Systolic inequalities on surfaces

Papadopoulos, K.

K. Papadopoulos

Systolic inequalities on surfaces

Bachelor thesis

30 June 2022

Thesis supervisor: dr. F. Pasquotto

Leiden University
Mathematical Institute

Contents

1 Introduction

A torus is most commonly known and recognised as a doughnut. But in fact, this is only one 'kind' of torus. One might be surprised that there is such a thing as a flat torus which is locally isometric to the euclidean plane - a flat doughnut if you will. A detailed description of the flat torus is given in paragraph *3 Quotient surfaces: Flat torus*.

In this thesis we are interested in studying the space of Riemannian metrics on tori. This can be done by comparing the length of a special class of curves on the surface, the systoles, and the total area of the surface with respect to the metrics. This leads to the definition of the systolic ratio and the study of systolic inequalities. Most likely, the first systolic inequality was given and proved by Loewner in 1949 [12], which is the main focus of this thesis (see Theorem 4.1).

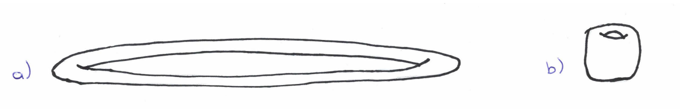

Figure 1: Two tori embedded in $\mathbb{R}^3$

We start by looking at the Riemannian metric and introducing the first fundamental form (induced by the Euclidean metric in $\mathbb{R}^3$) on surfaces (see paragraph *2.1 Riemannian metric and first fundamental form*). Next a short definition and description of the Gaussian curvature of surfaces is given in paragraph *2.2 Curvature*.

From this point onwards we look more specifically at tori. After a short introduction with regard to isometries, we first study the flat torus as a quotient surface in section *3 Quotient surfaces: Flat torus* and state the *Classification of flat tori [1]* and the *Uniformization [1]* theorem (see paragraph *3.1 Theorems concerning flat tori*). These two theorems give a great insight in the space of Riemannian metrics for tori as the *Classification of flat tori [1]* classifies the metrics of constant curvature of tori up to homothety and isometry and the *Uniformization [1]* theorem dictates that each metric is conformally equivalent to a metric of constant curvature.

Next we define the systoles of a surface and the systolic ratio (see paragraph *4 Systoles*). The word systoles comes from the Greek word for contraction and is used as a medical term for heart contraction (i.e. contraction of the heart to pump blood to the arteries) [2]. Funnily enough the systoles of a surface are in fact the *non*-contractible loops of minimal length...
In the same section we state and prove Loewner's theorem which gives an upper bound for the systolic ratio of two dimensional tori. This theorem was stated in 1949 and proved in a course of Riemannian Geometry at Syracuse University [12].

Lastly the systolic ratio and the Gaussian curvature at certain points is computed for two surfaces embedded in $\mathbb{R}^3$ (see paragraph paragraph *2.2 Curvature*).

The thesis ends with a recap of the interesting results and suggestions for possible further research, such as systolic inequalities for the real projective plane and the Klein bottle (see paragraph *6 Conclusion*).

2 Riemannian metric on a surface

In this thesis we will repeatedly speak of surfaces. When doing so we mean a compact, smooth, two dimensional manifold without boundary.

2.1 Riemannian metric and first fundamental form

In this paragraph the first fundamental form is defined and described based on Chapter 4 of [4] and paragraph 1.2 of [6], as well as paragraph 2.2 of [11] regarding the computations of the first fundamental form.

Let S be a smooth surface. A Riemannian metric g on S is a family of inner products of the tangent spaces of S. Specifically, g associates to each point $p \in S$ a positive definite symmetric bilinear form :

$$g_p : T_pS \times T_pS \to \mathbb{R}$$

When working with surfaces embedded in $\mathbb{R}^3$ the metric is the Euclidean metric restricted to the tangent vectors of the surface. This is called the induced metric or first fundamental form.

The first fundamental form I_p at a point p is the inner product on the tangent spaces of S at p:

$$I_p : T_pS \times T_pS \to \mathbb{R}$$

$$(v, w) \mapsto I_p(v, w) = \langle v, w \rangle$$

where $\langle -, - \rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^3$.
Let S be parametrised by $\xi(\phi, \theta)$:

$$\xi : D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$$

$$(\phi, \theta) \mapsto \xi(\phi, \theta) = (x(\phi, \theta), y(\phi, \theta), z(\phi, \theta))$$

The point p can be seen as a point of a parametrized curve γ on the surface S with:

$$\gamma(0) = p \text{ and } \gamma(t) = \xi(\phi(t), \theta(t)) \text{ with } t \in (-\epsilon, \epsilon)$$

So an element of T_pS is a tangent vector γ'. We have:

$$\gamma' = \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta'$$

where ξ_ϕ is the partial derivative of ξ with respect to ϕ and ξ_θ is the partial derivative with respect to θ.

Now we can compute the first fundamental form:

$$I_p(\gamma', \gamma') = \langle \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta', \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta' \rangle$$

$$= \langle \xi_\phi, \xi_\phi \rangle (\phi')^2 + \langle \xi_\theta, \xi_\theta \rangle (\theta')^2 + 2\theta'\phi' \langle \xi_\phi, \xi_\theta \rangle$$

with $g_{11} = \langle \xi_\phi, \xi_\phi \rangle$, $g_{12} = \langle \xi_\phi, \xi_\theta \rangle$ and $g_{22} = \langle \xi_\theta, \xi_\theta \rangle$. These are called the components of the metric and can be written as follows in a 2×2 matrix:

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

Because the inner product is symmetrical we have $g_{12} = g_{21}$. The metric is completely defined by these components. We introduce the following notation:

$$\tilde{g} = \det(g_{ij}) = g_{11}g_{22} - g_{12}^2$$

In the rest of this section we will explain the relation of the metric with lengths of curves and area of surfaces.

Let γ be a smooth curve on the surface S with the parametrization described above. Then the length of this curve for t between $t = a$ and $t = b$ is given by:

$$\int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{\gamma'(t) \cdot \gamma'(t)}dt = \int_a^b \sqrt{(\gamma'_\phi \phi'(t) + \gamma'_\theta \theta'(t))(\gamma'_\phi \phi'(t) + \gamma'_\theta \theta'(t))}dt =$$
$$= \int_a^b \sqrt{g_{11}\phi'(t)^2 + 2\phi'(t)\theta'(t)g_{12} + g_{22}\theta'(t)^2}dt$$

So the length is the integral of the square root of the first fundamental form:

$$l_g(\gamma) = \int_a^b \sqrt{I_{\gamma(t)}}\ dt$$

The area of a parametrized surface can be calculated with the following formula:

$$\text{Area } S = \iint_D \left| \frac{\partial \xi}{\partial \phi} \times \frac{\partial \xi}{\partial \theta} \right|\ d\phi\ d\theta$$

A cross product property gives us:

$$\left| \frac{\partial \xi}{\partial \phi} \times \frac{\partial \xi}{\partial \theta} \right|^2 = |\xi_\phi|^2 |\xi_\theta|^2 - \langle \xi_\phi, \xi_\theta \rangle = g_{11}g_{22} - g_{12} = \det(g_{ij}) = \tilde{g}$$

So we can express the area of the surface S via the metric:

$$\text{Area } S = \iint_D \sqrt{\tilde{g}}\ d\phi\ d\theta$$

Since the cross product is specific for $\mathbb{R}^3$ this formula is more general and holds for any surface with metric g.

Next we give the definition of an isometry for general surfaces:

Definition 2.1. *An isometry between two general surfaces S and S' with metrics g and g' is a diffeomorphism $f : S \to S'$ such that:*

$$\forall p \in S, \forall w_1, w_2 \in \mathbb{T}_p^2 S\ :\ g_p(w_1, w_2) = g'_{f(p)}(df(w_1), df(w_2))$$

For surfaces embedded in $\mathbb{R}^3$ we have the following definition of an isometry:

Definition 2.2 ([5])**.** *An isometry in $\mathbb{R}^3$ between two surfaces S and S' is a diffeomorphism $f : S \to S'$ such that:*

$$\forall p \in S, \forall w_1, w_2 \in \mathbb{T}_p^2 S\ :\ \langle w_1, w_2 \rangle = \langle df(w_1), df(w_2) \rangle$$

Theorem 2.1 (Chase, H.S., Theorem 1.3 [3]). *Let S and S' be two surfaces with metrics g and g'. S and S' are locally isometric if and only if the fundamental form is locally preserved.*

With "locally preserved" we mean that in a small neighbourhood of an arbitrary point $p \in S$ the first fundamental form is the same as in a small neighbourhood of $f(p) \in S$.
This theorem can be generalised for surfaces that are globally isometric. If S and S' are globally isometric there exists a diffeomorphism $f : S \to S'$ so that

$$\forall p \in S, \forall w_1, w_2 \in \mathbb{T}_p^2 S \ : \ g_p(w_1, w_2) = g'_{f(p)}(df(w_1), df(w_2))$$

and if $S, S \subset \mathbb{R}^3$:

$$\forall p \in S, \forall w_1, w_2 \in \mathbb{T}_p^2 S \ : \ \langle w_1, w_2 \rangle = \langle df(w_1), df(w_2) \rangle$$

So for $\gamma' \in T_p S$ an element of the tangent space at a point $p \in S$, then $df(\gamma') \in T_{f(p)} S'$ is an element of the tangent space at the point $f(p) \in S'$ and we have:

$$I_p(\gamma', \gamma') = \langle \gamma', \gamma' \rangle = \langle df(\gamma') \ , \ df(\gamma') \rangle = I_{f(p)}(df(\gamma'))$$

This means that the two surfaces will have the same metric components and thus we can conclude that they will have the same area.

