The winding number and the Fredholm index: The Toeplitz index theorem and an application

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1 Introduction

"Banach spaces are very wild." - dr. M. Roelands

And yes, Banach spaces are indeed very wild. Even though we have many tools for analysing properties of Banach spaces - such as the Hahn-Banach Theorem, Open-mapping Theorem and the Uniform boundedness principle - they can lack a lot of other properties present in spaces with more structure. For instance, Banach spaces do not have to be reflexive in general; we have no general notion of orthogonality; the closure of the space of finite rank operators does not have to be the compact operators and so on. Many Banach spaces however admit more structure than is encapsulated in the notion of a Banach space. On some, such as operator spaces, we have a natural notion of a vector multiplication.

Banach spaces with a multiplication that behaves ‘nicely’ with the topology inferred by the norm are called Banach algebras. Whenever there is a multiplicative unit present in our Banach algebra we can also define the spectrum of an element of said algebra, generalising the concept of spectra associated with linear operators. This will prove to be invaluable in analysing Banach algebras. Namely its properties can allow us to discern which algebras can be made into Banach algebras, giving us a better feel for the type of vector spaces involved. In this thesis the reader will no doubt also see the general framework of Banach algebras allows the spectrum to flourish in its applications to solving integral equations, such as the following example

$$K : \mathscr{C}([0,1]) \to \mathscr{C}([0,1]),$$

$$K(g)(t) = c \cdot g(t) + \int_0^1 k(x, g(t)) f(x)\, dx = 0.$$

Some problems we can not solve with this theory however such as proving if the equation

$$W : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+),$$

$$W(\phi)(t) = c\phi(t) + \int_0^\infty k(t-s)\phi(s)\, ds = 0, \quad t \geq 0,$$

has solutions for functions $g \in L^2(\mathbb{R}_+)$. Luckily we have not exhausted the structure of operator algebras $\mathscr{B}(V)$ as a source of inspiration: when we require the underlying space V to be Hilbert, we can also take adjoints of operators. Formalising this in a more general framework will give us a special type of Banach algebra, known as a C^*−algebra. Within C^*−algebras we can greatly strengthen some of the theorems known for Banach algebras and its unique structure makes way for new theorems of great beauty and of use to the problem above. In doing so we will demonstrate a deep connection between a certain operator algebra and the winding number which we can associate with elements of this algebra. The theorem stating this connection is called the *Toeplitz index Theorem* and will serve as the main theorem of this thesis.

The outline of this thesis is now as follows. In Section 2 we will define and display some of the properties of Banach algebras. We will do the same for C^*-algebras in Section 3 whilst making the distinction between Banach and C^*-algebras all the more clear, highlighting its special properties. In Section 4 we will formulate a comprehensive and clear definition of the winding number tailored to our needs. From Section 5 onward we will delve into operator theory, displaying the Toeplitz Index Theorem in Section 6 and applying it to integral equations in Section 7. The reader well-acquainted with $C^*-$and Banach algebras may want to skip ahead to Section 4 or 5.

2 Banach algebras

2.1 Introduction

Suppose K is a field. A K-algebra A is a K-vector space with a binary operation $(x, y) \mapsto xy$ that is both bilinear and associative. In this thesis we shall assume $K = \mathbb{C}$ and refer to $\mathbb{C}-$algebras simply as algebras. The material in this section is mainly based on [2] and lesser so on [8] and [10]. Often when dealing with vector spaces, concepts like distance and convergence naturally arise and hence we like to define a topology on the space. We do the same for algebras, where the multiplication is connected with the norm as follows.

Definition 2.1.1. Suppose we have an algebra A which is also a normed vector space A. We say A is a *normed algebra* if it satisfies

$$\|xy\| \leq \|x\| \cdot \|y\|, \quad \text{for all } x, y \in A. \tag{1}$$

Furthermore, if A is also complete under its norm, we say A is a Banach algebra. If the algebra A has a multiplicative unit we call it unital.

Examples of elementary (finite dimensional) algebras include matrix algebras $\mathrm{Mat}(n, \mathbb{C})$ over a fixed dimension $n \in \mathbb{N}$ with the natural multiplication, or simply $\mathbb{C}^n$. Even though finite dimensional algebras are not the main topic of study in functional analysis, they can serve as useful examples. The former is for example Banach under its associated operator norm and the latter the supremum norm over its coordinates.

A very important example of an infinite dimensional normed algebra is space of bounded operators $\mathscr{B}(V)$ on some normed space V equipped with the standard operator norm and with multiplication being composition. Another example is the space of continuous functions $\mathscr{C}(X)$ on some compact Hausdorff space X with pointwise product and supremum norm. In this case $\mathscr{C}(X)$ is a Banach algebra and should V be a Banach space, then $\mathscr{B}(V)$ is a Banach algebra also.

Sometimes multiple multiplications can be appropriate for vector spaces. For example if we look at the sequence space $\ell^1(\mathbb{Z})$ we can define a pointwise product $(a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} = (a_n b_n)_{n\geq 0}$ to make it a Banach algebra, but also by way of convolution

$$(ab)_n = \sum_{k=-\infty}^{\infty} a_n b_{n-k} \tag{2}$$

In a similar fashion, we can also define a convolution product on $\mathscr{L}^1(\mathbb{R})$, with $f, g \in \mathscr{L}(\mathbb{R})$ satisfying $(f \star g)(x) := \int_{-\infty}^{\infty} f(x) g(x-t)\, dt$. It can be shown that both $\ell^1(\mathbb{Z})$ and $\mathscr{L}^1(\mathbb{R})$ satisfy property (1) using Young's inequalityA.2.1. This product will indeed make $\mathscr{L}^1(\mathbb{R})$ into a Banach algebra. The algebras $\ell^1(\mathbb{Z})$ and $\mathscr{L}^1(\mathbb{R})$ are very different however as the former has a multiplicative identity while the latter does not. We will use convolution products later on to define integral operators on $\mathscr{L}^p$ spaces.

Remark 2.1.2. Multiplicative identities need not have norm 1 even though this is intuitively a desired property. This problem can be resolved, however, as in[2] where it is shown that any Banach algebra can be renormed to satisfy $\|1\| = 1$ under an equivalent norm. This is achieved by showing that any unital Banach algebra A is algebraically isomorphic to a subalgebra of the space $\mathscr{B}(A)$ of bounded operators, which happens to induce a norm with the desired properties. For this reason we will from now on assume that $\|1\| = 1$ holds.

2.2 The spectrum in Banach Algebras

A main topic of study within functional analysis are the spectra of (bounded) operators. Since we know $\mathscr{B}(V)$ to be a Banach algebra if V is a Banach space, it is natural to generalise this definition for general Banach algebras. We will phrase it a bit more generally however.

Definition 2.2.1. Given a unital normed algebra A the *spectrum* $\sigma(x)$ of an element $x \in A$ is given by

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ is not invertible }\}.$$

For the rest of this section we denote A to be a unital Banach algebra. Stepping away from operator algebras, the spectrum of an element in a general algebra is purely an algebraic construct as there are no further requirements regarding the continuity of such an inverse. This is in a sense also true in Banach operator algebras as bijective linear mappings are invertible by the Banach isomorphism theoremA.2.4.

On the other hand, we will see many properties of Banach algebras relating the spectrum of an element and the topology of the Banach algebra, which will allow us to exclude some important algebraic structures from being able to be made into a Banach algebra.

Before we delve into the properties of the spectrum, we require some basic information on the set of invertible elements.

Theorem 2.2.2. *For all $x \in A$, $\|x\| < 1$ the series $\sum_{n=0}^{\infty} x^n$ converges with limit $(1-x)^{-1}$. We also have the estimates*

$$\left\|(1-x)^{-1}\right\| \leq \frac{1}{1-\|x\|} \textit{ and } \left\|1-(1-x)^{-1}\right\| \leq \frac{\|x\|}{1-\|x\|}.$$

The proof is similar to what is taught in elementary courses on linear functional analysis and will not be included here. In the proof the completeness of A is used in the sense that absolute convergence of series implies convergence of series. This is on the most fundamental level the property which will allow us to build our theory for Banach algebras.

If we denote by $A^\times$ the set of invertible elements of A, we can show with the theorem above that the subset $A^\times$ of A is open and that the operation $x \mapsto x^{-1}$ is continuous as seen in [2, Theorem 1.5.3]. The proof uses the norm estimations of 2.2.2 and is quite technical. The fact that $A^\times$ is open allows us to prove our first property on the spectrum which later on will be strengthened significantly.

Lemma 2.2.3. *For all $x \in A$, $\sigma(x)$ is closed and we have $\sigma(x) \subseteq \{z \in \mathbb{C} : |z| \leq \|x\|\}$.*

Proof. Since $A^\times$ is open and the mapping $\mathbb{C} \to A$, $\lambda \mapsto x - \lambda 1$ is continuous, the set $\mathbb{C} \setminus \sigma(x)$ must be open and thus $\sigma(x)$ closed. If we assume $|\lambda| > \|x\|$ we can see that $x - \lambda 1 = -\lambda(1 - \lambda^{-1}x)$ and since $\frac{\|x\|}{|\lambda|} < 1$ we see that the latter should be invertible. □

Corollary 2.2.4. *There does not exist a norm $\|\_\|$ such that the algebra $\mathbb{C}[X]$ together with $\|\_\|$ is a Banach algebra.*

Proof. We know polynomials $f \in \mathbb{C}[X]$ to be invertible if and only if $\deg(f) = 0$. Fix $f \in \mathbb{C}[X]$, $\deg(f) > 0$. We can see that for $\lambda \in \mathbb{C}$, $f - \lambda 1$ will not be invertible, thus $\sigma(f) = \mathbb{C}$ which is clearly unbounded making $\mathbb{C}[X]$ unable to become a Banach algebra by Lemma 2.2.3. □

The following theorem by Gelfand is very useful in classifying a certain type of Banach algebras. In proving this we 'generalise' complex functions to admit a Banach algebra as codomain. We can then define the derivative analogously to the regular complex case—should the limit exist. It also displays nicely why Banach algebras are best discussed over a complex field.

Theorem 2.2.5. *For all $x \in A$ we have $\sigma(x) \neq \emptyset$.*

Proof. Suppose $x \in A$ and $\lambda_0 \notin \sigma(x)$. By Lemma 2.2.3 $\sigma(x)$ is closed and $x - \lambda 1$ is invertible with $\lambda \in \mathbb{C}$.

We can then see

$$(x-\lambda)^{-1} - (x-\lambda_0)^{-1} = (\lambda - \lambda_0)(x-\lambda)^{-1}(x-\lambda_0)^{-1}.$$

Dividing by $\lambda - \lambda_0$ and using $(x-\lambda)^{-1} \to (x-\lambda_0)^{-1}$ as $\lambda \to \lambda_0$ yields

$$\lim_{\lambda \to \lambda_0} \frac{(x-\lambda)^{-1} - (x-\lambda_0)^{-1}}{\lambda - \lambda_0} = (x-\lambda_0)^{-2}.$$

If we now assume $\sigma(x)$ to be empty, we can see this equality holds for every $\lambda \in \mathbb{C}$. Now if we take a bounded linear functional ρ on A we define $f(\lambda) = \rho((x-\lambda)^{-1})$ and $f'(\lambda) = \frac{\rho((x-\lambda)^{-1}) - \rho(x-\lambda_0)^{-1}}{\lambda - \lambda_0}$.

Using the earlier assertions and the linearity and continuity of ρ we can see that $f'(\lambda) = \rho((x-\lambda)^{-2})$ meaning that f is an entire function. We can use the previously found estimates on geometric series to conclude f is bounded also and by Liouville's theorem A.2.6 then constant. By the same estimates we can also see f will tend to zero as $\lambda \to \infty$, thus meaning f is constant zero.

Since ρ was chosen arbitrarily and by in [17, Corollary 5.22] we have that $(x-\lambda)^{-1} = 0$, which is clearly a contradiction since the element is invertible.

□

Remark 2.2.6. Should the Banach algebra A now be a division algebra also we can draw a remarkable conclusion. If we define $\theta : \mathbb{C} \to A$ with $\theta(\lambda) = \lambda 1$, then $\theta(\mathbb{C})$ is a subalgebra isometrically isomorphic to $\mathbb{C}$. For every element $x \in A$ there must be a $\lambda \in \mathbb{C}$ such that $x - \lambda 1$ is not invertible. Since A is a division algebra this can only happen when $x = \lambda 1$, thus A is isometrically isomorphic to the one-dimensional algebra $\mathbb{C}$.

This result excludes numerous algebras from being able to become Banach algebras. For example the space of rational functions as every non constant element has empty spectrum. The result itself is useful in building up our theory as we can see in Section 2.4.

To give the reader an idea of possible ways to strengthen the relation between the norm of an element and its spectrum, we state the following result without proof. An excellent proof can be below [2, Theorem 1.7.3] using similar techniques as used in proving Theorem 2.2.5.

Theorem 2.2.7. *For all $x \in A$ we have*

$$\lim_{n\to\infty} \left\| x^n \right\|^{1/n} = \sup\{\|\lambda\| : \lambda \in \sigma(x)\}$$

We call this value the *spectral radius* and denote it by $r(x)$.

Using Theorem 2.2.7 and Lemma 2.2.3 we can give bounds for the spectrum of an element in a Banach algebra, but we have as of yet not seen any general way to compute the spectrum. It is for instance not clear what the spectrum of a general element $x \in \ell^1(\mathbb{Z})$ under convolution product is. We shall soon see a powerful tool to achieve this, but first we require some knowledge on quotients of Banach algebras.

2.3 Quotients of Banach algebras

As we will see there exist many examples of algebras with a topology where taking quotients is not only useful but also natural to consider. For example, looking at the normed algebra $\ell^\infty(\mathbb{N})$ under pointwise multiplication, we can define $c_0(\mathbb{N})$ to be the space of sequences converging to zero. The space $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ contains classes of elements converging to the same element making it a suitable space for studying the *asymptotic behavior* of sequences. We shall see more examples later on. For the rest of this section A will once again denote a Banach algebra.

