Topology of enriched categories through lifting presheaves

The, J.C.

Citation

The, J. C. *Topology of enriched categories through lifting presheaves*.

Jorre The

Topology of enriched categories through lifting presheaves

Bachelor thesis

30 June 2022

Thesis supervisor: dr. D.S.T. Holmes

Leiden University
Mathematical Institute

Contents

1 Introduction

Category theory, though relatively young as a mathematical field, has undoubtably had an immense and far-reaching impact on much of modern mathematical research. Its power is largely found in how suprisingly effective it is at placing structures of almost all areas of mathematics in a very uniform framework. For many applications, one finds that categories are even more useful when endowed with additional structure. For example, in algebraic contexts, the hom-sets of a category might take on the structure of an abelian group. Such a category often behaves as a *pre-additive category*, a category whose hom-sets are abelian groups and for which composition of morphisms is bilinear. That is, given morphisms $f, f' : a \to b$ and $g, g' : b \to c$ we have

$$\begin{aligned} (g + g') \circ f &= g \circ f + g' \circ f \\ g \circ (f + f') &= g \circ f + g \circ f'. \end{aligned}$$

This and similar examples led Eilenberg and Kelly in [1] to develop a theory of categories whose hom-sets are not necessarily sets, but objects of some other monoidal category, for example abelian groups or pointed sets. Some importance was placed on the existence of an (often forgetful) functor from this other category $\mathcal{V}$ to the category **Set** of sets. Through this functor, these categories could be viewed as ordinary categories with some richer structure, which warrented them the name *enriched categories*.

It is therefore perhaps suprising that enriched categories can be the perfect tool to describe structures that seemingly bare no resemblance to ordinary categories. Possibly the best example of this, is Lawvere's observation in [2] that (generalized) metric spaces can be viewed as enriched categories with hom-objects in the category $\overline{\mathbb{R}}^+ = [0, \infty]$ of non-negative real numbers and infinity under addition with the categorical structure of an ordered set. These Lawvere metric spaces were generalized metric spaces in that they were not required to be symmetric or positive-definite. A *Lawvere metric space* is thus a set X equipped with for every $x, y \in X$ an element $X(x, y) \in \overline{\mathbb{R}}^+$ such that for all $x, y, z \in X$ we have

$$\begin{aligned} X(y, z) + X(x, y) &\geq X(x, z) \\ X(x, x) &= 0. \end{aligned}$$

To view Lawvere metric spaces as enriched categories, one lets the metric $X(-, -)$ take on the role of the hom-functor. The triangle inequality $X(y, z) + X(x, y) \geq X(x, z)$ then corresponds to composition of morphisms and the axiom $X(x, x) = 0$ corresponds to the existence of identity morphisms. Using this interpretation, many properties of metric spaces follow from purely (enriched) category theoretic arguments.

This thesis presents such a process. We will show that, when presented with a monoidal functor of monoidal categories, presheaves on an enriched category can be lifted through this functor in a way that is consistent with the change of enrichment. This category theoretical result is then applied to construct the topology on metric spaces. The crux is that this allows one to define a natural topology not only on metric spaces, but on a broad class of enriched categories.

For this thesis, basic knowledge of ordinary categories is assumed, but a complete treatment of the necessary enriched category theory is given. This covers only the concepts required for proving the results and some examples to aid the reader's comprehension of them. For a more general introduction to enriched category theory, we will refer to [3]. We will largely adopt the (non-standard) notation of Lawvere's [3] to strengthen the interpretation of enriched categories as generalized metric spaces.

2 Monoidal categories and quantales

As mentioned, enriched categories have hom-objects that are not sets but objects of some other monoidal category. We will therefore first give an overview of monoidal categories. We will subsequently focus our attention to a particular type of monoidal category, namely Flagg's value quantales. We will see how these allow for defining a natural generalization of Lawvere metric spaces.

2.1 Monoidal categories

The following definition and examples are from [3, §1.1].

Definition 2.1. A *monoidal category* is a category $\mathcal{V}$ together with

- a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, called the *monoid operation*,
- a distinguished object k in $\mathcal{V}$, called the *monoidal identity*,
- for all objects a, b and c in $\mathcal{V}$ a natural isomorphism

$$\alpha_{a,b,c} : a \otimes (b \otimes c) \to (a \otimes b) \otimes c,$$

- for every object a in $\mathcal{V}$ two natural isomorphisms

$$\lambda_a : k \otimes a \to a \qquad \text{and} \qquad \rho_a : a \otimes k \to a,$$

such that $\lambda_k = \rho_k$ and the following diagrams commute for all a, b, c and d:

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[r, "\alpha_{a,b,c\otimes d}"] \arrow[d, "\mathrm{id}_a \otimes \alpha_{b,c,d}"'] & (a \otimes b) \otimes (c \otimes d) \arrow[r, "\alpha_{a\otimes b,c,d}"] & ((a \otimes b) \otimes c) \otimes d \\
a \otimes ((b \otimes c) \otimes d) \arrow[rr, "\alpha_{a,b\otimes c,d}"] & & (a \otimes (b \otimes c)) \otimes d \arrow[u, "\alpha_{a,b,c} \otimes \mathrm{id}_d"']
\end{tikzcd}$$

$$\begin{tikzcd}
a \otimes (k \otimes b) \arrow[rr, "\alpha_{a,k,b}"] \arrow[dr, "\mathrm{id}_a \otimes \lambda_b"'] & & (a \otimes k) \otimes b \arrow[dl, "\rho_a \otimes \mathrm{id}_b"] \\
& a \otimes b &
\end{tikzcd}$$

We call $\mathcal{V}$ *symmetric* if for all objects a and b in $\mathcal{V}$ there exists a natural isomorphism $\sigma_{a,b} : a \otimes b \to b \otimes a$ such that $\sigma_{b,a} \circ \sigma_{a,b} = \mathrm{id}_{a \otimes b}$ and such that for all a, b and c the following diagrams commute:

$$\begin{tikzcd}
a \otimes (b \otimes c) \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "\mathrm{id}_a \otimes \sigma_{b,c}"'] & (a \otimes b) \otimes c \arrow[r, "\sigma_{a\otimes b,c}"] & c \otimes (a \otimes b) \arrow[d, "\alpha_{c,a,b}"] \\
a \otimes (c \otimes b) \arrow[r, "\alpha_{a,c,b}"] & (a \otimes c) \otimes b \arrow[r, "\sigma_{a,c} \otimes \mathrm{id}_b"] & (c \otimes a) \otimes b
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes a \arrow[rr, "\sigma_{a,b}"] \arrow[dr, "\lambda_a"'] & & a \otimes k \arrow[dl, "\rho_a"] \\
& a &
\end{tikzcd}$$

Example 2.2. The category **Set** of sets is a symmetric monoidal category with the cartesian product $\times$ as the monoid operation and a one-element set $k = \{*\}$ as the monoidal identity. In fact, every category that has all finite products can be given the structure of a monoidal category by taking the product $\times$ as the monoidal operation. Such monoidal categories are called *cartesian.* Another example of a cartesian category is the category **Cat** of small categories.

Example 2.3. The category **Ab** of abelian groups is a symmetric monoidal category with the tensor product $\otimes_{\mathbb{Z}}$ as the monoid operation and the group of integers $\mathbb{Z}$ as the monoidal identity.

Example 2.4. Define the *category of truth values* to be the category of two objects $\mathbf{2} = \{\text{false}, \text{true}\}$ with besides the identity morphisms only one morphism false $\to$ true. It is a (cartesian) symmetric monoidal category with conjunction $\wedge$ as the monoidal operation and true as the monoidal identity.

We will now introduce the notion of Bénabou cosmos ([4, p. 1]), which we will simply call cosmos.

Definition 2.5. A symmetric monoidal category $\mathcal{V}$ is called *closed* if for every object b in $\mathcal{V}$ the functor $- \otimes b : \mathcal{V} \to \mathcal{V}$ has a right adjoint $\mathcal{V}(b, -) : \mathcal{V} \to \mathcal{V}$, called the *internal hom-functor.* For $a, b \in \mathcal{V}$, the *evaluation morphism* $\varepsilon_{a,b} : \mathcal{V}(a, b) \otimes a \to b$ is the counit of this adjunction. A *cosmos* (plural: cosmoi) is a bicomplete closed symmetric monoidal category, that is, a closed symmetric monoidal category that has all small limits and colimits. A cosmos is called semi-cartesian if the monoidal identity is a terminal object.

All of the previous examples are in fact cosmoi. For **Set**, the internal hom-functor is given by taking $\mathbf{Set}(S, S')$ to be the set of functions $S \to S'$. Similarly $\mathbf{Ab}(A, A')$ is the abelian group of homomorphisms $A \to A'$ and $\mathbf{Cat}(C, C')$ is the category of functors $C \to C'$. For the category $\mathbf{2}$ of truth values, the internal hom-functor is given by the implication operation $\mathbf{2}(a, b) = (a \implies b)$.

2.2 Quantales

A very different class of examples of cosmoi is offered by quantales. Quantales and their interpretation as cosmoi are discussed in [5, Ch. II] We will first give the order-theoretic definition of a quantale.

Definition 2.6. Recall that a *complete lattice* is a poset $(L, \leq)$ such that every subset $S \subseteq L$ has a greatest lower bound $\bigwedge S$, called its *meet*, and a least upper bound $\bigvee S$, called its *join.* We call $0 = \bigwedge L$ the *bottom element* and $\infty = \bigvee L$ the *top element* of L.

A *(commutative) quantale* is a complete lattice Q with an associative and commutative binary operation $+ : Q \times Q \to Q$ with an identity element, such that for all $q \in Q$ and $S \subseteq Q$ we have

$$\bigwedge S + q = \bigwedge (S + q) \, .$$

Example 2.7. The set $\overline{\mathbb{R}}^+ = [0, \infty]$ of non-negative real numbers with infinity is a quantale with the usual ordering and addition.

Proposition 2.8. *A quantale is exactly a small thin cosmos with monoidal operation* $+$ *and a morphism* $a \to b$ *if and only if* $a \geq b$.

