Local Convergence of Random Graphs

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Bachelor thesis ASB

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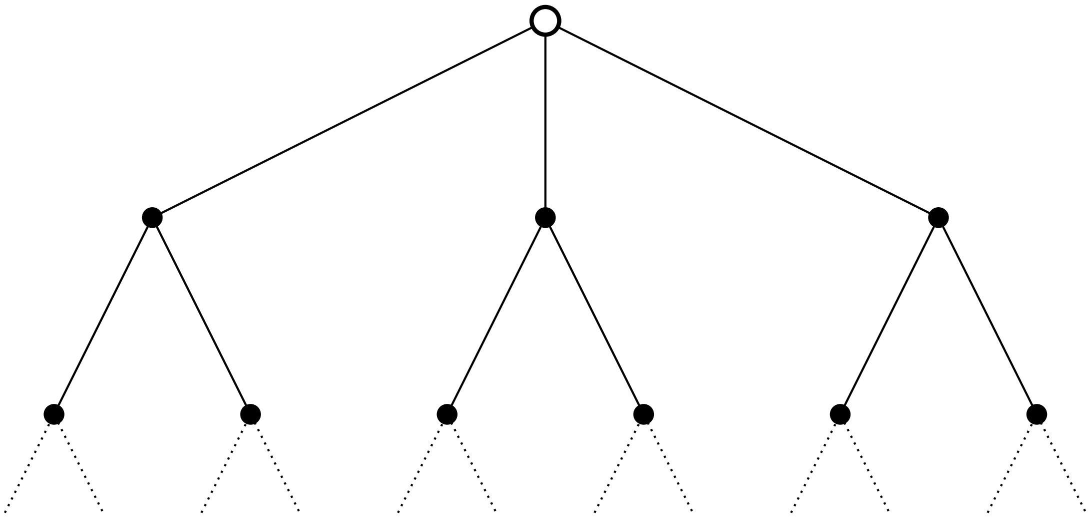

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1 Introduction

Local convergence is a type of convergence on graph sequences, which can be used for both deterministic and random graph sequences. It serves an important role in large scale graph models as the local convergence shows what a small part of large graphs looks like. Often a tree-like structure is observed in random graph models, giving a more precise view of these large graphs. This thesis serves as a brief introduction to the concept local convergence of random graphs and only requires some basic prior knowledge of probability theory. The topic is a relatively new part of mathematics as it is first introduced this century in 2000 by Benjamini and Schramm [2] and separately by Aldous and Steele [1] in 2004.

Local convergence requires graphs to be sparse, which are graphs that contain a relative low amount of edges. There are different types of convergence introduced for sparse graphs and we will cover two of them in detail. One is the local weak convergence and the other is local convergence in probability. There is a third type of convergence that has a similar construction to the other two, namely local almost sure convergence, which is not covered in this thesis. One can find more details about it in the book [6]. The aforementioned convergences are only for sparse graphs and when graphs are dense, the situation is completely different. Therefore other techniques must be used. For example graphons are often used to define limits of dense graph sequences, see [4]. Graphons are graph limits in the form of a function, giving each pair of vertices a certain probability of getting an edge between them.

This thesis is based on Chapter 2 of the book Random Graphs and Complex Networks Volume II by Remco van der Hofstad [6] and also uses some content of its predecessor, Volume I [5]. Volume II is recommended for anyone who wants to read more into local convergence or anyone who is interested other topics about random graphs as Chapter 2 is only a small part of the book. Be aware that definitions and notation can differ between this thesis and the book. Some concepts of the book do not translate well into a short paper and have been altered to better fit this thesis. On top of that, the book is not published yet, making it prone to changes. There are also alternative sources covering the same topic, such as Bordavene's notes [3], covering it in a more compact way.

The basics of this thesis are stated in Chapter 2 by defining both graphs and sequences for reference. Also the combination of these two concepts is introduced as a simple type of convergence of certain kinds of graphs, namely that of rooted graphs. This is supported by a metric on the rooted graph space. From this the actual local convergence of graph sequences is defined, which is done in Chapter 3. Two different types of local convergence are covered in this thesis: local weak convergence and local convergence in probability. Furthermore two theorems are proven that are very useful for determining if a graph sequence converges locally by reducing the problem in this chapter. Local convergence is based on real convergence of functions over graphs. The first theorem of the two reduces the amount of functions the convergence has to hold for to only a single class of graph functions. The second theorem shows that it can be simplified even further by limiting the set of functions dependent on the graph sequence. Some examples are given to clarify how local convergence works in practise.

The second half of this thesis consists of Chapter 4 and 5 in which concrete uses of the local convergence are shown by applying it on well known random graph models. Chapter 4 covers a special case of the configuration model that generates graphs with a constant degree. Chapter 5 is about the Erdös-Rényi model which generates graphs by drawing each edge independently with the same probability.

2 Convergence and graphs

In this section convergence and graphs are introduced, with which the convergence of rooted graph sequences will be defined. Even though these are two basic concepts, they require some good thoughts to properly combine them. The convergence introduced in this section is not the local convergence that will be covered in section 3.

Graphs are very important in this thesis. Therefore we give a detailed definition of graphs, including different types of graphs and their properties.

Definition 2.1 (Graph)**.** A graph $G = \big(V, E\big)$ consists of a set of vertices V and a set of edges E. Each edge connects two different vertices and is denoted by $e = \{v, w\} \subset V$. The degree d_v of a vertex $v \in V$ is the amount of adjacent edges

$$d_v = \#\big\{w \in V : \{v, w\} \in E\big\}. \tag{2.1}$$

If the degree of all vertices is finite, the graph is called locally finite. △

A graph can not contain two different edges between the same two vertices, nor can it contain an edge that connects a vertex to itself. If these so called double edges and self-loops are allowed, the object is called a multigraph.

Definition 2.2 (Subgraph)**.** A graph $H(V', E')$ is called the subgraph of graph $G(V, E)$ if $V' \subset V$ and $E' \subset E$. This is denoted as $H \subset G$. △

Definition 2.3 (Paths in a graph)**.** Let $G = (V, E)$ be a graph and $u, v \in V$ two vertices. A path P from u to v is a sequence of vertices $w_0, w_1, w_2, ..., w_n$ such that $w_0 = u, w_n = v$ and for all $0 \leqslant i < n$, $\{w_i, w_{i+1}\} \in E$. The length of path P is equal to n and the distance $d_G(u, v)$ is the length of the shortest path from u to v. △

Definition 2.4 (Connected graph)**.** Given a graph $G = (V, E)$, two vertices $v, w \in E$ are connected if there exists a path between v and w. If every pair of vertices in G is connected, the graph is called a connected graph.

A connected component C of graph G is a connected subgraph of G, such that there does not exist another connected subgraph D of G with the property $C \subset D \subset G$. △

Graphs can look very similar to each other, for example the two graphs in Figure 1. By rearranging the vertices, the graphs are equal to each other. These graphs are called isomorphic. A more formal, but less visual definition of isomorphic graphs is as follows:

Definition 2.5 (Graph isomorphism)**.** Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a bijection $\phi : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\big\{\phi(u), \phi(v)\big\} \in E_2$. This is written as $G_1 \simeq G_2$. Graph isomorphism defines an equivalence relationship in the set of graphs. △

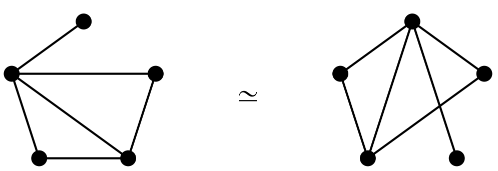

Figure 1: Two isomorphic graphs.

For convergence of graphs, the difference between isomorphic graphs is irrelevant and such graphs are treated as equal. The collection of all locally finite graphs up to isomorphism is denoted by $\mathcal{G}$. Convergence of sequences is a well known topic in mathematics, but extending this to graphs is not as trivial as it may sound. Before convergence of graph sequences is introduced, we look at a more basic example of convergence, namely that of real sequences. When we understand how convergence of real sequences works, we can extend that to graph sequences.

Definition 2.6 (Convergence of real numbers)**.** Let $(a_n)_{n\geqslant 0}$ be a real sequence. It is said to converge to the number $a \in \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that, for all $n \geqslant N$

$$|a_n - a| < \epsilon. \tag{2.2}$$

△

A real sequence $(a_n)_{n\geqslant 0}$ converges to a if the distance between a_n and a goes to 0 as seen in Definition 2.6. The key word here is distance. In Equation (2.2) the basic metric on $\mathbb{R}$ is used to measure the distance between two real numbers, which is the absolute value of their difference. To construct convergence on $\mathcal{G}$, a metric on $\mathcal{G}$ is needed, but here lies a problem. We do not have a natural metric on $\mathcal{G}$ that provides useful information, as it is difficult to define distance between two graphs in a meaningful way.

Remark 2.7. There do exist metrics on $\mathcal{G}$. For example, let $G, H \in \mathcal{G}$ be two graphs and set the metric d_0 as follows.

$$d_0(G,H) = \begin{cases} 0, & \text{if } G \simeq H. \\ 1, & \text{if } G \not\simeq H. \end{cases} \tag{2.3}$$

This metric does not give any structure on $\mathcal{G}$ and other metrics have similar problems. △

As the graph space $\mathcal{G}$ does not have a nice metric, which is needed for convergence, we introduce a new space that is very similar to $\mathcal{G}$. This space is the rooted-graph space, which does have a nice metric. The difference between the two spaces is that each graph in the rooted-graph space has exactly one marked vertex that is called the root. Isomorphism is also a bit different on rooted graphs as the root has to be taken into account.

Definition 2.8 (Rooted graph)**.** A rooted graph (G, o) is a graph $G = (V, E)$ together with a distinguished vertex $o \in V$ called the root. Two rooted graphs (G_1, o_1) and (G_2, o_2) are said to be isomorphic if there exists an isomorphism $\phi : V_1 \to V_2$, such that $\phi(o_1) = o_2$. We write this as $(G_1, o_1) \simeq (G_2, o_2)$. The space of all locally finite rooted connected graphs up to isomorphism is denoted by $\mathcal{G}_\star$ △

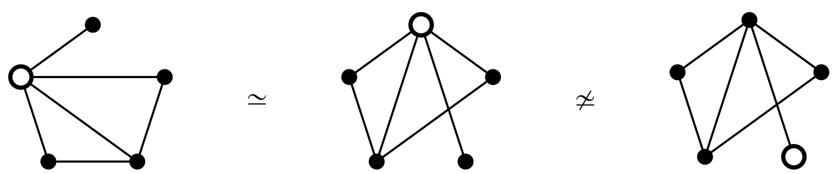

Figure 2: From left to right, two isomorphic rooted graphs and a rooted graph that is not isomorphic to the other two.