Remark 2.3.1. We do not know yet, however, how the topology will behave for quotient structures of A/I. In this context there are two important remarks to make. Firstly, it is easily seen the closure of an ideal is indeed an ideal. Furthermore given unital A and $I \subsetneq A$ a proper ideal we can prove the norm closure $\overline{I}$ to be proper also. Indeed, we know the intersection of I and A^{-1} to be empty. Since $A \setminus A^{-1}$ is closed 2.2.3 and $I \subseteq A \setminus A^{-1} \subsetneq A$ we can simply take the closure of I yielding a proper subset of A. From this, it also follows that any maximal ideal is closed.

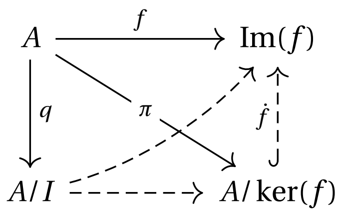
Secondly, if we endow the quotient A/I of a normed algebra A with the norm

$$\|x + I\| := \inf_{i \in I} \|x + i\|,$$

the quotient will again be a normed algebra. If we also assume A to be Banach and I to be closed then A/I is a Banach algebra. This last statement follows from the fact that for $x \in I$ with I closed there exists $j \in I$ for which $\inf_{i \in I} \|x + i\| = \|x + j\|$ holds. From this, we can see that any Cauchy sequence $(x_n + I) \subseteq A/I$ can be associated with a sequence $(x_n + i_n)_{n \geq 0}$ converging to $(x + i)$ for some $x \in A, i \in I$. We can then show $(x_n + I)_{n \geq 0} \to (x + I)$. After verifying that $\|xy + I\| \leq \|x + I\| \cdot \|y + I\|$ holds we can conclude that A/I is indeed a Banach algebra.

Now that these elementary results have been stated we shall state that the universal property of quotients is also valid for Banach algebras

Theorem 2.3.2 (Isomorphism theorem for Banach algebras)**.** *Let $f : A \to B$ be a bounded homomorphism between Banach algebras and $I \subset \ker(f)$ a closed ideal in A. The natural mappings $\pi : A \to A/I$, $q : A/I \to A/\ker(f)$ and f uniquely induce the dotted arrows seen in the commutative diagram below.*



Furthermore $\dot{f}$ is invertible and $\|\dot{f}\| = \|f\|$.

Proof. The fact that the diagram commutes and the mappings are well-defined are elementary results from algebra, see [14, Section 3.1]. For $\|\dot{f}\| = \|f\|$ note that $\|f\| = \|\dot{f} \circ \pi\| \leq \|\pi\| \cdot \|\dot{f}\| \leq \|\dot{f}\|$. Also, we have $f(x) = \dot{f}(x + I)$ so

$$\left\| \dot{f}(x+I) \right\| = \left\| f(x) \right\| = \left\| f(x+z) \right\| \le \left\| f \right\| \cdot \|x+z\|, \quad \text{for all } z \in \ker(f).$$

Taking the infimum over z and dividing

$$\left\| \dot{f}(x+I) \right\| \le \left\| f \right\| \cdot \|x+z\|$$

by $\inf_{z \in \ker(f)} \|x+z\| = \|\dot{x}\|$ yields the statement. □

2.4 The Gelfand spectrum of a Banach algebra

In this section we will discuss the Gelfand transform. This phenomenon will provide us with a powerful tool for classifying certain types of commutative unital Banach algebras as well as a way to compute spectra of elements of the spectrum. We write $\mathrm{Hom}(A, \mathbb{C})$ for the not necessarily bounded algebraic homomorphisms between A and $\mathbb{C}$. Throughout this section A will denote a commutative Banach algebra. The material is loosely based on [8, Section 7.8] and [2, Section 1.9].

Definition 2.4.1 (Gelfand spectrum)**.** The *Gelfand spectrum* of A is defined as the set

$$\mathrm{sp}(A) = \{\omega \in \hom(A, \mathbb{C}) : \omega \neq 0\}$$

of algebraic homomorphisms.

Firstly — before delving into the structure of $\mathrm{sp}(A)$ itself — we can note that multiplicativity of ω ensures that $\omega(1) = 1$. More so, we can note that all the $\omega \in \mathrm{sp}(A)$ are bounded. Indeed, if ω is nonzero then surely it is surjective. By Theorem 2.3.2 we then have a mapping,

$$\tilde{\omega} : A/\ker(\omega) \xrightarrow{\sim} \mathbb{C}.$$

Since $\mathbb{C}$ is a field we know $\ker(\omega)$ to be maximal and thus closed. From this we can see that the quotient mapping $A \to A/\ker(\omega)$ must be bounded also, meaning that ω is the composition of two linear mappings with norm 1, thus by (1) $\|\omega\| \le 1$. Since also $\omega(1) = 1$, we see that $\|\omega\| = 1$.

Having made these remarks we can also note that by Remark 2.2.6 any maximal ideal $M \in A$ will correspond with some homomorphism $\omega \in \hom(A, \mathbb{C})$. Indeed, A/M will be a division algebra thus isometrically isomorphic to $\mathbb{C}$. Writing $A/M \to \mathbb{C}$, $\lambda + M \mapsto \lambda$ and taking the composition of this with the quotient mapping will yield our homomorpshism. We can now conclude that there exists a bijection between the space of maximal ideals of A and elements $\mathrm{sp}(A)$.

Having made these remarks we now define the Gelfand transform as follows.

Definition 2.4.2 (Gelfand transform)**.** Given an element $x \in A$ the *Gelfand transform of* x is given by

$$\hat{x} : \mathrm{sp}(A) \to \mathbb{C}, \qquad \hat{x}(\omega) = \omega(x).$$

The map $A \to \mathrm{sp}(A), \quad x \mapsto \hat{x}$ is named *the Gelfand map.*

If we denote A' to be the dual space of A, we know that $\mathrm{sp}(A)$ is a subset of the unit ball $B_{A'}$ of A'. By Alaoglu A.2.3 we know that if $\mathrm{sp}(A)$ were to be weak-* closed in $B_{A'}$ it would be weak-* compact. The following lemma asserts this is the case.

Lemma 2.4.3. *The Gelfand spectrum* $\mathrm{sp}(A)$ *is a weak-* compact Hausdorff space.*

Proof. The Hausdorff property follows from the fact that $\mathrm{sp}(A)$ is a topological subspace of $B_{A'}$. The Gelfand spectrum $\mathrm{sp}(A)$ is weak-* closed since if $f_n \to f$ weak-*, then for all $x, y \in A$ we have

$$\begin{aligned} f(xy) &= \lim_{n\to\infty} f_n(xy) \\ &= \lim_{n\to\infty} f_n(x) f_n(y) \\ &= f(x) f(y). \end{aligned}$$

Moreover $f(1) = 1$ so clearly $f \in \mathrm{sp}(A)$. □

Before stating the main theorem of this section, we give a quick proof of a relation between the Gelfand spectrum of a Banach algebra and the spectrum of an element.

Theorem 2.4.4. *We have for a unital Banach algebra* A *that for all* $x \in A$

$$\sigma(x) = \{\hat{x}(\omega) : \omega \in \mathrm{sp}(A)\}.$$

Proof. Suppose $\omega \in \mathrm{sp}(A)$ and $\omega(x) = \lambda$ for some $\lambda \in \mathbb{C}$. Then $x - \lambda 1 \in \ker(\omega)$, thus $x - \lambda$ is contained in a proper ideal and thus not invertible. From the definition of the spectrum we conclude $\lambda \in \sigma(x)$.

Conversely for $\lambda \in \sigma(x)$ we have $x - \lambda 1$ not invertible meaning that $(x - \lambda 1)$ is a proper ideal in A, contained within a maximal ideal M (AC). By earlier remarks we know this maximal ideal to correspond with some $\omega \in \mathrm{sp}(A)$ with $(x - \lambda 1) \subseteq \ker(\omega)$. We then know $\omega(x) - \lambda = \omega(x - \lambda 1) = 0$ to hold thus $\omega(x) = \lambda$. In other words,

$$\sigma(x) \subseteq \{\hat{x}(\omega) : \omega \in \mathrm{sp}(A)\}$$

proving the statement. □

The following theorem can be used to classifies certain types of Banach algebras.

Theorem 2.4.5. *Let A be a commutative Banach algebra with Gelfand spectrum* $\mathrm{sp}(A)$ *and $a \in A$. The Gelfand map $A \to \mathscr{C}(\mathrm{sp}(A))$, $a \mapsto \hat{a}$ is a homomorphism of norm* 1 *and its kernel is the intersection of all maximal ideals. Moreover for each $a \in A$ we have $\|\hat{a}\|_\infty = r(a)$.*

To show the reader an application of this material we will now present a worked example.

Example 2.4.6. If we observe the unital Banach algebra $\ell^1(\mathbb{Z})$ with the natural 1-norm and with the convolution product we know $\ell^1(\mathbb{Z})' \cong \ell^\infty(\mathbb{Z})$ as seen in [17, Theorem 5.5]. Writing $a := (a_n)_{n\geq 0}$ for an element $a \in \ell^1(\mathbb{Z})$ we first show that every $\phi \in \ell^1(\mathbb{Z})'$ acts in the following manner

$$\phi(a) = \sum_{n=-\infty}^{\infty} b_n a_n, \text{ for some } (b_n)_{n\geq 0} \in \ell^\infty(\mathbb{Z}).$$

Let $z \in S^1$ and $(\dots, z^{-1}, 1, z, \dots) \in \ell^\infty$ and write $\omega : \ell^1(\mathbb{Z}) \to \mathbb{C}$ with $\omega(a) = \sum_{n\in\mathbb{Z}} a_n z^n$. For $a, d \in \ell^1(\mathbb{Z})$ can we see that

$$\begin{aligned}
\omega(a)\omega(d) &= (\sum_{n\in\mathbb{Z}} a_n z^n)(\sum_{n\in\mathbb{Z}} d_n z^n) \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} a_n d_k z^{n+k} \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{m-n} a_n z^m \\
&= \omega(a \cdot d)
\end{aligned}$$

holds, meaning this ω is indeed a homomorphism. We can also easily see that $\|\phi\| = |z| = 1$, making ω bounded.

Now in proving that *any* nonzero homomorphism is of this form we take $e_n \in \ell^1(\mathbb{Z})$ with $e_n(i) = \delta_{in}$. We then note that for every $n \in \mathbb{Z}$, $z_n := \omega(e_n) = \omega(e_1)^n$, so with ω we can associate a sequence $(\dots, z^{-1}, 1, z, z^2, \dots)$. Since we know ω is bounded it follows that $z \in S^1$ should hold. We have thus described the Gelfand spectrum of $\ell^1(\mathbb{Z})$ completely. Using Theorem 2.4.5 we can now see that

$$\sigma(a) = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n : z \in S^1 \right\}$$

holds.

3 C^*-algebras

The material in the sections 3.1-3.3 is mainly based on [2, Section 2], [8, Section 7].

3.1 Introduction

As has been hinted on throughout this text some Banach algebras have more structure than is enveloped in definition of Banach algebras. For instance if we look at the Banach algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$ we can define an involution operation $\_^* : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ mapping $f \in \mathscr{B}(\mathscr{H})$ to its adjoint f^* which is the unique bounded linear map satisfying $\langle f(x), y\rangle = \langle x, f^*(y)\rangle$ [17, Theorem 5.2]. We generalise this in the following definition.

Definition 3.1.1. Let A be a Banach algebra. If we equip A with a involution $\_^* : A \to A$ with $a \mapsto a^*$ satisfying that for all $a, b \in A$ and scalars $\lambda, \mu \in \mathbb{C}$

1. $a^{**} = a$
2. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$
3. $(ab)^* = b^* a^*$
4. $\|a^* a\| = \|a\|^2$

we name A to be a C^**-algebra*. Furthermore we call elements $a \in A$ satisfying $a^* a = aa^*$ *normal* and if $a^* = a$ holds we call them *self-adjoint*.

Note that all self-adjoint elements are normal. It can be shown that the adjoint on $\mathscr{B}(\mathscr{H})$ indeed satisfies these properties making $\mathscr{B}(\mathscr{H})$ a C^*-algebra. The Banach algebra $\mathscr{C}(X)$ introduced in section 2.1 is also a C^*-algebra when equipped with complex conjugation as involution. The reader might wonder if all Banach algebras can be equipped with an involution to become a C^*-algebra. This is certainly not the case. For instance, $\ell^1(\mathbb{Z})$ is not a C^*-algebra and Hilbert spaces $\mathscr{H}$ can be made into C^*-algebras iff $\mathscr{H}$ is finite dimensional. The latter will be proved in Section 3.4. First, we will show the involution is isometric.

Lemma 3.1.2. *Every $a \in A$ satisfies $\|a^*\| = \|a\|$.*

Proof. Note that $\|a\|^2 = \|a^* a\| \leq \|a^*\| \|a\|$ which implies $\|a\| \leq \|a^*\|$. Doing the same for a^* and using $a = (a^*)^*$ yields $\|a^*\| \leq \|a\|$, concluding the proof. □

The structure of C^*-algebras strengthens many of the principles we have proven for Banach algebras. For example, we can show with induction on n that a self-adjoint $a \in A$ satisfies $\|a^n\|^{\frac{1}{n}} = \|a\|$ for all $n \in \mathbb{N}$. By Lemma 3.1.2 we can see that $r(a) = \|a\|$. From this we can prove the following.

Lemma 3.1.3 (Continuity of *-homomorphisms)**.** *Let A, B be C^*-algebras and $\rho : A \to B$ a *-homomorphism (i.e. ρ algebraic homomorphism with $\rho(a^*) = \rho(a)^*$ for all $a \in A$). Then $\|\rho(a)\| \leq \|a\|$.*

Proof. Firstly, if for some $\lambda \in \mathbb{C}$ the element, $a - \lambda 1$ is invertible then $\rho(a - \lambda 1)\rho((a - \lambda 1)^{-1}) = \rho(1) = 1$ thus $\rho(a - \lambda 1)$ is also invertible. From this follows $\mathbb{C} \setminus \sigma(\rho(a)) \supseteq \mathbb{C} \setminus \sigma(a)$ which implies $\sigma(\rho(a)) \subseteq \sigma(a)$. We can then conclude $r(\rho(a)) \leq r(a)$.

Since $a^* a$ is self-adjoint we have

$$\|\rho(a)\|^2 = \|\rho(a^* a)\| = r(\rho(a^* a) \leq r(a^* a) = \|a^* a\| = \|a\|^2.$$

□

To contrast, it is not difficult to construct a discontinuous operator on a Banach algebra. If we take an arbitrary Banach algebra A with product defined as $(x, y) \mapsto 0$ the question boils down to finding an unbounded operator on *the space A* which we can do in general if we assume the axiom of choice (which we gladly do).

