Graph-Theoretical Methods for Ringel's Earth-Moon Problem

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GRAPH-THEORETICAL METHODS FOR RINGEL'S EARTH-MOON PROBLEM

Bachelor thesis

July 29 2022

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1 Introduction

A well-known problem in Graph Theory is the Four Color problem, which is believed to be first conjectured by Francis Guthrie in 1862. The problem states that all planar maps can be colored with four colors such that no country has the same color as one of its neighbors. Guthrie was a student of Augustus de Morgan at the time, who wrote to William Hamilton about his student's problem:

> "A student of mine [Guthrie] asked me today to give him a reason for a fact which I did not know was a fact—and do not yet. He says that if a figure be any how divided and the compartments differently colored so that figures with any portion of common boundary line are differently colored—four colors may be wanted but not more—the following is his case in which four colors are wanted.
>
> Query cannot a necessity for five or more be invented..." [17]

This letter would spark one of the most famous problems for mathematicians to solve since it was written, as described in problem 1.

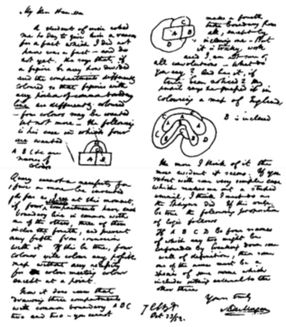


Figure 1: De Morgan's letter to Hamilton, 1852

Problem 1 (The Four-color Problem)**.** Can any planar map be properly colored using no more than four colors?

The problem turned out to be exceptionally difficult. Many false proofs of the statement would appear, such as one by Kampe in 1879 (Kampe's proof was later shown to be incorrect by Heawood in 1890). Finally, a proof was found only as late as the 1970's (See [2] and [11]). However, this proof was not without controversy, as it was one of the first computer-assisted proofs that was unfeasible for a human to check by hand. The problem's simplicity to understand, combined with its difficulty to prove made it notorious among mathematicians, and we will look at a generalization of the Four Color Theorem in this paper.

1.1 The Earth-Moon Problem

The Earth-Moon problem is similar to the Four Color problem, but with a simple generalization. Recall that the Four Color problem restricts its attention to planar graphs. The Earth-Moon problem asks the same, but for a map on two spheres (an Earth-Moon map): the earth and the moon. In this scenario, countries on the earth are allowed to have colonies on the moon, and we wish to color each country the same color as its colonies. In practice, this means drawing two separate maps next to one another and numbering the countries with the same numbers, as can be seen in Figure 2. The question now becomes: How many colors are sufficient to color any Earth-Moon map? The problem was first proposed by Gerhard Ringel [14], who originally believed the answer to be 8, and gave an example of an Earth-Moon map which required 8 colors.

As we will see in Subsection 1.2, there are examples that require more colors. We will see in Subsection 1.3 that there is an upper bound for the answer as well. As is customary when considering map coloring, we will use the tools given by graph theory to study this problem, as each Earth-Moon map can be thought of equivalently as a graph in which the vertices represent countries (and colonies) and an edge between two nodes represents the corresponding countries sharing a border. As such, in order to generalize the Earth-Moon problem and understand how Sulanke's Graph (See Subsection 1.2) was found, the following definition is necessary.

Definition 1 (Thickness)**.** A graph G=(V,E) has thickness t, denoted $\theta(G) = t$, if the edgeset E can be partitioned into t sets such that each of these sets induces a planar graph with vertex set V, and t is minimal.

It is clear to see that any planar graph has thickness 1, since all of its edges can be selected to form a planar graph. Any Earth-Moon graph has at most thickness 2, since the map of Earth and the map of the Moon can be viewed as 2 separate planar graphs on the same vertex set. This also allows us to easily generalise the Earth-Moon problem by allowing more moons or planets to be utilised. In that case, a problem with k planets is associated with coloring thickness k graphs. Another definition will also prove to be essential in subsequent sections.

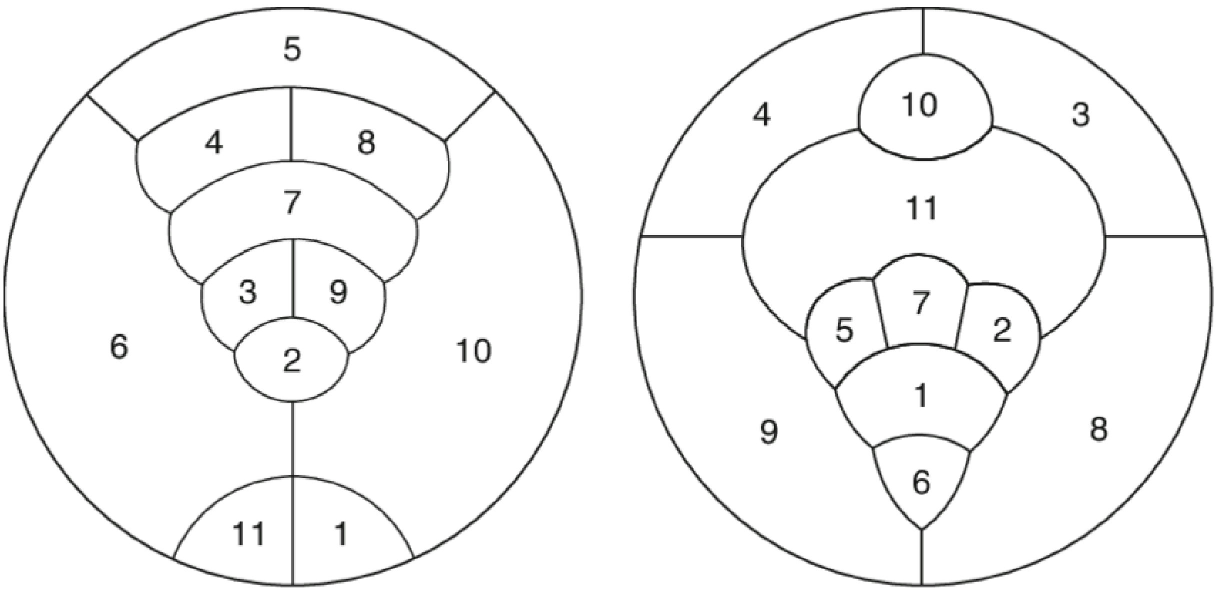
Definition 2 (Chromatic number)**.** A graph G has chromatic number n, denoted $\chi(G) = n$, if G can be colored using n colors such that no adjacent vertices have the same color and n is minimal.

