Ordering and pricing strategies for single and two-period inventory problems

Gent, K. van

K. van Gent

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Thesis supervisor: **dr. O. Kanavetas**

Leiden University
Mathematical Institute

Contents

Introduction

Inventory management and choosing the right price are key elements in optimizing the profit of your business or personal project. The biggest obstacle when determining the quantities one wants to have in stock and for which price to sell these is the unpredictability of the demand. When the exact demand is known, everyone can maximize their profit by ordering the exact demand for a given price. However, in general, this is not the case and there are only previous demands to rely on. In this thesis, we will discuss two models, that describe approximations of real-world problems, concerning inventory management.

The first model is a single-period model for perishable products with a price-dependent demand D with known probability density function ϕ_D. After a brief review of existing literature, which is done in Chapter 1, the single-period model will be analyzed in Chapter 2. This analysis consists of proving the existence of a unique optimal ordering quantity q_1^* and optimal price p^* and of finding, if possible, an explicit expression for them. Because of the complexity of the model, it is not possible to obtain an expression for general demand distribution, so multiple demand distributions are considered in Section 2.2. To determine q^* and p^* we will use the following roadmap:

1. Determine the general profit function, which has parameters q, p and D.

2. Determine the expected profit function, since the demand D is a random variable.

3. Assume p to be given and maximize the expected profit function for q^*. This will give an expression for q^*, which depends only on p.

4. Use the found expression for q^* to determine the expected profit function, given the optimal ordering quantity and maximize for p to find p^*.

The second model we will discuss is a two-period model for stable products, i.e., non-perishable products. As with the first model, this model considers a price-dependent demand D with a known probability density function ϕ_D. Here we add the assumption that unsatisfied demand from the first period can be met in the second period. Another important assumption is that the price p is the same in both periods. Again, specific probability density functions are assumed to solve this problem, which is analyzed in Chapter 4. The same roadmap as above can be used to determine the optimal quantities q_1^*, q_2^* for the first and second period respectively, and the optimal price p^*, with the only difference the parameters q_1 and q_2 instead of q.

Chapter 1

Single-period problem

Nowadays, running a company or store comes with many difficulties and challenges. One of these is inventory management, which requires one to make sure that there always are enough products to meet the demand and at the same time that there are not too many products in the inventory, as this will lead to more costs. Of course, the type of product is majorly important as it comes to determining how many products you want in your inventory. We distinguish two types of products: stable products and perishable products. Stable products are products that one can always sell, no matter how long they are stored. Perishable products, on the other hand, can only be sold for some time, after which they will become unsellable.

The first examples of perishable products that will probably come to mind are food, flowers, newspapers, or seasonal products such as winter clothing. But also such things as reservations or tickets are perishable products since they lose their value after the event has ended. The fact that these are perishable products explains the recurring sales in these industries.

Although sales is an effective method to clear your inventory and still make some profit, better inventory management will prevent this and increase your profit. But how can inventory management be improved? For the optimal inventory policy, both the probability of underordering, which causes revenue loss and overordering, which leads to unsellable leftovers, need to be considered.

One well-known model that describes this is the so-called Newsboy or single-period Problem. This model considers the inventory policy of perishable products for a single period. Newspapers fit this model very well, hence the name.

Assumptions of the model

1. It involves a single perishable product.
2. It involves a single time period.
3. There is no initial inventory.
4. The only decision to be made is the value of q, the number of units to order.
5. The *demand* for buying products is a random variable D with known probability distribution ϕ_D, and is independent of the price.
6. The goal is to minimize the costs, where the cost components are:

$$
\begin{aligned}
c &= \text{ the cost for ordering a single product} \\
s &= \text{ the salvage value per product leftover at the end of the period} \\
p &= \text{ the cost of unsatisfied demand}
\end{aligned}
$$

The single-period problem is, as stated earlier, well-known, and it comes as no surprise that this is already solved. As Hillier showed [1], the cost function, which we want to minimize, is

$$C(D,q) = c \cdot y + p \cdot \max\{0, D-q\} - s \cdot \max\{0, q-D\}. \tag{1.1}$$

We will not show the entire proof of Hillier here, but there is a part of it that is convenient to mention. In reality, the demand is a discrete random variable. However, since the probability distribution of this demand is often hard to find and will most probably lead to expressions, which are slightly more difficult to solve analytically, Hillier approximated the discrete demand D with a continuous random variable. We will follow his reasoning and, thus, unless otherwise stated, continuous demand is assumed in the rest of this thesis.

The maximization of the cost function results in an optimal ordering quantity q^*, which is the value which satisfies

$$\Phi_D(q^*) = \frac{p-c}{p-s}, \tag{1.2}$$

where $\Phi_D(q^*)$ is the probability density function of our demand D.

Chapter 2

Single-period model with price dependent demand

The Newsboy Problem gives us an optimal ordering quantity for a single period, assuming that the demand is independent of the price. In the next model, we will take away this assumption. Furthermore, instead of minimizing the costs, we will maximize our profit and, more importantly, we will do this for both the ordering quantity and the price. To be more precise, we will first determine the ordering quantity that maximizes the profit, given the price, and then we will maximize the profit function for the price.

Assumptions of the model

1. It involves a single perishable product.
2. It involves a single time period.
3. There is no initial inventory.
4. The decisions to be made are the value of q, the number of units to order, and p, the received price per sold unit.
5. The *demand* for buying products is a random variable D with known probability distribution ϕ_D, and is **dependent** on the price.
6. The goal is to maximize the profit.

2.1 The Optimal Quantity

The choice of the value of q depends for a great part on the probability distribution of our demand. For instance, choosing a value of q higher than the maximum possible demand would be far from optimal, as there certainly will be unsold items at the end of the period. As stated before, a balance of the risk of overordering and underordering is needed to maximize the profit. Aside from the demand, multiple factors play a role in this model.

We have the following variables:

p = the price per product

q = the quantity ordered at the beginning of the period

c = the costs to produce a single product

s = the *salvage value* per product, i.e. the price received per leftover

D = the demand per period

Note that the *salvage value* can also be negative, implying that costs are attached to getting rid of left-overs. This could be the case when products need to be destroyed, transported or when there exist taxes for these products. The *salvage value* is assessed at the end of the period since that is the only time when it is certain that they will not be sold anymore. However, in our model, we assume positive values of the *salvage value*.

The total profit of the model is equal to the revenue minus the costs. The revenue here consists of the revenue of the sold products and the unsold products.

The amount that is sold at the end of the period is given by

$$\min\{D,q\} = \begin{cases} D & \text{if } D < q \\ q & \text{if } D \geq q. \end{cases}$$

Similarly, the amount that is unsold at the end of the period is

$$\max\{0,q-D\} = \begin{cases} 0 & \text{if } D > q \\ q-D & \text{if } D \leq q. \end{cases}$$

So if we put these together we find that the profit function is given by

$$P_p(D,q) = p \cdot \min\{D,q\} + s \cdot \max\{0,q-D\} - c \cdot q. \tag{2.1}$$

The profit function can be rewritten as:

$$\begin{aligned} P_p(D,q) &= p \cdot \min\{D,q\} + s \cdot \max\{0,q-D\} - c \cdot q \\ &= p \cdot \big(D - \max\{0,D-q\}\big) + s \cdot \max\{0,q-D\} - c \cdot q \\ &= p \cdot D - C_p(D,q). \end{aligned}$$

Since $p \cdot D$ is independent of q, maximizing $P_p(D,q)$ for q is the same as maximizing $C_p(D,q)$, which is then equivalent with minimizing Eq. (1.1). With Hillier's result (Eq. (1.2)), we find that the optimal ordering quantity is the value which satisfies

$$\Phi_D(q^*) = \frac{p-c}{p-s}. \tag{2.2}$$

Of course, we can derive this solution ourselves by maximizing our expected profit function. As we discussed before, we assume the demand D to be a continuous random variable. To rewrite our profit function we use the fact that

$$c \cdot q = c \cdot \big(q - D\big) + c \cdot D = c \cdot \big(\max\{0,q-D\} - \max\{0,D-q\}\big) + c \cdot D.$$

This result can be used to show that

$$\begin{aligned} P_p(D,q) &= p \cdot \big(D - \max\{0,D-q\}\big) + s \cdot \max\{0,q-D\} - \big(c \cdot \big(\max\{0,q-D\} - \max\{0,D-q\}\big) + c \cdot D\big) \\ &= (p-c) \cdot D - (p-c) \cdot \max\{0,D-q\} - (c-s) \cdot \max\{0,q-D\}. \end{aligned}$$

Now that the profit function is in this form, the expected profit can be derived.

To derive this expected profit, we will apply the linearity of expectation, i.e., the expectation of a sum of random variables is equal to the sum of their individual expectations of these random variables, regardless of whether they are independent. The expected profit $P_p(q)$ is expressed as

$$\begin{aligned} P_p(q) &= \mathbb{E}[P_p(D,q)] \\ &= \mathbb{E}[(p-c)\cdot D - \big((p-c)\cdot(D-q)^+ + (c-s)\cdot(q-D)^+\big)] \\ &= (p-c)\cdot\mathbb{E}[D \mid p] - (p-c)\cdot\int_q^\infty (\xi - q)\cdot\phi_D(\xi)\, d\xi - (c-s)\int_0^q (q-\xi)\cdot\phi_D(\xi)\, d\xi \end{aligned} \tag{2.3}$$

where $\phi_D(x)$ denotes the density function of our demand D. Now that we have determined our expected profit function, maximizing it is just a matter of taking the derivative and finding the roots of this derivative. Since the goal is to maximize for q, the derivative should also be with respect to q which results in

$$\begin{aligned} \frac{dP_p(q)}{dq} &= (p-c)\cdot\int_q^\infty \phi_D(\xi)\, d\xi - (c-s)\cdot\int_0^q \phi_D(\xi)\, d\xi \\ &= (p-c)\cdot(1-\Phi_D(q)) - (c-s)\cdot\Phi_D(q). \end{aligned}$$

Solving for the roots of this function gives us that

$$\begin{aligned} \Phi_D(q^*)\big(-(p-c)-(c-s)\big) &= -(p-c) \\ \Phi_D(q^*) &= \frac{p-c}{p-s}. \end{aligned} \tag{2.4}$$

We also have that

$$\frac{d^2P_p(q)}{dq^2} = -(p-s)\cdot\phi_D(q) \le 0$$

for all q. Thus our found ordering quantity q^* indeed maximizes $P_p(q)$ and this is the same result as we saw before (Eq. (2.2)).

2.2 The Optimal Price

We have now determined the optimal ordering quantity, given the price. However, this price may not be optimal for the profit. Hence, the next step is to maximize the profit function for the price, given the optimal quantity. In this way, the profit will be maximized for both the ordering quantity and the price. To do so, we will consider three different probability distributions for the demand. The distributions we will consider are

1. Uniform Distribution
2. Exponential Distribution
3. Normal Distribution

Choosing specific distributions allows us to obtain an explicit formula for the optimal ordering quantity, which we will need to analytically prove the optimal price.

In real-world problems, the demand depends on the price of the product. The higher the price of a product, the lower the demand. To implement this in the model, the assumption is made that the *mean value* or *expected value* of the demand distribution is linearly decreasing with p, i.e.

$$\mathbb{E}[D|p] = \mu(p) = \left(\alpha - \beta p\right)^+,$$

with $\alpha, \beta > 0$. Note that this mean value must be non-negative since the demand can not be negative. This results in an important condition for our price p; it must be less than or equal to α/β. Also, since the goal is to maximize the profit, the price should be higher than or equal to our costs c. Hence, the condition for p expands to

$$c \leq p \leq \frac{\alpha}{\beta}.$$

So, on this interval $\mu(p) = \alpha - \beta p$.

2.2.1 Uniform Distribution

In this section, the demand is uniformly distributed with mean value $\mu(p)$ as defined above. Note that the boundaries of the interval, the minimum and maximum values of the distribution, are also dependent on p. Thus

$$D \sim U\left[a(p), b(p)\right].$$

Using the known values for the mean value and variance of the uniform distribution we can determine the values of these boundaries. The variance $\sigma^2(p)$ is by assumption constant.

$$\begin{aligned} \mu(p) = \mathbb{E}\left[D|p\right] = \alpha - \beta p &= \frac{b(p) + a(p)}{2} \\ a(p) + b(p) &= 2(\alpha - \beta p), \end{aligned}$$

$$\begin{aligned} \sigma^2(p) &= \frac{(b(p) - a(p))^2}{12} \\ (b(p) - a(p))^2 &= 12\sigma^2 \\ b(p) - a(p) &= 2\sigma\sqrt{3} \end{aligned}$$

Combining these two outcomes gives us a system of equations:

$$\begin{cases} a(p) + b(p) & = 2(\alpha - \beta p) \\ a(p) - b(p) & = 2\sigma\sqrt{3}. \end{cases}$$

Solving this system gives us that

$$a(p) = \alpha - \beta p - \sigma\sqrt{3} \tag{2.5}$$

$$b(p) = \alpha - \beta p + \sigma\sqrt{3}. \tag{2.6}$$

You can see that the minimum and maximum values of the uniform distribution behave as expected. They lie evenly far from the mean value and are, as stated before, dependent on p. So,

$$D \sim U\left[(\alpha - \beta p) - \sigma\sqrt{3}\,,\ (\alpha - \beta p) + \sigma\sqrt{3}\right].$$

The known probability density function of a uniform distribution is $1/w$, where w denotes the *width* of the distribution, i.e., the maximum value minus the minimum value for all the values in the interval and zero otherwise. This results in the density function

$$\phi_D(x) = \begin{cases} \frac{1}{2\sigma\sqrt{3}} & \text{for } a(p) \le x \le b(p) \\ 0 & \text{else,} \end{cases} \tag{2.7}$$

with the accessory cumulative density function

$$\Phi_D(x) = \begin{cases} \frac{x-a(p)}{2\sigma\sqrt{3}} & \text{for } a(p) \le x \le b(p) \\ 0 & \text{else.} \end{cases} \tag{2.8}$$

From Eq. (2.4) we can obtain our optimal ordering quantity q^*:

$$\Phi_D(q^*) = \frac{q^* - a(p)}{2\sigma\sqrt{3}} = \frac{p-c}{p-s}$$

$$q^* = a(p) + 2\sigma\sqrt{3} \cdot \frac{p-c}{p-s}. \tag{2.9}$$

Solving for the price

Now that we have found the optimal ordering quantity for the Uniform Distribution, we can continue with the optimal price. Now that the demand has a known probability density function, the expected profit can be written as

$$\begin{aligned} P_p(q) &= \mathbb{E}[P_p(D,q)] \\ &= C_u \cdot \mathbb{E}[D \mid p] - C_u \cdot \int_q^\infty (\xi - q) \cdot \phi_D(\xi)\, d\xi - C_o \int_0^q (q - \xi) \cdot \phi_D(\xi)\, d\xi \\ &= (p-c)\cdot(\alpha - \beta p) - (p-c) \cdot \int_q^{b(p)} (\xi - q) \cdot \frac{1}{2\sigma\sqrt{3}}\, d\xi - (c-s) \int_{a(p)}^q (q-\xi) \cdot \frac{1}{2\sigma\sqrt{3}}\, d\xi \\ &= (p-c)\cdot(\alpha - \beta p) - \frac{p-c}{2\sigma\sqrt{3}} \cdot \left[\frac{1}{2}\left(\xi - q\right)^2\right]_q^{b(p)} - \frac{p-s}{2\sigma\sqrt{3}} \cdot \left[-\frac{1}{2}\left(q-\xi\right)^2\right]_{a(p)}^q \\ &= (p-c)\cdot(\alpha - \beta p) - \frac{p-c}{4\sigma\sqrt{3}} \cdot (b(p) - q)^2 - \frac{c-s}{4\sigma\sqrt{3}} \cdot (q - a(p))^2. \end{aligned} \tag{2.10}$$

We can now substitute our optimal ordering quantity q^* (Eq. (2.9)) for q in Eq. (2.10) to find the function we want to maximize for p, since this q^* maximizes the profit function for any given p.

$$\begin{aligned}
P_p(q^*) :&= \mathbb{E}[P_p(D,q^*)] \\
&= (p-c)\cdot(\alpha-\beta p) - \frac{p-c}{4\sigma\sqrt{3}}\cdot\left(b(p)-a(p)-2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\right)^2 - \frac{c-s}{4\sigma\sqrt{3}}\cdot\left(a(p)+2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}-a(p)\right)^2 \\
&= (p-c)\cdot(\alpha-\beta p) - \frac{p-c}{4\sigma\sqrt{3}}\cdot\left(2\sigma\sqrt{3}-2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\right)^2 - \frac{c-s}{4\sigma\sqrt{3}}\cdot\left(2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\right)^2 \\
&= (p-c)\cdot(\alpha-\beta p) - (p-c)\cdot\sigma\sqrt{3}\cdot\left(1-\frac{p-c}{p-s}\right)^2 - (c-s)\cdot\sigma\sqrt{3}\cdot\left(\frac{p-c}{p-s}\right)^2 \qquad (2.11)
\end{aligned}$$