The question whether this holds for 'sensible' Banach algebras is much more intricate however. For example, as is said in the introduction of [20] if we want to construct a discontinuous homomorphism between $\mathscr{C}(X) \to B$ for some infinite compact Hausdorff space X and an arbitrary Banach algebra B we need more axioms than ZFC. In [9, theorem 1.1] a construction of a discontinuous homomorphism using the continuum hypothesis was made.

Before moving on to deeper results on C^*-algebras we will give another example displaying the special nature of the adjoint. From Section 6 onward, this example will be of great importance.

Example 3.1.4. As is well-known, the Hilbert space $\ell^2(\mathbb{N})$ admits a orthonormal basis $(e_1, e_2, ...)$ and on this we can define the *unilateral shift* $\sigma(e_k) = e_{k+1}$. It is an easy exercise to show that σ is bounded and the adjoint σ^* of σ is uniquely defined by,

$$s(e_1) = 0, \qquad \sigma(e_k) = e_{k-1}.$$

Now if we would observe the Banach algebra generated by σ we would simply get a structure having sums of powers of σ which is not terribly interesting in itself. For example, the projections on the basis elements of $l^2(\mathbb{N})$ are not present in this algebra.

If we instead observe the C^*-algebra generated by σ, we can note that $I - \sigma\sigma^*$ denotes the orthogonal projection on the first coordinate. Likewise we can write $s^n(I - \sigma\sigma^*)$ to obtain the mapping sending the first coefficient of the first coordinate to *nth* position. More generally, writing $s^n(I - \sigma\sigma^*)(s^*)^k$ sends the coefficient on the *kth* position to the *nth* position. From this we can readily deduce that $C^*(\sigma)$, the C^*-algebra generated by σ, contains *all finite rank operators.*

3.2 The Gelfand spectrum of C^*-algebras

Now that we have developed some basic results on C^*-algebras we can wonder how the Gelfand transform will behave on C^*-algebras. The proof is included in order to display what properties set a C^*-algebra apart from general Banach algebras. For a proof we refer to [2].

Theorem 3.2.1 (Gelfand Spectrum of commutative unital C^*-algebras)**.** *Let A be a commutative unital C^*-algebra and let $X = sp(A)$ be the Gelfand spectrum of A. The Gelfand map is an isometric *-isomorphism of A onto $\mathscr{C}(X)$.*

This statement solidifies the importance of the space $\mathscr{C}(X)$ and gives a complete characterisation of unital commutative C^*-algebras. We should note however that non-commutative C^*-algebras are very easily constructed. Banach matrix algebras over finite dimensional spaces are, for instance, not commutative if there exists a non-normal element or rather a non-diagonalisable matrix. Do note that if we observe the C^*-algebra generated by 1 and a normal (or self-adjoint) element the resulting algebra will be commutative. In most cases this is exactly the way this theorem is applied.

This technique is used in proving the theorem below. It strengthens both the spectral radius theorem *and* the spectral permanence theorem. The spectral permanence theorem for Banach algebras states any unital subalgebra B of unital A satisfies $\partial\sigma_B(x) \subseteq \sigma_A(x)$. For a proof we refer to [2]. In C^*-algebras however the following holds.

Theorem 3.2.2 (Spectral radius for C^*-algebras)**.** *Let A be a unital C^*-algebra and $B \subseteq A$ be a C^*-subalgebra of A that contains the unit of A. Then for every $x \in B$ we have $\sigma_B(x) = \sigma_A(x)$. In particular, for every self-adjoint $x \in A$ we have*

$$r(x) = \|x\|.$$

From the last statement we see that for arbitrary elements $x \in A$,

$$\|x\| = \sqrt{\|x^* x\|} = \sqrt{r(x^* x)}$$

holds, meaning that the norm — directly linked with the topology — is completely determined by the algebraic structure of the C^*-algebra. In particular, this means that if there exist multiple norms that can make a unital algebra into a unital Banach algebra A then no involution can be defined to make this algebra a C^*-algebra. This is shown in the example below.

Example 3.2.3. Consider the algebra of differentiable functions on the circle $\mathscr{C}'(S^1)$. Any norm associated with $\alpha \in \mathbb{R}_{>0}$, $\|f\|_{\alpha,\mathscr{C}'(S^1)} = \|f\|_\infty + \alpha\,\|f'\|$ can make $\mathscr{C}'(S^1)$ into a Banach algebra.

In proving this, we need to check that $\|f\|_{\alpha,\mathscr{C}'(S^1)}$ is a norm and that it satisfies the normed space property. The former is trivially satisfied and the latter is a simple verification,

$$\begin{aligned}\|fg\| &= \|fg\| + \alpha\,\|f'g + fg'\| \\ &\leq \|f\|\,\|g\| + \alpha\,\|f'\|\,\|g\| + \alpha\,\|f\|\,\|g'\| \\ &\leq \|f\|\,\|g\| + \alpha\,\|f'\|\,\|g\| + \alpha\,\|f\|\,\|g'\| + \alpha^2\,\|f'\|\,\|g'\| \\ &= \|f\|\,\|g\|.\end{aligned}$$

We can thus conclude $\mathscr{C}'(S^1)$ can not be made into a C^*-algebra.

The spectral theory on normal operators can be greatly expanded on as can be seen in [2][Section 2.4] and [8][Chapter 7] wherein notions such as diagonalisability for normal operators are displayed. This will be beyond the scope of this thesis however as we are primarily interested in the structure of C^*-algebras, and will develop only the tools necessary to prove the Toeplitz index theorem (see Section 6.3 and GN (see Section 3.4.2). The reader is however encouraged to delve into this theory.

3.3 Quotients on C^*-algebras

When dealing with C^*-algebras, it is natural to wonder whether the quotient structures associated with Banach algebras can be generalised to C^*-algebras. The key insight in this matter is that any norm closed ideal of a C^*-algebra is closed under involution. To prove this we require the following lemma, which we shall state without proof.

Lemma 3.3.1. *Let A be a unital C^*-algebra and let $J \subset A$ be a norm-closed ideal. For all $x \in J$ there exists a sequence $e_1, e_2, \ldots$ of self-adjoint elements of J such that $\sigma(e_n) \subset [0,1]$ and $\lim_{n\to\infty} \|xe_n - x\| = 0$.*

This sequence of self-adjoint elements with this special limiting behavior is often called a *approximate identity*. Having obtained such a sequence $(e_n)_{n\geq 0} \subset J$ for any norm-closed ideal $J \subseteq A$ observe that if $x \in J$ then $\|x^* - e_n x^*\| = \|(x - e_n x)^*\| = \|x - e_n x\| \to 0$ meaning that the norm closure of a closed ideal is indeed closed under involution also.

We can then define the involution on a coset $\dot{x} := x + J$ to be $(x+J)^* = x^* + J$. We can then use Theorem 2.3.2 simplifying the proof leaving only the C^*-properties as seen in Definition 3.1.1 to be checked. The proof itself is quite technical and we shall state the result without proof.

Theorem 3.3.2. *Given a C^*-algebra A and a closed ideal J of A the involution defined as above makes A/J into a C^*-algebra.*

3.4 C^*-Hilbert spaces

The examples of C^*-algebras we have seen thus far have been isometrically *-isomorphic to some sub-* algebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. For example we can show the map

$$\mathscr{C}(X) \to \mathscr{B}(\mathscr{L}^2(X)), \qquad f \mapsto M_f,$$

where $M_f(g)(x) = f(x)g(x)$ to be an isometric *-isomorphism. It turns out this holds in general as has been proven by Gelfand and Naimark in [13].

Theorem 3.4.1 (Gelfand-Naimark-Segal-theorem, representation of C^*-algebras)**.** *Every unital C^*-algebra A is isometrically *-isomorphic to some subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.*

The Hilbert space $\mathscr{H}$ in the proof given in [13] is constructed by associating a non-trivial Hilbert space $\mathscr{H}_x$ with every element $x \in A$ and taking the direct sum, meaning $\mathscr{H} = \bigoplus_{x \in A} \mathscr{H}_x$, which is likely to be much too large and too abstract to help us understand A. The proof is hence not too useful for our application and we will refer the reader to [2][Section 4.7-4.8] or [8][Section 8.5] for a full proof. It can however be proven that each separable C^*-algebra A (i.e. each A having a countable dense subset) are isometrically *-isomorphic to $\mathscr{B}(\mathscr{H})$ for some separable Hilbert space $\mathscr{H}$ and this is the instance of the GNS-Theorem 3.4.1 we shall use.

Looking at the above theorem a natural question to ask is whether C^*-algebras can have a topology induced by an inner product, or in other words whether C^*-algebras can also be Hilbert spaces. As a first step we can show the following for $\mathscr{B}(\mathscr{H})$. The proof is inspired by the blog post [15].

Theorem 3.4.2. *Given a separable Hilbert space $\mathscr{H}$. The C^*-algebra $\mathscr{B}(\mathscr{H})$ does not have a Hilbert space structure if $\mathscr{H}$ is infinite dimensional.*

Proof. Let $\mathscr{H}$ be a separable Hilbert space and suppose $(e_n)_{n \geq 0}$ is a orthonormal basis. Note that the Banach algebra $c_0(\mathbb{N})$ of sequences vanishing at infinity is non-reflexive. We will proceed to show this space can be embedded in $\mathscr{B}(\mathscr{H})$. For some $(s_i)_{i \in \mathbb{N}} \in c_0(\mathbb{N})$ we write

$$T_s : \mathscr{H} \to \mathscr{H}, \qquad \sum_{i=0}^{\infty} \lambda_i e_i \mapsto \sum_{i=0}^{\infty} \lambda_i s_i e_i.$$

We can see this operator to be bounded since

$$\left\| T_s(\sum_{i=0}^{\infty} \lambda e_i) \right\| = \left\| \sum_{i=0}^{\infty} \lambda_i s_i e_i \right\| \leq \|s\|_\infty \left\| \sum_{i=0}^{\infty} \lambda_i e_i \right\|.$$

More so, since s_n admits a maximum for some $n \in \mathbb{N}$, we can also see that $\|T_s(e_n)\| = |s_n e_n| = \|s\|_\infty$ meaning that the mapping $s \mapsto T_s$ will be an isometry (note that it is bilinear) meaning its image will be closed. This implies that $\mathscr{B}(\mathscr{H})$ has a closed non-reflexive subspace meaning $\mathscr{B}(\mathscr{H})$ is not reflexive and not a Hilbert space. $\square$

Remark 3.4.3. To show that any unital infinite dimensional C^*-algebra does not have a norm induced by an inner product requires a bit more theory than is developed thus far and we will hence give a sketch.

Suppose we have an infinite dimensional unital C^*-algebra A. Since we know Gelfand mapping on C^*-algebras to be injective, the remark below Theorem 5 in [18] states there exists a normal element $z \in A$ with infinite spectrum. The C^*-algebra B generated by z and 1 is then commutative and by Theorem 3.2.1 we know it can be represented as $C(X)$ with X the Gelfand spectrum of B.

More so we know X to be compact Hausdorff and by Proposition 2.3 in [8] we know there exists a homeomorphism between X and $\sigma(z)$. Knowing that the spectrum of z is infinite, we now know that X must be compact Hausdorff with infinitely many points, meaning that $C(X)$ will be infinite dimensional. By Proposition 4.3.11 in [1] we now know there exists a closed subspace in $C(X)$ isometrically isomorphic to c_0. By the same argument as in 3.4.2 we know A to be non-reflexive and thus not a Hilbert space.

The reader in need of intuition for the fact that c_0 is a closed subspace of $C(X)$ is recommended to look into Urysohn's Lemma [19] which most likely could be used to construct a closed subspace c_0 in $C(X)$ directly but the author of this thesis has not worked this out explicitly.

4 Winding number

Moving on to our main application of the presented theory, we discuss the winding number shortly. The winding number is an integer associated with a point and a closed continuous curve in a plane. Intuitively it is the amount of times the said curve 'winds around' the point. The winding number is studied in different areas of mathematics such as complex analysis, Riemannian geometry and topology and often has its own definition in each field.

In this section we will state a topological definition and prove its equivalence to the definition as given by complex analysis. Furthermore we give two important properties of the winding number. Namely, it is in a certain sense invariant under homotopy and it has an additive property.

Before defining the winding number we require some toplogical background. The punctured plane, here $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ is homotopically equivalent with the circle S^1 since the mappings

$$\begin{aligned} f : \mathbb{C}^\times &\to S^1 & g : S^1 &\to \mathbb{C} \setminus \{*\} \\ z &\mapsto \frac{z}{|z|} & z &\mapsto z \end{aligned} \tag{3}$$

denote homotopy equivalences. Indeed, we can see that $f \circ g = \mathrm{Id}|_{S^1}$ and $g \circ f = \frac{z}{\|z\|}$ hold and clearly $F_t(z) = \frac{tz}{\|z\|} + (1-t)z$ is homotopic to the identity on $\mathbb{C}^\times$.

In particular we can see that the fundamental group of $\mathbb{C}^\times$ and of the S^1 are isomorphic under the mapping $(g \circ f)_* : \pi_1(\mathbb{C}^\times)([\gamma]) \to \pi_1(S^1)$, $[\gamma] \mapsto [g \circ f(\gamma)]$. Since we know the fundamental group is generated by the mapping $z \mapsto z$ we now know that there is exactly one $n \in \mathbb{Z}$ such that $z \mapsto z^n$ is homotopic with γ. We define this value n to be the winding number $\mathrm{Wn}(\gamma)$.

Remark 4.0.1. There is one subtlety involved in taking a basis point of our fundamental group. Our above discussion strictly dealt with regular homotopies even though the fundamental group requires path homotopy, in other words we do not require our paths to have the same beginning- and endpoints. We deal with this issue as follows:

Say we have a curve $\gamma : S^1 \to \mathbb{C} \setminus \{*\}$. If we scale it as before we get the curve $\gamma/|\gamma| : S^1 \to S^1$, with beginning- and endpoint $z := \gamma(0)/|\gamma(0)| = \gamma(1)/|\gamma(1)$. We now write $\rho : [0,1] \to S^1$ for $\rho(t) = e^{\arg(z)t}$ and note ρ is a nullhomotopic path. Similarly we write ρ^{-1} for the path in the opposite direction. The curve $\rho^{-1} \circ \gamma \circ \rho : [0,1] \to S^1$ will now indeed be path homotopic to a $z \mapsto z^n$ for a certain $n \in \mathbb{Z}$. Moreover, by taking $\mathrm{Wn}(\gamma) := \mathrm{Wn}(\rho^{-1} \circ \gamma \circ \rho) = n$, we have defined the winding number independent of the choice of the basis point for our fundamental group.