Proof. Let Q be a quantale. Since $+$ is associative, commutative and has identity element k, the natural isomorphisms $\alpha_{a,b,c} : a+(b+c) \to (a+b)+c$, $\lambda_a : k+a \to a$, $\rho_a : a+k \to a$ and $\sigma_{a,b} : a+b \to b+a$ can be taken to be the identity maps, so that the coherence conditions are trivially satisfied. This means that Q is a symmetric monoidal category. For all $b, c \in Q$, define

$$Q(b,c) = \bigwedge\{q \in Q : q+b \geq c\}.$$

Then $Q(b,-) : Q \to Q$ is a functor for every $b \in Q$, since for $c, c' \in Q$ with $c \geq c'$ we have $Q(b,c) \geq Q(b,c')$. For $a, b, c \in Q$ with $a+b \geq c$ we have $a \in \{q \in Q : q+b \geq c\}$ and therefore $Q(b,c) \leq a$. Conversely, if $Q(b,c) \leq a$ then there exists some $q \in Q$ with $q+b \geq c$ and $q \leq a$, from which we get $a+b \geq q+b \geq c$. Hence, $a+b \geq c$ if and only if $Q(b,c) \leq a$ and it follows that $Q(b,-) : Q \to Q$ is right adjoint to $-+b : Q \to Q$. We find that Q is a small thin cosmos.

Conversely, a small thin bicomplete category Q is a lattice and the symmetric monoidal operation defines an associative and commutative binary operation $+ : Q \times Q \to Q$ with identity element. If Q is closed, the functor $(-)+q : Q \to Q$ is a left adjoint and therefore commutes with colimits, so that

$$q + \bigwedge S = \bigwedge (q+S)$$

for all $q \in Q$ and $S \subseteq Q$. □

Remark 2.9. Note that the above category structure on a quantale is opposite to the usual category structure on a poset. In fact, a quantale is often defined to have the top element as its monoidal identity. The top element is then denoted by 1 with the quantale written multiplicatively. The reason for using additive notation here is because our leading example is the quantale $\overline{\mathbb{R}}^+$.

Example 2.10. In the quantale $\overline{\mathbb{R}}^+$, the internal hom-functor is truncated subtraction:

$$\overline{\mathbb{R}}^+(b,c) = \begin{cases} 0 & b \geq c \\ c-b & b < c \end{cases}.$$

2.3 Flagg metric spaces

Of particular interest is a type of quantale that is suitable for a metric to take values in, namely value quantales. The following is taken from Flagg [6].

Definition 2.11. Let V be a quantale. For $a, b \in V$, we say that b is *well above* a and write $b \succ a$, if for any subset $S \subseteq V$ with $a \geq \bigwedge S$, there exists some $s \in S$ such that $b \geq s$. We call V *totally distributive* if for all $a \in V$, we have

$$a = \bigwedge\{b \in V : b \succ a\}.$$

We call V a *value quantale* if it is totally distributive, has identity element 0, satisfies $\infty \succ 0$ and if for all $a, b \in V$ with $a \succ 0$ and $b \succ 0$ we have $\bigwedge\{a,b\} \succ 0$.

Example 2.12. The quantale $\overline{\mathbb{R}}^+$ is a value quantale. We have $a \succ b$ if and only if $a > b$ for $a, b \in \overline{\mathbb{R}}^+$.

Definition 2.13. Let V be a value quantale. A *V-continuity space* is a set X with for all $x, y \in X$ an element $X(x, y) \in V$, such that for all $x, y, z \in X$, we have

$$X(y,z) + X(x,y) \geq X(x,z)$$
$$X(x,x) = 0.$$

We call X *positive-definite* if whenever $X(x, y) = 0$ we have $x = y$ and we call X *symmetric* if $X(x, y) = X(y, x)$ for all $x, y \in X$.

We will refer to V-continuity spaces in general as *Flagg metric spaces.* In particular, $\overline{\mathbb{R}}^{+}$-continuity spaces are *Lawvere metric spaces*, as in [2]. An ordinary metric space is thus a positive-definite symmetric Lawvere metric space.

Let X be a V-continuity space. The well-above-relation allows us to define *open balls* $B_\varepsilon(x) = \{y \in X : \varepsilon \succ X(x, y)\}$ of radius $\varepsilon \in V$ around $x \in X$. The topology on X is defined as follows: a subset $U \subseteq X$ is called *open* if for every $x \in U$ there exists some $\varepsilon \succ 0$ such that $B_\varepsilon(x) \subseteq U$.

Proposition 2.14. *Let X be a V-continuity space. Then for the topological closure $\overline{S}$ of S, we have*

$$\overline{S} = \left\{ y \in S : \bigwedge \{X(y,x) : x \in S\} = 0 \right\}.$$

For a proof, see Lemma 3.2 of [6]. Note that for $V = \overline{\mathbb{R}}^{+}$, this gives the familiar identity

$$\overline{S} = \left\{ y \in S : \inf_{x \in S} X(y,x) = 0 \right\}.$$

It will be this closure operator that we will generalize from value quantales to general cosmoi. For this, we must first interpret Flagg metric spaces as categories enriched in a value quantale.

3 Enriched categories

This entire section is based on Kelly [3].

3.1 $\mathcal{V}$-categories and $\mathcal{V}$-functors

Definition 3.1. Let $\mathcal{V}$ be a cosmos. A *category enriched in* $\mathcal{V}$ or simply $\mathcal{V}$*-category* is a class X together with

- for every pair of elements x and y in X an object $X(x,y)$ of $\mathcal{V}$, called the *hom-object*,
- for all elements x, y and z in X a morphism $M_{x,y,z} : X(y,z) \otimes X(x,y) \to X(x,z)$, called the *composition morphism*,
- for every element x of X a morphism $j_x : k \to X(x,x)$, called the *identity morphism*,

such that the following diagrams commute for all elements w, x, y and z of X:

$$\begin{tikzcd}
X(y,z) \otimes (X(x,y) \otimes X(w,x)) \arrow[rr, "\alpha_{X(y,z),X(x,y),X(w,x)}"] \arrow[d, "\mathrm{id}_{X(y,z)} \otimes M_{w,x,y}"'] & & (X(y,z) \otimes X(x,y)) \otimes X(w,x) \arrow[d, "M_{x,y,z} \otimes \mathrm{id}_{X(w,x)}"] \\
X(y,z) \otimes X(w,y) \arrow[dr, "M_{w,y,z}"'] & & X(x,z) \otimes X(w,x) \arrow[dl, "M_{w,x,y}"] \\
& X(w,z) &
\end{tikzcd}$$

$$\begin{tikzcd}
X(y,y) \otimes X(x,y) \arrow[r, "M_{x,y,y}"] & X(x,y) & X(x,y) \otimes X(x,x) \arrow[l, "M_{x,x,y}"'] \\
k \otimes X(x,y) \arrow[u, "j_y \otimes \mathrm{id}_{X(x,y)}"] \arrow[ur, "\lambda_{X(x,y)}"'] & & X(x,y) \otimes k \arrow[u, "\mathrm{id}_{X(x,y)} \otimes j_x"'] \arrow[ul, "\rho_{X(x,y)}"]
\end{tikzcd}$$

For $\mathcal{V}$-categories X and Y, a $\mathcal{V}$*-functor* is a function $F : X \to Y$ with for every pair of elements x and y in X a morphism $F_{x,y} : X(x,y) \to Y(F(x),F(y))$, such that for all elements x, y and z in X the following diagrams commute:

$$\begin{tikzcd}
X(y,z) \otimes X(x,y) \arrow[r, "M_{x,y,z}"] \arrow[d, "F_{y,z} \otimes F_{x,y}"] & X(x,z) \arrow[d, "F_{x,z}"] \\
Y(F(y),F(z)) \otimes Y(F(x),F(y)) \arrow[r, "M_{F(x),F(y),F(z)}"'] & Y(F(x),F(z))
\end{tikzcd}$$

$$\begin{tikzcd}
& X(x,x) \arrow[dd, "F_{x,x}"] \\
k \arrow[ur, "j_x"] \arrow[dr, "j_{F(x)}"] & \\
& Y(F(x),F(x))
\end{tikzcd}$$

We call $F : X \to Y$ *fully faithful* if every $F_{x,y}$ is an isomorphism.

Example 3.2. Suppose $\mathcal{V}$ is the monoidal category **Set** under $\times$. Then a **Set**-category X is exactly a category in the usual sense whose hom-sets are the sets $X(x, y)$ for $x, y \in X$. The composition of morphisms is then expressed by the maps

$$\circ = M_{x,y,z} : X(y, z) \times X(x, y) \to X(x, z)$$

and the identity morphism of $x \in X$ is the image of $j_x : k \to X(x, x)$. The commutative diagrams of Definition 3.1 are equivalent to composition of morphisms being associative and the identity morphism being an identity element of the composition.

A **Set**-functor is exactly a functor between ordinary categories, whose action on morphisms is given by the maps $F_{x,y} : X(x, y) \to Y(F(x), F(y))$. The commutative diagrams express exactly that functors preserve composition and identity morphisms. A functor being fully faithful means that the maps $F_{x,y} : X(x, y) \to Y(F(x), F(y))$ are bijective, which is exactly what it means for a functor to be full and faithful in the usual sense.

We will frequently return to this example to show that concepts defined for enriched categories coincide with the corresponding notions for ordinary categories in the case where $\mathcal{V} = \mathbf{Set}$.

Example 3.3. The case for $\mathcal{V} = \mathbf{Ab}$ is very similar and yields the notion of an additive category as previously announced. The hom-sets $X(x, y)$ are now abelian groups and the composition is given by maps $M_{x,y,z} : X(y, z) \otimes_{\mathbb{Z}} X(x, y) \to X(x, z)$ from the tensor product instead of the ordinary product, which is exactly equivalent to the composition being bilinear. An **Ab**-functor is an *additive functor*, that is, a functor $F : X \to Y$ whose action $F_{x,y} : X(x, y) \to Y(F(x), F(y))$ on hom-sets is a homomorphism of abelian groups.

Example 3.4. A (small) category enriched in $\mathcal{V} = \mathbf{2}$ is a *pre-ordered set* (we will never use the term **2**-category to refer to these, to avoid confusion with 2-categories described in the next example). Axiomatically, pre-ordered sets can be described as partially ordered sets except anti-symmetry is dropped. Let X be a category enriched in **2** and write $x \leq y$ whenever $X(x, y) = \text{true}$. Then the existence of the composition morphisms is equivalent to the transitivity condition

$$y \leq z \text{ and } x \leq y \implies x \leq z \text{ for all } x, y, z \in X$$

and the existence of the identity morphisms is equivalent to the reflexivity condition

$$x \leq x \text{ for all } x \in X.$$

Note that since **2** is thin, the diagrams of Definition 3.1 always commute.