Note that for general surfaces we can use the first fundamental form as long as we look at local coordinate charts.

2.2 Curvature

In this section a brief definition and description is given of the Gaussian curvature for surfaces in $\mathbb{R}^3$.

Definition 2.3 (Chase, H.S., Definition 3.5 [3]). *The principal curvatures at a point of a surface are the extreme values of the curvatures of the curves created by the intersection of S with planes containing the normal vector.*

For flat, straight curves (i.e. straight lines in a plane) the curvature is zero (see figure 2a), for curves that bend away from the normal vector on the point p the curvature is negative (see figure 2b) and positive for curves that bend in the direction of the normal vector [10].

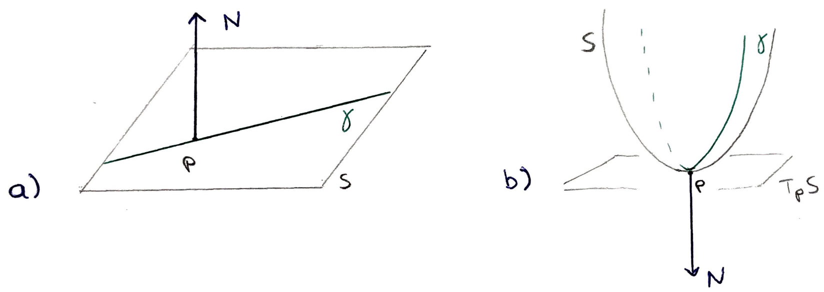

Figure 2: Different surfaces S and γ a curve created by the intersection of the surface with a plane containing a normal vector N.

Definition 2.4 (Chase, H.S., Definition 3.4 [3])**.** *The Gaussian curvature K in a point of a surface is the product of the two principal curvatures at that point.*

Example 2.1. *In the case of a cylinder we have only curves of negative curvature and of curvature 0. So the Gaussian curvature will be 0. This is expected since the cylinder is a quotient surface and locally isometric with the euclidean plane.*

A notable theorem concerning Gaussian curvature is the Gauss-Bonnet Theorem for compact surfaces, but first we need to define the genus and the Euler characteristic of a surface (Corollary 5.3, [14]).
The genus of a surface is informally speaking the amount of 'holes' the surface has. A precise definition is omitted as this explanation suffices for our use of the genus of a surface.

Definition 2.5. *The Euler characteristic of a surface S is $2 - 2h$, where h is the genus of the surface S.*

Theorem 2.2 (Gauss-Bonnet)**.** *Let S be a orientable compact surface. Then we have:*

$$\iint_S K \, dA = 2\pi\chi(S)$$

where K is the Gaussian curvature and $\chi(S)$ is the Euler Characteristic of S.

This theorem is also known as the Global Gauss-Bonnet theorem. It seems to appear partially in 1921 and fully in 1930 in work of Blaschke, but from Blaschke's way of writing Berger concludes that it was a "folk theorem" in the twenties [2].

There is an interesting result with regard to the Gaussian curvature of tori. The Euler characteristic of every torus is 1, so from the Gauss-Bonnet theorem we have that the total curvature is 0. So when we're looking at a torus with constant curvature, the curvature is 0 for every point of the surface and our torus is thus flat.

The Gaussian curvature can be explicitly computed using the first fundamental form and the second fundamental form. First one needs to take a parametrisation of the surface and next compute the first fundamental form. Then the Gaussian map n is defined and the Gaussian curvature of the surface is the determinant of the differential of the Gaussian map [11].
I choose to further illustrate Gaussian curvature by computing it for two surfaces in paragraph *5 Some explicit examples on the computation of the systolic ratio and curvature* and recommend *Gaussian Curvature and The Gauss-Bonnet Theorem* by Olfa Jaïbi for a full description and explanation behind the computation of the Gaussian curvature (see [11]).

3 Quotient surfaces: Flat torus

Before defining the flat torus as a quotient surface, we include a couple of preliminary definitions and statements concerning the isometries of the Euclidean plane.

Definition 3.1. *The euclidean distance d between any two points $p, q \in \mathbb{R}^2$ is given by:*

$$d(q,p) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$$

Definition 3.2. *An isometry of the Euclidean plane is a map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ that leaves distances unchanged:*

$$d(x,y) = d(\phi(x), \phi(y)) \ \ \forall x, y \in \mathbb{R}^2$$

The set of isometries of the Euclidean plane is noted as $\boldsymbol{I}(\mathbb{R})$.

In paragraph 1.5 of *Geometry of Surfaces* [16], John Stillwell presents the following result:

Theorem 3.1 (Classification of Euclidean Isometries)**.** *Any isometry of the Euclidean plane is either a rotation, a translation or a glide reflection.*

(Clarification: A glide reflection is the combination of a reflection with a translation in the direction of the line of reflection [16]).

The description of the flat torus as a quotient surface is based on Chapter 2 of *Geometry of Surfaces* by John Stillwell [16].
Let $\mathbb{R}^2/\Gamma$ be a quotient space where Γ is a group of isometries of the euclidean plane. There are different ways of choosing Γ but in order for $\mathbb{R}^2/\Gamma$ to describe a surface, it has to satisfy two conditions. Namely, Γ has to be:

- discontinuous: there exists no point $x \in \mathbb{R}^2$ whose neighbourhood contains infinitely many points of its equivalence class
- fixed-point free

The isometries that fulfil these conditions are translations and glide reflections. The way distance is defined on these quotient spaces (described further on in this paragraph) makes that these surfaces are locally isometric to the Euclidean plane. The resulting surfaces (quotient surfaces) are the cylinder, the twisted cylinder, the torus and the Klein bottle.

Since this thesis focuses on tori, we give a detailed description of the flat torus as a quotient surface and state related theorems (*Classification of flat tori [1]* and *Uniformization [1]*)

Definition 3.3. *A flat torus $\mathbb{T}^2$ is a two dimensional, compact, connected and orientable euclidean manifold without boundary of genus 1 with constant curvature 0.*

That the flat torus is a *euclidean* manifold refers to the fact that each point has a neighbourhood which is isometric to $\mathbb{R}^2$.

More specifically, we define the flat torus $\mathbb{T}^2$ as the quotient space $\mathbb{R}^2/\Gamma$, with Γ a subgroup of the isometries on the Euclidean plane generated by two linearly independent translations t_1 and t_2. This quotient space is often called *orbit space*. This group Γ generates a lattice on $\mathbb{R}^2$ (see Figure 3). That is a set of points that is reached by any linear combination of the translations t_1 and t_2.

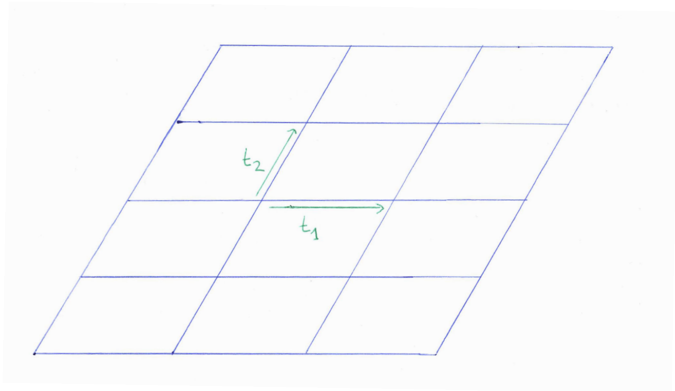

Figure 3: Lattice spanned on $\mathbb{R}^2$ by t_1 and t_2

The equivalence classes in the orbit space consist of points of the plane that are equivalent to each other. That is, the equivalence class of a point $x \in \mathbb{R}^2$ is the set of all the points of $\mathbb{R}^2$ which can be reached by repeatedly applying t_1 and t_2 to x. The equivalence class of x is called the Γ-orbit of x.

