in such a way that if we stand on the green dot and take a step towards the domain, we decrease on ℓ^1 distance.

Suppose that we stand on the red dot in Figure 18. Suppose that we take the path along the tadpole. Then both walks sit at the green dot. The confined simple random walk has only one possible step, which decreases the ℓ^1-distance, whereas the non-confined simple random walk has four available steps; two of these steps decrease the ℓ^1-distance towards the origin and two steps increase the ℓ^1 distance towards the origin. Thus the ℓ^1 distance of the confined simple random walk is stochastically less than or equal to the distance of the non-confined simple random walk. So, our domain is allowed to have tadpoles, as long as the step into the tadpole increases the ℓ^1 distance to the origin.

Thus our domain is allowed to have boundaries such as in Figure 5(a),5(b) and 5(c), but also boundaries such as in Figure 5(d),5(e) and 5(f), provided they are pointing away from the point of return.

Definition 27 (**The augmented $\star$-property**). *A bounded domain has the augmented $\star$-property if for every boundary point there are at least as many steps that lead inwards than outwards, plus tadpoles that increase the ℓ^1 distance.*

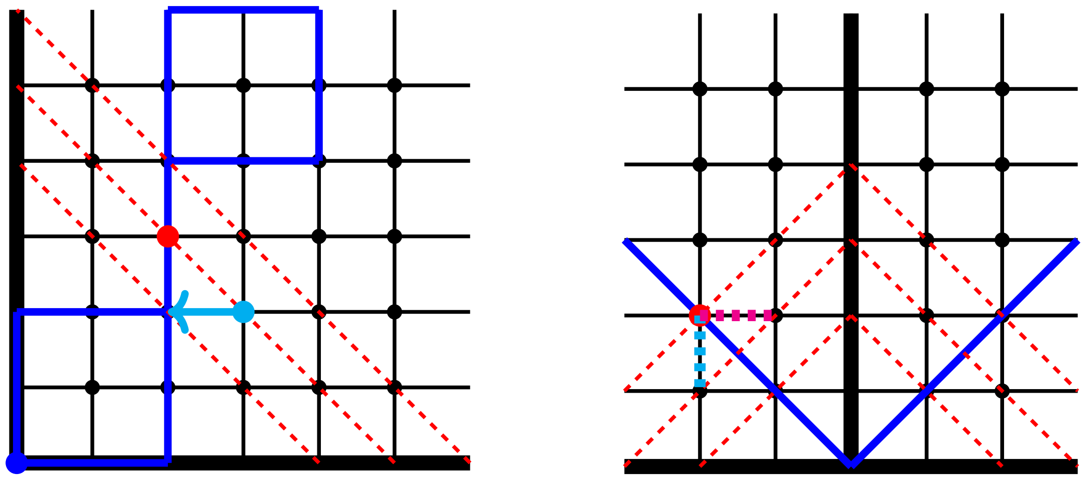
Thus we can rephrase Theorem 25 as:

Theorem 28 (**Recurrence in an augmented $\star$-property domain.**). *The confined simple random walk on an augmented $\star$-property domain is stochastically closer to the origin than the non-confined random walk under the ℓ^1-metric.*

You might ask: *"Why do we need such a complicated coupling?* The reason is that the coupling must be such that it keeps the two simple random walks stochastically ordered in ℓ^1-distance.

Assume in Figure 19a that we have omitted the rule that if the non-confined random walk decreases its ℓ^1-distance then the confined random walk also decreases its ℓ^1-distance. $\widehat{S}_n$ goes to the left with probability $\frac{1}{4}$. This move, however, is not an option for $\widehat{S}_n'$. Since $\widehat{S}_n'$ does not necessarily need to follow the decrease in ℓ^1 distance of $\widehat{S}_n$, the $\widehat{S}_n'$ increases its ℓ^1 distance with probability $\frac{1}{2}$, and it decreases its ℓ^1 distance also with probability $\frac{1}{2}$. Hence we cannot guarantee that the ℓ^1 distance of the confined simple random walk is always the same or less than the non-confined simple random walk. Hence we cannot omit this rule in the coupling.