In Figure 2 there are examples of isomorphic and non-isomorphic rooted graphs. Just as with normal graphs, isomorphism can be seen as two graphs being the same by moving the vertices around. The non-isomorphic rooted graph shows that the location of the root is relevant for determining if two graphs are isomorphic.

Note that the graph space $\mathcal{G}_\star$ only contains connected graphs, unlike $\mathcal{G}$ which does contain disconnected graphs. This small difference between the two spaces is important as the connectedness of $\mathcal{G}_\star$ is required for the metric that will be defined on this set. On top of that, disconnected graphs of $\mathcal{G}$ are used in later sections, making this difference between the two spaces necessary.

The neighbourhoods around the root are subgraphs that limit a graph to vertices and corresponding edges that are up to a certain distance apart from the root.

Definition 2.9 (Neighbourhood in a rooted graph)**.** Let (G, o) be a rooted graph with $G = (V, E)$ and $r \in \mathbb{N}$. The neighbourhood with radius r around root o in G is the rooted subgraph of (G, o) containing all vertices with a distance of at most r from the root o and the edges that are in a path starting from the root of length at most r. This neighbourhood is denoted as $B_r^{(G)}(o)$. A more formal definition is as follows:

$$V\big(B_r^{(G)}(o)\big) = \big\{x \in V : d_G(o, x) \leqslant r\big\} \tag{2.4}$$
$$E\big(B_r^{(G)}(o)\big) = \Big\{\{x, y\} \in E : x, y \in V, d_G(o, x) < r\Big\}$$
$$B_r^{(G)}(o) = \Big(\big(V\big(B_r^{(G)}(o)\big), E\big(B_r^{(G)}(o)\big)\big), o\Big).$$

△

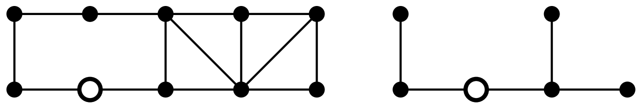

Figure 3: Left the rooted graph (G, o) and right the neighbourhood $B_2^{(G)}(o)$

The neighbourhood around the root makes it possible to compare how different two rooted graphs are, with which the metric can be constructed. For this we seek the biggest neighbourhood of two graphs where they are still the same.

Definition 2.10 (Metric on the rooted graph space)**.** For two rooted connected graphs (G_1, o_1) and (G_2, o_2), define the variable $R^\star$ and metric $d_\star$ on $\mathcal{G}_\star$ as follows:

$$R^\star\big((G_1, o_1), (G_2, o_2)\big) = \sup\Big\{r : B_r^{(G_1)}(o_1) \simeq B_r^{(G_2)}(o_2)\Big\}, \tag{2.5}$$

$$d_\star\big((G_1, o_1), (G_2, o_2)\big) = \frac{1}{R^\star + 1}. \tag{2.6}$$

△

The distance between two rooted graph ranges from 0 to 1. By definition of metrics, two isomorphic graphs have distance 0. Two rooted graphs that are only equivalent in the neighbourhood of radius 0 have distance of 1. In Figure 4 a non trivial example of the distance between two graphs is given.

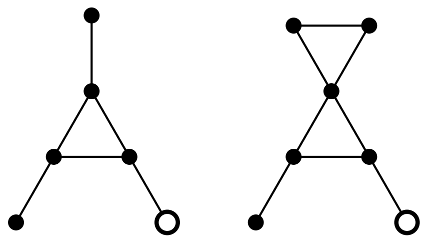

Figure 4: Two rooted graphs with $R^\star = 2$ and distance $\frac{1}{3}$ on the metric $d_\star$.

The metric $d_\star$ on the rooted graph space $\mathcal{G}_\star$ induces a notion of convergence on $\mathcal{G}_\star$. The definition is the same as Definition 2.6, but with a different space and metric.

Definition 2.11 (Convergence of rooted graphs)**.** Let $(G_n, o_n)_{n \geqslant 0}$ be a sequence of connected rooted graphs. It is said to converge to $(G, o) \in \mathcal{G}_\star$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that, for all $n \geqslant N$,

$$d_\star\big((G_n, o_n), (G, o)\big) < \epsilon. \tag{2.7}$$

△

Let us now look at some examples of rooted graph sequences that do and do not converge to get a better understanding of the concept. Let $(L_n, 0)$ be the rooted line graph that consists of $n+1$ vertices connected in a line with the root at one end as in Figure 5. The sequence $(L_n, 0)_{n\geqslant 0}$ converges to the rooted graph of $\mathbb{N}$ with the root at 0 that is called $(\mathbb{N}, 0)$. This can be proven by computing the distance between $(L_n, 0)$ and $(\mathbb{N}, 0)$. From vertex 0 up to vertex n, both graphs are isomorphic. The value n is also the last vertex of $(L_n, 0)$, unlike $(\mathbb{N}, 0)$ which is infinite. Therefore $R^\star\big((L_n, 0), (\mathbb{N}, 0)\big) = n$ and $(L_n, 0)$ converges to $(\mathbb{N}, 0)$ as n goes to infinity, since

$$d_\star\big((L_n, o_n), (\mathbb{N}, o)\big) = \frac{1}{R^\star + 1} = \frac{1}{n+1} \to 0. \tag{2.8}$$

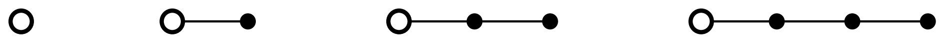

Figure 5: The rooted line graph $(L_n, 0)$ for n from 0 to 3.

Not all rooted graph sequences converge. For example, let (S_n, o_n) be the rooted star graph with $n+1$ vertices. In this rooted graph there exists a root and n other vertices that are connected only to the root, see Figure 6 for an example. In the sequence $(S_n, o_n)_{n\geqslant 0}$, the degree of the root is equal n, which is different for every single star graph. Due to this, the neighbourhood $B_1^{(S_n)}(o_n)$ is different for each n. Let $(G, o) \in \mathcal{G}_\star$ be any rooted graph. $R^\star\big((S_n, o_n), (G, o)\big)$ is equal to 0 for all star graphs, except for the star graph with $d_{o_n} = n = d_o$. For all $n > d_o$, the degree of o_n is greater than the degree of o, thus the distance between (S_n, o_n) and (G, o) is 1. This implies that the sequence of rooted star graphs does not converge.

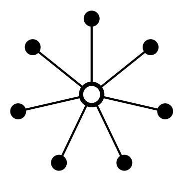

Figure 6: The rooted star graph (S_7, o_7).

The metric $d_\star$ also induces a notion of continuity for functions $h : \mathcal{G}_\star \to \mathbb{R}$. Bounded and continuous functions play an important role in local convergence. Whether a function on $\mathcal{G}_\star$ is bounded is analogous to the real case. Continuity on the other hand is quite different compared to functions over $\mathbb{R}$. To define continuity of functions from $\mathcal{G}_\star$ to $\mathbb{R}$, the metric of both spaces is required. On $\mathbb{R}$ we use the standard metric and on $\mathcal{G}_\star$ we use the metric $d_\star$ from Definition 2.10.

An example of a bounded and continuous function from $\mathcal{G}_\star$ to $\mathbb{R}$ is given by

$$\begin{array}{rccl} p_r^{(H,o')} : & \mathcal{G}_\star & \to & \mathbb{R} \\ & (G, o) & \mapsto & \mathbf{1}_{\left\{B_r^{(G)}(o) \simeq B_r^{(H)}(o')\right\}}, \end{array} \tag{2.9}$$

for $(H, o') \in \mathcal{G}_\star$ and $r \in \mathbb{N}$. The function is bounded and continuous for any $r \in \mathbb{N}$ and $(H, o') \in \mathcal{G}_\star$, see Lemma 2.12. The functions may look deceivingly discontinuous as similar real valued functions are indeed discontinuous. Due to the structure of $\mathcal{G}_\star$ the given functions are actually continuous. This function is will be used later and is denoted as $p_r^{(H,o')}$.

Lemma 2.12 (Continuous and bounded function)**.** For all $(H, o') \in \mathcal{G}$ and $r \in \mathbb{N}$, the function $p_r^{(H,o')} : \mathcal{G}_\star \to \mathbb{R}$ is bounded and continuous. △

Proof. Let $(H, o') \in \mathcal{G}$ and $r \in \mathbb{N}$ be given. It is clear that $p_r^{(H,o')}$ can only take on the values 0 and 1 and is therefore bounded.

The function $p_r^{(H,o')}$ is continuous if for all $(G, o_G) \in \mathcal{G}_\star$ and all $\epsilon > 0$, there exist a $\delta > 0$ such that all $(F, o_F) \in \mathcal{G}_\star$ with $d_\star\big((F, o_F), (G, o_G)\big) < \delta$ satisfy $\big|p_r^{(H,o')}(F, o_F) - p_r^{(H,o')}(G, o_G)\big| < \epsilon$.

Assume $\delta = \frac{1}{r}$ and let $(G, o_G) \in \mathcal{G}_\star$ be given. All rooted graphs (F, o_F) that follow the criteria satisfy $d_\star\big((F, o_F), (G, o_G)\big) = \frac{1}{R^\star+1} < \frac{1}{r}$. As $R^\star$ is a natural number or infinite it follows that $R^\star \geqslant r$. Thus by definition of $R^\star$, their neighbourhoods of radius r are isomorphic $B_r^{(F)}(o_F) \simeq B_r^{(G)}(o_G)$ and have the same image over $p_r^{(H,o')}$.

$$\Big|p_r^{(H,o')}(F, o_F) - p_r^{(H,o')}(G, o_G)\Big| = \Big|\mathbf{1}_{\big\{B_r^{(F)}(o_F)\simeq B_r^{(H)}(o')\big\}} - \mathbf{1}_{\big\{B_r^{(G)}(o_G)\simeq B_r^{(H)}(o')\big\}}\Big| = \tag{2.10}$$

$$\Big|\mathbf{1}_{\big\{B_r^{(G)}(o_G)\simeq B_r^{(H)}(o')\big\}} - \mathbf{1}_{\big\{B_r^{(G)}(o_G)\simeq B_r^{(H)}(o')\big\}}\Big| = 0$$

□

3 Local convergence of graphs

In this section the definition of two different local convergences of random graphs are covered. The aforementioned convergence from Definition 2.7 regards sequences of deterministic rooted graphs and does not state anything about sequences of random graphs. Therefore a different type of convergence is needed to cover randomness. We first discuss what the desired properties are for convergence of random graph sequences. Next the definition of both local weak convergence and convergence in probability is introduced. In between and at the end there are examples to further clarify the concept.

