Algebraic topology and real division algebras: The 1,2,4,8 Theorem

Nifterik, L.D.V. van

L. van Nifterik

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Thesis supervisor: dr. F. Arici

Leiden University
Mathematical Institute

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Introduction

After the discovery of the complex numbers, several mathematicians like Gauss, Weierstrass and Dedekind questioned the existence of even bigger number systems which would still have some of the basic properties of the complex numbers.
In the middle of the nineteenth century, the mathematician Hamilton discovered the quaternions, a four dimensional number system in which the properties of the complex numbers still persist, with the exception of the commutativity law [19]. This discovery was closely followed by the one of the octonion numbers, an eight dimensional number system which conserves most properties of the quaternions, but in which the associativity law does no longer hold [7].
Even if some structure was lost when forming those higher dimensional number systems, they all remain what we call division algebras, which roughly design spaces in which division can be performed. It is then natural to wonder whether it is possible to get divisions algebras over $\mathbf{R}$ of a even higher dimension that the one of the octonions.
Heinz Hopf put a first step into answering this question by stating in 1940 that if any higher dimensional real division algebra would exist, then its dimension would have to be a power of two, thereby reducing the field of possibilities. A bit later, Michel Kervaire and John Milnor proved that the only dimensions a division algebra over $\mathbf{R}$ can take are the dimensions 1,2,4 or 8, thereby excluding the possibility of a real division algebra whose dimension would be higher than the one of the octonions. This thesis aims at giving a proof of this theorem called the 1,2,4,8-Theorem.
The outline of this work is as follows:
In section 1 we will introduce the concept of real division algebras and describe the spaces of the quaternions and octonions.
In section 2 we will give a proof oh Hopf's theorem, with the use of some notions in algebraic topology. This theorem will give us that the dimension of a real division algebra has to be a power of two.
In section 3, we will introduce the concepts of vector bundles and complex K-theory and give some of their properties.
In section 4 we will use the theory of vector bundles and complex K-theory to introduce the notions of Hopf invariant and Adams operations in order to eventually prove that the only possible dimensions of a real division algebra are indeed given by 1,2,4 or 8.

1 Real division algebras

In this section, we will describe the concept of a finite dimensional real division algebra and in particular introduce the one of the quaternions and octonions. This will enable us to grasp the structure of real divisions algebras of different dimensions and thus give us an insight about the kind of objects they are. Our main references for this section will be given by [14] and [15].
We start by introducing the concept of a real algebra:

Definition 1.1. Let V be a vector space over the field $\mathbf{R}$. Suppose that V is provided with a multiplication $\cdot : V \times V \to V$, $(x, y) \mapsto x \cdot y$. Then V is said to be an algebra over $\mathbf{R}$ (or a real algebra), if this multiplication obeys the two following distributive laws:

- $(\lambda x + \mu y) \cdot z = \lambda(x \cdot z) + \mu(y \cdot z)$
- $x \cdot (\lambda y + \mu z) = \lambda(x \cdot y) + \mu(x \cdot z)$

  with $\lambda, \mu \in \mathbf{R}$ and $x, y, z \in V$.

Furthermore:

- if for all $x, y, z \in V$, the relation $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ holds, then V is an associative algebra.
- if for all $x, y \in V$, the relation $x \cdot y = y \cdot x$ holds, then V is an commutative algebra.

Definition 1.2. Let $\mathcal{A}$ be a real algebra. An element $e \in \mathcal{A}$ is called an identity (or unit) element if for all $x \in \mathcal{A}$, we have that $x \cdot e = e \cdot x = x$.

Proposition 1.1. *Let $\mathcal{A}$ be a real algebra. If there exists such an identity element e in $\mathcal{A}$, then it is unique.*

Definition 1.3. Let $\mathcal{A}$ be a real algebra and x an element of $\mathcal{A}$. Then x is called a *divisor of zero* (or *zero divisor*) if there exists $y \in \mathcal{A}$ with $y \neq 0$ such that the relation $x \cdot y = 0$ (left zero divisor) or $y \cdot x = 0$ (right zero divisor) holds. If $\mathcal{A}$ has no divisors of zero then the equation $x \cdot y = 0$ for $x, y \in \mathcal{A}$ implies that either x or y is zero. $\mathcal{A}$ is then said to have no zero divisors.

Several examples of real algebras can be given:

Example 1.1. Familiar examples of real algebras are given by the vector spaces $\mathbf{R}$ and $\mathbf{C}$ over $\mathbf{R}$, respectively of dimension 1 and 2. They are associative and commutative, contain 1 as identity element, and have no zero divisors.

Example 1.2. The $\mathbf{R}$-vector space $S(n, \mathbf{R})$ of symmetric $n \times n$-matrices with coefficients in $\mathbf{R}$ equipped with the symmetrical matrix product $(A, B) \mapsto \frac{1}{2}(AB + BA)$ forms a real commutative algebra. However, because of the non-commutativity of matrices in general, this algebra is non associative for $n > 1$.

Example 1.3. We define the space of Hamilton's quaternions, denoted $\mathbf{H}$, as the space of numbers of the form $a+bi+cj+dk$ where a, b, c, d are real numbers and i, j, k the so-called quaternions units. This space forms an extension of the complex numbers and was first described by the Irish mathematician William Rowan Hamilton in 1843 ([19]). In this space, the multiplication on the base $\{1, i, j, k\}$ is defined using the following table:

ab	**1**	**i**	**j**	**k**
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

where a is the element of the row and b the one of the column. By extension, this table gives us a multiplication on the whole space which turns the space of the quaternions into a four dimensional real algebra. Looking at the properties of the basis elements and then extending it, one can easily prove that this algebra has the element 1 as identity element, is associative and has no zero divisors.
Since the quaternion units does not commute with each other (for example $ij \neq ji$), this algebra is non-commutative.

Example 1.4. We define the space of Cayley octonions, denoted **O**, as the space of real linear combinations of the so called octonion units $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. This space was first described in 1843 by John T.Graves in a letter to William Rowan Hamilton ([7]) and later by Arthur Cayley ([5]) independently. It forms an extension of the quaternions just as the quaternion space forms an extension of the complex numbers. The multiplication of the octonion units is defined using the following table:

ab	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_0$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\mathbf{e}_1$	e_1	-e_0	e_3	-e_2	e_5	-e_4	-e_7	e_6
$\mathbf{e}_2$	e_2	-e_3	-e_0	e_1	e_6	e_7	-e_4	-e_5
$\mathbf{e}_3$	e_3	e_2	-e_1	-e_0	e_7	-e_6	e_5	-e_4
$\mathbf{e}_4$	e_4	-e_5	-e_6	-e_7	-e_0	e_1	e_2	e_3
$\mathbf{e}_5$	e_5	e_4	-e_7	e_6	-e_1	-e_0	-e_3	e_2
$\mathbf{e}_6$	e_6	e_7	e_4	-e_5	-e_2	e_3	-e_0	-e_1
$\mathbf{e}_7$	e_7	-e_6	e_5	e_4	-e_3	-e_2	e_1	-e_0

where a is the element of the row and b the element of the column. This table gives us by extension the multiplication on the whole space. Using this table, one can prove that the octonions form an eight dimensional real algebra with e_0 as identity element and with no zero divisors. Like the quaternions, we can see from the table that this algebra is non-commutative. Besides this algebra is also non-associative: we have for example that $(e_1e_2)e_4 = e_3e_4 = e_7$ but $e_1(e_2e_4) = e_1e_6 = -e_7$.

Definition 1.4. (**R**-algebra homomorphisms)
Let $\mathcal{A} = (V, \cdot)$ and $\mathcal{B} = (W, *)$ be real algebras. We say that a **R**-linear map $f : \mathcal{A} \to \mathcal{B}$ is an **R**-algebra homomorphism if for all $x, y \in V$, we have that $f(x \cdot y) = f(x) * f(y)$

Remark 1.2. The terms *monomorphism*, *epimorphism*, *endomorphism*, *isomorphism* ,*automorphism* are used when the **R**-linear map f is a morphism of the corresponding type.

Theorem 1.3. *Let $\mathcal{A} = (V, \cdot)$ be a one dimensional real algebra such that the multiplication $\cdot$ is not the zero mapping. Then $\mathcal{A}$ is isomorphic to the algebra* **R**.

Proof. Let $\mathcal{A}$ be a one-dimensional real algebra with a non-zero multiplication $(x,y) \mapsto x \cdot y$. Then $\mathcal{A} = \mathbf{R}a$, for some $a \in \mathcal{A} \setminus \{0\}$. Since the multiplication is not the zero mapping, we have that the product $x \cdot y$ with $x, y \in \mathcal{A}$ is not always zero. Let $x = \alpha a$ and $y = \beta a$, $\alpha, \beta \in \mathbf{R}$. Then the statement $x \cdot y \neq 0$ means that $\alpha\beta a^2 \neq 0$ and therefore a^2 must also be non-zero. Hence, we have that $\mathcal{A} = \mathbf{R}a^2$ as well and there exists an $\gamma \in \mathbf{R}$ such that $a = \gamma a^2$. Take now $e = \gamma a$. Then for all $x = \alpha a \in A$, we have that $e \cdot x = \gamma a \cdot \alpha a = \alpha\gamma a^2 = \alpha a = x$ and similarly for $x \cdot e$. Thus, e is the identity element of $\mathcal{A}$.
Take now the map $f : \mathbf{R} \to \mathcal{A}$ given by $\alpha \mapsto \alpha e$. Clearly this map is $\mathbf{R}$-linear and for all $\alpha, \beta \in \mathbf{R}$, we have $f(\alpha\beta) = \alpha\beta e = \alpha e \cdot \beta e = f(\alpha) \cdot f(\beta)$. Thus f is an $\mathbf{R}$-algebra morphism between two real algebras of dimension 1. Besides, f is clearly injective and therefore bijective. Thus, it yields an isomorphism between $\mathbf{R}$ and $\mathcal{A}$. □

Definition 1.5. (Division algebras)
Let $\mathcal{A} = (V, \cdot)$ be a non-zero algebra. We say that $\mathcal{A}$ is a *division algebra* if for all $a, b \in V$ such that $a \neq 0$ the two equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions in $\mathcal{A}$.

It is clear that every division algebras has no zero divisors. For finite-dimensional algebras, we have the following:

Criterion. *For a finite-dimensional algebra $\mathcal{A}$, the following statements are equivalent:*

1. *$\mathcal{A}$ has no zero divisors;*
2. *$\mathcal{A}$ is a division algebra;*

Proof. We only need to show that $1 \Rightarrow 2$. Let $\mathcal{A}$ be a finite-dimensional algebra with multiplication $(x,y) \mapsto x \cdot y$. Assume $\mathcal{A}$ has no zero divisors and take then $a \in \mathcal{A} \setminus \{0\}$.
Consider now the linear map $f : \mathcal{A} \to \mathcal{A}$ given by $x \mapsto a \cdot x$. By hypothesis the map is injective and therefore it is also bijective. This means that for all $b \in \mathcal{A}$, there is a unique $x \in \mathcal{A}$ such that $a \cdot x = b$. By doing the exact same proof for the linear map $y \mapsto y \cdot a$, we have that the equation $y \cdot a = b$ for any $b \in \mathcal{A}$ has a unique solution as well, which concludes the proof. □

Example 1.5. The real algebras $\mathbf{R}$ and $\mathbf{C}$, as well as the Hamilton's quaternions $\mathbf{H}$ and the Cayley octonions $\mathbf{O}$, as stated before, have no zero-divisors. Since those are finite-dimensional real algebras, the criterion gives us that they must be real division algebras.

Example 1.6. The four dimensional $\mathbf{R}$-vector space $S(2, \mathbf{R})$ of symmetric 2×2-matrices with coefficients in $\mathbf{R}$ equipped with the symmetrical matrix product as defined previously contains divisors of zero such as $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ since $A \cdot B = 0$ when we take, for instance, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore it is not a real division algebra.

We have stated in this section that all one-dimensional real algebra are isomorphic to $\mathbf{R}$. In fact, the existence, up to isomorphism, of finite dimensional real division algebras in general is quite restricted. The possibility of such an algebra to exist has been studied by several mathematicians in the twentieth century. It has given rise to a few theorems on the subject, which will constitute the topic of the following sections.