We have assumed that our random walk finds itself on the positive quadrant of $\mathbb{Z}^2$. You might wonder: *"Does the coupling also work on other quadrants?"* In Figure 19b we have rotated our boundary of the positive quadrant by 45 degrees and we can observe that our coupling also holds for rotations of a quadrant, hence it works for all quadrants.
Finally, we are going to show that $\widehat{S}_n'$ is always closer than or equal in distance to the origin than $\widehat{S}_n$ on the positive quadrant of $\mathbb{Z}^2$.



(a) An example where the confined simple random walk would not necessarily be less in distance then the non-confined simple random walk, when we do not add the clause: if the non-confined random walk decreases its ℓ^1-distance, then the confined random walk must decrease its ℓ^1-distance.

(b) A rotation of the positive quadrant of $\mathbb{Z}^2$. Having a point on the 4th quadrant and illustrate what the possible steps are for the confined random walk in the non-confined random walk makes a move that decreases its ℓ^1-distance.

Figure 19: The left figure illustrating why we need such a complicated coupling. The right figure is a rotation of the positive quadrant of $\mathbb{Z}^2$.

Proof. The proof is by induction. Assume that we have two simple random walks on the positive quadrant of $\mathbb{Z}^2$, both starting at the origin: $(\widehat{S}_n^{'})_{n=0}^{\infty}$ is a confined simple random walk and $(\widehat{S}_n)_{n=0}^{\infty}$ is a non-confined simple random walk. We have coupled these simple random walks as mentioned above.

Base Step: $n = 1$
Under the coupling, both simple random walks increase their ℓ^1-distance by 1. Hence the ℓ^1-distance of the confined simple random walk is the same as the one of the non-confined simple random walk.

Induction hypothesis
Assume that the ℓ^1-distance of the confined simple random walk is less than the ℓ^1-distance of the non-confined simple random walk for $0 \leq n \leq N$.

Induction step Look at the $(N+1)$st step and the different types of boundaries.
If both walks are in the interior of the domain and have not reached the domain yet, then under the coupling they make the exact same steps and thus have the same ℓ^1-distance.

If at the Nth step we reach the boundary ∂D of D, and the non-confined simple random walk leaves the domain in the $(N+1)$st step, then we look at the following boundaries (rotations excluded):

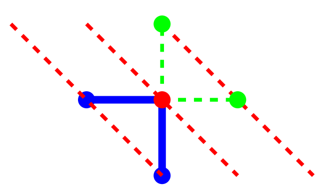


Figure 20: The two steps of this boundary point keep the random walk in the domain and decrease the ℓ^1-distance.

If the non-confined simple random walk makes one of the green steps with probability $\frac{1}{4}$, then it increases the ℓ^1-distance, whereas for the confined simple random walk only two available steps are possible, each with probability $\frac{1}{2}$. The coupling makes sure that these random walks are stochastically ordered.

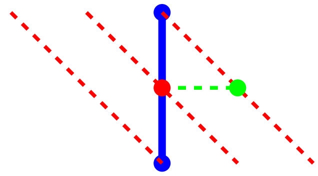


Figure 21: A boundary point where the confined random walk has three possible steps. Two of these steps decrease the ℓ^1-distance.

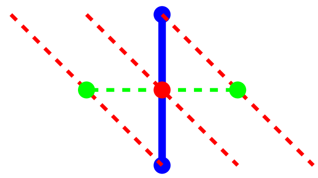


Figure 22: A boundary point that can be placed in different ways either two moves that decrease the ℓ^1 distance or one.

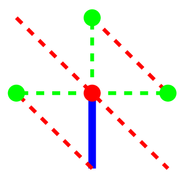


Figure 23: A boundary point that only has one move that keeps the confined random walk in the domain. This move decreases the ℓ^1-distance on the positive quadrant of $\mathbb{Z}^2$ if it is placed correctly.

If the non-confined simple random walk makes the step to the right with probability $\frac{1}{4}$, then it increases the ℓ^1-distance, whereas for the confined simple random walk, there are three steps possible, each with probability $\frac{1}{3}$. The coupling makes sure that the ℓ^1-distance of the confined random walk is smaller than the non-confined random walk.

If the non-confined simple random walk makes the step to the left with probability $\frac{1}{4}$, then it decreases the ℓ^1-distance, whereas for the confined simple random walk, there are two steps possible, each with probability $\frac{1}{2}$. The coupling makes sure that the ℓ^1-distance of the confined random walk is smaller than or the same distance as the non-confined random walk.