Eq. (2.11) shows the expected profit function $P_p(q^*)$, only depending on the known parameters and the variable p. Now we will prove that there exists a unique value $p^* \in [c, \alpha/\beta]$ that maximizes $P_p(q^*)$ in this interval. To do so, we will use the first derivative for proving that there exists a root and the second derivative for proving that there exists a maximum and that it is unique.

$$\begin{aligned}
\frac{dP_p(q^*)}{dp} &= (\alpha-\beta p) - \beta(p-c) - \sigma\sqrt{3}\cdot\left(1-\frac{p-c}{p-s}\right)^2 - (p-c)\cdot\sigma\sqrt{3}\left[\left(1-\frac{p-c}{p-s}\right)^2\right]' \\
&\quad - (c-s)\cdot\sigma\sqrt{3}\cdot\left[\left(\frac{p-c}{p-s}\right)^2\right]' \\
&= (\alpha-\beta p) - \beta(p-c) - \sigma\sqrt{3}\cdot\left(1-\frac{p-c}{p-s}\right)^2 - (p-c)\cdot\sigma\sqrt{3}\cdot -2\cdot\left(1-\frac{p-c}{p-s}\right)\cdot\frac{c-s}{(p-s)^2} \\
&\quad - (c-s)\cdot\sigma\sqrt{3}\cdot 2\cdot\frac{p-c}{p-s}\cdot\frac{c-s}{(p-s)^2} \\
&= (\alpha-\beta p) - \beta(p-c) - \sigma\sqrt{3}\cdot\left(\frac{c-s}{p-s}\right)^2 + 2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\cdot\left(\frac{c-s}{p-s}\right)^2 - 2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\cdot\left(\frac{c-s}{p-s}\right)^2 \\
&= (\alpha-\beta p) - \beta(p-c) - \sigma\sqrt{3}\cdot\left(\frac{c-s}{p-s}\right)^2 \qquad (2.12)
\end{aligned}$$

The values at the boundaries of the desired interval, so at $p=c$ and $p=\alpha/\beta$ can tell us a lot about the behaviour of $P'_p(q^*)$. At $p=c$ the first derivative is equal to

$$\begin{aligned}
P'_c(q^*) &= (\alpha-\beta c) - \sigma\sqrt{3} \\
&= a(c),
\end{aligned}$$

and at $p=\alpha/\beta$ it is

$$\begin{aligned}
P'_{\alpha/\beta}(q^*) &= (\alpha-\alpha) - (\alpha-\beta c) - \sigma\sqrt{3}\left(\frac{c-s}{\frac{\alpha}{\beta}-s}\right)^2 \\
&< -(\alpha-\beta c+\sigma\sqrt{3}) \\
&= -b(c).
\end{aligned}$$

To be certain that we have a non-negative derivative at $p=c$, we need the condition $a(c)\ge 0$, i.e., $c \le \alpha/\beta - \sigma\sqrt{3}$. Intuitively, the interval for the price should be sufficiently large. Note that the value $b(c)$ is by definition greater than zero, since $c \le \alpha/\beta$. Thus we have a negative first derivative at the right boundary.

Since $P'_p(q^*)$ is continuous for $p > s$, it also is continuous in the desired interval. Since $P'_c(q^*) \cdot P'_{\alpha/\beta}(q^*) \leq 0$, the Theorem of Bolzano tells us that there exists at least one $p \in [c, \alpha/\beta]$ such that $P'_p(q^*) = 0$.

We have now proven the existence of a root in the interval $[c, \alpha/\beta]$. The second derivative will give more information about whether this is a maximum or a minimum. The second derivative is

$$P''_p(q^*) := \frac{d^2 P_p(q^*)}{dp^2} = -2\beta + 2\sigma\sqrt{3} \cdot \frac{(c-s)^2}{(p-s)^3}. \tag{2.13}$$

If we determine the zeros of the second derivative, we only have one real zero p_z, which is

$$\begin{aligned} \sigma\sqrt{3} \cdot \frac{(c-s)^2}{(p_z-s)^3} &= \beta \\ (p_z - s)^3 &= \frac{(c-s)^2 \cdot \sigma\sqrt{3}}{\beta} \\ p_z &= \sqrt[3]{\frac{(c-s)^2 \cdot \sigma\sqrt{3}}{\beta}} + s. \end{aligned} \tag{2.14}$$

So this p_z is the only real root of our second derivative. We also have that

$$\frac{dP''_p(q^*)}{dp} = -3 \cdot 2\sigma\sqrt{3} \cdot \frac{(c-s)^2}{(p-s)^4}, \tag{2.15}$$

which is strictly negative for all $p \in \mathbb{R}$. So the second derivative is strictly decreasing on the whole of $\mathbb{R}$. So, for all $p > p_z$, we have a negative second derivative, i.e., the expected profit function is concave. More importantly, the function is convex for $p \leq p_z$. There are two options: either $p_z \leq c$ or $p_z > c$. We will now determine in which situations we encounter the different cases.

$$\begin{aligned} P''_c(q^*) &= 0 \\ \frac{1}{c-s} \cdot 2\sigma\sqrt{3} &= 2\beta \\ c - s &= \frac{\sigma\sqrt{3}}{\beta}. \end{aligned}$$

In the case that $c - s < \frac{\sigma\sqrt{3}}{\beta}$, then

$$p_z = \sqrt[3]{\frac{(c-s)^2 \cdot \sigma\sqrt{3}}{\beta}} + s > \sqrt[3]{(c-s)^3} + s = c.$$

Concluding this section, there are two possible cases: p_z as in Eq. (2.14) is either less than or equal to the cost c, in which case the expected profit function $P_p(q^*)$ is concave on the whole interval, or p_z is greater than the cost, which results in a convex and concave part.

In the last part of this section we will compare these two cases.

- $p_z \leq c$, which is equivalent to $c - s \geq \frac{\sigma\sqrt{3}}{\beta}$
  In this case, the expected profit function $P_p(q^*)$ is, as stated before, concave and since there exists a root of $P'_p(q^*)$, this is also the global maximum.

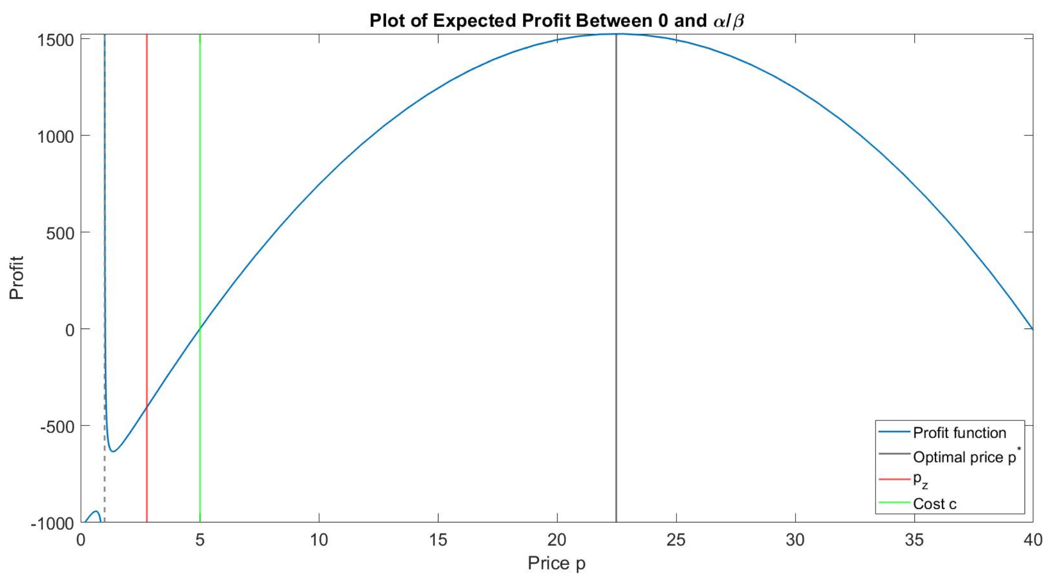
- $p_z > c$, which is equivalent to $c - s < \frac{\sigma\sqrt{3}}{\beta}$
  In this case, c lies in the convex part of the function, and $P'_c(q^*) > 0$. Combining these two facts, ensures that if there exists an extreme value, which is true, this value will be a global maximum.

Thus in both cases there exists a unique value p^* which maximizes the expected profit $P_P(q^*)$ in the interval $[c, \alpha/\beta]$.

Numerically solving for p^*

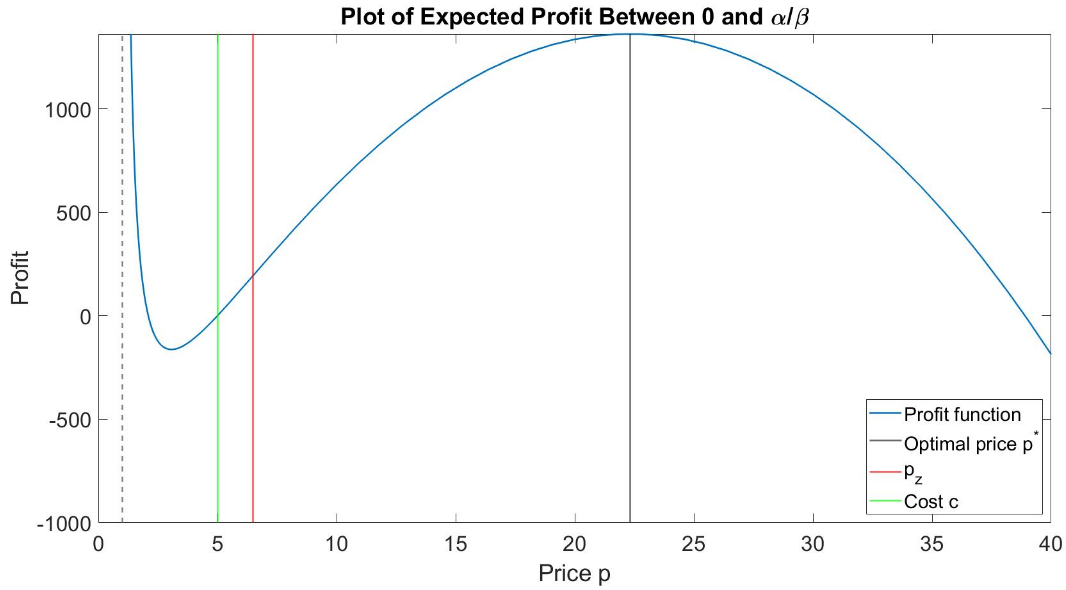
Now that we have proven the existence and uniqueness of an optimal price p^*, we still need to determine the value of this p^*. p^* is a root of Eq. (2.12) and, although this equation is solvable for specific values of the parameters, we can not solve it analytically. What we do know is that, when we numerically solve the roots of Eq. (2.12), p^* is the maximum (real) root. This is because if there exists a root $x \geq p$ of $P'_p(q^*)$, this would contradict the concaveness of the function for $p > p_z$.

We will consider two examples: one of each of the two possible cases described above. In the first example we have $\mu(p) = 200 - 5p$ with cost $c = 5$, salvage value $s = 1$ and variance $\sigma^2(p) = 1$. This gives the graph



and results in an optimal ordering quantity $q^* \approx 88,62$ and optimal price $p^* \approx 22,49$. In the plot, p^* is the maximum root of Eq. (2.12), which indeed is the maximum value here. Furthermore, it is easy to see that we are in the first case, namely $p_z \leq c$.

The second example has the exact same parameters as the first example, but a significant higher variance of $\sigma^2(p) = 900$. This gives the graph



and results in an optimal ordering quantity $q^* \approx 120,86$ and optimal price $p^* \approx 22,32$.

Sensitivity Analysis

In this next section, we will change the parameters one by one to see the sensitivity of these parameters. This will give us a good insight into which parameters have the greatest influence on the model.

TABLE 2.1: Changing α

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
100	5	5	1	1	38,13	12,48	276,73
150	5	5	1	1	63,44	17,49	776,00
200	5	5	1	1	88,62	22,49	1525,61
250	5	5	1	1	113,73	27,50	2525,37
300	5	5	1	1	138,81	32,50	3775,20

TABLE 2.2: Changing β

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
200	2	5	1	1	96,47	52,50	4506,11
200	5	5	1	1	88,62	22,49	1525,61
200	10	5	1	1	75,63	12,49	557,98
200	20	5	1	1	49,92	7,48	122,34
200	30	5	1	1	24,45	5,81	19,65

The results in Table 2.1 and 2.2 are as expected; increasing α leads to a higher optimal ordering quantity and optimal price, since the mean value $\mu(p)$ of the demand increases as α increases. In the same way, decreasing β leads to the same results, since $\mu(p)$ decreases as β decreases.

Table 2.3: Changing c

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
200	5	2	1	1	96,56	21,00	1803,35
200	5	5	1	1	88,62	22,49	1525,61
200	5	10	1	1	75,55	24,98	1115,26
200	5	20	1	1	49,83	29,93	488,68
200	5	30	1	1	24,40	34,87	117,69

Table 2.4: Changing s

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
200	5	5	0	1	88,50	22,49	1524,51
200	5	5	1	1	88,62	22,49	1525,61
200	5	5	2	1	88,74	22,50	1526,81
200	5	5	3	1	88,88	22,50	1528,14
200	5	5	4	1	89,05	22,50	1529,61

Table 2.5: Changing σ

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
200	5	5	1	1	88,62	22,49	1525,61
200	5	5	1	2	89,73	22,49	1519,97
200	5	5	1	5	93,08	22,47	1503,06
200	5	5	1	10	98,66	22,44	1474,88
200	5	5	1	20	109,78	22,38	1418,54

Furthermore, looking at the tables, one can see that changing α, β, or c has a significantly greater effect on the optimal ordering quantity q^*, as well as on the optimal price p^*, than changing s or σ. Therefore it also has a significantly greater effect on the expected profit, which the tables show.

The most interesting table, most likely, is Table 2.4, as it might not show what one would initially think. When we increase the salvage value s, i.e., we get more value for every unsold product, the risk of overordering decreases, since the loss is significantly less. However, since we still want to minimize this risk, increasing the salvage value only leads to a slight increase in the expected profit.

Lastly, we note that the variance σ has a completely different influence than the other parameters. In all the other cases, either both the optimal ordering quantity and the optimal price significantly change, or they both roughly stay the same. In Table 2.5 however, we can see that, as the optimal price stays basically the same, the optimal ordering quantity increases quite a bit more. Also, as the variance increases, the expected profit decreases, which intuitively is caused by the fact with a greater variance comes a greater range for the demand. This increases both the risk of overordering and the risk of underordering, which in its turn, causes the expected profit to decrease.

