The Friendship Paradox: “Why your friends on average have more friends than you.”

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“Why your friends on average have more friends than you.”

Bachelor thesis

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1 The Friendship Paradox

Network science in an interdisciplinary research area that aims to describe, analyse, control and optimise complex networks that form the backbone of modern society. Interconnectedness is the new paradigm that drives the common interest in such networks, together with the fact that large-scale computing allows for an advanced numerical study of their intrinsic complexity.

In this bachelor thesis we focus on one single topic in network science, namely, the Friendship Paradox. We will highlight several features of this remarkable phenomenon.

1.1 Introduction

"Your friends are more popular than you." It is a painful statement, but surprisingly, it is true. This phenomenon was discovered by the American sociologist Scott Feld in 1991 [1]. The statement is roughly as follows. Say that we have a group of n people, where n is fixed. For each person in this group, we can compute the difference between the average number of friends of all the friends of this person and the average number of friends of this person himself or herself. We are assuming that all the friendships in this group are mutual, i.e., if person i is a friend of person j, then person j is also a friend of person i. If we compute such numbers for all people in the group and take the difference, then we end up with a positive number, which should be interpreted as saying that on average a person is less popular than his or her friends.

The contents of this chapter are as follows. After a brief statement about a possible application of the Friendship Paradox, the reader will be introduced to the tools that will be used throughout this bachelor thesis. The Friendship Paradox will be explained in a more mathematical way and two different versions of the proof will be presented.

1.2 Application of the Friendship Paradox

The Friendship Paradox can be used to implement an effective way to slow down the spread of infectious diseases. Suppose that we have a group of people, with a corresponding "friendship graph" that is not known explicitly, and suppose that an infectious disease breaks out. Suppose also that we only have one vaccine at our disposal and we want to use this vaccine as effectively as possible. In other words, we want to use it on the person with the most friends or contacts. One way we could do this is to choose a random person in our group and give the vaccine to him or her. A different and wiser approach would be to select a person at random, and let him or her choose a friend to give the vaccine to. The more popular persons, i.e., the persons with the most contacts, are more likely to be chosen. Hence, the second approach is more effective in combating this disease.

1.3 Tools needed for the analysis

1.3.1 Type of graphs

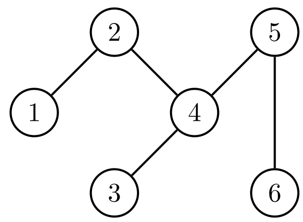
Since the Friendship Paradox is a graph-theoretical topic, we will study graphs as mathematical objects. A graph G is given by a set of vertices, denoted by V, and a set of edges, denoted by E. We will only consider simple graphs.

Definition 1. *An undirected graph $G = (V, E)$ with $V = [n] = \{1, 2, \ldots, n\}$ is called simple if it does not contain self-loops or multiple edges between any two vertices.*

1.3.2 The adjacency matrix

A graph G can be represented either by a drawing or a matrix, called the adjacency matrix A. The matrix element $A(i, j)$ equals the indicator of the event that i and j are connected by an edge.

Example 1. *Consider the following graph $G = (V, E)$ with 6 vertices:*



An equivalent way of denoting this graph is by using the adjacency matrix A, which is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Two facts can be noted. First, for undirected graphs G, adjacency matrices are symmetric (meaning that all friendships are mutual), i.e., $A(i, j) = A(j, i)$ for all $i, j \in V$. Second, the degrees of the vertices are the row sums of A:

$$\forall i \in V: \quad d_i = \sum_{j=1}^{n} A(i, j).$$

1.3.3 The friendship matrix

The friendship matrix F is an important tool for the proof of the Friendship Paradox. In terms of the adjacency matrix, it is given by

$$F(i, j) = A(i, j) \left(\frac{d_j}{d_i} - 1 \right),$$

where we assume that the degree of each vertex is strictly positive, i.e.,

$$\forall i \in V : d_i > 0.$$

1.4 Formulation of the Friendship Paradox

The Friendship Paradox concerns a connected and undirected simple graph $G = (V, E)$, which consists of a set with vertices $V = [n]$ and a set of edges $E = \{(i, j) \mid i, j \in V\}$. Two vertices are either connected via a single edge (they are friends) or are not connected (they are not friends). Self-loops do not occur. The adjacency matrix is:

$$A(i,j) = \begin{cases} 1 \text{ if } (i,j) \in E, \\ 0 \text{ if } (i,j) \notin E. \end{cases}$$

For such a graph G and a given vertex $i \in V$, we can compute the difference between the average degrees of the neighbours (friends) of i and the degree of i itself. This number, which we denote by Δ_i, is called the *friendship bias* of vertex i, and is given by

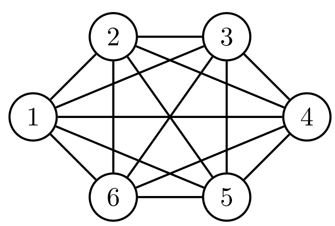
$$\Delta_i = \left[\frac{1}{d_i} \sum_{j=1}^{n} A(i,j) d_j \right] - d_i. \tag{1}$$

The equality only holds in the case where all the degrees are the same, i.e., all persons have exactly the same number of friends.

In Section 1.6, two versions of the proof are provided. In the first proof we use the properties of the adjacency and friendship matrix, while in the second proof we compare the expectation of the degree of a vertex $\bar{U}$ that is uniformly chosen, denoted by $\mathbb{E}[d_{\bar{U}}]$, with the expectation of the degree of a vertex U that is at the end of a uniformly chosen edge, denoted by $\mathbb{E}[d_U]$.

1.5 The Friendship Paradox in practice

Example 2. *Consider a group of 6 people, where every person is a friend of every other person in the group. The friendship graph is the complete graph on 6 vertices:*



The adjacency matrix of this graph is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The degree sequence is obtained by taking the row sums:

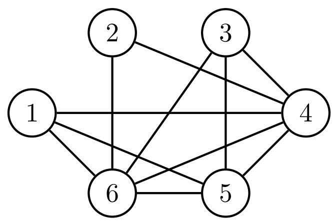
$$d = (d_1, d_2, d_3, d_4, d_5, d_6) = (5, 5, 5, 5, 5, 5)\,.$$

Note that all the degrees are the same, and hence

$$\Delta_i = \frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j = \frac{1}{5}\,(5+5+5+5+5) - 5 = 0, \quad i = 1, \ldots, 6.$$

Since each of the Δ_i is zero, the average value Δ is also zero.

Example 3. *In the group of friends we met in Example 2, there arises a quarrel. As a result, persons 1, 2 and 3 are no longer friends with each other, nor are persons 2 and 5. Now the friendship graph looks different:*



The adjacency matrix and degree sequence are also changed:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \qquad d = (3, 2, 3, 5, 4, 5).$$

Obviously, not every person has the same number of friends. In the next table the Δ_i-values are computed.

i	d_i	$\Delta_i = \left[\frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j\right] - d_i$
1	*3*	$\frac{1}{3}(5+4+5) - 3 = \frac{5}{3}$
2	*2*	$\frac{1}{2}(5+5) - 2 = 3$
3	*3*	$\frac{1}{3}(5+4+5) - 3 = \frac{5}{3}$
4	5	$\frac{1}{5}(3+2+3+4+5) - 5 = -\frac{8}{5}$
5	*4*	$\frac{1}{4}(3+3+5+5) - 4 = 0$
6	5	$\frac{1}{5}(3+2+3+5+4) - 5 = -\frac{8}{5}$

From here Δ can be computed:

$$\Delta = \frac{1}{6}\left(\frac{5}{3} + 3 + \frac{5}{3} - \frac{8}{5} + 0 - \frac{8}{5}\right) = \frac{47}{90} > 0.$$

In Examples 2 and 3 we have seen that Δ is zero if every person in the graph has the same number of friends, but greater than zero if this is not the case. The strictly positive number as outcome in Example 3 is not a coincidence: it is exactly what the Friendship Paradox has to offer. In the next section we move from examples to proofs. As stated before, two different version of the proof will be provided.

1.6 Proof of the Friendship Paradox

1.6.1 Vertex version of the proof

Theorem 1. *For any connected and undirected simple graph $G = (V, E)$, the Friendship Paradox holds:*

$$\Delta = \sum_{i=1}^{n} \Delta_i \geq 0,$$

where Δ_i is defined in equation (1)*.*

Proof. As a first step, we write out Δ in terms of A:

$$\begin{aligned}
\Delta = \frac{1}{n}\sum_{i=1}^{n}\Delta_i &= \frac{1}{n}\sum_{i=1}^{n}\left(\left[\frac{1}{d_i}\sum_{j=1}^{n}A(i,j)d_j\right] - d_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{d_i}\sum_{j=1}^{n}A(i,j)d_j - \sum_{j=1}^{n}A(i,j)\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}A(i,j)\left(\frac{d_j}{d_i} - 1\right).
\end{aligned}$$

If we exploit symmetry, then the above expression becomes

$$\begin{aligned}
\Delta = \frac{1}{n}\sum_{i=1}^{n}\Delta_i &= \frac{1}{2n}\sum_{i,j=1}^{n}A(i,j)\left(\frac{d_j}{d_i} + \frac{d_i}{d_j} - 2\right) \\
&= \frac{1}{2n}\sum_{i,j=1}^{n}A(i,j)\left(\sqrt{\frac{d_j}{d_i}} - \sqrt{\frac{d_i}{d_j}}\right)^2 \geq 0.
\end{aligned}$$

The only case in which $\Delta = 0$ is when the degree of each vertex the same, i.e., $d_i = d_j$ for all $i, j \in [n]$. In other words, all the persons in the graph have the same number of friends. □

1.6.2 Edge version of the proof

Theorem 2. *Let $G = (V, E)$ be an undirected and connected simple graph with vertex set $V = [n]$. Let $\bar{U}$ be a uniformly chosen vertex from $[n]$, and let $e = (U, V)$ be a uniformly chosen edge from E. Then $\mathbb{E}[d_U] \geq \mathbb{E}[d_{\bar{U}}] \geq 0$. The interpretation of this inequality is that the number of friends of an individual in a randomly chosen friendship is greater than or equal to that of a randomly chosen individual.*

Proof. The expected value of the degree of vertex $\bar{U}$ is

$$\mathbb{E}(d_{\bar{U}}) = \frac{1}{n} \sum_{i=1}^{n} d_i.$$

To calculate the expected value of the degree of vertex U, we write

$$P(d_U = k) = \sum_{i,j=1}^{n} P\left(e = (i,j)\right) \mathbb{1}_{\{d(i)=k\}}. \tag{2}$$

The probability in the sum can be expressed as

$$P(e = (i,j)) = \frac{1}{2|E|} \mathbb{1}_{\{(i,j) \text{ is an edge}\}} = \frac{A(i,j)}{2|E|}.$$

We can substitute this into (2), to get

$$\begin{aligned} P(d_U = k) = \sum_{i,j=1}^{n} \frac{A(i,j)}{2|E|} \mathbb{1}_{\{d(i)=k\}} &= \frac{n}{2|E|} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{d(i)=k\}} \sum_{j=1}^{n} A(i,j) \\ = \frac{n}{2|E|} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{d(i)=k\}} d(i) &= \frac{n}{2|E|} \frac{1}{n} \sum_{i=1}^{n} k \mathbb{1}_{\{d(i)=k\}} \\ = \frac{n}{2|E|} k \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{d(i)=k\}} &= \frac{n}{2|E|} k p_k. \end{aligned}$$

In the last line above, p_k is nothing other than the fraction of vertices with degree k. If we write

$$\frac{n}{2|E|} := c,$$

then we obtain the following:

$$1 = \sum_{k=1}^{\infty} P(d_U = k) = c \sum_{k=1}^{\infty} k p_k \Longrightarrow c = \frac{1}{\sum_{k=1}^{\infty} k p_k}.$$

With this notation, we observe that $P(d_U = k)$ can be written as

$$P(d_U = k) = \frac{k p_k}{\sum_{k=1}^{\infty} k p_k}.$$

We can say a bit more about the summation in the denominator, because this turns out to be nothing other than $\mathbb{E}[d_{\bar{U}}]$:

$$\begin{aligned}\sum_{k=1}^{\infty} kp_k &= \sum_{k=1}^{\infty} k\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{d(i)=k\}} = \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{\infty} k\mathbb{1}_{\{d(i)=k\}} \\ &= \frac{1}{n}\sum_{i=1}^{n} d(i) = \mathbb{E}[d_{\bar{U}}].\end{aligned}$$

