The Minimal Complete Cap Set Problem

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1 Introduction

In the year 1974, the card game SET was designed by Marsha Falco, and since its public release in 1991, this game has been a favourite of many mathematicians. This, clearly for the fact that, besides it being an exceptionally fun game to play, there is a lot of underlying math involved (for instance, the deck of cards is isomorphic to $\mathbb{F}_3^4$). This might be best summarized in the book *The Joy of Set*, By Liz McMahon, Gary Gordon, Hannah Gordon and Rebecca Gordon. [4] The authors take their time to show how you can interpret several facets of this game using geometry, linear algebra, probability and combinatorics. Most importantly, it discusses perhaps the most famous problem related to this game, the so called "cap-set" problem, or, more precisely, the "Maximal cap-set" problem. In the context of the game, this problem could be best explained as such: "What is the largest number of cards you could have such that this collection of cards contains no SETs? This question was answered originally in a paper published in 1970 (21 years before the game of SET was released to the public!) called "Sul Massimo ordine delle calotte in $S_{4,3}$" by the Italian mathematician Guiseppe Pellegrino. Since then mathematicians have occupied their time trying to improve the bounds for the size of a maximal cap for finite dimension n [3]. However, a relatively unexplored topic seems to be that of "minimal complete caps" Within the context of the card game, this question is: "What is the smallest number of cards such that this collection of cards contains no sets, and adding any other card would create a set in this collection?" In this thesis we give three proofs to show that a minimal complete cap for $n = 3$ consists of 8 elements, and discover a previously unknown complete cap for dimension $n = 4$, and construct a minimal complete cap for $n = 5$. We also construct a way to determine structure of caps using a counting argument previously used to determine cardinalities of maximal caps. [2]

2 Construction of a Complete Cap

As was briefly mentioned in the introduction, we can interpret this card game as the vectorspace $\mathbb{F}_3^4$. In this section we will establish the necessary definitions and theorems so that we can investigate these minimal complete caps. An important note to make is that we will be working with objects in $\mathbb{F}_3^n$, where $n \in \mathbb{Z}_{\geqslant 1}$. However, most of the results in this thesis have to do only with small n, namely $n \in \{1, 2, 3, 4, 5\}$.

Definition 2.1 (SET)**.** *A subset* $\{x, y, z\}$ *of* $\mathbb{F}_3^n$ *is called a* ***SET*** *if*

$$x + y + z = 0.$$

It is quite evident that any "SET" is uniquely determined by two points of the "SET". Clearly, if one has two points x and y, then the third point z is equal to $-(x + y)$. Perhaps more importantly, as $\mathbb{F}_3^n$ can be geometrically interpreted as an n-dimensional affine space with three points on a line, it should be noted that every "SET" is exactly a line in $\mathbb{F}_3^n$.

Definition 2.2 (Interset)**.** *We call* $B \subset \mathbb{F}_3^n$ *an* ***interset*** *if* $B = \{x_1.x_2, x_3, x_4\}$ *with*

$$x_1 + x_2 = x_3 + x_4$$

Whilst perhaps not immediately apparent, the definition of the interset implies that any interset does not contain a "SET". If it did, then x_1, x_2, x_3, x_4 would not be distinct.
Once again, for the sake of the reader's intuition, it is useful to interpret the interset geometrically. Since a SET is a line in $\mathbb{F}_3^n$, an interset $B = \{x_1, x_2, x_3, x_4\}$ is a set such that the line through x_1 and x_2 intersects the line through x_3 and x_4. And since any line in $\mathbb{F}_3^n$ consists of only three points, this implies that

$$x_1 + x_2 = x_3 + x_4$$

We can generalize this definition further:

Definition 2.3 (Multiple interset)**.** *For $r \in \mathbb{Z}_{\geqslant 3}$, an $(r-1)$-**tuple interset** in $\mathbb{F}_3^n$ is a set $\{x_1, x_2, ...x_{2r-1}, x_{2r}\} \subset \mathbb{F}_3^n$ that contains no SETS and for which*

$$x_1 + x_2 = \cdots = x_{2r-1} + x_{2r}.$$

We call the common value

$$-(x_1 + x_2), ..., -(x_{2r-1} + x_{2r})$$

*the **interset point** of $\{x_1, ..., x_{2r}\}$.*

We had to include the condition that $\{x_1, .., x_{2r}\}$ does not contain a SET, since this does not follow from the condition

$$x_1 + x_2 = \cdots = x_{2r-1} + x_{2r}.$$

As for the geometric interpretation, it is clearly a generalisation of the single case.

The SET and interset are the two key objects we need when constructing and identifying complete caps (which we will define shortly) for a quick example.

Example 2.4. *Let $x, y, z \in \mathbb{F}_3^n$ be pairwise distinct, such that $\{x, y, z\}$ is not a SET. Then, for any such x, y, z we find that*

- *$\{x, y, -(x + y)\}$ is a SET*
- *$\{x, y, z, x + y - z\}$ is an interset.*

An immediately apparent difference between a SET and an interset is, whilst a SET is uniquely determined by selecting two points, this is not true for the interset. As is evident in the example given above, any subset $\{x, y, z\}$ of $\mathbb{F}_3^n$ that is not a SET can be augmented to an interset by adding any of the points $x + y - z$, $x + z - y$ and $y + z - x$. This example gives rise to the following definition:

Definition 2.5. *Let $A \subset \mathbb{F}_3^n$. Then we define two corresponding sets*

- *$S_A := \{-(x + y)|x, y \in A, x \neq y\}$ if $|A| \geqslant 2$, and $S_A = \emptyset$ otherwise*
- *$C_A := \{x + y - z|x, y, z \in A$ pairwise distinct$\}$ if $|A| \geqslant 3$, and $C_A = \emptyset$ otherwise*

*where S_A is the **Solution Set** of A, and C_A is the **Coincidence Set** of A.*

It is then evident that for a given set A, S_A contains all "SET" points of A, and C_A contains all "interset" points of A. We observe that if A does contain a SET, then all points in this SET are also contained in S_A and C_A. However, the sets A we will be exploring in this thesis do not contain SETs, and thus, apart from mentioning it here, these observations will play no role in what follows.

We can use a counting argument to find upper bounds for $|S_A|$ and $|C_A|$ for a given set A. To make sure the following is always well-defined, we define the binomial as follows. For integers $x, y \geqslant 0$

$$\binom{x}{y} = \frac{x(x-1)\cdots(x-y+1)}{y!}$$

Lemma 2.6. *Let $A \subset \mathbb{F}_3^n$ be a cap with $|A| = m \in \mathbb{Z}_{\geqslant 0}$. Then*

- $|S_A| \leqslant \binom{m}{2}$
- $|C_A| \leqslant 3\binom{m}{3}$

with equality for S_A if and only if A contains no intersets.

Proof. Let $A \subset \mathbb{F}_3^n$ be a cap with $|A| = m$. For $m \in \{0, 1, 2, 3\}$ we have equality, we assume $m \geqslant 4$. Because elements of S_A are constructed by picking a subset $\{x, y\}$ of A and then taking $-(x+y)$ we find the upper bound $\binom{m}{2}$. In a similar fashion, because elements of C_A are constructed by picking three distinct elements of A, and, as shown in example 2.4, three elements in A give rise to exactly three elements in C_A, we find the upper bound $3\binom{m}{3}$.
Now suppose that $|S_A| < \binom{m}{2}$. That implies that there are four distinct elements $x_1, x_2, x_3, x_4 \in A$ such that

$$-(x_1 + x_2) = -(x_3 + x_4)$$

which is the same as saying that A contains an interset. For the other implication, assume A contains no intersets. Then, for any subset $\{x_1, x_2, x_3, x_4\}$ of A, we have

$$x_1 + x_2 \neq x_3 + x_4$$

this means that any pair of distinct elements $x_1, x_2 \in A$ gives rise to a unique element $-(x_1 + x_2) \in S_A$. Thus $S_A = \binom{m}{2}$. □

Definition 2.7. *Let $A \subset \mathbb{F}_3^n$ be a cap. We call B a proper $(r-1)-$tuple interset in A if there does not exist an $r-$tuple interset B' with $B \subset B' \subset A$. In the case $r = 2$ we call B a proper interset in A.*

For $r \geqslant 2$, Let $i(r)$ denote the number of proper $(r-1)-$tuple intersets in a cap A.