2.2.2 Exponential Distribution

In this section, the demand is exponentially distributed with mean value $\mu(p) = \alpha - \beta p$. The mean value of an exponential distribution with parameter λ is $1/\lambda$, thus

$$D \sim Exp\left[\frac{1}{\alpha - \beta p}\right].$$

Opposite to the uniform distribution, the exponential distribution has a probability density function on the interval $[0, \infty]$ for all mean values $\mu(p)$. This density function has a relatively high probability for all demands less than the mean value and is explicitly described in the following way:

$$\phi_D(x) = \begin{cases} \frac{1}{\alpha - \beta p} \cdot e^{-\frac{x}{\alpha - \beta p}} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{2.16}$$

with the accessory cumulative density function

$$\Phi_D(x) = \begin{cases} 1 - e^{-\frac{x}{\alpha - \beta p}} & x \geq 0 \\ 0 & x < 0. \end{cases} \tag{2.17}$$

From Eq. (2.4) we once again can obtain the optimal ordering quantity q^*.

$$\begin{aligned} \Phi_D(q^*) = 1 - e^{-\frac{q^*}{\alpha - \beta p}} &= \frac{p - c}{p - s} \\ e^{-\frac{q^*}{\alpha - \beta p}} &= 1 - \frac{p - c}{p - s} \\ -\frac{q^*}{\alpha - \beta p} &= \ln\left(\frac{c - s}{p - s}\right) \\ q^* &= -(\alpha - \beta p) \cdot \ln\left(\frac{c - s}{p - s}\right). \end{aligned} \tag{2.18}$$

Solving for the price

With the optimal ordering quantity q^* as in Eq. (2.18), we can determine the optimal price. Just as with the Uniform Distribution, a known probability density function leads to a simplified expected profit. We also still have the same expected profit function as before, so we find that:

$$\begin{aligned} P_p(q) &= \mathbb{E}[P_p(D, q)] \\ &= (p - c) \cdot \mathbb{E}[D \mid p] - (p - c) \cdot \int_q^\infty (\xi - q) \cdot \phi_D(\xi)\, d\xi - (c - s) \cdot \int_0^q (q - \xi) \cdot \phi_D(\xi)\, d\xi \\ &= (p - c) \cdot (\alpha - \beta p) - (p - c) \cdot \int_q^\infty (\xi - q) \cdot \frac{1}{\alpha - \beta p} e^{-\frac{\xi}{\alpha - \beta p}}\, d\xi - (c - s) \cdot \int_0^q (q - \xi) \cdot \frac{1}{\alpha - \beta p} e^{-\frac{\xi}{\alpha - \beta p}}\, d\xi. \end{aligned}$$

In order to further simplify the function, the two integrals in the above function need to be determined. Note that

$$I_1 = \int (\xi - q) \cdot \frac{1}{\alpha - \beta p} e^{-\frac{\xi}{\alpha - \beta p}}\, d\xi = -\int (q - \xi) \cdot \frac{1}{\alpha - \beta p} e^{-\frac{\xi}{\alpha - \beta p}}\, d\xi.$$

So it is sufficient to solve I_1. To do so, we will use integration by parts, where

$$f = \xi - q \implies f' = 1, \text{ and}$$
$$g' = \frac{1}{\alpha - \beta p} e^{-\frac{\xi}{\alpha-\beta p}} \implies g = -e^{-\frac{\xi}{\alpha-\beta p}}.$$

So we get that

$$\begin{aligned} I_1 &= -(\xi - q) \cdot e^{-\frac{\xi}{\alpha-\beta p}} + \int e^{-\frac{\xi}{\alpha-\beta p}} \, d\xi \\ &= -(\xi - q) \cdot e^{-\frac{\xi}{\alpha-\beta p}} - (\alpha - \beta p) \cdot e^{-\frac{\xi}{\alpha-\beta p}} \\ &= (q - \xi - (\alpha - \beta p)) \cdot e^{-\frac{\xi}{\alpha-\beta p}}. \end{aligned}$$

Now that this integral has been solved, the expected profit function is given by

$$\begin{aligned} P_p(q) &= (p-c)\cdot(\alpha-\beta p) - (p-c)\cdot\int_q^\infty (\xi - q)\cdot\frac{1}{\alpha-\beta p} e^{-\frac{\xi}{\alpha-\beta p}} \, d\xi - (c-s)\cdot\int_0^q (q-\xi)\cdot\frac{1}{\alpha-\beta p} e^{-\frac{\xi}{\alpha-\beta p}} \, d\xi \\ &= (p-c)\cdot(\alpha-\beta p) - (p-c)\cdot\left[(q-\xi-(\alpha-\beta p))\cdot e^{-\frac{\xi}{\alpha-\beta p}}\right]_q^\infty + (c-s)\cdot\left[(q-\xi-(\alpha-\beta p))\cdot e^{-\frac{\xi}{\alpha-\beta p}}\right]_0^q \\ &= (p-c)\cdot(\alpha-\beta p) - (p-c)\cdot(\alpha-\beta p)\cdot e^{-\frac{q}{\alpha-\beta p}} + (c-s)\cdot\left((-(\alpha-\beta p))\cdot e^{-\frac{q}{\alpha-\beta p}} - (q-(\alpha-\beta))\right) \\ &= (p-s)(\alpha-\beta p) - (p-s)\cdot(\alpha-\beta p)\cdot e^{-\frac{q}{\alpha-\beta p}} - (c-s)\cdot q. \end{aligned} \tag{2.19}$$

Although some computations were needed, the expected profit function $P_P(q)$ is simplified and looks a lot more manageable. The next step is, in a similar way as with the Uniform Distribution, to use the value of q^* (Eq. (2.18)) in $P_p(q)$:

$$\begin{aligned} P_p(q^*) &= (p-s)(\alpha-\beta p) - (p-s)\cdot(\alpha-\beta p)\cdot e^{-\frac{-(\alpha-\beta p)\cdot\ln\left(\frac{c-s}{p-s}\right)}{\alpha-\beta p}} - (c-s)\cdot\left(-(\alpha-\beta p)\cdot\ln\left(\frac{c-s}{p-s}\right)\right) \\ &= (p-s)(\alpha-\beta p) - (p-s)\cdot(\alpha-\beta p)\cdot e^{\ln\left(\frac{c-s}{p-s}\right)} + (c-s)\cdot(\alpha-\beta p)\cdot\ln\left(\frac{c-s}{p-s}\right) \\ &= (p-s)(\alpha-\beta p) - (p-s)\cdot(\alpha-\beta p)\cdot\frac{c-s}{p-s} + (c-s)\cdot(\alpha-\beta p)\cdot\ln\left(\frac{c-s}{p-s}\right) \\ &= (p-c)(\alpha-\beta p) + (c-s)\cdot(\alpha-\beta p)\cdot\ln\left(\frac{c-s}{p-s}\right). \end{aligned} \tag{2.20}$$

The expected profit function (Eq. (2.20)) depends only on the variable price p. Again, we want to find the value p^* that maximizes $P_p(q^*)$.

$$\begin{aligned} P'_P(q^*) = \frac{dP_p(q^*)}{dp} &= (\alpha-\beta p) - \beta(p-c) - \beta(c-s)\cdot\ln\left(\frac{c-s}{p-s}\right) + (c-s)(\alpha-\beta p)\cdot\left(\ln\left(\frac{c-s}{p-s}\right)\right)' \\ &= (\alpha-\beta p) - \beta(p-c) - \beta(c-s)\cdot\ln\left(\frac{c-s}{p-s}\right) + (c-s)(\alpha-\beta p)\cdot\left(-\frac{1}{p-s}\right) \\ &= (\alpha-\beta p) - \beta(p-c) - \beta(c-s)\cdot\ln\left(\frac{c-s}{p-s}\right) - (\alpha-\beta p)\cdot\frac{c-s}{p-s}. \end{aligned} \tag{2.21}$$

In particular, at $p = c$ we have that

$$\begin{aligned}
P'_c(q^*) &= (\alpha - \beta c) - \beta(c - c) - \beta(c - s) \cdot \ln\left(\frac{c-s}{c-s}\right) - (\alpha - \beta c) \cdot \frac{c-s}{c-s} \\
&= (\alpha - \beta c) - \beta(c - s) \cdot \ln 1 - (\alpha - \beta c) \\
&= 0.
\end{aligned}$$

Also, at $p = \alpha/\beta$, we have that

$$\begin{aligned}
P'_{\alpha/\beta}(q^*) &= \left(\alpha - \beta \cdot \frac{\alpha}{\beta}\right) - \beta\left(\frac{\alpha}{\beta} - c\right) - \beta(c-s) \cdot \ln\left(\frac{c-s}{\frac{\alpha}{\beta} - s}\right) - \left(\alpha - \beta \cdot \frac{\alpha}{\beta}\right) \cdot \frac{c-s}{\frac{\alpha}{\beta} - s} \\
&= -\beta\left(\frac{\alpha}{\beta} - c\right) - \beta(c-s) \cdot \ln\left(\frac{c-s}{\frac{\alpha}{\beta} - s}\right) \\
&= \beta(c-s) - \beta\left(\frac{\alpha}{\beta} - s\right) - \beta(c-s) \cdot \ln\left(\frac{c-s}{\frac{\alpha}{\beta} - s}\right) \\
&= \beta(c-s) + \beta(c-s) \cdot -\left(\frac{\frac{\alpha}{\beta} - s}{c-s}\right) - \beta(c-s) \cdot \ln\left(\frac{c-s}{\frac{\alpha}{\beta} - s}\right) \\
&\overset{*}{<} \beta(c-s) + \beta(c-s) \cdot \left(-\ln\left(\frac{\frac{\alpha}{\beta} - s}{c-s}\right) - 1\right) - \beta(c-s) \cdot \ln\left(\frac{c-s}{\frac{\alpha}{\beta} - s}\right) \\
&= \beta(c-s) - \beta(c-s) + \beta(c-s) \cdot \ln\left(\frac{c-s}{\frac{\alpha}{\beta} - s}\right) - \beta(c-s) \cdot \ln\left(\frac{c-s}{\frac{\alpha}{\beta} - s}\right) \\
&= 0,
\end{aligned}$$

where at $*$, we use that

$$\ln\{x\} < x - 1 \quad \text{for } x \neq 1,$$

which implies that $-\ln\{x\} - 1 > -x$ for $x \neq 1$. Since the first derivative at $p = c$ is zero, we have already proven the existence of a root in the interval $[c, \alpha/\beta]$. Furthermore, we have that

$$\begin{aligned}
P''_p(q^*) = \frac{d^2 P_p(q^*)}{dp^2} &= -\beta - \beta + \frac{\beta(c-s)}{p-s} + \frac{\beta(c-s)}{p-s} + \frac{(c-s)(\alpha-\beta)}{(p-s)^2} \\
&= -2\beta + 2\beta \cdot \frac{c-s}{p-s} + (\alpha - \beta p) \cdot \frac{c-s}{(p-s)^2}.
\end{aligned} \tag{2.22}$$

Note that Eq. (2.22) is monotonely decreasing for $p > s$. Recall that in the previous section, with the Uniform Distributed demand, there were two different cases: $P''_c(q^*) < 0$ or $P''_c(q^*) \geq 0$. However, here we have that

$$\begin{aligned}
P''_c(q^*) &= -2\beta + 2\beta \cdot \frac{c-s}{c-s} + (\alpha - \beta c) \cdot \frac{c-s}{(c-s)^2} \\
&= (\alpha - \beta c) \cdot \frac{1}{c-s},
\end{aligned}$$

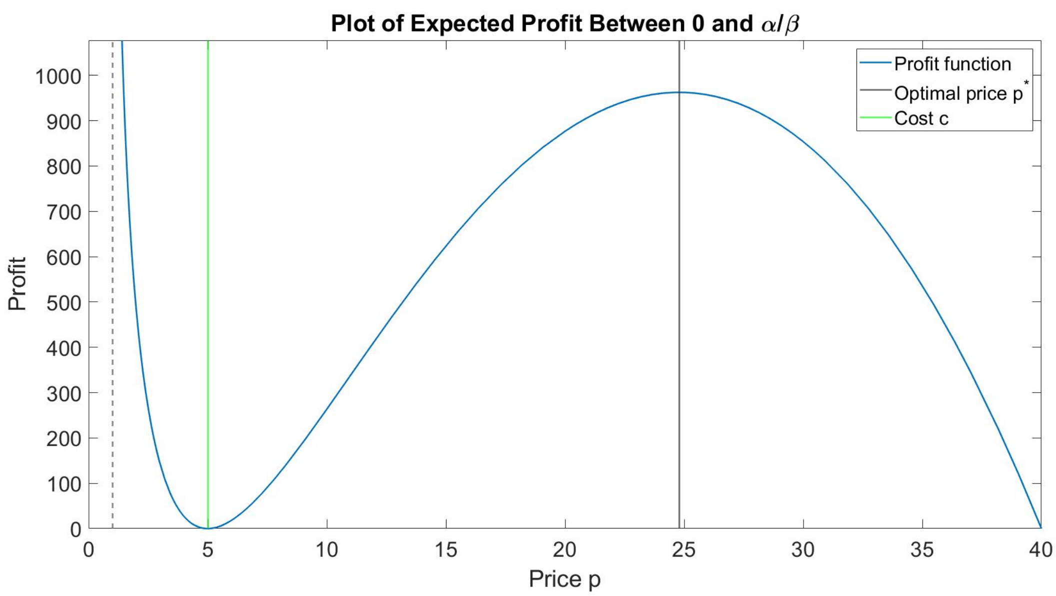
which is strictly positive for all choices of α, β, c and s. So, opposite to the Uniform Distribution, we know that, in all cases, c lies in the convex part. Thus the first derivative $P'_{c+\epsilon}(q^*)$ for $\epsilon \in \mathbb{R}_{>0}$ sufficiently small is strictly positive.

As $P'_p(q^*)$ is continuous for $p > s$, it also is continuous in the desired interval. Since $P'_{c+\epsilon}(q^*) \cdot P'_{\alpha/\beta}(q^*) < 0$, the Theorem of Bolzano tells us that there exists at least one $p^* \in (c, \alpha/\beta]$ such that $P'_p(q^*) = 0$. Since c lies in the convex part and $P'_c(q^*) = 0$, the expected profit function $P_p(q^*)$ has a minimum at c and thus $P_p(q^*)$ must have a global maximum at this root p^*. So in all cases we have a unique value p^* that maximizes the expected profit $P_p(q^*)$ in the interval $[c, \alpha/\beta]$.

Numerically solving for p^*

Now that we have proven the existence and uniqueness of an optimal price p^*, we still need to determine the value of this p^*. p^* is a root of Eq. (2.21) and, although this equation is solvable for specific values of the parameters, we can not solve it analytically. What we do know is that, when we numerically solve the roots of Eq. (2.21), p^* is the maximum (real) root. We know that there is a root at $p = c$, which lies in the convex part. For $p > p_z$, where p_z denotes the root of the second derivative with $p_z > c$ (the exact value of p_z is irrelevant), the expected profit function is concave, so there exists only one root (also for $p > \alpha/\beta$), which is our optimal price p^*.

We will consider an example with the same parameters as in the first case of the Uniform Distribution, thus $\mu(p) = 200 - 5p$ with cost $c = 5$, salvage value $s = 1$ and variance $\sigma^2(p) = 1$. This gives the graph



and results in an optimal ordering quantity $q^* \approx 66,23$ and optimal price $p^* \approx 27,90$.

Sensitivity Analysis

Just as with the uniformly distributed demand, we will perform a sensitivity analysis on the variables. The tables are listed on the next page and show similar results for α, β, and c. All three of them cause a significant change in the optimal ordering quantity, optimal price, and expected profit in the way one would expect. With the salvage value s however, it is different than with the uniformly distributed demand.

TABLE 2.6: Changing α

α	β	c	s	q^*	p^*	$P_{p^*}(q^*)$
100	5	5	1	35,74	13,89	128,60
150	5	5	1	80,88	19,40	439,68
200	5	5	1	135,62	24,79	962,65
250	5	5	1	197,44	30,10	1707,64
300	5	5	1	264,89	35,37	2680,54

TABLE 2.7: Changing β

α	β	c	s	q^*	p^*	$P_{p^*}(q^*)$
200	2	5	1	230,07	56,16	3565,44
200	5	5	1	135,62	24,79	962,65
200	10	5	1	71,47	13,89	257,19
200	20	5	1	21,59	8,13	30,59
200	30	5	1	4,20	6,08	2,18

TABLE 2.8: Changing c

α	β	c	s	q^*	p^*	$P_{p^*}(q^*)$
200	5	2	1	272,90	22,10	1526,04
200	5	5	1	135,62	24,79	962,65
200	5	10	1	66,23	27,90	486,78
200	5	20	1	18,79	32,62	108,57
200	5	30	1	3,53	36,52	11,12

TABLE 2.9: Changing s

α	β	c	s	q^*	p^*	$P_{p^*}(q^*)$
200	5	5	0	120,55	25,03	896,46
200	5	5	1	135,62	24,79	962,65
200	5	5	2	156,24	24,49	1042,78
200	5	5	3	187,29	24,10	1143,83
200	5	5	4	244,35	23,57	1281,21

As Table 2.9 shows, the salvage value has a greater influence than with the uniformly distributed demand. The reason for this is that with the uniform distribution, all demands (in the interval of the uniform distribution) are equiprobable, i.e., they all have the same chance of occurring. With the exponential distribution, however, the probability of a demand x lower than the mean value $\mu(p)$ is significantly higher than a demand x higher than $\mu(p)$. Since low demands have a higher probability, the risk of overordering is way higher than with the uniformly distributed demands and this causes that increasing the salvage value leads to a much higher expected profit. In other words, since we are more likely to overorder, we are more likely to lose $c-s$ per leftover, which of course decreases as s increases.