Using this definition, we can translate the Four Color theorem to a more modern setting by stating it as "For any planar graph G, $\chi(G) \leq 4$". This may not seem like a helpful formulation, but the utility of this definition will become apparent later. Using the above definitions, we can reformulate the Earth-Moon problem in a graph-theoretical setting.

Problem 2 (The Earth-Moon Problem)**.** What is the maximal chromatic number for any graph of thickness 2?

We will use the framework from Problem 2 in subsequent sections, as it allows us to use several useful techniques afforded to us by graph theory. Problem 2 is a rather complex one. It is known that identifying both the chromatic number and the thickness are NP-complete problems [4]. In some sense, this makes the central problem of this thesis doubly complicated. We will see in the upcoming sections that progress on this problem is limited, most likely due to this complexity (combined with a lack of general interest).

1.2 Sulanke's Graph



Thom Sulanke's configuration for the earth-Mars map problem

Figure 2: Thom Sulanke's Earth-Moon map as seen in [6]

Martin Gardner is a familiar household name among recreational mathematicians, and he had many fans across the world. One of these fans, Thom Sulanke, was particularly caught by a puzzle Gardner posed in the March 1975 issue of *Scientific American* in his popular column *Mathematical Games* called “The Colored Poker Chips”[1]. Sulanke had found the answer and wrote to Gardner. Sulanke included a construction for an Earth-moon map that required 9 colors (shown in Figure 2) asking whether it was a new result and worthy of publication. Gardner wrote back to him, and his reply can be seen in Figure 3. The example Sulanke provided was the first known Earth-moon map requiring 9 colors. It is not clear from the figure that this should be the case, but it can be deduced by viewing the corresponding graph in a different way. Figure 2 shows a map, side by side, of the earth and the moon. This makes for a nice visual representation of the graph, but it is impractical to work with generally. Here is where the introduction of graph theory becomes a helpful, and necessary, tool. We continue from Definition 3 onward by viewing the Earth-Moon problem as it is formulated in Problem 2, as a graph-theoretical problem, rather than a map-based one.

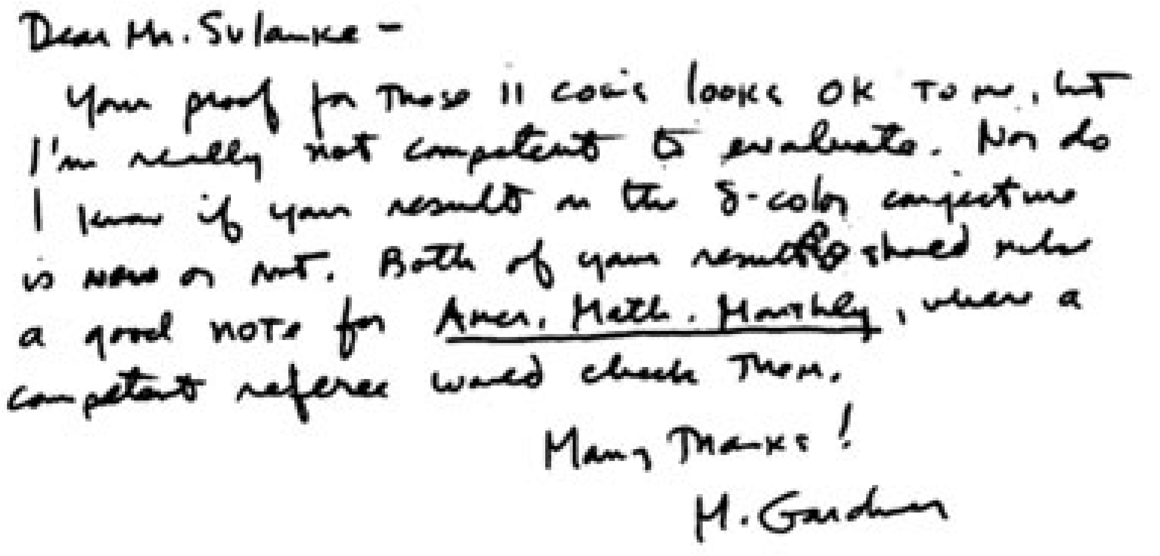

Dear Mr. Sulanke —
Your proof for those 11 coins looks OK to me, but I'm really not competent to evaluate. Nor do I know if your result on the 8-color conjecture is new or not. Both of your results should make a good note for Amer. Math. Monthly, where a competent referee would check them.
Many Thanks!
M. Gardner