Lemma 2.8. *Let $A \subset \mathbb{F}_3^n$ be a cap with $|A| = m$. Then*

$$|S_A| = \binom{m}{2} - \sum_{r \geqslant 2} i(r)(r-1)$$

Proof. For each $r \geqslant 2$ there are precisely $i(r)$ points $z \in S_A$ with the property that there are precisely r subsets $\{x, y\} \subset \mathbb{F}_3^n$ such that

$$-(x+y) = z.$$

Thus for all $r \geqslant 2$, we subtract $i(r)(r-1)$ from $\binom{m}{2}$. This gives us the desired result. □

We now have enough to define the complete cap:

Definition 2.9 (Cap)**.** *Let $A \subset \mathbb{F}_3^n$. We call A a **cap** if*

$$A \cap S_A = \emptyset.$$

*A cap A is **complete** if*

$$A \cup S_A = \mathbb{F}_3^n.$$

*A complete cap A is called **minimal** if there does not exist a complete cap A' such that $|A'| < |A|$.*

Since S_A is the set of "SET"-points of A, a set A is a cap if and only if it does not contain any SETs. Furthermore, a cap is complete if for any point not contained in A, adding it to this set would mean that A does contain a "SET". The purpose of this thesis is to study minimal complete caps in $\mathbb{F}_3^n$. That is, complete caps of minimal cardinality.

Example 2.10. *The subset $\{0,1\}^n \subset \mathbb{F}_3^n$ is a complete cap for every $n \in \mathbb{Z}_{\geqslant 1}$*

Because of the example above, it is clear that, for every $n \geqslant 1$ there exists a complete cap of cardinality 2^n in $\mathbb{F}_3^n$. And thus, the purpose of this thesis is to find complete caps of a smaller cardinality. With the next theorem, we narrow down the possibilities for cardinalities of complete caps, by defining a lower bound on the cardinality of such caps.

Theorem 2.11. *Let $A \subset \mathbb{F}_3^n$ be a complete cap of cardinality $|A| = m \in \mathbb{Z}_{\geqslant 1}$ Then*

$$m + \binom{m}{2} \geqslant 3^n \tag{1}$$

Proof. Let A be a complete cap of cardinality $m \in \mathbb{Z}_{\geqslant 1}$. Then, by Lemma 2.6, we have

$$|S_A| \leqslant \binom{m}{2}$$

and because A and S_A are disjoint, and their union equals $\mathbb{F}_3^n$, we find

$$3^n = |\mathbb{F}_3^n| = |A \cup S_A| = |A| + |S_A| \leqslant m + \binom{m}{2}.$$

□

Example 2.12. *For $n = 1$ we find that the only number m that satisfies* (1) *is* 2*. For $n = 2$ the smallest number that satisfies* (1) *is* 4*, which is also the maximal cardinality of a cap for $n = 2$. [4]*

Lemma 2.13. *We write*

$$\mathbb{F}_3^n = \mathbb{F}_3^m \oplus \mathbb{F}_3^k = \{(x,y)|x \in \mathbb{F}_3^m, y \in \mathbb{F}_3^k\}.$$

Let $A \subset \mathbb{F}_3^m$, $B \subset \mathbb{F}_3^k$ be complete caps of $\mathbb{F}_3^m$ respectively $\mathbb{F}_3^k$. Then the set

$$A \times B = \{(x,y)|x \in A, y \in B\}$$

is a complete cap of $\mathbb{F}_3^n$.

Proof. Assume A and B are complete caps of $\mathbb{F}_3^m$ respectively $\mathbb{F}_3^k$. Then $A \times B$ is a cap, we need to show that it is complete. Let $(x_1, y_1) \in \mathbb{F}_3^n \setminus (A \times B)$. Without loss of generality we can then assume that $x_1 \notin A$. Then there exist $x_2, x_3 \in A$ such that $-(x_2 + x_3) = x_1$. If y_1 is an element of B, then $(x_2, y_1), (x_3, y_1) \in A \times B$. Then

$$(x_1, y_1) + (x_2, y_1) + (x_3, y_1) = 0.$$

Thus $(x_1, y_1) \in S_{A \times B}$. Now assume $y_1 \notin B$, analogously, there exist $y_2, y_3 \in B$ such that $-(y_2 + y_3) = y_1$. Then $(x_2, y_2), (x_2, y_3), (x_3, y_2), (x_3, y_3) \in A \times B$, and thus

$$(x_1, y_1) + (x_2, y_2) + (x_3, y_3) = 0$$

and

$$(x_1, y_1) + (x_2, y_3) + (x_3, y_2) = 0.$$

thus $(x_1, y_1) \in S_{A \times B}$. Since $A \times B$ is a cap we can then conclude that $A \times B$ is a complete cap. $\square$

We require one more definition before we can start trying to construct complete caps, and that is the definition of affine transformations:

Definition 2.14 (Affine Transformation)**.** *A mapping $\phi : \mathbb{F}_3^n \to \mathbb{F}_3^n$ is called an* ***affine transformation*** *if there exist an invertible matrix $M \in GL(n, \mathbb{F}_3)$ and a vector $b \in \mathbb{F}_3^n$ such that for all $x \in \mathbb{F}_3^n$*

$$\phi(x) = Mx + b$$

It is then clear that an affine transformation is bijective. What should be evident to the reader is that an affine transformation maps SETs to SETs.

Indeed, let $\{x, y, z\} \subset \mathbb{F}_3^n$ be a SET, and ϕ an affine transformation. then

$$\phi(x) + \phi(y) + \phi(z) = Mx + b + My + b + Mz + b = M(x + y + z) + 3b = M \cdot 0 = 0.$$

Since an affine transformation maps lines to lines, it also maps planes to planes, and hyperplanes to hyperplanes. But, most importantly, it maps caps to caps, and complete caps to complete caps. For this to be useful we define the following relationship:

Definition 2.15 (Affine equivalence)**.** *We call two sets $A, B \subset \mathbb{F}_3^n$ **affinely equivalent**, or $A \sim_{aff} B$, if there exists an affine transformation ϕ such that*

$$\phi(A) = B.$$

It is easy to check that $\sim_{aff}$ is an equivalence relationship. We call an equivalence class of sets under affine equivalence an affine equivalence class.

Since affine transformations map caps to caps, we can now categorize different caps by affine equivalence classes, which will be done in the fourth chapter, also it will allow us to construct a minimal complete cap for $n = 3$ with relative ease in the next chapter.

3 A Minimal complete cap for $n = 3$

In this section we will prove that a minimal complete cap A of $\mathbb{F}_3^3$ has cardinality 8. We will give two proofs for this fact. In the first proof we try to construct a complete cap of a smaller cardinality point by point. The second proof was given by H. W. Lenstra, who gives a more direct approach.

3.1 The counting proof

For the first approach, we require a few Lemmas:

Lemma 3.1. *Let $A \subset \mathbb{F}_3^n$, with $|A| > 3$ be an interset-free cap.*
Let $\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\} \subset A$ be two distinct subsets of A such that

$$x_1 + x_2 - x_3 = x_4 + x_5 - x_6.$$

Then

$$|\{x_1, x_2, x_3\} \cap \{x_4, x_5, x_6\}| \leqslant 1.$$

Proof. The proof is by contradiction: Let $A \subset \mathbb{F}_3^n$ with $|A| > 3$ be an interset-free cap. Let $\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\} \subset A$ be two distinct subsets of A such that

$$x_1 + x_2 - x_3 = x_4 + x_5 - x_6$$

and assume that these subsets have two elements in common. We can then assume $x_1 = x_4$ and $x_2 = x_6$. By substituting these in the equation above, we find

$$-(x_2 + x_3) = x_5.$$

But A does not contain a SET, and thus we have our contradiction. □

Lemma 3.2. *Let $A \subset \mathbb{F}_3^n$, with $|A| \geqslant 3$, such that A is an interset-free cap. Let $x_1, x_2, x_3, x_4, x_5 \in A$ where x_1, x_2, x_3 are pairwise distinct, and $x_4 \neq x_5$ such that*

$$x_1 + x_2 - x_3 = -(x_4 + x_5)$$

Then $x_1, ..., x_5$ are pairwise distinct.