2.2.3 Normal Distribution

In this section, the demand is normally distributed with mean value $\mu(p) = \alpha - \beta p$ and with constant variance $\sigma^2(p) = \sigma^2$, i.e.,

$$D \sim \mathcal{N}\left[\alpha - \beta p, \sigma^2\right].$$

The Normal Distribution is known to have one particular property, namely the fact that all numbers that lie evenly far from the mean have equal probability. So a demand x higher than $\mu(p)$ is just as likely as a demand x lower than $\mu(p)$. The probability density function of the demand is given by

$$\phi_D(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\alpha+\beta p}{\sigma}\right)^2}$$

for all $x \in \mathbb{R}$. The accessory cumulative distribution function is

$$\phi_D(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt,$$

which is quite complicated compared to the cumulative distribution functions of the Uniform and Exponential Distribution. It is so complicated that there does not exist a closed-form expression for the inverse cumulative distribution function. For this reason, we will denote this inverse by Φ_{inv} in the following derivation of the optimal ordering quantity:

$$\begin{aligned}
\Phi_D(q^*) = \Phi\left(\frac{q^* - \alpha + \beta p}{\sigma}\right) &= \frac{p-c}{p-s} \\
\frac{q^* - \alpha + \beta p}{\sigma} &= \Phi_{inv}\left(\frac{p-c}{p-s}\right) \\
q^* &= (\alpha - \beta p) + \sigma \cdot \Phi_{inv}\left(\frac{p-c}{p-s}\right).
\end{aligned} \tag{2.23}$$

Note that since we have no explicit expression for our optimal quantity, it is impossible to solve this problem analytically. We can, however, find an expression for the expected profit function $P_p(q)$, which will allow us to solve it numerically.

$$\begin{aligned}
P_p(q) &= \mathbb{E}[P_p(D,q)] \\
&= (p-c)\cdot\mathbb{E}[D \mid p] - (p-c)\cdot\int_q^{\infty} (\xi - q)\cdot\phi_D(\xi)\, d\xi - (c-s)\cdot\int_{-\infty}^{q} (q-\xi)\cdot\phi_D(\xi)\, d\xi \\
&= (p-c)\cdot(\alpha-\beta p) - (p-c)\cdot\int_q^{\infty} \xi\cdot\phi_D(\xi)\, d\xi + (p-c)\cdot q\left(1-\Phi(q^*)\right) - (c-s)\cdot q\cdot\Phi(q^*) \\
&\quad + (c-s)\cdot\int_{-\infty}^{q} \xi\cdot\phi_D(\xi)\, d\xi \\
&= (p-c)\cdot(\alpha-\beta p) - (p-c)\cdot\int_q^{\infty} \xi\cdot\phi_D(\xi)\, d\xi + (p-c)\cdot q\left(\frac{c-s}{p-s}\right) - (c-s)\cdot q\cdot\frac{p-c}{p-s} \\
&\quad + (c-s)\cdot\int_{-\infty}^{q} \xi\cdot\phi_D(\xi)\, d\xi \\
&= (p-c)\cdot(\alpha-\beta p) - (p-c)\cdot\int_q^{\infty} \xi\cdot\phi_D(\xi)\, d\xi + (c-s)\cdot\int_{-\infty}^{q} \xi\cdot\phi_D(\xi)\, d\xi
\end{aligned}$$

$$= (p-c)\left(\int_{-\infty}^{\infty} \xi \cdot \phi_D(\xi)\, d\xi - \int_{q}^{\infty} \xi \cdot \phi_D(\xi)\, d\xi\right) + (c-s)\cdot \int_{-\infty}^{q} \xi \cdot \phi_D(\xi)\, d\xi$$

$$= (p-s)\cdot \int_{-\infty}^{q} \xi \cdot \phi_D(\xi)\, d\xi. \tag{2.24}$$

To further simplify $P_p(q)$ we need to determine the integral

$$I_1 = \int_{-\infty}^{q} \xi \cdot \phi_D(\xi)\, d\xi = \int_{-\infty}^{q} \xi \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\xi-\alpha+\beta p}{\sigma}\right)^2}\, d\xi.$$

To do so, we will split the integral into two integrals that are easier to solve.

$$\begin{aligned}
I_1 &= \sigma \cdot \int_{-\infty}^{q} \left(\frac{\xi-\alpha+\beta p}{\sigma}\right)\cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\xi-\alpha+\beta p}{\sigma}\right)^2}\, d\xi + (\alpha-\beta p)\cdot \int_{-\infty}^{q} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\xi-\alpha+\beta p}{\sigma}\right)^2}\, d\xi \\
&= \sigma \cdot \int_{-\infty}^{q} \left(\frac{\xi-\alpha+\beta p}{\sigma}\right)\cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\xi-\alpha+\beta p}{\sigma}\right)^2}\, d\xi + (\alpha-\beta p)\cdot \Phi(q^*) \\
&= \frac{\sigma}{\sqrt{2\pi}} \cdot \int_{-\infty}^{q} \left(\frac{\xi-\alpha+\beta p}{\sigma}\right)\cdot e^{-\frac{1}{2}\left(\frac{\xi-\alpha+\beta p}{\sigma}\right)^2}\, d\xi + (\alpha-\beta p)\cdot \frac{p-c}{p-s} \\
&= \frac{\sigma}{\sqrt{2\pi}} \cdot \left[`-e^{-\frac{1}{2}\left(\frac{\xi-\alpha+\beta p}{\sigma}\right)^2}\right]_{\infty}^{q} + (\alpha-\beta p)\cdot \frac{p-c}{p-s} \\
&= -\frac{\sigma}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{q-\alpha+\beta p}{\sigma}\right)^2} + (\alpha-\beta p)\cdot \frac{p-c}{p-s}.
\end{aligned} \tag{2.25}$$

If we replace I_1 with Eq. (2.25), we find that

$$\begin{aligned}
P_p(q) &= (p-s)\cdot I_1 \\
&= (p-c)\cdot(\alpha-\beta p) - (p-s)\cdot \frac{\sigma}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{q-\alpha+\beta p}{\sigma}\right)^2}.
\end{aligned} \tag{2.26}$$

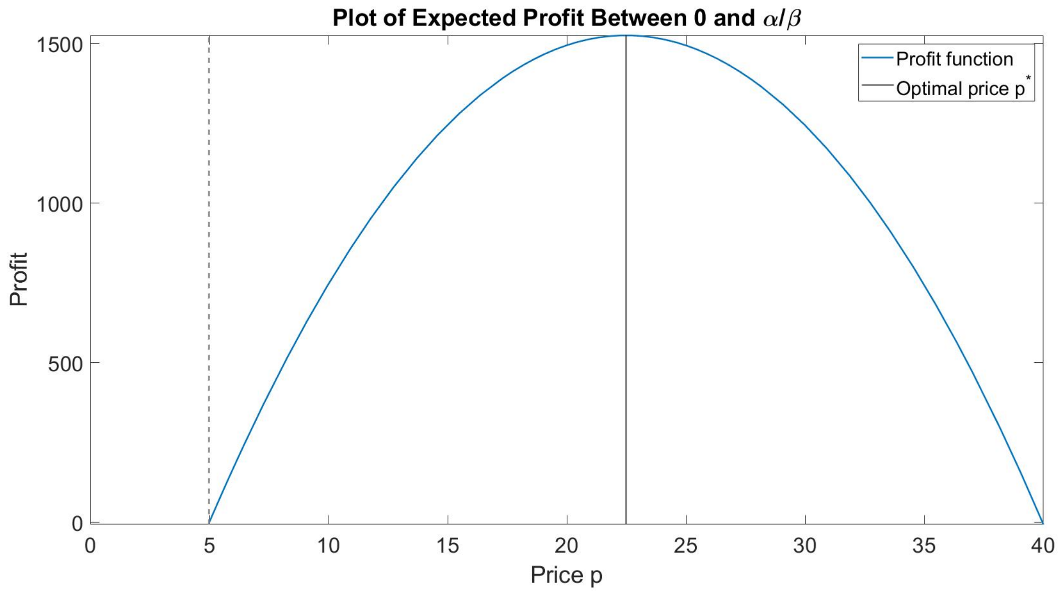
Substituting q^* as in Eq. (2.23) in the expected profit function in Eq. (2.25) results in

$$P_p(q^*) = (p-c)\cdot(\alpha-\beta p) - (p-s)\cdot \frac{\sigma}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\Phi_{inv}\left(\frac{p-c}{p-s}\right)\right)^2}. \tag{2.27}$$

Solving for the price numerically

As we stated before, we can not maximize Eq. (2.27) analytically. However, it is possible to solve this numerically, simply by looping over all possible values of p and determining the value p^* that maximizes $P_p(q^*)$.

For $\mu(p) = 200 - 5p$, cost $c = 5$, salvage value $s = 1$ and variance $\sigma^2(p) = 1$ we then have the graph



and results in optimal ordering quantity $q^* \approx 88,44$ and optimal price $p^* = 22,49$.

Sensitivity Analysis

As with the previous two distributions, we end with a section in which we analyze the sensitivity of the parameters of our model. The parameters of the normally distributed demand behave in a very similar way as with uniformly distributed demand. This is mostly due to the fact that both distributions have the property that all demands x from the mean are equiprobable.

In fact, the values in the tables on this and the next page differ with less than 1 on all entries with the uniformly distributed demand. When we consider this similarity, one could suggest using the found optimal ordering quantity and optimal price of the case with the uniformly distributed demand to find a good approximation for the normally distributed demand, since we were not able to find an explicit formula for these.

TABLE 2.10: Changing α

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
100	5	5	1	1	37,99	12,48	277,00
150	5	5	1	1	63,25	17,49	776,09
200	5	5	1	1	88,44	22,49	1525,49
250	5	5	1	1	113,58	27,49	2525,05
300	5	5	1	1	138,69	32,49	3774,69

TABLE 2.11: Changing β

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
200	2	5	1	1	96,44	52,49	4505,01
200	5	5	1	1	88,44	22,49	1525,49
200	10	5	1	1	75,49	12,49	558,25
200	20	5	1	1	49,90	7,49	122,51
200	30	5	1	1	24,45	5,82	19,61

TABLE 2.12: Changing c

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
200	5	2	1	1	96,64	21	1802,94
200	5	5	1	1	88,44	22,49	1525,49
200	5	10	1	1	75,47	24,97	1115,90
200	5	20	1	1	49,90	29,94	489,34
200	5	30	1	1	24,49	34,89	117,24

TABLE 2.13: Changing s

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
200	5	5	0	1	88,31	22,49	1524,55
200	5	5	1	1	88,44	22,49	1525,49
200	5	5	2	1	88,60	22,49	1526,55
200	5	5	3	1	88,77	22,5	1527,76
200	5	5	4	1	89,11	22,5	1529,22

TABLE 2.14: Changing σ

α	β	c	s	σ	q^*	p^*	$P_{p^*}(q^*)$
200	5	5	1	1	88,44	22,49	1525,49
200	5	5	1	2	89,38	22,48	1519,73
200	5	5	1	5	92,20	22,45	1502,47
200	5	5	1	10	96,89	22,4	1473,71
200	5	5	1	20	106,26	22,29	1416,28

Chapter 3

Two-period problem

Some companies may want to, for various reasons, plan for two periods instead of one. If, for example, it turns out that in the second period the company needs considerably fewer products, they might cut down on their production staff or machinery and in this reduce their costs. Another example is that sometimes companies have contracts with other companies that provide either products or materials that cover multiple periods.

Although it may seem reasonable, using the optimal single-period solution twice will, in most cases, not be the optimal solution for the two-period problem. Besides the fact of having two periods, the assumptions of this model are basically the same as in the single-period model.

Assumptions of the model

1. It involves a single stable product.
2. It involves a two-time period, where unsatisfied demand in period 1 is backlogged to be met in period 2. However, unsatisfied demand in period 2 can in no way be backlogged.
3. There is no initial inventory.
4. The only decisions to be made are the values of q_1 and q_2, the number of units to order at the beginning of period 1 and period 2.
5. The *demands* for buying products are *indepent and identically distributed* random variables D_1 and D_2 with known probability distribution ϕ_D, and are independent of the price.
6. The goal is to minimize the total costs for both periods, where the cost components are:

$$\begin{aligned} c &= \text{ the cost for ordering a single product} \\ s &= \text{ the salvage value per product leftover at the end of the period} \\ p &= \text{ the cost of unsatisfied demand} \end{aligned}$$

Note that the assumption is made that the two demand distributions are the same and that the values of the costs are also the same in both periods. In particular, we assume that the price p is the same in both periods.

As with the single-period problem, the optimal ordering quantity for the case where the demand is independent of the price has already been proven. Hillier [1] showed that the expected cost function for both periods, $C_1(D, y)$, is given by

$$C_1(x_1) = c \cdot q_1 + L(q_1) + \int_0^{q_1 - q_2^*} L(q_1 - \xi)\phi_D(\xi)\ d\xi + \int_{q_1 - q_2^*}^{\infty} [(q_2^* - q_1 + \xi) + L(q_2^*)]\phi_D(\xi)\ d\xi, \tag{3.1}$$

where

$$L(x) = p \cdot \int_x^{\infty} (\xi - x)\phi_D(\xi)\ d\xi - s \cdot \int_0^{x} (x - \xi)\phi_D(\xi)\ d\xi.$$

In Eq. (3.1) q_1 denotes the ordering quantity of period 1 and q_2^* denotes the optimal ordering quantity of period 2, which actually is the optimal ordering quantity of the single-period model, as in Eq. (1.2). The reason for not showing the whole proof of Hillier is the fact that in the next chapter we will, for our own model, prove the same results in a very similar way. If one wants to read the full length of Hillier's proof, it is available at [1].

The value of q_1 that minimizes Eq. (3.1), denoted by q_1^*, satisfies the equation

$$-p + (p - s) \cdot \Phi(q_1^*) + (c - p)\Phi(q_1^* - q_2^*) + (p - s) \int_0^{q_1^* - q_2^*} \Phi(q_1^* - \xi)\phi_d(\xi)\ d\xi = 0. \tag{3.2}$$

The accessory optimal ordering policy is to order q_1^* products at the beginning of period 1 and for period 2 ordering $\max\{0, q_2^* - x\}$, where x is the stock level at the beginning of period 2. Eq. (3.2) holds for general demand distributions ϕ_D and can not be solved explicitly. However, for some specific demand distributions, this expression leads to a bit simpler result.

When the demand is uniformly distributed over the range 0 to t, i.e.,

$$\phi_D(\xi) = \begin{cases} \frac{1}{t} & \text{if } 0 \le \xi \le t \\ 0 & \text{otherwise,} \end{cases}$$

q_1^* can be obtained from the expression

$$q_1^* = \sqrt{(q_2^*)^2 + \frac{2t(c[p)}{p - s} q_2^* + \frac{t^2[2p(p - s) + (c - s)^2}{(p - s)^2]}} - \frac{t(c - s)}{p - s}. \tag{3.3}$$

When the demand is exponentially distributed, i.e.

$$\phi_D(\xi) = \begin{cases} \alpha e^{-\alpha\xi} & \text{if } \xi \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

q_1^* satisfies the equation

$$(c - s)e^{-\alpha(q_1^* - q_2^*)} + (p - s)e^{-\alpha q_1^*} + \alpha(p - s)(q_1^* - q_2^*)e^{-\alpha q_1^*} = c - 2s. \tag{3.4}$$

Chapter 4

Two-period model with price dependent demand

Whereas the two-period problem assumed a price-independent demand, the model we will observe will have a price-dependent demand. Also, instead of minimizing the costs, we will maximize the profit. The rest of the assumptions remain basically the same.

Assumptions of the model

1. It involves a single stable product.
2. It involves a two-time period, where unsatisfied demand in period 1 is backlogged to be met in period 2. However, unsatisfied demand in period 2 can in no way be backlogged.
3. There is no initial inventory.
4. The only decisions to be made are the values of q_1 and q_2, the number of units to order at the beginning of period 1 and period 2.
5. The *demands* for buying products are *independent and identically distributed* random variables D_1 and D_2 with known probability distribution ϕ_D, and are **dependent** on the price.
6. The goal is to maximize the profit.

4.1 The Optimal Ordering Quantities

Just as with the single-period model, we will first determine the optimal ordering quantities for general demand D with probability density function ϕ_D. The variables of the two-period model are the same as the variables of the single-period model, with the small change of the demand and the ordering quantity, which are now defined per period. The rest of the variables are assumed to be the same in both periods. So we now have the variables:

p = the price per product
q_i = the quantity ordered at the beginning of period i
c = the costs to produce a single product
s = the *salvage value* per product, i.e. the price received per leftover at the end of period 2
D_i = the demand of period i

We will determine the optimal policy of this model by using dynamical programming, that is, first solving for the second period and then for the first period. So if we take a look at the second period, there is still a single period left, after which the products can no longer be sold. Thus, at the beginning of the second period, we are in the exact same situation as with the single-period model. So we can find our optimal ordering quantity q_2^* as before (Eq. (2.2)):

$$\Phi_D(q_2^*) = \frac{p-c}{p-s} \tag{4.1}$$

Note that there is a possibility that, at the beginning of period 2, we have a stock level of $x > 0$. However, the only difference in the profit function, which we want to maximize, is an extra $c \cdot x$ term. Since this term does not depend on the ordering quantity or the price, both the optimal ordering quantity and optimal price stay the same. Hence Eq.(4.1) shows the same optimal ordering quantity as in the single-period model.