Now we are able to express $P(d_U = k)$ in terms of $\mathbb{E}(d_{\bar{U}})$:

$$P(d_U = k) = \frac{kp_k}{\mathbb{E}[d_{\bar{U}}]} \tag{3}$$

For the numerator of (3), we write

$$\begin{aligned}\mathbb{E}[d_U] &= \sum_{k=1}^{\infty} kP(d_U = k) = \sum_{k=1}^{\infty} k\frac{kp_k}{\mathbb{E}[d_{\bar{U}}]} \\ &= \sum_{k=1}^{\infty} \frac{k^2 p_k}{\mathbb{E}[d_{\bar{U}}]} = \frac{1}{\mathbb{E}[d_{\bar{U}}]}\sum_{k=1}^{\infty} k^2 p_k.\end{aligned} \tag{4}$$

In order to compare $\mathbb{E}[d_U]$ with $\mathbb{E}[d_{\bar{U}}]$, we need to find an expression for $\mathbb{E}\left[(d_{\bar{U}})^2\right]$ in terms of p_k. This leads to

$$\begin{aligned}\mathbb{E}\left[(d_{\bar{U}})^2\right] &= \frac{1}{n}\sum_{i=1}^{n} d(i)^2 = \frac{1}{n}\sum_{i=1}^{n} d(i)^2 \sum_{k=1}^{\infty}\mathbb{1}_{\{d(i)=k\}} \\ &= \sum_{k=1}^{\infty} k^2 \frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{d(i)=k\}} = \sum_{k=1}^{\infty} k^2 p_k.\end{aligned}$$

We substitute this result into (4), to get

$$\mathbb{E}[d_U] = \frac{\mathbb{E}\left[(d_{\bar{U}})^2\right]}{\mathbb{E}[d_{\bar{U}}]}.$$

If we now use that for a random variable X the variance is defined as $\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$, then we can rewrite this fraction as

$$\mathbb{E}[d_U] = \frac{\mathrm{Var}(d_{\bar{U}}) + \mathbb{E}[d_{\bar{U}}]^2}{\mathbb{E}[d_{\bar{U}}]} = \mathbb{E}[d_{\bar{U}}] + \frac{\mathrm{Var}(d_{\bar{U}})}{\mathbb{E}[d_{\bar{U}}]}.$$

Since the variance is non-negative, this leads to the conclusion that $\mathbb{E}[d_U] \geq \mathbb{E}[d_{\bar{U}}]$. $\square$

1.7 Formulation of the problem

Up until now, only deterministic graphs were considered. It is more interesting to investigate how quantities like Δ_i and Δ behave when we consider random graphs. In that case, Δ_i and Δ are no longer fixed, because they become random variables! Different types of random graphs will be studied, for instance, the Erdős–Rényi random graph, both homogeneous (Chapter 2) and inhomogeneous (Chapter 3), and the Configuration Model (Chapter 3). For the homogeneous Erdős–Rényi random graph, a probability distribution for Δ_i can be derived, including its expectation. In Chapter 4, simulations in Matlab will be used to verify the correctness of some of the computations done in the previous chapters. Also, the behaviour of the eigenvalues of the friendship matrix will be investigated.

2 The Erdős–Rényi random graph model

In this chapter we move from deterministic graphs to random graphs. The first class of random graphs that will be considered is the one constructed in 1959 by Hungarian mathematicians Paul Erdős and Alfréd Rényi. In what follows, the probability theoretical properties of the degree of a vertex are discussed, followed by the specification of a slightly different random graph model.

In this setting, a probability distribution of Δ_i and its expectation can be computed. For the calculation of the expectation, we first assume that the degrees are all strictly greater than zero. This is not always the case, and therefore in a separate section we take care of this discrepancy.

2.1 Description of the Erdős–Rényi random graph model

Erdős and Rényi made the assumption that for a fixed group of people of size n, each pair of persons become mutual friends with probability $p \in [0, 1]$. This p is the same for every pair.

Definition 2. *An Erdős–Rényi random graph, denoted by $ER_n(p)$, is generated as follows: for a vertex set $V = [n] = \{1, 2, \ldots, n\}$, where the number of vertices n is fixed, each pair of vertices i and j with $i \neq j$ is connected via a single edge with probability p, independently of each other.*

In other words, each edge follows a Bernoulli distribution with parameter p. A consequence of this property is that the distribution of the degree of each vertex is the sum of $n - 1$ Bernoulli(p) random variables, i.e., a Binomial($n - 1, p$) random variable. Here we make the assumption that $A(i, i) = 0$ for all vertices $i \in V$, meaning that people cannot be friends with themselves.

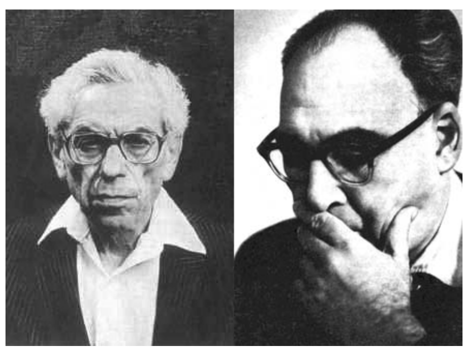


Figure 1: Paul Erdős (left) and Alfréd Rényi (right).

2.2 Behaviour of the degrees

Before proceeding with the search for a probability distribution, it is good to start with an investigation of the properties of the degrees of Erdős–Rényi random graphs. In this section the following quantities will be computed explicitly.

- The expectation and variance of d_i: $\mathbb{E}[d_i]$ and $\mathrm{Var}(d_i)$.
- The covariance of d_i and d_j: $\mathrm{Cov}(d_i, d_j)$.
- The correlation coefficient of d_i and d_j: ρ_{d_i,d_j}.

2.2.1 Expectation and variance

By the construction of Erdős–Rényi random graphs, the expected value of the degree is the same for every vertex in the graph:

$$\mathbb{E}[d_i] = \mathbb{E}\left[\sum_{\substack{j=1\\j\neq i}}^{n} A(i,j)\right] = \sum_{\substack{j=1\\j\neq i}}^{n} \mathbb{E}\left[A(i,j)\right] = (n-1)(1p+0(1-p)) = (n-1)p.$$

The variance is determined as follows. Again, it is not important which vertex is considered, because also the variance is the same for every vertex in the graph:

$$\mathrm{Var}(d_i) = \mathbb{E}\left(d_i^2\right) - \mathbb{E}(d_i)^2 = \mathbb{E}\left(d_i^2\right) - (n-1)^2p^2.$$

The hardest part of the computation is to determine the value of $\mathbb{E}(d_i^2)$. Write

$$\mathbb{E}\left(d_i^2\right) = \mathbb{E}\left[\left(\sum_{\substack{j=1\\j\neq i}}^{n} A(i,j)\right)^2\right] = \mathbb{E}\left[\sum_{\substack{j,j'=1\\j,j'\neq i}}^{n} A(i,j)A(i,j')\right] = \sum_{\substack{j,j'=1\\j,j'\neq i}}^{n} \mathbb{E}\left[A(i,j)A(i,j')\right].$$

We split the sum into two parts. For the first part we take $j = j'$ and for the second part $j \neq j'$. This gives

$$\begin{aligned}\mathbb{E}\left(d_i^2\right) &= \sum_{\substack{j=1\\j\neq i}}^{n} \mathbb{E}\left[A(i,j)^2\right] + \sum_{\substack{j,j'=1\\j,j'\neq i\\j\neq j'}}^{n} \mathbb{E}\left[A(i,j)A(i,j')\right]\\ &= (n-1)(1^2p+0^2(1-p)) + (n-1)(n-2)(1p^2+0p(1-p)+0(1-p)^2)\\ &= (n-1)p + (n-1)(n-2)p^2.\end{aligned}$$

Substituting this last expression into the expression for the variance, we end up with

$$\begin{aligned}\mathrm{Var}(d_i) &= (n-1)p + (n-1)(n-2)p^2 - (n-1)^2p^2\\ &= (n-1)p\left[1+(n-2)p-(n-1)p\right]\\ &= (n-1)p\left[1+np-2p-np+p\right]\\ &= (n-1)p(1-p).\end{aligned}$$

2.2.2 Covariance

The covariance between two different vertices i and j consists of three terms of which two are already known by now:

$$\mathrm{Cov}\left(d_i, d_j\right) = \mathbb{E}\left[d_id_j\right] - \mathbb{E}[d_i]\,\mathbb{E}\left[d_j\right] = \mathbb{E}\left[d_id_j\right] - (n-1)^2p^2. \tag{5}$$

The correlation coefficient has the same value for every pair of vertices in the graph. The remaining task is to determine the value of $\mathbb{E}\left[d_i d_j\right]$:

$$\mathbb{E}\left[d_i d_j\right] = \mathbb{E}\left[\sum_{\substack{k=1\\k\neq i}}^{n} A(i,k) \sum_{\substack{l=1\\l\neq j}}^{n} A(j,l)\right] = (\star)$$

We distinguish between four cases:

- $k = j$ and $l = i$
- $k = j$ and $l \neq i$
- $k \neq j$ and $l = i$
- $k \neq j$ and $l \neq i$.

Using this partitioning, we can split $\mathbb{E}\left[d_i d_j\right]$ into four parts:

$$\begin{aligned}
(\star) &= \mathbb{E}\left[A(i,j)A(j,i)\right] + \mathbb{E}\left[\sum_{\substack{l=1\\l\neq i,j}}^{n} A(i,j)A(l,j)\right] + \mathbb{E}\left[\sum_{\substack{k=1\\k\neq i,j}}^{n} A(i,k)A(j,i)\right] \\
&\quad + \mathbb{E}\left[\sum_{\substack{k=1\\k\neq i,j}}^{n} A(i,k) \sum_{\substack{l=1\\l\neq i,j}}^{n} A(l,j)\right] \\
&= \mathbb{E}\left[A(i,j)^2\right] + \sum_{\substack{l=1\\l\neq i,j}}^{n} \mathbb{E}\left[(A(i,j)A(l,j)\right] + \sum_{\substack{k=1\\k\neq i,j}}^{n} \mathbb{E}\left[(A(i,k)A(j,i)\right] \\
&\quad + \sum_{\substack{k=1\\k\neq i,j}}^{n} \mathbb{E}\left[(A(i,k)\right] \sum_{\substack{l=1\\l\neq i,j}}^{n} \mathbb{E}\left[(A(l,j)\right].
\end{aligned}$$

If we write out all the expectations and collect all the terms, then we obtain

$$\begin{aligned}
(\star) &= \left(1^2 p + 0^2(1-p)\right) + 2(n-2)\left(1p^2 + 0(1-p)2p + 0(1-p)^2\right) + \left((n-2)p\right)^2 \\
&= p + 2(n-2)p^2 + (n-2)^2p^2 = p + \left(2n - 4 + n^2 - 4n + 4\right)p^2 \\
&= p + \left(n^2 - 2n\right)p^2 = p + n(n-2)p^2.
\end{aligned}$$

Inserting this expression into (5), we find the desired covariance.

$$\begin{aligned}
\mathrm{Cov}\left(d_i, d_j\right) &= p + n(n-2)p^2 - (n-1)^2p^2 \\
&= p\left(1 + (n^2-2n)p - (n^2-2n+1)p\right) = p(1-p).
\end{aligned}$$

2.2.3 Correlation coefficient

Our next goal is to calculate the correlation coefficient ρ, which is defined and computed as follows:

$$\rho_{d_i,d_j} = \frac{\mathrm{Cov}\left(d_i, d_j\right)}{\sqrt{\mathrm{Var}(d_i)} \cdot \sqrt{\mathrm{Var}(d_j)}} = \frac{(1-p)p}{(n-1)p(1-p)} = \frac{1}{n-1}, i \neq j.$$

We note two facts. First, the correlation coefficient is strictly positive and does not depend on the connection probability, only on the number of vertices. Second, $\rho_{d_i,d_j} \downarrow 0$ as $n \to \infty$. The interpretation of this fact is that, asymptotically, the degrees of two vertices do not depend on one another.

2.3 From Binomial to Poisson

We previously mentioned that for the Erdős–Rényi random graph the degree of each vertex has a Binomial distribution with parameters $n-1$ and p. There are situations in which p depends on the number of vertices. A very natural choice is to set the connection probability equal to $p = \frac{\lambda}{n-1}$ for a given parameter $\lambda \in (0, \infty)$. This gives rise to the next theorem.