2 Hopf's theorem. Homology and cohomology with coefficients in $\mathbf{Z}_2$.

The first important theorem on the subject is called Hopf's theorem, stated and proved by the German mathematician Heinz Hopf in 1940([11]). The theorem in its general form gives a result about the dimension of particular unit spheres. By using some tools of algebraic topology, we will give a proof of this theorem, before looking more closely at one of its corollary which deals with the dimension of real division algebras.

2.1 Hopf's theorem and its corollary

In order to state the general theorem, we will firstly recall some definition:

Definition 2.1. We define the $(n-1)-$dimensional unit sphere S^{n-1} as the set of points:

$$S^{n-1} := \{x \in \mathbf{R}^n \quad | \quad ||x|| = 1\}$$

where $||\cdot||$ denotes the Euclidean norm on $\mathbf{R}^n$: $||(x_1, x_2, ..., x_n)|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$

As we have already dealt in the previous with the real division algebra of dimension 1, we will now consider in the rest of this article that n is an integer greater or equal to 2.

Definition 2.2. Let f be a map going from a product of spaces $X \times Y$ to a space Z. We say that f is odd if for all $(x, y) \in X \times Y$, $f(-x, y) = f(x, -y) = -f(x, y)$

With those two definitions, we can now state Hopf's theorem:

Theorem 2.1. *(Hopf's theorem)*
If there exists a continuous odd mapping of $S^{n-1} \times S^{n-1}$ into S^{n-1} then n is a power of 2.

From Hopf's theorem follows an interesting corollary about the dimension of real division algebras.

Corollary. *The dimension of a finite dimensional real division algebra is a power of two.*

Proof. Let A be a n-dimensional real division algebra. A is an algebra so there exist on A a multiplication $* : A \times A \to A$. Because A is a real division algebra of dimension n, we can find a vector space isomorphism of A onto $\mathbf{R}^n$ and transfer the multiplication $*$ on A to a multiplication $\cdot : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ where $x \cdot y$ with $x = (x_1, ..., x_n), y = (y_1, .., y_n) \in \mathbf{R}^n$ is equal to $(x_1y_1, ..., x_ny_n)$. Now by restricting to S^{n-1} we get a map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ defined as:

$$(x, y) \mapsto \frac{x \cdot y}{||x \cdot y||}$$

with $||\cdot||$ being the Euclidean norm.
Since A is a division algebra, we have that for all $a, b \in A$ non-zero, $ab \neq 0$ and therefore g is well-defined and continuous. Moreover, it is quite easy to notice that the coefficient-wise multiplication turns g into an odd map. Hence, we have constructed from our n-dimensional real division algebra an odd mapping g from $S^{n-1} \times S^{n-1}$ into S^{n-1}. It follows from Hopf's theorem that n is a power of two.
Thus the dimension of any finite-dimensional real division algebra has to be a power of two, which proves the corollary. □

We now proceed to prove the theorem with the help of algebraic topology.

2.2 Homology and cohomology

In order to prove Hopf's theorem, we will need to take a closer look at homology and cohomology with coefficients in $\mathbf{Z}_2$ on unit spheres and projective spaces. This will constitute the main topic of the following pages.

Definition 2.3. We define the real projective space $\mathbf{P}^{n-1}$ as the topological space of lines passing through the origin in $\mathbf{R}^n$. In other words, the real projective space $\mathbf{P}^{n-1}$ is given by the quotient of $\mathbf{R}^n \setminus \{0\}$ under the equivalence relation $x \sim \lambda x$, $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$.

For $x \in \mathbf{R} \setminus \{0\}$, one can always turns λx into a vector of norm 1 by taking $\lambda = \pm \frac{1}{||x||}$. Because the choice of lambda then only depends on its sign, it is possible to give an alternative definition of the real projective space $\mathbf{P}^{n-1}$, which we will more likely use in this section:

Definition 2.4. The real projective space $\mathbf{P}^{n-1}$ can be defined as the $(n-1)$-dimensional manifold obtained from the sphere S^{n-1} after identifying each point $z \in S^{n-1}$ with its antipodal point $-z$. Hence:

$$\mathbf{P}^{n-1} := S^{n-1}/\{\pm 1\}$$

We denote with $\alpha : S^{n-1} \to \mathbf{P}^{n-1}$ this identification.

The proof of Hopf's theorem requires singular homology and cohomology on projective spaces and spheres. The two points given below about homology groups are thus particularly relevant: Let X be a topological space. The q-th homology group on X is denoted $H_q(X)$.

1. Every closed path ω in X represents a homology class $|\omega|$ in $H_1(X)$
2. Every q-dimensional submanifold M of an n-dimensional manifold X represents a homology class $|M|$ in $H_q(X)$. For $q = n$, we have that $H_n(X) \cong \mathbf{Z}$ with $|X|$ being the non-zero element.

For the unit sphere and real projective spaces, the homology classes are given by the points above. This does however not hold in general. We have that:

$$H_q(S^n) = \begin{cases} \mathbf{Z} \text{ if } q = 0 \text{ or } q = n \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

For P an arbitrary point, $|P|$ is the non-zero element of $H_0(S^n)$ and in the case $q = n$, $|S^n|$ is the non zero element of $H_n(S^n)$

For the q-th homology of the real projective space $\mathbf{P}^n$, we get:

$$H_q(\mathbf{P}^n) = \begin{cases} \mathbf{Z} \text{ if } q = 0, n \\ \mathbf{Z}_2 \text{ if } 0 < q < n \\ 0 \text{ if } q > n \end{cases}$$

where $|\mathbf{P}^q|$ is the non-zero element of $H_q(\mathbf{P}^n)$ with $\mathbf{P}^q$ the q-th dimensional projective subspace of $\mathbf{P}^n$.

In the proof of Hopf's theorem, we will use homology with coefficients in $\mathbf{Z}_2$.By the universal coefficient theorem, the homology groups then have the following properties:

Every q-dimensional submanifold M of an n-dimensional oriented closed manifold X represents a homology class $|M|$ in $H_q(X, \mathbf{Z}_2)$. For $q = n$, we have that $H_n(X, \mathbf{Z}_2) \cong \mathbf{Z}_2$ with $|X|$ being the non-zero element.

The q-th homology group on the sphere is given by:

$$H_q(S^n, \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & \text{if } q = 0 \text{ or } q = n \\ 0 & \text{otherwise} \end{cases}$$

The q-th homology group on the real projective space is given by

$$H_q(\mathbf{P}^n, \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & \text{if } q \leq n \\ 0 & \text{if } q > n \end{cases}$$

We now have given the q-th homology classes for the unit sphere and real projective spaces with coefficients in $\mathbf{Z}$ and $\mathbf{Z}_2$. From now on, we will write R to denote either of the rings $\mathbf{Z}$ or $\mathbf{Z}_2$. Since Hopf's theorem actually deals with maps between unit spheres, we will also need the following proposition:

Proposition 2.2. *Any continuous map $f : X \longrightarrow Y$ between X, Y topological spaces induces a homomorphism $f_* : H_q(X, R) \longrightarrow H_q(Y, R)$ on the q-th homology defined as $f_*([\sigma]) = [f \circ \sigma]$. This map f_* is called the pushforward of f.*

Beside being a group homomorphism, this map f_* has the following properties:

Properties.

- *Preservation of the identity:* $(id_X)_* = id_{H_q(X,R)}$
- *Preservation of the composition: For $f : X \to Y$ and $g : Y \to Z$ continuous maps, we have that $(g \circ f)_* = g_* \circ f_*$ which then gives us a map $H_q(X, R) \longrightarrow H_q(Z, R)$*

Hence we can get a map between homology groups from a continuous maps between topological spaces. We can now continue with the next important topic of this section which is the notion of cohomology with coefficients in $\mathbf{Z}_2$.

Definition 2.5. The q-th cohomology group $H^q(X, \mathbf{Z}_2)$ of a topological space X with coefficients in $\mathbf{Z}_2$ is given by the dual of its homology group, in other words:

$$H^q(X, \mathbf{Z}_2) = \text{Hom}(H_q(X), \mathbf{Z}_2)$$

Notation. For $u \in H^q(X, \mathbf{Z}_2)$ and $x \in H_q(X, \mathbf{Z}_2)$, we will denote the value of u on x by $< u, x >$

We can now define a map on cohomology as well:

Definition 2.6. For $f_* : H_q(X, \mathbf{Z}_2) \to H_q(Y, \mathbf{Z}_2)$ with X, Y topological spaces, we have a dual homomorphism $f^* : H^q(Y, \mathbf{Z}_2) \to H^q(X, \mathbf{Z}_2)$ defined by $f^*(u) = u \circ f_*$ such that

$$< f^*(u), x >=< u, f_*(x) >$$

for $x \in X$

With this definition we can easily see that the properties of f_* translate into those corresponding ones for f^*:

Properties.

- *Preservation of the identity:* $(id_{H_q(X,\mathbf{Z}_2))})^* = id_{H^q(X,\mathbf{Z}_2)}$

- *Composition: For $f : X \to Y$ and $g : Y \to Z$ continuous maps, we have that $(g \circ f)^* = f^* \circ g^*$ which then gives us a map $H^q(Z, \mathbf{Z}_2) \longrightarrow H^q(, \mathbf{Z}_2)$*

We have now defined the concept of homological and cohomological spaces with coefficients in $\mathbf{Z}_2$ and see that we can construct maps between them from continuous maps between topological spaces. As the map defined in Hopf's theorem has a product of topological space as domain, the next step consist in constructing a product on the cohomology and homology groups.
It is possible to define a product $H^p(X, \mathbf{Z}_2) \times H^q(X, \mathbf{Z}_2) \to H^{p+q}(X, \mathbf{Z}_2)$ on the cohomology groups for X a topological space([8, Chapter 3.2]). Using this product, we can define the following ring:

Definition 2.7. Let X be a topological space. We define the cohomology ring with values in $\mathbf{Z}_2$ as the direct sum $H^*(X, \mathbf{Z}_2) = \bigoplus_{p \geq 0} H^p(X, \mathbf{Z}_2)$. This ring is associative and commutative.

To define the product on homology, we firstly need to use the following important theorem, which we will state but not prove. A proof can be found in [8, Section 3.3]

Theorem 2.3. *(Poincaré duality)*
For M a closed oriented topological manifold, there exists an isomorphism between the $(n-p)$-th homological space of M and the p-th cohomological space of M given by $\pi : H^p(M, \mathbf{Z}_2) \to H_{n-p}(M, \mathbf{Z}_2)$, $\alpha \mapsto |M| \frown \alpha$.

More information about the notations used in this definition can be found in the Appendix. Using this isomorphism, we can now define a product on homological spaces.

Definition 2.8. For X a topological space, the intersection product on $H_*(X, \mathbf{Z}_2)$ is defined degree by degree as the multiplication:

$$H_{n-p}(X, \mathbf{Z}_2) \times H_{n-q}(X, \mathbf{Z}_2) \to H_{n-(p+q)}(X, \mathbf{Z}_2)$$

Concerning the intersection product of projective spaces, which will be our main tool in the proof of Theorem 2.1, we can state those two important examples:

Example 2.1. We recall that q projective spaces of dimension $n-1$ in general position intersect in a projective subspace $\mathbf{P}^{n-q}$ of dimension $n-q$ where $0 \leq q \leq n$.

Let $t = \pi(|\mathbf{P}^{n-1}|)$ be the non-zero element of $H^1(\mathbf{P}^n, \mathbf{Z}_2)$ and $t^q = \pi(|\mathbf{P}^{n-q}|)$ the non-zero element of $H^q(\mathbf{P}^n, \mathbf{Z}_2)$.
Then

$$H^*(\mathbf{P}^n, \mathbf{Z}_2) = \mathbf{Z}_2[t] / (t^{n+1}),$$

with (t^{n+1}) denoting the ideal generated by the element t^{n+1}

Example 2.2. The homology classes $|\mathbf{P}^r \times \mathbf{P}^s|$ with $r + s = q$ with $0 \leq r, s \leq n$ form a basis of $H_q(\mathbf{P}^n \times \mathbf{P}^n, \mathbf{Z}_2)$.

With the intersection product $|\mathbf{P}^r \times \mathbf{P}^s||\mathbf{P}^k \times \mathbf{P}^l| = |\mathbf{P}^r \cap \mathbf{P}^k \times \mathbf{P}^s \cap \mathbf{P}^l|$ we have that the cohomology ring is equal to $\mathbf{Z}_2[t] / (u^{n+1}, v^{n+1})$ where $u = \pi(|\mathbf{P}^{n-1} \times \mathbf{P}^n|)$ and $v = \pi(|\mathbf{P}^n \times \mathbf{P}^{n-1}|)$

2.3 Proof of Hopf's theorem

We now have the necessarily knowledge to prove the theorem. The proof mainly consists in the proof ow two smaller claims that will then quite quickly lead us to Hopf's theorem.