A **2**-enriched functor $F : X \to Y$ is exactly an order preserving map, since the existence of the morphisms $F_{x,y} : X(x, y) \to Y(F(x), F(y))$ for all $x, y \in X$ is equivalent to

$$x \leq y \implies F(x) \leq F(y) \text{ for all } x, y \in X.$$

Then F is fully faithful if and only if it is an *order embedding*, that is

$$x \leq y \iff F(x) \leq F(y) \text{ for all } x, y \in X.$$

Example 3.5. Suppose $\mathcal{V}$ is the monoidal category **Cat** of small categories under $\times$. A category enriched in **Cat** is known as a 2-category. If X is a 2-category, the hom-sets $X(x, y)$ are themselves categories, so that there exist morphisms between morphisms, known as *2-morphisms*. The composition of 2-morphisms within $X(x, y)$ is called *vertical composition*. Given 2-morphisms η_1 between morphisms $f_1, g_1 \in X(y, z)$ and η_2 between morphisms $f_2, g_2 \in X(x,y)$, the functor $M_{x,y,z} : X(y,z) \times X(x,y) \to X(x,z)$ maps (f_1, f_2) to $f_1 \circ f_2$, (g_1, g_2) to $g_1 \circ g_2$ and maps (η_1, η_2) to a 2-morphism from $f_1 \circ f_2$ to $g_1 \circ g_2$, known as the *horizontal composite.*

A **Cat**-functor between 2-categories X and Y is called a *2-functor.* It maps objects of X to objects of Y, morphisms in $X(x,y)$ to morphisms in $Y(F(x), F(y))$ and 2-morphisms between $f, g \in X(x,y)$ to 2-morphisms between $F(f), F(g) \in Y(F(x), F(y))$.

Example 3.6. As discussed before, one of the examples we are most interested in is when $\mathcal{V}$ is a value quantale. In that case, a (small) $\mathcal{V}$-category X is a Flagg metric space, since the existence of the composition morphisms and identity morphisms are equivalent to the fact that for all $x, y, z \in X$, we have

$$\begin{aligned} X(y,z) + X(x,y) &\geq X(x,z) \\ 0 &\geq X(x,x). \end{aligned}$$

Again, since $\mathcal{V}$ is thin, the diagrams of Definition 3.1 always commute.

For a $\mathcal{V}$-functor $F : X \to Y$ the existence of morphisms $F_{x,y} : X(x,y) \to Y(F(x), F(y))$ for all $x, y \in X$ is equivalent to the condition that $X(x,y) \geq Y(F(x), F(y))$ for all $x, y \in X$. That is, $\mathcal{V}$-functors are distance-decreasing maps of Flagg metric spaces. Fully faithful $\mathcal{V}$-functors are distance-preserving maps.

Proposition 3.7. *The cosmos $\mathcal{V}$ is itself a $\mathcal{V}$-category with hom-objects $\mathcal{V}(a,b)$, whose composition morphisms $\mathcal{V}(c,d) \otimes \mathcal{V}(a,b) \to \mathcal{V}(a,c)$ correspond to*

$$(\mathcal{V}(b,c) \otimes \mathcal{V}(a,b)) \otimes a \xrightarrow{\alpha} \mathcal{V}(b,c) \otimes (\mathcal{V}(a,b) \otimes a) \xrightarrow{\mathrm{id}_{\mathcal{V}(b,c)} \otimes \varepsilon_{a,b}} \mathcal{V}(b,c) \otimes b \xrightarrow{\varepsilon_{b,c}} c$$

under the adjunction of $- \otimes a$ and $\mathcal{V}(a,-)$ and whose identity morphisms $j_a : k \to \mathcal{V}(a,a)$ correspond to $\lambda_a : k \otimes a \to a$.

Definition 3.8. For a $\mathcal{V}$-category X, we define the *opposite $\mathcal{V}$-category* X^{op} with the same objects as X, whose hom-objects are $X^{\mathrm{op}}(x,y) = X(y,x)$ and whose composition morphisms are the composition

$$X^{\mathrm{op}}(y,z) \otimes X^{\mathrm{op}}(x,y) \xrightarrow{\sigma_{X(z,y),X(y,x)}} X(y,x) \otimes X(z,y) \xrightarrow{M_{z,y,x}} X^{\mathrm{op}}(x,z).$$

Given $\mathcal{V}$-categories X and Y, we define the $\mathcal{V}$-category $X \otimes Y$, called the *tensor product* of X and Y, whose objects are pairs (x,y) indexed by $x \in X$ and $y \in Y$, whose hom-objects are

$$(X \otimes Y)((x,y),(x',y')) = X(x,x') \otimes Y(y,y')$$

and whose composition and identity morphisms are the compositions

$$\begin{tikzcd}
(X(x',x''), Y(y',y'')) \otimes ((X(x,x'), Y(y,y'))) \arrow[d] & \\
(X(x',x'') \otimes X(x,x')) \otimes (Y(y',y'') \otimes Y(y,y')) \arrow[r, "M_{x,x',x''} \otimes M_{y,y',y''}"] & X(x,x'') \otimes Y(y,y'')
\end{tikzcd}$$

$$k \cong k \otimes k \xrightarrow{j_x \otimes j_y} X(x,x) \otimes Y(y,y)$$

where the vertical morphism is obtained by any suitable composition of α and σ.

A $\mathcal{V}$-functor $T : X \otimes Y \to Z$ gives rise to *partial $\mathcal{V}$-functors* $T(-,y) : X \to Z$ and $T(x,-) : Y \to Z$ for all $x \in X$ and $y \in Y$. It may therefore be viewed as a $\mathcal{V}$-functor in two variables. This is also called a *$\mathcal{V}$-bifunctor.*

It can easily be verified that if $\mathcal{V} = \mathbf{Set}$, one recovers the usual notions of opposite category and product category.

3.2 The $\mathcal{V}$-category of $\mathcal{V}$-functors

In the category **Set** we can speak of individual elements of an object. If this object is a hom-set, these elements are morphisms. In the enriched case, we cannot speak about individual elements of an object of $\mathcal{V}$, and in particular we cannot speak of morphisms in a $\mathcal{V}$-category. Nevertheless, it is customary to think of morphisms $k \to v$ as *elements* of an object $v \in \mathcal{V}$. Indeed, if $\mathcal{V} = \mathbf{Set}$ then k is a one-element set and there is a natural bijection between elements of a set S and morphisms $k \to S$.

This notion of elements of objects of $\mathcal{V}$ leads to straightforward generalizations of concepts in **Set**-enriched categories to the $\mathcal{V}$-enriched case. For example, a *$\mathcal{V}$-natural tranformation* between $\mathcal{V}$-functors $F, G : X \to Y$ is defined to be a family of morphisms $\eta_x : k \to Y(F(x), G(x))$ indexed by $x \in X$ such that for all $x, y \in X$, the following diagram commutes:

$$\begin{tikzcd}
& k \otimes X(x,y) \arrow[r, "\eta_y \otimes F_{x,y}"] & Y(F(y),G(y)) \otimes Y(F(x),F(y)) \arrow[dr, "M_{F(x),F(y),G(y)}"] & \\
X(x,y) \arrow[ur, "\lambda^{-1}_{X(x,y)}"] \arrow[dr, "\rho^{-1}_{X(x,y)}"'] & & & Y(F(x),G(y)) \\
& X(x,y) \otimes k \arrow[r, "G_{x,y} \otimes \eta_x"] & Y(G(x),G(y)) \otimes Y(F(x),G(x)) \arrow[ur, "M_{F(x),G(x),G(y)}"'] &
\end{tikzcd}$$

Note that if $Y = \mathcal{V}$, the family of morphisms $\eta_x : k \to \mathcal{V}(F(x), G(x))$ gives rise to a family of morphisms $F(x) \to G(x)$ in $\mathcal{V}$ through the adjunction of $- \otimes F(x)$ and $\mathcal{V}(F(x), -)$. If all of these morphisms are isomorphisms, we speak of a *$\mathcal{V}$-natural isomorphism* between F and G. Just as in ordinary categories, one can define horizontal and vertical composition of $\mathcal{V}$-natural transformations, and this gives $\mathcal{V}-\mathbf{Cat}$ the structure of a 2-category (in the sense of Example 3.5).

The purpose of this paragraph will be to define an arguably more suitable structure on $\mathcal{V}-\mathbf{Cat}$. To start, $\mathcal{V}$-natural transformations are defined by a family of elements of $Y(F(x), G(x))$. However, a general principle in enriched categories is that the choice of k in the definition of elements $k \to v$ is somewhat arbitrary. It is often more useful to instead consider a *generalized element* of V, that is, a morphism $a \to v$ from any object a. This leads to the following definition.

Definition 3.9. Let a be an object of $\mathcal{V}$ and $T : X^{\mathrm{op}} \otimes X \to \mathcal{V}$ a $\mathcal{V}$-functor. A *$\mathcal{V}$-extranatural transformation* $\eta : a \overset{\cdot}{\to} T$ is a family of morphisms $\eta_x : a \to T(x,x)$ indexed by $x \in X$ such that for all $x, y \in X$ the following diagram commutes:

$$\begin{tikzcd}
X(x,y) \arrow[r, "T(x,-)_{x,y}"] \arrow[d, "T(-,y)_{y,x}"'] & \mathcal{V}(T(x,x), T(x,y)) \arrow[d, "\mathcal{V}(\eta_x, \mathrm{id}_{T(x,y)})"] \\
\mathcal{V}(T(y,y), T(x,y)) \arrow[r, "\mathcal{V}(\eta_y, \mathrm{id}_{T(x,y)})"] & \mathcal{V}(a, T(x,y))
\end{tikzcd}$$

It is a good exercise to check that a $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors $F, G : X \to Y$ is the same as a $\mathcal{V}$-extranatural transformation $k \overset{\cdot}{\to} Y(F-, G-)$.

Definition 3.10. Let $T : X^{\mathrm{op}} \otimes X \to \mathcal{V}$ be a $\mathcal{V}$-functor. An *end* of T is an object $\int_{x \in X} T(x,x)$ in $\mathcal{V}$ with a $\mathcal{V}$-extranatural transformation $\omega : \int_{x \in X} T(x,x) \overset{\cdot}{\to} T$ with the universal property that for any other $\mathcal{V}$-extranatural transformation $\eta : a \overset{\cdot}{\to} T$ there exists a unique morphism $f : a \to \int_{x\in X} T(x,x)$ making the following diagram commute for all $y \in X$:

$$\begin{tikzcd}
a \arrow[d, dashed, "f"] \arrow[dr, "\eta_y"] & \\
\int_{x\in X} T(x,x) \arrow[r, "\omega_y"] & T(y,y)
\end{tikzcd}$$

We call $\omega : \int_{x\in X} T(x,x) \overset{\cdot}{\to} T$ the *universal $\mathcal{V}$-extranatural transformation* of $\int_{x\in X} T(x,x)$.