The non-confined simple random walk makes any step with probability $\frac{1}{4}$. Two of the steps increase the ℓ^1-distance, and the other two steps decrease the ℓ^1-distance. The confined simple random walk only has one step available and takes that with probability 1. This step decreases the ℓ^1-distance. Hence the coupling again preserves the distance.

Thus, for all of the boundaries shown we see that the coupling preserves the fact that the ℓ^1-distance of the confined simple random walk is less than or equal to the ℓ^1-distance of the non-confined simple random walk. We can conclude by means of induction under the coupling the ℓ^1-distance of the confined simple random walk is always less than or equal to the distance of the non-confined simple random walk. □

Since the ℓ^1 distance is symmetric, the same argument applies to the other three quadrants.

Theorem 29 (**Recurrence of Simple Random walk with the augmented $\star$-property**). *The confined simple random walk on a domain D with the augmented $\star$-property is recurrent.*

Proof. Suppose that we have a simple random walk on a connected domain with boundaries such as in Definition 27. Then by Theorem 25, the confined simple random walk is stochastically closer to the origin than the non-confined random walk. Since it is known that the non-confined random walk on $\mathbb{Z}^2$ is recurrent with respect to the origin with probability 1. Then by Theorem 25 the confined simple random walk does so too. □

3.3 Complications

The coupling we build rests on the premise that it works for all domains that fulfill the augmented $\star$-property, (See Definition 27). In Figure 2a both simple random walks find themselves at a boundary point. The non-confined simple random walk, finds itself at the coordinate (4,0) with an ℓ^1-norm of 4. The confined simple random walk finds itself at the coordinate (2,3), with an ℓ^1-norm of 5. This situation would infringe on the argument in the proof of Theorem 25 . However, this placement of the two random walks can never occur. We might ask ourselves: *"How did the non-confined simple random walk get to the position it is in now?"* If we look one step in the past, then it had to come from the right or from above. Since the confined random walk, the green dot, follows the non-confined random walk, we see that the green dot can never be in the spot it is in now. Hence this situation is not possible.

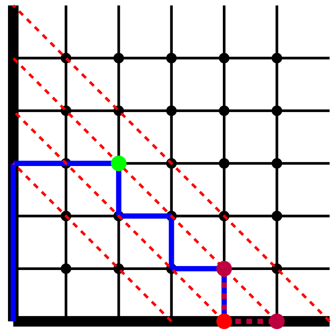


Figure 24: A situation that would impose a contradiction for our coupling. However, the situation where the non-confined simple random walk is on the x-axis with an ℓ^1-norm of 4 and the confined simple random walk is on the boundary with an ℓ^1-norm of 5 cannot arise. The non-confined random walk made a step from above or from the right to land in this position, which decreases the ℓ^1-distance.

4 Evolving Domains

We have shown that the one- and two-dimensional simple random walk are recurrent, we have looked at electrical networks and random walks and presented a proof that a random walk inside a domain with the augmented $\star$-property (See Definition 27) is recurrent. In this chapter we define rules for evolving domains, phrase and prove the main research question of this paper, and provide some open questions.

What exactly is an evolving domain? We are looking at a discrete-time Markov process, the random walk, which evolves at every time step t. We define an evolving domain to be a domain that alters at every time step t. The evolution of the domain must abide by the following rules:

- The domain must always contain the origin (0,0).
- The domain must be a connected domain.
- The domain must at every time step be an augmented $\star$-property domain.
- The domain evolves independently of the random walk.

Formalizing this we arrive at the following definition:

Definition 30 (Random walk on the evolving domain). *We call $(\hat{S}'_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ a confined simple random walk on an evolving domain if the connected, augmented $\star$-property domains $\mathbb{D}_n \subseteq \mathbb{Z}^2$ containing the origin are such that the confined simple random walk, defined on the time-space domain $\mathcal{D} := \{(n,x) \in \mathbb{N}\times\mathbb{Z}^2 : x \in \mathbb{D}_n\}$, is well-defined and is conditionally Markov.*

Note that he random walk has the Markov Property, not the domain.