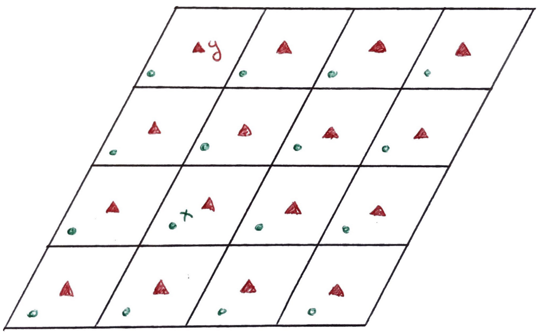

Figure 4: Γ orbit of point x in $\mathbb{R}^2$

Each parallelogram formed by the lattice over $\mathbb{R}^2$ contains exactly one representative of every Γ orbit in its interior and exactly two representatives on the border of each parallelogram since the opposite sides are identified. The only exception are the corner points of the parallelogram. These four points belong to the same Γ orbit. This makes it very convenient to focus only on one such parallelogram, which we call the *fundamental region* of the torus, instead of the whole $\mathbb{R}^2$ plane. The choice of the parallelogram doesn't matter since they are all equivalent.

Let $x, y \in \mathbb{R}^2$ arbitrarily given and Γx and Γy be their respective Γ orbits in the orbit space. We define distance d between these two orbits as follows:

$$d(\Gamma x, \Gamma y) = \min\{d_e(x', y')|\ x' \in \Gamma x,\ y' \in \Gamma y\}$$

with d_e the euclidean distance between two points in $\mathbb{R}^2$.

From this definition of distance we have that $\mathbb{R}^2/\Gamma$ is locally isometric to $\mathbb{R}^2$.

Since each point of a Γ orbit has the same set of distances to points of another Γ orbit, the above definition can be written as follows:

$$d(\Gamma x, \Gamma y) = \min\{d_e(x, y')|\ y' \in \Gamma y\}$$

The quotient map from $\mathbb{R}^2$ to $\mathbb{R}^2/\Gamma$ is called the orbit map $\Gamma\cdot$ which sends each point of the plane to its orbit in $\mathbb{R}^2/\Gamma$. For orbits Γx and Γy with $d(\Gamma x, \Gamma y) < \epsilon$, with ϵ sufficiently small, we have that the ϵ neighbourhood of Γx

$$D_\epsilon(\Gamma x) = \{\Gamma y \mid d(\Gamma x, \Gamma y) < \epsilon\}$$

is isometric to the ϵ neighbourhood of x (in $\mathbb{R}^2$) under the orbit map $\Gamma\cdot$.

This map sends every disc in the interior of a fundamental region isometrically to $\mathbb{R}^2/\Gamma$. So we can say that the orbit map is a local isometry. For points on the edge of the fundamental region we have that the section of the disc which would 'fall' out of the fundamental region appears on the opposite side around the equivalent point (see figure 5).

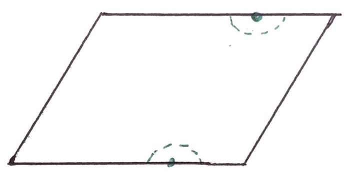

Figure 5: Illustration of a disc of a point on the edge of a fundamental region of a torus $\mathbb{R}^2/\Gamma$.

So, each flat torus is obtained by a parallelogram functioning as a fundamental region and the choice of the two translations which generate the subgroup Γ of isometries of the Euclidean plane fixes the metric of the torus. Therefore, along with the definition of distance on the quotient surface, we have that the area of a flat torus is the area of its fundamental region. This property will be used in the proof of Theorem 4.1 in section *4 Systoles*.

Classifying flat tori in relation to the fundamental regions is the next logical step and is described in the following paragraph.

3.1 Theorems concerning flat tori

In this section we will state the Classification theorem for flat tori and the Uniformization theorem. For the Classification theorem we will also provide a proof, but in order to do this we need to introduce the following lemma:

Lemma 3.2. *Let $\mathbb{T}_1^2$ be the flat torus obtained as the orbit space $\mathbb{R}^2/\Gamma_1$ with Γ_1 generated by two linearly independent translations t_1 and t_2. The flat torus $\mathbb{T}_2^2$ obtained as the orbit space $\mathbb{R}^2/\Gamma_2$ with Γ_2 generated by linear independent translations $t_1 + t_2$ and t_2 is the same torus $\mathbb{T}_1^2$.*

Proof. Let $x \in \mathbb{R}^2$ arbitrarily given. Its orbit $\Gamma_1 x$ consists of the points obtained by every linear combination of t_1 and t_2:

$$\Gamma_1 x = \{x + \lambda t_1 + \mu t_2 \ | \ \lambda, \mu \in \mathbb{Z}\}$$

The $\Gamma_2 x$ orbit consists of the points obtained by every linear combination of $t_1 + t_2$ and t_2:

$$\Gamma_2 x = \{x + \lambda'(t_1 + t_2) + \mu' t_2 \ | \ \lambda', \mu' \in \mathbb{Z}\}$$

$$= \{x + \lambda' t_1 + (\mu' + \lambda')t_2 \ | \ \lambda', \mu' \in \mathbb{Z}\}$$

Since $\mu', \lambda' \in \mathbb{Z}$ we have that $\mu' + \lambda' \in \mathbb{Z}$. Let $\nu = \mu' + \lambda'$. We have:

$$\Gamma_2 x = \{x + \lambda' t_2 + \nu t_2 \ | \ \lambda', \nu \in \mathbb{Z}\}$$

which is essentially $\Gamma_1 x$ and so

$$\Rightarrow \Gamma_1 x = \Gamma_2 x$$

Since x was arbitrarily given, we have that $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$ have the same equivalence classes and therefore are the same quotient space. In other words $\mathbb{T}^2_1$ and $\mathbb{T}^2_2$ are the same.

□

One can see that this lemma can be generalised for any two groups generated by linear combinations of the same two translations:

Lemma 3.3. *Let $\mathbb{T}^2_1$ be the flat torus obtained as the orbit space $\mathbb{R}^2/\Gamma_1$ with Γ_1 generated by two linearly independent translations t_1 and t_2. The flat torus $\mathbb{T}^2$ obtained by the orbit space $\mathbb{R}^2/\Gamma_2$ with Γ_2 generated by linear independent translations $a \cdot t_1 + bt_2$ and $c \cdot t_1 + d \cdot t_2$ is the same torus $\mathbb{T}^2_1$, with $a, b, c, d \in \mathbb{Z}$.*

The proof is the same as that of Lemma 3.2.

It is important to distinguish the difference between isometric flat tori and flat tori that are the same. Two flat tori are the same when, viewed as an orbit space, they have the same lattice, or in other words the same equivalence classes as shown in Lemma 3.3.
Two tori $\mathbb{T}^2_1$ and $\mathbb{T}^2_2$are globally isometric when there exists a bijection $f : \mathbb{T}^2_1 \to \mathbb{T}^2_2$ between the two which preserves distances. That is, for two points x and y of $\mathbb{T}^2_1$ holds:

$$d(\Gamma x, \Gamma y) = d(f(\Gamma x), f(\Gamma y))$$

Now we can prove the classification theorem of flat tori. A version of this theorem can be found in *A course in Arithmetic* by Jean-Pierre Serre. [15] (Theorem 1, page 78). We choose the version stated in Benedetti's lecture notes [1].