Since they are defined by universal properties, ends are unique up to a unique isomorphism. We will therefore speak of *the* end of a $\mathcal{V}$-functor $T : X^{\mathrm{op}} \otimes X \to \mathcal{V}$.

Example 3.11. Consider $\mathcal{V} = \mathbf{Set}$ and let $F, G : X \to Y$ be two functors. Elements of the set $\int_{x\in X} Y(F(x), G(x))$ are in bijection with maps $k \to \int_{x\in X} Y(F(x), G(x))$ and therefore in bijection with natural families of maps $k \to Y(F(x), G(x))$ indexed by $x \in X$, by the universal property of ends. Since $Y(F(x), G(x))$ is the set of morphisms $F(x) \to G(x)$, the set $\int_{x\in X} Y(F(x), G(x))$ is therefore in bijection with the set of natural families of morphisms $\eta_x : F(x) \to G(x)$, where the naturality condition is equivalent to the following diagram commuting for every morphism $f : x \to y$:

$$\begin{tikzcd}
F(x) \arrow[r, "\eta_x"] \arrow[d, "F(f)"] & G(x) \arrow[d, "G(f)"] \\
F(y) \arrow[r, "\eta_y"] & G(y)
\end{tikzcd}$$

This is exactly the definition of a natural transformation from F to G. We find that $\int_{x\in X} Y(F(x), G(x))$ is exactly the set of natural transformtations from F to G.

Recall that for **Set**-enriched categories X and Y, the *functor category* of X and Y is the **Set**-enriched category $[X, Y]$ whose objects are functors $X \to Y$ and whose hom-sets are the sets of natural transformations between pairs of functors. With the set of natural transformations between two functors described as in the previous example, this generalizes directly to the enriched setting.

Definition 3.12. Let X and Y be $\mathcal{V}$-categories. We define the *$\mathcal{V}$-category of $\mathcal{V}$-functors* $X \to Y$ to be the $\mathcal{V}$-category $[X, Y]$ whose objects are $\mathcal{V}$-functors $X \to Y$, whose hom-objects are given by the end $[X, Y](F, G) = \int_{x\in X} Y(F(x), G(x))$ for $F, G : X \to Y$ and whose composition and identity morphisms are the unique morphisms making the following diagrams commute for all $x \in X$:

$$\begin{tikzcd}
{[X,Y](G,H) \otimes [X,Y](F,G)} \arrow[r] \arrow[d] & {[X,Y](F,H)} \arrow[d] \\
Y(G(x), H(x)) \arrow[r, "M_{F(x),G(x),H(x)}"'] & Y(F(x), H(x))
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[r, "j_{F(x)}"] \arrow[dr] & Y(F(x), F(x)) \arrow[d] \\
& {[X,Y](F,F)}
\end{tikzcd}$$

Here, the vertical morphisms are given by the universal $\mathcal{V}$-extranatural transformations associated to the ends.

3.3 The Yoneda embedding

Recall from Proposition 3.7 that $\mathcal{V}$ itself can be given the structure of a $\mathcal{V}$-category. Given a $\mathcal{V}$-category X we may therefore speak of the $\mathcal{V}$-category of $\mathcal{V}$-functors $[X^{\mathrm{op}}, \mathcal{V}]$, called the *$\mathcal{V}$-category of presheaves on X*.

Example 3.13. A ring R can be viewed as a 1-element category $R = \{*\}$ enriched in **Ab**. The underlying additive group of R is the hom-object $R(*,*)$ and multiplication of elements of $R(*,*)$ is defined by the only composition morphism

$$M_{*,*,*} : R(*,*) \otimes_{\mathbf{Z}} R(*,*) \to R(*,*)$$

of R, that is, $M_{*,*,*}(s,r) = rs$. The fact that this is a homomorphism from the tensor product indicates the distributivity of multiplication over addition. The multiplicative unit of R is the image of $1 \in \mathbb{Z}$ under the homomorphism $j_* : \mathbb{Z} \to R(*,*)$. The associative and unital laws of multiplication in R are then given by the commutative diagrams of Definition 3.1.

Let $M : R^{\mathrm{op}} \to \mathbf{Ab}$ be a presheaf. This consists of a choice of abelian group, namely the image $M(*)$ of $*$ under M, and for every $r \in R(*,*)$ a choice of endomorphism $M(r)$ of $M(*)$. Since M is additive, we have $(r+s)(m) = r(m)+s(m)$ for all $r, m \in R(*,*)$ and $m \in M(*)$. Moreover, since M respects composition and identity morphisms, we have $(rs)(m) = s(r(m))$ and $1(m) = m$ for all $r, s \in R(*,*)$ and $m \in M(*)$. This means that M can equivalently be viewed as a right R-module.

In fact, the presheaf category $[R^{\mathrm{op}}, \mathbf{Ab}]$ is exactly the additive category of right R-modules and similarly, $[R, \mathbf{Ab}]$ is the additive category of left R-modules.

Let X be a $\mathcal{V}$-category. For every $x \in X$, we get a presheaf $X(-,x) : X^{\mathrm{op}} \to \mathcal{V}$ which on hom-objects is given as the adjunct

$$X^{\mathrm{op}}(y,z) \to \mathcal{V}(X(y,x), X(z,x))$$

of the composition morphism $M_{x,y,z} : X^{\mathrm{op}}(y,z) \otimes X^{\mathrm{op}}(x,y) \to X^{\mathrm{op}}(x,z)$ of X^{op}. Presheaves of this form are called *representable presheaves.* Similarly, we have for every $x \in X$ a *representable copresheaf* $X(x,-) : X \to \mathcal{V}$ which on hom-objects is given as the adjunct

$$X(y,z) \to \mathcal{V}(X(x,y), X(x,z))$$

of the composition morphism $M_{x,y,z} : X(y,z) \otimes X(x,y) \to X(x,z)$ of X.

Representable presheaves define a $\mathcal{V}$-functor $X \to [X^{\mathrm{op}}, \mathcal{V}]$ sending $x \in X$ to $X(-,x)$ and whose morphisms of hom-objects

$$X(x,y) \to [X^{\mathrm{op}}, \mathcal{V}](X(-,x), X(-,y))$$

are given by applying the universal property of ends to the $\mathcal{V}$-extranatural transformation

$$X(x,y) \overset{\cdot\cdot}{\to} \mathcal{V}(X(-,x), X(-,y))$$

whose components are the morphisms $X(z,-)_{x,y} : X(x,y) \to \mathcal{V}(X(z,x), X(z,y))$ of hom-objects of the representable copresheaf $X(z,-)$ for $z \in X$.

Just as in the case of $\mathcal{V} = \mathbf{Set}$, we will refer to this $\mathcal{V}$-functor $X \to [X^{\mathrm{op}}, \mathcal{V}]$ as the *Yoneda embedding* of X. We will adopt the notation of [7] and denote denote this $\mathcal{V}$-functor by よ_X, where よ is the Japanese kana for "yo" in Hiragana. The name "embedding" is warrented by よ_X being fully faithful, which follows from the *Yoneda Lemma.*

Lemma 3.14 (Yoneda Lemma). *Let X be a $\mathcal{V}$-category and $F : X^{\mathrm{op}} \to \mathcal{V}$ a presheaf. Then we have a $\mathcal{V}$-natural isomorphism*

$$F(x) \cong [X^{\mathrm{op}}, \mathcal{V}](X(-,x), F)$$

indexed by $x \in X$.

For a proof, see [3, §2.4]. In particular, for $x, y \in X$, the morphisms

$$(\text{よ}_X)_{x,y} : X(x,y) \to [X^{\mathrm{op}}, \mathcal{V}](X(-,x), X(-,y))$$

are actually isomorphisms in $\mathcal{V}$, giving us the following.

Corollary 3.15. *For a $\mathcal{V}$-category X, the Yoneda embedding $\text{よ}_X : X \to [X^{\mathrm{op}}, \mathcal{V}]$ is a fully faithful $\mathcal{V}$-functor.*

Example 3.16. Let X be a small category enriched in $\mathbf{2}$, that is, a pre-ordered set. A presheaf $F : X^{\mathrm{op}} \to \mathbf{2}$ can exactly be viewed as a downward-closed subset of X, where $x \in X$ is in the set if and only if $F(x) = \text{true}$. A subset $S \subseteq X$ is *downward-closed* if for every $x \in S$ and $y \in X$ with $y \leq x$ we also have $y \in S$, which for F is equivalent to the existence of morphisms $X^{\mathrm{op}}(x,y) \to \mathbf{2}(F(x), F(y))$ for all $x, y \in X$. The $\mathbf{2}$-enriched presheaf category $[X^{\mathrm{op}}, \mathbf{2}]$ can then be identified with the set of downward-closed subsets of X ordered by inclusion. The Yoneda embedding now gives an order embedding of X into this presheaf category by sending $x \in X$ to the downward-closed set $\{y \in X : y \leq x\}$.

Example 3.17. Revisiting Example 3.13, the Yoneda embedding $R \to [R^{\mathrm{op}}, \mathbf{Ab}]$ maps $* \in R$ to R viewed as a left R-module. The Yoneda lemma now states that a left R-module M is naturally isomorphic to the left R-module of R-module homomorphisms $R \to M$.

3.4 Tensor product of $\mathcal{V}$-functors

A final $\mathcal{V}$-category theoretical result that will become relevant later is that of the tensor product of a $\mathcal{V}$-functor by a presheaf. For this, we must define ends in general $\mathcal{V}$-categories and the dual concept of coends.