Figure 3: Gardner’s reply to Sulanke: ”Your proof for those 11 coins looks OK to me, but I’m really not competent to evaluate. Nor do I know if your result on the 8-color conjecture is new or not. Both of your results should make a good note for Amer. Math. Monthly, where a competent referee would check them. Many Thanks! M. Gardner” [6]

Definition 3. The join of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ (with V_G and V_H disjoint), denoted $G \vee H$, is the graph containing both G and H and all edges between nodes of G and nodes of H. Equivalently, $G \vee H := (V_G \cup V_H, E_{G\vee H})$ with

$$E_{G\vee H} = E_G \cup E_H \cup \{(i, j) \,|\, i \in G,\ j \in H\}.$$

By identifying each of the numbers in Sulanke’s graph with a node, and letting an edge represent a border between the corresponding countries, we can see that Sulanke’s graph is $C_5 \vee K_6$ with C_5 the cyclical graph on 5 vertices and K_6 the complete graph on 6 vertices. In Figure 2, numbers 1 through 5 represent C_5 and 6 through 11 represent K_6. Now we can use this to conclude the colorability of Sulanke’s graph.

[1]”What is the smallest number of poker chips that can be placed on a flat surface so that chips of three different colors are required to meet the condition that no two touching chips are the same color? [...] Our problem is to determine the smallest number of chips that can be placed flat on the plane so that chips of not three but four colors are necessary for meeting the same proviso”

Lemma 1.1. *For any two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ we have*

$$\chi(G \vee H) = \chi(G) + \chi(H).$$

Proof. Suppose $\chi(G \vee H) < \chi(G) + \chi(H)$ and let $G \vee H$ be properly colored. By the pigeonhole principle, there must exist a node $v \in V_G \subset V_{G \vee H}$ and $w \in V_H \subset V_{G \vee H}$ such that v and w are colored using the same color. However, by the definition of the join of G and H, v and w share an edge, so $G \vee H$ is not properly colored. We conclude $\chi(G \vee H) \geq \chi(G) + \chi(H)$. Since coloring $G \vee H$ by coloring G and H as subgraphs, using no overlapping colors, gives a proper coloring of $G \vee H$, we conclude $\chi(G \vee H) \leq \chi(G) + \chi(H)$. This proves $\chi(G \vee H) = \chi(G) + \chi(H)$. □

We can use Lemma 1.1 to find the chromatic number for Sulanke's Graph. Since $\chi(K_6) = 6$ and $\chi(C_5) = 3$ we find that $\chi(K_6 \vee C_5) = 9$. Sulanke's graph is a somewhat peculiar example, as it appears not to have any sort of structure that could be used to generalize this graph to find more examples needing 9 or even 10 colors. More examples of Earth-Moon graphs with chromatic number 9 have since been found in, for example, [7]. We will revisit these examples in addition to other methods of generating candidates for solving the Earth-Moon problem in later sections.

1.3 M-pires

Now that we know from Subsection 1.2 that the answer to the Earth-Moon problem is at least 9, we are interested in finding an upper bound. Thankfully, we can find one by viewing Earth-Moon graphs in yet another way. To this end, we generalize the Four-color problem from a different point of view. Instead of allowing colonies on other planets, we now allow countries to be disconnected. We call such countries empires. If it consists of precisely M disconnected parts, we call such a country an M-pire. A map in which each of the countries consists of at most M disconnected parts is called an M-pire map. M-pire graphs are defined analogously. This can be viewed as a graph in which each vertex is uniquely associated with at most $M - 1$ other independent vertices. Figure 4 shows an example of an M-pire. A proper coloring of an M-pire map is one that uses the same color for each of these families of M independent vertices. Now, we wonder what the chromatic number is for such graphs. Luckily, Heawood [9], who initially posed this colorability problem, was able to provide an upper bound.

Theorem 1.2. *For any M-pire graph $G = (V, E)$ with $M \geq 1$, $\chi(G) \leq 6M$.*

Proof. The proof can be found in [9]. A modern version of the proof is included here for clarity since the language used in [9] is outdated and can be difficult to understand. This proof is adapted from [10].

Let $G = (V, E)$ be an M-pire graph. We construct $G^* = (V^*, E^*)$ by merging the families of independent vertices into a single vertex. Any edge from the family of independent vertices is also an edge of the merged vertex, since any edge $e \in E$ connects vertices from different independent families and therefore belongs in E^*. We remove multiple edges so that G^* is simple. This way, we can observe $|V| \leq M|V^*|$ since the vertices of G^* are formed out of at most M vertices of G. Similarly, we obtain $|E^*| \leq |E|$.

Now, the average degree of G^*, D, is equal to $\frac{2|E^*|}{|V^*|}$. Note that, since G is a simple planar graph, we have $|E| \leq 3|V| - 6$ by Euler's polyhedron formula. From this, we get $D = \frac{2|E^*|}{|V^*|} \leq \frac{2|E|}{|V^*|} \leq \frac{2(3|V|-6)}{|V^*|} = \frac{6|V|-12}{|V^*|} \leq \frac{6M|V^*|-12}{|V^*|} = 6M - \frac{12}{|V^*|}$. Now, since $\frac{12}{|V^*|} > 0$, we have $D < 6M$. This also means G^* contains a vertex of degree at most $6M - 1$ since the average degree would be at least $6M$ if this was not the case.