Proof. Again, the proof is by contradiction: Assume that one of x_1, x_2 equals one of x_4, x_5. Without loss of generality, we assume $x_1 = x_4$. Then, by substituting this into the equation above, we find

$$x_2 + x_5 = x_1 + x_3.$$

If x_1, x_2, x_3, x_5 are distinct, then A contains an interset, if $x_2 = x_5$, then A contains a SET, and if we assume $x_3 = x_5$, then $x_1 = x_2$ which contradicts the assumption that x_1 and x_2 are distinct. Thus $x_1 \neq x_4$.
Next assume that x_3 equals one of x_4, x_5, say $x_3 = x_4$. Substituting this into the equation above gives

$$-(x_1 + x_2) = x_5.$$

If x_1, x_2, x_5 are distinct, then A contains a SET, against our assumption. Thus we can assume $x_1 = x_5$. However, this implies that $x_1 = x_2 = x_5$, which contradicts the assumption that $x_1 \neq x_2$, and thus $x_3 \neq x_4$.
We conclude that x_1, x_2, x_3, x_4, x_5 are pairwise distinct. $\square$

Theorem 3.3. *A minimal complete cap of* $\mathbb{F}_3^3$ *has cardinality* 8.

The proof will be by contradiction, Let $A \subset \mathbb{F}_3^3$ be a complete cap such that $|A| < 8$. Then, by Theorem 2.11, $|A| = 7$. More importantly, because

$$7 + \binom{7}{2} - 27 = 1$$

A contains exactly one proper interset by Lemma 2.8. This implies that there exists a cap of six elements that contains no intersets. In this proof we try to construct a complete cap of cardinality 7 by choosing elements one by one, in such a way that up to the sixth element, the cap is interset-free.

3.1.1 The first four elements

An interset-free cap of cardinality 4 is precisely a set such that no three points are collinear, and no four points are coplanar. In an affine space of dimension 3 such a set of elements is in *free position*, and any affine transformation is determined by how it maps elements in free position. Thus we can choose these four elements freely. We pick $A = \{0, e_1, e_2, e_3\}$ where e_i is the vector with a 1 in the $i-$th coördinate, and zeroes in any other.

3.1.2 The fifth element

Note that $A = \{0, e_1, e_2, e_3\}$ is an interset-free cap. Thus

- $|S_A| = \binom{4}{2} = 6$, by Lemma 2.6.
- Lemma 3.1 implies that $|C_A| = 3\binom{4}{3} = 12$, and Lemma 3.2 implies that $S_A \cap C_A = \emptyset$.

Hence we have

$$|\mathbb{F}_3^3 \setminus (A \cup S_A \cup C_A)| = 27 - (4 + 6 + 12) = 5$$

choices for a fifth element. Specifically, for a fifth element x_5,

$$x_5 \in \{(1,1,1), (-1,-1,-1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\} \tag{2}$$

Note that, by construction, $A = \{0, e_1, e_2, e_3, x_5\}$ is still an interset-free cap. Since $|A| = 5$, Lemma 3.2 does not imply anymore that $S_A \cap C_A = \emptyset$.

3.1.3 The sixth element

In this part we prove that if we augment an interset-free cap of five elements to a cap of six elements, the latter will always contain two intersets.

Lemma 3.4. *Let $A \subset \mathbb{F}_3^3$ be an interset-free cap of cardinality 5. Then we have*

$$|S_A \cap C_A| = 6$$

Proof. We will show this for one specific choice for x_5 from (1). The proof for the other cases is entirely similar. Take $x_5 = (1, 1, 1)$. This gives rise to the equation

$$e_1 + 0 - x_5 = -(e_2 + e_3)$$

where the left hand side of the equation is an element of C_A, and the right-hand side is an element of S_A. By moving the terms $e_1, 0, e_2$ and e_3 to the other side of the equation, this gives six equations of this form. Thus we find

$$|C_A \cap S_A| = 6$$

□

Note that, since A is still interset-free, by Lemma 2.6 we have

$$3\binom{5}{3} - 6 = 24$$

expressions $x_1 + x_2 - x_3$ with $\{x_1, x_2, x_3\} \subset A$, not contained in either A or S_A. Also note that, for an interset-free cap A of cardinality 5, we have

$$|\mathbb{F}_3^3 \setminus (A \cup S_A)| = 12.$$

This gives rise to the next lemma:

Lemma 3.5. *Let $y \in \mathbb{F}_3^3 \setminus (A \cup S_A)$. Then A has exactly two distinct subsets $\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}$ such that*

$$y = x_1 + x_2 - x_3 = x_4 + x_5 - x_6.$$

Proof. Assume that there exists a $y \in \mathbb{F}_3^3 \setminus (A \cup S_A)$, for which there exists at most one subset $\{x_1, x_2, x_3\}$ of A such that

$$y = x_1 + x_2 - x_3.$$

Because $C_A \cap A = \emptyset$, and $24 = 2 \cdot 12$, there then exists an element $z \in \mathbb{F}_3^3 \setminus (A \cup S_A)$ that has three expressions of this form, that is, there are pairwise distinct subsets $\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9\} \subset A$ such that

$$z = x_1 + x_2 - x_3 = x_4 + x_5 - x_6 = x_7 + x_8 - x_9.$$

Then by Lemma 3.1, the intersection of any pair of these subsets contains at most one element. However, because $|A| = 5$ this gives a contradiction. $\square$

This Lemma implies that any cap $A \subset \mathbb{F}_3^3$ of cardinality 6 contains at least two intersets. This means that a complete cap of 7 elements, which can only contain a single interset, does not exist. This proves Theorem 3.1.

3.2 The Second Proof

Here we give the proof proposed by H.W. Lenstra. Suppose there exists a complete cap $A \subset \mathbb{F}_3^3$ of cardinality 7. Then, A contains exactly one interset. Because of affine equivalence, replacing A by an affinely equivalent set, we may assume

$$\{x_1, -x_1, x_2, -x_2\} = B \subset A$$

where $x_1, x_2 \in \mathbb{F}_3^3 \setminus \{0\}$. Note that $V := B \cup S_B$ is a 2$-$dimensional linear subspace of $\mathbb{F}_3^3$.
Let $x_3 \in A \setminus B$. Because $x_3 \notin V$, we can express $\mathbb{F}_3^3$ as the disjoint union of three planes

$$V \cup (x_3 + V) \cup (-x_3 + V)$$

Let $x_4 \in A$ such that $x_4 \in (-x_3 + V)$. Then $x_3 + x_4 \in B$. But this implies that $-(x_3 + x_4) \in B \subset A$, which is a contradiction, because A is a cap. So

$$A \cap (-x_3 + V) = \emptyset.$$

Thus, there are three elements $x_3, x_4, x_5 \in A$ that are contained in $x_3 + V$. Since

$$|A \cap V| = 4 \text{ and } |A \cap (x_3 + V)| = 3$$

there are exactly 12 expressions $x + y$ such that $x \in A \cap V$ and $y \in A \cap (x_3 + V)$. Since any plane in $\mathbb{F}_3^n$ contains only nine points, this means that at least two of these expressions are equal to one another, which means A contains at least two intersets. This means that a subset $A \subset \mathbb{F}_3^3$ of cardinality 7 contains at least two intersets, and thus, a minimal complete cap of $\mathbb{F}_3^3$ has cardinality 8.

4 Equivalence Classes

In this section we will apply a counting argument originally used by Davis and Maclagan in a paper they wrote as graduate students in 2003 to prove that the cardinality of maximal caps for $n = 3, 4$ are respectively 9 and 20 [2]. We use this counting argument to determine the number of affine equivalence classes of complete caps for $n = 3$.

4.1 $n = 1, 2$

For the dimensions $n = 1, 2$, all complete caps have cardinality 2 and 4 respectively, thus these are maximal caps. Because for $n < 6$ all maximal caps are affinely equivalent, (see [1]) this means that, for these cases, there exists only a single affine equivalence class of complete caps.

4.2 $n = 3$

In this section we occasionally work with general n, it should be noted that we assume $n \geqslant 3$ throughout this section.

Theorem 4.1. *There exist exactly three affine equivalence classes of complete caps of* $\mathbb{F}_3^3$*. These are:*

- *The class* A_1*, consisting of all triple intersets. These are complete caps of cardinality* 8*; (e.g.* $\{0, 1\}^3$*).*
- *The class* A_2*, consisting of all complete caps that contain precisely two double intersets. These are complete caps of cardinality* 8*.*
- *The class* A_3*, consisting of all maximal caps. These are all caps of cardinality* 9*. These contain only single intersets.*

Before proving the above theorem we shall discuss the technique applied by Davis and Maclagan mentioned above, which we will use in a different context. To do so, we require one proposition of theirs. More importantly, if there exists another affine equivalence class of complete caps, then the elements of this equivalence class are complete caps of cardinality 8.