Definition 3.18. Let $T : X^{\mathrm{op}} \otimes X \to Y$ be a $\mathcal{V}$-functor. An *end* of T is an object $\int_{x \in X} T(x,x)$ in Y together with a $\mathcal{V}$-natural isomorphism

$$Y\left(y, \int_{x \in X} T(x,x)\right) \cong \int_{x \in X} Y(y, T(x,x))$$

indexed by $y \in Y$. Dually, a *coend* of T is an object $\int^{x \in X} T(x,x)$ (note the upper index) in Y together with a $\mathcal{V}$-natural isomorphism

$$Y\left(\int^{x \in X} T(x,x), y\right) \cong \int_{x \in X} Y(T(x,x), y).$$

To see that if $Y = \mathcal{V}$, this definition of ends coincides with Definition 3.10, note that the universal $\mathcal{V}$-extranatural transformation $\omega : \int_{x \in X} T(x,x) \dot{\to} T$ gives for every $y \in \mathcal{V}$ a universal $\mathcal{V}$-extranatural transformation

$$\mathcal{V}(y, \omega) : \mathcal{V}\left(v, \int_{x \in X} T(x,x)\right) \overset{\cdot}{\longrightarrow} \mathcal{V}(y, T(-,-))$$

and therefore a $\mathcal{V}$-natural isomorphism

$$\mathcal{V}\left(y, \int_{x\in X} T(x,x)\right) \cong \int_{x\in X} \mathcal{V}(y, T(x,x)).$$

We use coends to define the tensor product of $\mathcal{V}$-functors by presheaves over certain categories. The following defintion and examples are from [8].

Definition 3.19. A $\mathcal{V}$-category Y is called *tensored* over $\mathcal{V}$ if for all $a \in \mathcal{V}$ and $y \in Y$ there exists an object $a \bullet y$ in Y with a $\mathcal{V}$-natural isomorphism

$$Y(a \bullet y, z) \cong \mathcal{V}(a, Y(y,z))$$

indexed by $z \in Y$. In that case, the *tensor product* of a $\mathcal{V}$-functor $F : X \to Y$ by a presheaf $G : X^{\mathrm{op}} \to \mathcal{V}$ is the coend

$$G \bullet F = \int^{x\in X} G(x) \bullet F(x)$$

which is an object of Y.

Note that $\mathcal{V}$ is always tensored over itself with $\bullet$ being the tensor product $\otimes$ on $\mathcal{V}$, since the isomorphism

$$\mathcal{V}(a \otimes b, c) \cong \mathcal{V}(a, \mathcal{V}(b,c))$$

given by the adjunction of $- \otimes b$ and $\mathcal{V}(b, -)$ is actually $\mathcal{V}$-natural.

Example 3.20. Any cocomplete **Set**-category is tensored over **Set** with $S \otimes y = \bigsqcup_S y$ being the copower of y by S – the coproduct of S copies of $y \in Y$.

The following example explains the name “tensor product”.

Example 3.21. Recall from Example 3.13 that given a ring R viewed as a one-element **Ab**-enriched category, $\mathcal{V}$-functors $N : R \to \mathbf{Ab}$ and $M : R^{\mathrm{op}} \to \mathbf{Ab}$ are precisely left and right R-modules respectively. Their tensor product $M \bullet N$ as **Ab**-functors is then exactly their tensor product $M \otimes_R N$ as R-modules.

Definition 3.22. Given $\mathcal{V}$-functors $F : X \to Y$ and $G : Y \to X$, we say that F is *left $\mathcal{V}$-adjoint* to G and G is *right $\mathcal{V}$-adjoint* to F if there exists a $\mathcal{V}$-natural isomorphism

$$Y(F(x), y) \cong X(x, G(y))$$

indexed by $(x,y) \in X \otimes Y$. In that case, we speak of a $\mathcal{V}$-adjunction and write $F \dashv G$.

Note that a $\mathcal{V}$-category Y is tensored over $\mathcal{V}$ if and only if every representable copresheaf $Y(y, -) : Y \to \mathcal{V}$ has a left $\mathcal{V}$-adjoint.

4 Lifting presheaves along monoidal functors

To introduce the notion of change-of-enrichment, it is worth first looking deeper into the notion of elements of an object $v \in \mathcal{V}$, that is, morphisms $k \to v$.

We can associate to every object $v \in \mathcal{V}$ the set $\hom_{\mathcal{V}}(k, v)$ of elements of v. This defines a functor $\hom_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathbf{Set}$. Given a $\mathcal{V}$-category X, the *underlying category* of X is the **Set**-enriched category X_0 which has the same objects as X, but whose hom-sets are $\hom_{\mathcal{V}}(k, X(x, y))$ for $x, y \in X$. This is an instance of *change of enrichment.* As an example, the underlying category of an additive category is obtained by simply forgetting the additive structure on the hom-sets. But not every underlying category retains so much of the information of a $\mathcal{V}$-category. The underlying category of a Flagg metric space for example is simply a preordered set $(X, \preceq)$ with $x \preceq y$ if and only if $X(x, y) = 0$. In particular, the underlying category of a Flagg metric space with positive-definite metric is discrete. It will more or less be this change in enrichment, however, that will provide Flagg metric spaces with their topology.

4.1 Change of enrichment

To discuss the general change of enrichment, we must first define the appropriate functors between cosmoi that make this possible.

Definition 4.1. A *monoidal functor* between monoidal categories $(\mathcal{V}, \otimes, k)$ and $(\mathcal{W}, \otimes, \ell)$ is a functor $N : \mathcal{V} \to \mathcal{W}$ together with natural transformations

$$\begin{aligned} \tilde{N}_{a,b} &: N(a) \otimes N(b) \to N(a \otimes b) \\ N_0 &: \ell \to N(k) \end{aligned}$$

such that for all $a, b, c \in \mathcal{V}$ the following diagrams commute:

$$\begin{tikzcd}
N(a) \otimes (N(b) \otimes N(c)) \arrow[r, "\alpha_{N(a),N(b),N(c)}"] \arrow[d, "\mathrm{id}_{N(a)} \otimes \tilde{N}_{b,c}"'] & (N(a) \otimes N(b)) \otimes N(c) \arrow[d, "\tilde{N}_{a,b} \otimes \mathrm{id}_{N(c)}"] \\
N(a) \otimes N(b \otimes c) \arrow[d, "\tilde{N}_{a,b\otimes c}"'] & N(a \otimes b) \otimes N(c) \arrow[d, "\tilde{N}_{a\otimes b,c}"] \\
N(a \otimes (b \otimes c)) \arrow[r, "N(\alpha_{a,b,c})"] & N((a \otimes b) \otimes c)
\end{tikzcd}$$

$$\begin{tikzcd}
\ell \otimes N(a) \arrow[d, "N_0 \otimes \mathrm{id}_{N(a)}"'] \arrow[rrd, "\lambda_a"] & & & & N(a) \otimes \ell \arrow[d, "\mathrm{id}_{N(a)} \otimes N_0"] \arrow[lld, "\rho_a"'] \\
N(k) \otimes N(a) \arrow[r, "\tilde{N}_{k,a}"'] & N(k \otimes a) \arrow[r, "N(\lambda_a)"'] & N(a) & N(a \otimes k) \arrow[l, "N(\rho_a)"] & N(a) \otimes N(k) \arrow[l, "\tilde{N}_{a,k}"]
\end{tikzcd}$$

If all the morphisms $\tilde{N}_{a,b}$ and N_0 are isomorphisms, we call N *strong.*

Example 4.2. The functor $N = \hom_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathbf{Set}$ is monoidal with natural transformations $\tilde{N}_{a,b} : \hom_{\mathcal{V}}(k, a) \times \hom_{\mathcal{V}}(k, b) \to \hom_{\mathcal{V}}(k, a \otimes b)$ mapping a pair of morphisms $(f : k \to a, g : k \to b)$ to the composition

$$k \cong k \otimes k \xrightarrow{f \otimes g} a \otimes b$$

and with $N_0 : \ell \to \hom_{\mathcal{V}}(k, k)$ picking out the identity morphism $k \to k$.

Example 4.3. An example very similar to the previous, is the monoidal functor $\mathcal{V} \to 2$ that takes on the value true on $v \in \mathcal{V}$ if there exists a morphism $k \to v$ and false otherwise.

It turns out that a monoidal functor $\mathcal{V} \to \mathcal{W}$ naturally provides a $\mathcal{V}$-category with the structure of a $\mathcal{W}$-category, and similarly for $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. Moreover, this association is 2-functorial (as in Example 3.5).

Theorem 4.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be cosmoi and let $N : \mathcal{V} \to \mathcal{W}$ be a monoidal functor. Then the following defines a 2-functor $N_* : \mathcal{V}-\mathbf{Cat} \to \mathcal{W}-\mathbf{Cat}$:*

- *For a $\mathcal{V}$-category X, the $\mathcal{W}$-category N_*X has the same objects as X and hom-objects given as $N_*X(x, y) = N(X(x, y))$. The composition morphisms of N_*X are given by the composition*

$$N_*X(y,z) \otimes N_*X(x,y) \xrightarrow{\tilde{N}_{X(y,z),X(x,y)}} N(X(y,z) \otimes X(x,y)) \xrightarrow{N(M_{x,y,z})} N_*X(x,z)$$

  and the identity morphisms by the composition

$$\ell \xrightarrow{N_0} N(k) \xrightarrow{N(j_x)} N_*X(x,x).$$

- *For a $\mathcal{V}$-functor $F : X \to Y$, the $\mathcal{W}$-functor $N_*F : N_*X \to N_*Y$ is given on objects by $N_*F(x) = F(x)$ and on hom-objects by $(N_*F)_{x,y} = N(F_{x,y})$.*
- *For a $\mathcal{V}$-natural transformation η between $\mathcal{V}$-functors $F, G : X \to Y$, the $\mathcal{W}$-natural transformation $N_*\eta$ between N_*F and N_*G has components given by the composition*

$$\ell \xrightarrow{N_0} N(k) \xrightarrow{N(\eta_x)} N_*Y(F(x), G(x)).$$

This is exactly Theorem 4.2.4 of [9], where a detailed proof is given.

Example 4.5. Observe that for $\hom_{\mathcal{V}} : \mathcal{V} \to \mathbf{Set}$, we obtain the underlying category functor $(-)_0 : \mathcal{V}-\mathbf{Cat} \to \mathbf{Cat}$, described in detail in [3, §1.3].

Example 4.6. Similarly, the functor $N : \mathcal{V} \to 2$ of Example 4.3 determines a 2-functor $N_* : \mathcal{V}-\mathbf{Cat} \to \mathbf{Poset}$ associating to every $\mathcal{V}$-category X the underlying poset $(X, \preceq)$ with $a \preceq b$ if and only if there exists a morphism $k \to X(a, b)$.