Now we prove the Theorem by induction on $|V^*|$. The statement is trivially true if $|V^*| \leq 6M$ since the graph can be colored by assigning a different color to each vertex. Now, assume the theorem is true for all M-pire graphs with fewer than $k > 6M$ vertices and let $G^* = (V^*, E^*)$ be an M-pire graph constructed as above with $k = |V^*|$ vertices. We construct $\tilde{G} = (\tilde{V}, \tilde{E})$ by removing a vertex v of degree at most $6M - 1$ and its edges from G^*. $\tilde{G}$ is an M-pire graph since G^* can contain multiple empires of size M. By induction, $\tilde{G}$ can be $6M$ colored since it has fewer than k edges. We color $\tilde{G}$ as a subgraph of G^* in this way. Now, only v is left to be colored. However, since its degree was at most $6M - 1$ there is at least 1 color we can use to color v. Now, G^* is $6M$ colored and the theorem is proved. □

This bound was proven to be sharp for $M > 1$ in stages. Heawood provided an example for $M = 2$ requiring 12 colors in [9] (also, see Figure 4). Later, Taylor gave examples for $M = 3$ and $M = 4$ in 1981, which can be found in [8]. With a general construction, Ringel and Young were able to show the bound was sharp for all $M \geq 2$ in [15]. For more information on the class of M-pire graphs, [10] is a great source.

We can use Theorem 1.2 for the Earth-Moon problem since all Earth-Moon graphs are 2-pire graphs. However, not all 2-pire graphs are Earth-Moon graphs. We conclude that the chromatic number for any Earth-Moon graph is at most 12 by using this fact.

Intuitively, having more edges in a graph means the chromatic number is likely to be higher since more nodes are connected, meaning they are unable to share a color. Having more nodes that are unable to share colors increases the likelihood of requiring more colors to properly color the graph as a whole. Likewise, having fewer edges in a graph means the thickness is likely to decrease since it will be easier to group the edges together in planar subgraphs. Having the extra freedom of placing the two disconnected regions of a country on different planets (which is the case for Earth-Moon graphs) instead of in amongst other countries (which is the case for M-pires) should allow for a lower chromatic number since there will be fewer edges on average. It is with this hope that we set about trying to find an Earth-Moon map requiring more than 9 and at most 12 colors.

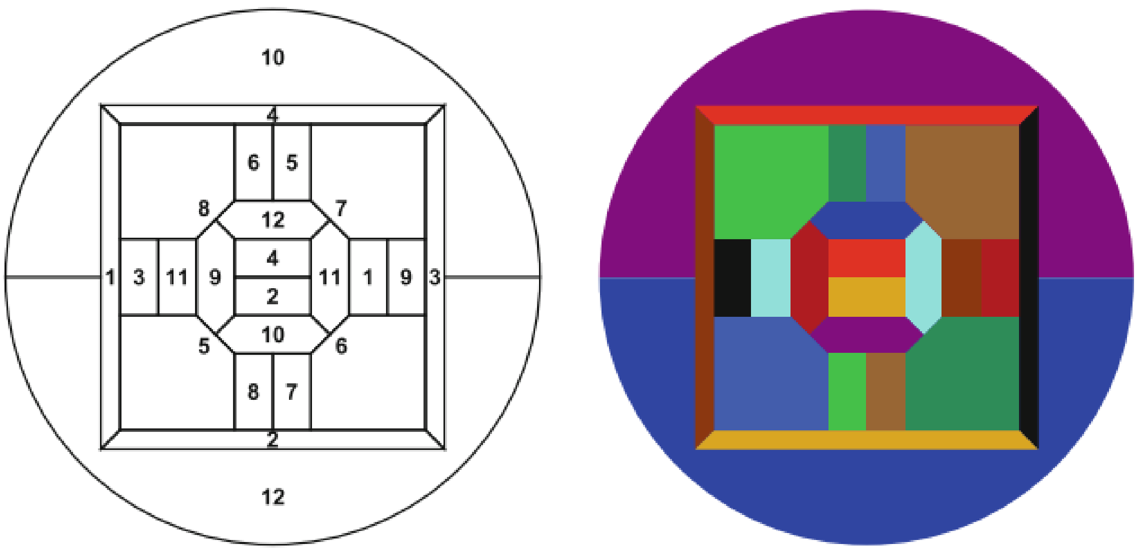


Figure 4: Gethner's visualisation of a 12-colorable 2-pire map, found by Scott Kim. [6]

2 The chromatic number approach

Now that we have a good way to construct thickness 2 graphs, we will also need some machinery to analyse the chromatic number of these graphs. In this section, we describe methods that provide information on the chromatic number of graphs.

2.1 Complement graphs

So far, the most promising way to obtain some knowledge of the chromatic number for Earth-Moon graphs is a proposition courtesy of [7]. For this, we introduce Proposition 2.1 which allows us to verify a lower bound for the chromatic number of a graph.