By theorem 3.3, no complete cap of cardinality 7 exists, and all maximal caps of $\mathbb{F}_3^3$ have cardinality 9 and are affinely equivalent to one another [1]. To prove Theorem 4.1 we only need to consider caps of cardinality 8.

Proposition 4.2. *Davis, Maclagen 2003 (Proposition 4) The number of hyperplanes containing a fixed k-dimensional linear variety in $\mathbb{F}_3^n$ equals*

$$\frac{3^{n-k}-1}{2}$$

Note that $\mathbb{F}_3^n$ can be expressed as the disjoint union of three hyperplanes. Moreover, using the Proposition, we find that there are exactly

$$\frac{3^{n}-1}{2}$$

ways to decompose $\mathbb{F}_3^n$ into three (parallel) pairwise disjoint hyperplanes. We can use this proposition to study how caps $A \subset \mathbb{F}_3^n$ are constructed. For this we need the following:

Definition 4.3. *A* ***Hyperplane Partition*** *of $\mathbb{F}_3^n$ is a tuple $\{H_1, H_2, H_3\}$ such that H_1, H_2, H_3 are hyperplanes in $\mathbb{F}_3^n$ and $\mathbb{F}_3^n = H_1 \sqcup H_2 \sqcup H_3$.*

Definition 4.4. *Let $A \subset \mathbb{F}_3^n$. A* ***partition*** *of A is a triple of integers (a, b, c) such that $a \geqslant b \geqslant c \geqslant 0$, and such that there is a Hyperplane Partition $\{H_1, H_2, H_3\}$ of $\mathbb{F}_3^n$ with $|A \cap H_1| = a$, $|A \cap H_2| = b$, $|A \cap H_3| = c$.*

Definition 4.5. *For an integer $n \geqslant 1$ we denote by m_n the maximal cardinality of a cap in $\mathbb{F}_3^n$.*

It is known that $m_1 = 2$, $m_2 = 4$, $m_3 = 9$, $m_4 = 20$, $m_5 = 45$ and $m_6 = 112$ (see [4]).

Corollary 4.6. *Let $A \subset \mathbb{F}_3^n$ be a cap, and (a, b, c) a partition of A. Then*

$$a \leqslant m_{n-1}.$$

Proof. Clearly, if $a > m_{n-1}$, A must contain a SET, and is therefore not a cap. □

Definition 4.7 (hyperplane decomposition)**.** *Let $A \subset \mathbb{F}_3^n$. Consider all triples of integers (a, b, c) with $3^{n-1} \geqslant a \geqslant b \geqslant c \geqslant 0$, $a + b + c = |A|$. Let $(a_1, b_1, c_1), ..., (a_k, b_k, c_k)$ be all these triples, in reverse lexicographic order. For $i = 1, .., k$ let x_i denote the number of (a_i, b_i, c_i) partitions in A. Then we call $x = (x_1, ..., x_k)$ the* ***Hyperplane Decomposition*** *(HD) of A.*

Corollary 4.6 implies that if A is a cap of $\mathbb{F}_3^n$, then $x_i = 0$ for each triple (a_i, b_i, c_i) with $a_i > m_{n-1}$. For this reason, we often abbreviate the hyperplane decomposition of a cap A as $(x_t, ..., x_k)$, where t is the smallest index i for which $a_i \leqslant m_{n-1}$.

Example 4.8. *The four possible partitions of a cap of cardinality* 4 *in* $\mathbb{F}_3^3$ *are:*

- $(4, 0, 0)$
- $(3, 1, 0)$
- $(2, 2, 0)$
- $(2, 1, 1)$

Then the hyperplane decomposition of an interset in $\mathbb{F}_3^3$ *is* $(1, 0, 6, 6)$ *and the hyperplane decomposition of a set of four points in free position is* $(0, 4, 3, 6)$*.*

Lemma 4.9. *Let* $A, B \subset \mathbb{F}_3^n$ *be affinely equivalent caps. Then* A *and* B *have the same hyperplane decomposition.*

Proof. This follows directly from the definition of an affine transformation. $\square$

Regrettably, whether the converse is true or not has not yet been proven. It is evident that for a cap $A \subset \mathbb{F}_3^n$ of cardinality m, we have

$$\sum_{i=t}^{k} x_i = \frac{3^n - 1}{2}.$$

Davis and Maclagan construct another equation in $(x_t, ..x_k)$, by counting $2-$marked hyperplanes, which are pairs $(H, \{x, y\} \subset A \cap H)$ where H is a hyperplane. Since any pair of points span a line in $\mathbb{F}_3^n$, there exist exactly $\frac{3^{n-1}-1}{2}$ such hyperplanes H that contain a distinct pair of points. For $x_i = \#(a_i, b_i, c_i)$, we write $\alpha_i = \binom{a_i}{2} + \binom{b_i}{2} + \binom{c_i}{2}$. Then we have the equation

$$\sum_{i=t}^{k} \alpha_i x_i = \frac{3^{n-1} - 1}{2} \binom{m}{2}.$$

We can do the same for $3-$marked hyperplanes, since any three points in a cap are not collinear. Thus for $\beta_i = \binom{a_i}{3} + \binom{b_i}{3} + \binom{c_i}{3}$ we find

$$\sum_{i=t}^{k} \beta_i x_i = \frac{3^{n-2} - 1}{2} \binom{m}{3}.$$

For most caps A, we cannot get such a simple equation for $4-$marked hyperplanes, because four distinct elements of a cap are not necessarily linearly independent.

For a complete cap, the kind of partitions that occur are restricted.

Lemma 4.10. *Let $A \subset \mathbb{F}_3^n$ be a complete cap, and let $a_1, a_2, a_3 \in \mathbb{Z}_{\geqslant 0}$ such that (a_1, a_2, a_3) is a partition of A. Then, for each permutation $(i, j, k) = (1, 2, 3)$ we have*

$$a_i \cdot a_j + a_k + \binom{a_k}{2} \geqslant 3^{n-1}.$$

Proof. Let A be a complete cap with partition (a_1, a_2, a_3). Then there exist three parallel pairwise disjoint hyperplanes H_1, H_2, H_3, such that

$$|V_1| = a_1, \quad |V_2| = a_2, \quad |V_3| = a_3,$$

where we write $V_i = A \cap H_i$ for simplicity.
Because the H_i are parallel hyperplanes, and A is a cap, we know that, for $(i, j, k) = (1, 2, 3)$ the set

$$-(V_i + V_j) := \{-(x + y) | x \in V_i, y \in V_j\}$$

is contained in $S_A \cap H_k$, and has cardinality at most $a_i \cdot a_j$. Moreover

$$|S_A \cap H_k| \leqslant a_i \cdot a_j + \binom{a_k}{2}$$

Hence, because A is complete

$$3^{n-1} = |H_k| = |V_k| + |S_A \cap H_k| \leqslant a_k + a_i \cdot a_j + \binom{x_k}{2}$$

□

This Lemma will be useful in the next section, where we use it to restrict the ways one can construct complete caps of small cardinality.

But we want to find an easy way to compute possible hyperplane decompositions. Luckily, we can use the counting argument used by Davis and Maclagan to define the following:

Definition 4.11. *Let $A \subset \mathbb{F}_3^n$ be a subset of cardinality m with k distinct partitions. Let $x_j = \#(a_j, b_j, c_j)$ for $j \in \{1, .., k\}$. We define the following objects*

1. *The* 3 *by k matrix Y with entries*

$$y_{ij} = \begin{cases} 1 & \text{if } i = 1 \\ \binom{a_j}{2} + \binom{b_j}{2} + \binom{c_j}{2} & \text{if } i = 2 \\ \binom{a_j}{3} + \binom{b_j}{3} + \binom{c_j}{3} & \text{if } i = 3 \end{cases}$$

2. *The column vector*

$$B = \begin{pmatrix} \frac{3^n-1}{2} \\ \frac{3^{n-1}-1}{2}\binom{m}{2} \\ \frac{3^{n-2}-1}{2}\binom{m}{3} \end{pmatrix}$$

3. *The set*

$$H_m^n = \{x^T \in \mathbb{Z}_{\geqslant 0}^k | Yx = B\}$$

Corollary 4.12. *Let $A \subset \mathbb{F}_3^n$ be a cap of cardinality m. Then the hyperplane decomposition x of A is an element of H_m^n.*

