Extension of Monotone Norms on Partially Ordered Vector Spaces

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1 Introduction

Within functional analysis exists the study of partially ordered vector spaces. Unsurprisingly, a partially ordered vector space is a vector space equipped with a partial order, which is respected by vector addition and scalar multiplication. Traditionally, the study of partially ordered vector spaces is done by embedding them in vector lattices, partially ordered vector spaces on which the supremum exists, since the theory of vector lattices is more developed.

When considering norms on partially ordered vector spaces, an interesting question is how this sense of size relates to the notion of size in the sense of the partial order. In this thesis, we will regard the properties of a norm being monotone, monotone*, symmetrically monotone, regular and having a full unit ball. Extension theorems exist for most of these properties, which, when embedding partially ordered vector spaces in vector lattices, allows us to construct extensions which share the properties of the original norms. Kalauch and Van Gaans note in [1, Example 3.3.22] that when extending norms this way, the resulting extensions may end up no longer being norms, but merely seminorms. Inspired by Zhang's result of [3, Lemma 2.1.6], this thesis aims to answer the question when norms do extend to norms.

The main result of this thesis covers many cases where the extension of norms do result in norms. In particular, this is true whenever the norm that is extended is greater than a scalar multiple of some regular norm. The proof of the main result uses many results from Kalauch and Van Gaans as well as Zhang.

In this thesis, we dedicate section 2 to a brief summary of the established theory of partially ordered vector spaces, section 3 to seminorms, and section 4 to the extension of seminorms. Then, in section 5 we bring everything together when stating the main results of this thesis. Finally, we conclude with section 6, where we state some exploratory research with some possible leads for further research.

2 Preliminaries

We begin by defining the spaces which we will be working with. In this section, and throughout this thesis, the vector spaces are over $\mathbb{R}$.

Definition 2.1. [1, Definition 1.1.1 and 1.1.6] Let X be a vector space.

i. A partial order $\leq$ on X is called a **vector space order** if

   (a) $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,

   (b) $x \in X$, $0 \leq x$ and $\lambda \in [0, \infty)$ imply $0 \leq \lambda x$.

   Instead of $x \leq y$, $x \neq y$ we write $x < y$. An element $x \in X$ is called **positive** if $0 \leq x$. For given $a, b \in X$, we denote

$$[a, b] := \{x \in X \mid a \leq x \leq b\}$$

   and call $[a, b]$ an **order interval**. For $M \subseteq X$ the sets of all upper and lower bounds are denoted by, respectively,

$$M^{\mathrm{u}} = \{x \in X \mid \forall m \in M : x \geq m\}$$

   and

$$M^{\mathrm{l}} = \{x \in X \mid \forall m \in M : x \leq m\}.$$

   We use abbreviations such as M^{ul} for $(M^{\mathrm{u}})^{\mathrm{l}}$, etc. Further, a set $M \subseteq X$ is called **full** if for every $a, b \in M$ we have $[a, b] \subseteq M$.

ii. A nonempty set $K \subseteq X$ is called a **wedge** in X if $x, y \in K$,$\lambda, \mu \in [0, \infty)$ implies $\lambda x + \mu y \in K$.

iii. If K is a wedge in X with the additional property $K \cap (-K) = \{0\}$, then K is called a **cone** in X.

iv. A subset $M \subseteq X$ is called **directed** if for every $x, y \in M$ there is $z \in M$ such that $x \leq z$, $y \leq z$.

The relation between cones and vector space orders is given by the following proposition.

Proposition 2.1. *[1, Proposition 1.1.2] Let X be a vector space.*

i. Let K be a cone in X and $\leq$ the binary relation on X defined by means of

$$x \leq y \iff y - x \in K. \tag{1}$$

   Then $\leq$ is a vector space order.

ii. Let $\leq$ be a vector space order on X. Then the set

$$K_0 := \{x \in X \mid 0 \leq x\} \tag{2}$$

   of all positive elements in X is a cone in X.

iii. Let K be a cone in X, $\leq$ the binary relation in (1), and K_0 the corresponding cone in (2). Then $K = K_0$.