Theorem 3. *For the case where $p = \frac{\lambda}{n-1}$, the binomial distribution with parameters $n-1$ and p converges in the limit as $n \to \infty$ to the Poisson(λ) distribution, i.e.,*

$$P\left(\mathrm{Binom}(n-1,p) = k\right) = \binom{n-1}{k} p^k \left(1-p\right)^{n-1-k} \longrightarrow \frac{\lambda^k}{k!} e^{-\lambda} = P\left(\mathrm{Poisson}(\lambda) = k\right).$$

Proof. For $p = \frac{\lambda}{n-1}$ we can write

$$\begin{aligned} P(d_i = k) &= \binom{n-1}{k} p^k (1-p)^{n-1-k} = \frac{(n-1)!}{(n-1-k)!k!} \left(\frac{\lambda}{n-1}\right)^k \left(1 - \frac{\lambda}{n-1}\right)^{n-1-k} \\ &= \frac{\lambda^k}{k!} \frac{(n-1)!}{(n-1-k)!(n-1)^k} \left(1 - \frac{\lambda}{n-1}\right)^{n-1} \left(1 - \frac{\lambda}{n-1}\right)^{-k}. \end{aligned} \tag{6}$$

Three of the four factors in the above expression depend on n. We check what happens with each of these terms when $n \to \infty$.

- $$\lim_{n\to\infty} \frac{(n-1)!}{(n-1-k)!(n-1)^k} = \lim_{n\to\infty} \frac{(n-1)(n-2)\dots(n-k)}{(n-1)^k} = 1$$

- $$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n-1}\right)^{n-1} = e^{-\lambda}$$

- $$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n-1}\right)^{-k} = 1$$

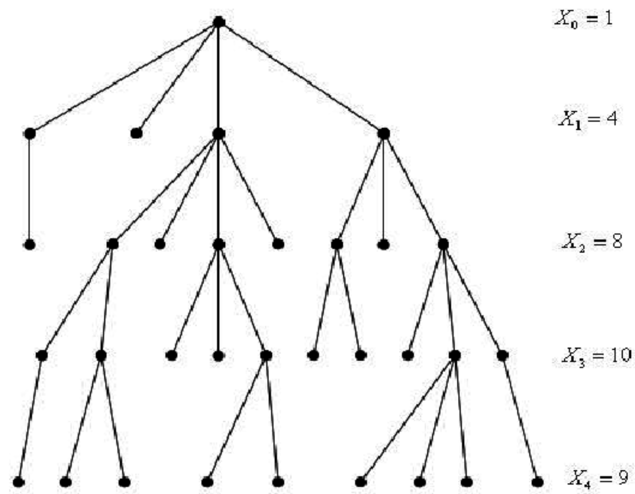
With the computation of these three limits, we can finish the proof by substituting them into (6):

$$\lim_{n\to\infty}(\star) = \frac{\lambda^k}{k!}e^{-\lambda} = P\left(\text{Poisson}(\lambda) = k\right).$$

□

In the limit as $n \to \infty$, we move from one random graph model to another. In particular, we move from the Erdős–Rényi random graph model to the Poisson random graph model, which looks locally tree-like. In fact, the latter corresponds to a Galton-Watson tree with offspring distribution Poisson(λ).

Example 4. *In this example, we consider a Galton-Watson tree. First draw the root of the tree. This root has a Poisson(λ) number of friends (here, 4). Each of these friends also generates a Poisson(λ) number of friends (here, respectively,* $1, 0, 4$ *and* 3*). Etc.*



2.4 Analysis of the distribution of the friendship bias

The knowledge obtained about the degrees in the Erdős–Rényi random graph allows us to start looking for the probability distribution of Δ_i. The ideas of the paper written by G. Cantwell, A. Kirkley and M. Newman [2] are used in this section. For now, only the n-dependent connection probability case is of interest to us, i.e.,

$$p = \frac{\lambda}{n-1}.$$

As an abbreviation, we will write p_k for the probability that a given vertex has degree equal to k (in the limit as $n \to \infty$, this is also the fraction of vertices with degree equal to k):

$$P(d_i = k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

For $n \to \infty$, we have already seen that Binomial$(n-1, p)$ converges to Poisson(λ), for which

$$p_k = \frac{\lambda^k}{k!} e^{-\lambda}.$$

The law of total probability allows us to write down the probability that Δ_i is equal to some rational number r:

$$P(\Delta_i = r) = \sum_k P\left(\Delta_i = r \mid d_i = k\right) P(d_i = k) = \sum_k p_k P\left(\Delta_i = r \mid d_i = k\right). \tag{7}$$

Given that $\Delta_i = r$ and $d_i = k$, the following equality can be obtained after we write out the definition of Δ_i:

$$r = \Delta_i = \frac{1}{k} \sum_{j=1}^{n} A_{ij} d_j - k \iff \sum_{j=1}^{n} A_{ij} d_j = rk + k^2 =: K_0.$$

Using this equality, we can rewrite (7) as

$$P(\Delta_i = r) = \sum_k p_k P\left(K_0 = rk + k^2\right). \tag{8}$$

We know that K_0 is an integer, with $K_0 \geq k$, since every neighbour of a vertex has to have a degree that is one or higher (we recall our assumption that there are no isolated vertices). Obviously, we cannot have all possible values of r into this formula. In particular, the only allowed values of r that satisfy $m + k = K_0 = rk + k^2$ are rationals of the form

$$r = 1 + \frac{m}{k} - k,$$

where m is a non-negative integer.

K_0 is the distribution of neighbouring degrees, which is not just p_k. By definition, a neighbour is a vertex at the end of an outgoing edge. Since a vertex of degree k is the final destination of k edges, the degree distribution is proportional to kp_k. Therefore, with the help of a normalizing constant N, the degree distribution for neighbouring degrees is of the form

$$q_l = P\left(K_0 = l\right) = \frac{lp_l}{\sum_j jp_j} = \frac{l\frac{\lambda^l}{l!}e^{-\lambda}}{N} = \frac{\lambda^l e^{-\lambda}}{(l-1)!N}.$$

The normalizing factor N can be computed by using that the sum of the q_l has to be equal to 1:

$$\sum_{l=1}^{\infty} q_l = 1 \Rightarrow \frac{1}{N}\sum_{l=1}^{\infty}\frac{\lambda^l e^{-\lambda}}{(l-1)!} = \frac{e^{-\lambda}}{N}\sum_{l=1}^{\infty}\frac{\lambda^l}{(l-1)!} = \frac{e^{-\lambda}\lambda}{N}\sum_{l=1}^{\infty}\frac{\lambda^{l-1}}{(l-1)!}.$$

After introducing a shift of index $l' = l-1$, we see that the summation is nothing other than $e^{-\lambda}$:

$$\sum_{l=1}^{\infty} q_l = \frac{e^{-\lambda}\lambda}{N}\sum_{l'=0}^{\infty}\frac{\lambda^{l'}}{l'!} = \frac{e^{-\lambda}\lambda e^{\lambda}}{N} = \frac{\lambda}{N} = 1 \Longrightarrow N = \lambda.$$

Now, an explicit expression for q_l can be found if $z = \lambda$ is substituted into the q_l term:

$$q_l = \frac{\lambda^l e^{-\lambda}}{(l-1)!\lambda} = \frac{\lambda^{l-1}e^{-\lambda}}{(l-1)!}.$$

We recognize this as a Poisson random variable shifted one unit to the right:

$$P\Big(d_j - 1 = k\Big) = P\Big(d_j = k+1\Big) = q_{k+1} = \frac{\lambda^k e^{-\lambda}}{k!}.$$

For the next part of the computation, the following theorem is needed, which is stated without proof.

Theorem 4. *Let $X_1, X_2, \ldots, X_n$ be independent Poisson random variables with parameters, respectively, $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then their sum $X = \sum_{i=1}^n X_i$ is again a Poisson random variable, with parameter $\lambda = \sum_{i=1}^n \lambda_i$.*

With Theorem 4 and the fact that the degrees do not depend on each other, the following chain of equalities can be derived:

$$\begin{aligned} P\Big(K_0 = rk + k^2\Big) &= P\left(\sum_{j=1}^{n} A(i,j)d_j = rk + k^2\right) \\ &= P\left(k + \sum_{j=1}^{n} A(i,j)(d_j - 1) = rk + k^2\right) \\ &= P\left(\sum_{j=1}^{n} A(i,j)(d_j - 1) = rk + k^2 - k\right) \\ &=^{\text{Thm.4}} P\Big(\text{Poi}\left(k\lambda\right) = rk + k^2 - k\Big) \\ &= \frac{(k\lambda)^{rk+k^2-k}e^{-k\lambda}}{(rk+k^2-k)!}. \end{aligned} \tag{9}$$

Here we use the fact that $d_i = k$, combined with the fact that:

$$\sum_{j=1}^{n} A(i,j)d_j = \sum_{j=1}^{n} A(i,j) + \sum_{j=1}^{n} \left(d_j - 1\right) = d_i + \sum_{j=1}^{n} A(i,j) \left(d_j - 1\right) = k + \sum_{j=1}^{n} A(i,j) \left(d_j - 1\right).$$

This leads to a formula for the probability distribution of Δ_i as an infinite sum (see (8)):

$$P\left(\Delta_i = r\right) = \sum_{k=1}^{\infty} p_k \frac{(k\lambda)^{rk+k^2-k} e^{-k\lambda}}{(rk+k^2-k)!}.$$

2.5 The expectation of the friendship bias

It is interesting to investigate the expectation of both Δ_i and Δ. In the first step of the following computation the law of total expectation is used, where we condition on the degree of a given vertex i. From there the definition of Δ_i is written out. Indeed,

$$\begin{aligned}
\mathbb{E}\left[\Delta_i\right] &= \sum_{k=1}^{\infty} p_k \mathbb{E}\left[\Delta_i \mid d_i = k\right] \\
&= \sum_{k=1}^{\infty} p_k \mathbb{E}\left[\frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j - d_i \mid d_i = k\right] \\
&= \sum_{k=1}^{\infty} p_k \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{n} A(i,j)d_j - k \mid d_i = k\right].
\end{aligned}$$

The expectation in the above line can be split into two parts, where we observe that the second part is a constant:

$$\begin{aligned}
\mathbb{E}\left[\Delta_i\right] &= \sum_{k=1}^{\infty} p_k \left(\mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{n} A(i,j)d_j \mid d_i = k\right] - \mathbb{E}\left[k \mid d_i = k\right]\right) \\
&= \sum_{k=1}^{\infty} p_k \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{n} A(i,j)d_j \mid d_i = k\right] - \sum_{k=1}^{\infty} k p_k. \qquad (10)
\end{aligned}$$

In the upcoming lines, we again use that k is a constant, so we can pull it out of the expectation. The infinite sum in (10) is just the expectation of a Poisson(λ) random variable, which is equal to λ. Furthermore, we add and subtract d_i in order to obtain an expectation that can be split up again:

$$\begin{aligned}
\mathbb{E}\left[\Delta_i\right] &= \sum_{k=1}^{\infty} \frac{p_k}{k} \mathbb{E}\left[\sum_{j=1}^{n} A(i,j)d_j \mid d_i = k\right] - \lambda \\
&= \sum_{k=1}^{\infty} \frac{p_k}{k} \mathbb{E}\left[d_i - d_i + \sum_{j=1}^{n} A(i,j)d_j \mid d_i = k\right] - \lambda \\
&= \sum_{k=1}^{\infty} \frac{p_k}{k} \mathbb{E}\left[d_i - \sum_{j=1}^{n} A(i,j) + \sum_{j=1}^{n} A(i,j)d_j \mid d_i = k\right] - \lambda \\
&= \sum_{k=1}^{\infty} \frac{p_k}{k} \mathbb{E}\left[d_i + \sum_{j=1}^{n} A(i,j)(d_j - 1) \mid d_i = k\right] - \lambda \\
&= \sum_{k=1}^{\infty} \frac{p_k}{k} \left(\mathbb{E}\left[d_i \mid d_i = k\right] + \mathbb{E}\left[\sum_{j=1}^{n} A(i,j)(d_j - 1) \mid d_i = k\right]\right) - \lambda. \qquad (11)
\end{aligned}$$

The next line contains two sums, because we split up (11). The first sum is nothing other than the number 1, while the expectation inside the second sum is the expectation of a Poisson random variable with parameter $k\lambda$:

$$\begin{aligned}
\mathbb{E}\left[\Delta_i\right] &= \sum_{k=1}^{\infty} \frac{p_k}{k} k + \sum_{k=1}^{\infty} \frac{p_k}{k} \mathbb{E}\left[\sum_{j=1}^{n} A(i,j)(d_j - 1) \mid d_i = k\right] - \lambda \\
&= 1 - \lambda + \sum_{k=1}^{\infty} \frac{p_k}{k} \mathbb{E}\left[\sum_{j=1}^{n} A(i,j)(d_j - 1) \mid d_i = k\right]. \qquad (12)
\end{aligned}$$

After observing that some terms in (12) cancel each other, we find a surprisingly 'easy-looking' result:

$$\mathbb{E}\left[\Delta_i\right] = 1 - \lambda + \sum_{k=1}^{\infty} \frac{p_k}{k} k\lambda = 1 - \lambda + \lambda \sum_{k=1}^{\infty} p_k = 1 - \lambda + \lambda = 1.$$

Using this result, we can also compute the expectation of Δ:

$$\mathbb{E}\left[\Delta\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} \Delta_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\Delta_i\right] = \frac{1}{n} n = 1.$$

2.6 About the (possibly) isolated vertices

In the previous computations, we assumed that the degree of a given vertex i is strictly positive, even though it is possible that a vertex under both the Erdős–Rényi and the Poisson random graph model ends up with zero friends.