Proof. H_m^n is constructed for this purpose. □

This can be used to show that there are only three affine equivalence classes of complete caps in $\mathbb{F}_3^3$. Let A be a cap of cardinality 8. Then its hyperplane decomposition is an element of H_8^3, we compute the elements of H_8^3. There are precisely six triples of integers (a, b, c) with $a \geqslant b \geqslant c \geqslant 0$, $a + b + c = 8$ and $a > m_2 = 4$. By Corollary 4.6, a cap A in $\mathbb{F}_3^3$ of cardinality 8 can have only the following four partitions

$$\begin{array}{cc} (4,4,0) & (4,2,2) \\ (4,3,1) & (3,3,2). \end{array}$$

Thus we may express the hyperplane decomposition of A as (x_7, x_8, x_9, x_{10}). By the comments above, this tuple must satisfy the following system in non-negative integers

$$\begin{cases} x_7 + x_8 + x_9 + x_{10} = 13 \\ 12x_7 + 9x_8 + 8x_9 + 7x_{10} = 112 \\ 8x_7 + 5x_8 + 4x_9 + 2x_{10} = 56. \end{cases}$$

One can easily verify that there exist but three non-negative integer solutions to this system. These are

- $(3, 0, 6, 4)$
- $(2, 4, 3, 4)$
- $(1, 8, 0, 4)$.

Then for any cap A of cardinality 8. One of these 4-tuples is its hyperplane decomposition.

We are now ready to prove Theorem 4.1. As mentioned before, we only have to consider complete caps of cardinality 8. We will show that if a cap of cardinality 8 is complete it is either an element of A_1, or an element of A_2. For this we require the following Lemmas:

Lemma 4.13. *Let $A \subset \mathbb{F}_3^3$ be a triple interset. Then A is a complete cap, and $A \sim_{aff} \{0,1\}^3$.*

Proof. Let A be a triple interset. Then A contains a double interset $B \subset A$. Because of affine equivalence, we can, without loss of generality, choose

$$B = \{\pm e_1, \pm e_2, \pm e_3\}.$$

Note that B does not contain any other proper intersets besides the "0" interset. Because A is a cap, and because $|B \cup S_B| = 19$, we have 8 choices for the seventh element. Because of our choice for B, we find that:

$$x_7 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$$

Where $\lambda_i \in \{\pm 1\}$ for all i. We apply the linear transformation $\phi : \mathbb{F}_3^3 \to \mathbb{F}_3^3$ that maps e_i to $\lambda_i e_i$ for $i = (1,2,3)$. ϕ maps B to B, and x_7 to $e_1+e_2+e_3$. Since A is a triple interset, $-(e_1+e_2+e_3)$ is also an element of A. Thus by applying a few affine transformations, A can be transformed to $\{\pm e_1, \pm e_2, \pm e_3, \pm(e_1+e_2+e_3)\}$. We conclude that all triple intersets $A \subset \mathbb{F}_3^3$ are affinely equivalent. Specifically, they are affinely equivalent to $\{0,1\}^3$. □

Lemma 4.14. *Let A be a complete cap of cardinality 8 with the following properties*

- *$A \not\sim_{aff} \{0,1\}^3$.*
- *A contains a double interset.*

Then A contains exactly two double intersets.

Proof. Let A be a complete cap with the above stated properties. Then, as before, by affine equivalence, we can choose a cap A that contains a double interset B such that

$$B = \{\pm e_1, \pm e_2, \pm e_3\}.$$

By applying the same argument as from the previous Lemma, we can assume that $e_1 + e_2 + e_3$ is also an element of A. Then, for the eighth element x_8, there are three choices

$$x_8 = e_1 + e_2 + e_3 + e_i$$

for $(i = 1,2,3)$. We can then apply a linear transformation which permutes the e_i. Clearly, such a transformation maps $\{\pm e_1, \pm e_2, \pm e_3, e_1 + e_2 + e_3\}$ to itself. This means we can choose

$$x_8 = e_1 + e_2 - e_3.$$

It is evident that the cap $\{\pm e_1, \pm e_2, \pm e_3, e_1 + e_2 + e_3, e_1 + e_2 - e_3\}$ contains two proper double intersets. Thus by applying a few affine transformations, any complete cap of cardinality 8 with $A \not\sim_{aff} \{0,1\}^3$, can be transformed into $\{\pm e_1, \pm e_2, \pm e_3, e_1 + e_2 + e_3, e_1 + e_2 - e_3\}$, which is a complete cap that contains two proper double intersets. $\square$

If there exists another equivalence class of complete caps, by Lemmas 4.13 and 4.14, the complete caps in this equivalence class contain only single intersets. With the following Lemmas, we can count the number of intersets of a cap in $\mathbb{F}_3^3$ by looking at its hyperplane decomposition.

Lemma 4.15. *Let $A \subset \mathbb{F}_3^3$ be a cap. For every (single) interset $B \subset A$, there exists a unique hyperplane H such that*

$$B = A \cap H.$$

Proof. For a cap A of $\mathbb{F}_3^3$, by Corollary 4.6, there does not exist another hyperplane H such that $|A \cap H| > 4$ because the maximal cardinality of a cap for $n = 2$ is 4. Assume $B \subset A$ is an interset. An interset is a set of four coplanar points. Then, by proposition 4.2 there exists exactly one hyperplane H such that $B \subset H$, which implies that $B = A \cap H$. $\square$

Lemma 4.16. *Let $A \subset \mathbb{F}_3^3$ be a cap and $H \subset \mathbb{F}_3^3$ a hyperplane such that*

$$|A \cap H| = 4.$$

Then $A \cap H$ is an interset. Moreover, assume H, H' are two hyperplanes such that

$$A \cap H = A \cap H'$$

Then $H = H'$.

Proof. Let $A \subset \mathbb{F}_3^3$ be a cap, and let H be a hyperplane such that $|A \cap H| = 4$. A hyperplane in $\mathbb{F}_3^3$ is a plane, and thus $A \cap H$ is a set of four coplanar points. Since A is a cap, this is an interset. By proposition 4.2 there does not exist another hyperplane H' such that $A \cap H = A \cap H'$, and thus H is unique. $\square$

We assumed A is a complete cap that contains only single intersets. This cap has one of three hyperplane decompositions. By Lemma 2.8, since $i(r) = 0$ for all $r \geqslant 3$, we can then conclude that A contains nine single intersets. However, given a hyperplane decomposition $x = (x_7, x_8, x_9, x_{10})$, by Lemmas 4.15 and 4.16, the number of single intersets of A equals

$$2x_7 + x_8 + x_9.$$

Since the hyperplane decomposition of a cap A of cardinality 8 is one of $(3,0,6,4), (2,4,3,4)$ or $(1,8,0,4)$, we find that for each of these hyperplane decompositions

$$2x_7 + x_8 + x_9 > 9.$$

Thus a complete cap A of cardinality 8 cannot contain only single intersets. By Lemmas 4.13 and 4.14, any complete cap A of cardinality 8 is either an element of A_1, or an element of A_2.

We conclude that there exist but three affine equivalence classes of complete caps for $n = 3$.

4.3 $n = 4$

Regrettably, this technique becomes significantly less useful for larger n. For instance, look at a maximal cap $M \subset \mathbb{F}_3^4$. Up to affine equivalence, there is only one such cap (see [1]). There are 36 integer triples (a, b, c) with $a \geqslant b \geqslant c \geqslant 0$, $a + b + c = 20$ and $a > m_3 = 9$. By Corollary 4.6 this leaves for M only the following eight partitions

$$\begin{array}{cc} (9,9,2) & (8,8,4) \\ (9,8,3) & (8,7,5) \\ (9,7,4) & (8,6,6) \\ (9,6,5) & (7,7,6). \end{array}$$

This gives rise to a hyperplane decomposition $(x_{37}, x_{38}, x_{39}, x_{40}, x_{41}, x_{42}, x_{43}, x_{44})$ of M, and this must be a solution in non-negative integers of the system

$$\begin{cases} x_{37} + x_{38} + x_{39} + x_{40} + x_{41} + x_{42} + x_{43} + x_{44} = 40 \\ 73x_{37} + 67x_{38} + 63x_{39} + 61x_{40} + 62x_{41} + 59x_{42} + 58x_{43} + 57x_{44} = 2470 \\ 168x_{37} + 141x_{38} + 123x_{39} + 114x_{40} + 116x_{41} + 101x_{42} + 96x_{43} + 90x_{44} = 4560 \end{cases}$$

Only one of these solutions of this system can correspond to a maximal cap of $\mathbb{F}_3^4$. However, a quick computer search shows that this system has 692 non-negative integer solutions.