Proof. (Hopf's theorem)
Let $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ be a continuous odd mapping. Define $G : \mathbf{P}^{n-1} \times \mathbf{P}^{n-1} \to \mathbf{P}^{n-1}$ as being the map induced by g on projective spaces through α, with α as given in Definition 2.4. Hence, the map G is given by $([x],[y]) \mapsto [g(x,y)]$ where $x, y \in S^{n-1}$ and $[\cdot]$ denotes the equivalence class for the equivalence relation

$$x \sim y, \text{ if } y = \pm x$$

which α is the quotient map.
Example 2.2. about intersection product of projective spaces gives us the following:

- There exist u and v such that:

$$H^*(\mathbf{P}^n \times \mathbf{P}^n, \mathbf{Z}_2) = {\mathbf{Z}_2[t]}\big/{(u^{n+1}, v^{n+1})}$$

  and the definitions of such u and v give us that $< u, |\mathbf{P}^1 \times \text{point}| >= 1$;
  $< u, |\text{point} \times \mathbf{P}^1| >= 0$ and $< v, |\mathbf{P}^1 \times \text{point}| >= 0$; $< v, |\text{point} \times \mathbf{P}^1| >= 1$.

- $H_1(\mathbf{P}^{n-1} \times \mathbf{P}^{n-1})$ has $|\mathbf{P}^1 \times \text{point}|$ and $|\text{point} \times \mathbf{P}^1|$ as basis.

Claim. *We have that:* $G_*(|\mathbf{P}^1 \times \textit{point}|) = G_*(|\textit{point} \times \mathbf{P}^1|) = |\mathbf{P}^1|$

Proof of the claim. We have to prove that $G_*(|\mathbf{P}^1 \times \text{point}|) = G_*(\text{point} \times \mathbf{P}^1)$ is the non-zero element of $H_1(\mathbf{P}^{n-1})$.

Let $|\omega| \in \mathbf{P}^{n-1}$ be the homology class of a closed path in $\mathbf{P}^{n-1}$. Then there exist a path $\tilde{\omega} \in S^{n-1}$ such that $\omega = \tilde{\omega}$. Because ω is closed, we have two possibilities for $\tilde{\omega}$.
Either:

- The path $\tilde{\omega}$ is already closed in S^{n-1} in which case $|\omega| = 0$, or;
- we may have, from the definition of $\mathbf{P}^{n-1}$ as quotient of S^{n-1} that $\tilde{\omega}$ is a path joining two antipodal points in S^{n-1} in which case $|\omega| \neq 0$. Indeed, $\tilde{\omega} \circ \tilde{\omega}$ then lifts to loop in S^{n-1} which can thus be homotoped to the trivial loop since $H_1(S^{n-1}) = 0$. Thus, the projection of $\tilde{\omega} \circ \tilde{\omega}$ through α is also zero, and, since $H_1(\mathbf{P}^{\mathbf{n-1}}) = \mathbf{Z}_2$ for $n > 2$, which is the only interesting case in the theorem, we have then that $|\omega|$ defines an element of order 2 in $H_1(\mathbf{P}^{n-1})$ and is thus non zero.

We have that $|\mathbf{P}^1 \times \text{point}|$ is a closed path $\omega \times \text{point}$ in $\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ and we take $\tilde{\omega}$ as being a path joining two antipodal points.
Since g is an odd map, we then get that $g(\tilde{\omega} \times \text{point})$ still joins two antipodal points on the circle. Under α, this path in S^{n-1} becomes $G(\omega \times \text{point})$ which then becomes $G_*(|\omega \times \text{point}|)$ on the homological spaces which is then non-zero,as stated before.
Because the same proof holds for $|\text{point}| \times \mathbf{P}^1$ with a closed path $\text{point} \times \omega$, the claim holds.

Take now t, u, v as defined before for the intersection product of projective spaces but this time on $\mathbf{P}^{n-1}$. Hence, we now have

$$H^*(\mathbf{P}^{n-1}, \mathbf{Z}_2) = {\mathbf{Z}_2[t]}\big/{t^n} \text{ and } H^*(\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}, \mathbf{Z}_2) = {\mathbf{Z}_2[t]}\big/{(u^n, v^n)}$$

.

Claim. *The following equation between t, u and v holds:*

$$G^*(t) = u + v$$

Proof of the claim. Let's compute $< G^*(t), |\text{point} \times \mathbf{P}^1| >$. We have that:

$$< G^*(t), |\text{point} \times \mathbf{P}^1| >=< t, G_*(|\text{point} \times \mathbf{P}^1|) >=< t, |\mathbf{P}^1| >= 1$$

as well as:

$$< u + v, |\text{point} \times \mathbf{P}^1| >= 1$$

and similarly for $|\mathbf{P}^1 \times \text{point}|$ But since a cohomology class is determined only by its value on the homology class, this gives us indeed that $G^*(t) = u + v$, which proves the claim.

Now that we have proven those two claims, the proof theorem follows rather easily: We know that $t^n = 0$ and that G^* is a homomorphism, hence:

$$0 = G^*(t^n) = (G^*(t))^n = (u + v)^n$$

Therefore:

$$0 = \sum_{k=1}^{n-1} \binom{n}{k} u^k v^{n-k}$$

since $u^n = 0$ and $v^n = 0$.
Thus, all the coefficients $\binom{n}{k}$ have to be 0. In $\mathbf{Z}_2$, this implies that n has to be a power of two, which proves the theorem. □

With Hopf's theorem, we now have proven that the dimension of a real division algebra is a power of two. In fact, we know since 1958 that this dimension has to be either 1, 2, 4 or 8. This result was proven for the first time by Kervaire [13] and Milnor [17] independently. The proof of this fact will be the main topic of the following parts of this article.

3 Vector bundles and complex K-theory.

To prove that the dimension of a real division algebra can only be either 1, 2, 4 or 8, we will in this article make use of complex K-theory. In order to define K-theory, we will firstly take a closer look at vector bundles:

3.1 Vector bundles

We start with the definition of a vector bundle:

Definition 3.1. Let E and B be topological spaces. An n-dimensional vector bundle ξ over a field F is a triple (E, p, B) with $p : E \to B$ a continuous bijection, together with a n-dimensional vector space structure over F on $p^{-1}(b)$ for each $b \in B$ such that the following local condition is satisfied: each point $b \in B$ has a neighbourhood U and a U-isomorphism $h : U \times F^n \to p^{-1}(U)$ such that for every $x \in U$, the restriction $x \times F^n \to p^{-1}(x)$ is a vector space isomorphism.

If $F = \mathbf{R}$, then (E, p, B) is called a real vector bundle. Similarly, it is called a complex vector bundle when $F = \mathbf{C}$.

The map h is called a local trivialization of the vector bundle. The space B is called the base space and the space E the total space. We will sometimes denotes a vector bundle only bi its total space E. The vector spaces $p^{-1}(b)$ for $b \in B$ are called the fibers.

From now on, we will consider the field F to be either $\mathbf{R}$ or $\mathbf{C}$. Several examples of vector bundles can be given:

Example 3.1. Let B be a topological space. Set $E = B \times F^n$ and let $p : E \to B$ define the projection onto the first factor. Then (E, p, B) forms a n-dimensional real or complex vector bundle called the *product* or *trivial* bundle. When the field is clear from the context, the trivial n-dimensional vector bundle over a fixed base space B will be denoted by ϵ^n.

Example 3.2. Consider the complex projective space $\mathbf{CP}^n$ as the space of complex lines through the origin. The tautological line bundle is given by the bundle $(H, p, \mathbf{CP}^n)$ where H is defined as:

$$H = \{(l, v) \in \mathbf{CP}^n \times \mathbf{C}^{n+1} \mid v \in l\}$$

and $p : H \to \mathbf{CP}^n$, $(l, v) \mapsto l$ the projection onto $\mathbf{CP}^n$.
Because each line in $\mathbf{CP}^n$ can be defined by a point z in $\mathbf{C}^{n+1}$ with a least one coefficient being non zero, we have that $\mathbf{CP}^n$ is covered by opens U_i where U_i is the set of points $(z_1, ..., z_{n+1}) \in \mathbf{C}^{n+1}$ such that $z_i \neq 0$. The local trivializations $h_i : p^{-1}(U_i) \to U_i \times \mathbf{C}^n$ are then given by $(l, v) \mapsto (l, z_i v)$.

Example 3.3. We define the tangent bundle TS^n of S^n in $\mathbf{R}^{n+1}$ as the bundle (E, p, B) where $E = \{(x, v) \in S^n \times \mathbf{R}^{n+1} \mid x \perp v\}$ and $p : E \to B$ is given by $(x, v) \mapsto x$. Let $x \in S^n$ and take U_x the open space consisting of the hemisphere containing x and bounded by the hyperplane passing through the origin and orthogonal to x. Let π_x be the orthogonal projection projecting a vector $v \in \mathbf{R}^{n+1}$ onto $p^{-1}(x)$. Then, since for $y \in U_x$, the projection π_x restricts to an isomorphism of $p^{-1}(y)$ onto $p^{-1}(x)$, we can define local trivializations $h_x : p^{-1}(U_x) \to U_x \times \mathbf{R}^n$ of the bundle as $(y, v) \mapsto (y, \pi_x(v))$.

Example 3.4. The normal bundle NS^n of S^n in $\mathbf{R}^{n+1}$ is given by the bundle (E, p, B) where $E = \{(x, v) \in S^n \times \mathbf{R}^{n+1} \mid v = tx, \text{for some } t \in \mathbf{R}\}$, i.e. E is the space of pairs $(x, v) \in S^n \times \mathbf{R}^{n+1}$ where v has to be perpendicular to the tangent plane to S^n at x, and $p : E \to S^n$ is given by $(x, v) \mapsto x$. Local trivializations $h_x : p^{-1}(U_x) \to U_x \times \mathbf{R}^n$ can then be given by using orthogonal projections of $p^{-1}(y)$ onto $p^{-1}(x)$ for $y \in U_x$, as in the previous example.

One can also induce a bundle from a continuous map between base spaces of vector bundles:

Definition 3.2. Let $\xi = (E, p, B)$ be a vector bundles and $f : B' \to B$ a continuous map. We define the pullback or induced bundle by f as the bundle (f^*E, π, B') where f^*E corresponds to the space $\{(b', e) \in B' \times E \mid f(b') = p(e)\} \subseteq B' \times E$ equipped with the subspace topology and π is given by $\pi : f^*E \to B'$, $(b', e) \mapsto b'$.

For a local trivialization $h : p^{-1}(U_i) \to F^n$ of ξ, we can define on f^*E a local trivialization $h' : \pi(f^*(U_i)) \to f^*U_i \times F^n$, $(b', e) \mapsto (b', pr_2(h(e)))$ where $pr_2(h(e))$ denotes the projection onto the second coordinate of $h(e)$.

We will now introduce two important notions on vector bundles that will be needed later on. The first one is the one of transition functions:

Definition 3.3. Let (E, p, B) be an n-dimensional vector bundle over $F = \mathbf{R}$ or $\mathbf{C}$. Then there exists an open covering U_i of B with $i \in I$ and I an index set, such that we get local trivializations:

$$h_i : p^{-1}(U_i) \to U_i \times F^n$$

such that for $i, j \in I$ the map :

$$h_{ij} = h_i \circ h_j^{-1} : U_i \cap U_j \to p^{-1}(U_i \cap U_j) \times F^n$$

is linear on each fibre. Hence we have that $h_{ij}(x, v) = (x, g_{ij}(x)(v))$ for some $g_{ij} : U_i \cap U_j \to GL(n.F)$. These functions (g_{ij}) are called the *transitions functions* of the vector bundle (E, p, B) with respect to the local trivializations (U_i, h_i).