The main theorem of this bachelor thesis is:

Theorem 31. *$(\hat{S}'_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ as in Definition 30 is recurrent.*

4.1 Natural Evolving Domains

Before we prove this theorem, the following question arises *"What is a natural way for the domain to evolve?"* A few ideas come to mind: A deterministic approach, i.e., the domain evolves in a certain way. A non-deterministic approach, i.e., at every different boundary point depending on the surroundings of the domain will evolve differently, a probabilistic approach, i.e., the boundary point evolves of a certain rate.

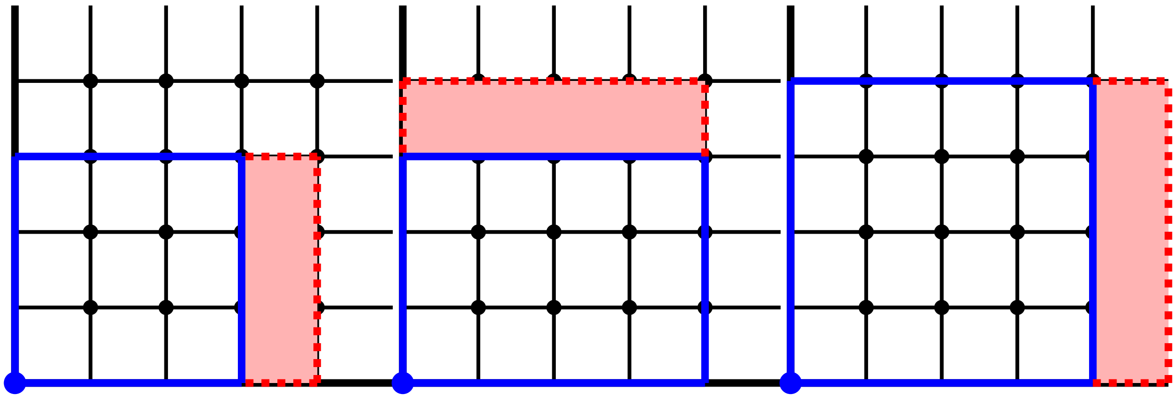
An example of a deterministic evolving domain is when we start with a domain of the form of a square on the positive quadrant of $\mathbb{Z}^2$, and we evolve in the following way: At an even time step we add one layer of height and at an odd time step we add one layer of width.

In chapter 2 we explained the conversion of a simple random walk into an electrical network, and now we ask ourselves *"Can we use that natural translation into electrical networks on evolving domains?"*

Unfortunately, The Dirichlet Principle does not hold for evolving domains. The Dirichlet Principle depends on the use of the total effective resistance. When our domain is evolving, the total effective resistance evolves as well, and thus the techniques used in chapter 2 do not apply to evolving domains.

To answer the question stated, *"What is a natural way for the domain to evolve?"*, there are several natural ways for a domain to evolve naturally. Since the simple random walk on a finite domain on $\mathbb{Z}^2$ is recurrent we will look at evolving domains that increase in size. We give an example of a deterministic evolving domain, a non-deterministic evolving domain and finally a probabilistic evolving domain.

A deterministic evolving domain evolves according to a set of rules. In this example we first expand the right boundary by 1 and then in the next time-step we expand the upper boundary by 1.

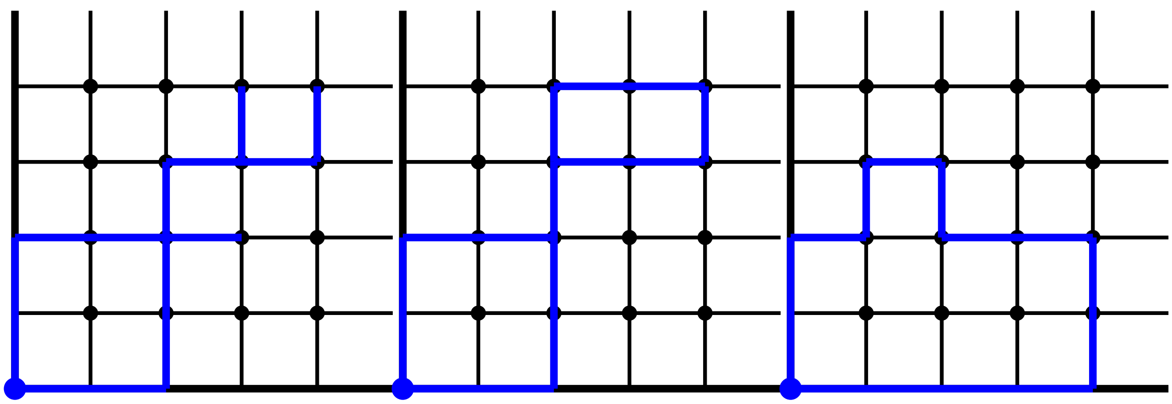


(a) A deterministic evolving domain at time step $t = 0$. Highlighted in red is the augmented part in the next step.