By definition, the probability that a vertex i has degree zero under the assumption that pairs of vertices have a connection probability p is

$$P\left(d_i = 0\right) = \left(1-p\right)^{n-1}.$$

For the case where the connection probability does not depend on the number of vertices, this probability tends to zero exponentially fast. For the n-dependent case, however, this is not the case:

$$P\left(d_i = 0\right) = \left(1 - \frac{\lambda}{n-1}\right)^{n-1} \overset{n\to\infty}{\longrightarrow} e^{-\lambda}.$$

If we want to compute Δ_i for a given graph, then dividing by the degree of vertex i is necessary. For this reason, Δ_i is no longer properly defined for isolated vertices. In order to still take these vertices into account, we employ a new definition of Δ_i, denoted by $\Delta_i^\star$:

$$\Delta_i^\star = \begin{cases} \frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j - d_i & \text{if } d_i > 0, \\ 1 & \text{if } d_i = 0. \end{cases}$$

With this new definition of the friendship bias, $\Delta^\star$ is defined as the arithmetic mean of the $\Delta_i^\star$. Assigning the value 1 to isolated vertices is natural, since this value was obtained in Section 2.4 for $\mathbb{E}\left[\Delta_i\right]$. So, essentially we give isolated vertices the same Δ_i as the average of Δ_i of the vertices that are connected. Our main interest is the value of the expectation of the friendship bias, both for the entire graph and that of a specific vertex i.

Theorem 5. *For the above choice of* $\Delta_i^\star$*,*

$$\mathbb{E}\left[\Delta^\star\right] = \mathbb{E}\left[\Delta_i^\star\right] = \begin{cases} 1-p & \text{for } \mathrm{ER}_n(p), \\ 1-\lambda e^{-\lambda} & \text{for Poisson}(\lambda). \end{cases}$$

Proof. In the proof, the law of total expectation is used by conditioning on the degree of a vertex. This idea originated from a paper written by S. Pal, F. Yu, Y. Novick, A. Swami and A. Bar-Noy [3], who did something similar on page 10 of their paper. The first step is to write out the definition of $\Delta_i^\star$ and split the expectation into three parts:

$$\begin{aligned} \mathbb{E}\left[\Delta_i^\star\right] &= \mathbb{E}\left[\left(\frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j - d_i\right)\mathbb{1}_{\{d(i)>0\}} + \mathbb{1}_{\{d(i)=0\}}\right] \\ &= \mathbb{E}\left[\left(\frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j - d_i\right)\mathbb{1}_{\{d(i)>0\}}\right] + P(d_i = 0) \\ &= \mathbb{E}\left[\frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j\mathbb{1}_{\{d(i)>0\}}\right] - \mathbb{E}\left[d_i\mathbb{1}_{\{d(i)>0\}}\right] + P(d_i = 0). \end{aligned}$$

Each of the three terms will be analysed separately.

We start with the probability that $d_i = 0$.

- For the Erdős–Rényi random graph this term vanishes, since it goes to zero exponentially fast as $n \to \infty$.
- For the Poisson random graph, this term converges to $e^{-\lambda}$.

Next, we move to the computation of the expectation of the sum. In order to do this, the introduction of extra notation comes in handy. Let N_i denote the set of neighbours of vertex i. Write $i \sim j$ if vertices i and j are connected. Then we can write

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{n} A(i,j) d_j \mid d_i = k\right] P\left(d_i = k\right) &= \sum_{k=1}^{\infty} \mathbb{E}\left[\frac{1}{k}\sum_{j\in N_i} d_j \mid d_i = k\right] P\left(d_i = k\right)\\ &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j\in N_i} \mathbb{E}\left[d_j \mid d_i = k\right] P\left(d_i = k\right). \qquad (13)\end{aligned}$$

To derive (13), we use once more that conditioning on the degree allows us to pull the factor $\frac{1}{k}$ out of the expectation, as well as the summation over the friends of vertex i. Furthermore, since we have the sum over the friends of vertex i, the d_j term inside the expectation is at least equal to 1, as i is a friend of j. For the other $n-2$ vertices $l \in V \setminus \{i,j\}$, we need to take the expectation of the sum of the indicator functions:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k} \sum_{j\in N_i} \mathbb{E}\left[d_j \mid d_i = k\right] P\left(d_i = k\right) &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j\in N_i} \mathbb{E}\left[1 + \sum_{l\in V\setminus\{i,j\}} \mathbb{1}_{\{l\sim j\}} \mid d_i = k\right] P\left(d_i = k\right)\\ &= \sum_{k=1}^{\infty} \frac{1}{k}\left[k + k(n-2)p\right] P\left(d_i = k\right).\end{aligned}$$

The factor k cancels, and we end up with

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k}\left[k + k(n-2)p\right] P\left(d_i = k\right) &= \left[1 + (n-2)p\right] \sum_{k=1}^{n} P\left(d_i = k\right)\\ &= \left[1 + (n-2)p\right] P\left(d_i > 0\right)\\ &= \left[1 + (n-2)p\right]\left[1 - P\left(d_i = 0\right)\right].\end{aligned}$$

- For the Erdős–Rényi random graph this term scales like $\left[1 + (n-2)p\right]$.
- For the Poisson random graph we observe that $(n-2)p$ converges to λ and so this term converges to $[1+\lambda]\left[1 - e^{-\lambda}\right]$.

What remains to be done is to compute the quantity $\mathbb{E}\left[d_i \mathbb{1}_{d_i>0}\right]$.

- For the Erdős–Rényi random graph, this is nothing other than the expectation of the binomial distribution with parameters $n-1$ and p, i.e.,

$$\begin{aligned}\mathbb{E}\left[d_i \mathbb{1}_{d_i>0}\right] &= \sum_{k=1}^{\infty} kP\left(d_i = k\right) = \sum_{k=1}^{\infty} k\binom{n-1}{k} p^k \left(1-p\right)^{n-1-k} \\ &= \sum_{k=0}^{\infty} k\binom{n-1}{k} p^k \left(1-p\right)^{n-1-k} = \mathbb{E}\left[\text{Binom}(n-1,p)\right] \\ &= (n-1)p.\end{aligned}$$

- For the Poisson random graph, this is just the expectation of a Poisson(λ) random variable, i.e.,

$$\mathbb{E}\left[d_i \mathbb{1}_{d_i>0}\right] = \sum_{k=1}^{\infty} kP\left(\text{Poi}(\lambda) = k\right) = \sum_{k=1}^{\infty} k\frac{e^{-\lambda}\lambda^k}{k!} = \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}$$

  After introducing a shift of index $k' = k-1$, we see that the summation also turns out to be a power of e:

$$\lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{k'=0}^{\infty}\frac{\lambda^{k'}}{k'!} = \lambda e^{-\lambda}e^{\lambda} = \lambda.$$

Now we have gathered all the terms that we need for the derivation of an explicit expression for both $\mathbb{E}\left[\Delta_i^\star\right]$ and $\mathbb{E}\left[\Delta^\star\right]$:

- $\text{ER}_n(p) : \mathbb{E}\left[\Delta^\star\right] = \mathbb{E}\left[\Delta_i^\star\right] = 1 + (n-2)p - (n-1)p = 1-p.$
- $\text{POI}(\lambda) : \mathbb{E}\left[\Delta^\star\right] = \mathbb{E}\left[\Delta_i^\star\right] = \left[1+e^{-\lambda}\right]\left[1-e^{-\lambda}\right] + e^{-\lambda} - \lambda = 1 - \lambda e^{-\lambda}.$

□

Note that for $p = 1$ every pair of vertices is connected, and every vertex has exactly $n-1$ friends. So all the degrees are the same and, by the definition of the statement of the Friendship Paradox, Δ must be equal to zero. This fits perfectly with the above expression.

3 Other random graph models

The Erdős–Rényi random graph has very interesting properties, but unfortunately it is not really realistic as a model for a network. This fact gave rise to the desire to study other random graph models. Although it might not be possible to compute the probability distribution for the friendship bias Δ for general models, computing the expectation $\mathbb{E}[\Delta]$ is worth spending time on, because it provides us with information on the average of the "gap" between the average number of friends of a person and the average number of friends of friends of this person.

The graph models that will be studied below are those with a pre-specified degree sequence (the Configuration Model), and those where the connection probability is not the same for all pairs of vertices.

3.1 The Configuration Model

In this section we will consider a random graph that is called the Configuration Model, founded by Béla Bollobás. See also Chapter 7 of [4]. Unlike the Erdős–Rényi random graph model, the degrees are given to us beforehand.

The best way to think about the pairing of the vertices into edges is via an algorithm. To each vertex we connect d_i "half-edges". Repeatedly two half-edges are chosen uniformly at random and are connected (to form a "full edge"). The resulting graph is called the Configuration Model with degree sequence d. The following example illustrates how this pairing of half-edges works.

Example 5. *Consider a graph with 4 vertices and degree sequence*

$$d = (d_1, d_2, d_3, d_4) = (2, 3, 2, 3)\,.$$

First, all the half-edges are drawn. After that, two different half-edges are chosen uniformly at random and are connected to each other. This pairing continues until all the half-edges are connected.

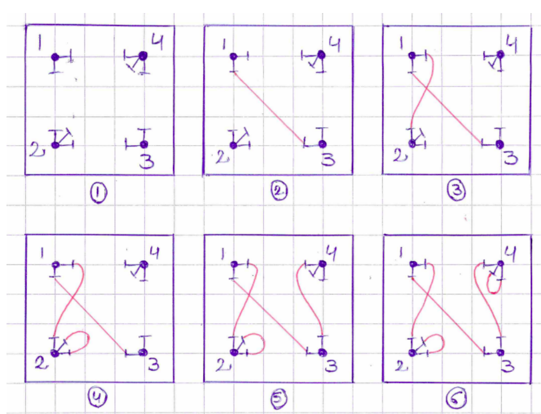


Figure 2: Simulation of the Configuration Model for a graph with 4 vertices and degree sequence $d = (2, 3, 2, 3)$.

A few remarks are in order here. First, the Configuration Model is well-defined only if the sum of the degrees of the graph is even. Otherwise, it is not possible to pair all the half-edges. Second, self-loops and multiple edges are allowed for the Configuration Model, in contrast to the Erdős–Rényi random graph model. Since we know the degrees of the vertices from the very beginning, there is no need to worry about isolated vertices. The hardest part of the computation of the expectation of the friendship bias is to find $\mathbb{E}\left[A(i,j)\right]$.

Theorem 6. *For the Configuration Model with degree sequence d, for* $i, j \in V = \{1, \ldots, n\} = [n]$ *the value of the expectation of the (i,j)-th entry of the adjacency matrix is equal to*

$$\mathbb{E}\left[A(i,j)\right] = \begin{cases} \frac{d_i d_j}{m_1 - 1} & \text{if } i \neq j, \\ \\ \frac{d_i(d_i-1)}{m_1-1} & \text{if } i = j, \end{cases}$$

where $m_1 = \sum_{i=1}^n d_i \in 2\mathbb{N}$.