4.4 More results using hyperplane decomposition

This method of determining equivalence classes of caps can also be used to find the largest cardinality for any interset-free cap. For instance

Lemma 4.17. *An interset-free cap $A \subset \mathbb{F}_3^3$ has at most cardinality* 5.

Proof. In section 3.1 we constructed such a cap. Assume $A \subset \mathbb{F}_3^3$ is an interset-free cap of cardinality 6. There are two integer triples (a, b, c) with $a \geqslant b \geqslant c \geqslant 0$, $a + b + c = 6$ and $a > m_2 = 4$. By Corollary 4.6, this leaves for A only the five partitions $(4, 2, 0), (4, 1, 1), (3, 3, 0), (3, 2, 1)$ and $(2, 2, 2)$. Since A is interset-free, by Lemma 4.16 we have $x_3 = x_4 = 0$. Hence, Corollary 4.12 implies that x_5, x_6, x_7 satisfy

$$\begin{cases} x_5 + x_6 + x_7 = 13 \\ 6x_5 + 4x_6 + 3x_7 = 60 \\ 2x_5 + x_6 = 20. \end{cases}$$

But the only solution to this system is $(x_5, x_6, x_7) = (1, 18, -6)$. So we can conclude that any interset-free cap $A \subset \mathbb{F}_3^3$ has at most cardinality 5. $\square$

This Lemma is thus another way to show that a complete cap in $\mathbb{F}_3^3$ of cardinality 7 does not exist. For $n = 4$ we can prove a similar statement, but first note that $A \subset \mathbb{F}_3^4$ with

$$A = \{0, e_1, e_2, e_3, e_4, e_1{+}e_2{+}e_3, -e_1{+}e_2{+}e_3{+}e_4, -e_2, +e_3{-}e_4, -e_1{+}e_2{-}e_3{-}e_4\}$$

is an interset-free cap of cardinality 9.

Lemma 4.18. *An interset-free cap $A \subset \mathbb{F}_3^4$ has at most cardinality* 9*.*

Proof. The existence of an interset-free cap of cardinality 9 is a given. It needs to be shown that one of cardinality 10 cannot exist. The possible partitions of a set of cardinality 10 are

$$\begin{array}{ccc}
(10,0,0) & (7,2,1) & (5,4,1) \\
(9,1,0) & (6,4,0) & (5,3,2) \\
(8,2,0) & (6,3,1) & (4,4,2) \\
(8,1,1) & (6,2,2) & (4,3,3) \\
(7,3,0) & (5,5,0) &
\end{array}$$

Since A is a cap, by Corollary 4.6 we have $x_1 = 0$. Since we assumed A is interset-free, by Lemma 4.17 we have $x_2 = \cdots = x_9 = 0$. Now Corollary 4.12 implies that $x_{10}, .., x_{14}$ satisfy

$$\begin{cases}
x_{10} + x_{11} + x_{12} + x_{13} + x_{14} = 40 \\
20x_{10} + 16x_{11} + 14x_{12} + 13x_{13} + 12x_{14} = 585 \\
20x_{10} + 14x_{11} + 11x_{12} + 8x_{13} + 6x_{14} = 480 \\
10x_{10} + 6x_{11} + 5x_{12} + 2x_{13} + x_{14} = 210.
\end{cases}$$

Note that, since we assumed A is interset-free, no four points in A are coplanar. This means that any four points in A are linearly independent. Moreover, by Proposition 4.2, for $n = 4$, this means any four points in A are contained in exactly one hyperplane. This allows us to, in this specific case, count $4-$marked hyperplanes (pairs $H, \{x_1, x_2, x_3, x_4\} \subset A \cap H$, where $H \subset \mathbb{F}_3^4$ is a hyperplane) in the same fashion as we do $2-$ and $3-$marked hyperplanes. This allows us to include the fourth equation.
This system has no non-negative integer solutions, every solution will give $x_{12} = -95$. We have our contradiction. We conclude that any interset-free cap $A \subset \mathbb{F}_3^4$ has at most cardinality 9. $\square$

5 Complete caps for $n > 3$

5.1 A 15 element complete cap for $n = 4$

In the previous sections the main focus has been finding and analysing complete caps for $n = 3$. In this section we will show that, by writing an algorithm to construct complete caps in a specific way, we have found a complete cap of cardinality 15 for $n = 4$ and a minimal complete cap for $n = 5$. One complete cap of cardinality 15 for $n = 4$ can be written as

$$A = \{0, e_1, e_2, e_3, e_4, (-1,1,0,0), (1,1,1,0), (-1,1,1,0), (-1,1,1,1), (0,0,-1,1),$$
$$(-1,-1,-1,1), (0,1,0,-1), (-1,-1,1,-1), (0,1,-1,-1), (-1,0,-1,-1)\}$$

So far, this is the only (under affine equivalence) known complete cap of cardinality 15.

5.1.1 Minimality

The question remains whether or not this new complete cap is indeed minimal. Theorem 2.11 allows for the existence of complete caps of cardinality 13 and 14, the inexistence of which has not been proven. However, by studying H^4_{13} and H^4_{14} we can deduce properties of these potential complete caps.

Theorem 5.1. *Let $A \subset \mathbb{F}_3^4$ be a complete cap of cardinality* 13. *Then the following are true*

1. *There exist* 2 *distinct hyperplane partitions H_1, H_2, H_3 such that*
$$\{|A \cap H_1|, |A \cap H_2|, |A \cap H_3|\} = \{7, 3, 3\}$$
2. *There exist* 24 *distinct hyperplane partitions H_1, H_2, H_3 such that*
$$\{|A \cap H_1|, |A \cap H_2|, |A \cap H_3|\} = \{6, 5, 2\}$$
3. *There exist* 2 *distinct hyperplane partitions H_1, H_2, H_3 such that*
$$\{|A \cap H_1|, |A \cap H_2|, |A \cap H_3|\} = \{6, 4, 3\}$$

Proof. Let A be a complete cap of cardinality 13. By Corollary 4.6 $x_1 = \cdots = x_6 = 0$. Hence a cap of this cardinality can have partitions

$$\begin{array}{ccc}
(9,4,0) & (8,3,2) & (6,6,1) \\
(9,3,1) & (7,6,0) & (6,5,2) \\
(9,2,2) & (7,5,1) & (6,4,3) \\
(8,5,0) & (7,4,2) & (5,5,3) \\
(8,4,1) & (7,3,3) & (5,4,4)
\end{array}$$

But since A is a complete cap, by Lemma 4.10 we have $x_7 = \cdots = x_{15} = 0$. Moreover, because of Lemma 3.5, we know that any cap of cardinality 6 in $\mathbb{F}_3^3$ is either a double interset, or it contains at least two proper intersets. This implies that, for a hyperplane $H \subset \mathbb{F}_3^4$ such that $|A \cap H| = 6$ we have $|S_{A\cap H}| \leqslant 13$. Since $13 + 6 + 6 \cdot 1 = 25 < 27$, by a special case of Lemma 4.10 $\{6, 6, 1\}$ is not a partition of a complete cap of cardinality 13. In other words, $x_{17} = 0$. By doing a computer search we find that there exist only seven elements of H_{13}^4 that satisfy these conditions, we write these tuples $x = (x_{16}, ..., x_{21})$.

- $(2, 0, 24, 14, 0, 0)$
- $(3, 0, 24, 14, 3, 0)$
- $(4, 0, 24, 6, 6, 0)$
- $(4, 0, 25, 6, 2, 3)$
- $(5, 0, 24, 2, 9, 0)$
- $(5, 0, 25, 2, 5, 3)$
- $(5, 0, 26, 2, 1, 6)$

Since, for all these elements, we have

- $x_{16} \geqslant 2$
- $x_{18} \geqslant 24$
- $x_{19} \geqslant 2$

this gives us the desired result. □

Theorem 5.2. *Let $A \subset \mathbb{F}_3^4$ be a complete cap of cardinality* 14. *Then there exists a hyperplane partition H_1, H_2, H_3 such that*

$$\{|A \cap H_1|, |A \cap H_2|, |A \cap H_3|\} = \{6, 6, 2\}$$

Proof. The proof is very similar to the previous theorem. Every element of H_{14}^4 that could be the hyperplane decomposition of a complete cap has a nonzero number of $\{6, 6, 2\}$ partitions. □

5.2 A minimal complete cap for $n = 5$

Let

$$V = \{e_1, e_2, e_3, e_4, e_5, (1, -1, 1, 1, 0), (1, 1, 0, 1, 1),$$
$$(-1, 1, 1, 0, 1), (-1, -1, -1, 1, 1), (0, -1, 1, -1, 1), (1, 0, -1 - 1, 1)\}$$

Then the cap $V \cup (-V)$ is a complete cap of cardinality 22. (See the attachments for a visual representation of this complete cap). This complete cap contains a single 10−tuple interset, and no other intersets.