Proposition 2.1. *Suppose $G = (V, E)$ is a graph with $|V| = n$. If G^c, the complement of G, does not contain K_m as a subgraph, then*

$$\chi(G) \geq \lceil \frac{n}{m-1} \rceil.$$

Proof. This proof is given in [7] and is included here for clarity.
Coloring the vertices of a graph is equivalent to partitioning the vertices into independent sets which in turn is equivalent to partitioning the vertices of the complement into complete subgraphs. Thus, if G^c does not contain K_m then at most $m-1$ vertices of G can be assigned the same color, in which case $\chi(G) \geq \lceil \frac{|V|}{m-1} \rceil$. □

Corollary 1. *Let $G = (V, E)$ be a graph.*
1. If $|V| \geq 17$ and G^c does not contain K_3 as a subgraph, then $\chi(G) \geq 9$.
2. If $|V| \geq 25$ and G^c does not contain K_4 as a subgraph, then $\chi(G) \geq 9$.
3. If $|V| \geq 33$ and G^c does not contain K_5 as a subgraph, then $\chi(G) \geq 9$.

Corollary 1 follows directly from Proposition 2.1 and is used in [7] to find many examples of thickness 2 graphs that require at least 9 colors. With this corollary in hand, graphs are generated with the required amount of vertices. Then, using Sulanke's Algorithm (see Subsection 3.2), the complement is made to be free of the accompanying complete graph as a subgraph. The graph is finally verified to be thickness 2 using software called *nauty* [13]. The precise way this method works, however, is unclear. Many key details in the modifications of *nauty* and an implementation of Sulanke's Algorithm are missing from the paper's methods. This makes it difficult to determine precisely how useful Proposition 2.1 is. However, we will see in Subsection 3.2 it may be key in solving the Earth-Moon problem.

2.2 Graph inflation

One possible way to construct candidate graphs for solving Problem 2 is called graph inflation. It is a simple construction that quickly results in large graphs of which a lower bound for the chromatic number is known.

Definition 4 (Graph inflation)**.** Let $G = (V, E)$ be a simple graph with $V = \{1, 2, ..., n\}$ and $\{H_1, H_2, ..., H_n\}$ a set of graphs. The inflated graph $G[H_1, H_2, ..., H_n]$ is obtained by replacing each vertex i of G by the graph H_i and each edge (i, j) of G by the edge set of the join of H_i and H_j.

If each H_i is a complete graph on r vertices we call the inflated graph the r-inflated graph and denote it by $G[r]$. We call $G[2]$ the clone of G. In the case that each H_i is a complete graph on r_i vertices (for some numbers r_i) we denote the inflated graph as $G[r_1, r_2, ..., r_n]$.

To illustrate this construction, Figure 5 shows an example of the 2-inflated graph $P_4[2]$. From this figure, it should be clear that even small graphs quickly turn into much larger graphs. This may not seem like much of an advantage, but it will become clear after the introduction of the independence number.

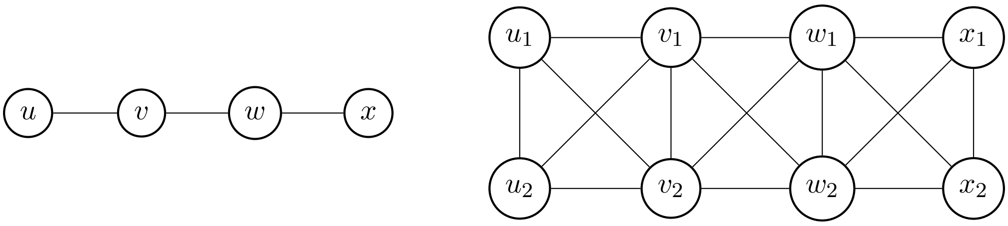


Figure 5: The path graph P_4 on the left, and its clone $P_4[2]$ on the right.

Definition 5 (Independence number)**.** The independence number of a graph $G = (V, E)$, denoted $\alpha(G)$, is the cardinality of a maximally sized set of independent vertices in G.

The meaning of independence in Definition 5 is one closely related to the chromatic number. Finding the independence number of a graph is called *the maximum independent set problem* and it is a strongly NP-hard problem [5]. Using the independence number, then, doesn't seem to be much of an upgrade to the NP-complete problems of the chromatic number and thickness we already have. However, the independence number can be found somewhat quickly (even by hand) for smaller graphs. The important part here is that graph inflation by complete graphs conserves the independence number as Lemma 2.2 shows. We also include another lemma and a theorem which will prove to be useful in using graph inflation to construct candidate graphs for Problem 2.

Lemma 2.2. *Let $G = (V, E)$ be a simple graph. For any sequence of natural numbers $r_1, r_2, ..., r_{|V|}$ we have $\alpha(G) = \alpha(G[r_1, r_2, ..., r_{|V|}])$.*

Proof. Let $G = (V, E)$ be a simple graph. Let $I \subset V$ be a maximal independent set of G. Now, for each $I_i \in I$, let R_i be a vertex from the complete graph in $G[r_1, r_2, ..., r_{|V|}]$ that contains I_i. Define $R = \bigcup_i R_i$. It is clear that $|I| = |R|$. R is a set of independent vertices of $G[r_1, r_2, ..., r_{|V|}]$ since if any R_i and R_j share an edge, then there must be an edge between I_i and I_j by the definition of graph inflation, but this is a contradiction with I_i and I_j being independent. In the same manner, this implies R is maximal, so we conclude $\alpha(G) = \alpha(G[r_1, r_2, ..., r_{|V|}])$. □

Theorem 2.3. *For any graph $G = (V, E)$, $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.*

Proof. This lemma can be found in [1], the proof of which is given in [16]. □

Lemma 2.4. *For any graph $G = (V, E)$ with thickness t we have:*

$$|E| \leq t(3|V| - 6).$$

Proof. By definition, G can be partitioned into t planar graphs. For each of these graphs we find the edge set to be no larger than $3|V| - 6$ due to Euler's polyhedron formula. We conclude $|E| \leq t(3|V| - 6)$. □

Using Lemma 2.2 and Theorem 2.3 we can find a way to create large graphs, of which we can find a lower bound for the chromatic number quite easily. To do this, we start with a graph $G = (V, E)$ with a known independence number. Using a graph with few nodes is preferable when keeping in mind Lemma 2.4. Then, we inflate this graph by some sequence of natural numbers $r_1, r_2, ..., r_{|V|}$. Using Theorem 2.3 we can find a lower bound for the chromatic number of this inflated graph. This eliminates the need for checking the chromatic number, but it still leaves the NP-complete problem of finding the thickness.