Proof. Let $i \in V$, which has degree d_i, and therefore d_i half-edges. Let $j \in V \setminus \{i\}$, with d_j the number of half-edges. For each half-edge of vertex i, say h_i, the probability that this half-edge will connect to a half-edge of vertex j is equal to d_j divided by the total number of half-edges minus one (the half-edge of vertex i):

$$P\left(h_i \sim j\right) = \frac{d_j}{m_1 - 1}.$$

Since this equality holds for all half edges of vertex i, we obtain the desired result:

$$\mathbb{E}\left[A(i,j)\right] = \sum_{\# h_i} P\left(h_i \sim j\right) = d_i \frac{d_j}{m_1 - 1} = \frac{d_i d_j}{m_1 - 1}.$$

Alternatively, suppose that $i = j$, and let h_i be an arbitrary half-edge of vertex i. The probability that this half-edge connects to another half-edge of vertex i (there are $d_i - 1$ many) is

$$P\left(h_i \sim \text{another half-edge of } i\right) = \frac{d_i - 1}{m_1 - 1}.$$

Since this probability is the same for all the d_i half-edges of vertex i, we obtain the desired result:

$$\mathbb{E}\left[A(i,j)\right] = \sum_{\# h_i} P\left(h_i \sim \text{another half-edge of } i\right) = d_i \frac{d_i - 1}{m_1 - 1} = \frac{d_i(d_i - 1)}{m_1 - 1}.$$

□

With the help of Theorem 6, both the expectation of Δ_i and Δ can be determined. Before showing these computations, it is again worth emphasising that the degrees are fixed. So, for example, it is allowed to pull terms out of the expectation:

$$\mathbb{E}\left[\Delta_i\right] = \mathbb{E}\left[\frac{1}{d_i} \sum_{j=1}^n A(i,j) d_j - d_i\right] = \frac{1}{d_i} \sum_{j=1}^n \mathbb{E}\left[A(i,j)\right] d_j - d_i.$$

We again distinguish between the two cases $i \neq j$ and $i = j$:

$$\begin{aligned}\mathbb{E}[\Delta_i] &= \frac{1}{d_i}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{d_i d_j}{m_1-1}d_j + \frac{1}{d_i}\frac{d_i(d_i-1)}{m-1}d_i - d_i = \sum_{\substack{j=1\\j\neq i}}^{n}\frac{d_j^2}{m-1} + \frac{d_i(d_i-1)}{m-1} - d_i\\ &= \frac{1}{m_1-1}\sum_{\substack{j=1\\j\neq i}}^{n} d_j^2 + \frac{d_i^2-d_i}{m_1-1} - d_i = \frac{m_2-d_i}{m_1-1} - d_i,\end{aligned}$$

where

$$m_2 = \sum_{j=1}^{n} d_j^2.$$

This result can be used for the calculation of the expectation of Δ:

$$\mathbb{E}[\Delta] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\Delta_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[\Delta_i] = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{m_2}{m_1-1} - \frac{d_i}{m_1-1} - d_i\right].$$

Inside the square brackets three terms appear, two of them depending on n. These terms can be rearranged such that we end up with

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{m_2}{m_1-1} - \frac{d_i}{m_1-1} - d_i\right] &= \frac{m_2}{m_1-1} - \frac{1}{n}\frac{m_1}{m_1-1} - \frac{m_1}{n} = \frac{m_2}{m_1-1} - \frac{m_1}{n}\left[\frac{1}{m_1-1}+1\right]\\ &= \frac{m_2}{m_1-1} - \frac{m_1}{n}\frac{m_1}{m_1-1} = \frac{m_2}{m_1-1} - \frac{m_1^2}{n(m_1-1)}.\end{aligned}$$

3.2 The Friendship Paradox for the Configuration Model

Let U be a vertex that is drawn uniformly from the vertex set V. We assume that there exists a non-negative random variable D such that

$$(a)\ \mathbb{E}[d_U] = \frac{1}{n}\sum_{i=1}^{n} d_i \longrightarrow \mathbb{E}[D] < \infty$$

and

$$(b)\ \mathbb{E}\left[(d_U)^2\right] = \frac{1}{n}\sum_{i=1}^{n} d_i^2 \to \mathbb{E}\left[D^2\right] < \infty.$$

With this way of writing, we have yet another way of showing that the expectation of Δ is non-negative. To that end, we state the following theorem.

Theorem 7. *Suppose that the Configuration model satisfies assumption (a) and (b), then we have that*

$$\mathbb{E}[\Delta] \to \frac{Var(D)}{\mathbb{E}[D]} \geq 0.$$

Proof.

$$\mathbb{E}[\Delta] = \frac{m_2}{m_1 - 1} - \frac{m_1^2}{n(m_1-1)} = \frac{m_2}{m_1-1} - \frac{m_1}{n}\frac{m_1}{m_1-1} = \frac{\frac{m_2}{n}}{\frac{m_1}{n} - \frac{1}{n}} - \frac{m_1}{n}\frac{m_1}{m_1-1}.$$

Assuming that all the degrees of the vertices are at least one, we have that the fraction $\frac{m_1}{m_1-1}$ tends to 1 in the limit as $n \to \infty$. Hence

$$\frac{\frac{m_2}{n}}{\frac{m_1}{n} - \frac{1}{n}} - \frac{m_1}{n}\frac{m_1}{m_1-1} \overset{n\to\infty}{\longrightarrow} \frac{\mathbb{E}\left[D^2\right]}{\mathbb{E}[D]} - \mathbb{E}[D] = \frac{\mathbb{E}\left[D^2\right] - \mathbb{E}[D]^2}{\mathbb{E}[D]} = \frac{\mathrm{Var}(D)}{\mathbb{E}[D]} \geq 0.$$

□

3.3 The inhomogeneous Erdős–Rényi random graph model

In Chapter 2 we assumed that each pair of vertices (or persons) has the same probability of becoming mutual friends (which we refer to as the homogeneous case). This is not realistic, and therefore it is worthwhile to study what happens when the connection probability is not the same for all pairs of vertices. In what follows, we will work with the following choice of p_{ij}:

$$p_{ij} = \frac{\lambda}{n}\, r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right).$$

Here, $r : [0,1] \to [0,\infty)$ is assumed to be Lipschitz continuous, i.e.,

$$\exists\, C \geq 0 : \forall x, y \in [0,1] : \left|r(x) - r(y)\right| \leq C\,|x-y|\,.$$

Moreover, define

$$m_k = \int_0^1 r(x)^k dx.$$

In the upcoming computations, we will substitute sums with integrals, this gives an error term of $O(\frac{1}{n^2})$. In order to show this, we need the Lipschitz continuity of r. Using that, the following inequality holds:

$$\left|\frac{1}{n}\sum_{k=1}^{n} r\left(\frac{k}{n}\right) - \int_0^1 r(x)dx\right| \leq \frac{1}{n}\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n}\left|r(x) - r\left(\frac{k}{n}\right)\right| dx. \tag{14}$$

Note that $\left|x - \frac{k}{n}\right| \leq \frac{1}{n}$, since $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \implies \left|r(x) - r\left(\frac{k}{n}\right)\right| \leq C\frac{1}{n}$. Using (14), we obtain the following estimation:

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n}\left|r(x) - r\left(\frac{k}{n}\right)\right| dx &\leq \frac{1}{n}\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} C\frac{1}{n}dx \\ &= \frac{C}{n^2}\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} 1dx \\ &= \frac{C}{n^2}\sum_{k=1}^{n}\frac{1}{n} = \frac{C}{n^2} = O\left(\frac{1}{n^2}\right). \end{aligned} \tag{15}$$

It is important to get an idea of what the resulting degree distribution looks like. This is no longer binomial, because the connection probabilities are no longer the same for all pairs of vertices. It turns out that the degree distribution converges to a Poisson random variable. The proof is given in Section 3.3.1. After this proof, the expectation of the friendship bias is computed, followed by a sanity check where we compare the computations of Chapter 2 with the computations of Chapter 3.

For the determination of the degree distribution, fix $i \in V$, and write $p_{ij} = p_j$. Then

$$P\left(d_i = k\right) = P\left(\sum_{j=1}^{n} A(i,j) = k\right),$$

where

$$A(i,j) \stackrel{d}{=} \text{Bernoulli}\left(\frac{\lambda}{n}\, r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right)\right) = \text{Bernoulli}(p_j).$$

Let $X = \sum_{j=1}^{n} A(i,j)$ and $Z \stackrel{d}{=} \text{Poisson}(\mu)$, where $\mu = \sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \left[-\log(1-p_j)\right]$. We will use the following asymptotic formula, which holds for small x:

$$-\log(1-x) \sim x \implies \mu_j = -\log(1-p_j) \sim \frac{\lambda}{n}\, r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right).$$

The sum of the μ_j contains the term m_1:

$$\mu = \sum_{j=1}^{n} \mu_j = \frac{\lambda}{n}\, r\left(\frac{i}{n}\right) \sum_{j=1}^{n} r\left(\frac{j}{n}\right) = \lambda r\left(\frac{i}{n}\right) \int_0^1 r(x)dx = \lambda r\left(\frac{i}{n}\right) m_1 + O\left(\frac{1}{n^2}\right).$$

The sum of the squares of the μ_j is of order $\frac{1}{n}$:

$$\begin{aligned}\sum_{j=1}^{n} \mu_j^2 &= \sum_{j=1}^{n} \left(\frac{\lambda}{n}\right)^2 r\left(\frac{i}{n}\right)^2 r\left(\frac{j}{n}\right)^2 = \frac{\lambda^2}{n^2}\, r\left(\frac{i}{n}\right)^2 \sum_{j=1}^{n} r\left(\frac{j}{n}\right)^2 \\ &= \frac{1}{n}\lambda^2 r\left(\frac{i}{n}\right)^2 \frac{1}{n} \sum_{j=1}^{n} r\left(\frac{j}{n}\right)^2 = \frac{1}{n} \underbrace{\lambda^2\, r\left(\frac{i}{n}\right) \int_0^1 r(x)^2 dx}_{\text{constant}} \;+\; O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{n}\right).\end{aligned}$$

This implies that $P\left(X = k\right) = P\left(Z = k\right) + O\left(\frac{1}{n}\right)$. Asymptotically, the degree distribution is Poisson(μ).

3.3.1 Convergence of the degree distribution

For the next part, let N_k denote the number of vertices in the graph with degree equal to k:

$$P_k = \frac{N_k}{n} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{d_i = k\}}.$$

Theorem 8. *As $n \to \infty$, P_k converges in probability to $p_k = \text{Poisson}(\mu)$.*

Proof. For the proof we follow the line of argument in the proof of Theorem 5.12 from Chapter 5 of the monograph by R. van der Hofstad [4]. This proof makes use of Chebyshev's theorem, and requires that two properties need to be shown as $n \to \infty$:
(a) $\mathbb{E}[P_k] \to p_k$,
(b) $\mathrm{Var}(p_k) \to 0$.

(a): $\mathbb{E}[P_k] \to p_k$:

$$\mathbb{E}[P_k] = \frac{1}{n}\sum_{i=1}^{n} P(d_i = k) = \frac{1}{n}\sum_{i=1}^{n} P(X = k) = \frac{1}{n}\sum_{i=1}^{n}\left(P(Z = k) + O\left(\frac{1}{n}\right)\right).$$

The $O\left(\frac{1}{n}\right)$ term can be pulled in front of the sum. This results in

$$\begin{aligned}\mathbb{E}[P_k] &= \frac{1}{n}\sum_{i=1}^{n} P(Z = k) + O\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} P\left(\text{Poisson}(\mu) = k\right) + O\left(\frac{1}{n}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{e^{-\mu}\mu^k}{k!} + O\left(\frac{1}{n}\right).\end{aligned}$$

In the next step, μ is written out as $\frac{\lambda}{n}\, r\left(\frac{i}{n}\right)\sum_{j=1}^{n} r\left(\frac{j}{n}\right)$, and we take the limit $n \to \infty$ to obtain an integral form:

$$\begin{aligned}\mathbb{E}[P_k] &= \frac{1}{n}\sum_{i=1}^{n} \frac{\exp\{-\lambda\, r\left(\frac{i}{n}\right)\frac{1}{n}\sum_{j=1}^{n} r\left(\frac{j}{n}\right)\}}{k!}\left(\frac{\lambda}{n}\right)^k r\left(\frac{i}{n}\right)^k \left(\sum_{j=1}^{n} r\left(\frac{j}{n}\right)\right)^k + O\left(\frac{1}{n}\right) \\ &= \int_0^1 \frac{\exp\{-\lambda r(x)\int_0^1 r(y)dy\}}{k!}\lambda^k r(x)\left(\int_0^1 r(y)dy\right)^k dx + O\left(\frac{1}{n}\right).\end{aligned}$$

Recall that $m_1 = \int_0^1 r(y)dy$, which allows us to simplify the above expression as

$$\mathbb{E}[P_k] = \int_0^1 \frac{\exp\{-\lambda r(x) m_1\}}{k!}\lambda^k r(x) m_1^k\, dx + O\left(\frac{1}{n}\right).$$

Define $U \sim \text{Uniform}[0,1]$ and let $W = r(U)$. Given W, let $Z' \stackrel{d}{=} \text{Poisson}(\lambda m_1 W)$. With this notation, it can be shown that (a) holds:

$$\begin{aligned}p_k &= P\left(Z' = k\right) = \mathbb{E}\left[P\left(Z' = k \mid W\right)\right] \\ &= \mathbb{E}\left[\frac{\exp\{-\lambda m_1 r(U)\}}{k!}\left(\lambda m_1 r(U)\right)^k\right] = \int_0^1 \frac{\exp\{-\lambda r(x) m_1\}}{k!}\lambda^k r(x) m_1^k\, dx.\end{aligned}$$

(b): $\mathrm{Var}(p_k) \to 0$:

With the choice of $P_k = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{d_i=k\}}$, the variance becomes

$$\begin{aligned}
\operatorname{Var}\left(P_k^n\right) &= \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n} \mathbb{1}_{\{d_i=k\}}\right)^2\right] - \frac{1}{n^2}\left(\sum_{i=1}^{n} P\left(d_i=k\right)\right)^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n} P\left(d_i=k\right) + \frac{1}{n^2}\sum_{\substack{i=1\\ i\neq j}}^{n} P\left(d_i=k, d_j=k\right) \\
&- \frac{1}{n^2}\sum_{i=1}^{n} P\left(d_i=k\right)^2 - \frac{1}{n^2}\sum_{\substack{i=1\\ i\neq j}}^{n} P\left(d_i=k\right) P\left(d_j=k\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \underbrace{\left(P\left(d_i=k\right) - P\left(d_i=k\right)^2\right)}_{\leq 2} + \underbrace{\frac{1}{n^2}\sum_{\substack{i=1\\ i\neq j}}^{n} \left[P\left(d_i=d_j=k\right) - P\left(d_i=k\right) P\left(d_j=k\right)\right]}_{=(\star)}.
\end{aligned}$$

Using the upper bound of the first sum (which is equal to 2), we can conclude that $2n$ is an upper bound for the sum, meaning that the first term is $O\left(\frac{1}{n}\right)$. What is left is to show is that this also holds for the second sum.