(b) A deterministic evolving domain at time step $t = 1$. Highlighted in red is the augmented part in the next step.

(c) A deterministic evolving domain at time step $t = 2$. Highlighted in red is the augmented part in the next step.

What does it mean to be a non-deterministic evolving domain? For the same input, we do not necessarily get the same output. Suppose we have the same input, Figure 26a, then we can get different outcomes such as in Figure 26b and 26c.



(a) A non-deterministic evolving domain at timestep $t = 0$.

(b) A non-deterministic evolving domain at time step $t = k$.

(c) A non-deterministic evolving domain at time step $t = k$.

4.2 Recurrence of Simple Random Walk on Evolving Domain

For proving the main theorem of the paper we again look at the two random walks and we want to show that they are stochastically ordered. We again use the coupling method for a confined and non-confined random walk in Definition 31. As a matter of fact, we use the same coupling, however, this time we use evolving domains, $\mathbb{D}_n$:

$$\begin{cases} X_i^{'}(\omega) = X_i(\omega) & \text{If } \widehat{S}_i^{'}(\omega) = \sum_{k=1}^{i-1} X_k^{'}(\omega) + X_i(\omega) \in \mathbb{D}_n. \\ \\ X_i^{'}(\omega) = \omega_i^{'} & \text{If } \widehat{S}_i^{'}(\omega) = \sum_{k=1}^{i-1} X_k^{'}(\omega) + X_i(\omega) \notin \mathbb{D}_n \\ & \text{If } ||(\omega_1, \cdots, \omega_i)||_1 > ||(\omega_1, \cdots \omega_{i-1})||_1, \\ & \text{then take step } \omega_i^{'} \text{ with probability } P(X_i(\omega_i^{'})) = \frac{1}{4} + \frac{1}{4}\frac{4-\mu}{\mu}. \\ & \text{If } ||(\omega_1, \cdots, \omega_i)||_1 < ||(\omega_1 \cdots \omega_{i-1})||_1, \\ & \text{then take step } \omega_1^{'} \text{ such that } ||(\omega_1, \cdots, \omega_i^{'})||_1 < ||(\omega_1 \cdots \omega_{i-1})||_1. \end{cases}$$

The confined simple random walk will follow the non-confined simple random walk. If the step chosen by the non-confined simple random walk is a possible step for the confined simple random walk, then the confined simple random walk will do the same step. If the step made by the non-confined simple random walk is not a possible step for the confined simple random walk, then there are two options if the step made by the non-confined simple random walk decreases the ℓ^1-distance with respect to the origin, then the confined simple random walk chooses a step that also decreases the ℓ^1-distance with respect to the origin. If the step made by the non-confined simple random walk increases the ℓ^1-distance, then the confined random walk chooses any available step.

We prove Theorem 31 on a case by case basis.