As an example, we study the graph $C_5[4,4,4,4,3]$, which Gethner gives as a hopeful candidate in [6]. It is easily verified that the independence number of C_5 is 2. $C_5[4,4,4,4,3]$ has 19 vertices, so by Lemma 2.2 and Theorem 2.3 we know that $\chi(C_5[4,4,4,4,3]) \geq \frac{19}{2}$ and since the chromatic number is a natural number, it must be at least 10 in this case. Using Lemma 2.4 we can also conclude that it is a valid candidate to have thickness 2, since $99 = |E_{C_5[4,4,4,4,3]}| \leq 2(3|V_{C_5[4,4,4,4,3]}| - 6) = 102$. If it can be verified that the thickness of $C_5[4,4,4,4,3]$ is indeed 2, it would improve the known lower bound for Problem 2 from 9 to 10. However, as Gethner describes, many attempts have been made to show $C_5[4,4,4,4,3]$ has thickness 2, with no luck thus far.

In conclusion, graph inflation gives a useful way of constructing candidate graphs for Problem 2, but it still requires some thinking about the thickness of such graphs. Some information on the thickness of certain inflated graphs exist, such as in [1], which is a generally useful source in understanding graph inflation more broadly. However, the theorems and propositions in [1] provide only a small step in understanding the potential of graph inflation for solving Problem 2. The hope for converting more of this potential into results comes from the apparent symmetry of inflated graphs. If this symmetry can be harnessed in some way to gather information on the thickness of these graphs, it would be a major improvement to the methodology of solving the Earth-Moon problem.

3 The thickness approach

Sulanke's graph (from Subsection 1.2) appears to have some structure and symmetry at first glance. Its construction also appears to come from some generalized structure for thickness 2 graphs. However, no such generalized structure is known. Sulanke's solution to the problem appears unique in its simplicity and symmetry. However, other Earth-Moon graphs with chromatic number 9 have since been found. In this section we discuss methods of generating graphs with thickness 2 or verifying their thickness.

Sulanke is central in the methods we discuss, as he has continued searching for solutions to the Earth-Moon problem since his first graph from 1973. Some of his examples were included in [3] and [7], which also highlight the methods we discuss in this section. After hearing a talk by another author of these papers, Ellen Gethner, Sulanke reached out to collaborate [6]. Using the methods we discuss in this section, they were able to construct a number of thickness 2 graphs with chromatic number 9, in addition to finding a construction for infinite families of these graphs.

3.1 Permuted Layer Graphs

The first method to consider is that of permuted layer graphs. Permuted layer graphs are a subclass of thickness 2 graphs that have two convenient properties for our intended purposes. Firstly, permuted layer graphs are easy to construct. Secondly, every maximal thickness 2 graph is a permuted layer graph (as we will see in this subsection). We begin by understanding the construction of permuted layer graphs, courtesy of [3].

Construction 1 (Permuted layer graph)**.** Let $H = (V_H, E)$ be a planar graph with $V_H = \{v_1, v_2, ..., v_n\}$ and σ a permutation of V_H. Construct the graph H' to be isomorphic to H with vertices labeled such that the vertex corresponding to v_i in H is labeled $\sigma(v_i)$ in H'. Identify H and H' at vertices of the same label and call the resulting graph $\tilde{G}$ (in other words: $\tilde{G} = H \cup H'$). If $\tilde{G}$ has multiple edges, we let G be the underlying simple graph. G is a *permuted layer graph with base graph* H. In the situation that $\tilde{G}$ has no multiple edges, we call G $(= \tilde{G})$ a *full permuted layer graph.*

To illustrate this construction, Figure 6 and Figure 8 show permuted layer representations of K_5 and K_6 respectively. Figure 7 gives a colorful representation of Figure 6 where the structure of K_5 and the way it is obtained from the permuted layer graph are more apparent. This also gives, by construction, a thickness 2 decomposition of both K_5 and K_6 (since it is known that K_5 and K_6 are not planar). These figures highlight an alternative way to view the construction of permuted layer graphs. This is done by viewing the underlying base graph H, adding the edge set $\{(\sigma(a), \sigma(b))|(a, b) \in E(H)\}$ and removing the multiple edges.

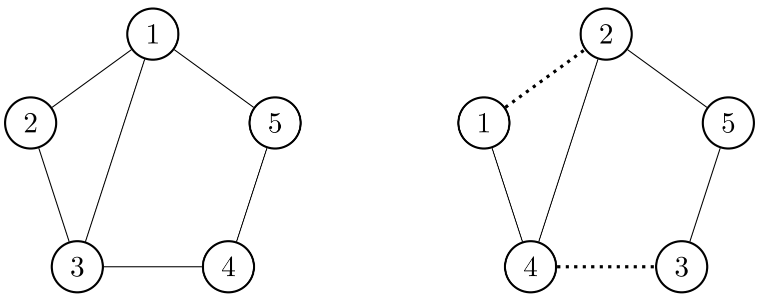


Figure 6: A permuted layer representation of K_5, with permutation (12)(34)(5). Edges (1,2) and (3,4) are dotted to indicate they are removed from the right layer to obtain the underlying simple graph.