Theorem 3.4 (Classification of flat tori [1])**.** *Every flat torus can be up to homothety and isometry uniquely obtained by a parallelogram with vertices $(0, 0), (1, 0), (x_0, y_0)$ and $(x_0 + 1, y_0)$ with*

$$(x_0, y_0) \in A := \{(x, y) \in \mathbb{R}^2 \ | \ x^2 + y^2 \geq 1, \ 0 \leq x \leq \frac{1}{2}, \ y \geq 0\}$$

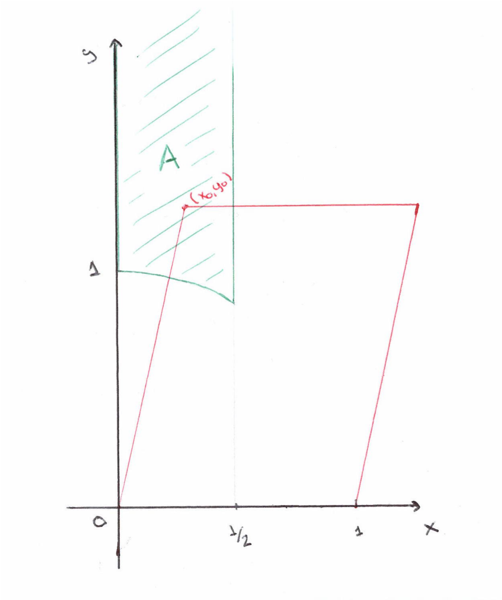

Figure 6: Domain vertice of parallelogram as fundamental region of a flat torus

Proof. Let $\mathbb{T}^2$ be a flat torus defined as the orbit space $\mathbb{R}^2/\Gamma$ with Γ generated by two linearly independent translations t_1 and t_2. Without loss of generality (because we are looking for a classification up to homothety) we choose t_1 to be of minimal length, namely that of the standard unit 1 . Then the length of t_2 must be larger than 1 and since we are working in $\mathbb{R}^2$ with the euclidean distance we get the first condition: $x^2 + y^2 \geq 1$

Because of Lemma 3.3 we know that each flat torus with fundamental region a parallelogram with sides t_1 and t_2 is identical to the flat torus with fundamental region a parallelogram with sides $t_1 + t_2$ and t_1 and to one with sides $t_2 - t_1$ and t_1. In the first case it is obvious that $||t_1 + t_2|| > ||t_1|| = 1$, but in the second case $||t_2 - t_1|| > 1$ isn't always true. So we get the restriction that (x, y) has to be on the circle with radius 1 and centre $(1, 0)$ or in its exterior.

We discern the following options for the x- coordinate x_0:

- $x_0 > 1$: In this case the torus is identical to one with vertices $(0,0),(1,0),(x_0',y_0)$ and $(x_0'+1, y_0)$ with $0 \leq x_0' \leq 1$ since we can subtract t_{1x} an arbitrary amount of times so that

  $$x_0' = x_0 - k \cdot t_{1x} = x_0 - k \cdot 1 = x_0 - k, \text{with } k \in \mathbb{Z} \setminus \{0\}$$

  So we can restrict ourselves to tori with fundamental region a parallelogram with vertices $(0,0),(1,0),(x_0,y_0)$ and (x_0+1,y_0) with $x_0 \in [0,1]$.

- $\frac{1}{2} < x_0 \leq 1$: Because of the previous point we can restrict x_0 so that $0 \leq x_0 \leq 1$ and look now specifically at tori with $\frac{1}{2} < x_0 \leq 1$.

  Let $\mathbb{T}^2$ be the torus obtained as the orbit space $\mathbb{R}^2/\Gamma$, where Γ is the group generated by two linear independent translations t_1 and t_2:

  $$t_1 : (a,b) \mapsto (a+1,b)$$
  $$t_2 : (a,b) \mapsto (a+x_0, b+y_0)$$

with fundamental region $(0,0),(1,0),(x_0,y_0)$ and (x_0+1,y_0) with $x_0 \in (\frac{1}{2},1]$.

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of points in the y axis and $p = (a,b) \in \mathbb{R}^2$ arbitrarily given. We have:

$$p' = \phi(p) = (-a,b)$$

We want to compare the orbit of p with the orbit of p after reflection, so, the orbit of p'.

Let Γ_r be the group generated by the two translations t_{1r} and t_{2r}:

$$t_{1r} : (a,b) \mapsto (a-1,b)$$
$$t_{2r} : (a,b) \mapsto (a-x_0,b+y_0)$$

We define the map $\tilde{\phi} : \mathbb{R}^2/\Gamma \to \mathbb{R}^2/\Gamma_r$ as follows:

$$\tilde{\phi}(\Gamma q) = \Gamma_r(\phi(q)) \quad , \quad \text{for } q \in \mathbb{R}^2 \text{ arbitrarily given}$$

We want to prove that $\tilde{\phi}$ is a bijective isometry and consequently that the two tori are isometric.

We will first prove injectivity.
Let q and p be two different points for which

$$\tilde{\phi}(\Gamma q) = \tilde{\phi}(\Gamma p)$$

or in other words

$$\Gamma_r(\phi(q)) = \Gamma_r(\phi(p))$$

By definition of $\mathbb{R}^2/\Gamma$ we have that $\Gamma q = \Gamma p$ is equivalent to $q - p \in \Gamma$.

$$\Gamma_r(\phi(q)) = \Gamma_r(\phi(p)) \Leftrightarrow \phi(q) - \phi(p) \in \Gamma_r$$

The map ϕ is linear (because it's a reflection in a line through the origin). So we have:

$$\Leftrightarrow \phi(q-p) \in \Gamma_r$$

This means that $\phi(q-p)$ is a linear combination of t_{1r} and t_{2r} (with $\lambda, \mu \in \mathbb{R}$):

$$\phi(q-p) = \lambda \cdot t_{1r} + \mu \cdot t_{2r} = \lambda(-1,0) + \mu(-x_0,y_0) = (-\lambda - \mu x_0, \mu y_0) = (-q_x + p_x, q_y - p_y)$$

So we also have

$$q - p = (q_x - p_x, q_y - p_y) = (-(-\lambda - \mu x_0), \mu y_0) = (\lambda + \mu x_0, \mu y_0) = \lambda \cdot t_1 + \mu t_2 \Leftrightarrow q - p \in \Gamma$$

And so $\tilde{\phi}$ is injective.

Where surjectivity is concerned, the map $\tilde{\phi}$ sends orbits of one orbit space to the other and since ϕ is surjective (isometries of the plane are bijective), we have that $\tilde{\phi}$ is also surjective.

Now the last thing to check is if $\tilde{\phi}$ is an isometry.
Let $p, q \in \mathbb{R}^2$ two different arbitrary points. We have:

$$d(\Gamma p, \Gamma q) = d(p, \Gamma q) = \min\{d_e(p,q') \mid q' \in \Gamma q\}$$

and

$$d(\tilde{\phi}\Gamma p, \tilde{\phi}\Gamma q) = d(\Gamma_r \phi(p), \Gamma_r \phi(q)) = \min\{d_e(\phi(p), \phi(q)') \ | \ \phi(q)' \in \Gamma_r p)\} =$$
$$= \min\{d_e(p, q') \ | \ q' \in \Gamma p)\}$$

This last step is possible due to the fact that ϕ is an isometry and to the bijectiveness of $\tilde{\phi}$ which sends orbits of $\mathbb{R}^2/\Gamma$ to orbits of $\mathbb{R}^2/\Gamma_r$. And so:

$$d(\tilde{\phi}\Gamma p, \tilde{\phi}\Gamma q) = \min\{d_e(p, q') \ | \ q' \in \Gamma p)\} = d(\Gamma p, \Gamma q)$$

So $\tilde{\phi}$ is an isometry and together with its bijectiveness we have proved that $\mathbb{R}^2/\Gamma$ and $\mathbb{R}^2/\Gamma_r$ are isometric tori.

So the torus $\mathbb{T}^2$ with fundamental region $(0,0),(1,0),(x_0,y_0)$ and (x_0+1,y_0) with $x_0 \in (\frac{1}{2}, 1]$ is isometric to the torus $\mathbb{T}^2_r$ with fundamental region $(0,0),(-1,0),(-x_0,y_0)$ and $(-x_0-1,y_0)$ (where Γ_r is generated by t_{1r} and t_{2r} as defined earlier). Because of Lemma 3.3, $\mathbb{T}^2_r$ is the same as the torus $\mathbb{T}^{2\prime}_r$ with fundamental region the parallelogram with vertices $(0,0),(1,0),(-x_0+1,y_0)$ and $(-x_0+3,y_0)$ where the group of translations has generators:

$$t'_{1r} = -t_{1r}$$

$$t'_{2r} = -t_{1r} + t_{2r}$$

Or in other words

$$t'_{1r} = t_1$$

$$t'_{2r} = t_1 + t_{2r}$$

Since $-x_0 \in [-1, -\frac{1}{2})$, we have that $-x_0 + 1 \in [0, \frac{1}{2})$. Therefore, we want to look at tori up to isometry, we can restrict ourselves to tori with fundamental region as described with $0 \leq x \leq \frac{1}{2}$.

□

Although looking at flat tori might seem restrictive, we will see that it will suffice. The reason behind this lies with the uniformization theorem.

Theorem 3.5 (Uniformization [1])**.** *Let S be a surface. Every Riemannian metric g on S is conformally equivalent to a Riemannian metric of constant curvature. There exists a Riemannian metric of constant curvature g_c and a positive function f such that g is isometric to $f^2 \cdot g_c$.*

These two theorems combined give us a better understanding of the space of Riemannian metrics on tori and is astounding.