Property. *(Cocycle condition) For $U_i \cap U_j \cap U_k$ with $i, j, k \in I$, the function $g_{ij}g_{jk}g_{ki}$ is the identity.*

Proof. For h_i a local trivialization as defined above, we have that $h_i \circ h_i^{-1}$ is the identity and for all $i, j, k \in I$ and $(x, v) \in U_i \cap U_j \cap U_k \times F^n$, this can be rewritten as:

$$(x, v) = (h_i \circ h_i^{-1})(x, v) = (h_i \circ h_j^{-1} \circ h_j \circ h_k^{-1} \circ h_k \circ h_i^{-1})(x, v)$$

$$= (h_i \circ h_j^{-1} \circ h_j \circ h_k^{-1})(x, g_{ki}(x)(v)) = (h_i \circ h_j^{-1})(x, g_{jk}g_{ki}(x)(v)) = (x, g_{ij}g_{jk}g_{ki}(x)(v))$$

□

Conversely, let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of B indexed by the set I and let $g_{ij} : U_i \cap U_j \to GL(n, F)$ be continuous maps satisfying the cocycle condition. Then we can take the disjoint union $\bigsqcup(U_i \times F^n)$ and use the maps g_{ij} to glue this to form an n-dimensional vector bundle over B with

$$E = \bigsqcup(U_i \times F^n)/\sim_{\mathcal{U}}$$

where $\sim_{\mathcal{U}}$ is here the equivalence relation given by $(x, v) \sim_{\mathcal{U}} (x, g_{ij}(x)(v))$.

Definition 3.4. Let $\xi = (E, p, B)$ be a vector bundle. We say that a map $s : B \to E$ is a section of the bundle ξ if it assigns to every $b \in B$ a vector $s(b)$ which is contained in the fiber $p^{-1}(b)$. In other words, the map $p \circ s$ is the identity on B.

It is quite clear that every vector bundle has a canonical section, by taking the section whose value is zero on each fiber. This section is called the zero section.

We will now define the concept of morphisms between vector bundles. One can broaden the definition of morphism between vector spaces to one between vector bundles as follows:

Definition 3.5. Let $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ be two vector bundles. We say that a $(u, f) : \xi \to \xi'$ is a *morphism* between the vector bundles ξ and ξ' if $u : E \to E'$ and $f : B \to B'$ are continuous maps such that $p'u = fp$, and for each $b \in B$, the restriction $u : p^{-1}(b) \to p'^{-1}(f(b))$ is a linear map.

When $B = B'$, this definition turns into the corresponding one:

Definition 3.6. Let $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ be two vector bundles over the same base space B. We say that a continuous map $u : \xi \to \xi'$ is a B-morphism between ξ and ξ' if $p'u = p$ and the restriction $u : p^{-1}(b) \to p'^{-1}(b)$ is linear for all $b \in B$.

This enables us to define the concept of isomorphism of vector bundles, which will be later required in complex K-theory:

Definition 3.7. Let $u : \xi \to \xi'$ be a B-morphism between two vector bundles. Then, u is an isomorphism if for all $b \in B$, the restriction $u : p^{-1}(b) \to p'^{-1}(b)$ is an isomorphism of vector spaces. Hence each fiber $p^{-1}(b)$ in E is sent to the corresponding fiber $p'^{-1}(b)$ in E' by a linear isomorphism.
If there exists an isomorphism u between two vector bundles ξ and ξ', then the vector bundles are said to be isomorphic.

Notation. We will use the notation $E \approx E'$ to indicate that the bundles (E, p, B) and (E', p', B) are isomorphic.

3.2 Operations on vector bundles

In order to define complex K-theory,we will need the concepts of direct sum, tensor product and exterior product of vector bundles as much as the one of isomorphisms.

Definition 3.8. Let $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ be two vector bundles over the same base space B. We define their *direct sum* $E \oplus E'$ as the vector bundle with total space:

$$E \oplus E' = \{(e, e') \in E \times E' \mid p(e) = p'(e')\}$$

with the projection $\pi : E \oplus E' \to B$ sending the pair $(e, e') \in E \oplus E'$ to the point $p(e) = p'(e')$ in B.

Proposition 3.1. *Let (E, p, B) and (E', p', B) be two vector bundles. Then $(E \oplus E', \pi, B)$ is again a vector bundle whose fibers are the direct sum of the fibers of E and E'.*

Proof. Let $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ be vector bundles and let A be a subset of B. Define now the restriction map $r : p^{-1}(A) \to A$ on A and the product map $q = p \times p' : E \times E' \to B \times B'$, $(e, e') \mapsto (p(e), p'(e')))$.

Then both maps r and q defines vector bundles $(p^{-1}(A), r, A)$ and $(E \times E', q, B \times B)$. This is quite trivially verified for the restriction. For the product, the fibers are then given by $p^{-1}(b) \times p'^{-1}(b')$ with $(b, b') \in B \times B'$ and the local trivialization property is verified by observing that for $h_1 : p^{-1}(U_1) \to U_1 \times F^m$ local trivialization for ξ and $h_2 : p'^{-1}(U_2) \to U_2 \times F^m$ local trivialization for ξ', we obtain that the product $h_1 \times h_2$ is a local trivialization for $(E \times E', q, B \times B')$.

The proof then follows from the remark that for vector bundles $(E.p, B), (E', p', B)$ with the same base space B, the direct sum $E \oplus E'$ corresponds to the restriction of the product map on the diagonal space $\Delta = \{(b, b), b \in B\}$ in B. □

We can give some examples of direct sums of vector bundles:

Example 3.5. Let ϵ^n and ϵ^m be the n and m-dimensional trivial vector bundles over a base space B. Then their direct sum is again a trivial vector bundle, of dimension $n + m$, for it is given by the space:

$$\{((b, v)(b, w)) \in (B \times F^n) \times (B \times F^m)\}$$

which we can then isomorphically send to $B \times F^{n+m}$ through $((b, v)(b, w)) \mapsto (b, (v, w))$.

Example 3.6. Consider the tangent bundle TS^n and the normal bundle NS^n of S^n as defined above. Their direct sum is then given by the space:

$$\{((x, v)(x, w)) \in (S^n \times \mathbf{R}^{n+1}) \times (S^n \times \mathbf{R}^{n+1}) \mid x \perp v, w = tx \text{ for some } t \in \mathbf{R}\}$$

which can be sent isomorphically to $S^n \times \mathbf{R}^{n+1}$ via the map $((x, v)(x, w)) \mapsto (x, v + w)$. Thus, the direct sum of TS^n and NS^n is trivial.

One can state several important properties about the direct sum of vector bundles. In order to state those properties, we first need to define the concept of inner product on vector bundles. We recall that a inner product on a vector space V over F is a positive definite bilinear form $< . >: V \times V \to F$ satisfying the property of conjugate symmetry.

Definition 3.9. Let $\xi = (E, p, B)$ be a vector bundle over F. An inner product on ξ is a map $< . >: E \oplus E \to F$ that restricts to an inner product on each fiber.

Proposition 3.2. *Let $\xi = (E, p, B)$ be a vector bundle over a compact Hausdorff topological space B. Then, there exists an inner product on ξ.*

Proof. Let $\xi = (E, p, B)$ be a n-dimensional vector bundle over F with B compact Hausdorff. On fibers $\{b\} \times F^n$, with $b \in B$, the inner product is given by $((b, v), (b, w)) \mapsto < v, w >$ where $< \cdot >$ is the standard inner product on F^n. This can be generalized on neighbourhoods U_b for $b \in U_b$, by using local trivializations $h_b : p^{-1}(U_b) \to U_b \times F^n$ and pulling back the inner product $< \cdot >$ of F^n to an inner product $< \cdot >_b$ on $p^{-1}(U_b)$. The U_b with $b \in U_b \subset B$ form a open cover of B.

Since B is compact Hausdorff, there exists a partition of unity subordinate to the collection of U_b's, i.e. a collection of maps $\varphi_\beta : B \to [0, 1]$ such that the support of φ_β is contained in U_b and $\sum_\beta \varphi_\beta = 1$. Using this partition of unity, we can glue together the different inner product on each $p^{-1}(U_b)$ to form a inner product on the whole bundle by setting

$$< v, w >= \sum_\beta \varphi_\beta(p(v)) < v, w >_{b_\beta}$$

where $\{\phi_\beta\}$ has its support contained in U_{b_β}. $\square$

Proposition 3.3. *Let $\xi = (E, p, B)$ be a vector bundle with B compact Hausdorff and let $E_0 \subset E$ be a vector subbundle. Then there exists a vector subbundle $E_0^\perp$ such that the direct sum $E_0 \oplus E_0^\perp$ is isomorphic to the total space E.*

Proof. Let $\xi = (E, p, B)$ be a n-dimensional vector bundle with B compact Hausdorff and $E_0 \subset E$ an m-dimensional vector subbundle of ξ. Define $E_0^\perp$ as the subspace of E that consists in each fiber of all vectors orthogonal to vectors in E_0, with respect to a chosen inner product on the bundle. The aim is to show that that $(E_0^\perp, p, B)$ is a vector bundle. Indeed, if this statement holds, then $E_0 \oplus E_0^\perp$ will then be isomorphic to E through the map $(v, w) \mapsto v + w$, according to Definition 3.6. and 3.7.

To do so, we have to verify the local triviality condition for $E_0^\perp$. As the question is local in B, we may here assume that $E = B \times F^n$. Since E_0 is an m-dimensional vector subbundle, there are, near each point $b_0 \in B$, m independent local sections $s_i : B \to B \times F^m$, $b \mapsto (b, s_i(b))$ and $m \leq n$. We now may enlarge this set of m independent local sections of E_0 to a set of n independent local sections $b \mapsto (b, s_i(b))$ on E. To do so, we choose sections $s_m, ..., s_n$ in the fiber $f^{-1}(b_0)$ and then take the same vectors for all nearby fibers: Indeed, since the determinant function is continuous and the sections $s_1, ..., s_m, s_{m+1}, ..., s_n$ are independent at b_0, they will remain independent in an neighbourhood of b in B.

We can now make the $s_1, ..., s_n$ orthogonal in each fiber by applying the Gram-Schmidt process. We call those new sections $s'_1, ..., s'_n$. Those are continuous, according to the Gram-Schmidt orthogonalization formula and $(s'_1, ..., s'_m)$ forms a basis of E_0 in each fiber.

Let U be an open in B and define the local trivializations as the maps $h : p^{-1}(U) \to U \times F^n$ sending $(b, s'_i(b))$ to the i-th standard basis vector of F^n. Since h sends E_0 to $U \times F^m$ and E_0 to $U \times F^{n-m}$, a local trivialization of $E_0^\perp$ is then given by the restriction of h on $E_0^\perp$, turning $E_0^\perp, p, B$ into a vector subbundle and therefore making $E_0 \oplus E_0^\perp$ isomorphic to E as stated before. $\square$

Proposition 3.4. *Let $\xi = (E, p, B)$ be a vector bundle with B compact Hausdorff. Then there exists a vector bundle (E', p, B) such that $E \oplus E'$ is the trivial bundle.*

Proof. Let $\xi = (E, p, B)$ be a n-dimensional vector bundle over F with B compact Hausdorff. Since ξ is a vector bundle, we have that every $b \in B$ has an open neighbourhood U_b over which E is trivial. Now, because B is compact Hausdorff we can construct an open cover map $\varphi_b : B \to [0, 1]$ that is 0 outside U_b and non zero on b ([9, Proposition 1.18]).

Then the sets $\{\varphi_b^{-1}(0, 1] \mid b \in B\}$ form an open cover of B. Since B is compact, this admits a finite open subcover $U_i = \varphi_i^{-1}(b)$ and relabel the φ_b on U_i as φ_i. We now look at the local trivializations $h_i : p^{-1}(U_i) \to U_i \times F^n$ and define $\pi_i : U_i \times F^n \to F^n$ the projection on F^n. Define maps $g_i : E \to F^n$, $e \mapsto \varphi_i(p(e))\pi_i(h_i(e))$ and let g be a map whose coordinates are given by the g_i's. Then $g : E \to F^N$ with F^N a product of copies of F^n, and, since the g_i's are linear injections on each fiber over $\varphi_i^{-1}((0,1])$, g is a linear injection on each fiber.