4.1 Definition of the winding number in complex analysis

The material is this subsection is based on [7]. We now turn to the definition of a winding number as given by complex analysis. We can define the winding number of a closed piecewise smooth curve $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ as

$$\mathrm{Wn}(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta}\, d\zeta. \tag{4}$$

The winding number $\mathrm{Wn}(\gamma)$ defined this way will indeed give an integer, which we can easily show for elementary curves of the form $\gamma(t) = e^{2\pi i t k}$, since

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta}\, d\zeta = \frac{1}{2\pi i} \int_0^1 \frac{1}{e^{2\pi i t k}} 2\pi i k e^{2\pi i t k}\, dt = \int_0^1 k\, dt = k \tag{5}$$

holds.

In proving the equivalence of this definition with the topological one given above we prove the fact that the integral is invariant under homotopic transformations of the underlying curve, as seen below.

Lemma 4.1.1. *For $Q = [0,1]^2$. Define a continuous function $H : Q \to D$ with $D \subseteq \mathbb{C}$ open and $f : D \to \mathbb{C}$ analytic. Then*

$$\int_{H|_{\partial Q}} f(\zeta) d\zeta = 0$$

holds.

Proof. Let $n \in \mathbb{N}_{\geq 1}$. We begin by partitioning Q is n^2 small squares $Q_{\mu\nu}$ with

$$Q_{\mu\nu} = \left\{ z \in Q : \frac{\mu}{n} x \leq \frac{\mu+1}{\nu}, \frac{\nu}{n} \leq y \leq \frac{\nu+1}{n} \right\}, (0 \leq \mu, \nu \leq n-1).$$

Since $H(Q) \subseteq D$ is compact there exists for large n an open disk $U_{\mu\nu} \subseteq \mathbb{C}$ with $H(Q_{\mu\nu}) \subseteq U_\mu \nu \subseteq D$. The Cauchy integral formula now states $\int_{H|_{\partial Q_{\mu\nu}}} f(\zeta)\, d\zeta = 0$. From this follows $\int_{H|_{\partial Q}} = \sum_{0 \leq \mu, \nu \leq n-1} \int_{H|_{\partial Q_{\mu\nu}}} = 0$ proving the theorem. □

If we now take $f(z) = \frac{1}{z}$ and H a homotopy between the elementary curve $z \mapsto z^n$ on the circle and an arbitrary curve $\gamma \in C(S^1)$ with $0 \notin \mathrm{Im}(\gamma)$ for some n we can state that by the above lemma

$$
\begin{aligned}
0 &= \int_{H|_{\partial Q}} f(\zeta)\,d\zeta \\
&= \int_{H(0,s)} f(\zeta)\,d\zeta + \int_{-H(1,s)} f(\zeta)\,d\zeta + \int_{H(t,1)} f(\zeta)\,d\zeta + \int_{-H(t,1)} f(\zeta)\,d\zeta \\
&= \int_{H(0,s)} f(\zeta)\,d\zeta - \int_{-H(1,s)} f(\zeta)\,d\zeta \\
&= \int_{\gamma} \frac{1}{z}\,d\zeta - \int_{\phi_n} \frac{1}{z}\,d\zeta
\end{aligned}
\tag{6}
$$

holds. Thus, the integrals $\int_\gamma \frac{1}{z}\,d\zeta$ and $\int_{\phi_n} \frac{1}{z}\,d\zeta$ agree and the two given definitions of the winding number coincide.

A nice property that will be of great use later on is the following.

Lemma 4.1.2. *Let $f, g \in \mathscr{C}(S^1)$, $0 \notin \operatorname{Im} f \cup \operatorname{Im} g$ be two curves. If we write fg for the pointwise product of f and g, then*

$$\mathrm{Wn}(fg) = \mathrm{Wn}(f) + \mathrm{Wn}(g)$$

holds.

Proof. Suppose $\mathrm{Wn}(f) = n, \mathrm{Wn}(g) = m$. If $F_t(z)$, $G_t(z)$ are homotopies between f and $z \mapsto z^n$, respectively, g and $z \mapsto z^m$, then we can see $F_t(z)G_t(z)$ to be continuous and $F_0(z)G_0(z) = f(z)g(z)$ and $F_1(z)G_1(z) = z^{n+m}$. In other words $f(z)g(z)$ is homotopic with z^{n+m} and thus $\mathrm{Wn}(fg) = \mathrm{Wn}(f) + \mathrm{Wn}(g)$ holds. □

5 Some operator theory

We have seen that any C^*-algebra can be viewed as a subalgebra of the bounded operators on some Hilbert space. In the following section we will classify some of these operators and develop theory necessary for stating and proving our main theorem in Section 6.3. The definitions will sometimes apply to general normed spaces as well, but in light of our application we will state them for no more general spaces than Banach spaces. Lastly, the reader unfamiliar with (co)kernels of linear operators is encouraged to visit the appendix A.1 for a compact outline. Throughout we fix a Hilbert space $\mathscr{H}$ and a Banach space V unless otherwise explicitly stated. The unit ball V is denoted by B_1. The material in this section is mainly based on [17] and [2].

5.1 Compact operators

Definition 5.1.1. Let X and Y be Banach spaces. An operator $T \in L(X,Y)$ is *compact* if the set $TB_1 := \{Tx : \|x\| \leq 1\}$ admits compact closure. We denote the set of compact operators from X to Y by $\mathscr{K}(X,Y)$ or simply $\mathscr{K}$ whenever the context is clear.

We can easily see that compact operators on general Banach spaces need to be bounded, since if they were not, we could construct a sequence $(Tx_n)_{n\geq 0} \subset TB_1$ of unit vectors without a convergent sub-sequence contradicting compactness of TB_1.

Intuitively compact operators can be viewed as *small.* This feeling is further strengthened by the lemma below. A proof can be found in [17, Theorem 7.12] and will be omitted here.

Lemma 5.1.2. *The space of compact operators $\mathscr{K}(\mathscr{H}) \subseteq \mathscr{B}(\mathscr{H})$ on some Hilbert space $\mathscr{H}$ is exactly the norm closure of the operators of finite rank.*

Example 5.1.3. If we take $\mathscr{H} = \ell^2(\mathbb{N})$ and $T_n \in \mathscr{B}(\mathscr{H})$ with

$$T_n(x_1, x_2, \ldots) = (x_1, 2^{-1}x_2, \ldots, 2^{-n}x_n, 0, \ldots)$$

for $n \in \mathbb{Z}_{\geq 1}$ we can easily see that the sequence $(T_n)_{n\geq 0}$ converges to an operator T of infinite rank. We will show without using Lemma 5.1.2 that $T(B_1)$ admits a compact closure.

Indeed if we let $\epsilon \in \mathbb{R}_{>0}$ be given and pick an $n \in \mathbb{Z}_{\geq 0}$ with $\epsilon > 2^{-n}$, we can define $P_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ to be the natural orthogonal projection

$$P_n((x_m)_{m\geq 0}) = \begin{cases} 0 & m \leq n \\ x_m & m > n \end{cases}.$$

The subset $P_n(T(B_1))$ can now be covered with one ball of size ϵ. The set $T(B_1) \setminus P_n(T(B_1))$ is a subset of the unit disk of a finite dimensional vector space so surely it can be covered by finitely many balls of size ϵ as well. Since ϵ was given arbitrarily we can conclude that $T(B_1)$ is totally bounded. Since it also complete it indeed admits compact closure.

As we can see above, proving that operators are compact without resorting to Lemma 5.1.2 is often more difficult. Unfortunately as has been proven by P. Enflo in 1972 [11] there exist Banach spaces where a result similar to Lemma 5.1.2 does not hold. The example below displays an argument that can be used to determine whether certain integral operators are compact. We will see more of these operators in Section 7.

Example 5.1.4. Consider the operator, $K : \mathscr{C}([0,1]) \to \mathscr{C}([0,1])$ given by

$$f \mapsto \int_0^1 k(x, \_) f(x)\, dx$$

for some continuous $k : [0,1]^2 \to \mathbb{C}$.

Note in particular that k is uniformly continuous. It can be seen easily that K is a well-defined bounded linear operator with norm

$$\|K\| = \left\| \int_0^1 k(x,s)\, ds \right\|_\infty$$

and that all $g \in KB_1$ are bounded by $\|K\|$. Furthermore if the collection of functions in KB_1 can be proven to be equicontinuous (i.e. we can "work with the same $\delta(\epsilon$ for every $f \in KB_1$") we can conclude that any sequence in KB_1 has a (uniformly) convergent subsequence due to the Arzelà-Ascoli Theorem, see A.2.2. This will prove KB_1 to be compact.

So let $\epsilon \in \mathbb{R}_{>0}$ be given and take $\delta \in \mathbb{R}_{>0}$ such that $|x-y| + |s-r| < \delta$ implies $\left\| k(x,s) - k(y,r) \right\| < \epsilon$. This we can do since k is uniformly continuous. We then have for any $f \in B_1$, $K(f) = \int_0^1 k(s, \_) f(s)\, ds \in KB_1$ and we have

$$|K(f)(x) - K(f)(y)| = \left| \int_0^1 (k(x,s) - k(y,s)) f(s)\, ds \right| \le \int_0^1 |k(x,s) - k(y,s)|\, ds < \int_0^1 \epsilon\, ds = \epsilon.$$

Thus all the $g \in KB_1$ are equicontinuous and compactness follows.

Considering the algebraic properties of $\mathscr{K}(\mathscr{H})$ we can note that finite linear combinations of compact operators are compact. More so, if we define $(T_n)_{n\ge0} \subseteq \mathscr{B}(\mathscr{H})$ to be sequences of finite rank operators converging to a compact T by 5.1.2, then for any bounded operator $G \in \mathscr{B}(\mathscr{H})$ we have

$$GT = \lim_{n\ge0} GT_n, \qquad TG = \lim_{n\ge0} T_n G \tag{7}$$

from which we can conclude that the compacts form a two-sided ideal within the C^*-algebra $\mathscr{B}(\mathscr{H})$ thanks to Theorem 5.1.2. We shall refer to the C^*-algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$ as the *Calkin algebra* and use it extensively later on. We define the norm on $\mathscr{B}(\mathscr{H})/\mathscr{K}$ as we did in section 3.3 for general quotient algebras, we shall denote it however by $\|.\|_e$ and call it the *essential norm* of $\mathscr{B}(\mathscr{H})/\mathscr{K}$ to set it apart from the norm of elements in $\mathscr{B}(\mathscr{H})$. The compacts in $\mathscr{B}(V)$ also form an ideal in Banach algebra $\mathscr{B}(V)$. This is not difficult to prove—see [17, Chapter 7]—but it is not needed for our main theorem and we will thus refrain from stating this.

Remark 5.1.5. Now that we have seen the set of compact operators within $\mathscr{B}(\mathscr{H})$ to be the closure of the finite rank operators, we can apply this result to Example 3.1.4 and conclude that the $C^*(\sigma)$ contains *even the compact operators on* $\ell^2(\mathbb{N})$.

We shall conclude our discussion on compact operators with a very deep result. The proof can be found in [2, Section 3.2].

Theorem 5.1.6. *(Fredholm alternative) Suppose $K \in \mathscr{K}(V)$ and $\lambda \in \mathbb{C}$. Then the kernel of $\lambda I - K$ is finite dimensional, the image of $\lambda I - K$ is a closed subspace of V of finite codimension, and we have*

$$\dim\ker(\lambda I - T) = \dim\operatorname{coker}(\lambda I - T).$$

The theorem states in particular that injectivity of $\lambda I - T$ is equivalent with surjectivity of $\lambda I - T$. By the Banach Isomorphism Theorem A.2.4 we then have that the spectrum of $\lambda I - T$ and its set of eigenvalues coincide. As an application, we can use this to give insight in the existence of solutions for certain integral equations, as is done in [2, Remark 3.2.4].

5.2 Fredholm operators

For this section we shall fix a Banach space V. Just like compact operators were considered *small* we can think of a corresponding way to define *large* operators. Intuitively operators $\lambda I - k$ with $k \in \mathscr{K}(V)$ seem to be a right choice by Theorem 5.1.6 but it is a bit too restrictive. For example the unilateral shift 5.2.3 fails to satisfy this definition but is still isometric and has a codimension of 1. Looking at the Fredholm alternative there are two approaches. Essentially instead of requiring a bounded operator $F : V \to V$ to be *equal* to the identity modulo compacts, we can require it to be *invertible* modulo compacts, i.e. there exists $G \in \mathscr{B}(\mathscr{H})$ such that
$FG = GF = I \mod \mathscr{K}(V)$, where T is invertible.

On the other hand we can show a bounded operator $T : V \to V$ with finite dimensional cokernel to have closed image, see [2, Page 95]. We can then consider relaxing

$\dim(\text{coker}(F)) = \dim(\ker(F))$ to merely requiring both the kernel and cokernel are finite dimensional. The main result of this section will state that these generalisations are in fact equivalent and we will call the resulting operators *Fredholm operators*.

Definition 5.2.1. An operator $T \in \mathscr{B}(V)$ is *Fredholm* if the value $\text{ind}(T) := \dim(\ker T) - \dim(\text{coker } T)$ is an integer. We call this value the *(Fredholm-)index associated with* T. The space of all Fredholm operators on V is denoted by $\mathscr{F}(V)$.

As was hinted before the following theorem holds true.

Theorem 5.2.2 (Atkinson's theorem)**.** *A bounded operator T is a Fredholm operator if and only if its equivalence class $\bar{T}$ is invertible in the Calkin algebra $\mathscr{B}(V)/\mathscr{K}(V)$.*

Instead of proving this theorem, we display its power in the next section by stating some corollaries.