4 Systoles

In this section we introduce systoles and the systolic ratio. We will denote the space of flat Riemannian metrics of the surface M being studied with $R(M)$.

Definition 4.1. *Let M be a compact, oriented surface without boundary. A systole of a metric $g \in R(M)$ is a non-contractible loop of minimal length which we denote with $sys(g)$.*

In 1898 Hadamard proved that if a manifold as described above is not simply connected (so for example the sphere S^2 does not comply), that this is a sufficient condition for the existence of a systole on the manifold. Since we are going to be looking at flat tori, which are not simply connected, we know that there will always be a systole.

Systoles give us a way to compare different metrics of the same (topologically) surface. Another way we can compare metrics is by looking at the total area of the surface with respect to the different metrics.

Definition 4.2. *Let M be a surface which is not simply connected and g a Riemannian metric of M. We define the systolic ratio $\sigma(g)$ as follows:*

$$\sigma(g) := \frac{sys^2(g)}{Area_g(M)}$$

Note that the systolic ratio is invariant under scaling. This follows from the result that the area of a surface is invariant under isometries (see paragraph 2.1) and that systoles, because they are geodesics, are also invariant under isometries.

To get a better understanding of the systolic ratio we include two examples.

Example 4.1. *Let T_1 be the flat torus with fundamental region the parallelogram with vertices $(0,0),(1,0),(\frac{1}{2},1)$ and $(1\frac{1}{2},1)$ and Riemannian metric g. It is obvious that the systole is a non contractible loop of length 1 (see Figure ...). The area of the torus is $1 \cdot 1 = \frac{1}{2}$. So*

$$\sigma(g) = \frac{1}{1} = 1$$

Example 4.2. *Let T_2 be the flat torus with fundamental region the parallelogram with vertices $(0,0),(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2})$ and $(1\frac{1}{2},\frac{\sqrt{3}}{2})$ and Riemannian metric g. It is obvious that the systole is a non contractible loop of length 1 (see Figure ...). The area of the torus is $1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$. So*

$$\sigma(g) = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

It turns out that this last torus or any flat torus obtained from a lattice spanned by two vectors that form an angle of $\frac{\pi}{3}$ ("equilateral lattice") is special, as we will see in the following theorem by Loewner [1].

Theorem 4.1 (Loewner, 1949). *Let $\mathbb{T}^2$ be a two dimensional torus arbitrarily with a Riemannian metric g. The following inequality holds:*

$$\sigma(g) \leq \frac{2}{\sqrt{3}}$$

Proof. Because of theorem 3.5 there exists a metric of constant curvature g_c and a smooth positive function f so that g and g_c are conformally equivalent and that g is isometric with $f^2 g_c$ and so $\mathbb{T}^2$ is isometric to the torus $\mathbb{T}^{2}$' with metric $f^2 g_c$. In paragraph *2.1 Riemannian metric and first fundamental form* we saw that globally isometric surfaces have the same area. From now on we will use $\mathbb{T}^2$ for both tori and only differentiate between them by specifying the metric. So we have

$$Area_g(\mathbb{T}^2) = Area_{f^2 g_c}(\mathbb{T}^2) \tag{1}$$

Let $\xi' : S \subset \mathbb{R}^2 \to \mathbb{R}^3$ a parametrization of the torus with metric $f^2 g_c$.

The first fundamental form of a point $p \in \mathbb{T}^2$ with metric $f^2 g_c$ is:

$$I_p = f^2(g_{c11}x'(t)^2 + 2x'(t)y'(t)g_{c12} + g_{c22}y'(t)^2)$$

We use the following notation for the metric components of $f^2 g_c$:

$$f^4 \tilde{g}_c = f^4 \det((g_c)_{ij})$$

And so we can calculate the area of the torus with metric g as follows:

$$Area_g(\mathbb{T}^2) = \iint_S \sqrt{f^4 \tilde{g}_c} = \iint_S f^2 \sqrt{\tilde{g}_c} \tag{1}$$

As mentioned in paragraph *2.2 Curvature*, a torus with a metric of constant curvature is necessarily a flat torus. For every point on this flat torus with metric g_c we can take the two standard perpendicular vectors of the plane as tangent vectors. Therefore the determinant of the metric components of g_c is 1 and we can rewrite the integral as follows:

$$\iint_S f^2(x,y)\sqrt{\tilde{g}_c} = \iint_S f^2(x,y)\; dxdy \tag{2}$$

Next we do a coordinate change to incorporate at length the systolic ratio of the flat metric g_c. The fundamental region of the flat torus with constant metric g_c is a parallelogram with vertices $(0,0), (1,0), (x_0, y_0)$ and $(1 + x_0, y_0)$, with (x_0, y_0) as defined in Theorem 3.4.
Let γ_t be the horizontal curves of this flat torus:

$$\gamma_t : s \mapsto (s + tx_0, ty_0)$$

with $t \in [0,1]$ and $s \in [0,1]$.

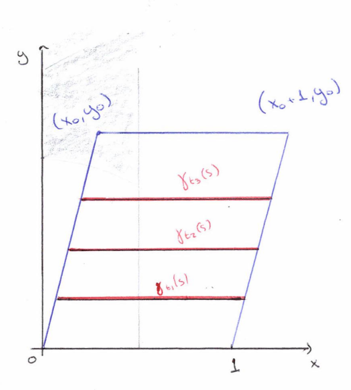

Figure 7: Curves γ_t for arbitrary values t_1, t_2 and t_3.

It is easy to see that for each t the curve γ_t has length 1 and is therefore a systole of the torus (see figure 7).
We can write $(x_0, y_0) = (r\cos\alpha, r\sin\alpha)$ with $\alpha \in [\frac{\pi}{3}, \frac{\pi}{2}]$ the angle enclosed by the vector (x_0, y_0) with the positive x-axis and $r \geq 1$ the length of that vector.

We parametrize the surface using the variables s, t via the transformation matrix T:

$$T = \begin{pmatrix} x(s,t) \\ y(s,t) \end{pmatrix} = \begin{pmatrix} s + tr\cos\alpha \\ tr\sin\alpha \end{pmatrix}$$

We have:

$$\det(J[T]) = \begin{vmatrix} 1 & r\cos\alpha \\ 0 & r\sin\alpha \end{vmatrix} = r\sin\alpha$$

Then (1) becomes:

$$Area_g(\mathbb{T}^2) = \int_0^1 \int_0^1 f^2(s + tx_0, y_0) r\sin\alpha \; dsdt = r\sin\alpha \int_0^1 \int_0^1 f^2(s + tx_0, y_0) \; dsdt$$

Using the Cauchy-Schwarz inequality for integrals (see *Appendix A*) we get the following:

$$Area_g(\mathbb{T}^2) \geq \frac{1 \cdot r\sin\alpha}{1^2} \int_0^1 \left(\int_0^1 f(s + tx_0, y_0) \; ds \right)^2 dt \tag{2}$$

The equality holds when f is a constant, meaning that g is homothetic to g_c. Since the flat torus is a quotient surface with a parallelogram as fundamental region and from the way that the distance between two points on such a torus is defined (see *3 Quotient surfaces: Flat torus*), the area of the torus equals the area of its fundamental region. We know that $y_0 = r\sin\alpha$ is the height of the fundamental region of the flat torus with metric g_c. The length of the matching base is 1 by construction and equal to the length of the systoles. So we have $Area_{g_c}(\mathbb{T}^2) = r\sin\alpha$ and we can rewrite (2):

$$Area_g(\mathbb{T}^2) \geq \frac{Area_{g_c}(\mathbb{T}^2)}{sys(g_c)} \int_0^1 \left(\int_0^1 f(s + tx_0, y_0) \; ds \right)^2 dt$$

$$= \frac{1}{\sigma(g_c)} \int_0^1 \left(\int_0^1 f(s + tx_0, y_0) \; ds \right)^2 dt \tag{3}$$

We have that the length of the curve γ_t with reference to g_c is

$$l_{g_c}(\gamma_t) = \int_0^1 \sqrt{x'(s)^2 + y'(s)^2} ds = \int_0^1 \sqrt{1^2 + 0^2} ds = \int_0^1 1 ds$$

Since g is isometric with $f^2 g_c$ we have:

$$l_g(\gamma_t) = l_{f^2 g_c}(\gamma_t) = \int_0^1 \sqrt{f^2(s + tx_0, ty_0)(x'(s)^2 + y'(s)^2)} ds =$$

$$= \int_0^1 f(s + tx_0, ty_0) \sqrt{1^2 + 0^2} ds = \int_0^1 f(s + tx_0, ty_0) ds$$