Let now $f : E \to B \times F^N$ be a map whose first coordinate is given by p and whose second coordinates is given by g. Because the projection of F^n onto the i-th component gives through g_i the second coordinate of a local trivialization over $\varphi_i^{-1}((0,1])$, the image of f is a subbundle of $B \times F^N$. Since f is an linear injection on each fiber, we thus get that E is isomorphic to a subbundle of $B \times F^N$, hence by the previous proposition, there exists a complementary subbundle such that $E \oplus E'$ is isomorphic to $B \times F^N$. □

In addition to the direct sum, one can also define the notions of tensor product and exterior power on vector bundles starting from the corresponding operations on vector spaces. We start with the latter:

Definition 3.10. Let $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ be two vector bundles over the same base space B. We define their tensor product as the vector bundle $(E \otimes E', \pi, B)$ where $E \otimes E'$ is the disjoint union $\bigsqcup_{x \in B} p^{-1}(x) \otimes p'^{-1}(x)$ of the vector spaces $p^{-1}(x) \otimes p'^{-1}(x)$ with x in B, and $\pi = (p, p')$.

This set is equipped with the following topology: Let U be an open set in B, let n_1 and n_2 be the dimensions of the vector bundles (E_1, p_1, B) and (E_2, p_2, B) respectively and choose isomorphisms $h_1 : p_1^{-1}(U) \to U \times F^{n_1}$ and $h_2 : p_2^{-1}(U) \to U \times F^{n_2}$ over which E_1 and E_2 are trivial. A topology $\mathcal{T}_u$ is then defined on $p_1^{-1}(U) \otimes p_2^{-1}(U)$ by letting the fiberwise tensor product map $h_1 \otimes h_2 : p_1^{-1}(U) \times p_2^{-1}(U) \to U \times (F^{n_1} \otimes F^{n_2})$ be a homeomorphism.

Proposition 3.5. *The topology $\mathcal{T}_U$ is well-defined and independent of the choice of the maps h_1 and h_2.*

Proof. Suppose we choose an isomorphism h_i' instead of h_i. Let U be defined as before. Then there are a continuous map $g_i : U \to GL_{n_i}(F)$ such that h_i' is the composition of h_i and isomorphisms of $U \times F^{n-1}$ of the form $(x, v) \mapsto (x, g_i(x)(v))$. Hence, we have that $h_1' \otimes h_2'$ is a composition of $h_1 \otimes h_2$ and isomorphisms of $U \times (F^{n_1} \otimes F^{n_2})$ of the form $(x, v) \mapsto (x, g_1 \otimes g_2(x)(v))$ where the $g_1 \otimes g_2$ are continuous maps $U \to GL_{n_1 n_2}(F)$, for the entries of the matrices $g_1(x) \otimes g_2(x)$ are the product of the entries of $g_1(x)$ and $g_2(x)$.

Suppose now we replace the open subset U bu an open subset V in B. Then, since the local trivializations over U restrict to local trivializations over V, we have that the topology on $p_1^{-1}(V) \otimes p_2^{-1}(V)$ induced by $\mathcal{T}_U$ is the same as the topology $\mathcal{T}_V$.

Therefore the topology on $E_1 \otimes E_2$ is well-defined and turns $E_1 \otimes E_2$ into a vector bundle over B. □

Similarly to the concept of tensor products, we can also define the exterior power on vector bundles. To do so, let us first recall the notion of exterior powers of vector spaces:

Definition 3.11. Let V be a vector space over a field F and k a positive integer. The exterior power $\lambda^k(V)$ of V is obtained by taking the quotient of the k-fold tensor product $V \otimes ... \otimes V$ by the subspace generated by vectors of the form $v_1 \otimes ... \otimes v_k - sgn(\sigma) v_{\sigma(1)} \otimes ... \otimes v_{\sigma(k)}$ where σ is any permutation of the set $\{1, ..., k\}$, and $sgn(\sigma) = \pm 1$ its sign. The sign of σ equals $+1$ for σ an even permutation and -1 for σ an odd permutation.

Using the fibers of a vector bundle, we can now translate this definition on vector spaces to one on vector bundles:

Definition 3.12. Let (E, p, B) be a vector bundle. We define the exterior power $\lambda^k(E)$ of E as the disjoint union of the exterior powers $\lambda^k(p^{-1}(b))$ of the vector spaces $p^{-1}(b)$, with $b \in B$.

To define a topology on this set, we take $(U_i)_{i\in I}$ for I an index set, an open cover of B. Because (E, p, B) is a vector bundle, we have local trivializations $U_i \times F^n$, where n is the dimension of the vector bundle, which are then glued together via the transitions functions $g_{ij} : U_{ij} \to Gl(n, F)$. Then, we may give in $\lambda^k(E)$ the local trivializations $U_i \times \lambda^k(F^n)$, glued together via $\lambda^k(g_{ij})$. The topology put on this set is then the one that makes those local trivializations continuous.

We can state a few properties about the exterior power of vector bundles, most of them resulting from the properties of exterior power on vector spaces:

Proposition 3.6. *Let E, E' be vector bundles and k be a positive integer. Then the following statements hold:*

(i) $\lambda^k(E \oplus E') = \bigoplus_i (\lambda^i(E) \otimes \lambda^{k-i}(E'))$

(ii) $\lambda^0(E) = \epsilon^0$*, where* ϵ^0 *is the trivial line bundle.*

(iii) $\lambda^1(E) = E$

(iv) $\lambda^k(E) = 0$ *when* k *is greater than the maximal dimension of the fibers of* E*.*

(v) $\lambda^k(f^*E) = f^*\lambda^k(E)$

with the last property following from uniqueness of the pullback bundle (see [3, chapter 1]).

3.3 Complex K-theory

Now that we have taken a look at some results from the theory of vector bundles, we are able to introduce the idea of complex K-theory. From now on, we will thus assume all vector bundles to be complex.

Definition 3.13. Let $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ be two vector bundles over the same connected base space B. We then say that those vector bundles *stably isomorphic*, with notation $E \approx_S E'$, if there exists an integer n such that $E \oplus \epsilon^n \approx E' \oplus \epsilon^n$. Similarly, we will write that $E \sim E'$, if there exist integers n and m such that $E \oplus \epsilon^m \approx E' \oplus \epsilon^n$.

Proposition 3.7. *Both relations $\approx_S$ and $\sim$ are equivalence relations.*

This proposition is trivially verified.

Definition 3.14. Let (E, p, B) and (E', p', B) be two vector bundles and let $[\cdot]_\sim$ be the equivalence class associated to the relation $\sim$. Then we can define the following operation on the equivalence classes of vector bundles:

$$[E]_\sim \oplus [E']_\sim = [E \oplus E']_\sim$$

From the definition of direct sum of vector bundles, it is quite easily verified that this operation is well-defined, associative and commutative. This gives us the following remarkable result:

Proposition 3.8. *Let X be a compact Hausdorff topological space. The set of $\sim$-equivalence classes of vector bundles over X forms an Abelian group with respect to the direct sum, and with the class of ϵ^0 as zero element. We denote this group by $\tilde{K}(X)$.*

Proof. The existence of inverses follows from Proposition 3.4. The other conditions being trivially verified. □

We can construct another Abelian group by using this time the equivalence relation $\approx_S$. This goes in a similar way as the construction of rational numbers out of integers:

Definition 3.15. Let X be a compact Hausdorff topological space. Let E_1, E_1', E_2, E_2' be vector bundles over X. We say that $E_1 - E_1' = E_2 - E_2'$ if and only if $E_1 \oplus E_2' \approx_S E_2 \oplus E_1'$.

Proposition 3.9. *The relation "=" in the previous definition is a equivalence relation.*

Proof. The relation = is trivially reflexive and symmetric. The transitive property of the relation follows from the cancellation property, which holds since X is compact. □

We can define on those equivalence classes the following addition rule:

Definition 3.16. Let E_1, E_1', E_2, E_2' be total spaces of vector bundles. The addition operation $+$ on the equivalence classes of formal differences $E_1 - E_1'$ and $E_2 - E_2'$ is given by the equivalence class of $(E_1 \oplus E_2) - (E_1' \oplus E_2')$, i.e. $(E_1 - E_1') + (E_2 - E_2') = (E_1 \oplus E_2) - (E_1' \oplus E_2')$.

It is quite clear that the addition rule $+$ as defined above is both commutative and associative as it follows from the properties of the equivalence relation $=$ and the direct sum. It is well-defined because for total spaces $E_1, E_1', E_2, E_2', E_3, E_3'$ such that $E_1 - E_1' = E_3 - E_3$, we then have that $E_2 \oplus E_3 \oplus E_2' \oplus E_1' \approx_S E_2 \oplus E_3' \oplus E_2' \oplus E_1$ since the condition $E_1 - E_1' = E_3 - E_3$ gives us that $E_1 \oplus E_3' \approx_S E_1' \oplus E_3$, meaning that indeed $(E_3 \oplus E_2) - (E_3' \oplus E_2') = (E_1 \oplus E_2) - (E_1' \oplus E_2')$

Proposition 3.10. *The set of formal differences $E - E'$ of vector bundles over a compact Hausdorff space X forms an Abelian group with respect to the sum $+$, and with zero element the equivalence class of the formal difference $E - E$ for any total space E of such vector bundle. We call this group $K(X)$.*

Proof. We only need to prove that any formal differences $E - E'$ has an inverse and this inverse is trivially given by $E' - E$. □

Proposition 3.11. *If X is a single point then $K(X) = \mathbf{Z}$*

Proof. Let X be a single point. Then a vector bundle over X is a vector space. Since those are only classified by their dimension, which is given by a positive integer, we get that the space of all the vector bundles over X is given by $\mathbf{Z}^+$. Hence the K group $K(X)$ is given by the space of integers $\mathbf{Z}$.

Proposition 3.12. *Let n be an integer and E a vector bundle over a compact Hausdorff space X. The map $K(X) \to \tilde{K}(X)$ sending $E - \epsilon^n$ to the $\sim$-equivalence class of E is a natural homomorphism. This homomorphism is surjective and its kernel is isomorphic to $\mathbf{Z}$.*

Proof. Let E, E' be vector bundles over a compact Hausdorff space X and n, m be integers.
Notice first we can rewrite every element $E - E'$ in $K(X)$ by adding a total space E'' to both E and E' such that $E' \oplus E'' = \epsilon^n$ (see Proposition 3.4). This gives us that $E - E' = (E \oplus E'') - (E' \oplus E'') = (E \oplus E'') - \epsilon^n$. Thus, by writing $\hat{E} = E \oplus E''$, we have that every element of $K(X)$ can be written as a formal difference of the form $\hat{E} - \epsilon^n$
Take now $E - \epsilon^n = E' - \epsilon^m$ in $K(X)$, we then have that $E \oplus \epsilon^n \approx_S E' + \epsilon^m$, thus the equivalence $E \sim E'$ holds, proving that the map in the proposition is indeed well-defined.
The map is clearly surjective and its kernel is equal to the space of formal differences $E - \epsilon^n$ such that the $\sim$-class of E is the one of ϵ^0. This means that there exist integers k, l such that

$E \oplus \epsilon^k \approx^0 \oplus \epsilon^l$. Hence E is isomorphic to ϵ^p for some integer p. Therefore the kernel of the map is equal to formal differences of the form $\epsilon^p - \epsilon^n$. Since the subgroup $\{\epsilon^p - \epsilon^n \mid n, p \in \mathbf{N}\}$ of $K(X)$ is isomorphic to $\mathbf{Z}$, the claim follows. □

In fact, for x_0 a basepoint of X, we have that the restriction of vector bundles to that basepoint x_0 defines a homomorphism $K(X) \to K(x_0) \approx \mathbf{Z}$. This homomorphism restricts to an isomorphism on the subgroup $\{\epsilon^n - \epsilon^m\}$ which gives us a splitting $K(X) \approx \tilde{K}(X) \oplus \mathbf{Z}$.
This means for instance that for X a single point, $\tilde{K}(X) = 0$ since $K(X)$ is isomorphic to $\mathbf{Z}$.