This allows us to define a partially ordered vector space as follows.

Definition 2.2. [1, page 3] Given a vector space X and a cone K on X, we can equip X with the vector space order $\leq$ as defined in Proposition 2.1 and call (X, K) a **(partially) ordered vector space**.

From [1, Example 1.1.5] we can see that many commonly known vector spaces can be equipped with a similar vector space order. From the vector space $\mathbb{R}$ with the cone $\mathbb{R}_+ := [0, \infty)$ one can use [1, Proposition 1.1.4] to construct partially ordered vector spaces $\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, \mathbb{R}^{\mathbb{R}}$, and their subspaces.

Now, if we require just a little more structure on partially ordered vector spaces, we get the following spaces.

Definition 2.3. [1, page 6] A partially ordered vector space (X, K) is called a **vector lattice** or **Riesz space** if for every $x, y \in X$ the set $\{x, y\}$ has a least upper bound (supremum) and a greatest lower bound (infimum), denoted by $x \vee y$ and $x \wedge y$, respectively.

Traditionally, the study of partially ordered vector spaces is done by embedding the partially ordered vector space into a Riesz space, since the added structure of having suprema is a useful property. This leads us to the following definitions.

Definition 2.4. [1, Definition 1.6.1] Let (X, K) be an ordered vector space and let D be a linear subspace of X.

i. D is called **majorizing** if for every $x \in X$ there is $y \in D$ such that $x \leq y$.

ii. D is called **order dense** if for every $x \in X$ one has that

$$x = \inf\{y \in D \mid y \geq x\}.$$

Definition 2.5. [1, Definition 2.2.1] A partially ordered vector space (X, K) is called **pre-Riesz** or a **pre-Riesz space** if for every $x, y, z \in X$ the inclusion $\{x + z, y + z\}^{\mathrm{u}} \subseteq \{x, y\}^{\mathrm{u}}$ implies $z \in K$.

Theorem 2.2. *[1, Theorem 2.4.5] Let X be a partially ordered vector space. The following statements are equivalent.*

i. X is a pre-Riesz space.

ii. There exist a Riesz space Y and a bipositive[1] linear map $i : X \to Y$ such that $i[X]$ is order dense in Y.

Definition 2.6. [1, Definition 2.4.6] A pair (Y, i) as in Theorem 2.2 is called a **vector lattice cover** of X.

[1]For every $x \in X$ one has $x \in K_X$ if and only if $i(x) \in K_Y$.

3 Seminorms on Partially Ordered Vector Spaces

Now that we have our setting, it is time to introduce the objects we will be studying. Again, X is a real vector space.

Definition 3.1. [1, Definition 3.1.1 and page 119] A map $p : X \to \mathbb{R}$ is called a **seminorm** on X if for every $x, y \in X$ and $\lambda \in \mathbb{R}$ we have $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = |\lambda|\, p(x)$. If we have the additional property that $p(x) = 0$ implies $x = 0$, then p is a **norm**. The **unit ball** of a seminorm p on X is the set $\{x \in X \mid p(x) \leq 1\}$.

Definition 3.2. Given two seminorms p, q on a vector space X, we say that q is **equivalent** to p if

$$\exists m, M \in \mathbb{R}_{>0}\ \forall x \in X : mp(x) \leq q(x) \leq Mp(x).$$

Note that this is the same definition as equivalence of norms, and likewise we have that equivalence of seminorms is an equivalence relation.

Now, we have two notions of the "size" of a vector. These do not have to relate to each other, but there are ways in which they can.