5.2.1 Minimality

This cap is indeed minimal. Note that

$$21 + \binom{21}{2} = 231 < 243 = 3^5.$$

By Theorem 2.11 we can then conclude that a complete cap of cardinality 22 is indeed minimal for $n = 5$. Moreover, due to the restrictions on partitions of complete caps, we can find its hyperplane decomposition.

Lemma 5.3. *Let $A \subset \mathbb{F}_3^5$ be a complete cap of cardinality 22. Then A has 66 $(10, 6, 6)$ partitions, and 55 $(9, 9, 4)$ partitions.*

Proof. Because A is a complete cap, by Corollary 4.6 and Lemma 4.10 the possible partitions of A are

$$\begin{array}{cc} (10, 6, 6) & (9, 7, 6) \\ (9, 9, 4) & (8, 8, 6) \\ (9, 8, 5) & (8, 7, 7) \end{array}$$

The other 46 integer triples (a, b, c) with $a \geqslant b \geqslant c \geqslant 0$, $a + b + c = 22$ do not give a partition of A. Thus, in the hyperplane decomposition of A we have $x_1 = \cdots = x_{46} = 0$, and by Corollary 4.12 the remaining entries $x_{47}, ..., x_{52}$ satisfy the linear system

$$\begin{cases} x_{47} + x_{48} + x_{49} + x_{50} + x_{51} + x_{52} = 121 \\ 75x_{47} + 78x_{48} + 74x_{49} + 72x_{50} + 71x_{51} + 70x_{52} = 9240 \\ 160x_{47} + 172x_{48} + 150x_{49} + 139x_{50} + 132x_{51} + 126x_{52} = 20020. \end{cases}$$

We find

$$\begin{cases} x_{50} = 2(-2x_{47} + x_{48} + 77) \\ x_{51} = 3x_{47} - 12x_{48} - 4x_{49} + 462 \\ x_{52} = 3(3x_{48} + x_{49} - 165) \end{cases}$$

and since all $x_i \geqslant 0$, we find

$$\begin{cases} x_{48} + 77 \geqslant 2x_{47} \\ 3x_{48} + x_{49} \geqslant 165 \\ 3x_{47} + 462 \geqslant 4(3x_{48} + x_{49}) \geqslant 660. \end{cases}$$

This implies that $x_{47} \geqslant 66$, and in turn $x_{48} \geqslant 55$. But $66 + 55 = 121$, which means that the only non-negative integer solution to this system is

$$(x_{47}, x_{48}, x_{49}, x_{50}, x_{51}, x_{52}) = (66, 55, 0, 0, 0, 0).$$

$\square$

5.3 The Algorithm

The algorithm, which the reader can find in the attachments section, attempts to construct a minimal complete cap in the same fashion as exemplified in the Counting Proof in section 3.1. It constructs the cap point by point in such a way that after each step, the number of intersets are minimized. The code is written in Python, and we describe how it works below.

- In class FPN, we define addition, multiplication and equality in the vector space $\mathbb{F}_p^n$.
- The function createSA creates S_A for a given cap A.
- The function adjoinSA takes a cap A, an element $x \in \mathbb{F}_3^n \setminus (A \cup S_A)$, constructs $S_{A\cup\{x\}} \setminus S_A$, and then adds the elements of this set to S_A.
- The function check_set asserts whether or not a set A is a cap or not.
- The function Teller takes an element $x \in \mathbb{F}_3^n \setminus (A \cup S_A)$ and assigns to it the value
$$\text{Teller}(x) = |S_{A\cup\{x\}}| - |S_A|.$$
- The construction of the cap starts with $n+1$ points in free position, just as in the counting proof, these points are
$$\{0, e_1, ..., e_n\}.$$
- The algorithm shuffles the order of the elements in $\mathbb{F}_3^n \setminus (A \cup S_A)$, and then adds the first element x (given this ordering) to the cap for which teller(x) is maximal.
- The if-statement checks if the cardinality of the cap A is larger than $22{\cdot}2^{n-5}$. If a cap is not complete and has cardinality larger than $22{\cdot}2^{n-5}$, it resets A to the starting state (the cap of $n+1$ points in free position). This condition exists because, by Lemma 2.13, the existence of a 22 complete cap for $n = 5$ gives a complete cap of cardinality $22 \cdot 2^{n-5}$ for $n > 5$. Thus we want to find a complete cap of smaller cardinality. If the reader wishes to construct a complete cap of a specific cardinality, one can simply adjust the bound in this if-statement.
- The algorithm continues to add points to A until A is complete. After which it prints the cardinality, and the specific elements contained in the complete cap.

It is interesting to note that as soon as we set $n = 5$, the cardinality of the output caps will vary. When finding the complete $22-$cap, we simply require that the output complete cap has cardinality less than 30 (since the existence of a complete cap of 15 elements for $n = 4$ means we can construct a complete cap of cardinality 30 for $n = 5$, by Lemma 2.13) and if the cap is not complete for cardinality greater than 29, the cap is set back to the starting state, that is a set of $(n+1)$ points in free position. For $n = 5$ specifically, the algorithm has constructed complete caps of cardinality 31, 30 and 22. But so far no complete caps of cardinalities 23 through 29 have been found.

6 Attachments

6.1 Code

```
 import random
import numpy as np


class FPN:
    def __init__(self, X, n=5, p=3):
        assert len(X) == n
        self.X = [X[i] % p for i in range(n)]
        self.dim = n
        self.p = p

    def __eq__(self, other):
        for i in range(self.dim):
            if self.X[i] != other.X[i]:
                return False
        return True

    def __ne__(self, other):
        return not self == other

    def __add__(self, other):
        return FPN([self.X[i] + other.X[i] for i in
        range(self.dim)], self.dim, self.p)

    def __sub__(self, other):
        return FPN([self.X[i] - other.X[i] for i in
        range(self.dim)], self.dim, self.p)

    def mul(self, scalar):
        return FPN([self.X[i] * scalar for i in
        range(self.dim)], self.dim, self.p)
```

```
    def __str__(self):
        returnstring = ""
        for tmp in self.X:
            returnstring += str(tmp)
        return returnstring

    def __hash__(self):
        return hash(self.__str__())

    def inc(self):
        new_arr = self.X.copy()
        for i in range(self.dim):
            new_arr[i] = (new_arr[i] + 1) % self.p
            if new_arr[i] != 0:
                break
        return FPN(new_arr, self.dim, self.p)

    def isunit(self):
        yes = False
        for y in self.X:
            if (y == 1 and yes) or y > 1:
                return False
            elif y == 1:
                yes = True
        return yes


def createSA(A):
    SA = set()
    for x in A:
        for y in A:
            if y != x:
                SA.add((x + y).mul(2))

    return SA


def adjoinSA(A, B, x):
    for y in A:
        B.add((y + x).mul(2))
    return B


def check_set(A):
    temp = list(A)
```

```
    zero = FPN(np.zeros(N, dtype=int), N, P)

    for i in range(len(temp)):
        for j in range(i + 1, len(temp)):
            for k in range(j + 1, len(temp)):
                x = temp[i]
                y = temp[j]
                z = temp[k]
                if (x + y + z) == zero:
                    return False
    return True


def teller(input, A):
    counter = 0
    temp = list(A)

    for i in range(len(temp)):
        if not SA.__contains__(input.mul(2) +
        temp[i].mul(2)):
            counter += 1

    return counter



N = 6
P = 3

if __name__ == "__main__":
    # random.seed(42)
    Z3 = set()

    zero = FPN(np.zeros(N, dtype=int), N, P)
    tmp = FPN(np.zeros(N, dtype=int), N, P)
    mapper = {}
    for x in range(P ** N):
        Z3.add(tmp)
        mapper[str(tmp)] = x
        tmp = tmp.inc()

    print(len(Z3))

    A = set()
    A.add(zero)
    for possible_unit in Z3:
```

```python
    if possible_unit.isunit():
        A.add(possible_unit)
print(len(A))
B = set()
B.add(zero)
for possible_unit in Z3:
    if possible_unit.isunit():
        B.add(possible_unit)
SA = createSA(A)
hi = True

while len(A.union(SA)) < 3 ** N:
    print(len(A))
    tempSet = Z3.difference(A.union(SA))
    assert len(A.intersection(SA)) == 0
    x1 = list(tempSet)[0]
    templist = list(tempSet)
    random.shuffle(templist)
    for x in templist:
        # print(teller(x, A))
        if teller(x, A) > teller(x1, A):
            x1 = x

    SA = adjoinSA(A, SA, x1)
    A.add(x1)
    if len(A) > 22 * 2 ** (N - 5):
        A = set()
        A.add(zero)
        for possible_unit in Z3:
            if possible_unit.isunit():
                A.add(possible_unit)
        SA = createSA(A)
assert check_set(A)
if len(A.union(SA)) >= 3 ** N:
    for x in A:
        print(str(x))
    print(len(A))
    print(len(SA))
```