Since the quantity that maximizes the profit in the second period stays the same, the optimal ordering policy is to stock untill this quantity is reached, i.e.,

$$\text{if} \begin{cases} x < q_2^* & \text{order } q_2^* - x \\ x \geq q_2^* & \text{order nothing,} \end{cases}$$

where x is the stock level at the beginning of period 2. The profit (of period 2) of this optimal policy can be expressed in the following way:

$$P_2(x) = \begin{cases} P_p(D_2, q_2^*) + c \cdot x & \text{if } x < q_2^* \\ P_p(D_2, x) + c \cdot x & \text{if } x \geq q_2^*. \end{cases} \tag{4.2}$$

Here, $P_p(D, q)$ is the profit function of the single-period model, as in Eq. (2.1). Note that $x = q_1 - D_1$ and thus that x is a random variable, as it is unknown at the beginning of period 1. With that knowledge we will compute the expected profit of the second period.

$$\begin{aligned} \mathbb{E}[P_2(q_1 - D_1)] &= \int_0^\infty P_2(q_1 - \xi)\phi_D(\xi)\, d\xi \\ &= \int_0^{q_1 - q_2^*} P_p(D_2, q_1 - \xi)\phi_D(\xi)\, d\xi + \int_{q_1 - q_2^*}^\infty P_p(D_2, q_2^*)\phi_D(\xi)\, d\xi + c \cdot (q_1 - \mathbb{E}[D_1|p]). \end{aligned}$$

Note that we here assume that $q_1 \geq q_2^*$, since any leftovers of the first period can still be sold the second period. If we now look at the expected profit of the two periods combined, we find that

$$\begin{aligned} P_p(q_1) &= \mathbb{E}[P_1(q_1)] \\ &= P_p(D_1, q_1) + \mathbb{E}[P_2(q_1 - D_1)] \\ &= P_p(D_1, q_1) + \int_0^{q_1 - q_2^*} P_p(D_2, q_1 - \xi)\phi_D(\xi)\, d\xi + \int_{q_1 - q_2^*}^\infty P_p(D_2, q_2^*)\phi_D(\xi)\, d\xi + c \cdot (q_1 - \mathbb{E}[D_1|p]). \end{aligned} \tag{4.3}$$

Thus Eq. (4.3) is the function that we want to maximize for q_1. To do so, we first take the derivative of this expected profit function with respect to q_1.

$$
\begin{aligned}
\frac{dP_p(q_1)}{dq_1} &= \frac{d}{dq_1}\left(P_p(D_1,q_1) + \int_0^{q_1-q_2^*} P_p(D_2,q_1-\xi)\phi_D(\xi)\, d\xi + \int_{q_1-q_2^*}^{\infty} P_p(D_2,q_2^*)\phi_D(\xi)\, d\xi + c\cdot(q_1 - \mathbb{E}[D_1|p])\right) \\
&= \frac{d}{dq_1}\big(P_p(D_1,q_1)\big) + \frac{d}{dq_1}\left(\int_0^{q_1-q_2^*} P_p(D_2,q_1-\xi)\phi_D(\xi)\, d\xi\right) + c. \qquad (4.4)
\end{aligned}
$$

We will determine the two derivatives in the above expression seperately. We have that

$$
\begin{aligned}
\frac{d}{dq}\big(P_p(D_1,q_1)\big) &= \frac{d}{dq_1}\left((p-c)\cdot\mathbb{E}[D_1|p] - (p-c)\cdot\int_{q_1}^{\infty}(\xi-q_1)\phi_D(\xi)\, d\xi - (c-s)\cdot\int_0^{q_1}(q_1-\xi)\phi_D(\xi)\, d\xi\right) \\
&= (p-c)\cdot\int_{q_1}^{\infty}\phi_D(\xi)\, d\xi - (c-s)\cdot\int_0^{q_1}\phi_D(\xi)\, d\xi \\
&= (p-c)\cdot(1-\Phi_D(q_1)) - (c-s)\cdot\Phi_D(q_1) \\
&= (p-c) - (p-s)\cdot\Phi_D(q_1),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{dq_1}\left(\int_0^{q_1-q_2^*} P_p(D_2,q_1-\xi)\phi_D(\xi)\, d\xi\right) \\
&= \int_0^{q_1-q_2^*} \frac{d}{dq_1}\left((p-c)\cdot\mathbb{E}[D_2|p] - (p-c)\cdot\int_{q_1-\xi}^{\infty}(\xi-q_1)\phi_D(\xi)\, d\xi - (c-s)\cdot\int_0^{q_1-\xi}(q_1-\xi)\phi_D(\xi)\, d\xi\right)\phi_D(\xi)\, d\xi \\
&= \int_0^{q_1-q_2^*} \big((p-c)\cdot(1-\Phi_D(q_1-\xi)) - (c-s)\cdot\Phi_D(q_1-\xi)\big)\phi_D(\xi)\, d\xi \\
&= \int_0^{q_1-q_2^*} \big((p-c) - (p-s)\cdot\Phi_D(q_1-\xi)\big)\phi_D(\xi)\, d\xi \\
&= (p-c)\cdot\Phi_D(q_1-q_2^*) - (p-s)\cdot\int_0^{q_1-q_2^*}\cdot\Phi_D(q_1-\xi)\phi_D(\xi)\, d\xi.
\end{aligned}
$$

Substituting the computed values of the two derivatives in Eq. (4.4) results in

$$
\frac{dP_p(q_1)}{dq} = p - (p-s)\cdot\Phi_D(q_1) + (p-c)\cdot\Phi_D(q_1-q_2^*) - (p-s)\cdot\int_0^{q_1-q_2^*}\cdot\Phi_D(q_1-\xi)\phi_D(\xi)\, d\xi. \qquad (4.5)
$$

It can be seen clearly now that the value q_1^* that maximizes the profit function satisfies the equation

$$p-(p-s)\cdot\Phi_D(q_1)+(p-c)\cdot\Phi_D(q_1-q_2^*)-(p-s)\cdot\int_0^{q_1-q_2^*}\cdot\Phi_D(q_1-\xi)\phi_D(\xi)\,d\xi=0, \tag{4.6}$$

since this value q_1^* should be a root of the first derivative. This result is the same as Hillier's (Eq. (3.2)).

4.2 The Optimal Price

We have now found the optimal ordering quantity q_2^* for the second period and the equation, which the optimal ordering quantity q_1 should satisfy. Both of these results were with a given price p, which might not be the best price. Thus the next step is to determine the optimal price for specific probability distributions for the demand. The two distributions we will consider are

1. Uniform Distribution
2. Exponential Distribution

Choosing a specific probability distribution for the demand will allow us to simplify the found equations and to obtain some explicit results. Just as with the single-period model we have that the *mean value* or *expected value* of the demand distributions is equal to

$$\mathbb{E}[D_1|p]=\mathbb{E}[D_2|p]=\mu(p)=(\alpha-\beta p)^+,$$

with $\alpha,\beta>0$. Again, there is the condition that

$$c\le p\le\frac{\alpha}{\beta},$$

which ensures that $\mu(p)$ is non-negative.

4.2.1 Uniform Distribution

In this section, the two demands D_1 and D_2 are uniformly distributed with mean value $\mu(p)$ as defined above. Recall the construction of the exact form of the Uniform Distribution in Section 2.2.1, which gave us that

$$D_1,D_2\sim U\left[(\alpha-\beta p)-\sigma\sqrt{3},(\alpha-\beta p)+\sigma\sqrt{3}\right].$$

The found probability density function ϕ_D of both D_1 and D_2 is

$$\phi_D(x)=\begin{cases}\frac{1}{2\sigma\sqrt{3}} & \text{for } a(p)\le x\le b(p)\\ 0 & \text{else,}\end{cases}$$

with the accessory cumulative density function (of both D_1 and D_2)

$$\Phi_D(x)=\begin{cases}\frac{x-a(p)}{2\sigma\sqrt{3}} & \text{for } a(p)\le x\le b(p)\\ 0 & \text{else.}\end{cases}$$

These two functions will allow us to significantly simplify Eq. (4.6) and even make it possible to find an explicit expression for our optimal ordering quantity q_1^* of the first period.

$$p-(p-s)\cdot\Phi_D(q_1^*)+(p-c)\cdot\Phi_D(q_1^*-q_2^*)-(p-s)\cdot\int_0^{q_1^*-q_2^*}\cdot\Phi_D(q_1^*-\xi)\phi_D(\xi)\;d\xi=0$$

$$p-(p-s)\cdot\frac{q_1^*-a(p)}{2\sigma\sqrt{3}}+(p-c)\cdot\frac{q_1^*-q_2^*}{2\sigma\sqrt{3}}-(p-s)\cdot\int_0^{q_1^*-q_2^*}\frac{q_1^*-\xi-a(p)}{2\sigma\sqrt{3}}\cdot\frac{1}{2\sigma\sqrt{3}}\;d\xi=0$$

$$p-(p-s)\cdot\frac{q_1^*-a(p)}{2\sigma\sqrt{3}}+(p-c)\cdot\frac{q_1^*-q_2^*}{2\sigma\sqrt{3}}-\frac{p-s}{12\sigma^2}\cdot\int_0^{q_1^*-q_2^*}\left(q_1^*-\xi-a(p)\right)\;d\xi=0$$

$$p-(p-s)\cdot\frac{q_1^*-a(p)}{2\sigma\sqrt{3}}+(p-c)\cdot\frac{q_1^*-q_2^*}{2\sigma\sqrt{3}}-\frac{p-s}{12\sigma^2}\cdot\left[(q_1^*-a(p))\cdot\xi-\frac{1}{2}\xi^2\right]_0^{q_1^*-q_2^*}=0$$

$$p-(p-s)\cdot\frac{q_1^*-a(p)}{2\sigma\sqrt{3}}+(p-c)\cdot\frac{q_1^*-q_2^*}{2\sigma\sqrt{3}}-\frac{p-s}{12\sigma^2}\cdot\left(\frac{1}{2}{q_1^*}^2+(q_2^*-q_1^*)\cdot a(p)-\frac{1}{2}{q_2^*}^2\right)=0.$$

Note that the left side of the equation is a polynomial of order 2. So to solve it, we can use the *abc*-formula. In order to do so, we will first group the ${q_1^*}^2, q_1^*$ and ${q_1^*}^0=1$ components.

$$\left(-\frac{p-s}{24\sigma^2}\right)\cdot{q_1^*}^2+\left(-\frac{p-s}{2\sigma\sqrt{3}}+\frac{p-c}{2\sigma\sqrt{3}}+\frac{p-s}{12\sigma^2}\cdot a(p)\right)\cdot q_1^*$$
$$+\left(p+\frac{p-s}{2\sigma\sqrt{3}}\cdot a(p)-\frac{p-c}{2\sigma\sqrt{3}}\cdot q_2^*-\frac{p-s}{12\sigma^2}\left(q_2^*\cdot a(p)-\frac{1}{2}{q_2^*}^2\right)\right)\cdot 1=0$$

$$\frac{1}{2}\cdot{q_1^*}^2+\left(2\sigma\sqrt{3}-2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}-a(p)\right)\cdot q_1^*$$
$$+\left(-\frac{p}{p-s}\cdot 12\sigma^2-2\sigma\sqrt{3}\cdot a(p)+2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\cdot q_2^*+q_2^*\cdot a(p)-\frac{1}{2}{q_2^*}^2\right)\cdot 1=0$$

$$\frac{1}{2}\cdot{q_1^*}^2+\left(2\sigma\sqrt{3}\cdot\frac{c-s}{p-s}-a(p)\right)\cdot q_1^*$$
$$+\left(-\frac{p}{p-s}\cdot 12\sigma^2-2\sigma\sqrt{3}\cdot a(p)+2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\cdot q_2^*+q_2^*\cdot a(p)-\frac{1}{2}{q_2^*}^2\right)\cdot 1=0.$$

The above equation can not be simplified any further, so we will now use the *abc*-formula to solve it. First, we will compute the discriminant D.

$$D = \left(2\sigma\sqrt{3}\cdot\frac{c-s}{p-s} - a(p)\right)^2$$
$$-4\cdot\frac{1}{2}\cdot\left(-\frac{p}{p-s}\cdot 12\sigma^2 - 2\sigma\sqrt{3}\cdot a(p) + 2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\cdot q_2^* + q_2^*\cdot a(p) - \frac{1}{2}{q_2^*}^2\right)$$
$$D = 12\sigma^2\cdot\frac{(c-s)^2}{(p-s)^2} - 4\cdot\sigma\sqrt{3}\cdot a(p)\cdot\frac{c-s}{p-s} + a(p)^2$$
$$+\left(\frac{2p}{p-s}\cdot 12\sigma^2 + 4\sigma\sqrt{3}\cdot a(p) - 4\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\cdot q_2^* - 2\cdot q_2^*\cdot a(p) + {q_2^*}^2\right)$$
$$D = 12\sigma^2\cdot\frac{(c-s)^2}{(p-s)^2} - 4\cdot\sigma\sqrt{3}\cdot a(p) + 4\cdot\sigma\sqrt{3}\cdot a(p)\cdot\frac{p-c}{p-s} + a(p)^2$$
$$+\left(\frac{2p}{p-s}\cdot 12\sigma^2 + 4\sigma\sqrt{3}\cdot a(p) - 4\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\cdot q_2^* - 2\cdot q_2^*\cdot a(p) + {q_2^*}^2\right).$$

Further simplification gives us that

$$D = 12\sigma^2\cdot\frac{(c-s)^2}{(p-s)^2} + a(p)^2 + \frac{2p}{p-s}\cdot 12\sigma^2 - 4\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\cdot(q_2^* - a(p)) - 2\cdot q_2^*\cdot a(p) + {q_2^*}^2$$
$$D = \frac{12\sigma^2\cdot((c-s)^2 + 2p(p-s))}{(p-s)^2} + \frac{4\sigma\sqrt{3}\cdot(c-p)}{p-s}\cdot(q_2^* - a(p)) + (q_2^* - a(p))^2. \tag{4.7}$$

Now that we have computed the discriminant D, it is easy to find the value of q_1^*.

$$q_1^* = \frac{\sqrt{D} - \left(\frac{(c-s)}{(p-s)}\cdot 2\sigma\sqrt{3} - a(p)\right)}{2\cdot\frac{1}{2}}$$
$$q_1^* = \sqrt{D} - \left(\frac{(c-s)}{(p-s)}\cdot 2\sigma\sqrt{3} - a(p)\right). \tag{4.8}$$

Note that, although the *abc*-formule returns two values of q_1^*, this is the only possible solution, since a negative sign before the root of the discriminant D would cause the optimal ordering quantity to be less than $a(p)$, i.e., less than the minimum demand. Also note that for $a(p) = 0$ and $2\sigma\sqrt{3} = t$, so in the case of a uniform distribution over the range 0 to t, Eq. (4.8) is the same as Eq. (3.3).

Since we now have a specific demand distribution, we can find an expression for q_2^* with Eq. (4.1). This expression will be the same as the expression of the single-period model with the uniform distribution (Eq. (2.9)) and is

$$q_2^* = a(p) + 2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}. \tag{4.9}$$

With Eq. (4.9) we can now simplify the discriminant D (Eq. (4.7)).

$$
\begin{aligned}
D &= \frac{12\sigma^2\cdot((c-s)^2+2p(p-s))}{(p-s)^2} + \frac{4\sigma\sqrt{3}\cdot(c-p)}{p-s}\cdot(q_2^*-a(p)) + (q_2^*-a(p))^2 \\
D &= \frac{12\sigma^2\cdot((c-s)^2+2p(p-s))}{(p-s)^2} + \frac{4\sigma\sqrt{3}\cdot(c-p)}{p-s}\cdot\left(2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\right) + \left(2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}\right)^2 \\
D &= \frac{12\sigma^2\cdot((c-s)^2+2p(p-s))}{(p-s)^2} - \frac{24\sigma^2\cdot(p-c)^2}{(p-s)^2} + \frac{12\sigma^2(p-c)^2}{(p-s)^2} \\
D &= \frac{12\sigma^2\cdot((c-s)^2+2p(p-s))}{(p-s)^2} - \frac{12\sigma^2\cdot(p-c)^2}{(p-s)^2} \\
D &= \frac{12\sigma^2\cdot((c-s)^2+2p(p-s)-(p-c)^2)}{(p-s)^2} \\
D &= \frac{12\sigma^2\cdot(c^2-2cs+s^2+2p^2-2ps-p^2+2pc-c^2)}{(p-s)^2} \\
D &= \frac{12\sigma^2\cdot((p-s)^2+2c(p-s))}{(p-s)^2}.
\end{aligned}
$$

With this simplified form of D we can determine the explicit expression of our optimal ordering quantity q_1^* of the first period, which is

$$
\begin{aligned}
q_1^* &= \sqrt{D} - \left(\frac{(c-s)}{(p-s)}\cdot 2\sigma\sqrt{3} - a(p)\right) \\
q_1^* &= \frac{2\sigma\sqrt{3}}{p-s}\cdot\sqrt{(p-s)^2+2c(p-s)} - \left(\frac{(c-s)}{(p-s)}\cdot 2\sigma\sqrt{3} - a(p)\right) \\
q_1^* &= \frac{2\sigma\sqrt{3}}{p-s}\cdot\left(\sqrt{(p-s)^2+2c(p-s)} - (c-s)\right) + a(p) \\
q_1^* &= 2\sigma\sqrt{3}\left(\sqrt{1+\frac{2c}{p-s}} - \frac{c-s}{p-s}\right) + a(p).
\end{aligned}
\tag{4.10}
$$

Solving for the price

Now that both optimal ordering quantities are obtained, the next step is to determine the optimal price. To do so, we will first determine the expected profit function $P_p(q_1^*)$ of the two periods combined. Eq. (4.3) shows the expected profit function for general demand distribution and general ordering quantity q_1. Since the demand distribution is now known, we can rewrite $P_p(q_1)$. Recall that

$$
\begin{aligned}
P_p(q_1^*) &= P_p(D_1, q_1^*) + \int_0^{q_1^*-q_2^*} P_p(D_2, q_1^*-\xi)\phi_D(\xi)\; d\xi + \int_{q_1^*-q_2^*}^{\infty} P_p(D_2, q_2^*)\phi_D(\xi)\; d\xi + c\cdot(q_1^* - \mathbb{E}[D_2|p]) \\
&= (1) + (2) + (3) + c\cdot(q_1^* - (\alpha - \beta p)).
\end{aligned}
$$

Since this will be quite a lengthy computation, we will determine the parts (1), (2) and (3) seperately.