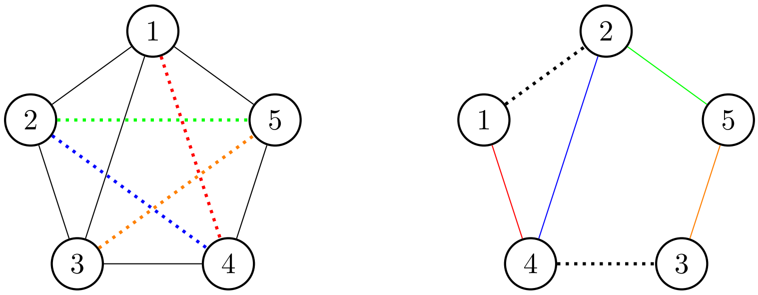


Figure 7: A permuted layer graph representation of K_5, such as in Figure 6. The colors indicate identical edges. Dotted edges do not belong to the graph they are in, but are drawn to represent they belong to the permuted layer graph.

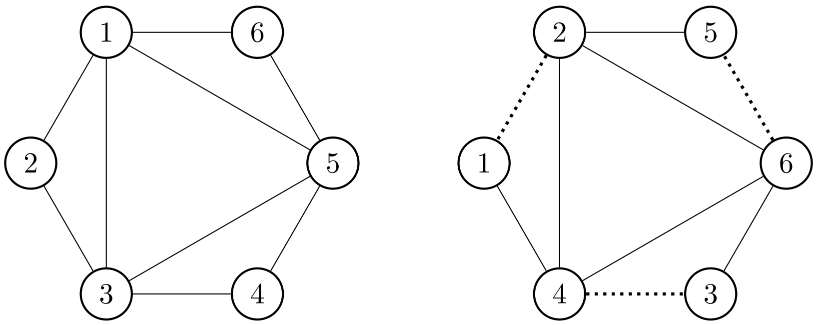


Figure 8: A permuted layer representation of K_6, with permutation (12)(34)(56). Edges (1,2), (3,4) and (5,6) are dotted to indicate they are removed from the right layer to obtain the underlying simple graph.

This method of constructing thickness 2 graphs (this construction of permuted layer graphs makes it impossible to construct a graph with thickness 3 or more) is close to the most intuitive way of constructing these, which is simply drawing two planar graphs next to one another and labeling the vertices such that the vertex set for each of the subgraphs is the same. This, however, is somewhat arbitrary. Which graphs does one select to use and why? Permuted layer graphs are in some sense half as arbitrary as it requires choosing only one base graph instead of two. However, this might also leave room for 'missing' options for candidate graphs of solving Problem 2, since being half as arbitrary might also mean being half as thorough. Luckily, this is not a concern. The permuted layer graph method is validated by the following fact stated in Proposition 3.1.

Proposition 3.1. *Every thickness 2 graph is a subgraph of a permuted layer graph.*

Proof. The proof can be found in [3]. □

The permuted layer graph construction is still somewhat arbitrary in its choice of the base graph, but it has the added benefit of symmetry over the intuitive construction method. This also provides an easier way for computations involving these graphs, since randomly generating only one planar graph as opposed to two makes traversing the search space more manageable. This can be used to speed up algorithms for finding Earth-Moon graphs. We discuss one such algorithm in Subsection 3.2.

3.2 Sulanke's Algorithm

Since finding thickness 2 decompositions of graphs is a complex problem, heuristics are a necessary part of finding solutions to Problem 2. To this end, Sulanke devised a modified version of simulated annealing, which we will discuss here. Simulated annealing is a probabilistic optimisation technique used in various contexts, mostly when the search space is discrete. It was introduced in various independent occasions, such as [12], where it was used to approximate a solution of the Traveling Salesman Problem. Sulanke's Algorithm uses a modified version of this technique where the objective is to minimise the amount of triangles in the complement of a graph. Combining this with Proposition 2.1 yields an interesting and possibly effective approach for finding candidate Earth-Moon graphs.

The aim of the algorithm is to reduce the number of triangles in the complement of a graph. This can be used alongside Proposition 2.1 to search for graphs with a desired chromatic number. The algorithm initialises with a thickness 2 graph. These two facts, combined, make the algorithm very suitable for finding solutions to Problem 2. The algorithm we introduce here relies heavily upon an operation of graphs called diagonal flips. A diagonal flip is performed by finding two triangles in a given graph that share a common edge. This edge is the diagonal, and the flip entails removing this diagonal and adding the complementary diagonal. Figure 9 illustrates this.