We will discuss some properties of change of enrichment. First of all, notice that a monoidal functor $N : \mathcal{V} \to \mathcal{W}$ between cosmoi gives rise to a $\mathcal{W}$-functor $N : N_*\mathcal{V} \to \mathcal{W}$, which on objects is the same as N and which on hom-objects is given as the adjunct $N_*\mathcal{V}(a, b) \to \mathcal{W}(N(a), N(b))$ of the composition

$$N(\mathcal{V}(a,b)) \otimes N(a) \xrightarrow{\tilde{N}_{\mathcal{V}(a,b),a}} N(\mathcal{V}(a,b) \otimes a) \xrightarrow{N(\varepsilon_{a,b})} N(b)$$

under the adjunction of $- \otimes N(a)$ and $\mathcal{W}(N(a), -)$.

Next, we will see that the association of a $\mathcal{V}$-functor F with a $\mathcal{W}$-functor N_*F is actually $\mathcal{W}$-natural.

Proposition 4.7. *Let $N : \mathcal{V} \to \mathcal{W}$ be a monoidal functor between cosmoi and X and Y two $\mathcal{V}$-categories. Then we have a $\mathcal{W}$-functor*

$$N_*- : N_*[X, Y] \longrightarrow [N_*X, N_*Y]$$

sending a $\mathcal{V}$-functor $F : X \to Y$ to the $\mathcal{W}$-functor $N_*F : N_*X \to N_*Y$. On hom-objects, it is given by applying the universal property of ends to the $\mathcal{W}$-extranatural transformation $N\omega : N_*[X, Y](F, G) \mathrel{\dot{\to}} N_*Y(F-, G-)$ obtained by applying N to the universal $\mathcal{V}$-extranatural transformation $\omega : [X, Y](F, G) \mathrel{\dot{\to}} Y(F-, G-)$.

Combining these two facts, we get a $\mathcal{W}$-functor

$$N \circ N_*- : N_*[X^{\mathrm{op}}, \mathcal{V}] \longrightarrow [N_*X^{\mathrm{op}}, \mathcal{W}]$$

from the category of presheaves on X to the category of presheaves on N_*X, given as the composition

$$N_*[X^{\mathrm{op}}, \mathcal{V}] \xrightarrow{N_*-} [N_*X^{\mathrm{op}}, N_*\mathcal{V}] \xrightarrow{N\circ-} [N_*X^{\mathrm{op}}, \mathcal{W}].$$

Note that on hom-objects, $(N \circ N_*-)_{F,G}$ is for $F, G \in N_*[X^{\mathrm{op}}, \mathcal{V}]$ the unique morphism making the following diagram commute for all $x \in X$:

$$\begin{tikzcd}
N_*[X^{\mathrm{op}}, \mathcal{V}](F, G) \arrow[r] \arrow[d, dashed, "(N\circ N_*-)_{F,G}"'] & N_*\mathcal{V}(F(x), G(x)) \arrow[d, "N_{F(x),G(x)}"] \\
{[N_*X^{\mathrm{op}}, \mathcal{W}](N \circ N_*F, N \circ N_*G)} \arrow[r] & \mathcal{W}(N(F(x)), N(G(x)))
\end{tikzcd}$$

Here, the horizontal morphisms are given by the universal extranatural transformations.

4.2 Lifting presheaves

To give context for the results that we will discuss next, we will briefly introduce the notion of *Kan lifts*. Given categories A, B and C and a functor $P : B \to C$, it is sometimes possible to lift a functor $A \to C$ to a functor $F : A \to B$ through P.

$$\begin{tikzcd}
 & B \arrow[d, "P"] \\
A \arrow[r, "F"] & C
\end{tikzcd}$$

Concretely, a *left Kan lift* of a functor $F : A \to C$ through P is a functor $\mathrm{Lift}_P(F) : A \to B$ equipped with a natural transformation ω from F to $P \circ \mathrm{Lift}_P(F)$, with the universal property that for any functor $G : A \to B$ with a natural transformation η from F to $P \circ G$, there exists a unique natural transformation ζ from F to G such that $\eta = (p \circ \zeta) \circ \omega$. If every functor $A \to C$ can be lifted in this way, then we obtain a functor $\mathrm{Lift}_P : [A, C] \to [A, B]$ that is left-adjoint to the functor

$$P \circ - : [A, B] \to [A, C]$$

given by post-composing with $P : B \to C$. It is worth remarking that for ordinary categories, Kan lifts rarely exist, as described in [7, Remark 2.1.5].

Now, we have defined the $\mathcal{W}$-functor

$$N \circ N_*- : N_*[X^{\mathrm{op}}, \mathcal{V}] \longrightarrow [N_*X^{\mathrm{op}}, \mathcal{W}]$$

given by post-composing with $N : N_*\mathcal{V} \to \mathcal{W}$. We will show that, under the condition that the $\mathcal{W}$-category $N_*\mathcal{V}$ is tensored over $\mathcal{W}$, presheaves in $[N_*X^{\mathrm{op}}, \mathcal{W}]$ can be lifted through this functor to presheaves in $N_*[X, \mathcal{V}]$, and that this lifting is $\mathcal{W}$-natural. This means that we will find a left $\mathcal{W}$-adjoint $\mathrm{Lift}_N : [N_*X^{\mathrm{op}}, \mathcal{W}] \to N_*[X^{\mathrm{op}}, \mathcal{V}]$ of $N \circ N_*-$.

To this end, we first have to describe how $N \circ N_*-$ interacts with representable presheaves and the Yoneda embedding.

Lemma 4.8. *Let X be a $\mathcal{V}$-category and $x \in X$. Then the representable presheaf $N_*X(-, x)$ of N_*X is given by applying $N \circ N_*-$ to the representable presheaf $X(-, x)$ of X.*

Proof. Let $x \in X$. By definition of hom-objects in N_*X, the presheaves $N \circ N_*(X(-, x))$ and $N_*X(-, x)$ are the same on objects. That they are the same on hom-objects is expressed by the commutativity of the following diagram:

$$\begin{tikzcd}
N_*X^{\mathrm{op}}(y, z) \arrow[r, "N(X(-,x)_{y,z})"] \arrow[dr, "N_*X(-,x)_{y,z}"'] & N_*\mathcal{V}(X(y, x), X(z, x)) \arrow[d, "N_{X(y,x),X(z,x)}"] \\
 & \mathcal{W}(N_*X(y, x), N_*X(z, x))
\end{tikzcd}$$

The commutativity of this diagram follows by applying the adjunction of $- \otimes N_*X(y, x)$ and $\mathcal{V}(N_*X(y, x), -)$ to the commutative diagram

$$\begin{tikzcd}
N_*X^{\mathrm{op}}(y, z) \otimes N_*X^{\mathrm{op}}(x, y) \arrow[r, "N(X(-,x)_{y,z})\otimes\mathrm{id}"] \arrow[d, "\tilde{N}_{X^{\mathrm{op}}(y,z),X^{\mathrm{op}}(x,y)}"'] & N_*\mathcal{V}(X(y, x), X(z, x)) \otimes N_*X(y, x) \arrow[d, "\tilde{N}_{\mathcal{V}(X(y,x),X(z,x)),X(y,x)}"] \\
N(X^{\mathrm{op}}(y, z) \otimes X^{\mathrm{op}}(x, y)) \arrow[r, "N(X(-,x)_{y,z}\otimes\mathrm{id})"] \arrow[dr, "N(M_{x,y,z})"'] & N(\mathcal{V}(X(y, x), X(z, x)) \otimes X(y, x)) \arrow[d, "N(\varepsilon_{X(y,x),X(z,x)})"] \\
 & N_*X(z, x)
\end{tikzcd}$$

Here, the square commutes by the naturality of $\tilde{N}$ and the triangle commutes because $X(-, x)_{y,z}$ is the adjunct of the composition morphism $M_{x,y,z}$ of X^{op} under the adjunction of $- \otimes X^{\mathrm{op}}(x, y)$ and $\mathcal{V}(X(y, x), -)$, whose counit is $\varepsilon_{X(y,x),X(z,x)}$.

Hence, $N \circ N_*(X(-, x)) = N_*X(-, x)$. □

Lemma 4.9. *For a $\mathcal{V}$-category X, the following diagram commutes:*

$$\begin{tikzcd}
 & N_*X \arrow[dl, "N_*\text{よ}_X"'] \arrow[dr, "\text{よ}_{N_*X}"] & \\
N_*[X^{\mathrm{op}}, \mathcal{V}] \arrow[rr, "N\circ N_*-"] & & [N_*X^{\mathrm{op}}, \mathcal{W}]
\end{tikzcd}$$

Proof. Lemma 4.8 tells us that $(N \circ N_*-) \circ N_*よ_X$ and $よ_{N_*X}$ are the same on objects. On hom-objects, $よ_{N_*X}$ is by definition given for $x, y \in X$ by the unique morphism making the following diagram commute for all $z \in X$:

$$\begin{tikzcd}
N_*X(x,y) \arrow[d, dashed, "(よ_{N_*X})_{x,y}"'] \arrow[dr, "N_*X(z,-)_{x,y}"] & \\
{[N_*X^{\mathrm{op}}, \mathcal{W}](N_*X(-,x), N_*X(-,y))} \arrow[r] & \mathcal{W}(N_*X(z,x), N_*X(z,y))
\end{tikzcd} \tag{4.1}$$

Here, the horizontal morphism is given by the universal $\mathcal{W}$-extranatural transformation. Now, applying Lemma 4.8 to the $\mathcal{V}$-category X^{op} we find that the morphisms $N_*X(z,-)_{x,y}$ are given as the composition

$$N_*X(x,y) \xrightarrow{N(X(z,-)_{x,y})} N_*\mathcal{V}(X(z,x), X(z,y)) \xrightarrow{N_{X(z,x),X(z,y)}} \mathcal{W}(N_*X(z,x), N_*X(z,y)).$$

But for all $z \in X$ the following diagram commutes:

$$\begin{tikzcd}
N_*X(x,y) \arrow[d, "(N_*よ_X)_{x,y}"'] \arrow[dr, "N(X(z,-)_{x,y})"] & \\
N_*[X^{\mathrm{op}}, \mathcal{V}](X(-,x), X(-,y)) \arrow[r] \arrow[d, "(N \circ N_*-)_{X(-,x),X(-,y)}"'] & N_*\mathcal{V}(X(z,x), X(z,y)) \arrow[d, "N_{X(z,x),X(z,y)}"] \\
{[N_*X^{\mathrm{op}}, \mathcal{W}](N_*X(-,x), N_*X(-,y))} \arrow[r] & \mathcal{W}(N_*X(z,x), N_*X(z,y))
\end{tikzcd}$$