Definition 3.3. [1, Definition 3.3.1 and 3.4.1] Let (X, K) be a partially ordered vector space and let p be a seminorm on X.

i. p is called **monotone** if for every $x, y \in X$ such that $0 \leq x \leq y$ one has that $p(x) \leq p(y)$.

ii. p is called **monotone*** if for every $x \in X$ and $u, v \in K$ such that $-u \leq x \leq v$ one has that $p(x) \leq p(u) + p(v)$.

iii. p is called **symmetrically monotone** if for every $x, y \in X$ such that $-y \leq x \leq y$ one has that $p(x) \leq p(y)$.

iv. For X directed, p is called **regular** if for every $x \in X$,

$$p(x) = \inf\{p(y) \mid y \in X, -y \leq x \leq y\}.$$

The last property of seminorms that we will be using is: having a full unit ball. These properties have a hierarchy.

Proposition 3.1. *Let (X, K) be a directed partially ordered vector space and let p be a seminorm on X. Regard the following properties of p:*

1. *p has a full unit ball.*
2. *p is symmetrically monotone.*
3. *p is monotone*.*
4. *p is monotone.*
5. *p is regular.*

The implications $1 \implies 2$, $2 \implies 3$, $3 \implies 4$, *and* $5 \implies 2$ *hold.*

Proof. For the implications $1 \implies 2$, $2 \implies 3$, and $3 \implies 4$, see [1, Proposition 3.3.4]. The implication $5 \implies 2$ is straightforwardly verified: if p is regular, then for any $x, y \in X$ with $-y \leq x \leq y$ we have

$$p(y) \geq \inf\{p(z) \mid z \in X, -z \leq x \leq z\} = p(x),$$

hence p is symmetrically monotone. □

It is useful to verify which properties are conserved under scalar multiplication. Luckily, it is all of them.

Lemma 3.2. *Let p be a seminorm on a vector space X and $C \in \mathbb{R}_+$, then*

1. *If p is monotone, then Cp is monotone.*
2. *If p is monotone*, then Cp is monotone*.*
3. *If p is symmetrically monotone, then Cp is symmetrically monotone.*
4. *If p has a full unit ball, then Cp has a full unit ball.*
5. *If p is regular, then Cp is regular.*

Proof.
1. Assume p is monotone, and let $x, y \in X$ such that $0 \leq x \leq y$, then since p is monotone, we have that $p(x) \leq p(y)$. Multiplying both sides by C, we get that $Cp(x) \leq Cp(y)$, hence Cp is also monotone.

2. and 3. are similarly verified as in 1.

4. Assume that p has a full unit ball B. Let $B_C := \{x \in X \mid Cp(x) \leq 1\}$ be the unit ball of Cp. Let $x, y \in B_C$ such that $x \leq y$. Now let $z \in [x, y]$.

Since p is a seminorm and C non-negative, we have that $1 \geq Cp(x) = p(Cx)$, and likewise $p(Cy) \leq 1$, or in other words $Cx, Cy \in B$. Since $x \leq z \leq y$, we have that $Cx \leq Cz \leq Cy$, or, in other words, $Cz \in [Cx, Cy] \subseteq B$, as B is full. So $Cp(z) = p(Cz) \leq 1$, hence $z \in B_C$.

5. Assume p is regular. Then for any $x \in X$:

$$\begin{aligned} Cp(x) &= C \inf\{p(y) \mid y \in X, -y \leq x \leq y\} \\ &= \inf\{Cp(y) \mid y \in X, -y \leq x \leq y\} \end{aligned}$$

which holds since C is non-negative. Hence, Cp is regular. □

For properties which are conserved when taking sums or suprema of seminorms, see [1, Proposition 3.3.7, Example 3.3.8 and Example 3.4.3]. These results will not be needed here.

4 Extension Formulae for Seminorms

We will be extending seminorms that are monotone* to seminorms that are again monotone*, and likewise for symmetrically monotone seminorms and seminorms that have a full unit ball. For this we need the following extension theorems.