Two commonly used types of convergence for sequences of random variables are convergence in distribution and in probability. The latter is the strongest convergence of the two as convergence in probability implies convergence in distribution.

Definition 3.1 (Convergence of random variables)**.** Let $(X_n)_{n\geqslant 0}$ be a sequence of random variables.

1. $(X_n)_{n\geqslant 0}$ converges in distribution to the random variable X if for all bounded and continuous functions $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}\big[f(X_n)\big] = \mathbb{E}\big[f(X)\big]. \tag{3.1}$$

   Convergence in distribution is written as $X_n \xrightarrow{d} X$. This convergence is also called local weak convergence.

2. The sequence of random variables $(X_n)_{n\geqslant 0}$ converges in probability to the random variable X if, for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\big(|X_n - X| > \epsilon\big) = 0. \tag{3.2}$$

   Convergence in probability is written as $X_n \xrightarrow{\mathbb{P}} X$.

△

Two types of local convergence are covered in this thesis and each one is comparable to one type of convergence of random variables as mentioned in Definition 3.1. The convergence of Definition 2.11 states whether rooted graph sequences from the space $\mathcal{G}_\star$ converge to a rooted graph. Local convergences on the other hand states whether unrooted graph sequences from the space $\mathcal{G}$ converge to a rooted graph.

Definition 3.2 (Local convergence of random graphs)**.** Let $(G_n)_{n\geqslant 0}$ be a sequence of finite graphs, with for each $n \in \mathbb{N}$, $G_n = (V_n, E_n) \in \mathcal{G}$.

1. $(G_n)_{n\geqslant 0}$ converges locally in distribution to the random rooted graph (G, o) with law μ if for all bounded and continuous functions $h : \mathcal{G}_\star \to \mathbb{R}$ the following holds:

$$\mathbb{E}\big[h(G_n, o_n)\big] = \frac{1}{|V_n|} \sum_{v \in V_n} \mathbb{E}\big[h(G_n, v)\big] \to \mathbb{E}_\mu\big[h(G, o)\big]. \tag{3.3}$$

   This is denoted as $(G_n, o_n) \xrightarrow{d} (G, o)$. The root o_n in equation (3.3) is an uniformly random chosen vertex of G_n. This is equal to taking the average of all $|V_n|$ possible choices for placing the root. It should converge to the expectation of $h(G, o)$ with respect to law μ.

2. $(G_n)_{n\geqslant 0}$ converges locally in probability to the random rooted graph (G, o) with law μ if for all bounded and continuous functions $h : \mathcal{G}_\star \to \mathbb{R}$ the following holds. The root o_n in Equation (3.4) is an uniformly random chosen vertex of G_n. This is also equal to taking the average of all $|V_n|$ possible choices for placing the root. It should converge in probability to the expectation of $h(G, o)$ with respect to law μ.

$$\mathbb{E}\big[h(G_n, o_n)\big|G_n\big] = \frac{1}{|V_n|} \sum_{v \in V_n} h(G_n, v) \xrightarrow{\mathbb{P}} \mathbb{E}_\mu\big[h(G, o)\big]. \tag{3.4}$$

△

Let us go into more detail of Definition 3.2, beginning with the local weak convergence as seen in Equation (3.3). The definition starts with an unrooted random graph sequence. For each graph G_n there is the expectation of the following. First, uniformly at random a vertex is chosen as the root. All components that do not contain the root are discarded, making (G_n, o_n) a connected rooted graph. Remember that the rooted graph space $\mathcal{G}_\star$ does not contain disconnected graphs and therefore we do not consider those. By choosing the root at random, the entire graph is sampled, even if the unrooted graph is disconnected. In total there are $\big|V_n\big|$ vertices where the root can be placed. To compute the expectation one can take the average over all possibles locations for the root and dividing everything by $\big|V_n\big|$. Next a function $h : \mathcal{G}_\star \to \mathbb{R}$ is applied to the now rooted random graph. The convergence must hold for every continuous and bounded function h. An example of such a function is $p_r^{(H_\star)}$ from Equation (2.9). The sequence should converge to the expectation of the same function h over the random rooted graph (G, o), which is distributed according to μ.

Convergence locally in probability (3.4) has one major difference with the local weak convergence. The random graph G_n is determined before the expectation is taken, creating a sequence of random variables opposed to the sequence of real numbers in the case of Equation (3.4). The convergence in probability of the graph sequence is the same convergence as in Definition 3.1. Local convergence in probability is the strongest of the two types of as it implies local convergence in distribution.

One might also expect a type of local convergence based on almost sure convergence in probability. Almost sure local convergence of random graphs also exists, but this variant is beyond the scope of this thesis. It becomes relevant when investigating deeper into this topic. The book Random Graphs and Complex Networks does cover this topic [6].

To get a better understanding of local convergence, some examples will be shown. The most basic case is a deterministic constant sequence $(G_n)_{n\geqslant 0}$. For each $n \in \mathbb{N}$, let the graph G_n be equal to C as seen in figure 7.

This deterministic constant graph sequence, converges locally in probability to the random graph (G, o) of Figure 8. This example also shows how local convergence handles disconnected graphs. Each component is represented with a probability equal to the probability that one of its vertices are chosen as the root. It is important that disconnected graphs are allowed in the original graph sequence for

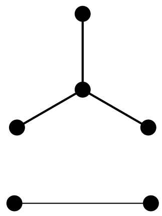

Figure 7: The unrooted disconnected graph C.

local convergence as random graph models are often able to generate disconnected graphs as seen in Section 5.

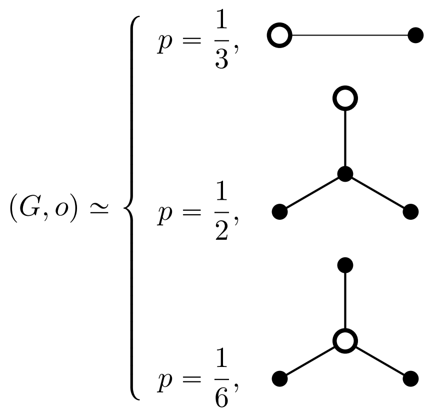

Figure 8: The distribution of the rooted random graph (G, o).

So far only the local convergence of basic graph sequences are covered. Proving local convergence of more complex graph sequences will use the following two theorems. These theorems will show that the two local convergences are much more manageable than it currently seems. Instead of showing that the convergence holds for all bounded and continuous function $h : \mathcal{G}_\star \to \mathbb{R}$, the theorems show that only checking a few functions h is sufficient in most cases.

Theorem 3.3 (Simplification of local convergence)**.** *Let $(G_n)_{n\geqslant 0}$ be a sequence of finite graphs, with for each $n \in \mathbb{N}$, $G_n = (V_n, E_n) \in \mathcal{G}$.*

1. *$(G_n)_{n\geqslant 0}$ converges locally weakly to the rooted graph (G, o) with law μ if and only if for all rooted graphs $(H, o') \in \mathcal{G}_\star$ and all $r \in \mathbb{N}$ the following holds:*

$$\mathbb{E}\big[p_r^{(H,o')}(G_n)\big] = \frac{1}{|V_n|} \sum_{v \in V_n} \mathbb{P}\big(B_r^{(G_n)}(v) \simeq B_r^{(H)}(o')\big) \to \mu\big(B_r^{(G)}(o) \simeq B_r^{(H)}(o')\big). \tag{3.5}$$

2. *$(G_n)_{n\geqslant 0}$ converges locally in probability to the rooted graph (G, o) with law μ if and only if for all rooted graphs $(H, o') \in \mathcal{G}_\star$ and all $r \in \mathbb{N}$ the following holds:*

$$p_r^{(H,o')}(G_n) = \frac{1}{|V_n|} \sum_{v \in V_n} \mathbf{1}_{\left\{B_r^{(G_n)}(v) \simeq B_r^{(H)}(o')\right\}} \xrightarrow{\mathbb{P}} \mu\big(B_r^{(G)}(o) \simeq B_r^{(H)}(o')\big). \tag{3.6}$$

Sketch of the proof. The proof of the theorem will be sketched for local convergence in distribution. Let $(G_n)_{n\geqslant 0}$ be a sequence of finite graphs and $(G, o) \in \mathcal{G}_\star$ a rooted graph. For the proof we need to show that $(G_n)_{n\geqslant 0}$ converges in distribution to the graph (G, o) if and only if for all $(H, o') \in \mathcal{G}_\star$ and all $r \in \mathbb{N}$, the convergence in Equation (3.5) holds.

Assume $(G_n)_{n\geqslant 0}$ converges locally in distribution to the graph (G, o) and let $(H, o') \in \mathcal{G}_\star$ and $r \in \mathbb{N}$. According to Lemma 2.12 the function $p_r^{(H,o')}(G_n)$ is bounded and continuous for all (H, o') and r. Because $(G_n)_{n\geqslant 0}$ converges locally in distribution, Equation (3.3) holds for all bounded and continuous functions h, thus it also holds for $h = p_r^{(H,o')}(G_n)$. For all functions of this form Equation (3.3) reduces to Equation (3.5).

For the implication the other way around we assume that the convergence of Equation (3.5) holds for all $(H, o') \in \mathcal{G}_\star$ and $r \in \mathbb{N}$. The sequence $(G_n)_{n\geqslant 0}$ is tight, meaning that every subsequence of $(G_n)_{n\geqslant 0}$ has a further subsequence that converges. This will not be proven in this thesis as it deviates too much from our topic. The book Complex Networks and Random Graphs 2 [6] has a detailed proof of the tightness on page 485. The limit along the converging subsequence of the subsequence for all $r \in \mathbb{N}$ converges to $B_r^{(G)}(o)$ up to radius r in distribution. For all r together this uniquely identifies the distribution of what the converging subsequence converges to, which is (G, o). As this holds for very subsequence of $(G_n)_{n\geqslant 0}$, it converges locally in distribution to (G, o) proving the implication.