So (3) becomes:

$$Area_g(\mathbb{T}^2) \geq \frac{1}{\sigma(g_c)} \int_0^1 l_g(\gamma_t)^2 \; dt$$

Applying the Max-Min theorem for integrals we get the following:

$$Area_g(\mathbb{T}^2) \geq \frac{1}{\sigma(g_c)} \min(l_g(\gamma_t)^2) = \frac{1}{\sigma(g_c)} sys(g)^2$$

or in other words:

$$\sigma(g) \leq \sigma(g_c)$$

The systolic ratio is maximal when the considered area is minimal, so $\sigma(g_c)$ is maximal when $Area_{g_c}(\mathbb{T}^2)$ is minimal. The fundamental regions of tori $\mathbb{T}^2$ with metric g_c are up to homothety and isometry parallelograms with base of length 1, so we're looking for the parallelograms with the smallest height. This is the case for $(x_0, y_0) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$. And so:

$$\sigma(g) \leq \sigma(g_c) \leq \frac{1^2}{1 \cdot \frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

And so the proof of the inequality

$$\sigma(g) \leq \frac{2}{\sqrt{3}}$$

is completed. □

Whilst this theorem concerns specifically two dimensional tori, Mikhael Gromov stated and proved that for any essential Riemannian manifold there is an upper bound for the systolic ratio. It is remarkable that the upper bound is the same for manifolds of the same dimension [7].

5 Some explicit examples on the computation of the systolic ratio and curvature

In this section we will compute the systolic ratio and Gaussian curvature of a sphere and a torus in $\mathbb{R}^3$. A full description on how to compute the Gaussian curvature can be found in the thesis of Olfa Jaïbi [11]. Therefore we omit detailed explanation and will proceed to computations.

5.1 Sphere

Let M be a sphere of radius 2: $\{(x, y, z) \mid x^2 + y^2 + z^2 = 4\}$
Let $\xi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of M with $D = \{(\phi, \theta) \mid 0 \leq \phi \leq \pi \ , \ 0 \leq \theta \leq 2\pi\}$

$$\xi : (\phi, \theta) \to (2 \sin \phi \cos \theta \ , \ 2 \sin \phi \sin \theta \ , \ 2 \cos \phi)$$

The first fundamental form I_p at a point p is the inner product of the tangent spaces of M at p. The point p can be seen as a point of a parametrized curve γ on the surface M with:

$$\gamma(0) = p \text{ and } \gamma(t) = \xi(\phi(t), \theta(t)) \text{ with } t \in (-\epsilon, \epsilon)$$

So an element of T_pM is a tangent vector γ'

$$I_p : T_pM \times T_pM \to \mathbb{R}$$

We have:

$$\gamma' = \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta'$$

where ξ_ϕ is the partial derivative of ξ with respect to ϕ and ξ_θ is the partial derivative with respect to θ.

$$\xi_\phi = (2 \cos \phi \cos \theta, 2 \cos \phi \sin \theta, -2 \sin \phi)$$
$$\xi_\theta = (-2 \sin \phi \sin \theta, 2 \sin \phi \cos \theta, 0)$$

And so we can compute the first fundamental form. The intermediate computations are ommited and left as an exercise to the reader.

$$I_p(\gamma', \gamma') = \langle \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta', \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta' \rangle$$
$$= \langle \xi_\phi, \xi_\phi \rangle (\phi')^2 + \langle \xi_\theta, \xi_\theta \rangle (\theta')^2 + 2\theta'\phi' \langle \xi_\phi, \xi_\theta \rangle$$
$$= (4 \cos^2 \phi \cos^2 \theta + 4 \cos^2 \phi \sin^2 \theta + 4 \sin^2 \phi)(\phi')^2 + (4 \sin^2 \phi \sin^2 \theta + 4 \sin^2 \phi \cos^2 \theta)(\theta')^2 +$$
$$2\theta'\phi'(-4 \cos \phi \sin \phi \cos \theta \sin \theta + 4 \cos \phi \sin \phi \sin \theta \cos \theta)$$
$$= 4(\phi')^2 + 4 \sin^2 \phi (\theta')^2$$
$$= 4 \left(\frac{d\phi}{dt} \right)^2 + 4 \sin^2 \phi \left(\frac{d\theta}{dt} \right)^2$$

We will now proceed to compute the Gaussian map $n : p \mapsto n(p) = \frac{\xi_\phi \times \xi_\theta}{||\xi_\phi \times \xi_\theta||}(p)$

We have:

$$\xi_\phi \times \xi_\theta = \begin{pmatrix} 4\sin^2\phi\cos\theta \\ 4\sin^2\phi\sin\theta \\ 4\sin\phi\cos\phi \end{pmatrix}$$

and

$$\begin{aligned} ||\xi_\phi \times \xi_\theta|| &= \sqrt{(4\sin^2\phi\cos\theta)^2 + (-4\sin^2\phi\sin\theta)^2 + (4\sin\phi\cos\phi)^2} = \\ &= 4\sqrt{\sin^4\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi\cos^2\phi} \\ &= 4\sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)} \\ &= 4\sin\phi \end{aligned}$$

So the Gaussian map is:

$$n(p) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \cdot p$$

The curvature K is equal to the determinant of the differential of the Gaussian map.

$$K = \det(Dn(p)) = \frac{\langle n, \xi_{\phi\phi}\rangle\langle n, \xi_{\theta\theta}\rangle - \langle n, \xi_{\phi\theta}\rangle^2}{\langle \xi_\phi, \xi_\phi\rangle\langle \xi_\theta, \xi_\theta\rangle - \langle \xi_\phi, \xi_\theta\rangle^2}$$

The terms in the denominator were calculated when computing the first fundamental form above. The terms of the numerator are as follows:

$$\langle n, \xi_{\phi\phi}\rangle = -2\sin^2\phi\cos^2\theta - 2\sin^2\phi\sin^2\theta - 2\cos^2\phi = -2\sin^2\phi - 2\cos^2\phi = -2$$

$$\langle n, \xi_{\theta\theta}\rangle = -2\sin^2\phi\cos^2\theta - 2\sin^2\phi\sin^2\theta = -2\sin^2\phi$$

$$\langle n, \xi_{\phi\theta}\rangle = -2\sin\phi\cos\phi\cos\theta\sin\theta + 2\sin\phi\cos\phi\cos\theta\sin\theta = 0$$

And so the curvature of the sphere with radius 2 in $\mathbb{R}^3$ is

$$K = \frac{-2\cdot(-2\sin^2\phi) - 0}{4\cdot 4\sin^2\phi - 0} = \frac{1}{4}$$

and thus constant as expected. Note that it is not possible to calculate the systolic ratio for the sphere since each loop is contractible and therefore not a systole.

5.2 Torus embedded in $\mathbb{R}^3$

Let us now compute the curvature of a torus embedded in $\mathbb{R}^3$.

Let M be a two dimensional torus in $\mathbb{R}^3$ constructed by rotating a circle of radius α and centre q on the xy-plane, over a circle on the xy plane with centre the origin and radius β. To exclude self intersection we have the restriction $\alpha < \beta$.
We take the parametrization $\xi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ with $D = \{(\phi,\theta) \mid 0 \le \phi \le 2\pi\ ,\ 0 \le \theta \le 2\pi\}$:

$$\xi : (\phi,\theta) \to (\cos\theta(\beta + \alpha\cos\phi)\ ,\ \sin\theta(\beta + \alpha\cos\phi)\ ,\ \alpha\sin\phi)$$

where θ refers to the position of the circle with radius α in the xy-plane and ϕ refers to the position of a point on the circle with radius α.