Remark 5.2.3. We can see he unilateral shift $\sigma : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ associated with an orthonormal basis $(e_1, e_2, \ldots)$ defined by $\sigma(e_i) = e_{i+1}$ is indeed Fredholm, and with index -1. Composing this operator with the adjoint σ^* which shifts in the opposite direction shows that $\sigma^*\sigma = I$ and $\sigma\sigma^* = I - P$ where P denotes the natural (compact) operator orthogonally projecting on the first coordinate.

The following theorem displays an analogy of the Fredholm index with the winding number. In Section 6 this will prove pivotal in the final steps of proving our main theorem. It is usually proven as a corollary of Theorem 5.2.2, but using a bit of algebra results in a much cleaner proof.

Theorem 5.2.4. *For Fredholm operators $F, G \in \mathscr{B}(V)$ we have*

$$\text{ind}(F) + \text{ind}(G) = \text{ind}(FG).$$

We proceed by using the snake lemma seen below and remind the reader that vector spaces are indeed modules over their Fields. For a proof see [14, Lemma 9.1].

Lemma 5.2.5. *Given modules A, B, C and homomorphisms $F : A \to B$, $G : B \to C$ then the sequence*

$$0 \to \ker f \to \ker gf \to \ker g \to \text{coker } f \to \text{coker } gf \to \text{coker } g \to 0 \tag{8}$$

is exact.

Proof of Theorem 5.2.4. Using the notation in the lemma above, if we now take F, G to be Fredholm operators on the Banach space $V = A = B = C$ we can see that all objects in the sequence are finite dimensional - even without using Atkinsons theorem.

In general we have for exact sequences $V_0 \to V_1 \to \dots \to V_n$ of finite dimensional space that $\sum_{i=1}^n (-1)^i \dim(V_i) = 0$ holds, which shows

$$\operatorname{coker} F - \ker GF + \ker G - \operatorname{coker} F + \operatorname{coker} GF - \operatorname{coker} G = 0 \tag{9}$$

and thus we see

$$\operatorname{ind} F + \operatorname{ind} G = \operatorname{ind} GF \tag{10}$$

to hold. □

5.2.1 Consequences of Atkinsons theorem

To conclude this chapter we will give three consequences of Atkinson's Theorem 5.2.2. One concerns the robustness of the Fredholm index, the other concerning the 'continuity' of the Fredholm index. Proofs can be found in [2, Section 3.3, 3.4]. We remind the reader we fix a Banach space V.

Lemma 5.2.6. *The set* $\mathscr{F}(V)$ *is open in* $\mathscr{B}(V)$

Proof. By Atkinson's theorem 5.2.2, we know $\mathscr{F}(V)$ to be exactly the set of invertible operators. By the text above Lemma 2.2.3 $\mathscr{F}(V)$ then to be open. □

Lemma 5.2.7 (Stability of the index)**.** *Given a Fredholm operator* $F \in \mathscr{F}(V)$ *and a compact operator* $K \in \mathscr{K}(V)$ *we have* $\operatorname{ind} A + K = \operatorname{ind} A$. *In particular we can define a Fredholm index on classes of Fredholm operators* $F \mod \mathscr{K}(V) \in \mathscr{B}(V)/\mathscr{K}$.

Theorem 5.2.8 (Continuity of the index)**.** *Given a Fredholm operator* $A \in \mathscr{F}(V)$, *let* $A_1, A_2, \dots$ *be a sequence of bounded operators with* $\lim_{n\to\infty} \|A_n - A\| = 0$. *Then there is a certain* n_0 *such that* A_n *is Fredholm for* $n \geq n_0$ *and* $\operatorname{ind} A_n = \operatorname{ind} A$.

Remark 5.2.9. Simply put, Theorem 5.2.8 states that the mapping $A \mapsto \operatorname{ind}(A)$ is continuous on $\mathscr{F}(V)$. If we now have a path $\gamma : [0,1] \to \mathscr{F}(V)$, then we know $t \mapsto \operatorname{ind}\gamma(t)$ to be continuous. Since this mapping takes values in $\mathbb{Z}$ we know this mapping to be constant. In other words, each element in the same path component of $\mathbb{F}(V)$ has the same Fredholm index.

Remark 5.2.10. The reader may wonder given the nice properties of Fredholm operators whether there exists a $*$-subalgebra of $\mathscr{B}(\mathscr{H})/\mathscr{K}$ which decomposes into path components consisting of Fredholm operators of the same index. Since every Fredholm operator is invertible in $\mathscr{B}(\mathscr{H})/\mathscr{K}$ however, we can see by Remark 2.2.6 this can only happen when $\mathscr{H}$ is finite dimensional and thus $\mathscr{B}(\mathscr{H})/\mathscr{K} \cong \mathbb{C}$ would need to hold.

6 Toeplitz Operators

Now — after some last few definitions and lemmas — we will move towards our main theorem. The object of interest will be a so called Toeplitz operator T_f on the space $L^2(S^1)$ associated with a certain curve $f \in \mathscr{C}(S^1)$ with $0 \notin \mathrm{Im}(f)$. The bulk of our work will be proving that T_f is a Fredholm operator. From this we will receive the equality $\mathrm{Wn}(f) = -\mathrm{ind}(T_f)$ in Section 6.3.

The result is in itself most remarkable. On the one hand we have the winding number of a curve which we can calculate easily using (complex) analysis and on the other hand an analytical property of an operator. We will apply this in Section 7 to prove existence of solutions for certain indefinite integral equations.

6.1 Algebraic and analytic properties

First we will take a closer look at the space $L^2(S^1)$. As is generally known, the Hilbert space $L^2(S^1)$ admits an orthonormal basis consisting of functions $z \mapsto z^n$ with $n \in \mathbb{Z}$, for a proof we refer [17, Section 3.5]. We then define the *Hardy space* $H^2 := H^2(S^1)$ as the closed subspace of $L^2(S^1)$ generated by the functions $z \mapsto z^n$ with $n \in \mathbb{Z}_{\geq 0}$. Note that the Hardy space consists exactly of those functions holomorphic on the open unit disk should their domains be extended. We denote the orthogonal projection from $L^2(S^1)$ on $H^2(S^1)$ by P.

We can now define Toeplitz operators.

Definition 6.1.1 (Toeplitz operator)**.** Given a curve $f \in \mathscr{C}(S^1)$ we define

$$\begin{aligned} T_f : H^2(S^1) &\to H^2(S^1) \\ g &\mapsto P(f \cdot g) \end{aligned}$$

to be the Toeplitz operator associated with f. We denote the vector space of Toeplitz operators by $\mathscr{T}(\mathscr{C}(S^1))$.

We start our discussion of with a lemma.

Lemma 6.1.2. *The set of invertible elements of* $\mathscr{C}(S^1)$ *is given by*

$$\mathscr{C}(S^1)^\times := \{f \in \mathscr{C}(S^1) : 0 \notin \mathrm{Im}(f)\}.$$

Proof. Suppose $0 \in \mathrm{Im}(f)$ then $f(s) = 0$ for some $s \in S^1$ meaning that $f(s)g(s) \neq 1$ for all $g \in \mathscr{C}(S^1)$. Furthermore if $0 \notin \mathrm{Im}(f)$ we have an inverse of f given by $z \mapsto \frac{1}{f(z)}$ proving the statement. □

Our main result connecting the winding number of $f \in \mathscr{C}(S^1)^\times$ and the index of the associated Toeplitz operator T_f will follow swiftly after proving T_f is Fredholm. Since the theory thus far has given us no tools for analysing the dimension of the kernel or cokernel of T_f we turn to Atkinson's Theorem 5.2.2 to use the characterisation of Fredholm operators as invertibile modulo compacts. Note, however, that in using this theorem, we need the T_f to be bounded whose proof is sketched below.

Lemma 6.1.3. *For every $f \in \mathscr{C}(S^1)$ and its associated Toeplitz operator T_f we have $\|T_f\| = \|T_f\|_e = \|f\|_\infty$, where, as before, $\|\_\|_e$ denotes the norm on the Calkin algebra $\mathscr{B}(H^2)/\mathscr{K}$. Also we have $T_f^* = T_{\overline{f}}$.*

The inequality

$$\|T_f\|_e \leq \|T_f\| \leq \|M_f\| = \|f\|_\infty$$

follows straight from the definition of $\|\_\|_e$ and the contracting nature of projections. The inequality $\|f\|_\infty \leq \|T_f\|_e$ makes use of a standard argument wherein it is used that the Laurent polynomials z, z^{-1} are dense in $L^2(S^1)$.

After having established the fact that $T_g \in \mathscr{B}(H^2)$ the statement $T_f^* = T_{\overline{f}}$ follows the definition of the adjoint. For a full proof see [10, Section 4.1].

In particular we can see from this lemma that no nonzero Toeplitz operator T_f is compact, for if it were, $0 = \|T_f\|_e = \|f\|_\infty$ would hold. Now having assured that our operators T_f are bounded and thus occur in the C^*-algebra $\mathscr{B}(H^2)$ we investigate the multiplicative and involutive behavoir of Toeplitz operators.

Lemma 6.1.4. *Given $h \in H^\infty := H^2 \cap L^\infty(S^1)$ the space H^2 is invariant under M_h i.e. $M_h(H^2) \subseteq H^2$. Thus we can see that $T_h = M_h|_{H^2}$. Moreover for every $g \in L^\infty$ and $h \in H^\infty$ we have*

$$T_g T_h = T_{gh} \quad \textit{and} \quad T_{\overline{h}} T_g = T_{\overline{h}g}.$$

Proof. Note that since $h \in H^\infty$ is bounded we have $h \cdot L^2 \subseteq L^2$. We require $h \cdot H^2 \subseteq H^2$. To prove this, note that h is analytic on the unit disk, i.e. $h = \sum_{n=0}^\infty a_n z^n$ for some $a_n \in \mathbb{C}$. We can then see for any $(z \mapsto z^n) \in H^2$ we have $z^n h \in H^2$ meaning that H^2 is indeed invariant for M_f.

We then have $T_g T_h = PM_g PM_h = PM_g M_h = PM_{gh} = T_{gh}$ and $T_{\overline{h}} T_g = (T_{\overline{g}} T_h)^* = T_{\overline{g}h}^* = T_{\overline{h}g}$, concluding the proof.

□

Remark 6.1.5. Note that we require the function h to be in H^2 to make use of its analytical properties and $h \in L^\infty(S^1)$ to make sure the Toeplitz operator is well-defined. To give an example of a function $f \in H^2$ with $f \notin H^\infty$, we can consider $\ln(1-z) = \sum_{n=0}^\infty -\frac{1}{n} z^n$. Clearly $\ln(1-z)$ goes to infinity as $z \to 1$ and its coefficients are indeed square-summable.

In spite of the previous lemma we can easily show that Toeplitz operators $T_f, T_g \in \mathscr{B}(H^2)$ are not 'multiplicative' in general. For if they would be multiplicative they would be commutative, since

$$T_f T_g = T_{fg} = T_{gf} = T_g T_f.$$

If we take $h(z) = \frac{1}{z}$ and $g(z) = z$ (both bounded on S^1), we can see that for
$1 := z \mapsto z^0 \in L^2(S^1)$

$$T_g T_h(1) = PM_z PM_{\frac{1}{z}}(1) = PM_z(0) \neq 1 = T_{hg}(1) = T_h T_g(1)$$

holds, meaning that Toeplitz operators are not in general commutative. We can, however, show that the Toeplitz operators *do* satisfy these properties over the Calkin algebra $\mathscr{B}(H^2)/\mathscr{K}$.

Lemma 6.1.6. *Suppose $f \in L^\infty(S^1)$. We then have $T_f T_z - T_z T_f$ is of rank at most one ($T_z = PM_z$ denotes the Toeplitz operator multiplying with $z \mapsto z$).*

Proof. We have

$$\begin{aligned} T_f T_z - T_z T_f &= PM_f PM_z - PM_z PM_f \\ &= PM_{fz} - PM_z PM_f \\ &= PM_z M_f - PM_z PM_f \\ &= PM_z(1-P)M_f \end{aligned}$$

For any $e_n := z \mapsto z^n, n \neq 0$ we can now see that $PM_z(1-P)e_n = 0$, meaning that $PM_z(1-P)$ has rank at most one. The result follows. □

We now clarify when Toeplitz operators on $\mathscr{B}(H^2)/\mathscr{K}$ are multiplicative and when they commute.

Theorem 6.1.7. *For all $g \in L^\infty(S^1)$ and continuous $f : S^1 \to \mathbb{C}$ we have*

$$T_f T_g - T_{fg} \in \mathscr{K}, \quad \textit{and} \quad T_g T_f - T_{gf} \in \mathscr{K}.$$

Proof. By the Stone-Weierstrass Theorem A.2.5 we can for any $\epsilon \in \mathbb{R}$ approximate any continuous function f on a nonempty compact subset of $\mathbb{C}$ with a certain polynomial $p(z) = \sum_{n=0}^N a_n z^n$ such that $\|f - p\|_\infty < \epsilon$. Furthermore we can see $PM_p(1-P)$ to be of finite rank, similar to $PM_z(1-P)$.

As in the proof of Lemma 6.1.6, we have $T_{fg} - T_f T_g = PM_f(1-P)M_g$ and

$$\|PM_f(1-P) - PM_p(1-P)\| = \|P(M_{f-p})(1-P)\| \leq \|M_{f-p}\|_\infty = \|f-p\|_\infty < \epsilon.$$

So there exists a sequence of finite rank operators converging to $T_{fg} - T_f T_g$ and since H^2 is Hilbert, we conclude $T_{fg} - T_f T_g$ to be compact.

Note that we can not use the same argument for $T_g T_f = T_{gf}$ as g is not necessarily continuous. We can however take the adjoint, resulting in

$$(T_g T_f - T_{fg})* = T_{\overline{f}} T_{\overline{g}} - T_{\overline{f}\overline{g}}$$

which is compact also. Since the ideal of compacts is closed under the adjoint (see Theorem 3.3.1) in $\mathscr{B}(H^2(S^1)$ we have that $T_g T_f - T_{fg}$ is compact.

□

6.2 Toeplitz algebras

Remember that we use $\mathscr{T}(\mathscr{C}(S^1))$ to denote the space of Toeplitz operators. We shall now combine the results from the previous section in the following theorem on the structure of $\mathscr{A} := \{T_f + K : T_f \in \mathscr{T}(\mathscr{C}(S^1)), K \in \mathscr{K}\}$.