Similarly the theorem can be proven for local convergence in probability too. □

To summarize Theorem 3.3, instead of checking the convergence in Equation (3.3) to all continuous and bounded functions h, it is sufficient to only check all functions of the form $p_r^{(H,o')}$. The expectations are explicitly written out in the theorem, which is computed by summing over all possible outcomes for the randomly chosen root v. Equation (3.5) is different from (3.6), due to the presence of a convergence in probability in the equation. This comes from the fact that the graph randomness is also averaged out for the local convergence in distribution, while the randomness of the graphs is still present in Equation (3.6).

On top of Theorem 3.3 it might be unnecessary to check the convergence for all functions of the form $p_r^{(H,o')}$. In fact the theorem below shows that for a graph sequence that converges locally to the random graph (G, o) with law μ, it is sufficient to only take into account all rooted graphs (H, o') that are isomorphic to graphs that μ can generate.

Theorem 3.4. *Let $(G_n, o_n)_{n\geqslant 0}$ be a sequence of random graphs and (G, o) a random rooted graph on $\mathcal{G}_\star$ having law μ. Denote $\mathcal{T}_\star$ as a subset of $\mathcal{G}_\star$ with the property $\mu\big((G, o) \in \mathcal{T}_\star\big) = 1$. Define the set $\mathcal{T}_\star(r) = \big\{B_r^{(H)}(o') : (H, o') \in \mathcal{T}_\star\big\}$. Then $(G_n, o_n)_{n\geqslant 0}$ converges locally in distribution to (G, o) if equation (3.5) holds for all $(H, o') \in \mathcal{T}_\star(r)$ and all $r \in \mathbb{N}$.*

This theorem also holds for local convergence in probability by applying it to equation (3.6).

Proof. The proof will be given for local convergence in distribution. Let $(G_n, o_n)_{n\geqslant 0}$ be a graph sequence that converges locally in distribution to the rooted graph (G, o), let $r \in \mathbb{N}$ and $\mathcal{T}_\star(r)$ as in the statement of the theorem.

First note that $\mathcal{T}_\star(r)$ is a countable set and note $\mu\big(B_r^{(G)}(o) \in \mathcal{T}_\star(r)\big) = 1$. From these facts combined follows that for every $\epsilon > 0$ there exists $m \in \mathbb{N}$ and a subset $\mathcal{T}_\star(r, m) \subset \mathcal{T}_\star(r)$ of size at most m such that $\mu\big(B_r^{(G)}(o) \in \mathcal{T}_\star(r, m)\big) \geqslant 1 - \epsilon$. For this subset the following bound can be made.

$$\mathbb{P}\Big(B_r^{(G_n)}(o_n) \notin \mathcal{T}_\star(r)\Big) = 1 - \mathbb{P}\Big(B_r^{(G_n)}(o_n) \in \mathcal{T}_\star(r)\Big) \leqslant 1 - \mathbb{P}\Big(B_r^{(G_n)}(o_n) \in \mathcal{T}_\star(r, m)\Big) \tag{3.7}$$

Next we take $n \to \infty$ in Equation (3.7).

$$\limsup_{n\to\infty} \mathbb{P}\Big(B_r^{(G_n)}(o_n) \notin \mathcal{T}_\star(r)\Big) \leqslant 1 - \liminf_{n\to\infty} \mathbb{P}\Big(B_r^{(G_n)}(o_n) \in \mathcal{T}_\star(r, m)\Big) = \tag{3.8}$$
$$1 - \mu\Big(B_R^{(G)}(o) \in \mathcal{T}_\star(r, m)\Big) \leqslant 1 - (1 - \epsilon) = \epsilon$$

Since $\epsilon > 0$ is arbitrary small, it follows that $\mathbb{P}\big(B_r^{(G_n)}(o_n) \notin \mathcal{T}_\star(r)\big) \to 0$ as $n \to \infty$. So for every $(H, o') \notin \mathcal{T}_\star(r)$ and $n \to \infty$ we have:

$$\mathbb{P}\Big(B_r^{(G_n)}(o_n) \simeq B_r^{(H)}(o')\Big) \to 0 = \mu\Big(B_r^{(G)}(o) \simeq B_r^{(H)}(o')\Big) \tag{3.9}$$

So for all $(H, o') \notin \mathcal{T}_\star(r)$ the convergence holds, making it unnecessary to check these (H, o') for local convergence in distribution and proving the theorem.

The theorem for local convergence in probability can be proven in a similar way. $\square$

Here is an example where Theorem 3.3 and 3.4 are applied. Let $(L_n)_{\geqslant 1}$ be the graph sequence of a line with n vertices, similar to Figure 5 but minus the root. $(L_n)_{\geqslant 1}$ converges locally in probability to the rooted $\mathbb{Z}$-graph with root 0, denoted as $(\mathbb{Z}, 0)$. Figure 9 shows a part of $(\mathbb{Z}, 0)$. Theorem 3.3 states that it is sufficient to only check that for all functions h of the form $p_r^{H_\star}(G_n)$ the convergence holds. This is further simplified by Theorem 3.4. As $(L_n)_{\geqslant 1}$ converges to the deterministic rooted graph $(\mathbb{Z}, 0)$, it is sufficient to only check the convergence for all functions of the form $p_r^{(\mathbb{Z},0)}(G_n)$. This results in the following convergence that has to be proven for all $r \in \mathbb{N}$.

$$p_r^{(\mathbb{Z},0)}(L_n) = \frac{1}{|V(L_n)|} \sum_{u \in V(L_n)} \mathbf{1}_{\left\{B_r^{(L_n)}(u) \simeq B_r^{(\mathbb{Z})}(0)\right\}} \xrightarrow{\mathbb{P}} \mu\Big(B_r^{(\mathbb{Z})}(0) \simeq B_r^{(\mathbb{Z})}(0)\Big) \tag{3.10}$$

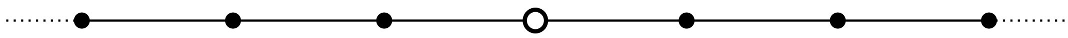

Figure 9: A part of the rooted $\mathbb{Z}$-graph $(\mathbb{Z}, 0)$

The right hand side of function (3.10) compares the graph $(\mathbb{Z}, 0)$ with itself. As a graph is always isomorphic with itself, the right hand side is equal to 1. On the left side the average is taken over if L_n with a random root is isomorphic to $(\mathbb{Z}, 0)$ within radius r. If $n \leqslant 2r$, there are not enough vertices such that L_n contains the graph $B_r^{(\mathbb{Z})}(0)$. In that case $\mathbf{1}_{\left\{B_r^{(L_n)}(u) \simeq B_r^{(\mathbb{Z})}(0)\right\}} = 0$. Else there are $n > 2r$ vertices, of which $2r$ are too close to the edge to be a root that locally looks like $B_r^{(\mathbb{Z})}(0)$, leaving $n - 2r$ vertices that suffice in the center. Indeed, for $n > 2r$,

$$p_r^{(\mathbb{Z},0)}(L_n) = \frac{1}{|V(L_n)|} \sum_{u \in V(L_n)} \mathbf{1}_{\left\{B_r^{(L_n)}(u) \simeq B_r^{(\mathbb{Z})}(0)\right\}} = \frac{n - 2r}{n}, \tag{3.11}$$

which converges to 1 as n goes to ∞, concluding the proof.

4 d-regular random graph

In this section a proof is given for the local convergence in probability for the random d-regular graph as the amount of vertices goes to infinity. The random d-regular graph exists of a set amount of vertices that all share the same degree d. A method of generating random d-regular graphs is required for which the configuration model is used.

Definition 4.1 (Regular configuration model). Fix $n, d \in \mathbb{N}$ such that nd is even, where n is the amount of vertices and d is the shared degree of all vertices. The d-regular configuration model $CM_{n,d}$ generates random graphs with n vertices that all have degree d. First all vertices are assigned d half-edges. Second a half-edge is chosen, which gets connected to another half-edge chosen uniformly at random. Together the two half-edges form an edge. This procedure is repeated until there are no half-edges left. This results in a random multigraph with the desired properties. $\triangle$

In Figure 10 is seen how a $CM_{6,3}$ graph gets generated. On the left all the vertices and half-edges are drawn. In the center six half-edges have been connected forming three full edges. The order in which half-edges are connected does not influence the outcome, but the model is easier to work with by starting the process from a single vertex. At the right the outcome is shown, which also displays the largest flaw of the regular configuration model. It can generate self-loops and double edges, which results in multigraphs.

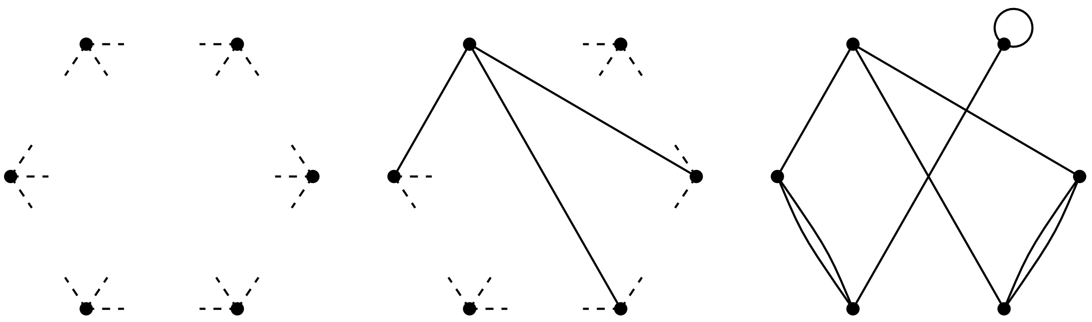

Figure 10: The process of generating a $CM_{6,3}$ graph

Theorem 4.2 (Local convergence of regular graphs)**.** *For a fixed degree $d \in \mathbb{N}_{>0}$ and a total amount of vertices $n \in \mathbb{N}_{>0}$, assume that dn is even. Let $G_n \in \mathcal{G}$ be the d-regular random graph that consists of n vertices, all with degree d. Then the sequence $(G_n)_{n\geqslant 0}$ converges locally in probability to the rooted d-regular tree, the infinite rooted tree consisting of vertices with constant degree d.*

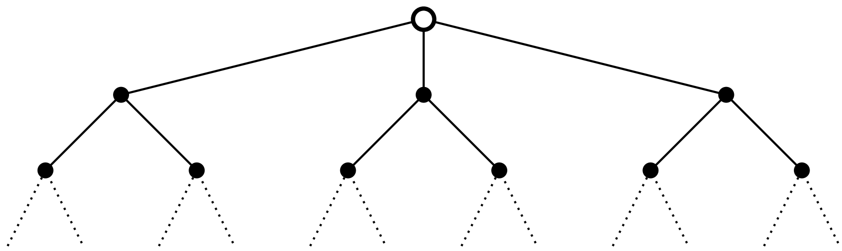

Figure 11: A part of the rooted 3-regular tree $\mathbb{T}_3$.