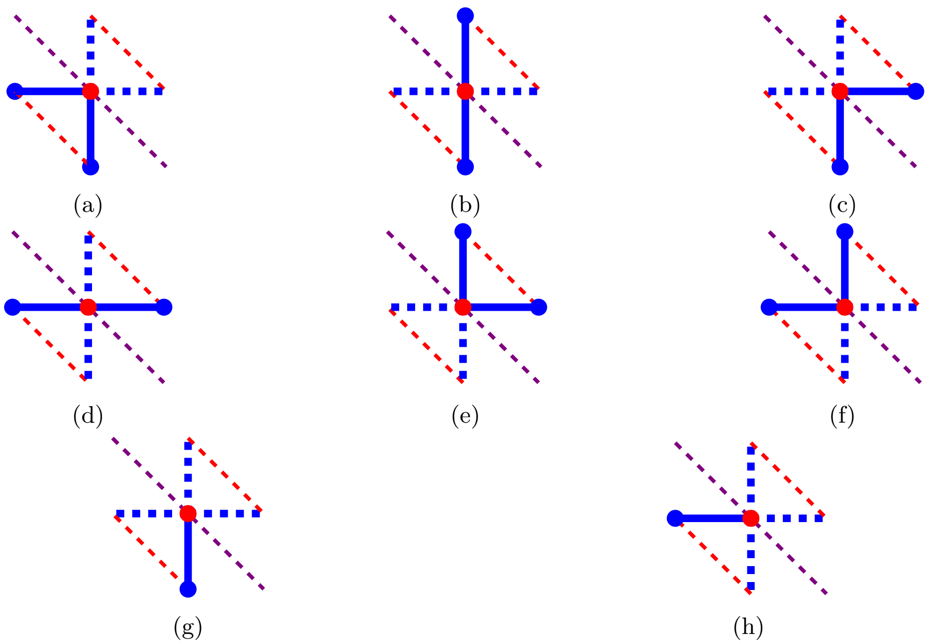


Figure 27: The permutations of the boundary points that satisfy the star property on the positive quadrant of $\mathbb{Z}^2$. The thin red, purple, dashed lines indicate different ℓ^1-distances. The red dot indicates where the confined simple random walk is.

Proof. Assume that we have two simple random walks on the positive quadrant of $\mathbb{Z}^2$, both starting at the origin: $(\widehat{S}_n')_{n=0}^{\infty}$ is a confined simple random walk and $(\widehat{S}_n)_{n=0}^{\infty}$ is a non-confined simple random walk. We coupled these simple random walks as mentioned in (4.2). We want to prove that, under this coupling, the ℓ^1-distances are ordered with probability 1 even when the domain evolves.

Base step: $n = 1$

Under the coupling both simple random walks increase their ℓ^1-distance by 1. Evolving the domain with boundaries such as in Figure 27, ensures that there is always a step to decrease the ℓ^1-distance. Hence our coupling holds and the confined simple random walk is stochastically closer in ℓ^1-distance than the non-confined simple random walk.

Induction hypothesis:

For all $0 \leq n \leq N$ the confined simple random walk on the evolving domain D_n, is stochastically closer in ℓ^1-norm.

We give the proof for time step $N + 1$. Assume that at time step N the confined random walk finds itself at the red dot in Figure 27a. Then by Theorem 28, the ℓ^1-distances of the two simple random walks are stochastically ordered. Then at time step $N + 1$ the confined simple random walk either moves to an interior point or another boundary point, such as in Figure 27.

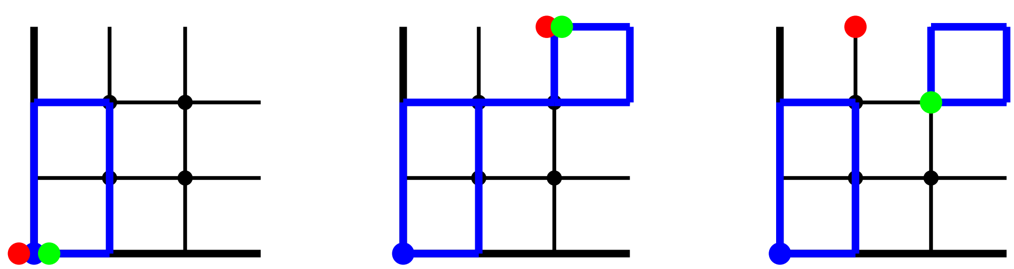
Case I: Interior point
Assume that at time step $N+1$ the domain D_{N+1} evolves in such a way that the position of the confined simple random walk at time step N becomes an interior position for the domain at time step $N+1$. Then our confined simple random walk can make four possible steps, the same steps as the non-confined simple random walk. Thus our coupling ensures that the simple random walks are stochastically ordered.

Case II : Becoming another boundary point
Assume that at time step $N+1$ the domain alters and that the boundary point in Figure 27a evolves into the boundary point as in Figure 27b, $\cdots$, Figure 27h. Since every boundary in Figure 27 satisfies the augmented $\star$-property, we have ensured that there is always a step where that decreases the ℓ^1-distance and thus our coupling always holds. Hence the two simple random walks are again stochastically ordered.