The first fundamental form I_p at a point p is the inner product of the tangent spaces of T at p. The point p can be seen as a point of a parametrized curve γ on the surface M with:

$$\gamma(0) = p \text{ and } \gamma(t) = \xi(\phi(t), \theta(t)) \text{ with } t \in (-\epsilon, \epsilon)$$

So an element of T_pM is a tangent vector γ'

$$I_p : T_pM \times T_pM \to \mathbb{R}$$

We have:

$$\gamma' = \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta'$$

where ξ_ϕ is the partial derivative of ξ with respect to ϕ and ξ_θ is the partial derivative with respect to θ.

$$\xi_\phi = (-\alpha \sin\phi \cos\theta, -\alpha \sin\phi \sin\theta, \alpha \cos\phi)$$
$$\xi_\theta = (-\sin\theta(\beta + \alpha\cos\phi)\ ,\ \cos\theta(\beta + \alpha\cos\phi)\ ,\ 0)$$

And so we can compute the first fundamental form. The intermediate computations are ommited and left as an exercise to the reader.

$$I_p(\gamma', \gamma') = \langle \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta', \xi_\phi \cdot \phi' + \xi_\theta \cdot \theta' \rangle$$
$$= \langle \xi_\phi, \xi_\phi \rangle (\phi')^2 + \langle \xi_\theta, \xi_\theta \rangle (\theta')^2 + 2\theta'\phi' \langle \xi_\phi, \xi_\theta \rangle$$
$$= (\alpha^2 \sin^2\phi \cos^2\theta + \alpha^2 \sin^2\phi \sin^2\theta + \alpha^2 \cos^2\phi)(\phi')^2 + (\sin^2\theta(\beta+\alpha\cos\phi)^2 + \cos^2\theta(\beta+\alpha\cos\phi)^2)(\theta')^2 + 2\theta'\phi' \cdot 0$$
$$= \alpha^2(\phi')^2 + (\beta + \alpha\cos\phi)^2(\theta')^2$$
$$= \alpha^2 \left(\frac{d\phi}{dt}\right)^2 + (\beta + \alpha\cos\phi)^2 \left(\frac{d\theta}{dt}\right)^2$$

We will now proceed to compute the Gaussian map $n : p \mapsto n(p) = \frac{\xi_\phi \times \xi_\theta}{||\xi_\phi \times \xi_\theta||}(p)$

We have:

$$\xi_\phi \times \xi_\theta = \begin{pmatrix} -\alpha\cos\phi\cos\theta(\beta + \alpha\cos\phi) \\ -\alpha\cos\phi\sin\theta(\beta + \alpha\cos\phi) \\ -\alpha\sin\phi(\beta + \alpha\cos\phi) \end{pmatrix}$$

and

$$||\xi_\phi \times \xi_\theta|| = \sqrt{(-\alpha\cos\phi\cos\theta(\beta + \alpha\cos\phi))^2 + (-\alpha\cos\phi\sin\theta(\beta + \alpha\cos\phi))^2 + (-\alpha\sin\phi(\beta + \alpha\cos\phi))^2} =$$

$$= \sqrt{\alpha^2\cos^2\phi(\beta + \alpha\cos\phi)^2 + \alpha^2\sin^2\phi(\beta + \alpha\cos\phi)^2}$$
$$= \sqrt{\alpha^2(\beta + \alpha\cos\phi)^2}$$
$$= \alpha(\beta + \alpha\cos\phi)$$

So the Gaussian map is:

$$n(p) = \begin{pmatrix} -\cos\phi\cos\theta \\ -\cos\phi\sin\theta \\ -\sin\phi \end{pmatrix} \cdot p$$

The curvature K is equal to the determinant of the differential of the Gaussian map:

$$K_p = \det(Dn(p)) = \frac{\langle n, \xi_{\phi\phi}\rangle\langle n, \xi_{\theta\theta}\rangle - \langle n, \xi_{\phi\theta}\rangle^2}{\langle \xi_\phi, \xi_\phi\rangle\langle \xi_\theta, \xi_\theta\rangle - \langle \xi_\phi, \xi_\theta\rangle^2}$$

The terms in the denominator were calculated when computing the first fundamental form above. The terms of the numerator are as follows:

$$\langle n, \xi_{\phi\phi}\rangle = \alpha\cos^2\phi\cos^2\theta + \alpha\cos^2\phi\sin^2\theta + \alpha\sin^2\theta = \alpha$$

$$\langle n, \xi_{\theta\theta}\rangle = \cos\phi\cos^2\theta(\beta + \alpha\cos\phi) + \cos\phi\sin^2\theta(\beta + \alpha\cos\phi) = \cos\phi(\beta + \alpha\cos\phi)$$

$$\langle n, \xi_{\phi\theta}\rangle = -\alpha\sin\phi\cos\phi\cos\theta\sin\theta + \alpha\cos\phi\sin\phi\cos\theta\sin\theta = 0$$

And so the curvature of a point $p \in M$ is computed by:

$$K_p = \frac{\alpha\cos\phi(\beta + \alpha\cos\phi) - 0}{\alpha^2(\beta + \alpha\cos\phi)^2 - 0} = \frac{\cos\phi}{\alpha(\beta + \alpha\cos\phi)}$$

We see that the curvature in this case is not constant and thus M is not flat (see paragraph *2.2 Curvature*) and is dependent on the position of the point with reference to the z-axis. So let M be the torus as described with $\alpha = 2$ and $\beta = 4$. For any point p with $\phi = \frac{\pi}{3}$, the curvature will be:

$$K_p = \frac{\frac{1}{2}}{2(4 + 2\cos\frac{\pi}{3})} = \frac{1}{4(4+1)} = \frac{1}{20}$$

On the same torus, for any point with $\phi = \frac{2\pi}{3}$, the curvature will be:

$$K_p = \frac{-\frac{1}{2}}{2(4 + 2\cos\frac{\pi}{3})} = \frac{1}{4(4-1)} = -\frac{1}{12}$$

The sign of the Gaussian curvature in these two points was to be expected since the points with $\phi = \frac{\pi}{3}$ are on the 'inside' of the torus and those with $\phi = \frac{2\pi}{3}$ are on the outside (see figure 8).

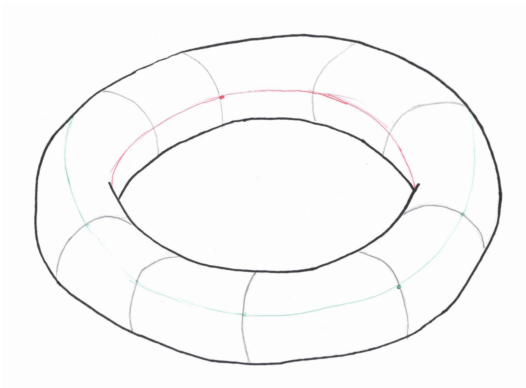

Figure 8: A torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$. The points on the red curve have $\phi = \frac{\pi}{3}$ and the points on the green curve have $\phi = \frac{2\pi}{3}$.

This doesn't come as a surprise as it is in fact impossible to embed a torus in $\mathbb{R}^3$ in a flat way [9].

Let us now calculate the systolic ratio of this specific torus. Systoles are the non-contractible loops of minimal length. By construction these will be the circles with radius $\alpha = 2$, in other words the circles created by the vertical cross sections of the torus (see black loops in figure 8). So: $sys_g(M) = 2\pi 2 = 4\pi$
The area of a torus is:

$$\iint_T ||\xi_\phi \times \xi_\theta|| d\phi \, d\theta = \int_0^{2\pi} \int_0^{2\pi} \alpha(\beta + \alpha \cos\phi) d\phi \, d\theta = \int_0^{2\pi} 2\pi\alpha\beta \, d\theta = 4\pi^2\alpha\beta$$

So, in our case we'll have:

$$Area_g(M) = 32\pi^2$$

The systolic ratio of M is:

$$\sigma(g) = \frac{4\pi}{32\pi^2} = \frac{1}{8\pi}$$

which is indeed smaller than $\frac{2}{\sqrt{3}}$.

6 Conclusion

The goal of the thesis was to study systolic inequalities on surfaces in order to understand the space of Riemannian metrics on surfaces. After polishing up on my knowledge of isometries and refreshing concepts of Analysis 2, we started by studying the flat torus as a quotient surface.

Constructing and giving an own proof of the Classification theorem turned out to be more challenging than thought at first, but thus all the more gratifying when at last figured out. The fact that the Classification of flat tori and the Uniformization theorem give us means to understand the space of Riemannian metrics on tori is intriguing. Due to lack of time, it wasn't possible to look at spheres or surfaces of higher genus (> 1) but this could be an interesting topic for further research.

After defining and understanding systoles, a great deal of the thesis was spent understanding the proof of Loewner's theorem (Theorem 4.1) given in the lecture notes by Benedetti (see [1]). This led to a more in depth study of the first fundamental form (see *2.1 Riemannian metric and first fundamental form*). Besides the fact that the statement that the systolic ratio of any two dimensional torus with a Riemannian metric is bounded by the systolic ratio of a specific Riemannian metric of constant curvature is very powerful, the proof is in itself also fascinating as it requires the combined knowledge acquired during the first part of the study.

Aside from Loewner's systolic upper bound for two dimensional tori there is also a theorem proved by Pu in 1952 (see [13] for the original publication), a Ph.D. student under Loewner, which states that the systolic ratio for the real projective plane has as upper bound $\frac{\pi}{2}$. There is also a similar inequality, first given by Bavart, which gives $\frac{\pi}{2\sqrt{2}}$ as the upper bound for the systolic ratio on the Klein bottle [12]. Of course, studying the proof of the theorem stating the existence of an upper bound for any essential Riemannian manifold can be just as interesting [7].