Using the tensor products on vector bundles, one can also endow $K(X)$ with a natural multiplication:

Definition 3.17. Let $E_1 - E_1'$ and $E_2 - E_2'$ be representatives for the equivalence classes of formal differences of vector bundles. We define their product as follows:

$$(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 + E_1' \otimes E_2' - (E_1 \otimes E_2' + E_1' \otimes E_2).$$

This product is well-defined: to see that, take $E_3 - E_3'$ such that $(E_3 - E_3') = (E_2 - E_2')$. Then $E_2 \oplus E_3' \approx E_2' \oplus E_3$. Hence, $(E_1 \otimes E_3 + E_1' \otimes E_3') \oplus (E_1 \otimes E_2' + E_1' \otimes E_2) \approx E_1 \otimes (E_2' \oplus E_3) + E_1' \otimes (E_2 \oplus E_3') \approx E_1 \otimes (E_2 \oplus E_3') + E_1' \otimes (E_2' \oplus E_3)$ meaning that $(E_1 - E_1')(E_2 - E_2') = (E_1 - E_1')(E_3 - E_3')$

Proposition 3.13. *The addition and product operation in $K(X)$ turn $K(X)$ into a commutative ring with the trivial line bundle ϵ^1 as identity element for the product operation.*

Proof. Let (E, p, X) be a vector bundle and E_1 and E_1' be total spaces of vector bundles. We have indeed that $(E_1 - E_1')(E - E) = E_1 \otimes E + E_1' \otimes E - (E_1 \otimes E + E_1' \otimes E)$ which is indeed the neutral element for the addition. Let $\epsilon^1 = (X \times \mathbf{C}, \pi, X)$ be the trivial line bundle and take $x \in X$. Because $\pi^{-1}(x)$ is given by $\{x\} \times \mathbf{C}$, the tensor product of $p^{-1}(x)$ with π^x is isomorphic to $p^{-1}(x)$, hence the tensor product of E with the trivial line bundle is again E which gives us that ϵ^1 is the identity element for the product operation. The others ring properties are trivially verified. □

Proposition 3.14. *Let X, Y be compact Hausdorff space and $f : X \to Y$ a map. Then f induces a ring homomorphism $f^* : K(Y) \to K(X)$, $E - E' \mapsto \tilde{f}^*(E) - \tilde{f}^*(E')$, where $\tilde{f}^*E$ and $\tilde{f}^*E'$ define here the pullback bundles induced by f.*
Moreover, f^* has the following properties:

(i) $(fg)^* = g^* f^*$ for $g : Z \to X$ with Z a compact Hausdorff space;

(ii) If id the identity map on X, then id^* is the identity map on $K(X)$;

(iii) If f is isomorphic to g, for g another map $X \to Y$, then $f^* = g^*$.

Proof. This properties follow from the ones on the pullback bundles:

(i) For (E, p, X) and (E', p', X) vector bundles and $f : X' \to X$ continuous, $\tilde{f}^*(E \oplus E') \approx f^*(E) \oplus f^*(E')$ and $f^*(E \oplus E') \approx f^*(E) \oplus f^*(E')$;

(ii) For $id : X \to X$, the identity map $id^*(E) \approx E$;

(iii) For $g : Z \to X'$ with Z compact Hausdorff, $(fg)^* = g^* f^*$;

(iv) For $g : X' \to X$ continuous, $f \simeq g \Rightarrow f^*E \approx g^*E$.

which themselves all follow quite easily from the definitions of pullback bundles, direct sums, tensor products and isomorphisms of vector bundles. □

We can now give the following theorem, called the *splitting principle* which will be important in the next section. This theorem will not be proven in this thesis, a proof can be found in [9, Section 2.3].

Theorem 3.15. *(The splitting principle)*
Let (E, p, X) be a vector bundle with X compact Hausdorff. Then there exists a compact Hausdorff space $F(E)$ and a map $\pi : F(E) \to X$ such that the induced map $\pi^* : K^*(X) \to K^*(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles.

We conclude by defining an external product in K-theory:

Definition 3.18. Let X, Y be compact Hausdorff spaces. Consider the product space $X \times Y$ and the projections map $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. The external product $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ is given by $a \otimes b \mapsto \pi_1^*(a)\pi_2^*(b)$.

Definition 3.19. Let X be a topological space and I the unit interval $[0, 1]$. We define the suspension SX of X as the quotient of the product space $X \times [0, 1]$ modulo the equivalence relation $\sim_I$ generated by $(x_1, 0) \sim_I (x_2, 0)$ and $(x_1, 1) \sim_I (x_2, 1)$. In other words, we obtain the suspension SX by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another one.

The definition and notation of the suspension is motivated by its use on the topological space S^n for n positive integer. Indeed, for $X = S^n$, we have that the suspension SX of X is given by $SS^n = S^{n+1}$ with the "suspension points" being the north and south pole $(0, 0, ..., 0, \pm 1)$ of S^{n+1}.

Theorem 3.16. *(Bott periodicity)*
Let X be a compact Hausdorff space. Then there exists a natural isomorphism $: \tilde{K}(X) \to \tilde{K}(S^2X)$ *where $S^2X = S(SX)$ is the double suspension of X.*

This isomorphism can be given explicitly but this goes beyond the scope of this thesis. Some formulas can be found for instance in [9] or in [18].
This isomorphism can be a great tool to compute some examples of K-groups, as done here below:

Computation of some K-theory groups.

Example 3.7. We have proven that for X a single point, the groups $K(X)$ and $\tilde{K}(X)$ are respectively given by $\mathbf{Z}$ and 0. This means that for the sphere S^0, we get $K(S^0) \approx \mathbf{Z} \oplus \mathbf{Z}$ and thus that $\tilde{K}(S^0) \approx \mathbf{Z}$. Using Bott periodicity, we then conclude that $\tilde{K}(S^n) \approx \mathbf{Z}$ for all n even.

For the odd spheres, we observe that every complex vector bundle over the circle S^1 is also trivial (see for instance [20, Example 4.9.1]) meaning that, similarly to the point, they only differ by dimension, which gives us that $K(S^1) \approx \mathbf{Z}$ and therefore that $\tilde{K}(S^1) = 0$. Hence $\tilde{K}(S^n) = 0$ for all n odd, by Bott periodicity.

Example 3.8. For $\mathbf{CP}^n$ n-dimensional complex projective space, the space $\tilde{K}(\mathbf{CP}^n)$ is isomorphic to $\mathbf{Z}^{n+1}$ and is generated by $1 - [H]$, where H is the tautological line bundle. The proof of this fact uses exact sequences in K-theory coming from the CW complex structure of $\mathbf{CP}^n$ and goes beyond the scope of this thesis. For more details, see [9, Proposition 2.24].

Those examples of K-groups will play an important role in the proof of the 1,2,4,8 theorem, which will be given in the next section.

4 K-Theory and the dimension of real division algebras

Now that we have introduced some properties of vector bundles and complex K-theory, as well as given some main examples of K-groups, we can finally move on to proving that the dimension of a real division algebra can only be 1, 2, 4 or 8. To do so, we first need some definitions:

Definition 4.1. A sphere S^{n-1} is called parallelizable if there exist $n-1$ tangent vector fields to S^{n-1} which are linearly independent at each point. Equivalently, the tangent bundle TS^{n-1} is trivial.

Definition 4.2. Let X be a connected topological space. We say that X is an H-space if there exists a continuous map $\mu : X \times X \to X$ with an identity element e such that for all $x \in X$ $\mu(e, x) = \mu(x, e) = x$

Lemma 4.1. *Let A be a real division algebra of dimension n or S^{n-1} be a parallelizable sphere. Then S^{n-1} is an H-space.*

Proof. If A is a real division algebra of dimension n with multiplication $\cdot$, then we can create a continuous map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$, $(x, y) \mapsto \frac{x \cdot y}{||x \cdot y||}$ where $|| \cdot ||$ is the Euclidean norm (see proof of the corollary of Hopf's theorem in section 2), thus S^{n-1} is indeed an H-space.

If S^{n-1} is parallelizable then there exists $n-1$ tangent vector fields $v_1, ..., v_{n-1}$ that are linearly independent in every point of S^{n-1}. Using the Gram–Schmidt process, we can assume the vectors $x, v_1(x), ..., v_{n-1}(x)$ to be orthonormal for all $x \in S^{n-1}$. For a first standard basis vector e_1, we may also assume, by changing the sign of v_{n-1} to preserve the orientation of the space and deforming the vector fields near e_1 if needed, that the vectors $v_1(e_1), ..., v_{n-1}(e_1)$ forms the others standard basis vectors $e_2, ..., e_n$. Let b_x be the linear map that sends the standard basis to $x, v_1(x), ..., v_{n-1}(x)$ and consider now the map $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$, $(x, y) \mapsto b_x(y)$.

The map μ is continuous and for $x = \sum_i \lambda_i e_i$ in S^{n-1}, we have that $\mu(x, e_1) = b_x(e_1) = x$ and $\mu(e_1, x) = b_{e_1}(x) = \lambda_1 e_1 + \lambda_2 v_1(e_1) + \lambda_3 v_2(e_1) + ... + \lambda_n v_{n-1}(e_1) = \sum_i \lambda_i e_i = x$. Hence, S^{n-1} is an H-space with the vector e_1 as identity element. □

The goal of this section is the proof of the following theorem:

Theorem 4.2. *Let n be a positive integer and $\mathcal{A}$ be an n-dimensional real algebra. Then the following statements are only true for $n = 1, 2, 4$ or 8.*

- *$\mathcal{A}$ is a division algebra*
- *The sphere S^{n-1} is parallelizable.*

As we have proven in Section 2 that the dimension of a division algebra has to be a power of two, we know in particular that n has to be even. The case $n = 1$ being already worked out in Section 1, we will from now on write n as $2m$, with m a positive integer. The proof of Theorem 4.2. requires the concept of the Hopf invariant. To be able to introduce it, we first have to define the following notions:

Definition 4.3. Let X be a topological space and I the unit interval. Then the cone CX of X is defined as the quotient:

$$CX = X \times I \big/ X \times \{0\}$$

CX is called the cone of the space X for its construction intuitively makes X into a cylinder with one end of this cylinder being collapsed into only one point.

Definition 4.4. Let X, X' be two topological spaces. Let $A \subset X'$ and f be a continuous map $A \to X$. We can then define a quotient space of $X \sqcup X'$ by identifying each point a in A with $f(a) \in X$. The resulting quotient space $X \sqcup_f X'$ is said to be the space X with X' attached along A via f.

This definition enables us to introduce the notion of a mapping cone for a map f:

Definition 4.5. Let X, Y be topological spaces and $f : X \to Y$ a map. We define the mapping cone C_f as follows:

$$C_f = Y \sqcup_f CX$$

where CX is the cone of X as in Definition 4.3. We attach this cone here to Y along $X \times \{1\}$ by identifying $(x, 1)$ with $f(x)$.

Definition 4.6. Let $G_1, ..., G_n$ be groups and $f_i : G_i \to G_{i+1}$ group homomorphisms for $i = 1, ..., n-1$. We say that the sequence of groups and group homomorphisms

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} ... \xrightarrow{f_{n-1}} G_n$$

is exact is for all $i = 1, ..., n-1$, $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$

Combining those definitions, we can finally introduce the Hopf invariant: consider a map $f : S^{4m-1} \to S^m$. We can form with this map the following mapping sequence:

$$S^{4m-1} \xrightarrow{f} S^{2m} \to C_f \to SS^{4m-1} \xrightarrow{Sf} SS^{2m},$$

where Sf denotes the map $S(S^{4m-1})) \to S(S^{2m})$ induced by f and given by $(x, t) \mapsto (f(x), t)$. This mapping sequence gives then rise to the following exact sequence of K-groups:

$$0 \leftarrow \tilde{K}(S^{2m}) \leftarrow \tilde{K}(C_f) \leftarrow \tilde{K}(S^{4m})$$

The exactness of this sequence of K-groups follows from the half-exactness of sequence in K-groups. More information about this sequence of K-groups can be found for instance in [12, Chapter 10] Let now $\alpha \in \tilde{K}(C_f)$ be the image of a generator of $\tilde{K}(S^{4m}) \approx \mathbf{Z}$ and β be an element of $\tilde{K}(C_f)$ that maps to a generator of $\tilde{K}(S^{2m}) \approx \mathbf{Z}$. Since the square of every element in $\tilde{K}(S^{2m})$ is 0, we get that $\beta^2 = 0$. By exactness we get that $\beta^2 = h\alpha$ for some integer h.

Definition 4.7. The integer h introduced above is called the *Hopf invariant* of f.