Case III: Becoming a boundary point from an interior point
Assume that at time step N the confined simple random walk is at an interior point and the domain evolves in such a way that the confined simple random walk is now at a boundary point as in Figure 27. Then at time step $N+1$ the confined simple random walk has less than 4 steps possible. The augmented $\star$-property again ensures that there is a step that decreases the ℓ^1-distance and hence the coupling ensures that the confined simple random walk is stochastically closer or in ℓ^1-distance.

Combining these cases with Theorem 28 and 29 we conclude that the confined simple random walk on an evolving, augmented $\star$-property domain on $\mathbb{Z}^2$ is recurrent. □

At the beginning of this chapter we have listed a few criteria that the evolving domain has to fulfill. You might wonder: *"Does our coupling not hold when we do not fulfill all these criteria?"* In the following example we ignore the criterion that the domain has to be connected and observe that the coupling does not hold.



(a) Two random walks at time step $t=0$

(b) Two random walks at time step $t=k$

(c) Two random walks at time step $t=k+1$

Figure 28: Two simple random walk, confined (green) and non-confined (red) on an evolving domain which fulfills the augmented star property, but does not adhere to the connected criterion. In this case we observe that our coupling does not hold. At timestep $k+1$, Figure 28c, we observe that the non-confined random walk cannot make a move that decreases the ℓ^1-distance.

For every criterion we can find an example where if we omit the criterion, then the coupling does not hold. In the beginning of the paper we listed that our domain had to be convex as a hypotheses. From this we can conclude that the domain need not be convex in order for the simple random walk to be recurrent.

Thus to conclude this chapter: A simple random walk on an evolving that has the augmented $\star$-property and contains the origin will be recurrent under this coupling with probability 1.

5 Conclusion

In this bachelor thesis our goal is to answer the question: Is the simple random walk on an evolving, connected domain, that abides by the augmented $\star$-property on $\mathbb{Z}^2$ recurrent? We have proven that this is indeed the case. We started with the one-dimensional simple random walk proving the recurrence on $\mathbb{Z}^1$. Then we moved onto the two-dimensional simple random walk by means of a decomposition of the two-dimensional simple random walk into two one-dimensional simple random walks.

In the second chapter we moved to electrical networks and to the conversion of simple random walks into an electrical network. This gave us another way to prove the recurrence of the simple random walk in two dimensions. Kirchoff's Current Law and Ohm's Law popped up and we used those laws to prove Dirichlet's Principle, which says that the dissipation energy is minimized by Ohm's law and Kirchoff's Current Law at the interior vertices. Before we could prove the Dirichlet Principle we needed to introduce the notion of the Weighted Laplacian and the Incidence Matrix. We proved the Dirichlet Principle in an attempt to use it for evolving domains, however, this turns out not to be possible.

In the first chapter we proved the recurrence of the simple random walk on the infinite lattice, however, we did not prove it for a confined simple random walk. In the third chapter we looked at the confined simple random walk and our goal was to prove the recurrence of the confined simple random walk. We used a technique called Markovian coupling and the notion of stochastic ordering to prove the recurrence. We needed to look at specific domains, that fulfill the augmented $\star$-property, and proved that under the coupling with probability one the confined simple random walk is stochastically closer to the origin then the non-confined simple random walk in ℓ^1-distance.

In the last chapter we looked at evolving domains. We defined what it means for a domain to be evolving and, using the same coupling as in the third chapter and the notion of stochastic ordering, we listed a set of criteria for the evolving domain to abide by and we proved the recurrence of the confined simple random walk subject to that list of criteria. Thus the simple random walk on an evolving, connected, augmented $\star$-property domain is recurrent.

6 Open Questions

What happens when we implement the coupling in a three-dimensional grid? From the literature it turns out that if the domain grows sufficiently slowly, then the simple random walk is less transient. How would the coupling impact that?

Can we implement the coupling on different grids? For example a triangular grid or an hexagonal grid? Does the coupling apply? In these grids there is a different coordinate system, so at the very least the ℓ^1-metric would have to be replaced by another metric.

What about the world of reinforced simple random walks? Can we get some insight with the knowledge we currently have about simple random walks in evolving domains, or is that so vastly different that we cannot apply our insight at all?

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