$$
\begin{aligned}
(1) &= P_p(D_1, q_1^*) \\
&= (p-c)\cdot\mathbb{E}[D_1|p] - (p-c)\cdot\int_{q_1^*}^{\infty}(\xi - q)\phi_D(\xi)\,d\xi - (c-s)\cdot\int_0^{q_1^*}(q_1^* - \xi)\phi_D(\xi)\,d\xi \\
&= (p-c)\cdot(\alpha-\beta p) - \frac{p-c}{4\sigma\sqrt{3}}\cdot(b(p)-q_1^*)^2 - \frac{c-s}{4\sigma\sqrt{3}}\cdot(q_1^* - a(p))^2.
\end{aligned}
$$

In the following two parts where we determine (2) and (3), there are two terms colored red. These are two $(p-c)(\alpha-\beta p)$ terms that both integrals have in common. Since they both have the same term, which is independent of ξ, multiplied with the density function, these will together result in an extra $(p-c)(\alpha-\beta)$ and are left out in the rest of the calculation for the purpose of simplification.

$$
\begin{aligned}
(2) &= \int_0^{q_1^*-q_2^*} P_p(D_2, q_1^* - \xi)\phi_D(\xi)\,d\xi \\
&= \frac{1}{2\sigma\sqrt{3}}\int_0^{q_1^*-q_2^*}\left(\color{red}{(p-c)\cdot(\alpha-\beta p)}\color{black} - \frac{p-c}{4\sigma\sqrt{3}}\cdot(b(p)-(q_1^*-\xi))^2 - \frac{c-s}{4\sigma\sqrt{3}}\cdot((q_1^*-\xi)-a(p))^2\right)d\xi \\
&= \frac{1}{24\sigma^2}\cdot\left(-(p-c)\cdot\int_0^{q_1^*-q_2^*}(b(p)-(q_1^*-\xi))^2\,d\xi - (c-s)\cdot\int_0^{q_1^*-q_2^*}((q_1^*-\xi)-a(p))^2\,d\xi\right) \\
&= \frac{1}{24\sigma^2}\cdot\left(-(p-c)\cdot\frac{1}{3}\left(\left(b(p)-q_2^*)\right)^3 - \left(b(p)-q_1^*\right)^3\right) + (c-s)\cdot\frac{1}{3}\left(\left(q_2^*-a(p)\right)\right)^3 - \left(q_1^*-a(p)\right)^3\right)\right).
\end{aligned}
$$

And lastly, we have that

$$
\begin{aligned}
(3) &= \int_{q_1^*-q_2^*}^{\infty} P_p(D_2, q_2^*)\phi_D(\xi)\,d\xi \\
&= \left(\int_{q_1^*-q_2^*}^{\infty}\phi_D(\xi)\,d\xi\right)\cdot P_p(D_2, q_2^*) \\
&= \left(\int_{q_1^*-q_2^*}^{2\sigma\sqrt{3}}\frac{1}{2\sigma\sqrt{3}}\,d\xi\right)\cdot P_p(D_2, q_2^*) \\
&= \left(\frac{2\sigma\sqrt{3}-(q_1^*-q_2^*)}{2\sigma\sqrt{3}}\right)\cdot\left(\color{red}{(p-c)\cdot(\alpha-\beta p)}\color{black} - (p-c)\cdot\sigma\sqrt{3}\cdot\left(1-\frac{p-c}{p-s}\right)^2 - (c-s)\cdot\sigma\sqrt{3}\left(\frac{p-c}{p-s}\right)^2\right) \\
&= \left(\frac{2\sigma\sqrt{3}-q_1^*+q_2^*}{2\sigma\sqrt{3}}\right)\cdot\left(-(p-c)\cdot\sigma\sqrt{3}\cdot\left(1-\frac{p-c}{p-s}\right)^2 - (c-s)\cdot\sigma\sqrt{3}\left(\frac{p-c}{p-s}\right)^2\right).
\end{aligned}
$$

So if we add this al together, we find the expected profit function

$$\begin{aligned}
P_p(q_1^*) &= 2\cdot(p-c)\cdot(\alpha-\beta p) - \frac{p-c}{4\sigma\sqrt{3}}\cdot(b(p)-q_1^*)^2 - \frac{c-s}{4\sigma\sqrt{3}}\cdot(q_1^*-a(p))^2 \\
&\quad + \frac{1}{72\sigma^2}\cdot\left(-(p-c)\cdot\left(\left(b(p)-q_2^*\right)^3-\left(b(p)-q_1^*\right)^3\right)+(c-s)\cdot\left(\left(q_2^*-a(p)\right)^3-\left(q_1^*-a(p)\right)^3\right)\right) \\
&\quad + \left(\frac{2\sigma\sqrt{3}-q_1^*+q_2^*}{2\sigma\sqrt{3}}\right)\cdot\left(-(p-c)\cdot\sigma\sqrt{3}\cdot\left(1-\frac{p-c}{p-s}\right)^2-(c-s)\cdot\sigma\sqrt{3}\left(\frac{p-c}{p-s}\right)^2\right)+c\cdot(q_1^*-(\alpha-\beta p)),
\end{aligned} \tag{4.11}$$

where q_1^* and q_2^* are the optimal quantities that we computed before. Since these values are known (Eq. (4.9) and Eq. (4.10)), the following equations can be used to simplify $P_p(q_1^*)$.

$$\begin{aligned}
q_1^*-a(p) &= 2\sigma\sqrt{3}\cdot\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)+a(p)-a(p) \\
&= 2\sigma\sqrt{3}\cdot\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right) \\
b(p)-q_1^* &= -(q_1^*-a(p))+2\sigma\sqrt{3} \\
&= 2\sigma\sqrt{3}\cdot\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right) \\
q_2^*-a(p) &= a(p)+2\sigma\sqrt{3}\cdot\frac{p-c}{c-s}-a(p) \\
&= 2\sigma\sqrt{3}\cdot\frac{p-c}{p-s} \\
b(p)-q_2^* &= -(q_2^*-a(p))+2\sigma\sqrt{3} \\
&= 2\sigma\sqrt{3}\cdot\left(1-\frac{p-c}{p-s}\right)
\end{aligned}$$

$$\begin{aligned}
-(q_1^*-q_2^*) &= a(p)+2\sigma\sqrt{3}\cdot\frac{p-c}{p-s}-\left(2\sigma\sqrt{3}\cdot\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)+a(p)\right) \\
&= 2\sigma\sqrt{3}\cdot\left(1-\sqrt{1+\frac{2c}{p-s}}\right).
\end{aligned}$$

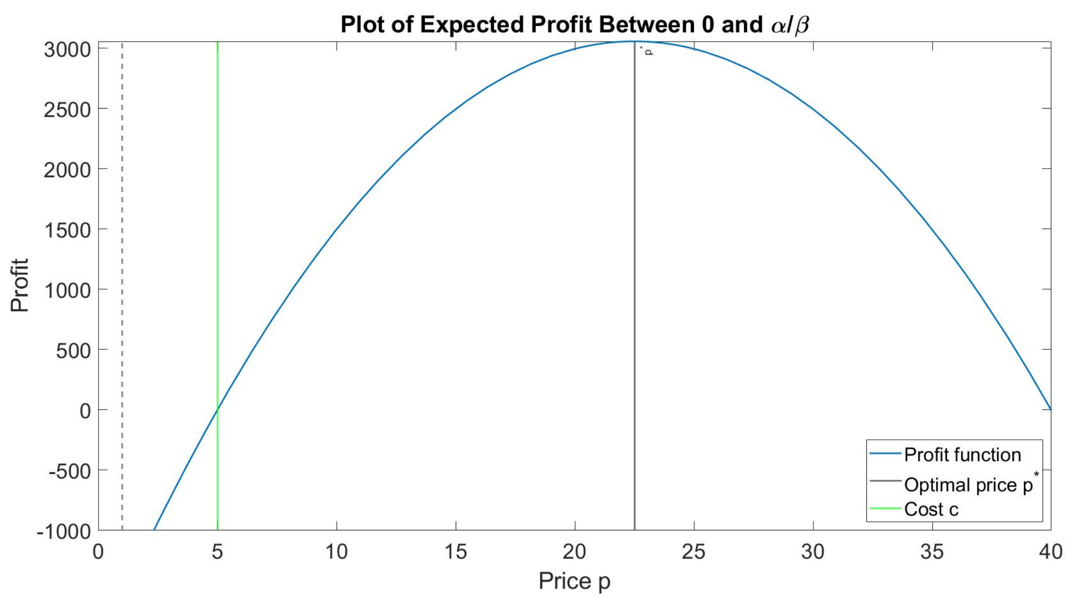
With these equations we find our final expression of the expected profit function, which is given by

$$
\begin{aligned}
P_p(q_1^*) = 2\cdot(p-c)\cdot(\alpha-\beta p) - \sigma\sqrt{3}\cdot(p-c)\cdot\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)^2 - \sigma\sqrt{3}\cdot(c-s)\cdot\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)^2 \\
-\frac{1}{3}\sqrt{3}\sigma\cdot(p-c)\cdot\left(1-\frac{p-c}{p-s}\right)^3+\frac{1}{3}\sqrt{3}\sigma\cdot(p-c)\cdot\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)^3 \\
+\frac{1}{3}\sqrt{3}\sigma(c-s)\cdot\left(\frac{p-c}{p-s}\right)^3-\frac{1}{3}\sqrt{3}\sigma(c-s)\cdot\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)^3 \\
+\left(2-\sqrt{1+\frac{2c}{p-s}}\right)\cdot\left(-(p-c)\cdot\sigma\sqrt{3}\cdot\left(1-\frac{p-c}{p-s}\right)^2-(c-s)\cdot\sigma\sqrt{3}\left(\frac{p-c}{p-s}\right)^2\right) \\
+c\cdot\left(2\sigma\sqrt{3}\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)-\sigma\sqrt{3}\right). \qquad (4.12)
\end{aligned}
$$

In comparison to the expected proft function of the single-period model, this is quite a complicated expression. Although it might be possible, finding and proving the optimal price p^* which maximizes this function analytically is extremely difficult. For this reason we will not cover this in this thesis. We can, however, solve it numerically and compare this to the found values of the single-period model.

Numerical Analysis

We will here consider one example, namely the first example we used for the uniform distribution with the single-period model. There we had $\mu(p) = 200-5p$ with cost $c = 5$, salvage value $s = 1$ and variance $\sigma^2(p) = 1$. This here gives the graph



and results in optimal ordering quantities $q_1^* \approx 89,32$, $q_2^* \approx 88,59$ and an optimal price $p^* \approx 2,50$. If we compare this to the results of the single-period model, where we had $q_1^* \approx 88,62$ and optimal price $p \approx 22,49$, we see that the optimal price is almost equal. Furthermore, as the optimal ordering quantity of the first period is slightly higher than the single period, the optimal ordering quantity of the second period is slightly less, which is what we would expect. Lastly, if we compare the graphs, we see that at the optimal price p^*, the expected profit of the two-period model is roughly twice as high as in the single-period model, which seems logical, since we have twice the amount of periods.

Of course, this is only a single example and this can not guarantee that this is the case for all examples. In the next section, we will see in more detail more what happens when we change certain parameters.

Sensitivity Analysis

Just as with the single-period model we will perform a sensitivity analysis on the two-period model.

TABLE 4.1: Changing α

α	β	c	s	σ	q_1^*	q_2^*	p^*	$P_{p^*}(q_1^*)$
100	5	5	1	1	39,35	38,08	12,49	559,45
150	5	5	1	1	64,32	63,39	17,5	1558,87
200	5	5	1	1	89,32	88,59	22,5	3058,54
250	5	5	1	1	114,31	113,71	27,5	5058,32
300	5	5	1	1	139,30	138,79	32,5	7558,17

TABLE 4.2: Changing β

α	β	c	s	σ	q_1^*	q_2^*	p^*	$P_{p^*}(q_1^*)$
200	2	5	1	1	96,46	96,78	52,5	9020,35
200	5	5	1	1	89,32	88,59	22,5	3058,54
200	10	5	1	1	76,80	75,53	12,5	1121,94
200	20	5	1	1	51,85	49,80	7,49	248,20
200	30	5	1	1	26,57	23,96	5,83	40,71

TABLE 4.3: Changing c

α	β	c	s	σ	q_1^*	q_2^*	p^*	$P_{p^*}(q_1^*)$
200	5	2	1	1	96,89	96,56	21	3610,16
200	5	5	1	1	89,32	88,59	22,5	3058,54
200	5	10	1	1	76,71	75,48	24,99	2241,28
200	5	20	1	1	51,49	49,61	29,97	986,69
200	5	30	1	1	26,27	13,97	34,96	235,86

TABLE 4.4: Changing s

α	β	c	s	σ	q_1^*	q_2^*	p^*	$P_{p^*}(q_1^*)$
200	5	5	0	1	89,16	88,46	22,5	3055,64
200	5	5	1	1	89,32	88,59	22,5	3058,54
200	5	5	2	1	89,49	88,73	22,5	3061,72
200	5	5	3	1	89,67	88,88	22,5	3065,23
200	5	5	4	1	89,88	89,04	22,5	3069,11

TABLE 4.5: Changing σ

α	β	c	s	σ	q_1^*	q_2^*	p^*	$P_{p^*}(q_1^*)$
200	5	5	1	1	89,32	88,59	22,5	3058,54
200	5	5	1	2	91,18	89,72	22,49	3054,58
200	5	5	1	5	96,63	92,99	22,49	3042,70
200	5	5	1	10	105,82	98,52	22,47	3022,90
200	5	5	1	20	124,08	109,47	22,45	2983,32

Since the analysis of the parameters is basically the same as with the single-period model, it is more interesting to compare the two models (See pages 14 and 15 for the single-period model). The first thing we note is that in all cases the optimal price p^* is similar to the optimal price in the single-period model. The biggest difference is a difference of nine cents, which occurs in the last row of Table 4.5.

Furthermore, again in all cases, the expected profit is higher than twice the expected profit of the single-period model. This justifies the statement that using the optimal solution of the single-period model twice is not optimal in all cases. In most cases, this is just slightly higher, around ten more. However, when the variance increases (Table 4.5), the expected profit of the two-period model is significantly higher than twice the single-period expected profit, up to almost 150 more. This phenomenon can be explained by the following reasoning.

As one can see, in all the other tables, the difference between the optimal ordering quantities q_1^* and q_2^* is quite small. In Table 4.5 however, we can see that this difference increases along with the variance. With a greater variance, there comes less predictability which in the single-period model decreases the expected profit by a lot. In the two-period model, however, where backlogging of unsatisfied demand of the first period can be met in the second period (so there is no risk of underordering), one can counter this with a higher optimal ordering quantity for the first period since this will cause the probability of underordering to be negligible and therefore increase the expected profit significantly.

4.2.1.1 Constraint on the Salvage Value

As we saw in the Sensitivity Analysis, the salvage value does not have a significant influence on the outcome of the model. However, when the salvage value comes too close to the costs, the optimal price drops extremely fast until it reaches the lowest possible price, which is c.

We would like to find some constraint for the salvage value, which would assure that we have a concave function, similar to the two cases with the single-period model on the top of page 13. To find this constraint, we first need to determine the first and second derivative of our expected profit function $P_p(q_1^*)$ as in Eq. (4.12). Since the computations are quite lengthy and trivial, we will only show the outcomes.