Proof. Denote the d-regular tree as $\mathbb{T}_d$ and model the random d-regular graph with the configuration model $CM_{n,d}$. Be aware that the configuration model can generate multigraphs which are not in $\mathcal{G}$, which causes two problems. First, due to the possible multigraphs, the configuration model is not the same as the d-regular random graph and second, our definition of local convergence in probability is not well defined for multigraphs. This is not a problem as we can still apply the same theory to multigraphs

To prove local convergence in probability, Theorem 3.3 can be applied. It states that it is sufficient to prove for all rooted graphs $(H, o') \in \mathcal{G}_\star$ and $r \in \mathbb{N}$ that

$$p_r^{(H_\star)}(G_n) = \frac{1}{|V(G_n)|} \sum_{u \in V(G_n)} \mathbf{1}_{\left\{B_r^{(G_n)}(u) \simeq B_r^{(H)}(o')\right\}} \xrightarrow{\mathbb{P}} \mu\big(B_r^{(\mathbb{T}_d)}(o) \simeq B_r^{(H)}(o')\big). \tag{4.1}$$

Equation (4.2) can be further simplified by applying theorem 3.4 as it is sufficient to only prove the convergence for graphs (H, o') that the d-regular graph converges to. We want to prove that $CM_{n,d}$ converges to the single graph $\mathbb{T}_d$, so only the following convergence has to be proven for all radii $r \in \mathbb{N}_{\geqslant 0}$:

$$p_r^{(\mathbb{T}_d)}(G_n) \xrightarrow{\mathbb{P}} \mu\big(B_r^{(\mathbb{T}_d)}(o) \simeq B_r^{(\mathbb{T}_d)}(o)\big). \tag{4.2}$$

On the right hand side of (4.2), two equal neighbourhoods are compared. These are always isomorphic, so we need that check that

$$p_r^{(\mathbb{T}_d)}(G_n) \xrightarrow{\mathbb{P}} 1. \tag{4.3}$$

The convergence can be proven with the Chebychev inequality, for which the first and second moment of $p_r^{(\mathbb{T}_d)}(G_n)$ are required. The first moment is as follows:

$$\mathbb{E}\Big[p_r^{(\mathbb{T}_d)}(G_n)\Big] = \mathbb{P}\Big[B_r^{(G_n)}(o_n) \simeq \mathbb{T}_d\Big] = \mathbb{P}\Big[B_r^{(G_n)}(o_n) \text{ is a tree}\Big] = 1 - \mathbb{P}\Big[B_r^{(G_n)}(o_n) \text{ is not a tree}\Big]. \tag{4.4}$$

The second equality from equation (4.4) comes from the fact that G_n and $\mathbb{T}_d$ are very similar as all their vertices have degree d. The only way $B_r^{(G_n)}(o_n)$ can be different from a d-regular tree is by having a cycle in radius r. That is equivalent to $B_r^{(G_n)}(o_n)$ not being a tree. Instead of computing the probability of $B_r^{(G_n)}(o_n)$ being a tree, the complement will be computed for this proof.

For the next step, an upper bound for the probability of $B_r^{(G_n)}(o_n)$ not being a tree will be given. Earlier in the proof is mentioned that the $CM_{n,d}$ model is used. Now the condition can be released as allowing multigraphs, increases the probability of $B_r^{(G_n)}(o_n)$ not being a tree, which is no problem for making an upper bound. This step is relevant, because releasing the condition makes the configuration model more manageable.

Before the entire upper bound is constructed, $B_1^{(G_n)}(o_n)$ not being a tree is first taken into account. After that the radius r will be increased. For the bound it is assumed that n is large, as if there are too few vertices the graph can never be a tree up to radius r. Furthermore $d = 3$ is assumed, so the proof can be visualized. The proof can easily be extended to general values of d. In total there are n vertices and $3n$ half-edges in the graph CM_n. In radius 1 around the root, only the 3 vertices of the root have to form a tree.

$$\mathbb{P}\Big[B_1^{(G_n)}(o_n) \text{ is not a tree}\Big] \leqslant \tag{4.5}$$
$$\mathbb{P}(\text{edge 1 fails}) + \mathbb{P}(\text{edge 2 fails} \mid \text{edge 1 did not fail}) + \mathbb{P}(\text{edge 3 fails} \mid \text{edge 1 and 2 did not fail})$$

With the phrase *edge m fails* in equation (4.5), the probability that the m-th edge ruins the tree is meant. This is under the assumption that already $m-1$ edges in the shape of a tree have been placed. Note that this is not an equality, as the probability of reaching edge m with an actual tree is not taken into account. For the sake of the upper bound, this does not matter and makes the computations less complex.

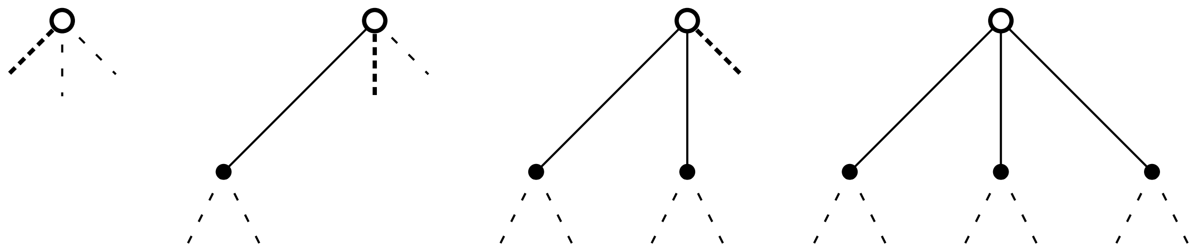

Figure 12: Process of connecting the first three edges of $CM_{n,3}$.

Connecting the first half-edge can only fail if it gets connected to one of the two other half-edges of the root. In total there are $3n$ half-edges, but the half-edge can not connect to itself. Therefore there are only $3n-1$ different options, making the probability of failure $\frac{2}{3n-1}$. Figure 12 shows the process of connecting the first three half-edges. The second edge is more likely to fail than the first as there are now three half-edges that would form a cycle. Also two half-edges have been used to form the first half-edge, making the probability that the second edge fails $\frac{3}{3n-3}$. Lastly the third edge fails with probability $\frac{4}{3n-5}$ which can be used with the other two probabilities to from the upper bound,

$$\mathbb{P}\Big[B_1^{(G_n)}(o_n) \text{ is not a tree}\Big] \leqslant \frac{2}{3n-1} + \frac{3}{3n-3} + \frac{4}{3n-5}. \tag{4.6}$$

There is a clear pattern in the probabilities that each half-edge fails to form a tree. This can be used to generalise the upper bound for all $r \in \mathbb{N}$. It takes two half-edges to form an edge, so every step that another edge is added, there are two fewer half-edges. By connecting a new vertex to the tree, a half-edge gets replaced by two new half-edges that are part of the tree. Therefore every step the amount of half-edges that would create a cycle in the three increases by 1. Lastly there are $3 \cdot 2^{r-1}$ edges in the tree up to radius r. This results in the following upper bound:

$$\mathbb{P}\Big[B_r^{(G_n)}(o_n) \text{ is not a tree}\Big] \leqslant \sum_{i=1}^{3\cdot 2^{r-1}} \frac{i+1}{3n-2i+1} \leqslant 3 \cdot 2^{r-1} \frac{3 \cdot 2^{r-1}+1}{3n - 6 \cdot 2^{r-1} + 1}. \tag{4.7}$$

For the limit of $n \to \infty$ equation (4.7) converges to 0, as only the denominator grows with n. Thus the original sequence goes to 0 too and the expectation that $B_r^{(G_n)}(o_n)$ is a tree goes to 1.

The second moment follows from the first moment. The second moment can be seen in (4.8), which is the probability that around two distinct roots $o_n^{(1)}$ and $o_n^{(2)}$ the graph looks like a tree.

$$\mathbb{E}\Big[p_r^{(\mathbb{T}_d)}(G_n)^2\Big] = \mathbb{P}\Big[B_r^{(G_n)}\big(o_n^{(1)}\big) \text{ and } B_r^{(G_n)}\big(o_n^{(2)}\big) \text{ are trees}\Big] \tag{4.8}$$

From (4.8),

$$\begin{aligned} 0 \leqslant 1 - \mathbb{E}\Big[p_r^{(\mathbb{T}_d)}(G_n)^2\Big] &= \mathbb{P}\Big[B_r^{(G_n)}\big(o_n^{(1)}\big) \text{ or } B_r^{(G_n)}\big(o_n^{(2)}\big) \text{ is not a tree}\Big] \leqslant \\ &\mathbb{P}\Big[B_r^{(G_n)}\big(o_n^{(1)}\big) \text{ is not a tree}\Big] + \mathbb{P}\Big[B_r^{(G_n)}\big(o_n^{(2)}\big) \text{ is not a tree}\Big] \end{aligned} \tag{4.9}$$

and this quantity converges to 0 as n goes to infinity. Therefore $\lim_{n\to\infty} \mathbb{E}\Big[p_r^{(\mathbb{T}_d)}(G_n)^2\Big] \to 0$.

Denote the random variable $X_n = p_r^{(\mathbb{T}_d)}(G_n)$. This part of the proof is also used in the proof of Theorem 5.4, hence the renaming of the random variables. For our proof we want to show

$$X_n \xrightarrow{\mathbb{P}} \mu\Big(B_r^{(\mathbb{T}_d)}(o) \simeq B_r^{(\mathbb{T}_d)}(o)\Big) = 1, \tag{4.10}$$

which is equivalent to proving that, for all $\epsilon > 0$,

$$\mathbb{P}\big(|X_n - 1| > \epsilon\big) \to 0. \tag{4.11}$$

We have already proven $\lim_{n\to\infty} \mathbb{E}[X_n] = 1$ and $\lim_{n\to\infty} \mathbb{E}[X_n^2] = 1$. Choose n large so that $\big|\mathbb{E}[X_n] - 1\big| < \frac{\epsilon}{2}$. Then

$$\mathbb{P}\big(|X_n - 1| > \epsilon\big) \leqslant \mathbb{P}\big(|X_n - \mathbb{E}[X_n]| > \frac{\epsilon}{2}\big). \tag{4.12}$$

Next we apply the Chebychev inequality to bound the probability above by

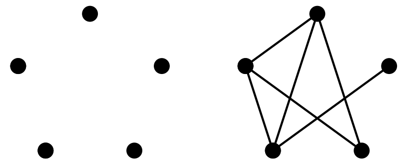

Figure 13: An example of an $ER_5(0.5)$ graph.

$$\mathbb{P}\big(\big|X_n - \mathbb{E}[X_n]\big| > \frac{\epsilon}{2}\big) \leqslant \frac{4\text{var}(X_n)}{\epsilon^2} \tag{4.13}$$

The variance converges to zero, $\lim_{n\to\infty} \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 = 1 - 1^2 = 0$. Therefore the upper bound is 0 and we conclude that X_n converges in probability to 1, finishing the proof.