Again, the horizontal arrows are given by the universal extranatural transformations. By the uniqueness of (4.1), we find that the following diagram commutes:

$$\begin{tikzcd}
N_*X(x,y) \arrow[r, "(N_*よ_X)_{x,y}"] \arrow[dr, "(よ_{N*X})_{x,y}"'] & N_*[X^{\mathrm{op}}, \mathcal{V}](X(-,x), X(-,y)) \arrow[d, "(N \circ N_*-)_{X(-,x),X(-,y)}"] \\
 & {[N_*X^{\mathrm{op}}](N_*X(-,x), N_*X(-,y))}
\end{tikzcd}$$

Hence, $(N \circ N_*-) \circ N_*よ_X$ and $よ_{N_*X}$ are also the same on hom-objects. ☐

Theorem 4.10. *Let $N : \mathcal{V} \to \mathcal{W}$ be a monoidal functor and suppose that $N_*\mathcal{V}$ is tensored over $\mathcal{W}$. Then for any $\mathcal{V}$-category X, the $\mathcal{W}$-functor*

$$N \circ N_*- : N_*[X^{\mathrm{op}}, \mathcal{V}] \to [N_*X^{\mathrm{op}}, \mathcal{W}]$$

has a left $\mathcal{W}$-adjoint

$$\begin{aligned} \mathrm{Lift}_N : [N_*X^{\mathrm{op}}, \mathcal{W}] &\to N_*[X^{\mathrm{op}}, \mathcal{V}] \\ F &\mapsto F \bullet N_*よ_X \end{aligned}$$

given by tensoring with the Yoneda embedding $よ_X : X \to [X^{\mathrm{op}}, \mathcal{V}]$.

Proof. Note that $N_*[X^{\mathrm{op}}, \mathcal{V}]$ is copowered over $\mathcal{W}$ with copowers computed pointwise, so that

$$F \bullet N_*\text{よ}_X = \int^{x \in N_* X^{\mathrm{op}}} F(x) \bullet N_*\text{よ}_X(x)$$

Thus, by definition of $\bullet$, ends and hom-objects of $\mathcal{W}$-functors we have the following $\mathcal{W}$-natural isomorphism indexed by $F \in [N_*X^{\mathrm{op}}, \mathcal{W}]$ and $G \in N_*[X^{\mathrm{op}}, \mathcal{V}]$:

$$\begin{aligned}
N_*[X^{\mathrm{op}}, \mathcal{V}](\mathrm{Lift}_N F, G) &= N_*[X^{\mathrm{op}}, \mathcal{V}](F \bullet N_*\text{よ}_X, G) \\
&= N_*[X^{\mathrm{op}}, \mathcal{V}] \left(\int^{N_* X^{\mathrm{op}}} F(x) \bullet N_*\text{よ}_X(x), G \right) \\
&= \int_{N_* X^{\mathrm{op}}} N_*[X^{\mathrm{op}}, \mathcal{V}] \big(F(x) \bullet N_*\text{よ}_X(x), G \big) \\
&\cong \int_{N_* X^{\mathrm{op}}} \mathcal{W} \big(F(x), N_*[X^{\mathrm{op}}, \mathcal{V}](N_*\text{よ}_X(x), G) \big) \\
&= [N_*X^{\mathrm{op}}, \mathcal{W}] \big(F, N_*[X^{\mathrm{op}}, \mathcal{V}](N_*\text{よ}_X(-), G) \big) .
\end{aligned}$$

The Yoneda lemma gives a $\mathcal{V}$-natural isomorphism

$$G \cong [X^{\mathrm{op}}, \mathcal{V}](\text{よ}_X(-), G) = \text{よ}_{[X^{\mathrm{op}}, \mathcal{V}]}(G) \circ \text{よ}_X$$

indexed by $G \in [X^{\mathrm{op}}, \mathcal{V}]$. The 2-functoriality of N_* described by Theorem 4.4 and applying Lemma 4.9 therefore yields the following $\mathcal{W}$-natural isomorphism indexed by $G \in N_*[X^{\mathrm{op}}, \mathcal{V}]$:

$$N \circ N_* G \cong N \circ N_*(\text{よ}_{[X^{\mathrm{op}}, \mathcal{V}]}(G)) \circ N_*\text{よ}_X = \text{よ}_{N_*[X^{\mathrm{op}}, \mathcal{V}]}(G) \circ N_*\text{よ}_X = N_*[X^{\mathrm{op}}, \mathcal{V}](N_*\text{よ}_X(-), G)$$

We conclude that we have a $\mathcal{W}$-natural isomorphism

$$\begin{aligned}
N_*[X^{\mathrm{op}}, \mathcal{V}](\mathrm{Lift}_N F, G) &\cong [N_*X^{\mathrm{op}}, \mathcal{W}] \big(F, N_*[X^{\mathrm{op}}, \mathcal{V}](N_*\text{よ}_X(-), G) \big) \\
&\cong [N_*X^{\mathrm{op}}, \mathcal{W}](F, N \circ N_* G)
\end{aligned}$$

and therefore a $\mathcal{W}$-adjunction $\mathrm{Lift}_N \dashv N \circ N_* -$. ☐

4.3 Topology of enriched categories

We will now apply the adjunction $\mathrm{Lift}_N \dashv N \circ N_* -$ to the monoidal functor $N : \mathcal{V} \to \mathbf{2}$ described in Example 4.3 to define a topology on enriched categories. To introduce this, we will first discuss the case where $\mathcal{V}$ is a value quantale.

Example 4.11. Let $\mathcal{V}$ be a value quantale and X a small $\mathcal{V}$-category. Note that $N_*\mathcal{V}$ is the opposite of the underlying poset of $\mathcal{V}$ and that it is tensored over $\mathbf{2}$ with $\mathrm{true} \bullet v = v$ and $\mathrm{false} \bullet v = \infty$ for all $v \in \mathcal{V}$.

Assume that the Flagg metric on X is positive definite, that is, $X(x, y) = 0$ if and only if $x = y$. Since the existence of a morphism from $k = 0$ to $X(x, y)$ is equivalent to $X(x, y) = 0$ and therefore to $x = y$, we know that $N_*X(x, y) = \mathrm{true}$ if and only if $x = y$. Thus, the set of presheaves $[N_*X^{\mathrm{op}}, \mathbf{2}]$ can naturally be identified with the powerset $\mathcal{P}(X)$ of X by sending a presheaf $F : N_*X^{\mathrm{op}} \to \mathbf{2}$ to the set of $x \in X$ for which $F(x) = \mathrm{true}$. Now, the map of pre-ordered sets

$$N \circ N_* - : N_*[X^{\mathrm{op}}, \mathcal{V}] \longrightarrow [N_*X^{\mathrm{op}}, \mathbf{2}] = \mathcal{P}(X)$$

sends a presheaf $G : X^{\mathrm{op}} \to \mathcal{V}$ to the set $\{x \in X : G(x) = 0\} \in \mathcal{P}(X)$.

We will compute the tensor product $S \bullet N_*\text{よ}_X$ for a subset $S \in \mathcal{P}(X) = [N_*X^{\mathrm{op}}, \mathbf{2}]$. Note that for $x \in X$, we have $N_*\text{よ}_X(x) = X(-, x)$ and therefore, using our description of the tensor product of $N_*\mathcal{V}$, the map $S(x) \bullet N_*\text{よ}_X(x) : N_*X^{\mathrm{op}} \to N_*\mathcal{V}$ is given by

$$(S(x) \bullet N_*\text{よ}_X(x))(y) = \begin{cases} X(y,x), & \text{if } x \in S \\ \infty, & \text{if } x \notin S \end{cases}.$$

Since $N_*\mathcal{V}$ is the opposite ordered set to the underlying ordered set of $\mathcal{V}$, taking the coend is equivalent to taking the meet in $\mathcal{V}$, so that

$$\mathrm{Lift}_N(S) = S \bullet N_*\text{よ}_X = \int^{x \in N_*X^{\mathrm{op}}} S(x) \bullet N_*\text{よ}_X(x) = \bigwedge \{X(-,x) : x \in S\}$$

is the presheaf in $[X^{\mathrm{op}}, \mathcal{V}]$ sending $y \in X$ to the point-set distance from y to S.

Thus, post-composing this with $N \circ N_*-$ we obtain the map

$$\begin{aligned} (N \circ N_*-) \circ \mathrm{Lift}_N : \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ S &\longmapsto \Big\{ y \in X : \bigwedge \{X(y,x) : x \in S\} = 0 \Big\}. \end{aligned}$$

This sends a subset $S \subseteq X$ to its closure $\overline{S}$ in the topology of the Flagg metric space X. We conclude that $(N \circ N_*-) \circ \mathrm{Lift}_N$ is exactly the closure operator of X.

We return to the general case where $\mathcal{V}$ is a cosmos. Then $N_*\mathcal{V}$ is again tensored over $\mathbf{2}$ with true $\bullet\, v = v$ and false $\bullet\, v = i$ the initial object of $\mathcal{V}$ for all $v \in \mathcal{V}$. We will use $(N \circ N_*-) \circ \mathrm{Lift}_N$ to construct the closure operator of a topology on $\mathcal{V}$-enriched categories. For this, the Kuratowski closure axioms ([10]) are useful.

Proposition 4.12 (Kuratowski closure axioms)**.**
Let X be a set and suppose $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a map of sets such that

(i) $\mathrm{cl}(\emptyset) = \emptyset$,

(ii) $A \subseteq \mathrm{cl}(A)$ *for all* $A \in \mathcal{P}(X)$,

(iii) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ *for all* $A \in \mathcal{P}(X)$,

(iv) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ *for all* $A, B \in \mathcal{P}(X)$.

Then there exists a unique topology on X whose closure operator is cl, *that is,* $\overline{S} = \mathrm{cl}(S)$ *for all* $S \in \mathcal{P}(X)$.