In order to do this, a coupling argument is needed. Define

$$Z_i := \sum_{k\neq i,j} X_{ik}, \text{ where } X_{ij} \sim \text{Bernoulli}\left(\frac{\lambda}{n}\, r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right)\right).$$

The degrees can now be written as $d_i = Z_i + X_{ij}$ and $d_j = Z_j + X_{ij}$, which gives

$$\left(d_i, d_j\right) \stackrel{d}{=} \left(Z_i + X_{ij}, Z_j + X_{ij}\right).$$

Let $\hat{X}_{ij}$ be such that $\hat{X}_{ij} \stackrel{d}{=} X_{ij}$. Hence, $(Z_i + \hat{X}_{ij}, Z_j + \hat{X}_{ij})$ are marginally the same. An upper bound for

$$P\left(d_i = d_j = k\right) - P\left(d_i = k\right) P\left(d_j = k\right). \tag{16}$$

is now given by:

$$(16) \leq \underbrace{P\left(\left(Z_i + X_{ij}, Z_j + X_{ij}\right) = (k,k), \left(Z_i + \hat{X}_{ij}, Z_j + \hat{X}_{ij}\right) \neq (k,k)\right)}_{=(\star\star)}.$$

The latter event can only happen if $\hat{X}_{ij} \neq X_{ij}$. Two cases are distinguished:

- $X_{ij} = 0 \implies \hat{X_{ij}} = 1$ and $Z_i = Z_j = k$.
- $X_{ij} = 1 \implies \hat{X}_{ij} = 0$ and $Z_i = Z_j = k-1$.

This observation gives rise to yet another upper bound:

$$(\star\star) \leq P\left(\hat{X}_{ij} = 1, d_j = k-1\right) + P\left(\hat{X}_{ij} = 0, d_j = k\right).$$

The probabilities in the above expression can be factorized because of the independence. When this last expression, after factorization, is substituted into $(\star)$, we indeed find that $(\star)$, and thus $\text{Var}\left(P_k^n\right)$, is $O\left(\frac{1}{n}\right)$:

$$\begin{aligned}(\star) &\leq \frac{1}{n^2}\sum_{\substack{i=1\\ i\neq j}}^{n} P\left(\bar{X}_{ij} = 1\right) P\left(d_j = k-1\right) + P\left(\bar{X}_{ij} = 0\right) P\left(d_j = k\right)\\ &\leq \frac{2}{n^2}\sum_{\substack{i=1\\ i\neq j}}^{n} \frac{\lambda}{n}\, r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right) = \frac{2\lambda}{n^3}\sum_{\substack{i=1\\ i\neq j}}^{n} r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right) = O\left(\frac{1}{n}\right).\end{aligned}$$

Now that we have shown that both (a) and (b) hold, the proof that P_k converges in probability to $p_k = \text{Poisson}(\mu)$ is complete. □

3.3.2 Convergence of the friendship bias

For the inhomogeneous Erdős–Rényi random graph model, we again use our definition of $\Delta_i^\star$, where we take into account that the degree of a vertex i can be zero:

$$\Delta_i^\star = \begin{cases} \frac{1}{d_i}\sum_{j=1}^{n} A(i,j) d_j - d_i & \text{if } d_i > 0,\\ 1 & \text{if } d_i = 0.\end{cases}$$

As mentioned before, $\Delta^\star$ is the arithmetic mean of the Δ_i:

$$\Delta^\star = \sum_{i=1}^{n} \Delta_i^\star.$$

Theorem 9. *Consider an inhomogeneous Erdős–Rényi random graph G with*

$$p_{ij} = \frac{\lambda}{n}\, r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right),$$

and r Lipschitz continuous. With the above definition of $\Delta_i^\star$ and $\Delta^\star$, we have

- $\mathbb{E}\left[\Delta_i^\star\right] \to 1 - \lambda m_1 r\left(\frac{i}{n}\right)\left(m_2\left(e^{-\lambda}-1\right)+1\right)$.
- $\mathbb{E}\left[\Delta^\star\right] \to 1 - \lambda m_1^2\left(m_2\left(e^{-\lambda}-1\right)+1\right)$.

Proof. Recall the first steps of the calculation of $\mathbb{E}\left[\Delta_i^\star\right]$:

$$\begin{aligned}
\mathbb{E}\left[\Delta_i^\star\right] &= \mathbb{E}\left[\left(\frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j - d_i\right)\mathbb{1}_{\{d(i)>0\}} + \mathbb{1}_{\{d(i)=0\}}\right] \\
&= \mathbb{E}\left[\left(\frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j - d_i\right)\mathbb{1}_{\{d(i)>0\}}\right] + P(d_i=0) \\
&= \underbrace{\mathbb{E}\left[\frac{1}{d_i}\sum_{j=1}^{n} A(i,j)d_j\mathbb{1}_{\{d(i)>0\}}\right]}_{=(\star)} - \mathbb{E}\left[d_i\mathbb{1}_{\{d(i)>0\}}\right] + P(d_i=0).
\end{aligned}$$

Similar to the computations in Chapter 2, we start by conditioning on the degree of vertex i:

$$\begin{aligned}
(\star) &= \sum_{k=1}^{n}\mathbb{E}\left[\frac{1}{k}\sum_{j=1} A(i,j)d_j \mid d_i=k\right] P\left(d_i=k\right) \\
&= \sum_{k=1}^{n}\mathbb{E}\left[\frac{1}{k}\sum_{j=1} A(i,j)\sum_{\substack{l=1\\ l\neq j}}^{n} A(j,l) \mid d_i=k\right] P\left(d_i=k\right).
\end{aligned}$$

The sum over l is written out, and from there the sum over k is split into two parts:

$$\begin{aligned}
(\star) &= \sum_{k=1}^{n}\mathbb{E}\left[\frac{1}{k}\sum_{j=1} A(i,j)\left(A(j,i)+\sum_{\substack{l=1\\ l\neq i,j}}^{n} A(j,l)\right) \mid d_i=k\right] P\left(d_i=k\right) \\
&= \sum_{k=1}^{n}\frac{1}{k}\ \mathbb{E}\left[\sum_{j=1}^{n} A(i,j)A(j,i) \mid d_i=k\right] P\left(d_i=k\right) \\
&+ \sum_{k=1}^{n}\frac{1}{k}\ \mathbb{E}\left[\sum_{j=1}^{n} A(i,j)\sum_{\substack{l=1\\ l\neq i,j}}^{n} A(j,l) \mid d_i=k\right] P\left(d_i=k\right).
\end{aligned}$$

Note that $A(i,j)A(j,i) = A(i,j)$, since the adjacency matrix is symmetric and each element is either 0 or 1. Hence

$$
\begin{aligned}
(\star) &= \sum_{k=1}^{n} \frac{1}{k}\ \mathbb{E}\left[d_i \mid d_i = k\right] P\left(d_i = k\right) \\
&+ \sum_{k=1}^{n} \frac{1}{k}\ \mathbb{E}\left[\sum_{j=1}^{n} A(i,j) \sum_{\substack{l=1 \\ l \neq i,j}}^{n} A(j,l) \mid d_i = k\right] P\left(d_i = k\right) \\
&= 1 - P\left(d_i = 0\right) + \underbrace{\sum_{k=1}^{n} \frac{1}{k}\ \mathbb{E}\left[\sum_{j=1}^{n} A(i,j) \sum_{\substack{l=1 \\ l \neq i,j}}^{n} A(j,l) \mid d_i = k\right] P\left(d_i = k\right)}_{(\star\star)}.
\end{aligned}
$$

As a first step towards computing $(\star\star)$, we pull the sum over j out of the expectation and split up the expectation, which is possible because of the independence:

$$
\begin{aligned}
(\star\star) &= \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n} \mathbb{E}\left[A(i,j) \sum_{\substack{l=1 \\ l \neq i,j}}^{n} A(j,l) \mid d_i = k\right] P\left(d_i = k\right) \\
&= \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n} \mathbb{E}\left[\sum_{\substack{l=1 \\ l \neq i,j}}^{n} A(j,l)\right] \mathbb{E}\left[A(i,j) \mid d_i = k\right] P\left(d_i = k\right) \\
&= \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n} \sum_{\substack{l=1 \\ l \neq i,j}}^{n} \frac{\lambda}{n}\ r\left(\frac{j}{n}\right) r\left(\frac{l}{n}\right) \mathbb{E}\left[A(i,j) \mid d_i = k\right] P\left(d_i = k\right) \\
&= \lambda \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n} r\left(\frac{j}{n}\right) \frac{1}{n} \sum_{\substack{l=1 \\ l \neq i,j}}^{n} r\left(\frac{l}{n}\right) \mathbb{E}\left[A(i,j) \mid d_i = k\right] P\left(d_i = k\right).
\end{aligned}
$$

In the limit as $n \to \infty$, the last sum turns into an integral. This integral will be denoted by m_1 and can be pulled in front of the first two sums:

$$
\begin{aligned}
(\star\star) &= \lambda \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n} r\left(\frac{j}{n}\right) \int_0^1 r(x)dx\ \mathbb{E}\left[A(i,j) \mid d_i = k\right] P\left(d_i = k\right) + O\left(\frac{1}{n}\right) \qquad (17) \\
&= \lambda m_1 \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n} r\left(\frac{j}{n}\right) \mathbb{E}\left[A(i,j) \mid d_i = k\right] P\left(d_i = k\right) + O\left(\frac{1}{n}\right).
\end{aligned}
$$

Since (17) contains a sum from $j = 1$ to n, the error term is not $O(\frac{1}{n^2})$ as stated in (15), but $O(\frac{1}{n})$.

Next we proceed with the calculation of $\mathbb{E}\left[A(i,j) \mid d_i = k\right]$, which is the only remaining unknown term. After this quantity is computed, we can find an explicit expression for the expectation of the friendship bias. Write

$$\begin{aligned}
\mathbb{E}\left[A(i,j) \mid d_i = k\right] &= P\left(A(i,j) = 1 \mid d_i = k\right) \\
&= \frac{P\left(A(i,j) = 1, \sum_{\substack{m=1\\ m\neq i}}^{n} A(i,m) = k\right)}{P\left(d_i = k\right)} \\
&= \frac{P\left(A(i,j) = 1, \sum_{\substack{m=1\\ m\neq i,j}}^{n} A(i,m) + A(i,j) = k\right)}{P\left(d_i = k\right)}.
\end{aligned}$$

Since $A(i,j)$ appears also in the first part of the probability in the numerator, it is allowed to write the second part of the probability as the sum of the remaining $A(i,m)$ being equal to $k-1$. Then what remains is a probability that can be split due to the independence:

$$\begin{aligned}
\mathbb{E}\left[A(i,j) \mid d_i = k\right] &= \frac{P\left(A(i,j) = 1, \sum_{\substack{m=1\\ m\neq i,j}}^{n} A(i,m) = k-1\right)}{P\left(d_i = k\right)} \\
&= \frac{P\left(A(i,j) = 1\right) P\left(\sum_{\substack{m=1\\ m\neq i,j}}^{n} A(i,m) = k-1\right)}{P\left(d_i = k\right)}.
\end{aligned}$$

First, the two probabilities in the numerator are

$$P\left(A(i,j) = 1\right) = \frac{\lambda}{n}\, r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right),$$

$$P\left(\sum_{\substack{m=1\\ m\neq i,j}}^{n} A(i,m) = k-1\right) \sim P\left(\text{Poisson}(\lambda') = k-1\right).$$

The paramater of the Poisson random variable, denoted by λ', turns out to be asymptotically the same as μ:

$$\lambda' = \frac{\lambda}{n} \sum_{\substack{m=1 \\ m \neq i,j}}^{n} r\left(\frac{i}{n}\right) r\left(\frac{m}{n}\right) = \frac{\lambda}{n}\, r\left(\frac{i}{n}\right) \sum_{\substack{m=1 \\ m \neq i,j}}^{n} r\left(\frac{m}{n}\right)$$
$$= \lambda r\left(\frac{i}{n}\right) \int_0^1 r(x)dx + O\left(\frac{1}{n^2}\right) = \lambda r\left(\frac{i}{n}\right) m_1 + O\left(\frac{1}{n^2}\right) = \mu + O\left(\frac{1}{n^2}\right).$$

The probability in the denominator is nothing other than the probability that the degree is equal to k, which is already known: $P\left(d_i = k\right) = P\big(\text{Poisson}(\mu) = k\big)$.