□

5 Erdös-Rényi model

The last local convergence of a graph sequence that will be convered is the Erdös-Rényi model. This model generates random graphs by connecting each pair of vertices with the same probability.

Definition 5.1 (Erdös-Rényi model)**.** For $n \in \mathbb{N}$ and $p \in [0,1]$, the Erdös-Rényi model $ER_n(p)$ is a random graph model defined as follows. Let $V_n = [n]$ be the set of vertices. Then for each pair of vertices $v, w \in V_n$, independently add an edge between the two vertices with probability p. △

Figure 13 shows an example of this process for $ER_5(0.5)$. The left graph shows the 5 vertices being placed and the right graph shows the result after a coin has been tossed for each pair of vertices to determine which edges should be added to the graph. The Erdös-Rényi model can generate disconnected graphs, but no multigraphs.

Lets take a closer look at how the $ER_n(p)$ model behaves. As all vertices $v \in V$ have a probability of p to be connected to each of the $n-1$ other vertices, the degree d_v is distributed according to $\text{Bin}(n-1, p)$,

$$\mathbb{P}(d_v = k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}. \tag{5.1}$$

Fix a vertex $v \in V$. In this case the degree d_v diverges almost surely if $p > 0$ is fixed as n grows. To generate sparse graphs for large n with the Erdös-Rényi model, the probability that p that two vertices are connected has to be dependent on n. For example, fix $\lambda > 0$, $n \in \mathbb{N}_{>0}$ and set $p = \frac{\lambda}{n}$. For $n \to \infty$, the binomial distribution of the degree distribution of d_v converges to the Poisson distribution with parameter λ

$$\mathbb{P}(d_v = k) \to \lambda^k \frac{e^{-\lambda}}{k!}. \tag{5.2}$$

The limit of the degree sequence has mean λ. We will prove that $ER_n\big(\frac{\lambda}{n}\big)$ converges locally in distribution to the so called rooted Galton-Watson tree, as seen in the next definition.

Definition 5.2 (Galton-Watson tree)**.** Fix $\lambda > 0$. The Galton-Watson tree with mean offspring λ and root o is a random tree and is denoted as $GW_\lambda(o)$. A Galton-Watson tree can be constructed in generations. Generation 1 of the tree consists of only 1 vertex, which is the root. Given the vertices of the n-th generation of a Galton-Watson tree, each vertex is independently connected to $\text{Poi}(\lambda)$ vertices in generation $n+1$. This is called the offspring of a vertex. △

Figure 14 shows a rooted Galton-Watson tree for $\lambda = 0.9$. Note that any rooted tree can be generated by $GW_\lambda(o)$ for any $\lambda > 0$, thus Figure 14 could have been generated by $GW_{0.1}(o)$ or $GW_{10}(o)$ for example. The parameter λ influences the probability of generating each tree.

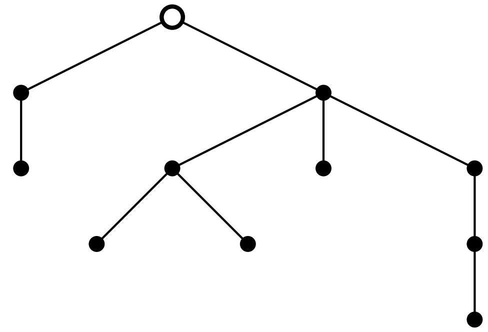

Figure 14: A rooted Galton-Watson tree with mean offspring $\lambda = 0.9$.

The last definition that is is required for the main proof of this chapter is the definition of ordered graphs. It is a method of labelling vertices in rooted graphs, which is used to compare rooted graphs with each other. It allows us to compare graphs in a stronger way than isomorphism, but weaker than equality.

Definition 5.3 (Ordered graphs)**.** Let $(G, o) \in \mathcal{G}_\star$ be a rooted graph with $G = (V, E)$. An ordering in G is a function $\psi : V \to \mathbb{N}$ with the following properties.

- ψ is injective.
- For all $v_1, v_2 \in v$ with $d_G(o, v_1) \leqslant d_G(o, v_2)$ we have $\psi(v_1) \leqslant \psi(v_2)$.
- If V is finite with t elements, then the first t elements of $\mathbb{N}$ are the image of ψ.
- If V is infinite, then ψ is also surjective.

Denote $(\overline{G}, o)$ as an ordered rooted graph of (G, o). This is (G, o) with the addition that each vertex of V is labelled according to a function ψ. Two rooted graphs $(G, o) \neq (G', o')$ are isomorphic if and only if $(\overline{G}, o) \simeq (\overline{G'}, o')$ as the ordering does not alter the structure of the graph. When determining if two ordered graphs are equal, only the ordering of V and the edges are taken into account. Therefore for two isomorphic rooted graphs $(G, o) \neq (G', o')$ it could be that $(\overline{G}, o) = (\overline{G'}, o')$ depending on how they are ordered. △

The function ψ from Definition 5.3 orders the vertices of a graph by distance from the root, beginning by $\psi(o) = 1$ as the root is the closest to itself. Note that a function ψ of a rooted graph (G, o) is not uniquely determined as vertices at the same distance from the root can be interchanged. Figure 15 shows two rooted ordered graphs. Two equivalent rooted graphs can only disagree between their ordered versions on vertices that are the same distance from the root.

Theorem 5.4 (Local convergence of Erdös-Rényi model)**.** *Fix $\lambda > 0$. For $n \to \infty$, the Erdös-Rényi model $ER_n\left(\frac{\lambda}{n}\right)$ converges locally in probability to the Galton-Watson tree $GW_\lambda(o)$.*

Proof. The proof is similar to that of Theorem 4.2. First the equation we want to prove is set up, then the first and second moment will be computed and lastly the two moments together are used to finish the proof.

Setting up.
Denote $G_n = ER_n\left(\frac{\lambda}{n}\right)$ and $(G, o) = GW_\lambda(o)$. The set of all possible rooted Galton-Watson trees consists of all rooted trees $\mathbb{T}_\star \subset \mathcal{G}_\star$. By applying both Theorem 3.3 and 3.4, $G_n = (V_n, E_n)$ converges locally in probability to (G, o) if for all $(T, a_1) \in \mathbb{T}_\star$ and $r \in \mathbb{N}$ the following holds:

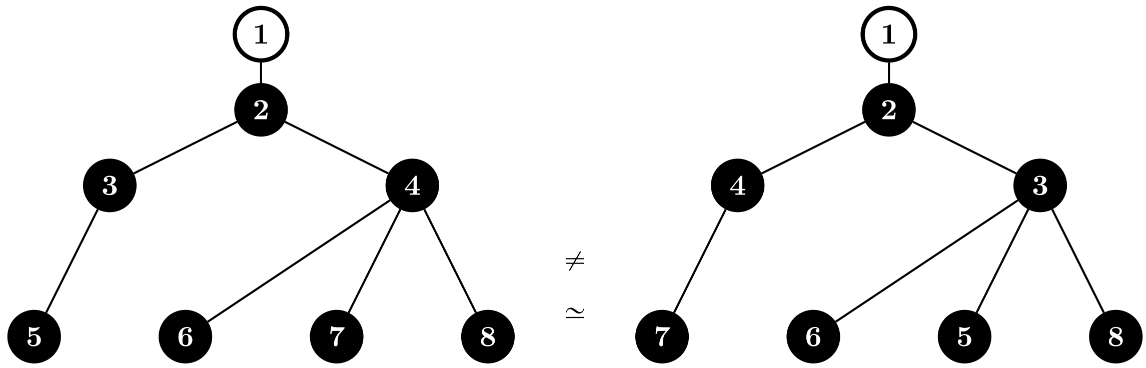

Figure 15: Two equivalent rooted ordered graphs that are not equal.

$$p_r^{(T,a_1)}(G_n) = \frac{1}{n} \sum_{v \in V_n} \mathbf{1}_{\left\{B_r^{(G_n)}(v) \simeq B_r^{(T)}(a_1)\right\}} \xrightarrow{\mathbb{P}} \mu\Big(B_r^{(G)}(o) \simeq B_r^{(T)}(a_1)\Big). \tag{5.3}$$

The rooted tree (T, a_1) plays a large role and a lot of variables depending on (T, a_1) are used throughout the proof. Here is a list of all the variables of (T, a_1) that will be used.

- $T = (V_T, E_T)$, which shows the names of the vertex and edge set.
- $\{a_1, a_2, ..., a_t\} = V_T$ is the set of all the t vertices.
- For $a_i \in V_T$, x_i is the size of the offspring of $a_i \in V_T$.
  - The offspring of the root is $x_1 = d_{a_1}$.
  - All other vertices x_i with $i \neq 1$, have offspring of size $x_i = d_{a_i} - 1$.
- For $a_i \in V_T$, $d_T(a_1, a_i)$ is the distance between the root and vertex a_i.
- $(\overline{T}, a_1)$ is the ordered rooted graph of $(\overline{T}, a_1)$ with ordering $\psi(a_i) = i$ for all $1 \leqslant i \leqslant t$.

Computing the left and right hand side of Equation (5.3) is difficult, because the amount of graphs isomorphic to $B_r^{(T)}(a_1)$ varies heavily depending on (T, a_1). An equivalent convergence in probability is constructed, based on if the ordered version of the rooted graphs are equal. Denote the number of ordered trees that are isomorphic to (T, a_1) by $\#(T, a_1)$.