Theorem 4.1. *[1, Theorem 3.3.15] Let (X, K_X) and (Y, K_Y) be directed partially ordered vector spaces such that there exists a bipositive linear map $i : X \to Y$ with $i[X]$ majorizing in Y. Let p be a seminorm on X. Define for $y \in Y$,*

$$p_1(y) := \inf\{p(x) + p(u) + p(v) \mid x \in X, u, v \in K_X, -i(u) \le y - i(x) \le i(v)\}.$$

Then

i. For every $y \in K_Y$ one has $p_1(y) = \inf\{p(w) \mid w \in K_X, i(w) \ge y\}$.

ii. p_1 is the greatest monotone seminorm on Y with $p_1 \circ i \le p$.*

iii. $p_1 \circ i = p$ if and only if p is monotone.*

Theorem 4.2. *[1, Theorem 3.3.17] Let (X, K_X) and (Y, K_Y) be directed partially ordered vector spaces such that there exists a bipositive linear map $i : X \to Y$ with $i[X]$ majorizing in Y. Let p be a seminorm on X. Define for $y \in Y$,*

$$p_2(y) := \inf\{p(x) + p(u) \mid x, u \in X - i(u) \le y - i(x) \le i(u)\},$$
$$p_3(y) := \inf\{p(u) \vee p(v) \mid u, v \in X, i(u) \le y \le i(v)\},$$

and let p_1 be as in Theorem 4.1. Then

i. For every $y \in K_Y$ we have $p_2(y) = \inf\{p(w) \mid w \in K_X, i(w) \ge y\}$ and $p_3(y) = \inf\{p(w) \mid w \in K_X, i(w) \ge y\}$.

ii. p_2 is the greatest symmetrically monotone seminorm on Y with $p_2 \circ i \le p$.

iii. $p_2 \circ i = p$ if and only if p is symmetrically monotone.

iv. p_3 is the greatest seminorm with a full unit ball on Y with $p_3 \circ i \le p$.

v. $p_3 \circ i = p$ if and only if p has a full unit ball.

vi. $p_3 \le p_2 \le p_1 \le 2p_2$ and $p_1 \le 3p_3$. In particular, p_1, p_2, and p_3 are equivalent.

vii. If p is monotone, then $p \le \frac{3}{2}p_1 \circ i$, $p \le 2p_2 \circ i$, and $p \le 3p_3 \circ i$.

Corollary 4.2.1. *[1, Corollary 3.3.20] Let (X, K) be a directed partially ordered vector space and let p be a monotone seminorm on X. Define for $y \in X$,*

$$p_1(y) := \inf\{p(x) + p(u) + p(v) \mid x \in X, u, v \in K, -u \le y - x \le v\},$$
$$p_2(y) := \inf\{p(x) + p(u) \mid x, u \in X, -u \le y - x \le u\},$$
$$p_3(y) := \inf\{p(u) \vee p(v) \mid u, v \in X, u \le y \le v\}.$$

Then p_1 is a monotone seminorm, p_2 is a symmetrically monotone seminorm, and p_3 is a seminorm with a full unit ball. The seminorms p_1, p_2, and p_3 are equivalent to p and $p_1, p_2, p_3 \le p$.*

In the main result of this thesis, we also need a similar extension theorem for regular seminorms.

Theorem 4.3. *[1, Theorem 3.4.4] Let (X, K_X) and (Y, K_Y) be directed partially ordered vector spaces and let $i : X \to Y$ be a bipositive linear map such that $i[X]$ is majorizing in Y. Let p be a seminorm on X. Define for $y \in Y$,*

$$\rho(y) := \inf\{p(x) \mid x \in X, -i(x) \leq y \leq i(x)\}.$$

Then

i. ρ *is the greatest regular seminorm on* Y *with* $\rho \circ i \leq p$ *on* K_X*.*

ii. $\rho \circ i = p$ *on* K_X *if and only if* p *is monotone.*

iii. $\rho \circ i \geq p$ *on* X *if and only if* p *is symmetrically monotone.*

iv. $\rho \circ i = p$ *on* X *if and only if* p *is regular.*

5 Extension to a Norm

In this section, we present our main result, Theorem 5.4. It answers, for many cases, whether norms on pre-Riesz spaces extend to norms on their vector lattice covers. From the following example, which we cite from Kalauch and Van Gaans, we see that this is not always the case, i.e. that there are examples of norms which only extend to seminorms.