The computation of $\mathbb{E}[d_i \mathbb{1}_{\{d(i)>0\}}]$ is very similar to the one we have seen in Chapter 2 and is just the same as the expectation of a Poisson(μ) random variable:

$$\mathbb{E}\left[d_i \mathbb{1}_{\{d(i)>0\}}\right] = \mathbb{E}\left(\text{Poisson}(\mu)\right) = \frac{\lambda}{n}\, r\left(\frac{i}{n}\right) \sum_{j=1}^{n} r\left(\frac{j}{n}\right)$$
$$= \lambda\, r\left(\frac{i}{n}\right) \int_0^1 r(x)dx + O\left(\frac{1}{n^2}\right) = \lambda\, r\left(\frac{i}{n}\right) m_1 + O\left(\frac{1}{n^2}\right).$$

Putting the above results together, we get that the expectation of $\Delta_i^\star$ can be written out as

$$\mathbb{E}\left[\Delta_i^\star\right] = \mathbb{E}\left[\frac{1}{d_i} \sum_{j=1}^{n} A(i,j) d_j \mathbb{1}_{\{d(i)>0\}}\right] - \mathbb{E}\left[d_i \mathbb{1}_{\{d(i)>0\}}\right] + P(d_i = 0)$$
$$= 1 - P\left(d_i = 0\right) + \lambda m_1 \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n} r\left(\frac{j}{n}\right) \mathbb{E}\left[A(i,j) \mid d_i = k\right] P\left(d_i = k\right)$$
$$- \mathbb{E}\left[d_i \mathbb{1}_{\{d(i)>0\}}\right] + P(d_i = 0) + O\left(\frac{1}{n^2}\right).$$

Note that $P\left(d_i = 0\right)$ cancels out. The same holds in the next line for $P\big(\text{Poisson}(\mu) = k\big)$:

$$\mathbb{E}\left[\Delta_i^\star\right] = 1 + \lambda m_1 \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{n} r\left(\frac{j}{n}\right) \frac{\frac{\lambda}{n}\, r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right) P\big(\text{Poisson}(\mu) = k-1\big)}{P\big(\text{Poisson}(\mu) = k\big)}\, P\big(\text{Poisson}(\mu) = k\big)$$
$$- \lambda r\left(\frac{i}{n}\right) m_1 + O\left(\frac{1}{n^2}\right)$$
$$= 1 - \lambda^2 m_1 r\left(\frac{i}{n}\right) \sum_{k=1}^{n} \frac{1}{k} \underbrace{\frac{1}{n} \sum_{j=1}^{n} r\left(\frac{j}{n}\right)^2}_{= m_2} P\big(\text{Poisson}(\mu) = k-1\big) - \lambda r\left(\frac{i}{n}\right) m_1 + O\left(\frac{1}{n^2}\right)$$
$$= 1 - \lambda^2 m_1 m_2 r\left(\frac{i}{n}\right) \sum_{k=1}^{n} \frac{1}{k} P\big(\text{Poisson}(\mu) = k-1\big) - \lambda r\left(\frac{i}{n}\right) m_1 + O\left(\frac{1}{n^2}\right) = (\star\star\star).$$

The sum can be written out as

$$\sum_{k=1}^{\infty} \frac{1}{k} P\left(\text{Poisson}(\mu) = k-1\right) = \sum_{k=1}^{\infty} \frac{1}{k} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} = \frac{1}{\lambda} e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = \frac{1}{\lambda} e^{-\lambda} \left(-1 + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right).$$

The last sum starts from $k = 0$ and is equal to $e^{-\lambda}$:

$$\sum_{k=1}^{\infty} \frac{1}{k} P\left(\text{Poisson}(\mu) = k-1\right) = \frac{1}{\lambda} e^{-\lambda} \left(-1 + e^{-\lambda} \right) = \frac{1}{\lambda} \left(-1 + e^{-\lambda} \right).$$

Substituting this result into $(\star\star\star)$, we obtain our final expression for $\mathbb{E}\left[\Delta_i^\star\right]$:

$$\begin{aligned}
\mathbb{E}\left[\Delta_i^\star\right] &= 1 - \lambda^2 m_1 m_2 r \left(\frac{i}{n}\right) \frac{1}{\lambda} \left(e^{-\lambda} - 1\right) - \lambda r \left(\frac{i}{n}\right) m_1 + O\left(\frac{1}{n^2}\right) \\
&= 1 - \lambda m_1 m_2 r \left(\frac{i}{n}\right) \left(e^{-\lambda} - 1\right) - \lambda m_1 r \left(\frac{i}{n}\right) + O\left(\frac{1}{n^2}\right) \\
&= 1 - \lambda m_1 r \left(\frac{i}{n}\right) \left(m_2 \left(e^{-\lambda} - 1\right) + 1\right) + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Fortunately, the calculation of the expectation of $\Delta^\star$ is by now easy. In its final form, there will not be a loose $r\left(\frac{i}{n}\right)$ term. The only terms that contains r are integral terms:

$$\begin{aligned}
\mathbb{E}\left[\Delta^\star\right] &= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\Delta_i\right] = \frac{1}{n} \sum_{i=1}^{n} 1 - \lambda m_1 r \left(\frac{i}{n}\right) \left(m_2 \left(e^{-\lambda} - 1\right) + 1\right) + O\left(\frac{1}{n^2}\right) \\
&= 1 - \lambda m_1 \left(m_2 \left(e^{-\lambda} - 1\right) + 1\right) \underbrace{\frac{1}{n} \sum_{i=1}^{n} r\left(\frac{i}{n}\right)}_{=m_1} + O\left(\frac{1}{n^2}\right) \\
&= 1 - \lambda m_1^2 \left(m_2 \left(e^{-\lambda} - 1\right) + 1\right) + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

□

3.3.3 Sanity check

After having completed the computation, we perform a sanity check. If we move back to the homogeneous case, meaning that $r \equiv 1$ and consequently $m_1 = m_2 = 1$, then do we obtain the same results as stated in Theorem 5? Recall that the statement of this theorem is that $\mathbb{E}\left[\Delta_i^\star\right] = \mathbb{E}\left[\Delta^\star\right] = 1 - \lambda e^{-\lambda}$. The answer is yes:

$$\mathbb{E}\left[\Delta_i^\star\right] = 1 - \lambda m_1 r \left(\frac{i}{n}\right) \left(m_2 \left(e^{-\lambda} - 1\right) + 1\right) = 1 - \lambda \times 1 \times 1 \left(1 \left(e^{-\lambda} - 1\right) + 1\right) = 1 - \lambda e^{-\lambda}.$$

We proceed with the sanity check for $\mathbb{E}\left[\Delta^{\star}\right]$:

$$\mathbb{E}\left[\Delta^{\star}\right] = 1 - \lambda m_1^2 \left(m_2 \left(e^{-\lambda} - 1 \right) + 1 \right) = 1 - \lambda \times 1^2 \left(1 \left(e^{-\lambda} - 1 \right) + 1 \right) = 1 - \lambda e^{-\lambda}.$$

This shows that the computation made in Chapter 3 is in line with that of Chapter 2.

This is as far as the mathematical computations go for this bachelor thesis. In the next chapter the correctness of the computations are checked via simulations in Matlab.

4 Simulations

In this chapter the outcomes of the computations in the previous chapters are "double checked" via the use of simulations.

4.1 Simulation of the Erdős–Rényi random graph

In Chapter 2, the following definition of $\Delta_i^\star$ was provided:

$$\Delta_i^\star = \begin{cases} \frac{1}{d_i}\sum_{j=1}^n A(i,j)d_j - d_i & \text{if } d_i > 0, \\ 1 & \text{if } d_i = 0. \end{cases}$$

For this choice of the friendship bias of vertex i, the expectation of $\Delta_i^\star$ was found to be $1-p$. To check whether these computations are right, simulations in Matlab are needed. After implementing the Erdős–Rényi random graph and simulating everything that comes with it (the adjacency matrix, the degree vector, the computation of all the Δ_i, of Δ itself and of the friendship matrix), we made a table. In this table, the absolute value of the error between the theoretical and experimental value is listed for some choices of the connection probability p. For these simulations, we chose the number of vertices to be equal to 10.000.

p	$\mathbb{E}[\Delta^\star]$	$\Delta^\star$	$\left\lvert\mathbb{E}[\Delta^\star] - \Delta^\star\right\rvert$
0.10	0.90	0.8907	$9.3 \cdot 10^{-3}$
0.20	0.80	0.8011	$1.1 \cdot 10^{-3}$
0.30	0.70	0.7025	$2.5 \cdot 10^{-3}$
0.40	0.60	0.6089	$8.9 \cdot 10^{-3}$
0.50	0.50	0.4981	$1.9 \cdot 10^{-3}$
0.60	0.40	0.3994	$6.0 \cdot 10^{-4}$
0.70	0.30	0.3000	0
0.80	0.20	0.2023	$2.3 \cdot 10^{-3}$
0.90	0.10	0.1007	$7.0 \cdot 10^{-4}$

The error term $\Delta^\star \quad \left\lvert\mathbb{E}[\Delta^\star] - \Delta^\star\right\rvert$ appears to be of order 10^{-3}. Altough we did not persue the computation, it is likely to be true that this error term is so small because of the fact that the variance is small.

4.2 Simulation of the inhomogeneous Erdős–Rényi random graph

From the simulation of the homogeneous Erdős–Rényi random graph, it is a small step to go to the inhomogeneous setting. Only the connection probabilities are different. For the inhomogeneous setting, we compute the $\Delta_i^\star$-values in exactly the same way as for the homogeneous setting.

In order to double-check the computations of Chapter 3, we must choose a specific function r and also a specific value of λ. Let us pick $r(x) = x$. Then we obtain the following value for m_1 and m_2:

$$m_1 = \int_0^1 r(x)dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2},$$

$$m_2 = \int_0^1 r(x)^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

These two values result in an expression for $\mathbb{E}[\Delta^\star]$ where λ is the only variable left:

$$\mathbb{E}[\Delta^\star] = 1 - \lambda m_1^2 \left(m_2 \left(e^{-\lambda} - 1\right) + 1\right) = 1 - \lambda \frac{1}{4}\left(\frac{1}{3}\left(e^{-\lambda} - 1\right) + 1\right) = 1 - \frac{1}{12}\lambda e^{-\lambda} - \frac{1}{6}\lambda.$$

In the table below, the outcomes of the simulations are compared with the theoretical values. Again, the number of vertices was chosen to be 10.000.

λ	$\mathbb{E}[\Delta^\star]$	$\Delta^\star$	$\lvert\mathbb{E}[\Delta^\star] - \Delta^\star\rvert$
1	0.8027	0.8134	$1.07 \cdot 10^{-2}$
2	0.6441	0.7400	$9.59 \cdot 10^{-2}$
3	0.4876	0.7375	$2.50 \cdot 10^{-1}$
4	0.3272	0.7835	$4.56 \cdot 10^{-1}$
5	0.1639	0.8206	$6.57 \cdot 10^{-1}$

As λ gets larger, the error term also gets larger. The main reason for this is the fact that we used an approximation in the calculations of Chapter 3. For instance, we used that

$$-\log\left(1 - p_{ij}\right) \sim p_{ij}$$

for small values of p_{ij}. For $n = 10.000$ and i and j close to 10.000, this approximation is not so accurate anymore. This leads to a larger error term. Moreover, for large values of λ, $\mathbb{E}\left[\Delta_i^\star\right]$ behaves like $1 - \frac{1}{6}\lambda$. For $\lambda > 6$, the theoretical values of the expectation of $\Delta_i^\star$ therefore become smaller than zero, which is in contradiction to the statement of the Friendship Paradox.