A map $\mathcal{P}(X) \to \mathcal{P}(X)$ satisfying these conditions is called a *Kuratowski closure operator.* Note that in particular, Kuratowski closure operators are order-preserving. If the reader is familiar with the terminology, they will be able to recognize that a Kuratowski closure operator is equivalently a monad on $\mathcal{P}(X)$ that commutes with finite coproducts. This is because coproducts in $\mathcal{P}(X)$ are unions and $\emptyset$ is the initial object of $\mathcal{P}(X)$, which is the empty coproduct. The following is then a direct consequence of the well-known category-theoretical facts that adjunctions give rise to monads and that left adjoints commute with colimits (see [11, Thm. 0.3.3, 0.3.6] for elementary proofs in the case of posets).

Proposition 4.13. *Let X be a set, P a pre-ordered set and $G : P \to \mathcal{P}(X)$ an order preserving map. Suppose that G commutes with finite coproducts and has a* $\mathbf{2}$*-enriched left adjoint $F : \mathcal{P}(X) \to P$. Then $G \circ F : \mathcal{P}(X) \to \mathcal{P}(X)$ is a Kuratowski closure operator.*

In the general case, we can no longer identify $[N_*X^{\mathrm{op}}, \mathbf{2}]$ with $\mathcal{P}(X)$, but only with the set $\downarrow X$ of downward-closed subsets of N_*X, as in Example 3.16. The inclusion map $I : \downarrow X \hookrightarrow \mathcal{P}(X)$ of ordered sets now has a **2**-enriched left adjoint

$$\begin{aligned} \downarrow : \mathcal{P}(X) &\longrightarrow \downarrow X \\ S &\longmapsto \{y \in X : \exists x \in S : y \leq x\} \end{aligned}$$

It is this **2**-enriched adjunction that lets us translate the operator $(N \circ N_*-) \circ \mathrm{Lift}_N$ on $[N_*X^{\mathrm{op}}, \mathcal{W}] = \downarrow X$ to an operator on $\mathcal{P}(X)$.

Theorem 4.14. *Let $\mathcal{V} \to \mathbf{2}$ be a monoidal functor that commutes with finite coproducts and let X be a small $\mathcal{V}$-category. Then there exists a unique topology on X whose closure operator is given as the composite*

$$\mathcal{P}(X) \xrightarrow{\downarrow} \downarrow X = [N_*X^{\mathrm{op}}, \mathbf{2}] \xrightarrow{\mathrm{Lift}_N} N_*[X^{\mathrm{op}}, \mathcal{V}] \xrightarrow{N\circ N_*-} [N_*X^{\mathrm{op}}, \mathbf{2}] = \downarrow X \xrightarrow{I} \mathcal{P}(X).$$

Proof. Since $\downarrow$ is left-adjoint to I and Lift_N is left-adjoint to $N \circ N_*-$, the composite $\mathrm{Lift}_N \circ \downarrow$ is left-adjoint to $I \circ (N \circ N_*-)$. By Propositions 4.12 and 4.13 it is therefore enough to show that $I \circ (N \circ N_*-)$ commutes with finite coproducts. Clearly, the inclusion $I : \downarrow X \hookrightarrow \mathcal{P}(X)$ commutes with finite coproducts, and all that is left is to show that $N \circ N_*- : N_*[X^{\mathrm{op}}, \mathcal{V}] \to [N_*X^{\mathrm{op}}, \mathbf{2}]$ does as well.

Let $F_1, \dots, F_n \in N_*[X^{\mathrm{op}}, \mathcal{V}]$ be presheaves on X. Then we can define a presheaf $F : X^{\mathrm{op}} \to \mathcal{V}$ which on objects is given as $F(x) = F_1(x) \cup \cdots \cup F_n(x)$ and on hom-objects is given as the adjunct of the universal morphism

$$X^{\mathrm{op}}(x, y) \otimes (F_1(x) \cup \cdots \cup F_n(x)) \longrightarrow F_1(y) \cup \cdots \cup F_n(y)$$

induced by the morphisms

$$X^{\mathrm{op}}(x, y) \otimes F_i(x) \xrightarrow{(F_i)_{x,y}\otimes \mathrm{id}} \mathcal{V}(F_i(x), F_i(y)) \otimes F_i(x) \xrightarrow{\varepsilon} F_i(y) \longrightarrow F_1(y) \cup \cdots \cup F_n(y).$$

Given $\mathcal{V}$-natural transformations η_i from F_i to $G : X^{\mathrm{op}} \to \mathcal{V}$ for every i, we have a family of morphisms $\eta_x : F_1(x) \cup \cdots \cup F_n(x) \to G(x)$ indexed by $x \in X$, given by the universal property of coproducts in $\mathcal{V}$. This defines a $\mathcal{V}$-natural transformation η, since pre-composing the η_x with the morphisms $F_i(x) \to F_1(x) \cup \cdots \cup F_n(x)$, which define a $\mathcal{V}$-natural transformation, yields η_i.

We find that F is the coproduct of $F_1, \dots, F_n$ in $N_*[X^{\mathrm{op}}, \mathcal{V}]$. Since N preserves coproducts, we have

$$\begin{aligned} N \circ N_*F(x) &= N(F_1(x) \cup \cdots \cup F_n(x)) \\ &= N(F_1(x)) \cup \cdots \cup N(F_n(x)) \\ &= N \circ N_*F_1(x) \cup \cdots \cup N \circ N_*F_n(x) \end{aligned}$$

for all $x \in X$ and this order-preserving map $N_*X^{\mathrm{op}} \to \mathbf{2}$ is clearly exactly the coproduct of the $N \circ N_*F_i$. □

Example 4.15. The reasoning of Example 4.11 shows that when $\mathcal{V}$ is a value quantale, Theorem 4.14 gives the topology of Flagg metric spaces, even when the metric is not positive-definite. For $\mathcal{V} = \overline{\mathbb{R}}^+$, the adjoint pair $\mathrm{Lift}_N \circ \downarrow \dashv I \circ (N \circ N_*-)$ is also given in [12, Prop. 6.1].

Non-example 4.16. There does not exist a monoidal functor $\mathbf{Ab} \to \mathbf{2}$ that commutes with finite coproducts. To see this, note that for a monoidal functor $N : \mathbf{Ab} \to \mathbf{2}$ we have a morphism $N_0 : \text{true} \to N(0)$ and therefore $N(0) = \text{true}$. But then N does not preserve finite coproducts, since the initial object $0 \in \mathbf{Ab}$ does not map to the initial object $\text{false} \in \mathbf{2}$. Thus, Theorem 4.14 does not define a topology on additive categories.

Example 4.17. For $\mathcal{V} = \mathbf{2}$, the monoidal functor $N : \mathbf{2} \to \mathbf{2}$ is the identity functor, so that $N \circ N_*- : [X^{\mathrm{op}}, \mathbf{2}] \to [X^{\mathrm{op}}, \mathbf{2}]$ is also the identity functor. Since Lift_N is left-adjoint to $N \circ N_*-$, it is also the identity functor and the closure operator of Theorem 4.14 is simply the composite

$$\mathcal{P}(X) \xrightarrow{\downarrow} [X^{\mathrm{op}}, \mathbf{2}] \xrightarrow{I} \mathcal{P}(X).$$

The closed sets of the induced topology are therefore exactly the downward-closed subsets of X. This topology on a pre-ordered set is known as the *Alexandroff topology.*

Example 4.18. Let $\mathcal{V} = \mathbf{Set}$ and consider a small $\mathbf{Set}$-enriched category X. Then given a downward-closed subset $S \in [N_* X^{\mathrm{op}}, \mathbf{2}]$, the presheaf $\mathrm{Lift}_N(S)$ is the coproduct $\bigsqcup_{x \in S} X(-, x)$ in the category of presheaves on X. Coproducts of presheaves in categories are computed point-wise and therefore $(N \circ N_*-)(\mathrm{Lift}_N(S))$ is the downward-closed subset of all $y \in X$ for which there exists a morphism

$$k \to \bigsqcup_{x \in S} X(y, x)$$

from the one-element set $k = \{*\}$. But this morphism exists if and only if $\bigsqcup_{x \in S} X(y, x)$ is non-empty, that is, if and only if there exists some $x \in S$ with a morphism $y \to x$ in X. Since S is downward-closed, this is equivalent to $y \in S$, so that $(N \circ N_*-) \circ \mathrm{Lift}_N$ is again the identity functor. The closure operator of Theorem 4.14 is therefore again given as

$$\mathcal{P}(X) \xrightarrow{\downarrow} [X^{\mathrm{op}}, \mathbf{2}] \xrightarrow{I} \mathcal{P}(X).$$

Thus, a subset $S \subseteq X$ is closed in the topology of Theorem 4.14 if and only if it is downward-closed, that is, if and only if for every morphism $y \to x$ with $x \in S$ we have $y \in S$.

References

[1] S. Eilenberg and G.M. Kelly. *Closed categories.* Proceedings of the Conference on Categorical Algebra, 1966.

[2] F.W. Lawvere. Metric spaces, generalized logic, and closed categories. *Theory and applications of categories*, 10(1):1, 2002.

[3] G.M. Kelly. *Basic Concepts of Enriched Category Theory.* Number 10. Reprints in Theory and Applications of Categories, 2005 [1982].

[4] R. Street. Elementary cosmoi I. In *Category Seminar*, Lecture Notes in Mathematics, pages 134–180. 1974.

[5] G.J. Seal D. Hofmann and W. Tholen. *Monoidal Topology: A Categorical Approach to Order, Metric, and Topology*, volume 153 of *Encyclopedia of mathematics and its applications.* Cambridge University Press, Cambridge, 2014.

[6] B. Flagg and R. Kopperman. Continuity spaces: Reconciling domains and metric spaces. *Theoretical computer science*, 177(1):111–138, 1997.

[7] F. Loregian. *(Co)end Calculus.* London Mathematical Society Lecture Note Series. Cambridge University Press, 2021.

[8] E. Riehl. *Categorical Homotopy Theory.* New Mathematical Monographs. Cambridge University Press, 2014.

[9] G.S.H. Cruttwell. *Normed spaces and the change of base for enriched categories.* PhD thesis, Dalhousie University, 2008.

[10] K. Kuratowski. Sur l'opération Ā de l'analysis situs. *Fundamenta mathematicae*, 3:182–199, 1922.

[11] K. Keimel, D.S. Scott, K.H. Hofmann, J.D. Lawson, G. Gierz, and M. Mislove. *Compendium of Continuous Lattices.* Springer, 2012.

[12] M.M. Bonsangue, F. van Breugel, and J.J.M.M. Rutten. Generalized metric spaces: Completion, topology, and powerdomains via the yoneda embedding. *Theoretical Computer Science*, 193(1):1–51, 1998.