Proposition 4.3. *The Hopf invariant is well-defined, and independent of the choice of β, Moreover $\alpha\beta = 0$.*

Proof. First of all, we see in the definition that the choice of β is unique up to adding up a multiple of α. Choose now $\beta' = \beta + k\alpha$. Then $\beta'^2 = (\beta + k\alpha)^2 = \beta^2 + 2k\alpha\beta + \alpha^2 = 2k\alpha\beta$ for α^2 is also zero, as it is the image of a generator of $\tilde{K}(S^{4m})$. Because α maps to 0 in $\tilde{K}(S^{2m})$ by exactness, the product $\alpha\beta$ maps to zero as well. This implies that $\alpha\beta = c\alpha$ for some integer c. This gives us:

$$\alpha\beta = c\alpha \Rightarrow \alpha\beta^2 = c\alpha\beta \Rightarrow \alpha(h\alpha) = c\alpha\beta \Rightarrow h\alpha^2 = c\alpha\beta \to c\alpha\beta = 0$$

for $\alpha^2 = 0$. Because $\alpha\beta$ belongs to the image of the infinite cyclic group $\tilde{K}(S^{4m})$ in $\tilde{K}(C_f)$, the equation $c\alpha\beta = 0$ implies that $\alpha\beta = 0$. It follows that $\beta'^2 = 2m\alpha\beta$ is zero as well. $\square$

Let $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ be a map and let $D^n = \{x \in \mathbf{R}^n \mid ||x|| \le 1\}$ be unit disk in $\mathbf{R}^n$. Its boundary ∂D^n is nothing but the sphere S^{n-1}. Let D^n_+ be the points of D^n whose last coordinate is positive and D^n_+ be the points of D^n whose last coordinate is negative.
We rewrite the sphere S^{2n-1} as $\partial D^{2n} = \partial D^n \times D^n = \partial D^n \times D^n \cup D^n \times \partial D^n$, and considering the sphere S^n as the union of D^n_+ and D^n_- with their boundary identified.

Definition 4.8. We define the associated map $\tilde{g}$ to g as the map $\hat{g} : S^{2^n-1} \to S^n$ given by $(x, y) \mapsto ||y|| \cdot g(x, y/||y||) \in D^n_+$ on $\partial D^n \times D^n$, and by $(x, y) \mapsto ||x|| \cdot g(x/||x||, y) \in D^n_-$ on $D^n \times \partial D^n$.

The map $\hat{g}$ is then well-defined and continuous, even for $||x|| = 0$ or $||y|| = 0$, on which $\hat{g}$ simply takes the value zero, and it is trivially verified that its values coincide with the ones of g on $S^{n-1} \times S^{n-1}$.
We can state the following lemma, whose proof we will omit:

Lemma 4.4. *([9, Lemma 2.18]) Consider an H-space multiplication $g : S^{2m-1} \times S^{2m-1} \to S^{2m-1}$. Then the map associated $\hat{g} : S^{4m-1} \to S^{2m}$ has Hopf's invariant ± 1.*

Our next step in the proof of Theorem 4.2 is showing that if there exists a map $f : S^{4m-1} \to S^{2m}$ with Hopf invariant ± 1, then $m = 1, 2$ or 4. To do so, we will have to introduce some useful and remarkable ring homomorphisms called the Adams operations:

Theorem 4.5. *Let X be a compact Hausdorff space and $k \ge 0$ an integer. There exist ring homomorphisms $\psi^k : K(X) \to K(X)$ satisfying the following properties:*

1. *For all maps $f : X \to Y$, $\psi^k f^* = f^* \psi^k$. (naturality)*
2. *For L a line bundle, $\psi^k(L) = L^k$.*
3. *$\psi^k \circ \psi^l = \psi^{kl}$.*
4. *For p prime, $\psi^p(\alpha) \equiv \alpha^p \pmod p$.*

Proof. Let (E, p, X) and (E', p', X) be vector bundles over the same compact Hausdorff base X. To be able to give such homomorphisms, we will have to make use of the exterior power of vector bundles and its properties in Proposition 3.6.
Define a polynomial λ_t such that $\lambda_t(E) = \sum_i \lambda^i(E)t^i \in K(X)$. By Property (iv) of Proposition 3.6, this sum is finite and using Property (i) and (iv), we also get that $\lambda_t(E \oplus E') = \lambda_t(E)\lambda_t(E')$.
Consider now the direct sum of n line bundles $L_1 \oplus ... \oplus L_n$ with n a positive integer. By Property 1 above we want to construct maps ψ^k such that $\psi^k(L_1 \oplus ... \oplus L_n) = L_1^k + ... + L_n^k$. Using the fact that $\lambda_t(E \oplus E') = \lambda_t(E)\lambda_t(E')$ as well as properties (ii), (iii) and (iv) of exterior powers, we now get $\lambda_t(L_1 \oplus ... \oplus L_n) = \prod_{i=1}^n (1 + L_i t)$. For polynomials of the form $\prod_{i=1}^n (1 + L_i t)$, the j-th coefficient of t^j is given by the j-th elementary symmetric functions σ_j of the L_i's, hence:

$$\lambda^j(L_1 \oplus ... \oplus L_n) = \sigma_j(L_1, .., L_n)$$

Now, by the fundamental theorem of symmetric polynomial, every symmetric polynomial in the variables $x_1, ..., x_n$ can be written as a unique polynomial in the elementary symmetric functions $\sigma_1, ..., \sigma_k$. In particular the polynomial $x_1^k + ... + x_n^k$ is given by a polynomial $s_k(\sigma_1, ..., \sigma_k)$. s_k is called the Newton polynomial, given recursively by:

$$s_k = \sum_{i=1}^{k-1} ((-1)^{i-1} \sigma_i s_{k-i}) + (-1)^{k-1} k \sigma_k$$

Define now ψ^k as $\psi^k(E) = s_k(\lambda^1(E), ..., \lambda^n(E))$. Then we have indeed that:

$$\psi^k(L_1 \oplus ... \oplus L_n) = s_k(\sigma_1(L_1, .., L_n), ..., \sigma_k(L_1, ..., L_n)) = L_1^k + ... + L_n^k$$

and so we constructed an operation with the desired property on $L_1 \oplus ... \oplus L_n$. Indeed, with this definition of ψ^k, we have that the first property $\psi^k f^* = f^* \psi^k$ follows from the fact that $\tilde{f}^*(\lambda^i(E)) = \lambda^i(\tilde{f}^*(E))$.

For the other properties, an additive property $\psi^k(E_1 \oplus E_2) = \psi^k(E_1) + \psi^k(E_2)$ of the ψ^k on vector bundles follows by the splitting principle (Theorem 3.15.) when we first pullback to split E_1 and then pullback to split E_2. This additive property of ψ^k on vector bundles then induces an additive operation on $K(X)$ by defining $\psi^k(E_1 - E_1') = \psi^k(E_1) - \psi^k(E_1')$. Indeed, if we take $E_1 - E_1' = E_3 - E_3'$ with E_1, E_1', E_3, E_3' total spaces of vector bundles, we have that $E_1 \oplus E_3' \approx_S E_1' \oplus E_3$, which means that $\psi^k(E_1) + \psi^k(E_3') = \psi^k(E_1') + \psi^k(E_3)$ which implies that $\psi^k(E_1) - \psi^k(E_1') = \psi^k(E_3) - \psi^k(E_3')$ making this well-defined.

A multiplicative property can also be deduced from the splitting principle. Indeed, if E is a sum of line bundle L_i and E' a sum of line bundles E_j', with i, j some indices, then $E \otimes E'$ is a sum of line bundles $L_i \otimes L_j'$. Hence we get:

$$\psi^k(E \otimes E') = \sum_{i,j} \psi^k(L_i \otimes L_j') = \sum_{i,j} (L_i \otimes L_j')^k = \sum_{i,j} L_i^k \otimes (L_j)'^k$$

$$= \sum_i L_i^k \sum_j (L_j')^k = \psi^k(E)\psi^k(E')$$

which gives us a multiplication on vector bundles. It then also induces a multiplication on elements of $K(X)$ in a similar way as for the additive property.

Using the splitting principle and the additive property, we can now deduce Property (3), since for a line bundle L, we get:

$$\psi^k(\psi^l(L)) = \psi^k(L^l) = L^{kl} = \psi^{kl}(L)$$

In a similar way, for $E = L_1 \oplus ... \oplus L_n$ with the L_i's, $i = 1, ..., n$ being line bundles, we have that for p a prime number:

$$\psi^p(E) = L_1^p + ... + L_n^p \equiv (L_1 + ... + L_n)^p \pmod p \equiv E^p \pmod p$$

which then gives us Property (4) by the splitting principle and the additive property. □

Proposition 4.6. *Let $k \geq 0$. The map $\psi^k : \tilde{K}(S^{2m}) \to \tilde{K}(S^{2m})$ corresponds to the multiplication with k^m*

Proof. Let $m = 1$ and take u the generator of $\tilde{K}(S^2) = \mathbf{Z} = {}^{\mathbf{Z}[u]}\!/_{(u^2)}$ given by $u = H - 1$. with H the tautological line bundle over $\mathbf{CP}^1 = S^2$. Then, using Property 2, we get:

$$\psi^k(u) = \psi^k(H - 1) = H^k - 1 = (u + 1)^k - 1 = 1 + ku - 1 = ku$$

because $u^k = 0$ for $k \geq 1$. Since ψ^k is additive, this gives us that $\psi^k : \tilde{K}(S^2) \to \tilde{K}(S^2)$ indeed corresponds to the multiplication by k.
For $m \geq 2$, we know that $\tilde{K}(S^{2m}) \approx \mathbf{Z} \approx {}^{\mathbf{Z}[\beta]}\!/_{(\beta^2)}$ for some generator β. Consider now the map $\tilde{K}(S^{2m}) \to \tilde{K}(S^2 \times .... \times S^2)$. Since, for X a topological space the map $f : S^2X \to SX$ is surjective, we then get that the induced map $f^* : \tilde{K}(SX) \to \tilde{K}(S^2X)$ is injective because

$F : X \to \tilde{K}(X)$ is a contravariant functor (see [12, Section 9.3.] for more information). This turns the map $\tilde{K}(S^{2m}) \to \tilde{K}(S^2 \times .... \times S^2)$ into a monomorphism. The image of β under this morphism is then given by a product $\alpha_1...\alpha_m$ with α_i is a generator of $\tilde{K}(S^2)$ for every $i = 1, ..., m$. Hence, the image of $\psi^k(\beta)$ through the morphism is given by $\psi^k(\alpha_1)...\psi^k(\alpha_m) = k^m\alpha_1...\alpha_m$ which is k^m times the image of β. Because the map $\tilde{K}(S^{2m}) \to \tilde{K}(S^2 \times .... \times S^2)$ is injective, this gives indeed that $\psi^k : \tilde{K}(S^{2m}) \to \tilde{K}(S^{2m})$ corresponds the multiplication with k^m. □

We are now able to give a proof of the following desired theorem:

Theorem 4.7. *If there exists a map $f : S^{4m-1} \to S^{2m}$ with Hopf invariant ± 1, then $m = 1, 2$ or 4.*

Proof. Take α, β elements of $\tilde{K}(C_f)$ as in the definition of the Hopf invariant for $f : S^{4m-1} \to S^{2m}$. Because α is the image of a generator of $\tilde{K}(S^{4m})$, we have that $\psi^k(\alpha) = k^{2m}\alpha$. Moreover the definition of β gives us that $\psi^k(\beta) = k^m\beta + \mu_k\alpha$ with μ_k an integer. Hence,

$$\psi^k(\psi^l(\beta)) = \psi^k(l^m\beta + \mu_l\alpha) = k^m l^m\beta + l^m\mu_k\alpha + k^{2m}\mu_l\alpha$$

and

$$\psi^l(\psi^k(\beta)) = \psi^l(k^m\beta + \mu_k\alpha) = l^m k^m\beta + k^m\mu_l\alpha + l^{2m}\mu_k\alpha$$

Using $\psi^k\psi^l = \psi^l\psi^k$, we thus get that:

$$l^m\mu_k\alpha + k^{2m}\mu_l\alpha = k^m\mu_l\alpha + l^{2m}\mu_k\alpha$$

$$\Rightarrow l^m\mu_k + k^{2m}\mu_l = k^m\mu_l + l^{2m}\mu_k$$

$$\Rightarrow (k^{2m} - k^m)\mu_l = (l^{2m} - l^m)\mu_k$$

Moreover, $\psi^k(\beta^2) \equiv \beta^2 \bmod 2$, by Property 4. of Theorem 4.6. Since $\psi^2(\beta) = 2^m\beta + \mu_2\alpha$, the equation $\beta^2 \equiv \mu_2\alpha \bmod 2$ holds. Because $\beta^2 = h\alpha$ with h the Hopf invariant, we have $h\alpha \equiv \mu_2\alpha \bmod 2$ which implies that $\mu_2 \equiv h \bmod 2$. By Lemma 4.4, $h = \pm 1$, therefore, μ_2 is a odd number. Besides, the equation $(k^{2m} - k^m)\mu_l = (l^{2m} - l^m)\mu_k$ for $k = 2$ and $l = 3$ gives us $(2^{2m} - 2^m)\mu_3 = (3^{2m} - 3^m)\mu_2 \Rightarrow 2^m(2^m - 1)\mu_3 = 3^m(3^m - 1)\mu_2 \Rightarrow 2^n | 3^m(3^m - 1)\mu_2$. This implies that 2^m has to divide $3^m - 1$ because both 3^m and μ_2 odd.