By swapping out the rooted graphs with their ordered counterparts in Equation (5.3), nothing changes as rooted graphs are isomorphic to their ordered versions. Replacing the isomorphism with an equality does alter the outcome as it does not take all graphs isomorphic to $(\overline{T}, a_1)$ into account. To compensate for this, the equation should be multiplied by the total number graphs isomorphic to (T, a_1) denoted as $\#(T, a_1)$.

$$p_r^{(T,a_1)}(G_n) = \frac{1}{n} \sum_{v \in V_n} \mathbf{1}_{\left\{B_r^{(G_n)}(v) \simeq B_r^{(T)}(a_1)\right\}} = \frac{1}{n} \sum_{v \in \overline{V_n}} \mathbf{1}_{\left\{B_r^{(\overline{G_n})}(v) \simeq B_r^{(\overline{T})}(a_1)\right\}} = \frac{\#(T, a_1)}{n} \sum_{v \in \overline{V_n}} \mathbf{1}_{\left\{B_r^{(\overline{G_n})}(v) = B_r^{(\overline{T})}(a_1)\right\}} \tag{5.4}$$

On the right hand side of Equation (5.3) a similar equality can be created by using the ordered version of the graphs and replacing the isomorphism with an equality. Multiplying by $\#(T, a_1)$ compensates for the loss of using equality. The probability of the two ordered graphs to be the same can be computed. For this every vertex a_i up to radius r should have exactly an offspring of x_i. In a Galton-Watson tree, the offspring of a vertex is Poisson distributed. So the total probability of an ordered

Galton-Watson tree being equal to $(\overline{T}, a_1)$ is the product of all the Poisson random variables for each vertex. Note that vertices at exactly distance r of the root do not contribute to the probability, as the edges of their offspring have a distance of $r+1$ from the root. Therefore they are out of scope for the neighbourhood of radius r around the root.

$$\mu\Big(B_r^{(G)}(o) \simeq B_r^{(T)}(a_1)\Big) = \mu\Big(B_r^{(\overline{G})}(o) \simeq B_r^{(\overline{T})}(a_1)\Big) = \#(T, a_1)\, \mu\Big(B_r^{(\overline{G})}(o) = B_r^{(\overline{T})}(a_1)\Big) = \tag{5.5}$$

$$\#(T, a_1) \prod_{i \in \mathbb{N}\,:\, a_i \in \overline{V_T},\, d_T(a_1, a_i) < r} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!}$$

Substituting Equation (5.4) and (5.5) in the local convergence in probability (5.3), results in this equal equation.

$$\frac{\#(T, a_1)}{n} \sum_{v \in \overline{V_n}} \mathbf{1}_{\left\{B_r^{(\overline{G_n})}(v) = B_r^{(\overline{T})}(a_1)\right\}} \xrightarrow{\mathbb{P}} \#(T, a_1) \prod_{i \in \mathbb{N}\,:\, a_i \in \overline{V_T},\, d_T(a_1, a_i) < r} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \tag{5.6}$$

The unknown constant value $\#(T, a_1)$ can be divided out of the equation, negating the requirement of computing $\#(T, a_1)$ explicitly.

$$\frac{1}{n} \sum_{v \in \overline{V_n}} \mathbf{1}_{\left\{B_r^{(\overline{G_n})}(v) = B_r^{(\overline{T})}(a_1)\right\}} \xrightarrow{\mathbb{P}} \prod_{i \in \mathbb{N}\,:\, a_i \in \overline{V_T},\, d_T(a_1, a_i) < r} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \tag{5.7}$$

All that is left to do is to compute the left hand side of Equation (5.7) and show that it converges in probability to the constant on the right. The right hand side is a constant as it only depends on (T, a_1) and r who are both fixed. The same strategy will be used as in the proof for local convergence in probability of the d-regular graph, see Theorem 4.2. By computing the first and second moment the variance can be derived which is used to finalize the proof.

First moment.
We start with the first moment. We compute the expectation and note that,

$$\mathbb{E}\left[\frac{1}{n} \sum_{v \in \overline{V_n}} \mathbf{1}_{\left\{B_r^{(\overline{G_n})}(v) = B_r^{(\overline{T})}(a_1)\right\}}\right] = \frac{1}{n} \sum_{v \in \overline{V_n}} \mathbb{P}\Big(B_r^{(\overline{G_n})}(v) = B_r^{(\overline{T})}(a_1)\Big) = \mathbb{P}\Big(B_r^{(\overline{G_n})}(v) = B_r^{(\overline{T})}(a_1)\Big), \tag{5.8}$$

since the distribution of the neighbourhood $B_r^{(\overline{G_n})}(v)$ does not depend on v.

When are the two ordered neighbourhoods around the root equal as in Equation (5.8)? The offspring of all the vertices $v \in \overline{V_n}$ must be exactly equal to their corresponding x_i and $B_r^{(\overline{G_n})}(v)$ must be a rooted tree. For this we introduce the random variables X_i and Y_i for all $i \in \mathbb{N}$ with $a_i \in \overline{V_T}$ and $d_T(a_1, a_i) < r$. Denote v_i as the vertex $v \in \overline{V_n}$ with $\psi(v) = i$. Let X_i be the offspring of v_i and let Y_i be the amount of edges of v_i that do not connect to a new vertex. Y_i checks if in the ordered rooted graph $B_r^{(\overline{G_n})}(v)$ the offspring of v_i creates any loops and therefore making the ordered rooted a graph a non-tree. It is possible that there does not exist a vertex $v \in \overline{V_n}$ with $\psi(v) = i$. This happens if $B_r^{(\overline{G_n})}(v)$ contains fewer vertices than $B_r^{(\overline{T})}(v)$. To prevent undefined variables, set $X_i = 0, Y_i = 0$. In this case these values do not matter as they will not be used. If $X_i = x_i$ for all random variables X_i, then both ordered neighbourhoods have the same offspring. If $Y_i = 0$ for all random variables Y_i, then $B_r^{(\overline{G_n})}(v)$ is a tree. If both are true, then the two ordered neighbourhoods are equal. This implies

$$\mathbb{P}\Big(B_r^{(\overline{G_n})}(v) = B_r^{(\overline{T})}(a_1)\Big) = \mathbb{P}\big((X_i, Y_i) = (x_i, 0)\ \forall i \in \mathbb{N} \text{ with } a_i \in \overline{V_T} \text{ and } d_T(a_1, a_i) < r\big). \tag{5.9}$$

With conditional probability, Equation (5.9) can be written as (5.10) by taking the product over the probabilities $\mathbb{P}\big((X_i, Y_i) = (x_i, 0)\big)$ given that for all previous i the equality holds.

$$\mathbb{P}\Big(B_r^{(\overline{G_n})}(v) = B_r^{(\overline{T})}(a_1)\Big) = \prod_{i \in \mathbb{N}\,:\, a_i \in \overline{V_T},\ d_T(a_1, a_i) < r} \mathbb{P}\big((X_i, Y_i) = (x_i, y_i) \mid \forall j < i (X_j, Y_j) = (x_j, 0)\big) \tag{5.10}$$

In Equation (5.10) the probability that the two ordered neighbourhoods are equal is computed by checking vertex for vertex what the probability is that the equality holds, beginning from the root. To get the exact value, two new variables are introduced. Denote s_i as the number of vertices that are in the same component of the root, but for which we did not check their offspring. So $s_1 = 1$ for $i = 1$ as in the first step we are going to check to which vertices the root is connected. After that for $i > 1$, $s_i = s_{i-1} + x_{i-1} - 1$ as in each step vertex v_i adds x_i new vertices to the graph by checking what the offspring is of v_i. Denote n_i as the number of vertices that are not in the same component as the root in the i-th step. This is equal to $n_i - s_i - i + 1$, because there are n vertices in total, but $s_i + 1$ vertices get added to the component every i-th step. Figure 16 and 17 show an example of this process and the corresponding variables, visualizing their values.

For the i-th vertex, assuming that all previous vertices are connected correctly, there are n_i vertices that are not part of the tree yet. We want that vertex v_i gets connected to exactly x_i of them. The probability that this happens is distributed binomialy: $\mathbb{P}\Big(\mathrm{Bin}\Big(n_i, \frac{\lambda}{n}\Big) = x_i\Big)$. On top of that, v_i should not be connected to any vertices that are already part of the rooted tree, thus not forming a cycle. There are currently s_i vertices in the rooted tree for which the offspring has not been computed yet. v_i is one of those vertices and it can not connect to itself, leaving $s_i - 1$ troublesome vertices left to check. The probability of not creating a cycle in this step is also distributed binomialy: $\mathbb{P}\Big(\mathrm{Bin}\Big(s_i - 1, \frac{\lambda}{n}\Big) = 0\Big)$. The product of these probabilities for all i generate the the probability of getting an equality.

$$\begin{aligned}\mathbb{P}\Big(B_r^{(\overline{G_n})}(v) = (\overline{T}, a_1)\Big) &= \prod_{i \in \mathbb{N}\,:\, a_i \in \overline{V_T},\ d_T(a_1, a_i) < r} \mathbb{P}\Big(\mathrm{Bin}\Big(n_i, \frac{\lambda}{n}\Big) = x_i\Big) \cdot \prod_{i \in \mathbb{N}\,:\, a_i \in \overline{V_T},\ d_T(a_1, a_i) < r} \Big(1 - \frac{\lambda}{n}\Big)^{s_i - 1} \\ &\to \prod_{i \in \mathbb{N}\,:\, a_i \in \overline{V_T},\ d_T(a_1, a_i) < r} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} = \mu\Big(B_r^{(\overline{G})}(o) = (\overline{T}, a_1)\Big)\end{aligned} \tag{5.11}$$

The limit for $n \to \infty$ is taken in the last step of Equation (5.11). In this case the binomial distribution in the first product converges to the Poisson distribution as seen in Equation (5.2). In the second product, $1 - \frac{\lambda}{n}$ converges to 1. s_i is constant and the amount of vertices a_i is constant too, therefore the entire product converges to 1. This results in the first moment being equal to the right hand side of Equation (5.7).

i	x_i	y_i	s_i	n_i
1	1	0	1	11
2	2	0	1	10
3	1	0	2	8
4	3	0	2	7

Figure 16: Different variables used in Figure 17.