6.2 Visual representation of Caps

In this section we use the capbuilder applet [5] to give visualisations of complete caps. But first we will show how one should interpret these figures.

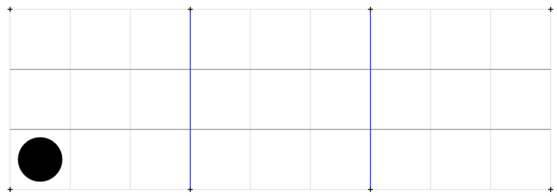

Figure 1: the point $0 \in \mathbb{F}_3^3$

Here we have three 3×3 grids. These represent planes in $\mathbb{F}_3^3$. The black dot represents the point 0. Points in the grid denoted by a black dot represent elements of the cap. When adding another element we get figure 2:

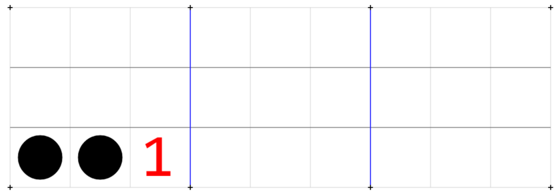

Figure 2: 0 and e_1

Now the point e_1 is also contained in our cap. the grid point to the right of e_1 has the number 1 in it. This means that the point $-e_1$ completes one SET in the cap $\{0, e_1\}$. Then we add e_2 and e_3:

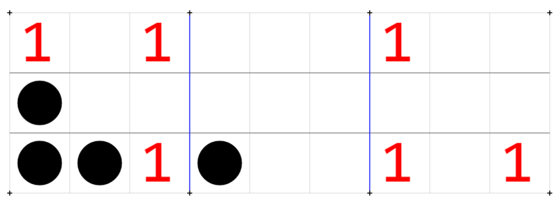

Figure 3: $A = \{0, e_1, e_2, e_3\}$

where e_2 is the point directly above 0, and e_3 is the point in the plane to the right of the other points. We shall now present these complete caps:

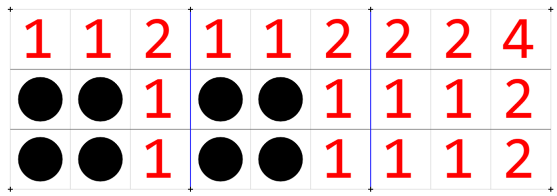

Figure 4: The equivalence class A_1 with HD $(3, 0, 6, 4)$

Note that in the top right box, the one that represents the point $(-1, -1, -1)$, the number 4 appears. This means that this cap contains a triple-interset.

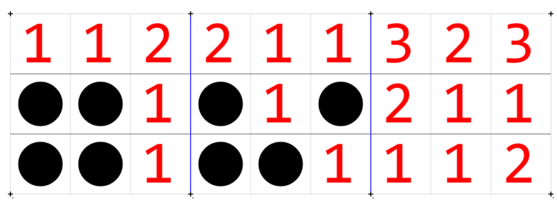

Figure 5: The equivalence class A_2 with HD $(2, 4, 3, 4)$

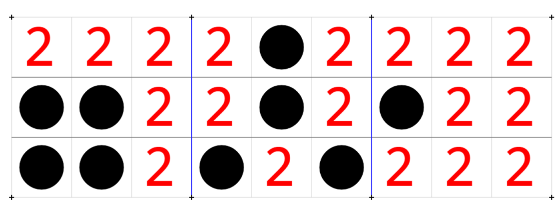

Figure 6: The maximal cap with HD $(9, 0, 4)$

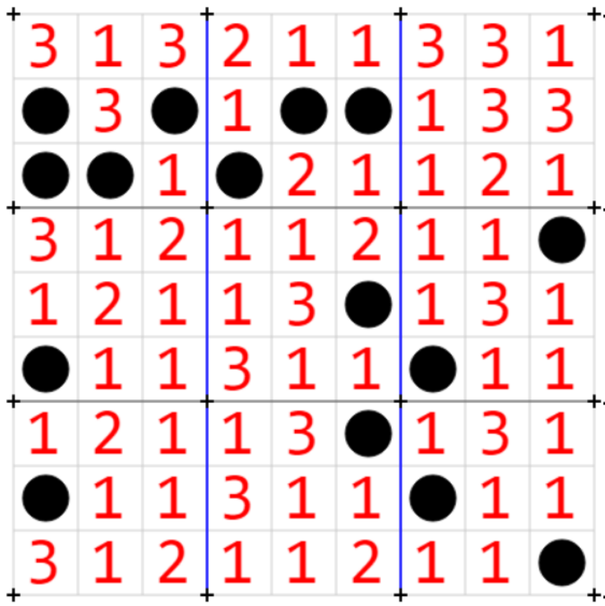

Figure 7: A 15 element complete cap of $\mathbb{F}_3^4$

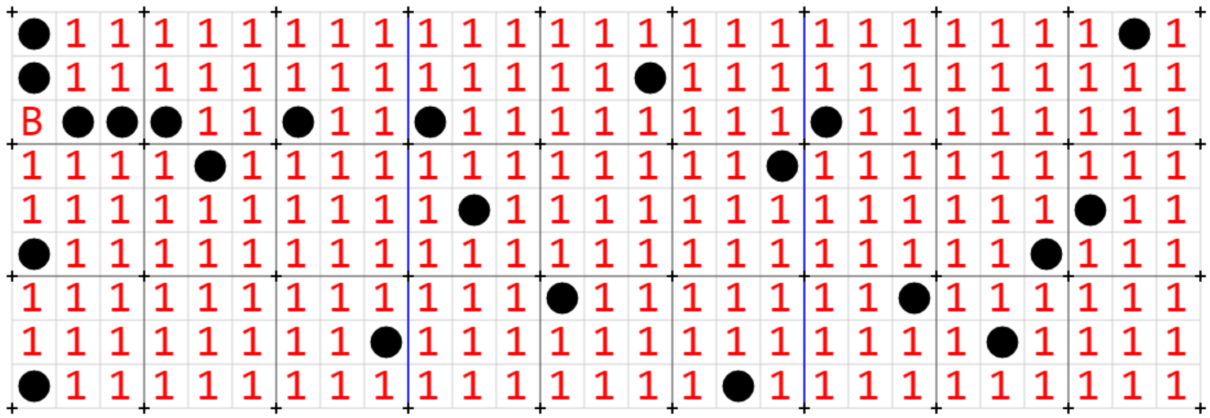

Figure 8: A minimal complete cap of $\mathbb{F}_3^5$

This is hexadecimal, the *B* represents the number 11, which means that this complete cap contains a 10-tuple interset.

References

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[2] Benjamin Lent Davis and Diane Maclagan. “The card game SET”. In: *The Mathematical Intelligencer* 25.3 (2003), pp. 33–40.

[3] Jordan S Ellenberg and Dion Gijswijt. “On large subsets of with no three-term arithmetic progression”. In: *Annals of Mathematics* (2017), pp. 339–343.

[4] Liz McMahon et al. *The joy of SET: The many mathematical dimensions of a seemingly simple card game*. Princeton University Press, 2019.

[5] *capbuilder*. https://webbox.lafayette.edu/ mcmahone/capbuilder.html.