The first derivative is given by

$$
\begin{aligned}
P'_p(q_1^*) = {} & 2\alpha - 4\beta p + 2\beta c - \sigma\sqrt{3}\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)^2 \\
& - 2\sigma\sqrt{3}(p-c)\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right) \\
& + 2\sigma\sqrt{3}(c-s)\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right) \\
& - \frac{1}{3}\sqrt{3}\sigma\left(\frac{c-s}{p-s}\right)^3 + \sqrt{3}\sigma\frac{p-c}{p-s}\left(\frac{c-s}{p-s}\right)^2 + \frac{1}{3}\sigma\sqrt{3}\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)^3 \\
& + \sqrt{3}\sigma(p-c)\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)^2\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right) \\
& + \sigma\sqrt{3}(c-s)\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)^2\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right) \\
& - \sigma\sqrt{3}\left(2-\sqrt{1+\frac{2c}{p-s}}\right)\left(\frac{c-s}{p-s}\right)^2 - (p-c)\cdot\sigma\sqrt{3}\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}\left(\frac{c-s}{p-s}\right)^2 \\
& - (c-s)\cdot\sigma\sqrt{3}\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}\left(\frac{c-s}{p-s}\right)^2 - 2c\sigma\sqrt{3}\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{p-c}{(p-s)^2}\right). \qquad (4.13)
\end{aligned}
$$

The second derivative is given by

$$\begin{aligned}
P_P''(q_1^*) = &-4\beta - 4\sigma\sqrt{3}\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)\cdot\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right)\\
&-2\sigma\sqrt{3}(p-c)\cdot\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right)^2\\
&+2\sigma\sqrt{3}(p-c)\cdot\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)\cdot\left(-\frac{2(c-s)}{(p-s)^3}+\frac{2c}{(p-s)^3\sqrt{1+\frac{2c}{p-s}}}-\frac{c^2}{(p-s)^4\left(1+\frac{2c}{p-s}\right)^{\frac{3}{2}}}\right)\\
&-4\sigma\sqrt{3}(c-s)\cdot\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)\cdot\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right)^2\\
&-2\sigma\sqrt{3}(c-s)\cdot\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)\cdot\left(-\frac{2(c-s)}{(p-s)^3}+\frac{2c}{(p-s)^3\sqrt{1+\frac{2c}{p-s}}}-\frac{c^2}{(p-s)^4\left(1+\frac{2c}{p-s}\right)^{\frac{3}{2}}}\right)\\
&+2\sigma\sqrt{3}\cdot\frac{(c-s)^3}{(p-s)^4}-4\sigma\sqrt{3}\cdot\frac{(p-c)(c-s)^3}{(p-s)^5}+2\sigma\sqrt{3}\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)^2\cdot\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right)\\
&+2\sigma\sqrt{3}(p-c)\cdot\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)\cdot\left(\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}-\frac{c-s}{(p-s)^2}\right)^2\\
&+\sigma\sqrt{3}(p-c)\cdot\left(1-\sqrt{1+\frac{2c}{p-s}}+\frac{c-s}{p-s}\right)^2\cdot\left(\frac{2(c-s)}{(p-s)^3}-\frac{2c}{(p-s)^3\sqrt{1+\frac{2c}{p-s}}}+\frac{c^2}{(p-s)^4\left(1+\frac{2c}{p-s}\right)^{\frac{3}{2}}}\right)\\
&+2\sigma\sqrt{3}\cdot\frac{(p-c)(c-s)^3}{(p-s)^5}-2\sigma\sqrt{3}\cdot\frac{(p-c)^2(c-s)^2}{(p-s)^5}\\
&-\sigma\sqrt{3}(c-s)\cdot\left(\sqrt{1+\frac{2c}{p-s}}-\frac{c-s}{p-s}\right)^2\cdot\left(-\frac{2(c-s)}{(p-s)^3}+\frac{2c}{(p-s)^3\sqrt{1+\frac{2c}{p-s}}}-\frac{c^2}{(p-s)^4\left(1+\frac{2c}{p-s}\right)^{\frac{3}{2}}}\right)\\
&-2\sigma\sqrt{3}\cdot\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}\cdot\left(\frac{c-s}{p-s}\right)^2+2\sigma\sqrt{3}\cdot\left(2-\sqrt{1+\frac{2c}{p-s}}\right)\cdot\frac{(c-s)^2}{(p-s)^3}\\
&+\sigma\sqrt{3}(p-c)\cdot\left(\frac{2c}{(p-s)^3\sqrt{1+\frac{2c}{p-s}}}-\frac{c^2}{(p-s)^4\left(1+\frac{2c}{p-s}\right)^{\frac{3}{2}}}\right)\cdot\left(\frac{c-s}{p-s}\right)^2\\
&+4\sigma\sqrt{3}(p-c)\cdot\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}\cdot\left(1-\frac{p-c}{p-s}\right)\cdot\frac{(c-s)^2}{(p-s)^3}-4\sigma\sqrt{3}(c-s)\cdot\frac{c}{(p-s)^2\sqrt{1+\frac{2c}{p-s}}}\cdot\frac{(p-c)(c-s)}{(p-s)^3}\\
&+2\sigma\sqrt{3}\cdot c\cdot\left(-\frac{2(c-s)}{(p-s)^3}+\frac{2c}{(p-s)^3\sqrt{1+\frac{2c}{p-s}}}-\frac{c^2}{(p-s)^4\left(1+\frac{2c}{p-s}\right)^{\frac{3}{2}}}\right).
\end{aligned} \tag{4.14}$$

From analyzing the plots of the two-period model with the uniform distributed demand, the observation was made that, as in the single-period model, the concaveness at $p = c$ is a sufficient condition for the function to be concave on the whole interval, i.e., the second derivative is non-increasing on our interval.

Thus, we want to determine in which cases the second derivative is negative at $p = c$. Luckily, for $p = c$, the second derivative becomes significantly simpler, since there are many factors $(p-c)$. As we stated before, we are looking for a condition for the salvage value, i.e.,

$$s \leq \gamma \cdot c,$$

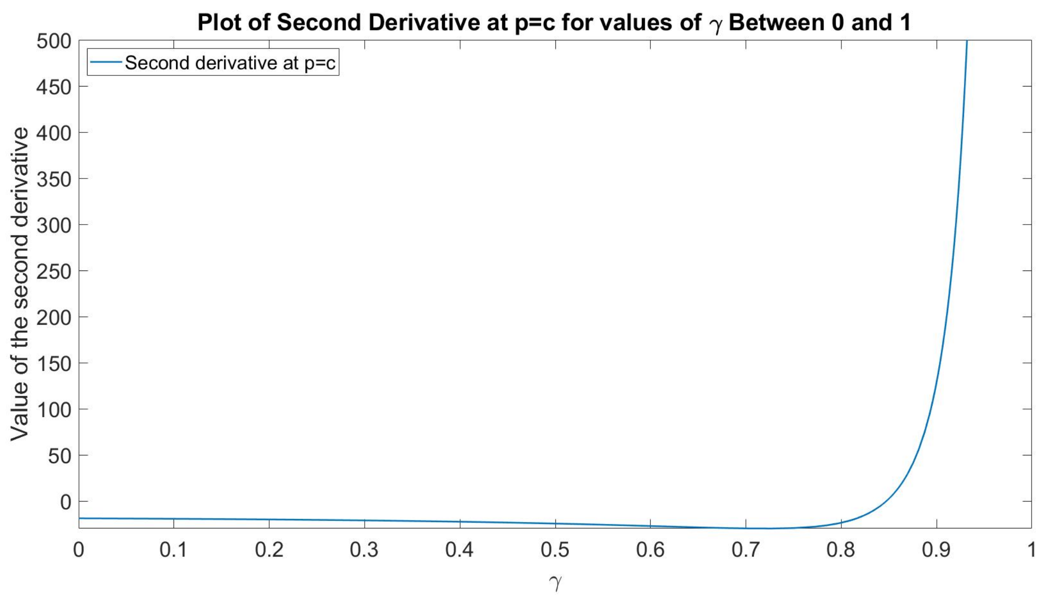
for $\gamma \in [0,1]$. To find this condition we will substitute γc for s in our second derivative. Altogether this results in

$$\begin{aligned}
P_c''(q_1^*) = -4\beta - 4\sigma\sqrt{3}\cdot &\left(\frac{2}{(1-\gamma)^2 c\sqrt{\frac{3-\gamma}{1-\gamma}}} - \frac{2}{(1-\gamma)c} - \frac{1}{(1-\gamma)^2 c} + \frac{\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)c} \right) \\
&- 4\sigma\sqrt{3}\cdot \left(\frac{1}{(1-\gamma)^3 c\frac{3-\gamma}{1-\gamma}} - \frac{2}{(1-\gamma)^2 c} + \frac{\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)c} - \frac{1}{(1-\gamma)^3 c\left(\frac{3-\gamma}{1-\gamma}\right)} + \frac{2}{(1-\gamma)^2 c\sqrt{\frac{3-\gamma}{1-\gamma}}} - \frac{1}{(1-\gamma)c} \right) \\
&- 2\sigma\sqrt{3}\cdot \left(-2\frac{\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)c} + \frac{2}{(1-\gamma)^2 c} - \frac{1}{(1-\gamma)^3 c\left(\frac{3-\gamma}{1-\gamma}\right)} + \frac{2}{(1-\gamma)c} - \frac{2}{(1-\gamma)^2 c\sqrt{\frac{3-\gamma}{1-\gamma}}} + \frac{1}{(1-\gamma)^3 c\left(\frac{3-\gamma}{1-\gamma}\right)^{\frac{3}{2}}} \right) \\
&+ 2\sigma\sqrt{3}\frac{1}{(1-\gamma)c} + 2\sigma\sqrt{3}\cdot \left(\frac{4}{(1-\gamma)^2 c\sqrt{\frac{3-\gamma}{1-\gamma}}} - \frac{4}{(1-\gamma)c} - \frac{4}{(1-\gamma)^2 c} + \frac{4\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)c} + \frac{\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)^2 c} - \frac{3-\gamma}{(1-\gamma)^2 c} \right) \\
&- \sigma\sqrt{3}\cdot \left(-\frac{8}{(1-\gamma)c} + \frac{8}{(1-\gamma)^2 c\sqrt{\frac{3-\gamma}{1-\gamma}}} - \frac{4}{(1-\gamma)^3 c\left(\frac{3-\gamma}{1-\gamma}\right)^{\frac{3}{2}}} + \frac{8\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)c} - \frac{8}{(1-\gamma)^2 c} + \frac{4}{(1-\gamma)^3 c\left(\frac{3-\gamma}{1-\gamma}\right)} \right) \\
&- \sigma\sqrt{3}\cdot \left(-\frac{2(3-\gamma)}{(1-\gamma)^2 c} + \frac{2\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)^2 c} - \frac{1}{(1-\gamma)^3 c\sqrt{\frac{3-\gamma}{1-\gamma}}} \right) - 2\sigma\sqrt{3}\cdot \frac{1}{(1-\gamma)^2 c\sqrt{\frac{3-\gamma}{1-\gamma}}} \\
&+ 2\sigma\sqrt{3}\cdot \left(\frac{2}{(1-\gamma)c} - \frac{\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)c} \right) + 2\sigma\sqrt{3}\cdot \left(-\frac{2}{(1-\gamma)^2 c} + \frac{2}{(1-\gamma)^3 c\sqrt{\frac{3-\gamma}{1-\gamma}}} - \frac{1}{(1-\gamma)^4 c\left(\frac{3-\gamma}{1-\gamma}\right)^{\frac{3}{2}}} \right).
\end{aligned} \tag{4.15}$$

This then further simplifies into

$$P_c''(q_1^*) = -4\beta + \sigma\sqrt{3}\cdot\left(-\frac{14}{(1-\gamma)^2 c\sqrt{\frac{3-\gamma}{1-\gamma}}} + \frac{14}{(1-\gamma)c} - \frac{6\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)c} + \frac{4}{(1-\gamma)^2 c} + \frac{2}{(1-\gamma)^3 c\left(\frac{3-\gamma}{1-\gamma}\right)}\right.$$
$$\left. + \frac{2}{(1-\gamma)^3 c\left(\frac{3-\gamma}{1-\gamma}\right)^{\frac{3}{2}}} + \frac{1}{(1-\gamma)^3 c\sqrt{\frac{3-\gamma}{1-\gamma}}} - \frac{2}{(1-\gamma)^4 c\left(\frac{3-\gamma}{1-\gamma}\right)^{\frac{3}{2}}}\right). \tag{4.16}$$

Although, compared to Eq. (4.14), this equation is significantly simpler, we still need to solve it numerically. If we plot Eq. (4.16) for values of γ, with the variables of the example on page 32, we find that,



as we expected, for high values of γ, thus the salvage value close to the cost, we have a positive derivative. If we numerically compute the root of the second derivative, we find that

$$\gamma = 0.8470 \qquad \text{(with steps of 0.001)}$$

for this specific example. In general the constraint on the salvage value is

$$s \leq \gamma \cdot c, \tag{4.17}$$

where γ satisfies the equation

$$\sigma\sqrt{3}\cdot\left(-\frac{14}{(1-\gamma)^2c\sqrt{\frac{3-\gamma}{1-\gamma}}}+\frac{14}{(1-\gamma)c}-\frac{6\sqrt{\frac{3-\gamma}{1-\gamma}}}{(1-\gamma)c}+\frac{4}{(1-\gamma)^2c}+\frac{2}{(1-\gamma)^3c\left(\frac{3-\gamma}{1-\gamma}\right)}\right.$$
$$\left.+\frac{2}{(1-\gamma)^3c\left(\frac{3-\gamma}{1-\gamma}\right)^{\frac{3}{2}}}+\frac{1}{(1-\gamma)^3c\sqrt{\frac{3-\gamma}{1-\gamma}}}-\frac{2}{(1-\gamma)^4c\left(\frac{3-\gamma}{1-\gamma}\right)^{\frac{3}{2}}}\right)=4\beta. \tag{4.18}$$

4.2.2 Exponential Distribution

In this section the demands D_1 and D_2 are exponentially distributed with mean value $\mu(p)=\alpha-\beta p$. Thus the probability density function is given by

$$\phi_D(x)=\begin{cases}\frac{1}{\alpha-\beta p}\cdot e^{-\frac{x}{\alpha-\beta p}} & \text{for } x\geq 0\\ 0 & \text{else,}\end{cases} \tag{4.19}$$

with the accessory cumulative density function

$$\Phi_D(x)=\begin{cases}1-e^{-\frac{x}{\alpha-\beta p}} & \text{for } x\geq 0\\ 0 & \text{else.}\end{cases} \tag{4.20}$$

Recall that the optimal ordering quantity q_1^* satisfies the equation

$$p-(p-s)\cdot\Phi_D(q_1)+(p-c)\cdot\Phi_D(q_1-q_2^*)-(p-s)\cdot\int_0^{q_1-q_2^*}\cdot\Phi_D(q_1-\xi)\phi_D(\xi)\,d\xi=0. \tag{4.21}$$

With Eq. (4.19) and (4.20) we can simplify this equation.

$$p-(p-s)\cdot\Phi_D(q_1^*)+(p-c)\cdot\Phi_D(q_1^*-q_2^*)-(p-s)\cdot\int_0^{q_1^*-q_2^*}\cdot\Phi_D(q_1^*-\xi)\phi_D(\xi)\,d\xi=0$$

$$p-(p-s)\cdot\left(1-e^{-\frac{q_1^*}{\alpha-\beta p}}\right)+(p-c)\cdot\left(1-e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}\right)-(p-s)\cdot\int_0^{q_1^*-q_2^*}\left(1-e^{-\frac{q_1^*-\xi}{\alpha-\beta p}}\right)\phi_D(\xi)\,d\xi=0$$

$$p-(p-s)\cdot\left(1-e^{-\frac{q_1^*}{\alpha-\beta p}}\right)+(p-c)\cdot\left(1-e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}\right)$$
$$-(p-s)\cdot\int_0^{q_1^*-q_2^*}\phi_D(\xi)\,d\xi+(p-s)\cdot\int_0^{q_1^*-q_2^*}e^{-\frac{q_1^*-\xi}{\alpha-\beta p}}\cdot\frac{1}{\alpha-\beta p}\cdot e^{-\frac{\xi}{\alpha-\beta p}}\,d\xi=0$$

$$p-(p-s)\cdot\left(1-e^{-\frac{q_1^*}{\alpha-\beta p}}\right)+(p-c)\cdot\left(1-e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}\right)$$

$$-(p-s)\cdot\left(1-e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}\right)+(p-s)\cdot\int_0^{q_1^*-q_2^*}\frac{1}{\alpha-\beta p}\cdot e^{-\frac{q_1^*}{\alpha-\beta p}}\,d\xi=0$$

$$p-(p-s)\cdot\left(1-e^{-\frac{q_1^*}{\alpha-\beta p}}\right)-(c-s)\cdot\left(1-e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}\right)+(p-s)\cdot(q_1^*-q_2^*)\cdot\frac{1}{\alpha-\beta p}\cdot e^{-\frac{q_1^*}{\alpha-\beta p}}=0$$

$$(p-s)\cdot e^{-\frac{q_1^*}{\alpha-\beta p}}+(c-s)e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}+(p-s)\cdot(q_1^*-q_2^*)\cdot\frac{1}{\alpha-\beta p}\cdot e^{-\frac{q_1^*}{\alpha-\beta p}}=c-2s. \tag{4.22}$$

With Eq.(4.1) and the fact that the demand is exponentially distributed demand, we can find an expression for the optimal ordering quantity q_2^*. This expression will be the same as the optimal ordering quantiy of the single-period model in Eq.(2.18), which is

$$q_2^*=-(\alpha-\beta p)\cdot\ln\left(\frac{c-s}{p-s}\right). \tag{4.23}$$

Solving for the price

Although we do not have an explicit expression for the optimal ordering quantity q_1^*, the next step is to determine the expected profit function. q_1^* can be computed numerically and then used in the expected profit function later. The expected profit function for general demand distribution (see Eq. (4.3)) is given by

$$\begin{aligned}P_p(q_1^*)&=P_p(D_1,q_1^*)+\int_0^{q_1^*-q_2^*}P_p(D_2,q_1^*-\xi)\phi_D(\xi)\,d\xi+\int_{q_1^*-q_2^*}^{\infty}P_p(D_2,q_2^*)\phi_D(\xi)\,d\xi+c\cdot(q_1^*-\mathbb{E}[D_1|p])\\&=(1)+(2)+(3)+c\cdot(q_1^*-(\alpha-\beta p)).\end{aligned}$$

We will determine the parts (1), (2) and (3) seperately. (1) can be obtained from the single-period model for general ordering quantity (Eq. (2.20)), which is given by

$$\begin{aligned}(1)&=P_p(D_1,q_1^*)\\&=(p-s)(\alpha-\beta p)-(p-s)(\alpha-\beta p)e^{-\frac{q_1^*}{\alpha-\beta p}}-(c-s)\cdot q.\end{aligned}$$