In the last paragraph we hinted to the fact that a flat two dimensional torus cannot be embedded in $\mathbb{R}^3$. For further study it could be interesting to understand why that is the case, seeing as a flat torus can be easily embedded in $\mathbb{R}^4$ [8].

A Cauchy-Schwarz inequality applied on integrals

The discrete form of the Cauchy-Schwarz inequality is formulated as follows:

For $a_i, b_i \in \mathbb{R}^2$ and $i \in \{1, 2, 3...\}$ the following inequailty holds:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

In this appendix I will give the proof of the Cauchy-Schwarz inequality for integrals following the one given by Tolsted E. in [17] which uses Young's inequality.

Theorem A.1 (Young's inequality)**.** *Let $\phi(x)$ and $\phi^{-1}(x)$ be continuous, strictly increasing functions which vanish at the origin and are inverse to each other. For $a, b \in \mathbb{R}^2_{\geq 0}$ we have:*

$$ab \leq \int_0^a \phi(x)dx + \int_0^b \phi^{-1}(x)dx \tag{3}$$

We choose $\phi(x) = x^\alpha$ $\phi^{-1}(x) = x^{\frac{1}{\alpha}}$ and (3) becomes:

$$ab \leq \int_0^a x^\alpha dx + \int_0^b x^{\frac{1}{\alpha}} dx$$

$$ab \leq \frac{a^{\alpha+1}}{\alpha+1} + \frac{b^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha}+1} \tag{4}$$

Define now $r = \alpha + 1$ and $r' = \frac{1}{\alpha} + 1$ and substitute in (4):

$$ab \leq \frac{1}{r}a^r + \frac{1}{r'}b^{r'} \tag{5}$$

Let $f, g : [a, b] \to \mathbb{R}^2$ be two Riemann integrable functions which are not zero over the whole interval. We define the following

$$S := \left(\int_c^d f^r(x)\, dx\right)^{\frac{1}{r}} \text{ and } T := \left(\int_c^d g^{r'}(x)\, dx\right)^{\frac{1}{r'}}$$

and choose:

$$a = \frac{f(x)}{S} \text{ and } b = \frac{g(x)}{T}$$

Filling this in (5) gives us:

$$\frac{f(x)}{S}\frac{g(x)}{T} \leq \frac{1}{r}\frac{f^r(x)}{S^r} + \frac{1}{r'}\frac{g^{r'}(x)}{T^{r'}}$$

Since S, T are definite integrals and thus a constant, integrating both sides of the inequality gives us

$$\frac{1}{ST}\int_c^d f(x)g(x)dx \leq \int_c^d \left(\frac{1}{r}\frac{f^r(x)}{S^r} + \frac{1}{r'}\frac{g^{r'}(x)}{T^{r'}}\right) dx$$

Linearity of integration gives us:

$$\frac{1}{ST}\int_c^d f(x)g(x)dx \leq \frac{1}{r}\frac{\int_c^d f^r(x)dx}{S^r} + \frac{1}{r'}\frac{\int_c^d g^{r'}(x)dx}{T^{r'}}$$

$$\frac{1}{ST}\int_c^d f(x)g(x)dx \leq \frac{1}{r} + \frac{1}{r'} \tag{6}$$

Note that:

$$\frac{1}{r} + \frac{1}{r'} = \frac{1}{\alpha+1}\frac{1}{\frac{1}{\alpha}+1} = \frac{\alpha+1+\frac{1}{\alpha}+1}{(\frac{1}{\alpha}+1)(\alpha+1)} = \frac{1+2\alpha+\alpha^2}{\alpha(\frac{1}{\alpha}+1)(\alpha+1)} = \frac{(\alpha+1)^2}{(\alpha+1)(\alpha+1)} = 1$$

So (6) becomes:

$$\frac{1}{ST}\int_c^d f(x)g(x)dx \leq 1$$

$$\int_c^d f(x)g(x)dx \leq \left(\int_c^d f^r(x)\,dx\right)^{\frac{1}{r}} \left(\int_c^d g^{r'}(x)\,dx\right)^{\frac{1}{r'}}$$

This is in fact the Hölder inequality for Riemann integrals. If we now take $r = r' = 2$ we get the Cauchy-Schwarz inequality:

$$\int_c^d f(x)g(x)dx \leq \left(\int_c^d f^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_c^d g^2(x)\,dx\right)^{\frac{1}{r'}} \tag{7}$$

To apply this in the proof of Loewner's theorem look at the special case for $g = 1$ and $c = 0, d = 1$. We get:

$$\int_0^1 f(x)dx \leq \left(\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_0^1 1\,dx\right)^{\frac{1}{r'}}$$

$$\int_0^1 f(x)dx \leq \left(\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}} (1)^{\frac{1}{r'}}$$

$$\int_0^1 f(x)dx \leq \left(\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}}$$

$$\left(\int_0^1 f(x)dx\right)^2 \leq \int_0^1 f^2(x)\,dx$$

References

[1] Gabriele Benedetti. First steps into the world of systolic inequalities: From riemannian to symplectic geometry. *arXiv preprint arXiv:2103.09356*, 2021. URL: https://www.few.vu.nl/~trt800/ddtg/benedetti_DDTG.pdf.

[2] Marcel Berger. *Panoramic View of Riemannian Geometry*. Springer Berlin / Heidelberg, Berlin, Heidelberg, 2012. URL: https://link-springer-com.ezproxy.leidenuniv.nl/book/10.1007/978-3-642-18245-7.

[3] Hunter S. Chase. Fundamental forms of surfaces and the gauss-bonnet theorem. *University of Chicago, Chicago*, 2012. URL: https://math.uchicago.edu/~may/REU2012/REUPapers/Chase.pdf.

[4] Greg Galloway. Introduction to differential geometry, 2018. Lecture notes. URL: https://www.math.miami.edu/~galloway/dgnotes/chpt4.pdf.

[5] Herman Gluck. Intrinsic geometry of surfaces, 2012. Lecture notes. URL: https://www2.math.upenn.edu/~shiydong/Math501X-4-IntrinsicGeometry.pdf.

[6] Claudio Gorodski. An introduction to riemannian geometry. URL: https://www.ime.usp.br/~gorodski/teaching/mat5771-2016/gorodski-riem-geom-2012.pdf.

[7] Mikhael Gromov. Filling riemannian manifolds. *Journal of differential geometry*, 18(1):1–147, 1983. URL: https://doi.org/10.4310/jdg/1214509283.

[8] Qing Han and Jia-Xing Hong. *Isometric embedding of Riemannian manifolds in Euclidean spaces*, volume 13. American Mathematical Soc., 2006. URL: https://catalogue.leidenuniv.nl/primo-explore/fulldisplay?vid=UBL_V1&lang=en_US&docid=TN_cdi_askewsholts_vlebooks_9781470413576&context=PC.

[9] Howard Jacobowitz. Isometric embedding of a compact riemannian manifold into euclidean space. *Proceedings of the American Mathematical Society*, 40(1):245–246, 1973. URL: https://www.ams.org/journals/proc/1973-040-01/S0002-9939-1973-0375173-3/S0002-9939-1973-0375173-3.pdf.

[10] Jeff Jauregui. Math 660: Principal curvatures, 2011. Lecture notes. URL: https://www.math.union.edu/~jaureguj/principal_curvatures.pdf.

[11] Olfa Jaïbi. Gaussian curvature and the gauss-bonnet theorem. 2013. URL: https://www.universiteitleiden.nl/binaries/content/assets/science/mi/scripties/jaibibach.pdf.

[12] Mikhail Gersh Katz. *Systolic geometry and topology*. Mathematical surveys and monographs ; volume 137. 2007. URL: https://www-ams-org.ezproxy.leidenuniv.nl/books/surv/137/.

[13] P. M. Pu. Some inequalities in certain nonorientable riemannian manifolds. *Pacific journal of mathematics*, 2(1):55–71, 1952. URL: https://msp.org/pjm/1952/2-1/pjm-v2-n1-p06-s.pdf.

[14] Grant Rotskoff. The gauss-bonnet theorem. 2010.

[15] Jean-Pierre Serre. *A course in Arithmetic*. Springer New York, NY, 1973. URL: https://www.math.purdue.edu/~jlipman/MA598/Serre-Course%20in%20Arithmetic.pdf.

[16] John Stillwell. *Geometry of surfaces*. Universitext. Springer, New York [etc.], 1992.

[17] Elmer Tolsted. An elementary derivation of the cauchy, hölder, and minkowski inequalities from young's inequality. *Mathematics magazine*, 37(1):2–12, 1964. URL: https://doi.org/10.1080/0025570X.1964.11975469.