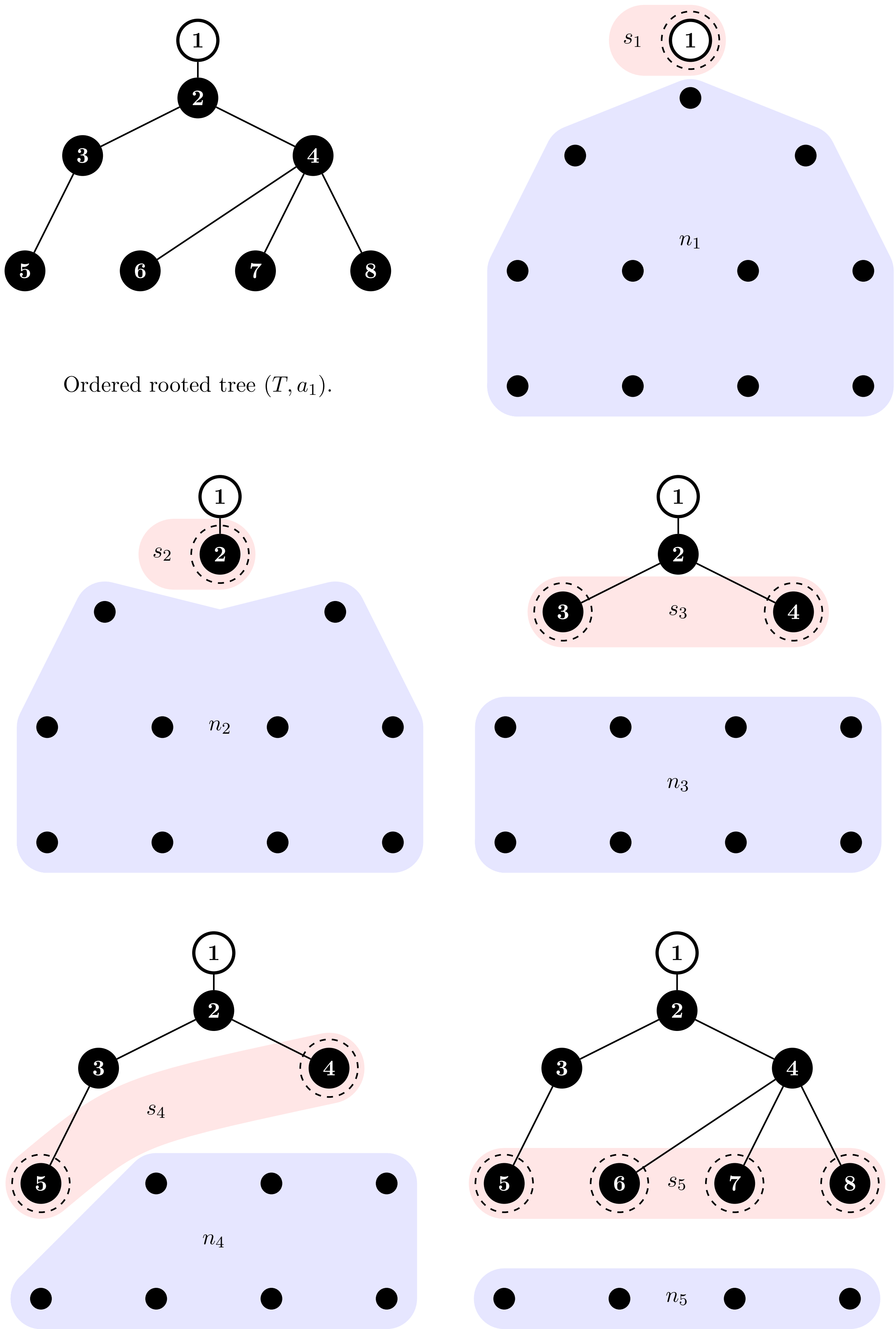

Figure 17: Visualization of the probability that the ordered rooted tree (T, a_1) (top left) is equal to the ordered $ER_{12}\left(\frac{\lambda}{12}\right)$ in radius $r = 3$. Vertices that contribute to either s_i or n_i are highlighted.

Computing the second moment.
Now the second moment will be computed. This can written similar to Equation (5.8), revealing that it can be split in two parts.

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{v\in V_n}\mathbf{1}_{\left\{B_r^{(G_n)}(v)=B_r^{(T)}(a_1)\right\}}\right)^2\right] = \frac{1}{n^2}\sum_{u,v\in V_n}\mathbb{P}\Big(B_r^{(G_n)}(u) = B_r^{(T)}(a_1), B_r^{(G_n)}(v) = B_r^{(T)}(a_1)\Big) \tag{5.12}$$

$$= \frac{1}{n^2}\left(n\mathbb{P}\Big(B_r^{(G_n)}(v) = B_r^{(T)}(a_1)\Big) + n(n-1)\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1)\Big)\right)$$

$$= \frac{1}{n}\mathbb{P}\Big(B_r^{(G_n)}(v) = B_r^{(T)}(a_1)\Big) + \frac{n-1}{n}\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1)\Big)$$

where u_1 and u_2 are two distinct uniformly chosen vertices.

The left side is $\frac{1}{n}$ times the first moment, which goes to 0 for $n \to \infty$, leaving only right side left to compute. For the right half, we split the equation up again. This is done by conditioning on the distance the two random roots have.

$$\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1)\Big) = \tag{5.13}$$
$$\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) \leqslant 2r\Big) +$$
$$\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big)$$

We will prove that the term conditioned on the two vertices u_1, u_2 being within a distance of $2r$ from each other from equation (5.13) goes to zero. This is done by giving an upper bound that goes to 0 too. The first upper bound is made by only taking the probability of u_1 and u_2 being at most $2r$ apart from each other. The second upper bound is a trivial one that simplifies the problem.

$$\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) \leqslant 2r\Big) \leqslant \tag{5.14}$$
$$\mathbb{P}\Big(d_{G_n}(u_1,u_2) \leqslant 2r\Big) = \mathbb{E}\left[\frac{\left|B_{2r}^{(G_n)}(u_1)\right| - 1}{n-1}\right] \leqslant \frac{1}{n}\mathbb{E}\left[\left|B_{2r}^{(G_n)}(v)\right|\right]$$

Now we will prove that this last quantity indeed converges to 0. For $n \to \infty$, λ is the mean offspring of a vertex in the Erdös-Rényi model. For n finite the mean offspring is a bit lower. Therefore,

$$\mathbb{E}\left[\left|B_0^{(G_n)}(u_1)\right|\right] = 1, \qquad \mathbb{E}\left[\left|B_1^{(G_n)}(u_1)\right|\right] \leqslant 1+\lambda \quad \text{and} \quad \mathbb{E}\left[\left|B_k^{(G_n)}(u_1)\right|\right] \leqslant \sum_{i=0}^{k}\lambda^i \leqslant k\lambda^k. \tag{5.15}$$

This can be verified by induction on k. Thus an upper bound for Equation (5.14) is as follows, which goes to 0 for $n \to \infty$ as λ and r are constant.

$$\frac{1}{n}\mathbb{E}\left[\left|B_{2r}^{(G_n)}\right|\right] \leqslant \frac{1}{n}2r\lambda^{2r} \to 0 \tag{5.16}$$

The only term of the second moment that does not go to 0 is the part of Equation (5.13) that is conditioned on the random vertices u, v being further than $2r$ apart. With conditional probability it can be split into two factors.

$$\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big) = \tag{5.17}$$
$$\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big) \times$$
$$\mathbb{P}\Big(B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1) \Big| B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big)$$

The left hand factor of Equation (5.17) can be derived from earlier statements. By Equation (5.14) the probability that two random vertices u_1 and u_2 are more than $2r$ apart from each other goes to 1, thus

$$\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big) \to \mu\Big(B_r^{(G)}(o) = B_r^{(T)}(a_1)\Big). \tag{5.18}$$

For the right hand factor of Equation (5.17) note that u_1 and u_2 being $2r$ apart from each other, is equivalent to $B_r^{(G_n)}(u)$ and $B_r^{(G_n)}(u)$ being disjoint. Thus $B_r^{(G_n)}(u)$ does not contain the t vertices of $B_r^{(T)}(a_1)$ and is part of an rooted Erdös-Rényi graph with only $n-t$ vertices and parameter $\frac{\lambda}{n}$.

$$\mathbb{P}\Big(B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1) \Big| B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big) = \tag{5.19}$$
$$\mathbb{P}\Big(B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1) \text{ in } ER_{n-t}(\tfrac{\lambda}{n})\Big)$$

As t is a constant, Equation (5.19) is also equal to the first moment for $n \to \infty$.

$$\mathbb{P}\Big(B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1) \Big| B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big) \to \mu\Big(B_r^{(G)}(o) = B_r^{(T)}(a_1)\Big) \tag{5.20}$$

Equation (5.18) and (5.20) can be substituted in Equation (5.17) to obtain

$$\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big) \times \tag{5.21}$$
$$\mathbb{P}\Big(B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1) \Big| B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big) \to \mu\Big(B_r^{(G)}(o) = B_r^{(T)}(a_1)\Big)^2.$$

We conclude the second moment is as follows. The fraction $\frac{n-1}{n}$ goes to 1 for $n \to \infty$.

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{1}{n}\sum_{v\in V_n} \mathbf{1}_{\left\{B_r^{(G_n)}(v) = B_r^{(T)}(a_1)\right\}}\right)^2\right] = \tag{5.22}$$
$$\lim_{n\to\infty} \frac{n-1}{n}\mathbb{P}\Big(B_r^{(G_n)}(u_1) = B_r^{(T)}(a_1), B_r^{(G_n)}(u_2) = B_r^{(T)}(a_1), d_{G_n}(u_1,u_2) > 2r\Big) \to$$
$$\mu\Big(B_r^{(G)}(o) = B_r^{(T)}(a_1)\Big)^2$$

Finishing the proof.
Denote the random variable

$$X_n = \frac{1}{n}\sum_{v\in V_n} \mathbf{1}_{\left\{B_r^{(G_n)}(v) = B_r^{(T)}(a_1)\right\}}. \tag{5.23}$$

We now have a similar situation as in the proof of Theorem 4.2, starting from Equation (4.10). The first and second moment of X_n are

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mu\Big(B_r^{(\overline{G})}(o) = B_r^{(\overline{T})}(a_1)\Big) \qquad \text{and} \qquad \lim_{n\to\infty} \mathbb{E}[X_n^2] = \mu\Big(B_r^{(\overline{G})}(o) = B_r^{(\overline{T})}(a_1)\Big)^2. \tag{5.24}$$

With this the variance of X_n can be computed.

$$\lim_{n\to\infty} \operatorname{var}(X_n) = \lim_{n\to\infty} \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 = 0 \tag{5.25}$$

As the variance is 0, the same steps as in the last part of the proof of Theorem 4.2 can be done and we conclude that $ER_n\left(\frac{\lambda}{n}\right)$ indeed converges to $GW_\lambda(o)$.

□

6 Bibliography

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