Example 5.1. [1, Example 3.3.22] "We provide a majorizing Riesz subspace X of a Riesz space Y and a monotone norm p on X such that every monotone seminorm on Y that extends p is not a norm.

Let $Y = B[0,1]$ and let $X = C[0,1]$. Let $p = \|\cdot\|_1$ on X. Then X is a majorizing Riesz subspace of the Riesz space Y and p is a monotone (even a Riesz) norm on X. Since every monotone seminorm on Y is monotone* due to Proposition 3.3.6, the greatest monotone seminorm on Y that extends p is the seminorm p_1 given by Theorem 3.3.15. If we let $y \in Y$ be given by $y(t) = 0$ for $t \in (0,1]$ and $y(0) = 1$, then $p_1(y) = 0$, so that p_1 is not a norm on Y. Because p_1 is the greatest monotone seminorm on Y extending p, no monotone seminorm on Y extending p is a norm."

Now, before we proceed, we need a few lemmas. A crucial part in the proof of Theorem 5.4 is that we obtain a seminorm on the bigger space that is a norm. For this we use the following result.

Lemma 5.1. *[1, Proposition 3.4.6] Let (X, K) be a pre-Riesz space and let (Y, i) be a vector lattice cover of X. Let p be a regular norm on X and let ρ be the greatest Riesz seminorm on Y with $\rho \circ i = p$. Then ρ is a norm if and only if K is p-closed[2].*

And for a slight variant on the main result, we need the following lemma.

Lemma 5.2. *[1, Corollary 3.4.13ii] Let (X, K) be a directed partially ordered vector space with a monotone norm p such that K is p-closed. If X is p-complete, then p is equivalent to a regular norm.*

Before we proceed to the main result, we need only one last simple observation.

Lemma 5.3. *If p is a norm and q a seminorm on a vector space X such that $p \leq Cq$ for some $C \in \mathbb{R}_+$, then q is also a norm.*

Proof. Suppose $x \in X$ is such that $q(x) = 0$, then

$$0 \leq p(x) \leq Cq(x) = C \cdot 0 = 0$$

hence $p(x) = 0$, and since p is a norm, it follows that $x = 0$. □

Corollary 5.3.1. *If p and q are equivalent seminorms, then p is a norm if and only if q is a norm.*

Finally, we are ready to bring everything together and state the main result.

Theorem 5.4. *Let (X, K_X) be a pre-Riesz space, (Y, i) a vector lattice cover of X, and p a monotone norm on X. Furthermore, assume $Cp \geq r$ where $C \in \mathbb{R}_{>0}$ and r is a regular norm on X, and that K_X is r-closed.*

Then,

1. *p_1, the greatest monotone* seminorm on Y with $p_1 \circ i \leq p$ as described in Theorem 4.1, is also a norm.*
2. *p_2, the greatest symmetrically monotone seminorm on Y with $p_2 \circ i \leq p$ as described in Theorem 4.2, is also a norm.*
3. *p_3, the greatest seminorm with a full unit ball on Y with $p_3 \circ i \leq p$ as described in Theorem 4.2, is also a norm.*

[2] p-closed means closed in the topology determined by the norm p. In the same vein, we use the term p-complete if every p-Cauchy sequence p-converges.

Proof.

1. Since r is regular, we can define the regular extension, ρ, of r on Y as in Theorem 4.3 such that $\rho \circ i = r$. Furthermore, by Lemma 5.1, ρ is a norm, as K_X is r-closed.

   By assumption, we have that $r \leq Cp$ for some $C \in \mathbb{R}_{>0}$, hence $\rho \circ i = r \leq Cp$. Thus, $(\frac{1}{C}\rho) \circ i = \frac{1}{C}(\rho \circ i) \leq p$, and since ρ is regular, it is also monotone*, and by Lemma 3.2 it follows that $\frac{1}{C}\rho$ is monotone*.