4.3 Analysis of the eigenvalues of the friendship matrix

In Chapter 1, the friendship matrix was introduced:

$$F(i,j) = A(i,j)\left(\frac{d_j}{d_i} - 1\right).$$

The structure of this matrix is such that all the eigenvalues of F are complex with real part equal to 0. In this section, the friendship matrix is used to look at 'friends of friends' rather than at the difference between the average number of friends of friends of person i minus the number of friends of person i. This gives rise to a new friendship matrix:

$$\hat{F}(i,j) = A(i,j)\frac{d_j}{d_i}.$$

It is interesting to make this matrix symmetric. By doing this, we obtain only real eigenvalues. The most natural choice would be as follows:

$$\hat{F}(i,j) = A(i,j)\left(\frac{d_j}{d_i} + \frac{d_i}{d_j}\right).$$

If we assume that we are in the homogeneous Erdős–Rényi setting, then each element of the adjacency matrix follows a Bernoulli distribution with the connection probability p as parameter. The expectations of all the degrees are equal to $(n-1)p$. For large n, this results in the fact that an element of $\hat{F}$ roughly behaves like

$$\hat{F}(i,j) = A(i,j)\left(\frac{d_j}{d_i} + \frac{d_i}{d_j}\right) \approx A(i,j)\left(\frac{(n-1)p}{(n-1)p} + \frac{(n-1)p}{(n-1)p}\right) \sim 2A(i,j).$$

It would be helpful if the eigenvalues of $\hat{F}$ would be centered at zero. In order to achieve this, we subtract $2p$ from each entry of $\hat{F}$, which brings us to yet another friendship matrix, denoted by $\widetilde{F}$:

$$\widetilde{F}(i,j) = A(i,j)\left(\frac{d_j}{d_i} + \frac{d_i}{d_j}\right) - 2p.$$

4.4 Behaviour of the eigenvalues of the symmetrised centered friendship matrix

Something interesting happens when we assume that $np \to \infty$ as $n \to \infty$. In the upcoming example, $n = 10.000$, and the eigenvalues are computed via simulation. If these eigenvalues are scaled by two times the variance of a Bernoulli(p) random variable, i.e., $2p(1-p)$, then we obtain the following histogram of the eigenvalues of $\widetilde{F}$.

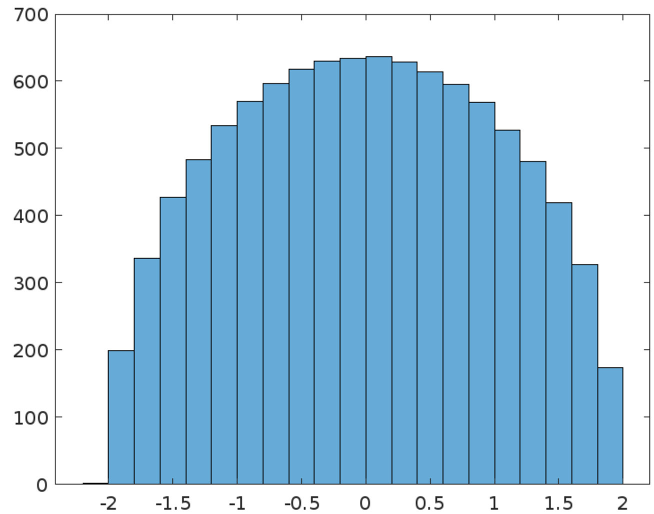


Figure 3: Simulation of the histogram of the eigenvalues of $\widetilde{F}$, scaled by the factor $2\sqrt{p(1-p)}$, with $n = 10.000$.

Since the eigenvalues of $\widetilde{F}$ are random, it is worth studying their cumulative distribution function:

$$G_n(x)(\omega) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{\lambda_i(\omega)<x\}}.$$

The semi-circular shape of Figure 3 gives rise to a conjecture:

Conjecture 1. *As $n \to \infty$,*

$$G_n(x) \to G(x) \textit{ almost surely, for all } x,$$

where λ_i are the eigenvalues of $\widetilde{F}$ and

$$G(x) = \frac{1}{\pi}\int_{-\infty}^{x} \sqrt{4-t^2}\,\mathbb{1}_{\{|t|<2\}} dt.$$

If we want to show that this conjecture is true, then we have to perform the same steps as in the proof of Theorem 7: the almost surely convergence can be proved by showing that $\mathbb{E}\left[G_n(x)\right] \to \mathbb{E}\left[G(x)\right]$ and $\mathrm{Var}\big(G_n(x)\big) \to 0$ in the limit as $n \to \infty$. This procedure is beyond the scope of this thesis.

Note that the assumption that $np \to \infty$ is necessary. If this assumption would be dropped, then the histogram of the eigenvalues would no longer look like a semi-circle, but more like a bell-sloped curve. In the next three figures, the connection probability for vertices i and j is

$$p_{ij} = \lambda r\left(\frac{i}{n}\right) r\left(\frac{j}{n}\right) = \lambda \frac{i}{n}\frac{j}{n} = \lambda \frac{ij}{n^2}$$

with $\lambda \in \{0.05, 0.5, 2\}$ The eigenvalues in each of the three pictures are scaled with the value of λ. It is obvious that the larger the value of λ gets, the leaner the shape of the histograms becomes.

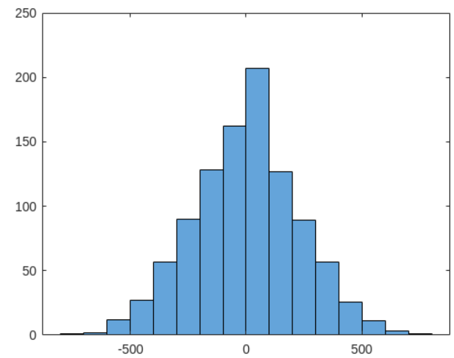


Figure 4: Scaled eigenvalues when $\lambda = 0.05$.

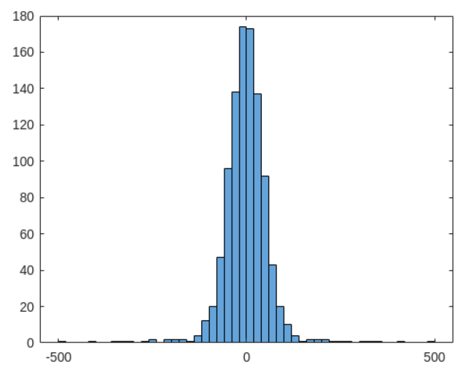


Figure 5: Scaled eigenvalues when $\lambda = 0.5$.

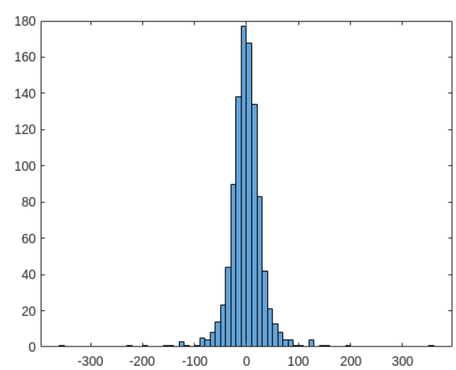


Figure 6: Scaled eigenvalues when $\lambda = 2$.

4.5 Conclusion

In Chapter 1, the Friendship Paradox was introduced, both in a non-mathematical and a mathematical way. Along with this, an example and an application were provided. To understand the mathematical notions of the Friendship Paradox, definitions of some tools were necessary, like that of simple graphs and corresponding adjacency matrices. As a conclusion of this chapter, two different proofs of the Friendship Paradox were given.

The Erdős–Rényi random graph was introduced and analyzed in Chapter 2, i.e., our interest shifted from deterministic graphs to random graphs. Before determining the values of $\mathbb{E}\left[\Delta_i^\star\right]$ and $\mathbb{E}\left[\Delta^\star\right]$ (which turned out to be the same), the distribution of the degree of a vertex was studied. For the case where the connection probability depended on n, a specific limiting probability distribution for Δ_i was found. From here, the two expectations were calculated, both for the case where p depended on n and p did not depend on n. Both values ($1 - \lambda e^{-\lambda}$ and $1 - p$) where found to be non-negative, which is in accordance with the statement of the Friendship Paradox.

Further graph models were introduced in Chapter 3, including the Configuration Model, where all the degrees of the vertices are fixed beforehand. Unfortunately, finding the probability distribution for Δ_i and Δ turned out too be to difficult: $\mathbb{E}\left[\Delta_i\right]$ and $\mathbb{E}\left[\Delta\right]$ (here also non-negative) are all we could compute. Subsequently, we moved to the inhomogeneous Erdős–Rényi random graph, where different pairs of vertices are connected with different connection probabilities. The computations were heavier than in Chapter 2, but with the right approximations and coupling arguments the degree distribution and expectations of Δ_i and Δ could still be found in the limit as $n \to \infty$. Moreover, they were in accordance with the outcomes of the computations made in Chapter 2.

In Chapter 4 we moved from mathematical computations to simulations, which were used to show the correctness of the computations in the previous chapters. The calculations were more accurate for the homogeneous Erdős–Rényi random graph than for the inhomogeneous Erdős–Rényi random graph, mainly because we used an approximation for the latter. Also, the eigenvalues were studied for a given (modified) friendship matrix $\widetilde{F}$, and under the assumption that $np \to \infty$ as $n \to \infty$, the graph of the distribution function of these eigenvalues was found to look like a semi-circle.

4.6 Open questions

A few open questions are worth mentioning:

1. For the graph models considered in this bachelor thesis, the variance of both Δ_i and Δ (or $\Delta_i^\star$ and $\Delta^\star$) was not computed. It would be interesting to obtain an idea of how large these variances are, and how they depend on the number of vertices. Is there a concentration around the mean, i.e., is the value of $\Delta^\star - \mathbb{E}\left(\Delta^\star\right)$ small for large n? The two tables in Chapter 4 give a rough idea, but an explicit computation in necessary to identify the order of magnitude of the variance.

2. Can one derive a more general result for graphs that look locally like a Poisson tree, i.e., graphs where the degrees follow approximately a Poisson(α) distribution for some $\alpha \in (0, \infty)$?

3. How far can one extend friendships? For the friendship paradox, we compared the number of friends of a given person with the average number of friends of friends of this person, with

the conclusion that on average a person is less popular than his or her friends. Does this also hold when we look at "friends of friends of friends" and, if so, can we come up with a proof?

4. For the computation of Δ_i we used the adjacency matrix and the degree vector (which contains information about the number of friends of a vertex). In the paper written by G. Cantwell, A. Kirkley and M. Newman, alternative Δ_i's are computed using quantities other than just the degrees, e.g.

$$\Delta_i^{(x)} = \frac{1}{d_i} \sum_{j=1}^{n} A(i,j) x_j - x_i,$$

where $x = (x_1, \dots, x_n)$ is some vector associated with the graph, e.g. x_i is the number of triangles vertex i lies in.

References

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[2] A. Kirkley G. Cantwell and M. Newman. The friendship paradox in real and model networks. *Journal of Complex Networks*, 9(2), 2021.

[3] Y. Novick A. Swami S. Pal, F. Yu and A. Bar-Noy. A study on the friendship paradox–quantitative analysis and relationship with assortative mixing. *Applied Network Science*, 4(1):1–26, 2019.

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A Appendix: Matlab code

```
%Stochastic matrix and its eigenvalues
n = input('n = ');
A = zeros(n,n);
%All vertices are not connected to itself.
for i = 1:length(A)
    A(i,i) = 0;
end
%Generating the Erdos-R nyi random graph
p = 0.5; %connection probability
for (i = 1:length(A))
    for (j = i+1:length(A))
        r = unifrnd(0,1);
        if (r <= p)
            A(i,j) = 1;
            A(j,i) = 1;
        end
    end
end

%Friendship matrix and its eigenvalues
F = zeros(n,n);
d = zeros(1,n);
%Creating the degree vector.
for (i=1:n)
    for (j=1:n)
        d(i) = d(i) + A(i,j);
    end
end
%The computation of all the \Delta_{i}'s:
deltavector = zeros(1,n);
for (i=1:n)
    if (d(i) == 0)
        deltavector(i) = 1;
    else
        sumfof = 0; %sum of the friends of friends.
        for (j=1:n)
            sumfof = sumfof + A(i,j)*d(j);
        end
        sumfof = sumfof/d(i);
        deltavector(i) = sumfof - d(i);
    end
end
Delta = sum(deltavector)/n;
%Generating the Friendship matrix
%We make it symmetric in order to have real eigenvalues, while not
%changing the properties of Delta (it is still non-negative).
for (i=1:n)
    for (j=1:n)
        if (d(i) == 0 | d(j) == 0)
            F(i,j) = 0;
        else
            F(i,j) = A(i,j)*(d(i)/d(j) + d(j)/d(i))-2*p;
        end
    end
end
f = eig(F)/(2*sqrt(n*p*(1-p)));
```

```
%Output
%A %Adjacency matrix
%d %Degree vector
%F %Friendship matrix
%f % Eigenvalues of F
plot(f,'.');
h = histogram(f) %Histogram of the eigenvalues of F
%deltavector %The vector with all the \Delta_{i}'s
Delta %The arithmatic mean of the \Delta_{i}'s
```