To complete the proof, we will need one last lemma from number theory:

Lemma 4.8. *Suppose that 2^m divides $3^m - 1$. Then $m = 1, 2$ or 4.*

Proof of the lemma. Write m as $m = 2^l k$, with l, k positive integers and k odd. Our aim is to find the highest power of two that divides $3^m - 1$. We proceed by induction on l:
Let $l = 0$. Then $m = k$. Since $3 \equiv -1 \bmod 4$, we get that $3^k \equiv -1 \bmod 4$ because k is odd. Thus $3^m - 1 = 3^k - 1 \equiv 2 \bmod 4$ and the highest power of 2 diving $3^m - 1$ in this case is 2.

Take now $l = 1$. Then $m = 2k$ and $3^m - 1 = 3^{2k} - 1 = (3^k - 1)(3^k + 1)$. We have already proven that the highest power of 2 dividing the first factor is 2. For the second factor, we start by noticing that $3^2 \equiv 1 \bmod 8$ meaning that $3^k \equiv 3 \bmod 8$ since k is odd. Hence, $3^k + 1 \equiv 4 \bmod 8$ so the highest power of 2 dividing $3^k + 1$ is 4. Together, we have that the highest power of 2 dividing $3^m - 1 = (3^k - 1)(3^k + 1)$ is thus 8.

For $l \geq 1$, we see that if we pass from l to $l + 1$, we then pass from m to $2m$ as m is equal to $2^l k$. Write now $3^{2m} - 1$ as the product $(3^m - 1)(3^m + 1)$. Since $l \geq 1$, we now have that m which means that $3^m \equiv 1 \bmod 4$, thus $3^m + 1 \equiv 2 \bmod 4$.
Hence the highest power of 2 dividing $3^{2m} - 1 = (3^m - 1)(3^m + 1)$ is equal to 2^{l+2} as it is two times the highest power of two dividing $3^m - 1$ which is 2^l by the induction hypothesis.
In conclusion, the highest power of two dividing $3^m - 1$ is 2 when $l = 0$ and 2^{l+2} if $l > 0$. So

if 2^m divides $3^m - 1$, then $m \leq l + 2$, which implies that $2^l \leq 2^l k = m \leq l + 2$. This is only possible when $l \leq 2$ and $m \leq 4$. Thus the only possibilities for m is $1, 2, 3$ or 4.
Because 2 divides 2 ($m = 1$), 4 divides 8 ($m = 2$) and 16 divides 80 ($m = 4$) but 8 does not divide 26 ($m = 3$), we get at then end that m can only be either 1, 2 or 4.

□

Hence, we have that the integer m has to be either 1, 2 or 4 which means that for $n \geq 2$, $n = 2m$ can only take the values 2, 4 or 8. As we have already dealt with the case $n = 1$ for division algebras in the first section, we finally get that the dimension of a real division algebra can only be either $1, 2, 4$ or 8 and they correspond to the spaces **R**, **C**, **H** and **O** as seen is Section 1.

Conclusion

After having defined the notion of real division algebra and in particular introduced the examples of the Hamilton quaternions and the Cayley octonions, we have proven Hopf's theorem, which implies that the dimension of any division algebra over the real numbers must be a power of two. The proof uses homology and cohomology with coefficients in $\mathbf{Z}_2$ that we introduced before.

With the help of vector bundles complex K-theory, we were eventually able to define the Hopf invariant and the Adams operations that we then used to give a proof that those power of two could actually only be either 1, 2, 4 or 8, which tells us in particular that there exists no more finite dimensional real division algebra beyond the Cayley octonions.

Once Hopf's theorem is proven, there can be another way to prove the 1,2,4,8 theorem which involves characteristic classes of vector bundles, in particular Stiefel-Whitney classes, instead of K-theory. The British Mathematician J.F. Adams, for instance, proved the theorem first using characteristic classes in [1] and later with the use of K-theory [2].

It is also possible to prove the theorem using methods from K-theory without stating first that the dimension has to be a power of two. However, in this case, one still has to prove at some point that the dimension of a real division algebra has to be at least even, when greater than one. In this thesis, the choice was made to incorporate Hopf's theorem in order to give a better look at the historical steps that were made in the solving of this problem.

An equivalent statement can be given about normed algebras over the real numbers (see [6] and [16]), the proof of it being done using this time analysis and operator theory instead of algebraic topology.

Appendix

Singular homology

Definition. Let $v_0, ..., v_p$ be points in $\mathbf{R}^n$ with n positive integer. We define their convex hull as the following set:

$$\left\{ \begin{array}{c} [v_0, ..., v_p] = \\ \sum_{i=0}^{p} \lambda_i v_i \mid \lambda_i \geq 0, \sum_{i=0}^{p} \lambda_i = 1 \subset \mathbf{R}^n \end{array} \right\}$$

Take now the standard basis $(e_0, ..., e_p)$ of $\mathbf{R}^{p+1}$:

Definition. We define the standard p-simplex as the space:

$$\Delta^p = [e_0, ..., e_p]$$

Definition. Let $v_0, ..., v_p$ be points in $\mathbf{R}^n$. We can define the following continuous map between complex hulls:

$$< v_0, ..., v_p >: \Delta^p \to [v_0, ..., v_p], \quad \sum_{i=0}^{p} \lambda_i e_i \mapsto \sum_{i=0}^{p} \lambda_i v_i$$

The previous map is an example of a so-called singular p-simplex in $\mathbf{R}^n$. For X a general topological space, a singular p-simplex in X is defined as follows:

Definition. Let X be a topological space and p a positive integer. A singular p-simplex in X is a continuous map $\sigma : \Delta^p \to X$.

Notation. We denote by $\Sigma_p(X)$ the set of p-simplices in X.

Definition. Let X be a topological space and p a positive integer. We define the group of singular p-chains $C_p(X)$ as the free abelian group on $\Sigma_p(X)$:

$$C_p(X) = \mathbf{Z}^{\Sigma_p(X)}.$$

Definition. Let $p \geq 1$. We then can define a group homomorphism

$$\partial : C_p(X) \to C_{p-1}(X), \quad \sigma \mapsto \sum_{i=0}^{p} (-1)^i \sigma \circ < e_0, ..., \hat{e_i}, ..., e_p >$$

where $< e_0, ..., \hat{e_i}, ..., e_p >: \Delta^{p-1} \Delta^p$ is the map defined by $e_0 \mapsto e_0, ..., e_{i-1} \mapsto e_{i-1}, e_i \mapsto e_{i+1}, ..., e_{p-1} \mapsto e_p$

This map has the following property:

Proposition. *Let ∂ be defined as above. Then $\partial \circ \partial = 0$*

This proposition can be easily verified by writing out the ∂'s.

Definition. Let $\sigma \in C_p(X)$ be a singular p-chain. We say that σ is a cycle if $\partial \sigma = 0$.

Using this map, we are now able to define the p-singular homology group of a topological space:

Definition. Let X be a topological space and $p \geq 1$. Define

$$Z_p(X) = \mathrm{Ker}(\partial : C_p(X) \to C_{p-1}(X)) \text{ and } B_p(X) = \mathrm{Im}(\partial : C_{p+1}(X) \to C_p(X))$$

. Then the singular homology group of X is given by:

$$H_p(X) = {}^{Z_p(X)}\!/\!_{B_p(X)}$$

This quotient is well-defined since the previous proposition gives us that $B_p(X) \subseteq Z_p(X)$

Definition. Let X be a topological space and G an Abelian group. We define the singular p-cochains with coefficients in G $C^p(X, G)$ as the dual group $\mathrm{Hom}(C_p(X), G)$ of the p-singular chain group $C_p(X)$).

Definition. The coboundary map $\delta : C^p(X, G) \to C^{p+1}(X, G)$ is given by the dual of the map $\partial : C_{p+1}(X) \to C_p(X)$. Thus, for a map $\varphi \in C^p(X, G)$, its coboundary φ is given by the composition:

$$C_{p+1}(X) \xrightarrow{\partial} C_p(X) \xrightarrow{\varphi} G$$

For a $p+1$-simplex $\sigma : \Delta^{p+1} \to X$, we thus get that:

$$\delta\varphi(\sigma) = \sum_{i=0}^{p+1} \varphi(\sigma_{|[v_1,\ldots,\hat{vi},\ldots,v_{p+1}]})$$

Because $\partial \circ \partial = 0$ and δ is the dual of ∂, we also have that $\delta \circ \delta$. Hence, we can define a new group in the same way as we defined the homology group:

Definition. Let X be a topological space and G an Abelian group. Define

$$Z^p(X) = \mathrm{Ker}(\delta : C^p(X) \to C^{p+1}(X)) \text{ and } B^p(X) = \mathrm{Im}(\delta : C^{p-1}(X) \to C^p(X))$$

. Then the singular cohomology group of X with coefficients in G is given by:

$$H^p(X, G) = {}^{Z^p(X)}\!/\!_{B^p(X)}$$

We can now give the definition of the cap product $\frown$, which is for instance used in the Poincaré duality theorem:

Definition. Let X be a topological space and R a coefficient ring. Let k, l be positive integers with $k \geq l$. We can define an R-bilinear cap product:

$$\frown : C_k(X, R) \times C^l(X, R) \to C_{k-l}(X, R), \quad (\sigma, \varphi) \mapsto \varphi(\sigma_{|[v_0,\ldots,v_l]})\sigma_{|[v_l,\ldots,v_k]},$$

where $\sigma : \Delta^k \to X$ and $\varphi \in C^l(X, R)$.

This cap product induces a cap product in homology and cohomology by using the formula $\partial(\sigma \frown \varphi) = (-1)^l(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$ which can be checked by writing out the different terms. Because $\partial(\sigma \frown \varphi) = \pm(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$, we have that the cap product of a chain and a cochain is again a chain.
Moreover, if $\partial\sigma = 0$, then $\partial(\sigma \frown \varphi) = \pm\sigma \frown \delta\varphi$ and if $\delta\varphi = 0$, then $\partial(\sigma \frown \varphi) = \pm\partial\sigma \frown \varphi$, hence the cap product of a cycle and a coboundary is a boundary and the cap product of a boundary and a cocycle is again a boundary.
This yields an induced cap product

$$H_k(X, R) \times H^l(X, R) \to H_{k-l}(X, R),$$

which is R-linear in each variable.

Topological manifolds

Definition. Let X be a topological space, n a positive integer and I an index set. An n-dimensional atlas for X is a set:

$$\mathcal{A} = \{(U_i, h_i, V_i) \mid mi \in I\}$$

such that for every $i \in I$:

- U_i is an open subset of X and $X = \bigcup_{i \in I} U_i$
- V_i is an open subset of $\mathbf{R}^n$
- $h_i : U_i \to V_i$ is a homeomorphism.

This enables us to give the definition of a topological n-dimensional manifold.

Definition. Let X be a topological space. An n-dimensional topological manifold is a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is an n-dimensional atlas for X.

In this thesis, since we only deal with compact connected topological spaces, we will define orientable topological manifolds as follows:

Definition. Let M be an n-dimensional topological manifold. We say that M is orientable if the n-th singular homology group $H_n(M)$ is isomorphic to $\mathbf{Z}$.

This definition is usually given as a theorem for compact connected topological manifolds and is therefore not the usual definition of an oriented topological manifold, which normally involves local homology (see for instance the definition in Hatcher's book about Algebraic Topology [8]). However, it is a sufficient definition here.
Since the homology group $H_n(M)$ as in the previous definition is then given by the integers, only two orientations on the manifold can be possible, namely the one given by the negative integers and the one given by the positive ones.

Definition. Let M be a n-dimensional topological manifold. We say that M is oriented if M is an orientable manifold with a chosen orientation fixed on it.

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l.d.v.van.nifterik@umail.leidenuniv.nl