   Since p_1 is the greatest monotone* seminorm on Y with $p_1 \circ i \leq p$, it follows that $\frac{1}{C}\rho \leq p_1$, hence $\rho \leq Cp_1$. By Lemma 5.3 we conclude that p_1 is a norm.

2. This proof is completely analogous to the proof of 1. Since ρ is regular, it is also symmetrically monotone, then by applying Theorem 4.2 it follows that $\rho \leq Cp_2$.

3. Let ρ be as in the proof of 1, then again ρ is a regular norm such that $\rho \circ i = r$. By Corollary 4.2.1, ρ is equivalent to another norm ρ_f on Y which has a full unit ball.

   Then we have that $\rho_f \leq m\rho$ for some constant $m > 0$, and so $\frac{1}{m}\rho_f \leq \rho$, which must also hold if we restrict $\frac{1}{m}\rho_f$ and ρ to $\mathrm{Im}(i)$, hence $\frac{1}{m}\rho_f \circ i \leq \rho \circ i$.

   Finally, we find $\frac{1}{m}\rho_f \circ i \leq \rho \circ i = r \leq Cp$, giving us $\frac{1}{mC}\rho_f \circ i \leq p$, and since p_3 is the greatest seminorm on Y with a full unit ball such that $p_3 \circ i \leq p$, we obtain $\frac{1}{mC}\rho_f \leq p_3$. Again, since ρ and ρ_f are equivalent, $\rho \leq M\rho_f$ for some constant $M > 0$. We find $\rho \leq mCMp_3$, and conclude that p_3 is a norm.

□

Remark. The reason we take a little detour with the proof for 3 is that a seminorm being regular does not imply that it has a full unit ball. Note that this adjusted proof would also have worked for 1 and 2.

Remark. In this proof we could have used Zhang's result in [3, Lemma 2.1.6] instead of Lemma 5.1 to obtain our norm ρ on the bigger space. This would allow us to assume pervasiveness instead of K being r-closed. Note, however, that this would result in a strictly weaker statement, since from Lemma 5.1 we know that ρ being a norm implies K being r-closed anyway.

Remark. Many examples of monotone* and symmetrically monotone norms are a sum of a regular norm with some seminorm, this gives us the condition for the theorem with $C = 1$. For example, one could consider $X = C^1[0,1], K_X = \{f \in C^1[0,1] \mid \forall x \in [0,1] : f(x) \geq 0\}$ with $Y = C[0,1]$ as a vector lattice cover. Then $p(f) := \|f\|_\infty + |f(0) + f(1)|$ or $q(f) := \|f\|_1 + \left|\int_0^1 f(x)dx\right|$ are examples of symmetrically monotone norms, since the terms $\|f\|_\infty$ and $\|f\|_1$ are regular norms and the terms $|f(0) + f(1)|$ and $\left|\int_0^1 f(x)dx\right|$ are symmetrically monotone seminorms.

For many other examples we have the following corollary.

Corollary 5.4.1. *Let (X, K_X) be a pre-Riesz space, (Y, i) a vector lattice cover of X and p a monotone norm on X such that K_X is p-closed and X is p-complete. Then by Lemma 5.2, p is equivalent to a regular norm p_r, and by the equivalence of p and p_r we know K_X is p_r-closed. Thus we can apply Theorem 5.4 and obtain that p_1, p_2, p_3 are norms.*

Revisiting the case of Example 5.1, we see that our results indeed do not apply, since the vector y given in the example is a limit point of the cone of X, but is not a continuous function itself, hence the cone is not closed.

6 Exploration

With our main result and its corollary, we have covered quite a few bases. However, we can still attempt to find more cases where norms extend to norms. In this section, let (X, K_X) be a pre-Riesz space, (Y, i) a vector lattice cover of X, p a monotone norm on X, and let $\hat{\cdot}$ denote the norm-completion of the object, i.e. $\hat{X}$ is the norm-completion of X, $\hat{p}$ is the norm on $\hat{X}$ extending p et cetera.