For the second and third part we use the same simplification as in the section with the uniform distributed demand. These terms $(p-s)(\alpha-\beta p)$ are highlighted in red.

$$\begin{aligned}(2)&=\int_0^{q_1^*-q_2^*}P_p(D_2,q_1^*-\xi)\phi_D(\xi)\,d\xi\\&=\int_0^{q_1^*-q_2^*}\left((p-s)(\alpha-\beta p)-(p-s)(\alpha-\beta p)e^{-\frac{q_1^*-\xi}{\alpha-\beta p}}-(c-s)\cdot(q_1^*-\xi)\right)\phi_D(\xi)\,d\xi\\&=-(p-s)(\alpha-\beta p)\cdot\int_0^{q_1^*-q_2^*}e^{-\frac{q_1^*-\xi}{\alpha-\beta p}}\cdot\frac{1}{\alpha-\beta p}e^{-\frac{\xi}{\alpha-\beta p}}\,d\xi-(c-s)\cdot\int_0^{q_1^*-q_2^*}(q_1^*-\xi)\frac{1}{\alpha-\beta p}e^{-\frac{\xi}{\alpha-\beta p}}\,d\xi\end{aligned}$$

$$= -(p-s)(\alpha-\beta p)\frac{1}{\alpha-\beta p}e^{-\frac{q_1^*}{\alpha-\beta p}} \cdot \int_0^{q_1^*-q_2^*} 1\, d\xi - (c-s)\cdot q_1^* \cdot \Phi_D(q_1^*-q_2^*) + (c-s)\cdot \left[-(\xi+\alpha-\beta p)e^{-\frac{\xi}{\alpha-\beta p}} \right]_0^{q_1^*-q_2^*}$$

$$= -(p-s)e^{-\frac{q_1^*}{\alpha-\beta p}}(q_1^*-q_2^*) - (c-s)\cdot q_1^*\left(1-e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}\right) - (c-s)(q_1^*-q_2^*+\alpha-\beta p)e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}} + (c-s)(\alpha-\beta p)$$

$$= -(p-s)e^{-\frac{q_1^*}{\alpha-\beta p}}(q_1^*-q_2^*) - (c-s)(-q_2^*+\alpha-\beta p)e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}} + (c-s)(\alpha-\beta p-q_1^*).$$

$$(3) = \int_{q_1^*-q_2^*}^{\infty} P_p(D_2, q_2^*)\phi_D(\xi)\, d\xi$$

$$= \int_{q_1^*-q_2^*}^{\infty} \left((p-s)(\alpha-\beta p) - (p-s)(\alpha-\beta p)e^{-\frac{q_2^*}{\alpha-\beta p}} - (c-s)\cdot q_2^* \right)\phi_D(\xi)\, d\xi$$

$$= \left(-(p-s)(\alpha-\beta p)e^{-\frac{q_2^*}{\alpha-\beta p}} - (c-s)\cdot q_2^* \right) \cdot \int_{q_1^*-q_2^*}^{\infty} \phi_D(\xi)\, d\xi$$

$$= \left(-(c-s)(\alpha-\beta p) - (c-s)\cdot q_2^* \right) \cdot e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}.$$

Combining the three parts results in the expected profit function

$$\begin{aligned} P_p(q_1^*) = {} & 2\cdot(p-s)(\alpha-\beta p) - (p-s)(\alpha-\beta p)e^{-\frac{q_1^*}{\alpha-\beta p}} - (c-s)\cdot q \\ & - (p-s)e^{-\frac{q_1^*}{\alpha-\beta p}}(q_1^*-q_2^*) - (c-s)(-q_2^*+\alpha-\beta p)e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}} + (c-s)(\alpha-\beta p-q_1^*) \\ & - (c-s)\cdot q_2^* \cdot e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}} + c\cdot(q_1^*-(\alpha-\beta p)) + c*(q_1^*-(\alpha-\beta p)), \end{aligned} \tag{4.24}$$

where q_1^* and q_2^* are the quantities as before. This further simplifies into

$$\begin{aligned} P_p(q_1^*) = {} & 2\cdot(p-s)(\alpha-\beta p) - 2\cdot(p-s)(\alpha-\beta p)e^{-\frac{q_1^*}{\alpha-\beta p}} - 2\cdot(c-s)\cdot q_1^* \\ & - (p-s)e^{-\frac{q_1^*}{\alpha-\beta p}}(q_1^*-q_2^*) + (c-s)(\alpha-\beta p)\left(1-e^{-\frac{q_1^*-q_2^*}{\alpha-\beta p}}\right) + c*(q_1^*-(\alpha-\beta p)) \end{aligned} \tag{4.25}$$

Solving this equation for both q_1^* and p will lead to the optimal policy.

Conclusion

In this thesis, we have studied a single-period model for perishable products as well as a two-period model for stable products, both with (known) price-dependent demand.

For the single-period model, we have found and proven an expression for the optimal ordering quantity q^* (Eq. (2.4)) for general demand distribution ϕ_D. For uniformly distributed demand, which we discussed in Section 2.2.1, we were not able to find an exact expression for the optimal price p^*. However, we have proven the existence and uniqueness of an optimal price and the code in Appendix A provides a fast way to find p^* for specific situations. As with the uniformly distributed demand, we have proven the existence and uniqueness of an optimal price p^* for an exponentially distributed demand. Again we have provided a code (see Appendix A) to find the exact value of p^* for certain situations since we did not find an expression for p^* for general situations. Lastly, we analyzed the model with normally distributed demand. Opposite to the two other probability distributions, the inverse normal distribution does not have an explicit form. Due to this, we were not able to solve the optimal ordering quantity and optimal price analytically. However, we provided a numerical analysis as well as a MatLab code (See Appendix A) to find q^* and p^* for specific values of the parameters.

For the two-period model with uniformly distributed demand, we found and proved explicit formulas (Eq. (4.9) and Eq.(4.10)) for the optimal ordering quantities q_1^* and q_2^* as well as an explicit formula for the expected profit function (Eq. (4.12)). With the code in Appendix B, we have provided a quick way to find the values of these optimal ordering quantities and the optimal price. In the exponentially distributed case, we only found implicit formulas for the optimal ordering quantity for the first period and the optimal price. If a fast way is found to optimize the profit function for both q_1 and p this problem will also be solved for specific situations.

These models are of course at most an approximation of real-life situations and there is no guarantee using these strategies will lead to the best profit since the chance of having exactly these specific demand distributions is very small. However, if previous demands are known and all very similar or the demand can be approximated in any other way, the results in this thesis can be used to find a good pricing and stocking policy.

References

[1] F. S. Hillier and G. J. Lieberman. *Introduction to Operatoins Research.* Mcgraw-Hill Education-Europe, 2001.

Appendix A

Codes single-period model

A.1 Uniformly distributed demand

```
syms a b c d D p P s S q Q x y Z C0 C1 C2 C3
a = 200; %alpha
b = 5; %beta
c = 5;
s = 1;
S = 1; %sigma
%Since p>s we can multiply Eq. (2.12) by (p-s)^2 and this results in a
%polynomial of degree 3
C0 = a*s^2 + b*c * s^2 - S * sqrt(3) *(c-s)^2; %Value of coefficient of
    p^0
C1 = -2*a*s - 2 * b*s^2 -2 * b * c *s ; %Value of coefficient of p^1
C2 = a + 4*b*s + b * c; %Value of coefficient of p^2
C3 = - 2 * b; %Value of coefficient of p^3

d = [C3 C2 C1 C0]; %Polynomial of our first derivative
Q = roots(d); %Solve for the roots of our equation
x = Q(imag(Q) == 0); %Filter real solutions
p = max(x); %Take the maximum root
q = a - b*p - (S * sqrt(3)) + 2 * S *sqrt(3) * ((p-c)/(p-s)); %Optimal
    ordering quantity
P = (p-c)*(a - b*p) - (p-c)*S*sqrt(3)*(1 -(p-c)/(p-s))^2 - (c-s)*S*sqrt
    (3)*((p-c)/(p-s))^2; %Profit
disp(['p = ',num2str(p)]);
disp(['q = ',num2str(q)]);
disp(['Profit = ', num2str(P)]);

Z = (((c-s)^2 * S*sqrt(3))/b)^(1/3) + s; %p_z

%Plotting our profit function on c < p < a/b
```

```
D = (y - c) * (a - b*y) - (y-c)*S*sqrt(3)* ( 1- ((y-c)/(y-s)))^2 - (c-s
    )* S * sqrt(3)* ((y-c)/(y-s))^2; %Profit function
fplot(D,'LineWidth',1.5);
xlim([0,(a/b)]);
xline(p, '-','LineWidth',1.5);
xline(Z, '-r','LineWidth',1.5);
xline(c, '-g','LineWidth',1.5);
legend({'Profit function','Optimal price p^*', 'p_z', 'Cost c'},'
    Location','southeast');
xlabel('Price p');
ylabel('Profit');
title('Plot of Expected Profit Between 0 and \alpha/\beta');
ax = gca;
ax.FontSize = 20;
```

A.2 Exponentially distributed demand

```
syms a b c d D p P s S q Q x y
a = 200; %alpha
b = 5; %beta
c = 5;
s = 1;

d = (a - b*p) - b*(p-c) - b*(c-s)*(log(c-s) - log(p -s)) - (a - b*p)*((
    c-s)/(p-s)) == 0; %First derivative == 0
Q = vpasolve(d, p); %Solve for the roots of our equation
Q = double(Q);
x = Q(imag(Q) == 0); %Filter real solutions
p = max(x); %Take the maximum root
q = -(a - b*p) * -(log((p-s)/(c-s))); %Optimal ordering quantity
p = round(p,4);
q = round(q,4);
P = (p-c)*(a - b*p) + (c-s)*( a- b*p) * log((c-s)/(p-s)); %Profit
disp(['p = ',num2str(p)]);
disp(['q = ',num2str(q)]);
disp(['Profit = ', num2str(z)]);

%Plotting our profit function on c < p < a/b
D = (y - c)*(a - b*y) + (c-s)*(a - b*y)* (log((c-s)/(y-s))); %Profit
    function
fplot(D, 'LineWidth',1.5); %Plotting
xlim([0,(a/b)]);
xline(p, '-', 'LineWidth',1.5);
xline(c, '-g', 'LineWidth',1.5);
legend({'Profit function','Optimal price p^*', 'Cost c'},'Location','
    northeast');
```

```
xlabel('Price p');
ylabel('Profit');
title('Plot of Expected Profit Between 0 and \alpha/\beta');
ax = gca;
ax.FontSize = 20;
```

A.3 Normally distributed demand

```
syms a b c q F G p P q s S v V W y
a = 200; %alpha
b = 5; %beta
c = 5;
s = 1;
S = 1; %sigma

F = @(y) (y-c) * (a - b*y)- (y-s) * (S/ (sqrt(2*pi)))* exp( - 1/2 * norminv((y-c)/(y-s)).^2); %Expected profit function
G = @(y) norminv((y-c)/(y-s))*S + (a - b*y); %Optimal quantity given the price

P = 0;
V = 0;
v = c; %Starting value of our price
q = (floor(a/b - c)) * 100; %Number of steps of 0.01 between c and a/b
for i = 1:q
    v = v + 0.01; %We take steps of 0.01, since these are steps of 1 cents, which is the smallest step in valuta
    if F(v) > V
        V = F(v);  %We update our maximal value
        P = v;     %We update our optimal price
    end
end
Q = G(P); %Computing our optimal quantity

%Plotting our profit function on c < p < a/b
fplot(F, 'LineWidth',1.5);
xlim([0,(a/b)]);
xline(P, '-', 'LineWidth',1.5);
legend({'Profit function','Optimal price p^*', 'Cost c'},'Location','northeast');
xlabel('Price p');
ylabel('Profit');
title('Plot of Expected Profit Between 0 and \alpha/\beta');
ax = gca;
ax.FontSize = 20;
disp(['p = ',num2str(P)]);
disp(['q = ',num2str(Q)]);
disp(['Profit = ', num2str(V)]);
```

Appendix B

Codes two-period model

B.1 Uniformly distributed demand

```
syms a A A1 b B B1 c p P P1 P2 P3 P4 q1 q2 Q1 Q2 s S
a = 200; %alpha
b = 5; %beta
c = 5;
s = 1;
S = 1; %sigma

A = @(y) a - b*y - (S * sqrt(3)); %a(p)
B = @(y) a - b*y + (S * sqrt(3)); %b(p)
Q2 = @(y) A(y) + 2 * S *sqrt(3) * ((y-c)/(y-s)) %Optimal ordering quantity for the second period
Q1 = @(y) ((2*S*sqrt(3))/(y-s))*( sqrt((y-s)^2 + 2*c*(y-s)) - (c-s)) + A(y); %Optimal ordering quantity for the first period
P1 = @(y)  (y-c)* (a - b*y) - ((y-c)/(4*S*sqrt(3))) * (B(y) - Q1(y))^2 - ((c-s)/(4*S*sqrt(3)))*(Q1(y) - A(y))^2; %(1) of the computation
P2 = @(y) (1/(72*S^2))* (-(y-c) * ((B(y) - Q2(y))^3 - ( B(y) - Q1(y))^3) + (c-s) *((Q2(y) - A(y))^3 - (Q1(y) - A(y))^3)); %(2) of the computation
P3 = @(y) ((2*S*sqrt(3) - Q1(y) + Q2(y))/(2*S*sqrt(3)))*(-(y-c)*S*sqrt(3) * (1 - (y-c)/(y-s))^2 - (c-s)*S*sqrt(3) * ((y-c)/(y-s))^2); %(3) of the computation
P4 = @(y) c * (Q1(y) - (a - b*y)) + (y-c)* (a - b*y); %Rest
P5 = @(y) P1(y) + P2(y) + P3(y) + P4(y); %Total profit function

P = 0;
V = 0;
v = c - 0.01; %Starting value of our price
q = (floor(a/b -c)) * 100; %Number of steps of 0.01 between c and a/b
for i = 1:q
```

```
    v = v + 0.01; %We take steps of 0.01, since these are steps of 1 cents, which is the smallest step in valuta
    if P5(v) > V
        V = P5(v);  %We update our maximal value
        P = v;      %We update our optimal price
    end
end
A1 = a - b*P - (S * sqrt(3)); %a(p) for optimal price
B1 = a - b*P + (S * sqrt(3)); %b(p) for optimal price
q2 = A1 + 2 * S *sqrt(3) * ((P-c)/(P-s)); %Optimal ordering quantity for the second period for optimal price
q1 = ((2*S*sqrt(3))/(P-s))*( sqrt((P-s)^2 + 2*c*(P-s)) - (c-s)) + A1; %Optimal ordering quantity for the first period for optimal price

%Plotting our profit function on c < p < a/b
fplot(P5,'LineWidth',1.5);
xlim([0,(a/b)]);
xline(P, '-', 'p^*','LineWidth',1.5);
xline(c, '-g','LineWidth',1.5);
legend({'Profit function','Optimal price p^*', 'Cost c'},'Location','southeast');
xlabel('Price p');
ylabel('Profit');
title('Plot of Expected Profit Between 0 and \alpha/\beta');
ax = gca;
ax.FontSize = 20;

disp(['p = ',num2str(P)]);
disp(['q1 = ',num2str(q1)]);
disp(['q2 = ',num2str(q2)]);
disp(['Profit = ', num2str(V)]);
```

B.2 Constraint on the salvage value

```
syms a A1 b c p P Q1 Q2 q1 q2 s S
a = 200; %alpha
b = 5; %beta
c = 1;
s = 1;
S = 1; %sigma

A1 = @(y) - 4*b + S*sqrt(3) * ( - (14/((1-y)^2*c*sqrt((3-y)/(1-y)))) + (14/((1-y)*c)) - (6 * sqrt((3-y)/(1-y))/((1-y)*c)) + (4/((1-y)^2*c)) + (2/((1-y)^3*c*((3-y)/(1-y)))) + (2/((1-y)^3*c*((3-y)/(1-y))^(3/2))) + (1/((1-y)^3*c*sqrt((3-y)/(1-y)))) - (2/((1-y)^4*c*((3-y)/(1-y))^(3/2)))); %Second derivative at p=c
```

```

%Plotting the second derivative at c for 0 < gamma < 1
fplot(A1,'LineWidth',1.5);
xlim([0,1]);
legend({'Second derivative at p=c'},'Location','northwest');
xlabel('\gamma');
ylabel('Value of the second derivative');
title('Plot of Second Derivative at p=c for values of \gamma Between 0
    and 1');
ax = gca;
ax.FontSize = 20;

P = 0;
V = 0;
v = -0.001; %Starting value of gamma
for i = 1:1000
    v = v + 0.001; %We take steps of 0.01, since these are steps of 1
        cents, which is the smallest step in valuta
    if A1(v) < 0
        V = A1(v);  %We update our maximal value
        P = v;      %We update our gamma
    end
end
disp(['gamma = ',num2str(P)]);
```