Theorem 6.2.1. *The set* $\mathscr{A} := \{T_f + K : T_f \in \mathscr{T}(\mathscr{C}(S^1)), K \in \mathscr{K}\}$ *is a* C^**-subalgebra of* $\mathscr{B}(L^2(S^1))$

Proof. By Lemma 6.1.3 and since compacts are bounded we know $\mathscr{A} \subseteq \mathscr{B}(L^2(S^1))$ to hold, so $\mathscr{A}$ satisfies the norming properties characteristic to C^*-algebras. Furthermore, we can easily show $\mathscr{A}$ to be closed under algebraic operations as for $T_f + K, T_g + K' \in \mathscr{A}$ it follows that

$$\begin{aligned}(T_f + K)(T_g + K') &= T_f T_g + K T_f K' + K T_g + K K' \\ &= T_{fg} + K'' + K T_g + K T_g + K K' \\ &= T_{fg} + K'''\end{aligned}$$

holds for some $K'', K''' \in \mathscr{K}(H^2)$, since the compacts form an ideal within $\mathscr{B}(L^2(S^1))$, as explained in the text surrounding Equation (7). It can be shown similarly that $\mathscr{A}$ is indeed closed under linear combinations. Lastly since $(T_f + K)^* = T_{\overline{f}} + K^*$ and by Theorem 3.3.2 we know $\mathscr{A}$ to be closed under taking adjoints as well. The property that $\mathscr{T}(\mathscr{C}(S^1))$ is norm-closed is a bit more subtle however and we will make use of the previously established relation $\|T_f\| = \|T_f\|_e = \|f\|_\infty$ in Theorem 6.1.3.

Let $(T_{f_n} + K_n)_{n\geq 0} \subset \mathscr{T}(\mathscr{C}(S^1))$ be a (Cauchy) sequence with limit $X \in \mathscr{B}(S^1)$. We can see that

$$\|f_n - f_m\| = \|T_{f_n - f_m}\| = \|T_{f_n} - T_{f_m}\|_e \leq \|T_{f_n} + K_n - (T_{f_m} + K_m)\| \to 0$$

holds, where the inequality follows straight from the definition of the essential norm. From this we can see that the associated sequence $(f_n)_{n\geq 0}$ converges to some $f \in \mathscr{C}(S^1)$. Hence $\|T_{f_n} - T_f\| = \|f_n - f\| \to 0$, so T_{f_n} converges to T_f. Since we know $K_n \to X - T_f$, $X - t_f$ must be compact by Lemma 5.1.2. So $\mathscr{A}$ is indeed a C^*-algebra. □

Corollary 6.2.2. *The C^*-algebra $\mathscr{A}/\mathscr{K}$ is a commutative unital C^*-algebra isometrically-*-isomorphic with $\mathscr{C}(S^1)$.*

Proof. We know by Theorem 3.3.2 that $\mathscr{A}/K$ is indeed a unital C^*-algebra which is commutative by Lemma 6.1.7. Lastly, since we can easily see the mapping $f \mapsto T_f$ to be linear, multiplicative by Lemma 6.1.4, isometric by Lemma 6.1.3 and surjective by definition 6.1.1 we know it to be invertible as well by the Banach isomorphism theorem. Lastly it maintains adjoints as well by Lemma 6.1.3, meaning it is indeed an isometric *-isomorphic between $\mathscr{C}(S^1)$ and $\mathscr{A}/\mathscr{K}$. □

Before moving to our main theorem we now have a final remark to make on the structure of $\mathscr{A} = \{T_f + K : T_f \in \mathscr{T}(\mathscr{C}(S^1)), K \in \mathscr{K}\}$.

Theorem 6.2.3. *For the C^*-algebra generated by $T_z \in \mathscr{A}$ we have $C^*(T_z) = \mathscr{A}$. Moreover we have*

$$C^*(T_z) \cong \mathscr{K} \oplus \mathscr{C}(S^1).$$

Proof. We already know $\mathscr{A}$ to be a C^*-algebra and $T_z \in \mathscr{A}$. The only thing left to prove now is $\mathscr{A} \subseteq C^*(T_z)$.

First we take $T_f \in \mathscr{T}(\mathscr{C}(S^1)$, associated with $f \in \mathscr{C}(S^1)$. Since S^1 is a compact Hausdorff space it follows from Stone-Weierstrass's theorem A.2.5 that for every $\epsilon \in \mathbb{R}_{>0}$ there exists a polynomial $p(z) = \sum_{i=0}^n a_n z^n \in \mathscr{C}(S^1)$ such that $\|f - p\| < \epsilon$.

Now suppose $\epsilon \in \mathbb{R}_{>0}$. For $\sum_{i=0}^n a_i (T_z)^i \in C^*(T_z)$ we can see

$$\begin{aligned} \sum_{i=0}^n a_i (T_z)^i &= \sum_{i=0}^n a_i T_{z^i} + a_i K_i && \text{with } K_i \in \mathscr{K}(S^1) \\ &= PM_{\sum_{i=0}^n a_i z^i} + \tilde{K} && \text{with } \tilde{K} = \sum_{i=0}^n a_i K_i \\ &= T_p + \tilde{K} && (11) \end{aligned}$$

to hold. Should the compact operators be in $C^*(T_z)$ we can then easily deduce $\mathscr{T}(\mathscr{C}(S^1)) \subset C^*(T_z)$.

We will first prove that all finite rank operators are in $C^*(T_z)$. The idea is related to Remark 5.2.3 on shift operators on the $\ell^2(\mathbb{N})$. Since $z \mapsto z \in H^\infty$, T_z acts on the natural orthonormal basis of H^2 as the unilateral shift σ acts on the natural orthonormal basis of $\ell^2(\mathbb{N})$, by Lemmas 6.1.4 and 6.1.3. More specifically: there is an isometric-*-isomorphism between $C^*(T_z)$ and $C^*(\sigma)$.

Using Example 3.1.4 we can then readily deduce that all finite rank operators on H^2 are present in $C^*(T_z)$ and, more so, all compacts since H^2 is a Hilbert space by 5.1.2 as seen in Remark 5.2.3. From (11) it then follows that $T_p \in C^*(T_z)$. Thus we have $\mathscr{T}(\mathscr{C}(S^1)) \subseteq C^*(T_z)$. Combining these two facts yields $C^*(T_z) = \mathscr{A}$ as required.

Finally we wish $C^*(T_z) \cong \mathscr{K} \oplus \mathscr{C}(S^1)$ as (C^*-algebras) to hold. Since we have $\mathscr{A} \cong C^*(T_z)$ and $\mathscr{A}/\mathscr{K} \cong \mathscr{C}(S^1)$ by Corollary 6.2.2 the statement is equivalent to proving that the short exact sequence

$$0 \to \mathscr{K} \to \mathscr{A} \to \mathscr{A}/\mathscr{K}$$

admits a continuous section. We can see this property holds, since, if we write $q : \mathscr{A} \to \mathscr{A}/\mathscr{K}$ and $s(\overline{T_f}) = T_f$ we know $(q \circ s)(T_f) = q(\overline{T_f}) = T_f$, which is a composition of *-homomorphisms between C^*-algebras so clearly this map is continuous. We can then apply the splitting lemma and obtain our result.

□

6.3 The Toeplitz index theorem for a continuous symbol

Now all the ingredients are in place to phrase and discuss our main result.

Theorem 6.3.1. *(Toeplitz Index Theorem) Let $f \in \mathscr{C}(S^1)$. The operator T_f is Fredholm if and only if $0 \notin \mathrm{Im}(f)$. Furthermore if T_f is Fredholm its index satisfies*

$$\mathrm{ind}(T_f) = -\mathrm{Wn}(f).$$

Proof. By Theorem 5.2.2 and Corollary 6.2.2 we know $T_f + K$ to be Fredholm for $K \in \mathscr{K}$ if and only if f is invertible. Assume so and thus $0 \notin \mathrm{Im}\, f$. In Section 4 we saw that f is homotopic to $z \mapsto z^n$, for some $n \in \mathbb{Z}$. Denote this homotopy by $F : [0,1] \times S^1 \to \mathbb{C}$, with $F(0,s) = f(s)$ and $F(1,s) = s^n$. We can then see that the mapping $s \mapsto T_{F(t,s)}$ is a continuous mapping as a composition of two continuous functions (note $f \mapsto T_f$ is continuous due to Corollary 6.2.2 and Lemma 3.1.3). Now, due to 5.2.3, and the fact that T_{z^n} acts as shifting n places on the basis of $H^2(S^1)$ we have $\mathrm{Wn}(f) = \mathrm{Wn}(z \mapsto z^n) = -\mathrm{ind}(T_{z^n}) = -\mathrm{ind}(T_f)$ yielding our main result. □

To give the reader a bit of context why this theorem is so cherished in pure mathematics we note it is a special case of the Atyiah-Singer index theorem. This theorem was first pulbished in the Paper [3] in 1963 by Michael Atiyah and Isadore Singer. It roughly relates a topological index, here the winding number, to an analytical index, here the Fredholm index. The discovery of this theorem paved the way for an entire new field of mathematics called index theory which has applications in fields such as geometry, topology as well as physics see [4]. As promised we shall see one particular application in the next section.

7 Application: Wiener–Hopf operators

Having build up the preceding theory we now demonstrate an application of Toeplitz operators to integral equations. In doing so, we view Toeplitz operators as a specific case of so-called Wiener–Hopf operators. This more general definition allows certain integral operators to be defined and we will display an interesting connection between these and Toeplitz operators. The material is loosely based on chapter 9 of [6] but will feature explicit calculations, explanation and examples not present in this book.

Definition 7.0.1 (Wiener–Hopf operators)**.** Let a vector space V a linear operator $A : V \to V$ and a projection $P : V \to V$ (a mapping satisfying $P^2 = P$) satisfying $\operatorname{Im} P = U$ be given. The *Wiener–Hopf operator associated with A and P* is given by

$$W : U \to U, \qquad W = PA.$$

Clearly, Toeplitz operators are Wiener–Hopf operators, where P represents the projection on the Hardy space, and A represents a multiplication operator.

If we take in the above definition $V = L^p(\mathbb{R})$, $1 \le p \le \infty$, $P : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ defined by

$$P(f)(x) = \begin{cases} 0 & x \le 0 \\ f(x) & x > 0 \end{cases}$$

and $A : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ with $A(\phi) = c\phi + \int_0^\infty k(\_ - s)\phi(s)\,ds$ we can define the following operator.

Definition 7.0.2 (Wiener–Hopf integral operators)**.** The Wiener–Hopf integral operator associated with $c \in \mathbb{C}$ and a real-valued $k \in L^1(\mathbb{R})$ is defined as a linear operator on $L^p(\mathbb{R}_+) := \{f \in L^p(\mathbb{R}) : f(x) = 0, \text{ for } x < 0\}$ by

$$W(\phi)(t) = \begin{cases} c\phi(t) + \int_0^\infty k(t-s)\phi(s)\,ds & t \ge 0 \\ 0 & t < 0 \end{cases}. \tag{12}$$

Note we defined $U = \operatorname{Im}(P)$ in this instance of a Wiener–Hopf operator. We write $L^p(\mathbb{R}_+) := \operatorname{Im} P$ and this is the space on which W acts.

Having stated this definition some remarks are in place as to it being well defined. Firstly the convergence of the integral follows directly from Young's inequality A.2.1, which states that for any $s, q, r \in \mathbb{R}$, $1 \le s, q, r \le \infty$ satisfying $1/s + 1/q = 1/r + 1$, we have for $f \in L^s, g \in L^q$ that $f \star g \in L^r$ and $\|f \star g\|_r \le \|f\|_s \|g\|_q$. In the setting of 12 we can see we have $s = 1$ and $q = r = p$, indeed satisfying this inequality.

Since Wiener–Hopf integral operators W themselves are difficult to analyse — we will for example not be able to use the Fredholm alternative 5.1.6 as we will see in 7.1.3 — we would rather analyse a simpler object closely related to W.

Definition 7.0.3. Given a Wiener–Hopf integral operator W as above. We define $a : \mathbb{R} \to \mathbb{C}$ to be the bounded continuous function with

$$a(\xi) := c + \int_{-\infty}^{\infty} e^{i\xi x} k(x)\, dx =: c + Fk(\xi). \tag{13}$$

We call this the *symbol* of W and shall write $W_a := W$.

Remark 7.0.4. Note that the symbol — which will play a crucial role in the section to come — is nothing more than the Fourier transform $F : L^1(\mathbb{R}) \to C(\mathbb{R})$ of the function k translated with a scalar c. From [16, Theorem 9.6], we can indeed see that $Fk \in L^\infty(\mathbb{R})$ and $||Fk||_\infty \le \|k\|_1$. Furthermore Fk will converge to 0 on both positive and negative infinity. For this reason we can see $\lim_{\xi \to \pm\infty} a(\xi) = c$. Finally, to explore the link between a symbol and its associated integral operator we state the following for the special case $p = 2$.

Lemma 7.0.5. *Denote by $S \subseteq L^\infty(\mathbb{R})$ the space of bounded continuous functions of the form $a(\xi) = c + Fk(\xi)$, for $c \in \mathbb{C}$, and $k \in L^1(\mathbb{R})$. Then the mapping*

$$\Psi : S \to \mathscr{B}(L^2(\mathbb{R}_+)), \qquad a \mapsto W_a$$

is a linear isometry of Banach spaces.

Proof. Verifying the linearity of Ψ is trivial. We will show that $\|a\|_\infty = \|W_a\|_\infty$ holds. Using the Fourier-Plancherel theorem (see [16, Theorem 9.13]) we have a map $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfying $\mathscr{F}|_{L^1} = F|_{L^2}$ and $\left\|\mathscr{F} f\right\|_2 = \sqrt{2\pi}\left\|f\right\|_2$, for $f \in L^2(\mathbb{R})$. We can then firstly derive for $f \in L^2(\mathbb{R})$

$$2\pi \left\|W_a f\right\|_2^2 = \left\|\mathscr{F}(W_a f)\right\|_2^2 = \left\|c\mathscr{F} f + \mathscr{F}(k \star f)\right\|_2^2.$$

Note that by Young's inequality we have $k \star f \in L^2(\mathbb{R})$. In [16, Theorem 9.2] it is shown that the Fourier transform F on $L^1(\mathbb{R})$ satisfies the property that $F(h \star l) = F(h)F(l)$, for $h, l \in L^1(\mathbb{R})$ and we can show $\mathscr{F}(k \star f) = F(k)\mathscr{F}(f)$ to hold, by using that $L^1 \cap L^2$ is dense in both L^1 and L^2 (under their respective topologies).