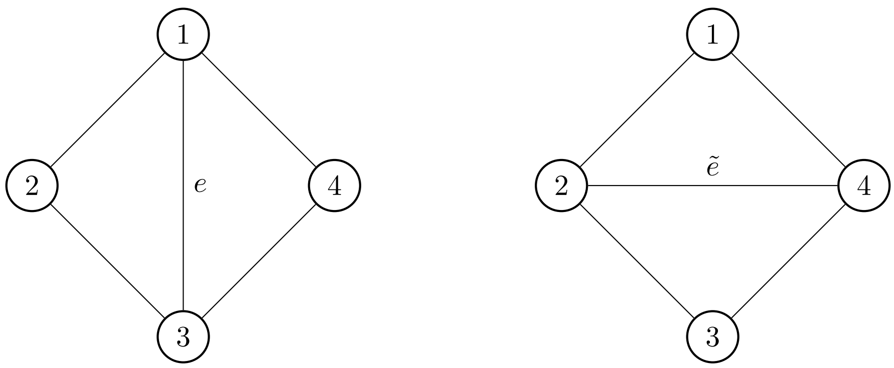


Figure 9: An illustration of a diagonal flip. Diagonal e is flipped to its complementary diagonal, $\tilde{e}$.

A few important parts of Algorithm 1 require further explanation. Most importantly, the temperature T is updated according to a temperature function. Such a function can be chosen when the algorithm is implemented. The temperature should decrease over time. The temperature is of great importance to the probability that a flip is accepted even if it increases the amount of triangles in the complement. This is done so that the algorithm does not get stuck in a local minimum. The probability function may also be chosen when the algorithm is implemented.

Despite its successful use in finding 9-critical thickness 2 graphs in [7], little is known about the effectiveness of Algorithm 1. For instance, it is not known how effective diagonal flips are as a means of finding a triangle-free complementary graph. The algorithm also seems to generalise to finding a K_m-free complementary graph for $m \geq 4$ according to [7], but it is unknown how this generalization is made. All in all, Sulanke's algorithm appears to be quite promising for finding candidate graphs for Problem 2 but more research into the algorithm is required. This research is complicated by the fact that sources on the subject are limited. The algorithm given here is based on [6], [7] and private communication between myself and Sulanke.

Algorithm 1 Sulanke's Algorithm

<u>INITIALIZATION</u>

Let $G = G_1 \cup G_2$ be a graph, where G_1 and G_2 are edge-maximal planar graphs on n vertices.

If G^c is triangle-free, then return G.

Set a maximal amount of flips k_{max}.

<u>IMPROVING LOOP</u>

for $k = 0$ through k_{max} (exclusive) **do**

  $T \leftarrow Temperature(1 - (k+1)/k_{max})$.

  Randomly select a diagonal edge e with complementary edge $\tilde{e}$.

  if The amount of triangles in G^c is reduced or maintained by flipping e to $\tilde{e}$ **then**

    Flip e to $\tilde{e}$.

  else if The amount of triangles in G^c is increased by flipping e to $\tilde{e}$ **then**

    Flip e to $\tilde{e}$ with probability $P(G, G_{\tilde{e}}, T)$.

  end if

  if G^c is triangle-free **then**

    return G

  end if

end for

4 Conclusion

This thesis is aimed at providing a clear overview of the tools and methods currently available for solving the Earth-Moon problem. In particular, it should serve as a guidebook to any brave soul attempting to find a thickness 2 graph with chromatic number 10 or higher. Since the problem is doubly NP-complete (see Section 1), possibly combined with a general lack of interest, progress on the problem has been slow. Nevertheless, there are quite a few known methods for finding candidate graphs such as graph inflation or permuted layer graphs. An overview of these methods with the purpose of applications on the Earth-Moon problem is given in each of the eponymous subsections.

Another issue one might face when attempting to take on the Earth-Moon problem is a general lack of transparency in the literature. Many of the sources available below contain a noticeable lack of clarity and thoroughness in explaining the topic at hand. This is not to say the sources are bad pieces of literature, it is to say that the sources often do not provide any insight on how they relate to the Earth-Moon problem. In most cases, this is to be expected, since the purpose of a source is usually not to provide an overview of the Earth-Moon problem, but rather to explain one of the methods or tools described in the subsections. Researching the topic can be quite challenging due to this fact. For example, sources on Sulanke's Algorithm contain no actual step-by-step description of the algorithm, only overviews of the general idea, or worse, only a description of the aim of the algorithm. In another case, many key theorems and lemmas lack a proof, or even a brief description of why they are true. In order not to contribute to this issue, I have attempted to clearly state or refer to proofs of every bit of reasoning applied in this thesis. To my knowledge, some of these proofs and descriptions are novel.

I believe there are many steps one could take in contributing to this problem, which are detailed in Subsection 4.1. Due to time restraints, answering these questions is outside the scope of this thesis. Most importantly, I hope this thesis serves as a good jumping-off point for those who choose to venture into the complexities of the Earth-Moon problem.

4.1 Open problems

I will provide an overview of questions and problems one might wish to tackle in order to contribute to solving the Earth-Moon problem. Each of these is prefaced by an indicator of which subsection is most appropriate to start researching first.

- (Graph inflation) Can the symmetry of Graph inflation be used in some way to determine the thickness of inflated graphs?
- (Permuted layer graphs) Is there some way to characterize which combinations of graphs and permutations give a pair of edge-maximal planar graphs when using the permuted layer graph construction? This would help in the initialisation of Sulanke's Algorithm.
- (Permuted layer graphs) Can anything be said about the chromatic number of permuted layer graphs based on the base graph and permutation?
- (Sulanke's Algorithm) How effective are diagonal flips at finding a triangle-free complement graph if one exists given an input graph?
- (Sulanke's Algorithm) How can Sulanke's Algorithm be generalized to finding K_m-free complement graphs with $m \geq 4$?

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