To this end, we could consider Corollary 5.4.1 of the main result and see what happens if instead of assuming norm-completeness, we try to take the norm-completion of X. In that case, we could use the Corollary 5.4.1 on $\hat{X}$, and then we would have to relate the result to the original norm p or its extension on Y.

The first question is whether $\hat{X}$ is an ordered vector space. For this, we use [2, Proposition 3.50] and find the cone of $\hat{X}$ by taking the closure of K_X in $\hat{X}$. Since in Corollary 5.4.1 we assume we have a monotone norm and that K_X is closed by this norm, we find that this indeed gives us a vector space order on $\hat{X}$ that extends the one on X.

Next, in order to use Corollary 5.4.1, we need to know if $\hat{p}$ retains the properties of being monotone*, being symmetrically monotone, or having a full unit ball. See [2, Theorem 3.51i][3] to know that this is indeed the case for symmetrical monotonicity. Adapting the proof given by Van Gaans for monotone* norms does not seem to yield the same result. Since norms with a full unit ball are often quite different from monotone* and symmetrically monotone norms, we would not expect adapting the proof to work in this case either. Indeed, it does not.

Next, in order to see if we can relate the extension of $\hat{p}$ to the extension of p on Y, we ask whether a vector lattice cover of $\hat{X}$ is also a vector lattice cover of X. We approach this question a bit backwards, considering a vector lattice cover Y of X, and taking the norm-completion of Y, $\hat{Y}$. We can take the closure of X in $\hat{Y}$, then the closure $\bar{X}$ is automatically the norm-completion of X and if X is order dense in $\hat{Y}$, $\hat{Y}$ is also a vector lattice cover of $\bar{X}$. This however, does not yield any conclusions, only more questions, like how to properly define i on $\hat{X} \setminus X$.

Finally, we can ask whether the norm-completion of a pre-Riesz space is still pre-Riesz, since that is a crucial premise for the main result. It is sufficient for a space to be directed and Archimedean, in order for it to be pre-Riesz.

Claim. If a norm $\|\cdot\|$ is monotone and the cone K_X is $\|\cdot\|$-closed, then the norm-completion of a pre-Riesz space (X, K_X) w.r.t. $\|\cdot\|$ is Archimedean.

Proof. Assume $y, x \in \hat{X}$ are such that $nx \leq y$ for all $n \in N$. Then $\frac{1}{n}y - x \geq 0$ for all $n \in \mathbb{N}$. We have that

$$\lim_{n\to\infty} \left\| (\frac{1}{n}y - x) - (-x) \right\| = \lim_{n\to\infty} \left\| \frac{1}{n}y \right\| = \lim_{n\to\infty} \left| \frac{1}{n} \right| \|y\| = 0$$

meaning the sequence $\frac{1}{n}y - x$ norm-converges to $-x$. And since all $\frac{1}{n}y - x \in \hat{K}$, we find that $-x \in \bar{\hat{K}} = \hat{K}$ and so $x \in -\hat{K}$. □

However, the neccesary condition of being directed is not always guaranteed. See [2, Proposition 3.55v] for a counterexample, and $(C[0,1], \|\cdot\|_1)$ on [2, page 62] for a concrete instance of this counterexample. Thus we conclude that the norm-completion of a pre-Riesz space is not neccesarily a pre-Riesz space.

Thus concludes the exploration of the extension of norms. Clearly, the approach using norm-completions is going to need some work. Besides this, one could attempt to generalise the main result by seeing what happens with the extension of norms that are not greater than any scalar multiple of any regular norms, or finding another way to guarantee the extension is a norm without assuming the cone is closed.

[3] In this work, Van Gaans refers to the property of a seminorm being symmetrically monotone with the term Fremlin.

References

[1] Anke Kalauch and Onno van Gaans. *Pre-Riesz Spaces.* De Gruyter, 2018.

[2] Onno van Gaans. *Seminorms on Ordered Vector Spaces.* PhD thesis, Katholieke Universiteit Nijmegen, 1999.

[3] Feng Zhang. *Extension of Operators on Pre-Riesz Spaces.* PhD thesis, Leiden University, 2018.