Having obtained this, we derive

$$\begin{aligned}
2\pi \left\|c\mathscr{F} f + \mathscr{F}(k \star f)\right\|_2^2 &= \int_{-\infty}^{\infty} |c(\mathscr{F} f)(t) + (Fk)(t)(\mathscr{F} f)(t)|^2\, dt \\
&= \int_{-\infty}^{\infty} |(c + (FK)(t))|^2 |(\mathscr{F} f)(t)|^2\, dt \\
&\le \|c + (Fk)\|_\infty^2 \int_{-\infty}^{\infty} (\mathscr{F} f)(t)^2\, dt \\
&= \|a\|_\infty^2 \left\|\mathscr{F} f\right\|_2^2 \\
&= 2\pi \|a\|_\infty^2 \left\|f\right\|_2^2,
\end{aligned}$$

which yields $\|W_a\| \leq \|a\|_\infty$.

For the opposite inequality $\|W_a\| \geq \|a\|_\infty$ we construct a sequence $(f_n)_{n\geq 0} \subseteq L^2(\mathbb{R}_+)$ satisfying $\|f_n\|_2 = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|W_a f_n\|_2 = \|a\|_\infty$. To do so, first note that by Remark 7.0.4 a is a continuous bounded function on $\mathbb{R}$ which tends to zero near $\pm\infty$. We then know that $\max_{\xi\in\mathbb{R}} a(\xi) = a(\tilde{t})$ exists for some $\tilde{t} \in \mathbb{R}$.

The idea is to define $(f_n)_{n\geq 0}$ in such a way that $(f_n(\tilde{t}))_{n\geq 0}$ converges to infinity while still having $\|f_n\|_2 = 1$ for all $n \in \mathbb{N}$. Write thus

$$x \mapsto f_n = \sqrt{\frac{1}{2\pi}}\mathscr{F}^{-1}\left(\frac{1}{n\sqrt{\pi}}e^{-\frac{1}{2}(\frac{x}{n})^2}\right) =: \sqrt{\frac{1}{2\pi}}\mathscr{F}^{-1}(\delta_n).$$

making f_n essentially the inverse Fourier-Plancherel transform a normal distribution δ_n centered around 0 scaled to have a 2-norm of 1. We indeed note that $f_n \in L^2(\mathbb{R})$ and $\|f_n\|_2 = 1$, and see that

$$\begin{aligned}
\|W_a f_n\|_2^2 &= \frac{1}{2\pi}\left\|W_a\mathscr{F}^{-1}\delta_n + \int_0^\infty (\mathscr{F}^{-1}\delta_n)(t)k(_ - t)\,dt\right\|_2^2 \\
&= \left\|c\delta_n + (\mathscr{F}\mathscr{F}^{-1}\delta_n \cdot Fk\right\|_2^2 \\
&= \int_{-\infty}^\infty \delta_n(s)^2|c + (Fk)(s)|^2\,ds \\
&\to |(c + Fk)(\tilde{t})|^2 = ||a||_\infty
\end{aligned}$$

holds, meaning that indeed $\|W_a\|_2 \geq \|a\|_\infty$. □

Remark 7.0.6. As has been said, the existence of the mapping $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ we used, stems from the Fourier-Plancherel Theorem [16, Theorem 9.13], and we will refer to it as the *Fourier-Plancherel transformation*. For our application we need this map instead of the Fourier transformation $F : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$. Even though the mappings F and $\mathscr{F}$ agree on $L^1 \cap L^2$, we do not have an explicit form for $\mathscr{F}$ contrasting its counterpart F, as seen in (13). Usually we resolve this by first looking at $L^1 \cap L^2$ before using the fact that $\overline{L^1 \cap L^2} = L^2$. In Theorem 7.1.2 below we will simply prove the statement for functions $L^1 \cap L^2$ and the rest to the reader to avoid more technicalities.

Lemma 7.0.5 shows writing W_a for the *Wiener–Hopf integral operator assoicated with symbol a* 'makes sense' and we will henceforth fix $p = 2$. Do note that the mapping $a \mapsto W_a$ is not multiplicative in general, making us unable to describe the spectrum of W_a completely through a.

The rest of this section will have the following outline. First we show the connection between Wiener–Hopf operators and Toeplitz operators whereafter we use the index theorem 6.3.1 to state results on the existence and uniqueness of solutions of $W_a(\phi) = g$, with $\phi, g \in L^2(\mathbb{R}_+)$. Finally we will present a worked example with a particularly nice geometrical interpretation.

7.1 The correspondence between a Toeplitz operator and a Wiener–Hopf operator

The main aim of this subsection is to construct an isometric isomorphism mapping W_a to some Toeplitz operator T_f. Since T_f is uniquely defined by f we wonder what a suitable pick for $f \in \mathscr{C}(S^1)$ would be. In light of the discussion above, we turn to the symbol $a : \mathbb{R} \to \mathbb{C}$, yet it is not defined on S^1. We can resolve this problem using some domain transformations which we will state shortly. The necessary properties are summarised in the next lemma.

Lemma 7.1.1. *If we define U and $U_\#$ as follows,*

$$U : L^2(S^1) \to L^2(\mathbb{R}), \quad (U\phi)(x) = \frac{\sqrt{2}}{i+x}\phi\left(\frac{i-x}{i+x}\right), \qquad x \in S^1, \tag{14}$$

$$U_\# : L^\infty(S^1) \to L^\infty(\mathbb{R}), \quad (U_\# a)(x) = a\left(\frac{i-x}{i+x}\right), \qquad x \in \mathbb{R}, \tag{15}$$

the following statements hold true.

1. *The mapping U is an isometric isomorphism of vector spaces with inverse*

$$(U^{-1}\phi)(t) = \frac{i\sqrt{2}}{i+t}\phi\left(i\frac{1-t}{1+t}\right), \quad \textit{for } \phi \in L^2(\mathbb{R}), t \in S^1 \setminus \{-1\}.$$

2. *The mapping $U_\#$ is an isometric isomorphism of C^*-algebras with inverse*

$$(U_\#^{-1} a)(t) = a\left(i\frac{i-t}{i+t}\right), \quad \textit{for } a \in L^\infty(\mathbb{R}), t \in \mathbb{R}.$$

3. *The restriction $U_\#|_{\mathscr{C}(S^1)}$ defines an isometric isomorphism between $\mathscr{C}(S^1)$ and $\mathscr{C}(\mathbb{R} \cup \{\infty\})$.*

4. *We have $UH^2(S^1) = \mathscr{F}L^2_+ = H^2(\mathbb{R})$ and $U_\# H^\infty(S^1) = H^\infty(\mathbb{R})$.*

The properties 1-3 are easily satisfied and the reader is encouraged to look at Appendix A.3 for proofs and issues of well-definedness. The transformations U and $U_\#$ stem from the so called *Cayley transformations* on which more can be found in Chapter 6 of [12]. Property 4 with a proof can be found in [16, Chapter 9].

Now we can make the relation between the integral and Toeplitz operators clear. Firstly we note that $\frac{1}{\|\mathscr{F}\|}U^{-1}F$ is an isometric isomorphism between $L^2(\mathbb{R}_+)$ onto $H^2(S^1)$, with inverse $\frac{1}{\|\mathscr{F}^{-1}\|}\mathscr{F}^{-1}U$. This leads us to the following theorem, and we will give a sketch of its proof.

Theorem 7.1.2. *Let W_a be a Wiener–Hopf integral operator associated with symbol a. Then we have the following equality,*

$$W_a = \mathscr{F}^{-1} U T_{U_\#^{-1} a} U^{-1} \mathscr{F}, \tag{16}$$

where $T_{U_\#^{-1} a}$ denotes the Toeplitz operator associated with $U_\#^{-1} a$. Moreover if $a(\xi) \neq 0$ for all $\xi \in \mathbb{R} \cup \{\pm\infty\}$ we have

$$\dim\ker W_a = \dim\ker T_{U_\#^{-1} a}, \quad \dim\operatorname{coker} W_a = \dim\operatorname{coker} T_{U_\#^{-1} a}.$$

Proof. Suppose $a(\xi) = c + Fk(\xi)$ for some $k \in L^1(\mathbb{R})$. We want to prove that for any $\phi \in H^2(S^1)$ we have $T_{U_\#^{-1} a}(\phi) = U^{-1} F W_a F^{-1} U(\phi)$. As said in Remark 7.0.6 we will first consider $\phi \in H^2(S^1)$ with $U(\phi) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}_+)$.

Instead of introducing new notation, we will denote $(F^{-1} f)(x) := \int_{-\infty}^{\infty} e^{-ixs} f(s)\, ds$ for any $f \in L^1(\mathbb{R})$ even if f is not in the range of F. We then have that $(F^{-1} U)(\phi)(x) = \int_{-\infty}^{\infty} e^{-ixs} U(\phi)(s)\, ds$, and that $F^{-1} U\phi \in L^2(\mathbb{R}_+)$ by Statement (4) of Lemma 7.1.1. Composing with W_a yields

$$\begin{aligned}
W_a F^{-1} U(\phi)(x) &= cF^{-1} U\phi(x) + \int_0^\infty \int_{-\infty}^\infty e^{-tsi} U(\phi)(s)\, ds k(x-t)\, dt \\
&= cF^{-1} U\phi(x) + \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-tsi} U(\phi)(s)\, ds k(x-t)\, dt \\
&= cF^{-1} U\phi(x) + \int_{-\infty}^\infty \int_{-\infty}^\infty e^{(v-x)si} U(\phi)(s) k(v)\, dv\, ds \\
&= cF^{-1} U\phi(x) + \int_{-\infty}^\infty e^{-xsi} \int_0^\infty e^{vsi} k(v)\, dv U(\phi)(s)\, ds \\
&= cF^{-1} U\phi(x) + \int_{-\infty}^\infty e^{-xsi} U(\phi)(s)(Fk)(s)\, ds \\
&= cF^{-1} U\phi(x) + F^{-1}(U(\phi)(Fk))(x).
\end{aligned}$$

$$\begin{aligned}
\text{Hence } U^{-1} F W_a F^{-1} U(\phi)(x) &= (c + (Fk)(x))\phi(x) \qquad \textit{after applying } U^{-1} F \\
&= T_{U_\#^{-1}(c+(Fk))} \phi(x)
\end{aligned}$$

where in the last step we used that the function $W_a F^{-1} U(\phi) \in L^2(\mathbb{R}_+)$ yielding by Lemma 7.1.1 that $U^{-1} F W_a F^{-1} U(\phi) \in H^2(\mathbb{R})$. Note that the domains of the integrals could be adjusted since $FU^{-1}\phi \in L^2(\mathbb{R}_+)$.

We get

$$W_a(\psi) = F^{-1} U T_{U_\#^{-1} a} U^{-1} F(\psi)$$

for every $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}_+)$ with $F\psi \in L^1(\mathbb{R})$, hence we obtain

$$W_a(\psi) = \mathscr{F}^{-1} U T_{U_\#^{-1} a} U^{-1} \mathscr{F}(\psi), \tag{17}$$

for every $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}_+)$ with $F\psi \in L^1(\mathbb{R})$.

As said in Remark 7.0.6, it can be shown that the set of these ψ is dense in $L^2(\mathbb{R}_+)$. Since the operators on the left and right hand side in (17) are all bounded on $L^2(\mathbb{R}_+)$, we obtain (17) for general $\psi \in L^2(\mathbb{R}_+)$.

Finally by the Toeplitz index theorem 6.3.1 we have

$$\dim\ker T_{U_\#^{-1}a} < \infty, \qquad \dim\operatorname{coker} T_{U_\#^{-1}a)} < \infty$$

and since $\frac{1}{\|F\|}U^{-1}F$ is an isometric isomorphism with inverse $\frac{1}{\|F^{-1}\|}F^{-1}U$ we have the same properties for the kernel and cokernel of W_a. □

Now that we have established this connection we can see that by the Toeplitz index theorem for a continuous symbol a, the corresponding Wiener–Hopf operator W_a will be Fredholm if and only if $0 \notin \operatorname{Im}(U_\#^{-1}a) = \operatorname{Im}(a)$. As a small corollary the equation

$$W_a(\phi)(t) = c\phi(t) + \int_0^\infty k(t-s)\phi(s)\,ds = 0$$

will have a solution for non trivial ϕ only if the winding number associated with its symbol will be negative.

One could wonder whether we could not simply use the Fredholm alternative 5.1.6 to analyse Wiener–Hopf integral operators.

Corollary 7.1.3. *For $k \in L^1(\mathbb{R})$, $k \neq 0$, the integral operator $T : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$*

$$T(f(t)) = \int_0^\infty k(t-s)f(s)\,ds$$

is not compact.

Proof. If T were compact, for any $c \in C$ the corresponding Wiener–Hopf operator would be of the form $cI - T$ with $T \in \mathcal{K}(L^2(\mathbb{R}_+))$. However using Theorem 7.1.2 this would imply that we can associate a Toeplitz operator of the similar form $cI - \tilde{K}$, for some $\tilde{K} \in \mathcal{K}(H^2(S^1))$, implying that $\tilde{K}$ would be a Toeplitz operator meaning that $0 = \tilde{K}$ as corollary of Lemma 6.1.3. Thus $T = 0$ contradicting with the fact that $k \neq 0$, $\mu - a.e.$ □

Remark 7.1.4. Corollary 7.1.3 might seem counter-intuitive as we have proved in 5.1.4 that some similarly shaped integral operators are compact. We must note however we can not use Arzela-Ascoli A.2.2 to prove since the integration domain in 7.0.1 is clearly not compact.

7.2 A worked example

Having established this theory we turn to a worked example. Let $c, z \in \mathbb{C}$ and observe the following Wiener–Hopf integral operator

$$W_a f(t) = cf(t) + \int_0^\infty f(s) e^{-(t-s-z)^2/2} ds, t \geq 0, \tag{18}$$

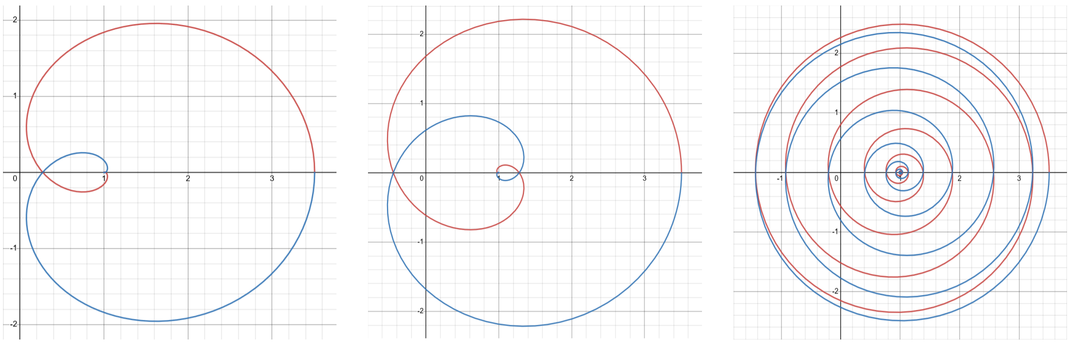
associated with symbol

$$a(\xi) = c + \int_{-\infty}^{\infty} e^{i\xi s} e^{-\frac{1}{2}(s-z)^2} ds = c + \sqrt{2\pi} e^{i\xi z} e^{-\frac{\xi^2}{2}}. \tag{19}$$

Clearly, we have taken $k(t) = e^{-\frac{(t-z)^2}{2}}$. In deriving (19), one uses the fact that $e^{-\frac{1}{2}t^2}$ is an eigenvector for Fourier transform F with eigenvalue $\sqrt{2\pi}$ along with a simple derivation. It is easy to see for nonzero $c, z \in \mathbb{C}$, that, $0 \notin \text{Im}(a)$ meaning that $T_{U_\#^{-1} a}$ will indeed be Fredholm which yields $U_\#^{-1} a$ will possess finite winding number around zero.

Having stated this, we wish to get some insight in the winding number associated with the Toeplitz operator $T_{U_\#^{-1} a}$. It may seem tempting to use complex analysis to calculate the winding number trying to solve $\int_{S^1} \frac{1}{U^{-1} a(\zeta)} d\zeta$ but this is ill-advised as the resulting integral may be very difficult to handle.

Instead noting that $\text{Im}(a) = \text{Im}(U_\#^{-1} a)$ We can thus also look at $a(\xi)$ directly. The shape of $a(\xi)$ suspects that as ξ goes to $\pm\infty$, $a(\xi)$ will converge to c. In doing so, the factor $e^{i\xi z}$ will ensure that a will spin around this point infinitely many times. This is not a problem however as we are as before only interested in the winding number around zero. It behavior is best seen in the following pictures.



For $z = 2$, $c = 1$. For $z = 3$, $c = 1$. For $z = 13$, $c = 1$.

The red lines denote the positive values for ξ whereas the blue lines denote the negative values of ξ. Starting at $c = 1$ the curve thus moves counterclockwise as $\xi \to \infty$. From the pictures it is clear that the winding number of $a(\xi)$ will be 0, 2, 6 for

$z = 2,\ z = 3,\ z = 13$, respectively. To prove this in a more formal fashion we proceed as follows.

Assuming z to be an integer and since we assume c to be positive we are only interested in the amount of times $a(\xi)$ intersects with the real axis while a is negative real. In the spirit of this we substitute $\xi = \frac{k\pi}{z}$ and derive as follows:

$$\begin{aligned}
a(\xi) = c + \sqrt{2\pi}e^{i\xi z}e^{-\frac{1}{2}\xi^2} && < 0 \\
\Leftrightarrow c + \sqrt{2\pi}e^{i\xi z}e^{-\frac{k^2\pi^2}{2z^2}} && < 0 \\
\Leftrightarrow c - \sqrt{2\pi}e^{-\frac{1}{2}\xi}e^{-\frac{k^2\pi^2}{2z^2}} && < 0 \\
\Leftrightarrow \frac{c}{\sqrt{2\pi}} && < e^{-\frac{1}{2}\xi}e^{-\frac{k^2\pi^2}{2z^2}} \\
\Leftrightarrow \frac{-2z^2\ln(\frac{c}{\sqrt{2\pi}})}{\pi^2} && > k^2.
\end{aligned}$$

We can now explicitly state for which k we have solutions. As we can see in Figure 7.2, we note that the winding number always must be an integer multiple of two. Also note that the condition $c < \sqrt{2\pi}$ must always be satisfied, regardless of our value of z.

We now know that for example $z = 3$ we indeed have a winding number of 2, meaning that by the Toeplitz index theorem 6.3.1 we obtain

$$\dim\operatorname{coker} W_a - \dim\ker W_a = -\operatorname{ind}(W_a) = \operatorname{Wn}(a) = 2.$$

This means that our integral operator has a nontrivial cokernel of at least dimension 2 meaning that for at least two linear independent $g \in L^2(\mathbb{R}_+)$,

$$cf(t) + \int_0^\infty f(s)e^{\frac{-(t-s-3)^2}{2}}\,ds = g(t)$$

has no solution $f \in L^2(\mathbb{R})$.

Should we have taken $\tilde{z} = -z$ instead, we would have obtained a symbol

$$\tilde{a}(\xi) = c + \sqrt{2\pi}e^{-i\xi z}e^{-\frac{\xi^2}{2}}$$

with the same image but rotating in the opposite direction. This would give us a Wiener–Hopf integral equation with index at least 2 meaning that

$$cf(t) + \int_0^\infty f(s)e^{\frac{-(t-s+3)^2}{2}}\,ds = 0$$

has at least solution 2 linearly independent solutions. More generally if

$$cf(t) + \int_0^\infty f(s)e^{\frac{-(t-s+3)^2}{2}}\,ds = g(t)$$

would have solutions, at least two would be linearly independent.

7.3 Further notes

Note that in Section 7.2 we could only give bounds on the amount of linearly independent solutions since the Fredholm index theorem 6.3.1 only makes a statement on the index, not on $\dim\ker W_a$ and $\dim\operatorname{coker} W_a$ separately. Strengthening the Fredholm index theorem to make statements $\dim\ker W_a$ and $\dim\operatorname{coker} W_a$ would be difficult, however as any winding number can correspond to multiple Toeplitz operators with different sizes of kernels and co-kernels.

For example if we write $f(z) = z^n$ and if we once again describe T_f as a unilateral shift on an orthonormal basis we can see $\dim\ker(T_f T_{f^{-1}}) = n$ and $\dim\operatorname{coker}(T_f T_{f^{-1}}) = n$, which leaves us with an associated Wiener–Hopf integral operator using 7.1.2 with a kernel and co-kernel of dimension n. On the other hand, $T_{f^{-1}} T_f$ will simply be the identity with trivial kernel and co-kernel which carries over to its associated integral operator. The curves $f \odot f^{-1}$ and $f^{-1} \odot f$ are still homotopic however.

In order to resolve this problem, we would need to find a way to count winding numbers *as we are stepping through the curve*, but it is not obvious as to how this should be done. For instance, we know any curve with nonzero winding number should pass the positive real axis and should self-intersect but the converse is not true in general, indicating this information is not sufficient to determine the winding number of our curve. Giving a formal proof of the conjecture that in example 7.2 we can replace "at least" with "exactly" will thus be difficult.

The reader could also wonder if we can make statements on the spectrum of Wiener–Hopf integral operators defined on general L^p spaces. We, of course, have no correspondence with Toeplitz operators as the Fredholm index theorem heavily relied on the fact that H^2 was a Hilbert space. Furthermore, it is not clear the Wiener–Hopf integral operators form a normed algebra or what the closure of such an algebra would look like, so using techniques like 2.4.4 will be difficult. Lastly, note we can not use the symbol either to deduce spectral properties since the mapping in 7.0.5 is of course not multiplicative.

A Appendix

A.1 Kernels and Cokernels

Below is a short discussion on kernels and cokernels and their place in functional analysis. If we view the space of bounded operators $\mathscr{B}(V)$ on a Banach space V we know the space $V/\ker(T)$ to be a Banach space since bounded operators always have closed kernels.

If we require the image of our operator T to be closed we can also define the Banach space $\operatorname{coker} T := V/\operatorname{Im}(T)$ which we call *cokernel* of T. We say $\dim \operatorname{coker} T$ to be the *codimension* of T. We can note the whereas injectivity of T is equivalent with having a trivial kernel, surjectivity is equivalent with having a trivial cokernel.

In functional analysis we can consider $\operatorname{coker}(T)'$, the space dual to the cokernel and note that it is isomorphic to the *annihilator* of $\operatorname{Im}(T)$, the set $\operatorname{Ann}(\operatorname{Im}(T)) = \{f \in V^* : \operatorname{Im}(T) \subseteq \ker(f)\}$.

Indeed, if we have a $f \in \operatorname{coker}(T)$ and denote $q : V \to V/(\operatorname{Im}(T))$ as the quotient mapping, we know $f \circ q \in \operatorname{Ann}(\operatorname{Im}(T))$. Likewise for any $g \in \operatorname{Ann}(\operatorname{Im}(T))$ we know $\operatorname{Im}(T) \subseteq \ker(g)$ thus we can factor through the cokernel of T as in

$$\begin{array}{ccc} V & \xrightarrow{\;g\;} & \mathbb{C} \\ & \searrow^{q} & \uparrow{\scriptstyle h} \\ & & \operatorname{coker}(T) \end{array}$$

yielding a function $h \in \operatorname{coker}(T)'$. Checking that the mappings $f \mapsto q \circ f$ and $g \mapsto h$ are linear and inverses of one another is now a straightforward excersise. In case $\dim(\operatorname{coker}(T)) < \infty$ we have immediately from the definition of the adjoint $T' : V' \to V'$ of T that $\operatorname{Ann}(T) = \ker(T')$ thus,

$$\dim(\ker T') = \dim(\operatorname{coker}(T))$$

giving yet another reason why it makes sense to call $V/\operatorname{Im}(T)$ a cokernel in functional analysis.

A.2 Supposed known theorems

This appendix subsection consists of theorems assumed to be known to the reader, but are included for sake of completeness nevertheless

Theorem A.2.1 (Young's convolution inequality)**.** *Let G be $\mathbb{R}$, S^1 or $\mathbb{Z}$ and suppose $f \in L^p(G), g \in L^q(G)$, with*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

and $1 \leq p, q, r \leq \infty$. *Then*

$$\|f \star g\|_r \leq \|f\|_p \|g\|_q.$$

where $(f \star g)(x) := \int_G f(x-t)g(t)\,dt$

For a proof see [5].

Theorem A.2.2 (Arzela-Ascoli Theorem on compactness)**.** *If X is compact and $U \subset \mathscr{C}(X)$ then U is compact iff U is closed bounded and equicontinuous.*

For a proof [8].

Theorem A.2.3 (Alaoglu's Theorem for Banach spaces)**.** *Given a Banach space V then the set*

$$\{f \in V' : \|f\|_\infty \leq 1\}$$

is weak- compact.*

For a proof see [8].

Theorem A.2.4 (Banach Isomorphism Theorem)**.** *Given a bounded bijective linear operator $T : U \to V$ of Banach spaces U and V. Then T is invertible.*

For a proof see [17].

Theorem A.2.5 (Stone-Weierstrass Theorem)**.** *If X is compact and A is a unital closed subalgebra of $C(X)$ such that for all $x, y \in X, x \neq y$ there exists $f \in A$ such that $f(x) \neq f(y)$ and for all $f \in A$ we have $\bar{f} \in A$, then $A = C(X)$ holds.*

In particular, the polynomials $f : X \to \mathbb{C}$ are dense in $C(X)$.

Theorem A.2.6 (Liouville's theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is a bounded and holomorphic function on $\mathbb{C}$ then it is constant.*

For a proof see [8].

A.3 Transforming the Hardy space

This section will be dedicated to working out some details on the transformations used in Section 7.1. We denote the mapping U by

$$U : L^2(S^1) \to L^2(\mathbb{R}), \quad (U\phi)(x) = \frac{\sqrt{2}}{i+x}\phi\left(\frac{i-x}{i+x}\right), \qquad x \in \mathbb{R}.$$

We can see that this mapping is invertible by U^{-1} defined as

$$U^{-1} : L^2(\mathbb{R}) \to L^2(S^1), \quad (U^{-1}\phi)(t) = \frac{i\sqrt{2}}{1+t}\phi\left(i\frac{1-t}{1+t}\right), \quad t \in T \setminus \{-1\}.$$

Note that L^p spaces are defined as classes of functions that agree $\mu - a.e.$, hence we do not fret over the fact that $U^{-1}(\phi)(t) \in L^2(S^1)$ is not defined for $t = -1$. For sake of convention can we simply define $U^{-1}(\phi)(-1) := 0$. Furthermore it is easy to verify these mapping U, U^{-1} are isometric isomorphisms on their respective vector spaces.

We will show the isometric property. We have for $\phi \in L^2(S^1)$,

$$\|U\phi\|_2^2 = \int_{-\infty}^{\infty} \left| \frac{\sqrt{2}}{i+x} \phi\left(\frac{i-x}{i+x}\right) \right| dx = \int_{S^1 \setminus \{-1\}} \left|\phi(u)^2\right| du = \|\phi\|_2^2$$

where we used the substitution $u = \frac{i-x}{i+x}$.

Lastly, for $\phi(i\frac{1-t}{1+t})$ to be well defined, we need $i\frac{1-t}{1+t}$ to be real for $t \in S^1$. Taking $t = \cos\theta + i\sin\theta$, $\theta \in (-\pi, \pi)$ we can indeed see that $i\frac{1-t}{1+t} = \frac{sin\theta}{1+cos\theta} = \tan(\theta/2) \in \mathbb{R}$ for $\theta \in (-\pi, \pi)$, yielding a bijection between $S^1 \setminus \{-1\}$ and $\mathbb{R}$.

For $a \in L^\infty(S^1)$ we also define $(U_\# a)(x) = a\left(\frac{i-x}{i+x}\right), x \in \mathbb{R}$ and similarly note it is an isometric *-isomorphism between C^*-algebras $L^\infty(S^1)$ and $L^\infty(\mathbb{R})$ as well as $C(S^1)$ and $C(\dot{\mathbb{R}})$, where $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

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